As any graduate mechanics text will reveal, in the late nineteenth and early twentieth centuries a major focus of theoretical physics was expressing fundamental laws in a Hamiltonian form. This formulation allowed for a rigorous notion of energy to be defined for various systems, and allowed beautiful linkages between symmetries and conserved quantities through Noether's theorem. More practically it also allowed for the identification of invariants with which to calculate stability, as well as simplifying various types of perturbation theory [7]. For example, in finite dimensions if one can find an invariant of a system such that a desired fixed state of the system locally extremizes the invariant, then the desired fixed state is non-linearly stable [9]. In the late twentieth century up to today there has been a concerted effort to extend these results to the various partial differential equations of physics through field-theoretical Poisson brackets (as in [6]); these are the infinite dimensional analog to the Poisson brackets of finite dimensional Hamiltonian systems.

Along this line Morrison recently showed that it is possible to give the compressible Euler equations a Poisson bracket formulation $(L^2(\mathbb{R}), \{\cdot,\cdot\}_M)$ where

$$\{F,G\}_M = \int_{\mathbb{R}^3} M_i \left(\frac{\delta G}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta F}{\delta M_i} - \frac{\delta F}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta G}{\delta M_i}\right)\ \mathrm{d}^3x + \int_{\mathbb{R}^3} \rho\left(\frac{\delta G}{\delta \mathbf{M}}\cdot\boldsymbol{\nabla}\frac{\delta F}{\delta\rho} - \frac{\delta F}{\delta \mathbf{M}}\cdot\boldsymbol{\nabla}\frac{\delta G}{\delta\rho}\right)\ \mathrm{d}^3x,$$

and where the Hamiltonian is physical [7]. Nguyen et al. [10] proposed extending this bracket to the incompressible case by the Dirac bracket method. A realistic test of extending Morrison's bracket to the incompressible case, while currently outside the limits of mathematical rigor in general, can be given in the context of the certain specific model of Riemannian ellipsoids [8], for which stability results are available using other methods.

In 1861 Riemann wrote a seminal paper investigating the problem of self gravitating fluids, with the restriction to incompressible inviscid fluids with a linear velocity profile confined in an ellipsoidal shape, now called Riemann ellipsoids. In investigating the equilibria of the resultant ellipsoids, he found that the system had two free parameters that separated the ellipsoids into five classes. In studying the problem of stability for the five classes of ellipsoids he found, Riemann used an energy criterion to investigate nonlinear stability of the ellipsoids [11]. Later, Chandrasekhar tested these same stability classes with linear stability analysis and found linear stability in many regimes Riemann concluded were unstable [2]. Of course, linear stability does not imply non-linear stability, but Lebovitz subsequently showed that Riemann's criterion to conclude instability was incorrect [5]. Thus, there was suspicion Chandrasekhar's results could be extended to show non-linear stability. To this end Morrison et al. defined the moment of inertia tensor Σ and the virial tensor $\mathcal{M}$ so that

$$\mathcal{M}_{ij} = \int_V X_i M_j\ \mathrm{d}^3X \text{ and } \Sigma_{ij} = \int_V \rho X_i X_j\ \mathrm{d}^3X,$$

where X is the inertial frame coordinate and $M = \rho U$ is the fluid's inertial frame momentum measured in the rotating coordinate system of the ellipsoid. They then noted his bracket could be reduced to $(\mathbb{R}^{18}, \{\cdot,\cdot\}_{RD})$ given in [8], provided one used the Dirac bracket method to make volume and divergence-free flow Casimirs of the restricted bracket $\{\cdot,\cdot\}_{RD}$. After transforming to the frame rotating with the ellipsoid and giving the bracket the suitably transformed physical Hamiltonian,

$$H(\mathcal{M},\Sigma) = \frac{1}{2}\mathrm{Tr}(\mathcal{M}^T\Sigma^{-1}\mathcal{M}) + V(\Sigma),$$

where $V(\Sigma)$ is the gravitational potential energy, it was then shown that the restricted bracket correctly produced the equations of motion for the Riemann ellipsoids [8]. This gave confidence in Morrison's bracket but, subsequently, Morrison and Benavides [1] wanted to evaluate the stability question.

This reduced bracket allowed one to determine the Casimir invariants of the ellipsoids: volume (C^1), divergence-free flow (C^2), and the circulation of the flow (C^3). There was also the additional invariant of the magnitude of the angular momentum (C^4), arising through a symmetry of the particular Hamiltonian above. The latter was omitted from [1]. There appeared in their results a further, small unexplained discrepancy with previously found stability results in [2].

In order to investigate the stability of the ellipsoids, I, under the mentorship of Lebovitz, used an Energy-Casimir function, F, which combined the invariants of the system so that the gradient of F would be zero in a parameter regime where Chanrasekhar and Riemann disagreed. This was done by the Lagrange multiplier method so that:

$$\begin{aligned} F(\mathcal{M},\Sigma) =& H + c_1\ln(\det(\Sigma)) + c_2\mathrm{Tr}(\Sigma^{-1}\mathcal{M}) \\ &+ c_3\mathrm{Tr}(\Sigma^{-1}\mathcal{M}\Sigma\mathcal{M}^T - \mathcal{M}^2) + c_4\|\epsilon\mathcal{M}\|^2, \end{aligned}$$

where c_i, are constants for each conserved quantity, and $\epsilon\mathcal{M}$ is the vector $\epsilon_{ijk}\mathcal{M}_{jk}$. The parameter regime of disagreement was chosen to be where the linearly moving fluid's angular velocity vector was aligned with the angular velocity vector of the bulk ellipsoid. Upon ensuring that the gradient of F was zero in this parameter regime, it was found that the restrictions on F ensured that the two free parameters were restricted to a single free parameter, chosen to be a_2, the length of the second principal axis of the ellipsoid in its rotating frame.

