Numerical Integration of ODEs: Single and Double Pendulum

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November 11, 2012

We attempt to numerically solve the equation of motion of a simple pendulum using Euler's method, leapfrog method, and fourth-order Runge-Kutta method. We assess the stability of the used methods and choose a recommended method for general oscillatory problems with damping. We use our chosen method to solve the equation of motion of a double pendulum. We assess the effects that varying pendulum masses, damping and time steps have on the system. We note that methods are stable with lower Δt and that RK4 is the most stable method. We also note that errors in RK4 deviate exponentially above a certain threshold, which can be determined by inspection. Therefore, we conclude that with Δt below the stability limit allows for a stable evaluation of double pendulum systems, though higher order Runge-Kutta methods and lower values of Δt would increase the accuracy further.

1 Introduction and Theory

1.1 Single Pendulum

The equation of motion of a single pendulum of mass m, length l, at an angle θ to the vertical can be written as

$$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta - \gamma\frac{d\theta}{dt} \tag{1}$$

where γ is the damping coefficient. By using a time scale (substitute in $\tilde{t} = \alpha t$) and some rearranging, we get:

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}0 & 1\\ a & b\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix}, \tag{2}$$

where $\omega = \frac{d\theta}{dt}$, $a = -g\left(\alpha^2 l\right)^{-1}$, $b = -\gamma\left(m\alpha^2 l\right)^{-1}$, and α is a time scale. This is achieved by using a small angle approximation where $\sin\theta \approx \theta$ for $\theta \ll 1$. Converting the equation to a pair of coupled ODEs as shown in equation 2 allows for the usage of various numerical methods.

We implement Euler's method by using step operations as shown in equation 3, where $f_n = \left(\frac{d\theta}{dt}\right)_n = \omega_n$ for $u_n = \theta_n$, for example. Errors in Euler's method are proportional to Δt.

$$u_{n+1} = u_n + f_n\Delta t \tag{3}$$

The leapfrog method can be implemented in a similar way using a step equation as shown in equation 4, where errors are proportional to $(\Delta t)^2$. Due to the requirement of values from 2 steps before, the first step in the leapfrog method must be calculated using Euler's method.

$$u_{n+1} = u_{n-1} + 2f_n\Delta t \tag{4}$$

The Runge-Kutta method of order 4 requires a slightly more complex set of equations as shown below:

$$u_{n+1} = u_n + \frac{1}{6}\left(k_{u1} + 2k_{u2} + 2k_{u3} + k_{u4}\right), \tag{5}$$

$$\begin{aligned}
k_{u1} &= f_{un}\left(u,v\right)\Delta t,\\
k_{u2} &= f_{un}\left(u+\frac{k_{u1}}{2},v+\frac{k_{v1}}{2}\right)\Delta t,\\
k_{u3} &= f_{un}\left(u+\frac{k_{u2}}{2},v+\frac{k_{v2}}{2}\right)\Delta t,\\
k_{u4} &= f_{un}\left(u+k_{u3},v+k_{v3}\right)\Delta t.
\end{aligned}$$

A Runge-Kutta method of order 4 yields errors in the order of $(\Delta t)^4$. To have a good measure of stability, we calculate the kinetic (T) and potential (V) energies of the pendulum. These can be found to be:

$$T = \frac{1}{2}ml^2\omega^2,\ V = mgl\left(1-cos\theta\right) \approx \frac{1}{2}mgl\theta^2. \tag{6}$$

where small angle approximation is used again to estimate V ($\cos\theta \approx 1-\frac{1}{2}\theta^2$ for $\theta \ll 1$). The total energy, which should theoretically be constant when $\gamma = 0$, would be:

$$E = T + V = \frac{1}{2}ml\left(l\omega^2 + g\theta^2\right). \tag{7}$$

As we can expect E to stay constant, any deviations can be considered as being due to instabilities in the method and initial conditions.

