Lecture 2
NUMERICAL QUADRATURE

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Five Part Lecture Series

Picard-Chebyshev Numerical Integration Applications in Astrodynamics

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Spring 2017

FIVE PART LECTURE SERIES

Lecture	Title	Presenter
1	**Orthogonal Approximation**	Junkins
2	**Numerical Quadrature**	Junkins
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NEWTON-COTES AND GAUSSIAN QUADRATURE

- **Newton-Cotes Formula** for approximating Integral has the form of a weighted average:

$$\mathscr{I}(f(x), a, b, c_1, c_2, \ldots, c_M, x_1, x_2, \ldots, x_M) \cong \int_a^b f(x)dx \cong \sum_{i=1}^{M} c_i f(x_i)$$

  - In the classical Newton-Cotes formulation, the nodes are uniformly spaced, and this rules out taking advantage of the truth that higher accuracy can be obtained via judicious use of non-uniform nodes.
  - Note, there are $2M$ unknowns $(c_1, c_2, \ldots, c_M, x_1, x_2, \ldots, x_M)$, and the resulting equations for a given $f(x)$ contain the $(x_1, x_2, \ldots, x_M)$ non linearily, so in general, we may anticipate solving a set of $2M$ coupled nonlinear algebraic equations.

- **Gaussian Quadrature** … is a special case of the Newton-Cotes approach wherein, we select judicious nodes ("quadrature points") chosen such that a polynomial of a given degree is integrated exactly by the Newton-Cotes formula. This will lead to non-uniform nodes for each degree polynomial.
  - Other intervals than $[a,b]$ can be used, and we can subdivide the given interval into many smaller intervals in order to improve accuracy, as needed.
  - Usually, the theoretical developments are done on the interval [-1,1], and of course, we can always transform $[a,b]$ onto [-1,1] with a linear variable change.

GAUSSIAN QUADRATURE

Consider the Newton-Cotes formula

$$\mathscr{I}(f(x),-1,1,c_1,c_2,\ldots,c_M,x_1,x_2,\ldots,x_M) \cong \int_{-1}^{1} f(x)dx \cong \sum_{i=1}^{M} c_i f(x_i)$$

Let us set M=2, and seek to determine the four unknowns (c_1, c_2, x_1, x_2) such that the Newton-Cotes formula is exact for the four polynomials:

$$(1)\ \ f(x)=1; \quad (2)\ \ f(x)=x; \quad (3)\ \ f(x)=x^2; \quad (4)\ \ f(x)=x^3$$

Note that these four requirements will give us four equations in four unknowns:

$$(1)\ \ f(x)=1 \Rightarrow \int_{-1}^{1} 1dx = 2 \qquad \Rightarrow \qquad c_1 + c_2 = 2$$

$$(2)\ \ f(x)=x \ \Rightarrow \int_{-1}^{1} xdx = 0 \qquad \Rightarrow \qquad c_1x_1 + c_2x_2 = 0$$

$$(3)\ \ f(x)=x^2 \Rightarrow \int_{-1}^{1} x^2dx = \frac{2}{3} \qquad \Rightarrow \qquad c_1x_1^2 + c_2x_2^2 = \frac{2}{3}$$

$$(4)\ \ f(x)=x^3 \Rightarrow \int_{-1}^{1} x^3dx = 0 \qquad \Rightarrow \qquad c_1x_1^3 + c_2x_2^3 = 0$$

These equations are satisfied by $c_1 = 1, c_2 = 1, x_1 = -\frac{1}{\sqrt{3}}, x_2 = \frac{1}{\sqrt{3}}$, so this **Gaussian Quadrature formula exactly integrates a cubic polynomial over the interval** [-1,1] :

$$\Rightarrow \ \ \mathscr{I}(f(x)) = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$$

Likewise a Gaussian Quadrature formula that exactly integrates a quantic polynomial over the interval [-1,1] can be shown to be:

$$\Rightarrow \ \ \mathscr{I}(f(x)) = \frac{5}{9} f(-\sqrt{\frac{3}{5}}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{\frac{3}{5}})$$

