Lorentz Transformations

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1 Lorentz Transformations

Lorentz Transformations are defined as isometries of the Minkwoski metric, i.e. for every Lorentz transformation Λ, we have that

$$g_{\mu\nu} = \Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}g_{\rho\sigma}. \tag{1}$$

Note that this is the vector representation of the Lorentz group, which is also serving as the defining representation. For an infinitesimal Lorentz transformation, we can expand Λ to first order to obtain

$$\Lambda^{\rho}{}_{\mu} = \delta^{\rho}{}_{\mu} + \omega^{\rho}{}_{\mu}. \tag{2}$$

We can see that ω is antisymmetric, i.e. $\omega_{\rho\mu} = -\omega_{\mu\rho}$, by inserting (2) into (1). Hence, we infer that the Lie algebra of the Lorentz group has dimension 6. The first three generators are given by

$$(A_1)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (A_2)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (A_3)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

and are referred to as boost generators[1]. The last three generators read

$$(B_1)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (B_2)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (B_3)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

and are called generators of rotations. A finite Lorentz transformation can be obtained by exponentiating a linear combination of the generators, i.e.

$$\Lambda = \exp\left(\vec{\beta}\vec{A} + \vec{\phi}\vec{B}\right). \tag{5}$$

However, we can also choose a different basis for the Lie algebra. We know that in four vector representation, an element of the Lie algebra can be written as

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \beta_3 \\ -\beta_1 & 0 & \theta_3 & -\theta_2 \\ -\beta_2 & -\theta_3 & 0 & \theta_1 \\ -\beta_3 & \theta_2 & -\theta_1 & 0 \end{pmatrix} \tag{6}$$

[1] Because we are raising the first index, the matrices are not antisymmetric in this representation

From this, we can read off that

$$\omega_{0i} = \beta_i \quad \omega_{ij} = \epsilon_{ijk}\theta_k, \tag{7}$$

or the other way round

$$\beta_i = \omega_{0i} \quad \theta_i = \frac{1}{2}\epsilon_{ijk}\omega_{jk}. \tag{8}$$

We can redefine the Lie algebra elements as

$$\omega^{\mu}{}_{\nu} = -\frac{\mathrm{i}}{2}\omega_{\alpha\beta}\left(J^{\alpha\beta}\right)^{\mu}{}_{\nu} \tag{9}$$

with

$$\left(J^{\alpha\beta}\right)_{\mu\nu} = \mathrm{i}(\delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu} - \delta^{\beta}{}_{\mu}\delta^{\alpha}{}_{\nu}). \tag{10}$$

It is not hard to see that $J^{\alpha\beta}$ is antisymmetric in α and β. Furthermore, with little effort, one can derive the commutation relations

$$\left[J^{\mu_1\mu_2}, J^{\mu_3\mu_4}\right] = \mathrm{i}(g^{\mu_2\mu_3}J^{\mu_1\mu_4} - g^{\mu_1\mu_3}J^{\mu_2\mu_4} - g^{\mu_2\mu_4}J^{\mu_1\mu_3} + g^{\mu_1\mu_4}J^{\mu_2\mu_3}). \tag{11}$$

We can then define the generators as

$$J^{0i} = K_i \quad J^{ij} = \epsilon_{ijk}L_k, \tag{12}$$

or the other way round[2]:

$$K_i = J^{0i} \quad L_k = \frac{1}{2}\epsilon_{ijk}J^{ij}. \tag{13}$$

Complexifying the Lie algebra, we can write the generators as

$$J_i^{\pm} = \frac{L_i \pm \mathrm{i}K_i}{2}. \tag{14}$$

The Lie algebra than decouples into two separate ideals as we can see by looking at the commutation relations

$$[J_i^+, J_j^+] = \mathrm{i}\epsilon_{ijk}J_k^+ \tag{15}$$
$$[J_i^-, J_j^-] = \mathrm{i}\epsilon_{ijk}J_k^- \tag{16}$$
$$[J_i^+, J_j^-] = 0, \tag{17}$$

revealing the isomorphism

$$\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}. \tag{18}$$

By studying the representations of the complexified Lie algebra, we can retrieve the representations of the real Lie algebra by employing

$$L_i = J_i^+ + J_i^- \quad K_i = -\mathrm{i}(J_i^+ - J_i^-). \tag{19}$$

We know that the irreducible representations of $\mathfrak{su}(2)_{\mathbb{C}}$ are characterized by the highest weight j. We can then uniquely label a representation (up to similarity) of the Lorentz algebra with the tuple (j_+, j_-). The irreducible representations can be obtained using

$$\rho_{j_+j_-}(J_i^+) = \rho_{j_+}(J_i^+) \otimes \mathbb{I}_{(2j_-+1)\times(2j_-+1)} \tag{20}$$
$$\rho_{j_+j_-}(J_i^-) = \mathbb{I}_{(2j_++1)\times(2j_++1)} \otimes \rho_{j_-}(J_i^-). \tag{21}$$

[2] They are related to the previously defined generators by $K_i = \mathrm{i}A_i$ and $L_i = \mathrm{i}B_i$.

