Understanding Bond Yields

February 2017

1 Bond Mathematics pre-requisite: Understanding The compounding effect

Consider a hypothetical world where an institutional investor buys a financial product that will pay him a fixed rate of return (a fixed income) of 10% per year over a term of 3 years. The investor buys $1,000,000 worth of this product. At the end of 3 years how much money will she have?
To make this clear we consider this scenario first year by year and then we will arrive at an important mathematical formula which we can use for other examples:
Because the investor earns 10% per year, at the end of the first year she will have earned 10% of the $1,000,000 she invested:

$$10\% \cdot \$1,000,000 = \$100,000$$

So she now has $1,100,000. This process will repeat for the 2nd year as well so by the end of the 2nd year the investor will have:

$$\$1,100,000 + 10\% \cdot \$1,100,000 = \$1,100,000 + \$110,000 = \$1,100,000 \cdot (1 + 10\%) = \$1,210,000$$

At the end of the 3rd year she will have:

$$\$1,210,000 \cdot (1 + 10\%) = \$1,331,000$$

This calculation can be summarized in one line:

$$\$1,000,000 \cdot (1 + 10\%) \cdot (1 + 10\%) \cdot (1 + 10\%) = \$1,000,000 \cdot (1 + 10\%)^3 = \$1,331,000$$

The investor will end up with $31,000 more than if she had received a flat 30% over the 3 year period. This effect is called the compounding effect, or compound interest.
We can apply the same method for an arbitrary rate of return, r, time period Δt, and principal investment with present value $V_{present}$. We can say that the future value of the investment is given by:

$$V_{future} = V_{present}(1 + r)^{\Delta t}$$

Understanding this formula is key to understanding how bonds work. If we know 3 of the 4 components in the equation we will be able to work out the 4th.

1.1 Example 1. Calculating the implied rate of return of a single future cash flow.

In this example we will consider a financial product that will payout \$100 to the investor in 2 years. The price is \$90. The question is, what is the annualised rate of return for this financial product? The formula is $V_{future} = V_{present}(1+r)^{\Delta t}$. If we put in the values we have:

$$\$100 = \$90(1+r)^2$$

If we rearrange the equation to make the annualised rate of return the subject we get the answer:

$$r = (\frac{\$100}{\$90})^{\frac{1}{2}} - 1$$

$$r = 0.054 = 5.4\%$$

1.2 Example 2. Calculating the implied rate of return for more than one future cash flow.

In this slightly more involved example we will consider another financial product. This one pays the investor \$2 at the end of 1 year and at the end of 2 years she will receive another \$2 as well as and extra \$100. The investor pays \$95 for this product. What is the implied annual rate of return? To see how we do this let us express the formula with future value, $V_{present}$ as the subject:

$$V_{present} = \frac{V_{future}}{(1+r)^{\Delta t}}$$

The accepted interpretation of this formula is that it tells us how the value of cash in the future relates to the its value today.
For the problem take this interpretation one step further and work on the premise that the present value of the future cash flows will be equal to the sum of the present value of each individual future cash flow:

$$V_{present} = \frac{\$2}{(1+r)^1} + \frac{\$2}{(1+r)^2} + \frac{\$100}{(1+r)^2} = \frac{\$2}{(1+r)^1} + \frac{\$102}{(1+r)^2} = \$95$$

If you know a little algebra we can see this is a quadratic polynomial in (1+r):

$$(1+r)^2 - \frac{2}{95}(1+r) - \frac{102}{95} = 0$$

The relevant solution to this is $(1+r) = 1.0467$ or $r = 4.67\%$

2 What is the Yield and why is it useful?

The bond yield is a quantity that is implied from bond prices. It is the implied fixed annual rate of return throughout the remaining term of the bond. It tells financial market participants (investors, bankers and issuers) the implied annual rate of return associated with a particular bond. This is important because it allows participants to compare bonds with different coupons and maturities.

For example if we know that a 3 year \$100 2% coupon bond has a price a \$97.17 and a 2 year \$100 zero coupon bond had a price of \$96.11, it is hard to see immediately which bond will yield a better return. But if we calculate the yields of each bond we can easily see that the market is implying that the coupon bond will return 3% per year and the zero only returns 2% per year which is much easier to think about. Because the bond yield has units of return per year it you can compare bonds with differing maturities. A very important concept in bond markets is the yield curve, this is where we take bond yields from the same issuer and plot them against their maturities.

2.1 The Yield of a 2 year annual Coupon bond.

We have a 2 year bond with an annual coupon of c and with an unknown yield of y.

$$p = \frac{c}{(1+y)} + \frac{c}{(1+y)^2} + \frac{1}{(1+y)^2}$$

This is because if we were to add the present values of all future cash flows, we would end up with the market value or purchase price of the bond. Here c represents the coupon payment and is a value usually between 0.01 and 0.50 (bond coupons are usually less than 50% of principal). We can now derive y as follows.

$$p(1+y)^2 = c(1+y) + c + 1$$

$$py^2 + (2p-c)y + p - 2c - 1 = 0$$

$$y = \frac{c - 2p + \sqrt{4p^2 - 4pc + c^2 - 4p^2 + 8pc + 4p}}{2p}$$

$$y = \frac{c - 2p + \sqrt{c^2 + 4pc + 4p}}{2p}$$

3 Graph showing Yield vs Price