PATH APPROXIMATION VS NUMERICAL QUADRATURE

In a significant family of problems, we have a differential equation of the form

$$\dot{x} = \frac{dx}{dt} = f(t), \qquad x(t_0) = a$$

Obviously, for an integrable $f(t)$, the differential equation is fully equivalent to the integral equation

$$x(t) = a + \int_a^t f(\tau)d\tau$$

Thus we see that solving a differential equation where $f(t)$ is a general function (or an accurate approximation of a given single valued general function) is closely related to the foregoing developments on quadrature. However, it is significant that we may have a variable upper limit, rather than a fixed limit, and that instead of just a ***numerical value to approximate the integral over fixed limits***, we may desire an analytically attractive approximation of the ***trajectory*** $x(t)$ expressed in terms of "nice" algebraic functions.

Obtaining not just a numerical value of the integral (i.e., numerical quadrature), but rather an algebraic expression that accurately approximates the integral for all infinity of t values in the interval of interest is central to the path approximation methods we discuss below. Obviously, specific t values can be substituted and a path approximation formula will give numerical quadrature result over any interior subinterval as a special case.

PATH APPROXIMATION VIA CHEBYSHEV POLYNOMIAL QUADRATURE

Consider the problem posed on the previous page with the interval $[-1 \le t \le 1]$, we have a differential equation of the form

$$\dot{x} = f(t),\ x(-1) = x_0 \qquad \Rightarrow \qquad x(t) = x_0 + \int_{-1}^{t} f(\tau)d\tau$$

Suppose $f(t)$ is smooth and single-valued, but is a messy ("ugly") function not found in your standard integral tables (or is not integrable by your favorite symbol manipulator). We can approximate the integrand in an infinity of ways as a linear combination of basis functions that are easily integrated. If we do least square approximation to determine the coefficients, of the basis functions we in general must invert a matrix that may be high dimensioned if we desire high accuracy. To avoid this dilemma, we can utilize any set of orthogonal functions. One attractive set are the Chebyshev orthogonal polynomials.

So we write the approximation of the integrand and integrate term-by-term to get

$$f(\tau) = \sum_{k=0}^{N-1} a_k T_k(\tau) \quad \Rightarrow \quad x(t) = x_0 + \int_{-1}^{t} f(\tau)d\tau = x_0 + \sum_{k=0}^{N-1} a_k \int_{-1}^{t} T_k(\tau)d\tau$$

Now, we know from Lecture 1 that

$$\boldsymbol{a} = \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{Bmatrix} = \begin{bmatrix} N \times M \\ \text{Least Square} \\ \text{Operator Matrix} \end{bmatrix} \begin{Bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_M) \end{Bmatrix}, \text{ and } \int_{-1}^{t} T_k(\tau)d\tau = \frac{1}{2}\left(\frac{T_{k+1}(\tau)}{k+1} - \frac{T_{k-1}(\tau)}{k-1}\right)\Bigg|_{-1}^{t},\ k>1; \begin{cases} \int_{-1}^{t} T_0(\tau)d\tau = [T_1(\tau)]_{-1}^{t} \\ \int_{-1}^{t} T_1(\tau)d\tau = \frac{1}{4}[T_2(\tau)+1]_{-1}^{t} \end{cases}$$

PATH APPROXIMATION VIA CHEBYSHEV POLYNOMIAL QUADRATURE

As developed in Lecture 1, the resulting series for $x(t)$ can also be written as a Chebyshev polynomial

$$x(t) = x_0 + \int_{-1}^{t} f(\tau)d\tau \cong x_0 + \sum_{k=0}^{N-1} a_k \int_{-1}^{t} T_k(\tau)d\tau \equiv \sum_{k=0}^{N} \beta_k T_k(\tau)$$

