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Quantitative Engineering Analysis III
12/6/24

Bottle Project Lab Report

I. Please include a picture of your bottle in your lab report.

II. Please fill in the following table of values, which should be included in your lab report:

Quantity name Value (units)	Value	(Unit)
Density of Water	1000.0	Kilogram/Meters^3
Density of Air	1.225	Kilogram/Meters^3
Heat Capacity Ratio of Air	1.4	
Ambient Pressure	101325	Pascals
Neck Diameter	0.01730	Meters
Neck Length	0.06930	Meters
Empty Bottle Mass	0.91000	Kilograms
Filled Bottle Mass	1.68900	Kilograms
Cross-Sectional Area of Bottle Neck	0.00024	Meters^2
Neck Volume	0.0000163	Meters^3
Neck Air Mass	0.0000200	Kilograms
Bottle Volume	0.000779	Meters^3
Empty Cavity Volume	0.000763	Meters^3

III. Separate from your lab report, make sure to upload a spreadsheet file of the raw data that you collected (mass and frequency measurements), as well as the MATLAB script that you wrote to analyze the data.

Uploaded in drive as well as submitted with assignment!

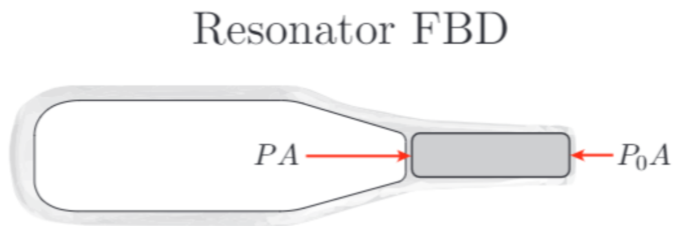
Raw Data Spreadsheet:
https://docs.google.com/spreadsheets/d/19q4bccu9adkUn4ZV6qbAOGRMLboOQGyA/edit?usp=sharing&ouid=102994437450710690259&rtpof=true&sd=true

Matlab Script:
https://drive.google.com/file/d/1zlj3-9DG6qsHWdh5jvu9LS8HqUUAW2CQ/view?usp=sharing

IV. Please include a summary of your derivation in which you applied Newton's second law to generate the nonlinear differential equation, linearized the system to find the equivalent mass-spring system (for small displacements),

and the used the effective mass and stiffness of the linearized system to compute the resonant frequency.

The first step of the derivation is to apply Newton's second law to generate the nonlinear differential equation. This is done by analyzing the forces that occur within the system.



We can see that air in the neck of the bottle has force applied on it both from the inside of the bottle (Pressure Inside Cavity x Cross Sectional Area of Neck) and exterior of the bottle (Atmospheric Pressure x Cross Sectional Area of Neck).

Therefore, the governing equation for our net total force is:

$$F_{net} = P(t)A - P_0A$$
$$P(t) = pressure\ in\ cavity$$
$$P_0 = atmospheric\ pressure$$
$$A = cross\ sectional\ area\ of\ neck$$

Using Newton's Second Law, F = ma, we can substitute the net force for the mass and acceleration of the air inside the bottle neck.

$$ma = P(t)A - P_0A$$

Next, while A and P_0 are given constants, $P(t)$ is a variable function dependent on external variables. As given in equation 24.9, $P(t)$ is:

$$P(t) = P_0(1 + \frac{A}{V_0}\Delta x(t))^{-\gamma}$$

Substituting into our equation for mass and acceleration of the neck air, we get:

$$ma = P(t)A - P_0A$$
$$ma = (P_0(1 + \frac{A}{V_0}\Delta x(t))^{-\gamma})A - P_0A$$
$$ma = P_0A((1 + \frac{A}{V_0}\Delta x(t))^{-\gamma}) - 1)$$

We can now substitute acceleration for the second derivative of position and rearrange for our full nonlinear governing differential equation:

$$m\frac{d^2\Delta x(t)}{dt^2} = P_0A((1 + \frac{A}{V_0}\Delta x(t))^{-\gamma}) - 1)$$

Now, in order to make this system easier to analyze, we want to linearize our governing equation around the equilibrium point $\Delta x(t) = 0$. To linearize, we will look at the right hand side of our equation as a function of Δx. We will take the first derivative of this function and see how it reacts to small changes around the equilibrium point.

