Ex 1) Soient X, Y v.a. $\to \mathbb{R}$ $(\Omega, \mathcal{A}, \mathbb{P})$ ①

Soit $\mu \in \mathbb{R}$. M.q.

X et Y sont independantes $\Longleftrightarrow$ $X+\mu$, $Y+\mu$ le sont aussi.

Si X et Y sont ind. $\boxed{\Longrightarrow}$

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x) \cdot \mathbb{P}(Y \leq y)$$

On a donc, $\mathbb{P}(X+\mu \leq x, Y+\mu \leq y)$

$$= \mathbb{P}(X \leq x-\mu, Y \leq y-\mu) \underset{\substack{\uparrow \\ \text{ind.} \\ X,Y}}{=} \mathbb{P}(X \leq x-\mu) \cdot \mathbb{P}(Y \leq y-\mu)$$

$$= \mathbb{P}(X+\mu \leq x) \cdot \mathbb{P}(Y+\mu \leq y).$$

Si $X+\mu$ et $Y+\mu$ sont ind. $\boxed{\Longleftarrow}$

$$\mathbb{P}(X+\mu \leq x, Y+\mu \leq y) = \mathbb{P}(X \leq \overbrace{x-\mu}^{:= x'}, Y \leq \overbrace{y-\mu}^{:= y'})$$

$$= \mathbb{P}(X \leq x', Y \leq y') \underset{\substack{\uparrow \\ X+\mu,\ Y+\mu \\ \text{independantes}}}{=} \mathbb{P}(X \leq x') \cdot \mathbb{P}(Y \leq y').$$

$\square$

Ex 2)

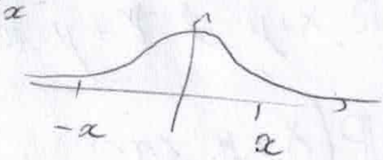
$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

$$2\cdot\phi(x) - 1 = 2\cdot\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt - 1$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-t^2/2}\, dt - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\cdot\int_{-\infty}^{x} e^{-t^2/2} - \underbrace{\int_{-\infty}^{\infty} e^{-t^2/2}\, dt}_{= \int_{-\infty}^{x} e^{-t^2/2} + \int_{x}^{\infty} e^{-t^2/2}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{x} e^{-t^2/2} - \int_{x}^{\infty} e^{-t^2/2} \right]$$



$$= \frac{1}{\sqrt{2\pi}} \left[\cancel{\underbrace{\int_{-\infty}^{-x} e^{-t^2/2}}_{= \int_{x}^{\infty} e^{-t^2/2}\, dt}} + \int_{-x}^{x} e^{-t^2/2} - \cancel{\int_{x}^{\infty} e^{-t^2/2}} \right]$$

(symetric) $= \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-t^2/2}\, dt$ $\square$

3) $X \sim N(0,1)$

③

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t \cdot e^{-t^2/2} dt = \frac{-1}{\sqrt{2\pi}} \cdot e^{-t^2/2} \Big|_{-\infty}^{\infty}$$

$$= \frac{-1}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}} = 0.$$

$$\int f' \cdot g = f \cdot g - \int f \cdot g'$$

$$E[X^2] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} t^2 \cdot e^{-t^2/2} dt$$

$$\Leftrightarrow \sqrt{2\pi} \cdot E[X^2] = \int \underbrace{t}_{g} \cdot \underbrace{t \cdot e^{-t^2/2}}_{f'} dt$$

$$= -t \cdot e^{-t^2/2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-t^2/2} \cdot 1 = -t \cdot e^{-t^2/2} \Big|_{-\infty}^{\infty} + \sqrt{2\pi}$$

$$= -\infty - (-\infty) + \sqrt{2\pi} = \sqrt{2\pi}$$

$$\Leftrightarrow E[X^2] = 1.$$

$$E[X^3] \cdot \sqrt{2\pi} = \int_{-\infty}^{\infty} \underbrace{t^2}_{g} \underbrace{t \cdot e^{-t^2/2}}_{f'} dt = -t^2 e^{-t^2/2} \Big| + \int e^{-t^2/2} \cdot 2t$$

$$= -t^2 e^{-t^2/2} \Big|_{-\infty}^{\infty} - 2 \cdot E[X] = 0.$$

Ex 6) $p = \frac{19}{37}$, $1-p = q = \frac{18}{37}$ (4)

i)

On cherche n_0 t.q. $\mathbb{P}(1000 \leq S_n) = \frac{1}{2}$ où S_n est la somme algébrique des pertes et des gains du casino sur n parties. $S_n = \sum_{k=1}^{n} X_k$ où $P(X_k = 1) = p$

$P(X_k = -1) = 1-p$.

$$E[X_k] = \sum x_i \cdot \underbrace{p_i}_{= P(X_k = x_i)} = 1 \cdot p + (-1) \cdot (1-p) = \frac{1}{37}$$

$$\sqrt{Var(X_k)} = \sigma = \sqrt{p \cdot (1-p)} = \sqrt{\frac{342}{37^2}} = \sqrt{\frac{342}{1369}} \cong \sqrt{0,25} \cong 0,5.$$

Via thm. central limite on sait que

$$\mathbb{P}(a < S_n^* < b) \underset{n \to \infty}{\simeq} \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}$$, donc en centrant et réduisant E.