1.2 Double Pendulum

For the double pendulum case, the set of coupled first-order ODEs representing the system becomes:

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \phi\\ \omega\\ \nu\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -(R+1) & R & -G & 0\\ (R+1) & -(R+1) & G\left(1-R^{-1}\right) & -GR^{-1}\end{pmatrix}\begin{pmatrix}\theta\\ \phi\\ \omega\\ \nu\end{pmatrix}, \tag{8}$$

where ϕ and ν are values for the newly attached 2nd pendulum, corresponding to θ and ϕ. The length of the attached pendulum is l. The Runge-Kutta method briefly shown in equation 5 requires a slight change, where for example k_{u2} would become:

$$k_{u2} = f_{un}\left(u+\frac{k_{u1}}{2},o+\frac{k_{o1}}{2},v+\frac{k_{v1}}{2},p+\frac{k_{p1}}{2}\right)\Delta t, \tag{9}$$

where o and p correspond to ϕ and ν.

The energy equations need to be changed as well. Taking into consideration the fact that pendulum 2's speed is relative to that of pendulum 1, we can work out:

$$\begin{aligned}
T &= \frac{1}{2}ml^2\omega^2 + \frac{1}{2}Ml^2\left(\left(\omega\cos\theta+\nu\cos\phi\right)^2 + \left(\omega\sin\theta+\nu\sin\phi\right)^2\right)\\
V &= mgl\left(2-\cos\theta\right) + Mgl\left(2-\cos\theta-\cos\phi\right)
\end{aligned} \tag{10}$$

which with our continued usage of small angle approximation, become,

$$\begin{aligned}
T &= \frac{1}{2}l^2\left(\left(m+M\right)\omega^2 + M\nu^2 + 2M\omega\nu\left(1-\left(\theta-\phi\right)^2\right)\right),\\
V &= gl\left(m+\frac{1}{2}m\theta^2+\frac{1}{2}M\left(\theta^2+\phi^2\right)\right)
\end{aligned} \tag{11}$$

2 Method

2.1 Single Pendulum

We implemented our methods using the programming language, Go[1]. An Ubuntu Linux environment was used, and a graphing library named Plotinum[2] was used to plot all graphs. Code for both pendulum cases can be compiled and run via the bash scripts named $run\_single.sh$ and $run\_double.sh$.

When implementing the methods outlined in section 1, we took care to keep our code reusable by splitting bits of code into their relevant functions. Examples include f_n of equation 3 (fu, fv in code), the calculations for potential and kinetic energies (KE, PE in code), and the numerical integration method itself ($Euler$, $Leapfrog$, $RK4$ in code) which uses these other functions.

For both an undamped ($\gamma = 0$) and damped case ($\gamma = 0.2$), we plot and observe graphs of θ, ω, T, V, and E with respect to Δt. We re-calculate the values for a and b in equation 2 to make sure our altered initial conditions are applied in calculations (function $updateab$ in code).

To determine the stability of a method, we first plotted the total energy E, at a certain time t_f for varying Δt and inspected the relationships shown. It is important to use a limiting time value t_f, instead of a set number of steps as this ensures that the pendulum calculations are for the same point in time, where the total energy, E should ideally be the same for all methods. We plot the $\log_{10}$ values as deviations can increase exponentially, making it difficult to identify the amount of deviation of more stable methods. Using a $\log_{10}$ plot also enables us to see small deviations better.

As evident in figure 1a, it is immediately obvious that E starts at approximately equal values for all three methods, after which E diverges away as h increases. We make sure to have a good assessment over Δt by starting from a low Δt of $1e-5$. A lower value could have been chosen, but would have taken too long to process, without giving a more detailed picture. A limit to the deviation was chosen by inspection, with the corresponding Δt value being a limiting Δt for stability in the method and its initial conditions.

We also observed the effect of different damping coefficients, specifically $\gamma = 0$ and $\gamma = 0.2$. Changes in stability is noted for the different values of γ and different methods used.