$$f(\Delta x) = P_0A((1 + \frac{A}{V_0}\Delta x(t))^{-\gamma}) - 1)$$

$$f'(\Delta x) = -\gamma P_0A\frac{A}{V_0}(1 + \frac{A}{V_0}\Delta x(t))^{-\gamma-1}$$

We want to see how $f(\Delta x)$ reacts when a very small change (δ) is made close the the equilibrium:

$$f(\Delta x + \delta) \approx f(\Delta x) + \delta f'(\Delta x)$$

$$f(0 + \delta) \approx f(0) + \delta f'(0)$$

$$f(0) = P_0A((1 + \frac{A}{V_0}(0))^{-\gamma}) - 1) = P_0A(1 - 1) = 0$$

$$f'(0) = -\gamma P_0A\frac{A}{V_0}(1 + \frac{A}{V_0}(0))^{-\gamma-1} = -\gamma P_0A\frac{A}{V_0}(1) = -\frac{\gamma P_0A^2}{V_0}$$

$$f(0 + \delta) \approx f(0) + \delta f'(0) = 0 + \delta(-\frac{\gamma P_0A^2}{V_0}) = -\delta\frac{\gamma P_0A^2}{V_0} \approx -\Delta x\frac{\gamma P_0A^2}{V_0}$$

Finally, we can substitute this linearization back into the right side of our equation:

$P_0A((1 + \frac{A}{V_0}\Delta x(t))^{-\gamma}) - 1)$ linearized to $-\Delta x\frac{\gamma P_0A^2}{V_0}$

$$m\frac{d^2\Delta x(t)}{dt^2} = P_0A((1 + \frac{A}{V_0}\Delta x(t))^{-\gamma}) - 1)$$

$$m\frac{d^2\Delta x(t)}{dt^2} = -\Delta x\frac{\gamma P_0A^2}{V_0}$$

Lastly, we can rearrange our equation into the undamped second-order system form $m\Delta\ddot{x} + k\Delta x = 0$:

$$m\frac{d^2\Delta x(t)}{dt^2} + \frac{\gamma P_0A^2}{V_0}\Delta x = 0$$

The mass of air in the neck is equal to the product of its density and volume:

$$m = \rho \times V_{neck}$$

Assuming the neck is prismatic, its volume is the product of its length and cross-sectional area:

$$V_{neck} = Length \times Area$$

Plugging in, we see that the mass of air inside the neck is given by:

$$m = \rho V_{neck} = \rho LA$$

Therefore, we have:

$$m\frac{d^2\Delta x(t)}{dt^2} + \frac{\gamma P_0 A^2}{V_0}\Delta x = 0,$$

$$m = \rho LA$$

$$k = \frac{\gamma P_0 A^2}{V_0}$$

Lastly, we need to compute the resonant frequency, $\omega_n = \sqrt{\frac{k}{m}}$, as given in equation 24.12.