The generators can be reexpressed as

$$-\frac{\mathrm{i}}{2}\omega_{\mu\nu}J^{\mu\nu} = -\mathrm{i}\sum_i(\beta_i K_i + \theta_i L_i) = -\sum_i\left(J_i^+(\theta_i - \mathrm{i}\beta_i) + J_i^-(\theta_i + \mathrm{i}\beta_i)\right), \tag{22}$$

leaving us with

$$\rho_{j_+j_-}(\Lambda) = \exp\left(-\mathrm{i}\sum_i\left(J_i^+(\theta_i - \mathrm{i}\beta_i) + J_i^-(\theta_i + \mathrm{i}\beta_i)\right)\right). \tag{23}$$

2 $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ Representation

With our knowledge about the structure of the Lorentz algebra, let us now investigate the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. We know that the 2 dimensional irreducible representations of J_i^+ and J_i^- are given by the Pauli matrices, i.e.

$$\rho_{\frac{1}{2}}(J_1^+) = \rho_{\frac{1}{2}}(J_1^-) = \frac{1}{2}\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{24}$$

$$\rho_{\frac{1}{2}}(J_2^+) = \rho_{\frac{1}{2}}(J_2^-) = \frac{1}{2}\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{25}$$

$$\rho_{\frac{1}{2}}(J_3^+) = \rho_{\frac{1}{2}}(J_3^-) = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{26}$$

The identity matrices in (21) just become 1 for the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. The 1 dimensional representation of $\mathfrak{su}2$ is trivially given by 0. One might think that this is making the $(\frac{1}{2}, 0)$ representation unfaithful, but this is not the case. We can see this by employing (19) showing that

$$\rho_{1/2,0}(L_i) = \rho_{1/2,0}(J_i^+ + J_i^-) = \frac{1}{2}\sigma_i \tag{27}$$

$$\rho_{1/2,0}(K_i) = \rho_{1/2,0}(-\mathrm{i}(J_i^+ + J_i^-)) = -\frac{\mathrm{i}}{2}\sigma_i \tag{28}$$

When we go back to the real Lie algebra, we are only allowed to use real components for the elements of the Lie alebra. Hence, in the sense of $\mathfrak{so}(3,1)$, we get $\ker\rho_{1/2,0} = \{0\}$. The $(\frac{1}{2}, 0)$ representation of the Lorentz group is then given by

$$\rho_{1/2,0}(\Lambda) = \exp\left(-\frac{1}{2}\vec{\sigma}(\vec{\theta} - \mathrm{i}\vec{\beta})\right). \tag{29}$$

Analogously, for the $(0, \frac{1}{2})$ representation, we obtain

$$\rho_{0,1/2}(\Lambda) = \exp\left(-\frac{1}{2}\vec{\sigma}(\vec{\theta} + \mathrm{i}\vec{\beta})\right). \tag{30}$$

3 Spinors

Spinors are defined as elements of the vector space on which the reducible $(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})$ representation acts. You may wonder why physicists care more about this representation than the seemingly more fundamental $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations. This has to do with parity and will be discussed in the next section. For now, let us write down the spinor representation of the Lorentz group, given by

$$\rho_{\mathrm{S}}(\Lambda) = \rho_{(1/2,0)}(\Lambda) \oplus \rho_{(0,1/2)}(\Lambda) = \begin{pmatrix} \exp\left(-\frac{1}{2}\vec{\sigma}(\vec{\theta} - \mathrm{i}\vec{\beta})\right) & 0 \\ 0 & \exp\left(-\frac{1}{2}\vec{\sigma}(\vec{\theta} + \mathrm{i}\vec{\beta})\right) \end{pmatrix}. \tag{31}$$

4 Dirac Spinors and Parity

Two questions that might come to ones mind when thinking about spinors is 1. why do we have to of them (a left and a right version) and 2. why is the bispinor representation interesting, even though it is not irreducible? The answer to both of these questions is parity. Parity is defined by its 4-vector representation