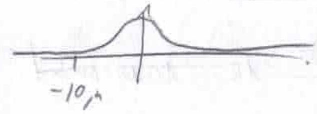
$$\mathbb{P}(1000 \leq S_n < \infty) = \mathbb{P}\left(\frac{1000 - \frac{n_0}{37}}{\sqrt{n} \cdot 0,5} \leq \underbrace{\frac{S_n - n \cdot \frac{1}{37}}{\sqrt{n} \cdot 0,5}}_{= S_n^*} \right)$$

$$= \mathbb{P}\left(\frac{1000 - \frac{n_0}{37}}{\sqrt{n_0} \cdot 0,5} \leq S_n < \infty \right) \underset{\text{thm. central limite}}{\simeq} \frac{1}{\sqrt{2\pi}} \int_{\frac{1000 - \frac{n_0}{37}}{\sqrt{n_0} \cdot 0,25}}^{\infty} e^{-x^2/2}$$

$$= \underbrace{\phi(\infty)}_{=1} - \phi\left(\frac{1000 - \frac{n_0}{37}}{\sqrt{n_0} \cdot 0,5} \right) = \frac{1}{2} \iff \phi\left(\frac{1000 - \frac{n_0}{37}}{\sqrt{n_0} \cdot 0,5} \right) = \frac{1}{2}$$

$$\Rightarrow \frac{1000 - \frac{n_0}{37}}{\sqrt{n_0} \cdot 0,5} = 0. \iff 37 \cdot 1000 = n_0 = 37'000.$$

5)



) On cherche $P(S_{n_0} < 0)$ avec $n_0 = 37\,600$.

Cela revient à $\mathbb{P}\left(-\infty < S_{n_0}^* < \underbrace{\frac{\frac{-n_0}{37}}{\sqrt{n_0}\cdot 0{,}5}}_{\simeq -10{,}40}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-10{,}4} e^{-x^2/2} = \phi(-10{,}4)$

$= 1 - \underbrace{\phi(10{,}4)}_{\approx 1} \approx 0.$

$\phi(10{,}4)$ est extrêmement proche de 1, le casino est sûr de ne pas perdre, sinon le business plan serait pas profitable, tout jeu de chance est basé sur une certitude de profit par celui qui organise.

Ex 4) $X, Y \sim N(0,1)$, définissons $Z := X + Y$

$E[Z] = E[X] + E[Y] = 0.$

$Var(Z) = Var(X) + Var(Y) = 2$, donc $\sigma = \sqrt{2}$.

$Z^* = \frac{Z - 0}{\sqrt{2}}$, ainsi via thm. central limite.

$$\mathbb{P}(a < \underbrace{Z^*}_{= \frac{X+Y}{\sqrt{2}}} < b) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx$$

Donc, Z suit une loi normale.

Ex 5) On cherche le nominal t.q.

$$\mathbb{P}(|F_n - p| > \alpha) = \beta \Rightarrow \begin{cases} x > -\alpha \Rightarrow x > -\alpha \\ x > \alpha \end{cases}$$

$|x| > \alpha$ $\alpha > 0$

cond. dominante.

$$\Leftrightarrow \mathbb{P}(|F_n - p| \leq \alpha) = 1 - \beta$$

On a $X_n \sim B(1,p)$ $\sqrt{Var(X_n)} = \sqrt{p\cdot(1-p)} = \sigma$

Et $E[X_n] = p$. Le thm. central limite dit que

$$\mathbb{P}(a < S_n^* < b) = \frac{1}{\sqrt{2\pi}} \cdot \int_a^b e^{-x^2/2} dx \quad \text{où } S_n = n \cdot F_n$$

$E[n\cdot F_n] = n \cdot \frac{p}{n}$ et $Var(n \cdot F_n) = \sigma = \sqrt{p(1-p)}$

On a donc :

$$\mathbb{P}(-\alpha < F_n - p) = \mathbb{P}(p - \alpha < F_n)$$

$$= \mathbb{P}\left(\frac{n\cdot(p-\alpha) - p}{\sqrt{p(1-p)}} < \frac{n \cdot F_n - p}{\sqrt{p(1-p)}}\right) = \frac{1}{\sqrt{2\pi}} \cdot \int_{\frac{n(p-\alpha)-p}{\sqrt{p(1-p)}}}^{\infty} e^{-x^2/2} dx = \beta$$

Avec $\beta = 0{,}05$, $\alpha = 0{,}01$,

On a : $1 - \phi\left(\frac{n(p-\alpha)-p}{\sqrt{p(1-p)}}\right) = 0{,}05 \Leftrightarrow \phi(\omega) = 0{,}95$ $0{,}9$

$$\Leftrightarrow \frac{n\cdot(p-\alpha) - p}{\sqrt{p(1-p)}} = 1{,}65 \Leftrightarrow \frac{1{,}65 \cdot (p(1-p))^{1/2} + p}{(p - 0{,}01)} = n.$$

N'ayant pas p, on ne peut pas explicitement trouver n.