Sigma Calculus

Homework solution

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1 Problem

For any two terms b and c if $b \leftrightarrow c$ then there exists such d that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$.

2 Solution

2.1 Old definitions

Firstly, let's recall Church-Rosser property and definition of $\rightarrow$ relation.

Theorem 1 (Church-Rosser property)**.** *If $a \twoheadrightarrow b$ and $a \twoheadrightarrow c$, then there exists d such that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$.*

Definition. $a \rightarrow b$ if $\exists_{a',\, b',\, C} \quad a \equiv C[a'] \wedge b \equiv C[b'] \wedge a' \rightarrowtail b'$

Later, we will use this fact, and call it Fact:

Fact. If $a \twoheadrightarrow b$ then $\exists_d a \twoheadrightarrow d \wedge b \twoheadrightarrow d$ (for $d = b$).

2.2 Useful lemma

Now, let's prove useful lemma.

Lemma. Given a and b, if there exists C, a', b' such that $a = C[a']$, $b = C[b']$ and $a' \twoheadrightarrow b'$, then $a \twoheadrightarrow b$.

Proof. Simple induction on $a' \twoheadrightarrow b'$ path.

Case 1 (Induction basis, for reflexive step)**.** Assume that $a' \twoheadrightarrow a'$, and there exists context C such that $a = C[a']$. Then obviously (from reflexivity of $\twoheadrightarrow$) $a \twoheadrightarrow a$.

Case 2 (Induction basis, for single reduction)**.** Assume that there exists C, a' and b' such that $a \equiv C[a']$, $b \equiv C[b']$ and $a' \rightarrow b'$. From definition of $\rightarrow$, there exists C', a'' and b'' such that $a' \equiv C'[a'']$, $b' \equiv C'[b'']$ and $a'' \rightarrowtail b''$. Let's define $C^\clubsuit = C[C'[-]]$ (the fact that we can define such nested context is pretty obvious and left without a proof). Thus, $C^\clubsuit[a''] \equiv a$, $C^\clubsuit[b''] \equiv b$ and $a'' \rightarrowtail b''$, so $a \rightarrow b$.

Case 3 (Induction step, for paths longer than one step)**.** Assume that there exists C, a' and b' such that $a \equiv C[a']$, $b \equiv C[b']$ and $a' \twoheadrightarrow b'$. Let's split our $a' \twoheadrightarrow b'$ into $a' \twoheadrightarrow c' \rightarrow b'$. Let's $c = C[c']$. From inductive hypothesis, $a \twoheadrightarrow c$. From anothr application of inductive hypothesis, $c \twoheadrightarrow b$. Thus, from transitivity of the relation $\twoheadrightarrow$, $a \twoheadrightarrow b$.

$\square$

2.3 Main theorem proof

Proof. Induction on derivation tree of $a \leftrightarrow b$.

Case 1 (Induction basis - **Eq Refl** rule)**.** The rule:

$$\frac{}{x \leftrightarrow x}$$

Proof goes straightforward from reflexivity of $\twoheadrightarrow$ and Fact.

Case 2 (Induction basis - **Eval Select** rule)**.** The rule:

$$\frac{}{o.l_j \leftrightarrow b_j\{|x_j \leftarrow o|\}}$$

Using the top-level invocation reduction and empty context ($C[-] = -$), we have that:

$$o.l_j \rightarrow b_j\{|x_j \leftarrow o|\}$$

From Fact, we have proven this case too.

Case 3 (Induction basis - **Eval Update** rule)**.** The rule:

$$\frac{}{o.l_j \Leftarrow \varsigma(x)b \leftrightarrow [l_j = \varsigma(x)b, l_i = \varsigma(x_i)b_i^{i \in \{1,..,n\}-\{j\}}]} \quad \text{given} \quad o \equiv [l_i = \varsigma(x_i)b_i^{i \in \{1,..,n\}}]$$

Using the top-level update reduction and empty context we have that:

$$o.l_j \twoheadrightarrow \varsigma(x)b \leftrightarrow [l_j = \varsigma(x)b, l_i = \varsigma(x_i)b_i^{i \in \{1,..,n\}-\{j\}}]$$

Again, from the Fact we can conclude that this case of induction basis holds.

