Higher Order Calculus

Homework solution

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1 Problem

Theorem 1 (Equivalence to a normal form). *If* $E \vdash A :: K$*, then* $E \vdash A \leftrightarrow A^{nf} :: K$

2 Analysis

2.1 Transformations to normal form

Let's look at the transformations, which induce normal form.

1. $X^{nf} = X$
2. $\mathrm{Top}^{nf} = \mathrm{Top}$
3. $[l_i v_i : B_i^{i \in 1..n}]^{nf} = [l_i v_i : B_i^{nf\ i \in 1..n}]$
4. $(\forall(X <: A :: K)B)^{nf} = (\forall(X <: A^{nf} :: K)B^{nf})$
5. $(\mu(X)A)^{nf} = \mu(X)A^{nf}$
6. $(\lambda(X :: K)B)^{nf} = \lambda(X :: K)B^{nf}$
7. $(B(A))^{nf} =$ if $B^{nf} \equiv \lambda(X :: K)C\{X\}$ for some X, K, C, then $(C\{A\})^{nf}$ else $B^{nf}(A^{nf})$

3 Proof

Let's imagine, that we have a tree which represents how we are transforming our A^{nf} to regular constructor. We have an assumption that $E \vdash A :: K$, so our tree exists and is finite. We will prove a theorem by induction on that tree. Transformation X is the transformation associated with the number on the list above.

Proof.

Case 1 (Induction basis - Transformation 1 & 2).
Both transformations are easy. Let's consider Transformation 1.

$$X^{nf} = X$$

We also have an assumption that

$$E \vdash X :: K$$

Thus, equivalence between X and X^{nf} is obvious (from rule **Con Eq X**).
For Transformation 2, proof looks exactly the same.

Case 2 (Induction step - Transformation 3).
If our constructor has shape $[l_i v_i : B_i^{i \in 1..n}]$, we have to consider transformation 3.

$$[l_i v_i : B_i^{i \in 1..n}]^{nf} = [l_i v_i : B_i^{nf\ i \in 1..n}]$$

We can conclude equivalence from the judgment **Con Eq Object**.

$$\dfrac{\dfrac{\text{from induction hypothesis}}{E \vdash B_i \leftrightarrow B_i^{nf}} \qquad \forall i \in 1..n}{E \vdash [l_i v_i : B_i^{i \in 1..n}] \leftrightarrow [l_i v_i : B_i^{nf\ i \in 1..n}]} \text{(Con Eq Object)}$$

Case 3 (Induction step - Transformation 4).
If our constructor has shape $\forall(X <: A :: K)B$, we have to consider transformation 4.

$$(\forall(X <: A :: K)B)^{nf} = (\forall(X <: A^{nf} :: K)B^{nf})$$

We can conclude equivalence from the judgment **Con Eq All**.

$$\dfrac{\dfrac{\text{from induction hypothesis}}{E \vdash A \leftrightarrow A^{nf} :: K} \qquad \dfrac{\text{from induction hypothesis}}{E, X <: A :: K \vdash B \leftrightarrow B^{nf}}}{E \vdash \forall(X <: A :: K)B \leftrightarrow \forall(X <: A^{nf} :: K)B^{nf}} \text{(Con Eq Object)}$$

Case 4 (Induction step - Transformation 5).
If our constructor has shape $\mu(X)A$, we have to consider transformation 5.

$$(\mu(X)A)^{nf} = \mu(X)A^{nf}$$

We can conclude equivalence from the judgment **Con Eq Rec**.

$$\dfrac{\dfrac{\text{from induction hypothesis}}{E, X \vdash B \leftrightarrow B^{nf}}}{E \vdash \mu(X)A \leftrightarrow \mu(X)A^{nf}} \text{(Con Eq Object)}$$

Case 5 (Induction step - Transformation 6).
If our constructor has shape $\lambda(X :: K)B$, we have to consider transformation 6.

$$(\lambda(X :: K)B)^{nf} = \lambda(X :: K)B^{nf}$$

We can conclude equivalence from the judgment **Con Eq Abs**:

$$\dfrac{\dfrac{\text{from induction hypothesis}}{E, X :: K \vdash B \leftrightarrow B^{nf} :: L}}{E \vdash \lambda(X :: K)B \leftrightarrow \lambda(X :: K)B^{nf} :: K \Rightarrow L} \text{(Con Eq Abs)}$$

Case 6 (Induction step - Transformation 7)**.**
If our constructor has shape $B(A)$, we have to consider transformation 7.

$$(B(A))^{nf} = \text{if } B^{nf} \equiv \lambda(X :: K)C\{X\} \text{ for some } X,\ K,\ C, \text{ then } (C\{A\})^{nf} \text{ else } B^{nf}(A^{nf})$$

Now we have two subcases.

Subcase 1 ($B^{nf} \equiv \lambda(X :: K)C\{X\}$ for some X, K, C)**.** We can conclude equivalence from the judgment **Con Eval Beta** and transitivity.

$$\dfrac{\dfrac{\dfrac{\text{from Assumption}}{E, X :: K \vdash C\{X\} :: L} \quad \dfrac{\text{from Assumption}}{E \vdash A :: K}}{E \vdash (\lambda(X :: K)C\{X\})(A) \leftrightarrow C\{A\} :: L}\text{(Con Eq Abs)} \quad E \vdash C\{A\} \leftrightarrow (C\{A\})^{nf} :: L}{E \vdash (\lambda(X :: K)C\{X\})(A) \leftrightarrow (C\{A\})^{nf} :: L}$$

Subcase 2 (else)**.** We can conclude equivalence from the judgment **Con Eq Appl**.

$$\dfrac{\dfrac{\text{from induction hypothesis}}{E \vdash B \leftrightarrow B^{nf} :: K \Rightarrow L} \quad \dfrac{\text{from induction hypothesis}}{E \vdash A \leftrightarrow A^{nf} :: K}}{E \vdash B(A) \leftrightarrow B^{nf}(A^{nf}) :: L}\text{(Con Eq Abs)}$$

$\square$