As discussed in the previous section, if F could be

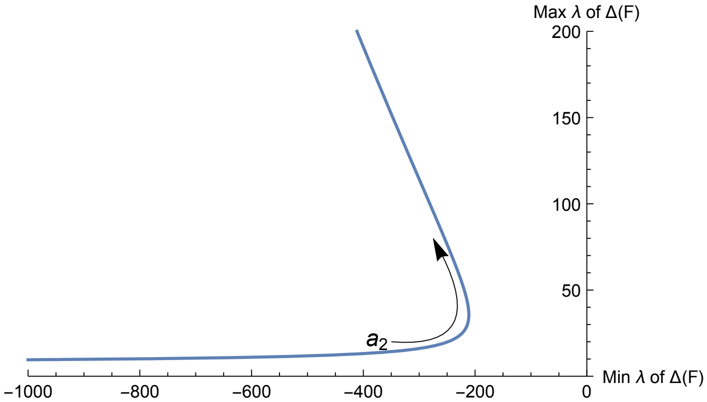

Figure 1: Plot of min. eigenvalue (λ) of $\Delta(F)$ versus max. λ, with arrow pointing in the direction of increasing a_2

shown to have a local extremum for this parameter regime, then it could be concluded the regime was nonlinearly stable as well as linearly stable, vindicating Chandrasekhar. Therefore $\Delta(F)$, the Hessian of F, was computed by hand as an 18×18 matrix. Mathematica code was written to numerically evaluate the various integrals necessary to find $\Delta(F)$ as a function of the single parameter a_2. Figure 1 shows a plot of the results of this analysis, revealing $\Delta(F)$ was indefinite over the whole parameter regime.

Various attempts were made to modify F such that its gradient would remain zero in the parameter domain, but such that $\Delta(F)$ would become definite. The largest was investigating the theoretical and computational effects on $\Delta(F)$ derived from adding arbitrary functions of the given Casimirs to F, as these would still be invariant. The addition of such functions only added terms proportional to $\nabla C^i \otimes \nabla C^i$ to $\Delta(F)$, and it was shown that such terms affect the definiteness in a complicated way dependent on ∇C and $\Delta(F)$. For lack of a useful theoretical result, further Mathematica code was implemented that demonstrated clearly that taking functions of invariants would not change the definiteness of $\Delta(F)$.

Given the results, it seems likely that Riemann came to an equivalent F through physical reasoning, and his only mistake was concluding nonlinear instability when his $\Delta(F)$ was indefinite. To verify this, a careful deconstruction of his methodology in [11] needs to be done in the original German. Additionally, Fassò et al. recently used advanced numerical techniques that showed each class of ellipsoids appears to have extremely intricate regions of stability and instability within the two-parameter family. To do so, they set up their system using a Hamiltonian formulation; therefore, their simulations might give intuition of how to modify F so as to prove nonlinear stability in the desired parameter regime [4]. We also anticipate that this study will explain the small discrepancy uncovered in [1], as remarked above.

Even if the Hamiltonian formulation of inviscid fluid mechanics does not settle the stability questions of the Riemann ellipsoids, its rigorous production of their equations of motion still gives confidence that the Poisson bracket is in totality rigorous. More study is required here, and just as much of my preliminary research focused on understanding Poisson geometry in a rigorous way so as to investigate the question of restricting $\{\cdot,\cdot\}_M$ to $\{\cdot,\cdot\}_{RD}$, I am currently studying global analysis to understand infinite dimensional Poisson geometry and address the rigorous formulation of a field-theoretical bracket for Euler's equations. This focuses on defining a manifold as locally diffeomorphic to a given inner product space and assigning it a Poisson structure. Certain results from the finite dimensional case carry over, but not all [3].

Finally, not only would an understanding of the formulation of such a Poisson structure lead to better understanding of the more applied realms of perturbation theory and stability theory (as we have seen), but it could also lead to important results in understanding the geometric properties of Euler's equations. As mentioned, this facilitates a better understanding of the invariants and symmetries of the equations. Since many of the proofs of regularity for PDEs rest on invariants of the PDE considered, formulating/finding invariants in a geometric way could lead to results about the regularity of solutions to Euler's equations. As these are the Navier-Stokes equations with viscosity $\nu = 0$, advances on this front bring us one step closer to solving one of the biggest open problems in PDE.

[1] Benavides, S. (2015). Unpublished manuscript, The University of Texas at Austin. Retrieved from http://hdl.handle.net/2152/29738.

[2] Chandrasekhar, S. (1969). *Ellipsoidal Figures of Equilibrium.* Yale University Press.

[3] Dunajski, M. (2012). Integrable systems [lecture notes].

[4] Fassò, F. and Lewis, D. (2001). *Arch. Rational Mech. Anal.*, **158**:259-292

[5] Lebovitz, N. R. (1966). *Ap. J.*, **145**:878-885.

[6] Marsden, *et al.* (1986). *Ann. Phys.*, **169**:29-47.

[7] Morrison, P. J. (1998). *Rev. Mod. Phys.*, **70**:468-519.

[8] Morrison, *et al.* (2009). *Ann. Phys.*, **324**:1747-1762.

[9] Musser, S. (2015). Unpublished manuscript, The University of Chicago. Retrieved from math.uchicago .edu/~mathreu/REU2015/.

[10] Nguyen, S. and Turski, L. (1999). *Physica A.*, **272**:48-55.

[11] Riemann, B. (1861). *Abh. d. Königl. Gesell. der Wis. zu Göttingen*, **9**:168-197.