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{32}$$

The adjoint action of P is given by

$$PK_iP^{-1} = -K_i \quad PL_iP^{-1} = L_i. \tag{33}$$

Note that this implies

$$PJ_i^{\pm}P^{-1} = J_i^{\mp}. \tag{34}$$

Taking a look at (23), we see that the effect of P on a representation of the Lorentz group is given by

$$P\rho_{j_+j_-}(\Lambda)P^{-1} = \rho_{j_-j_+}(\Lambda). \tag{35}$$

5 Vector Representation $(\frac{1}{2}, \frac{1}{2})$

In the vector representation, we get the generators

$$\rho_{1/2,1/2}(J_i^+) = \frac{1}{2}\sigma_i \otimes \mathbb{I}_{2\times 2} \tag{36}$$

$$\rho_{1/2,1/2}(J_i^-) = \frac{1}{2}\mathbb{I}_{2\times 2} \otimes \sigma_i, \tag{37}$$

explicitly given by (leaving out the $\rho_{1/2,1/2}$ for brevity):

$$\left(J_1^+\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix} \quad \left(J_2^+\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \left(J_3^+\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{38}$$

$$\left(J_1^-\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \quad \left(J_2^-\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \left(J_3^-\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{39}$$

These matrices do not coincide with the matrices given in the appendix, but you can verify yourself that they are related by the similarity transformation

$$T = \begin{pmatrix} 0 & \mathrm{i} & 1 & 0 \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 & -\mathrm{i} \\ 0 & -\mathrm{i} & 1 & 0 \end{pmatrix}. \tag{40}$$

6 Tensor Representations

A rank two contravariant tensor is given by the tensor product of two four vectors and hence transforms under the tensor product of two vector representations. In mathematical terms, this means that an arbirary tensor $T^{\mu\nu}$ transforms as

$$T^{\mu\nu} \to T'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma}. \tag{41}$$

We know that we can rewrite any rank two contravariant tensor as a sum of a trace, an antisymmetric and a traceless symmetric part, i.e.

$$T^{\mu\nu} = \frac{1}{4}g^{\mu\nu}T^{\rho}{}_{\rho} + T^{[\mu\nu]} + T^{(\mu\nu)}. \tag{42}$$

It is easy to check that the trace part has one degree of freedom, wheras the antisymmetric and traceless symmetric parts have 6 and 9. We can see that these parts furnish invariant subspaces under the Lorentz group, i.e. they do not transform into each other.

$$\frac{1}{4}g^{\mu\nu}T^{\rho}_{\rho} \to \frac{1}{4}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g^{\alpha\beta}T^{\rho}_{\rho} = \frac{1}{4}g^{\mu\nu}T^{\rho}{}_{\rho}. \tag{43}$$

The antisymmetric part transforms into an antisymmetric part again:

$$T^{[\mu\nu]} \to T'^{[\mu\nu]} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{[\alpha\beta]} = -\Lambda^{\mu}{}_{\beta}\Lambda^{\nu}{}_{\alpha}T^{[\alpha\beta]} = -T'^{[\nu\mu]}. \tag{44}$$

An interesting side note is that the antisymmetric part does not furnish an irreducible representation for isometries of all kinds of metrics. In fact, for a euclidean metric and $SO(4)$, the antisymmetric part reduces into two three dimensional irreducible representations, called the self-dual and anti-self-dual parts. The fact that this does not hold in the case of the Lorentz group is merely an artifact of the hyperbolic strucutre of spacetime. Last but not least, the traceless symmetric part transforms into a traceless symmetric part:

$$T^{(\mu\nu)} \to T'^{(\mu\nu)} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{(\alpha\beta)} = \Lambda^{\mu}{}_{\beta}\Lambda^{\nu}{}_{\alpha}T^{(\alpha\beta)} = T'^{(\nu\mu)}. \tag{45}$$

We hence see that the tensor representation can be decomposed as

$$\underbrace{\mathbf{4}}_{\text{index 1}} \otimes \underbrace{\mathbf{4}}_{\text{index 2}} = \underbrace{\mathbf{1}}_{\text{trace}} \oplus \underbrace{\mathbf{6}}_{\text{asym}} \oplus \underbrace{\mathbf{9}}_{\text{sym}}. \tag{46}$$

This procedure can be repeated for higher rank tensors. The general idea of creating new representations by taking tensor products of known, irreducible representation and decomposing them into invariant subspaces is called Clebsch-Gordan decomposition.