The coefficient vector $\boldsymbol{\beta}(t)$ can be compactly expressed in matrix notation as

$$\boxed{\boldsymbol{\beta} = \boldsymbol{X}_0 + [P_1][A]\boldsymbol{f},}$$

where

$$\boldsymbol{X}_0^T = [x_0 \quad 0 \quad \cdots \quad 0],$$

$$\boldsymbol{a} = \left(T^{\mathrm{T}}WT\right)^{-1} T^{\mathrm{T}}W\boldsymbol{f} = VT^{\mathrm{T}}W\boldsymbol{f} = A\boldsymbol{f}$$

$$\boxed{A \equiv VT^{\mathrm{T}}W}$$

$$[L] = \begin{bmatrix} T_0 & T_1(-1) & \cdots & T_N(-1) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Indefinite Integral Operator:

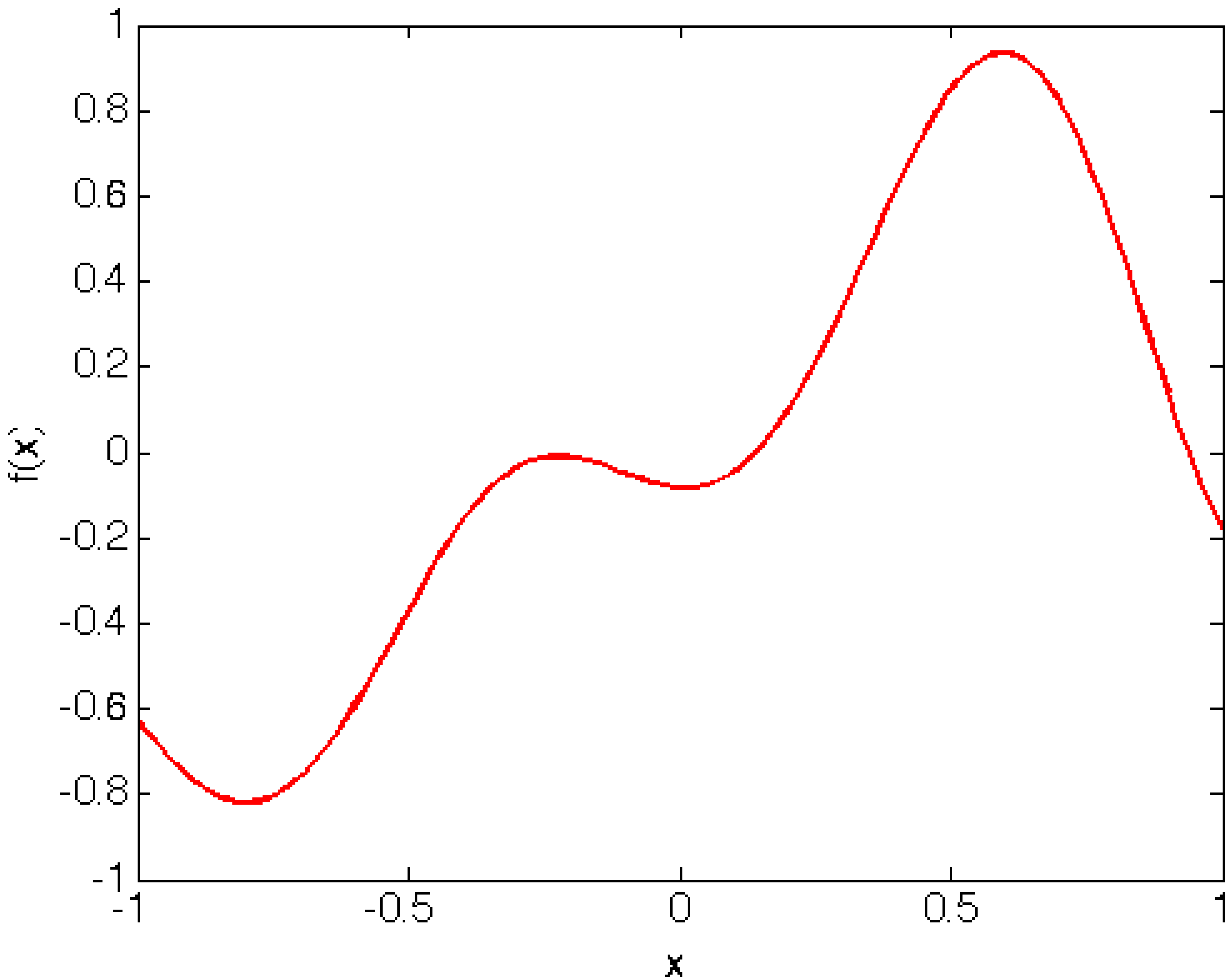
$$[S] = \boxed{\begin{bmatrix} 1/4 & 0 & & & \cdots & & 0 \\ 1 & 0 & -1/2 & 0 & & & \\ 0 & 1/4 & 0 & -1/4 & 0 & \vdots & \vdots \\ & 0 & 1/2k & 0 & -1/2k & 0 & \\ & & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dfrac{1}{2(N-2)} & 0 & \dfrac{-1}{2(N-2)} \\ & & & & 0 & \dfrac{1}{2(N-1)} & 0 \\ 0 & & & \cdots & & 0 & \dfrac{1}{2N} \end{bmatrix}}$$