Case 4 (Induction step - **Eq Trans** rule)**.** The rule:

$$\frac{a \leftrightarrow b \quad b \leftrightarrow c}{a \leftrightarrow c}$$

From inductive hypothesis we have that:

$$\exists_{d_1} a \twoheadrightarrow d_1 \wedge b \twoheadrightarrow d_1$$
$$\exists_{d_2} b \twoheadrightarrow d_2 \wedge c \twoheadrightarrow d_2$$

From Church-Russer property on d_1, d_2 and transitivity of $\twoheadrightarrow$, we have that $\exists_d a \twoheadrightarrow d \wedge c \twoheadrightarrow d$.

Case 5 (Induction step - **Eq Select** rule)**.** The rule:

$$\frac{a \leftrightarrow a'}{a.l \leftrightarrow a'.l}$$

From inductive hypothesis we have that:

$$\exists_d a \twoheadrightarrow d \land a' \twoheadrightarrow d$$

Using the context $C[-] = -.l$, relation above and Lemma, we may conclude that

$$a.l \twoheadrightarrow d.l \land a'.l \twoheadrightarrow d.l$$

Case 6 (Inductive step - **Eq Symm** rule)**.** The rule:

$$\frac{a \leftrightarrow b}{b \leftrightarrow a}$$

Obvious case, from inductive hypothesis and from the fact that $\land$ is commutative.

Case 7 (Inductive step - **Eq Update** rule)**.** The rule:

$$\frac{a \leftrightarrow a' \quad b \leftrightarrow b'}{a.l \Lleftarrow \varsigma(x)b \leftrightarrow a'.l \Lleftarrow \varsigma(x)b'}$$

From inductive hypothesis, we know that:

$$\exists_d a \twoheadrightarrow d \land a' \twoheadrightarrow d$$
$$\exists_c b \twoheadrightarrow c \land b' \twoheadrightarrow c$$

Then

$$a.l \Lleftarrow \varsigma(x)b \quad \overset{1}{\twoheadrightarrow} \quad d.l \Lleftarrow \varsigma(x)b \quad \overset{2}{\twoheadrightarrow} \quad d.l \Lleftarrow \varsigma(x)c$$

1. Using the Lemma on $a \twoheadrightarrow d$ and $C[-] = -.l \Lleftarrow \varsigma(x)b$
2. Using the Lemma on $b \twoheadrightarrow c$ and $C[-] = d.l \Lleftarrow \varsigma(x)-$

We could proceed with analogous proof that $a'.l \Lleftarrow \varsigma(x)b' \twoheadrightarrow d.l \Lleftarrow \varsigma(x)c$.

Case 8 (Inductive step - **Eq Object** rule)**.** The rule:

$$\frac{b_i \leftrightarrow b'_i \qquad \forall i \in \{1, \ldots, n\}}{[l_i = \varsigma(x_i)b_i] \leftrightarrow [l_i = \varsigma(x_i)b'_i] \qquad i \in \{1, \ldots, n\}}$$

From inductive hypothesis we know that:

$$\forall_i \exists_{c_i} b_i \twoheadrightarrow c_i \land b'_i \twoheadrightarrow c_i$$

Now, we can chain $\twoheadrightarrow$ in that way:

$$[l_i = \varsigma(x_i)b_i^{\forall 0<i<n}] \quad \overset{1}{\twoheadrightarrow} \quad [l_1 = \varsigma(x_1)c_1, l_i = \varsigma(x_i)b_i^{\forall 1<i<n}] \quad \twoheadrightarrow . \overset{2}{.} . \twoheadrightarrow \quad [l_i = \varsigma(x_i)c_i^{\forall 0<i<n}]$$

1. Using the Lemma and fact that $b_i \twoheadrightarrow c_i$, with context $C[-] = [l_1 = \varsigma(x_1)-, l_i = \varsigma(x_i)b_i^{\forall 1<i<n}]$.
2. Iteratively, we can perform similar conversion for all b_i.

$\square$