7 Field Representation (?, ?)

So far, we have only taken a look at finite dimensional representations of the Lorentz group. Because the Lorentz group is non compact, all of its finite dimensional representations are non unitary. However, we do have the possibility to construct infinite dimensional representations that are unitary. In order to do so, let's have a look at the transformation behaviour of a scalar field $\phi(x) \in C^{\infty}(V)$ (where $V = \mathbf{R}^4$ is the set of all 4-vectors) depending on the spacetime coordinates x living in the 4-vector representation. As you can check, a representation on scalar fields is given by

$$\rho_{\mathrm{S}}(\Lambda)\phi(x) = \phi(\Lambda^{-1}x). \tag{47}$$

In order to retrieve the corresponding representation of the Lie algebra, we consider an infinitesimal Lorentz transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$. The inverse transformation is the given by $(\Lambda^{-1})^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu}$. Inserting this in (47), we obtain

$$\phi(\Lambda^{-1}x) = \phi(x) - \omega^{\mu}{}_{\nu} x^{\nu} \partial_{\mu} \phi(x) = \left(1 - \frac{\mathrm{i}}{2} \omega_{\mu\nu} \mathrm{i} \left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right)\right) \phi(x). \tag{48}$$

Comparing this to the exponential map, we see that the field representation of the Lorentz algebra is given by

$$S^{\mu\nu} = \mathrm{i} \left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right). \tag{49}$$

We thus managed to find a representation of the Lorentz algebra on $C^{\infty}(V)$. Note that this representation is unitary with respect to the L^2 inner product.

8 Tensor Product with the Field Representation

Physical fields are given as the tensor product of a finite dimensional representation of the Lorentz group and the $C^{\infty}(V)$ representation. For example, we have the following decompositions:

$$\phi(x) \to (0,0) \otimes (?,?) \tag{50}$$

$$A^{\mu}(x) \to \left(\frac{1}{2}, \frac{1}{2}\right) \otimes (?,?) \tag{51}$$

$$\psi(x) \to \left(\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right) \otimes (?,?) \tag{52}$$

where the question marks denote the (j_+, j_-) representation of the $C^{\infty}(V)$ representation for which I could not find the exact values yet (maybe (∞, ∞)?). From the section about tensor representations, we know that tensor products of irreducible representations of the same algebra can be Clebsch-Gordan decomposed into irreducible representations. However, I still know to little about the $C^{\infty}(V)$ representation to give more information about how this works in this context. However, any field can now be transformed by

$$\rho_{\mathrm{R}}(\Lambda) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\left(S^{\mu\nu} + J^{\mu\nu}\right)\right), \tag{53}$$

where $S^{\mu\nu}$ is the representation on the internal space and $J^{\mu\nu}$ the infinite dimensional representation on $C^{\infty}(V)$.

9 Explicit Matrix Representations

The vector representation of the $J^{\mu\nu}$ matrices defined in (10) are given by

$$\left(J^{01}\right)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \left(J^{02}\right)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \left(J^{03}\right)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{54}$$

$$\left(J^{12}\right)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \left(J^{13}\right)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix} \quad \left(J^{23}\right)^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}. \tag{55}$$

From this, we obtainthe boost and rotation generators defined in (12) as

$$\left(K^1\right)^\mu{}_\nu = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \left(K^2\right)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \left(K^3\right)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{56}$$

$$\left(L^1\right)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \quad \left(L^2\right)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \quad \left(L^3\right)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{57}$$

Using (14), we arrive at the following $J_i^\pm$ matrices:

$$\left(J_1^+\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \quad \left(J_2^+\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \quad \left(J_3^+\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{58}$$

$$\left(J_1^-\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \quad \left(J_2^-\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \quad \left(J_3^-\right)^\mu{}_\nu = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{59}$$

10 Summary of Representations

To sum up, here is a table of the most important irreducible representations of the Lorentz algebra, including the corresponding generators $S^{\mu\nu}$. Recall that in order to retrieve a representation of the Lorentz group, we have to exponentiate the generators, i.e.

$$\rho_{\mathrm{R}}(\Lambda) = \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\rho_{\mathrm{R}}(S^{\mu\nu})\right), \tag{60}$$

where R is the representation we are interested in.

Name	(j_+, j_-)	$S^{\mu\nu}$
Scalar	$(0,0)$	0
Left Weyl Spinor	$(\frac{1}{2},0)$	$S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k, S^{0i} = -\frac{\mathrm{i}}{2}\sigma^i$
Right Weyl Spinor	$(0,\frac{1}{2})$	$S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k, S^{0i} = \frac{\mathrm{i}}{2}\sigma^i$
Dirac Spinor	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$\frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]$
Vector	$(\frac{1}{2},\frac{1}{2})$	$S^{\mu\nu}{}_{\alpha\beta} = \mathrm{i}\left(\delta^\mu{}_\alpha\delta^\nu{}_\beta - \delta^\mu{}_\beta\delta^\nu{}_\alpha\right)$
Field	$(?,?)$	$\mathrm{i}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)$