Definite Integral Operator:

$$\boxed{[P_1] = [[I]-[L]][S]}$$

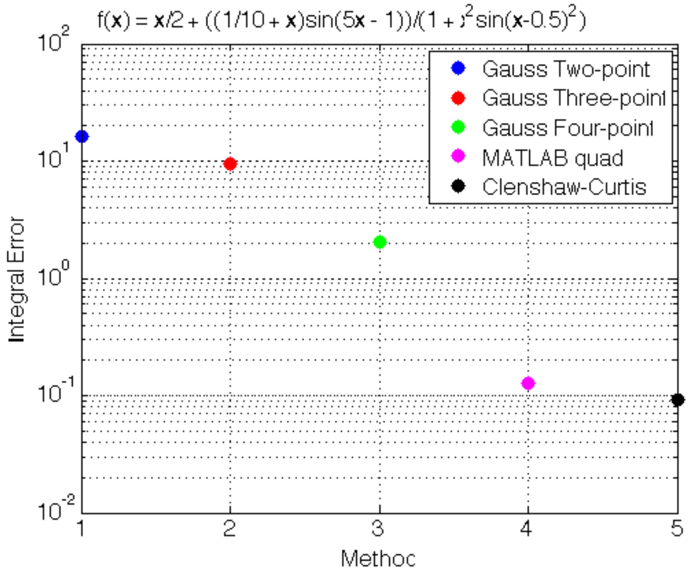
THE "UGLY" TEST FUNCTION 1

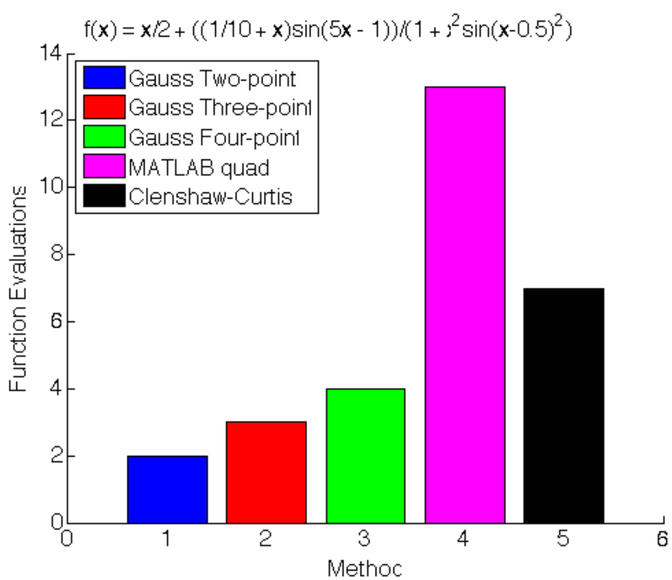
$$f(x) = \frac{x/2 + ((1/10 + x)\sin(5x+1)}{(1+x^2)\sin^2(x-1/2)}$$