$$\omega_n = \sqrt{\frac{k}{m}}$$

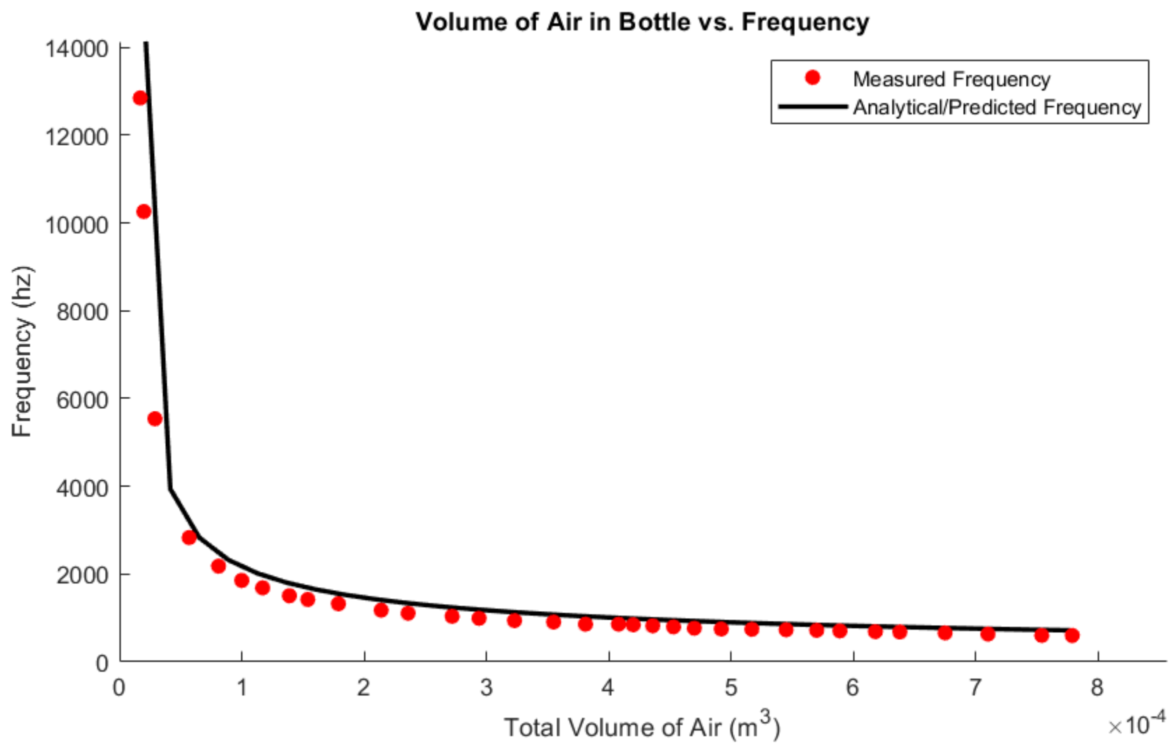
$$\omega_n = \sqrt{\frac{\gamma P_0 A^2}{V_0 \rho LA}}$$

$$\omega_n = \sqrt{\frac{\gamma P_0 A}{V_0 \rho L}}$$

With:

γ = heat capacity ratio
P_0 = ambient room pressure
A = cross sectional area of neck
V_0 = volume of air inside cavity
ρ = density of air
L = length of the bottle neck

V. Please include a plot comparing the predicted and measured values of the sound frequency as a function of the total volume of air inside the bottle. The plot should have labeled axes (with units), a descriptive title, and a legend.



VI. Please include a written description of your experimental procedure.

We began by measuring the mass of the bottle when completely empty and completely full using a precise scale. These values were recorded as baseline metrics along with the corresponding frequencies produced by the bottle in these states. Next, we proceeded to add controlled amounts of water, approximately 15–25 mL per trial, ensuring an even spread of data points. After each increment of water, we measured and recorded the frequency of the sound produced by the bottle. This process was repeated until the bottle was nearly full. All measurements, including mass, water volume, and frequency, were carefully documented for analysis.

VII. Please include a discussion of your analysis of the data. This description should include any relevant equations that you used to compute the system parameters from your measurements.

The data seems to be correlating well with the model predicted value. We attribute that to a clean set of data and well-measured system parameters. We also see the resonant frequency exponentially decay as the air inside the cavity increases. This is accurate according to our equation:

$$\omega_n = \sqrt{\frac{\gamma P_0 A}{V_0 \rho L}}$$

Relevant equations to computer system parameters include:

- $Area = \pi r^2$ to calculate the cross-section area of the neck
- $Volume = Area \times Height$ to calculate the air volume

Additionally, we used the internet to source values for the density of water, air, and the average ambient pressure in Needham, MA.

VIII. Is the Helmoholtz resonator model accurate? In which regime of air volumes is it good at predicting the resonant frequency? Where does it start to break down?

The Helmoholtz resonator model is mostly accurate, and for our model, very accurate. The model excels at low frequencies when there is an abundance of air in the bottle. However, as the air volume starts to decrease and the frequency turns exponentially higher, the model begins to lose accuracy and make worse predictions as to the frequency of the resonator.

IX. When deriving our model, are there any assumptions that we made about the system that aren't actually true? Come up with at least two potential sources of error that might explain mismatches between the measured and predicted frequencies.

When deriving our model, we made several assumptions that are not true. As a wise man once said, all models are wrong; some models are useful.

1. The assumption that all of our measured values were exactly correct could throw off our measurements. This could range from mismeasuring the neck of the bottle by a few millimeters, to having other interfering sounds in the room while taking tests. When taking measurements, our system will never behave ideally, leading to slight differences in measured and predicted frequencies.

2. The assumption that air will act as a perfect spring with no loss of energy when vibrating. This could have led to the predicted pitch to be higher than the actual pitch as the model does not take into consideration the loss of energy when vibrating.