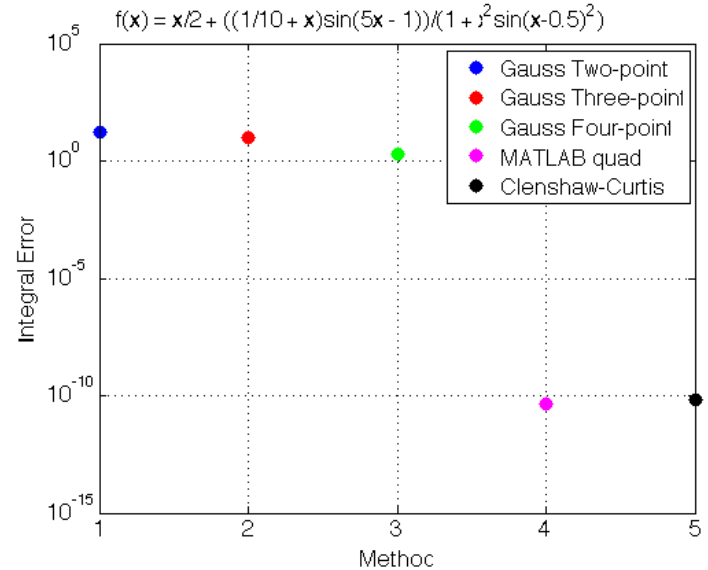
QUADRATURE COMPARISON 1 – LOW AND HIGH ACCURACY

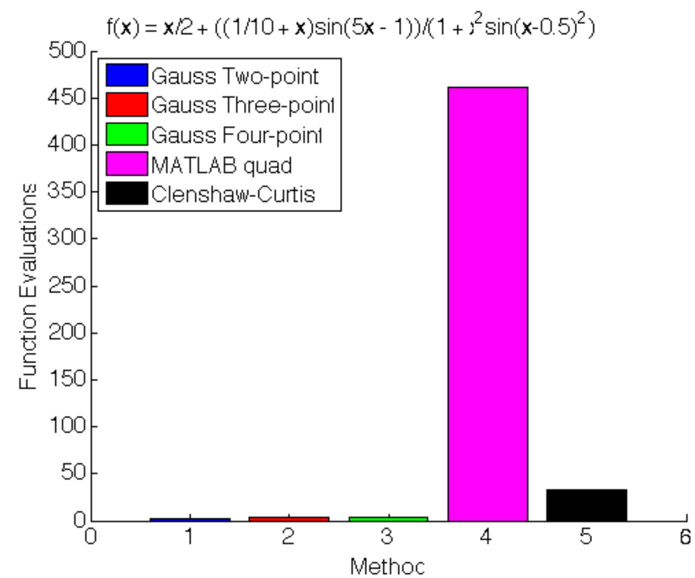
Low Accuracy





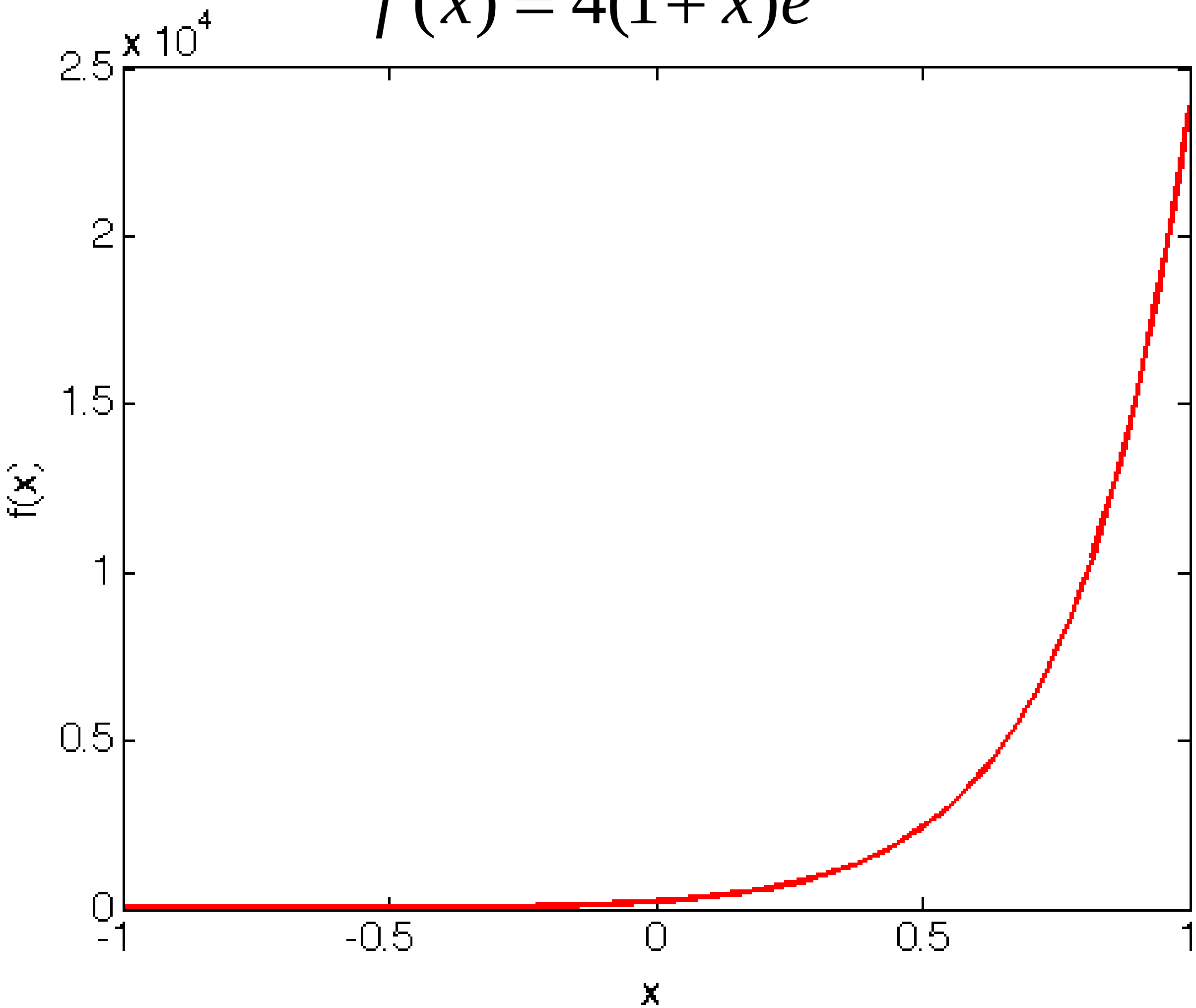
High Accuracy





QUADRATURE COMPARISON 2: EASIER FUNCTION

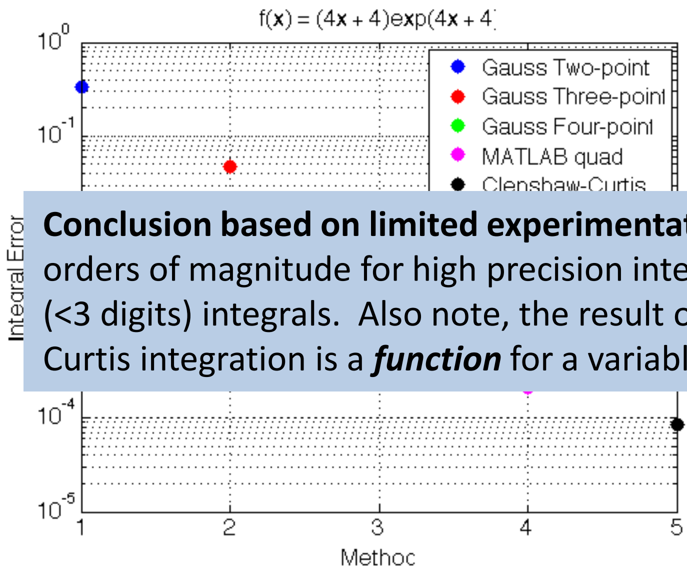
$$f(x) = 4(1+x)e^{4(1+x)}$$

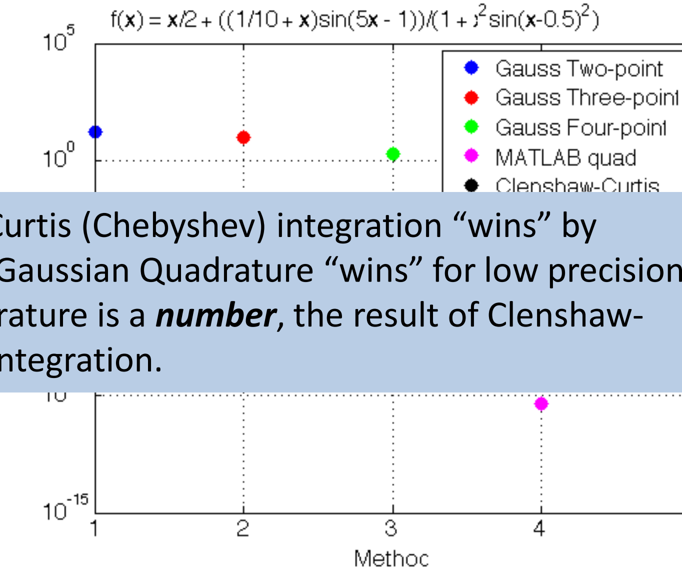


QUADRATURE COMPARISON 2 – LOW AND HIGH ACCURACY

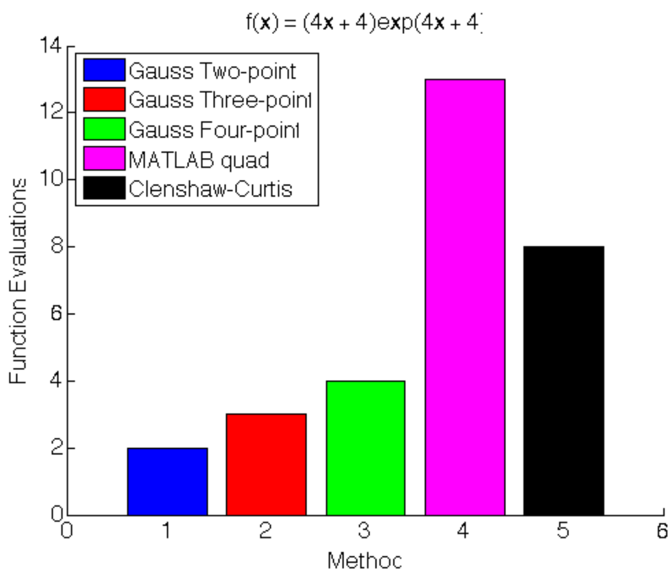
Low Accuracy

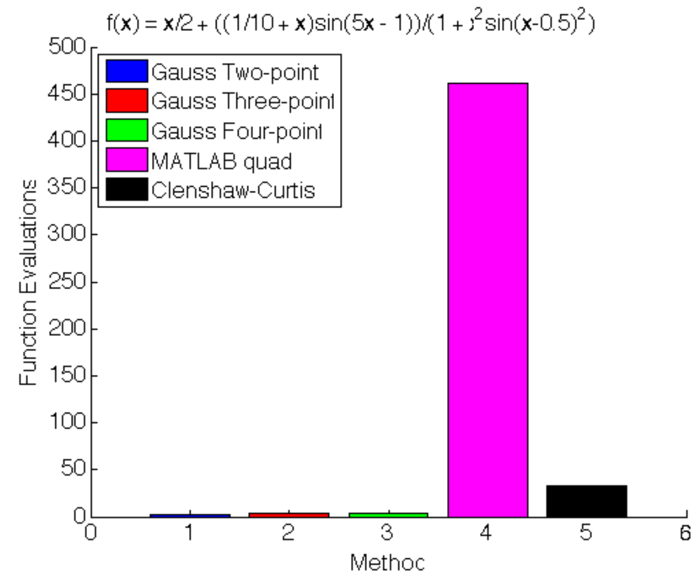
High Accuracy





Conclusion based on limited experimentation: Clenshaw-Curtis (Chebyshev) integration "wins" by orders of magnitude for high precision integrals (>6 digits), Gaussian Quadrature "wins" for low precision (<3 digits) integrals. Also note, the result of Gaussian Quadrature is a ***number***, the result of Clenshaw-Curtis integration is a ***function*** for a variable upper limit of integration.





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