2.2 Double Pendulum

The code for double pendulum is similar to that of the single pendulum. Changes include those mentioned in section 1.2 as well as adjustments to t_f. Adjustments to t_f had to be made as the Runge-Kutta method takes more time per iteration compared to the case of the single pendulum, as calculations are more than doubled.

3 Results and Discussion

3.1 Single Pendulum

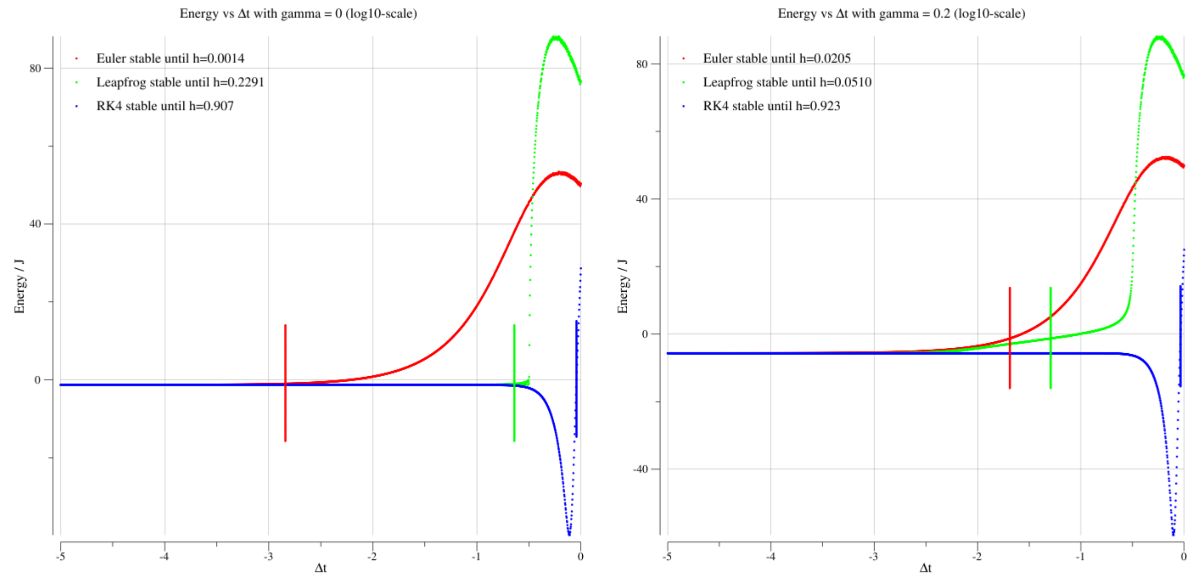
A plot of total energy, E against time step, Δt with $\gamma = 0$ is shown in figure 1a. The point at which deviations are greater than 0.05 are marked as vertical lines. It can be seen that the Euler method's total energy starts deviating at lower Δt, but the leapfrog method's E deviation rate is higher after approximately $\Delta t = 0.2$. We can see that E in the RK4 method drops slightly, then rises up slightly less quickly compared to leapfrog. Note that while the drop in E seems to be much, the plot is in $\log_{10}$.

The observed stability of our methods is in accordance with the broad expectation that the error of our methods will descend in the order: RK4 (order 4), Leapfrog (order 2), Euler (order 1).

In the damped situation shown in figure 1b, where $\gamma = 0.2$ was used, we can see that the graph is quite similar to figure 1a. However, one interesting point to note is that E for the leapfrog method seemingly starts deviating away at an earlier stage. This is mainly due to the fact that the plot is in $\log_{10}$-scale. We do notice however, that the limiting values of Δt for stability converge for Euler's method and leapfrog method. This can be explained by taking a look at the leapfrog method. As the first step in the leapfrog method is Euler's method, and damping reduces the contribution from subsequent steps, effects from the 2nd step onwards will be reduced as γ increases.

From the observations above, we can say that for a damped oscillating system, the leapfrog method will not be much more reliable than Euler's method, as γ increases. Therefore, the Runge-Kutta method of order 4 would be recommended.

[1] http://golang.org/
[2] http://code.google.com/p/plotinum/



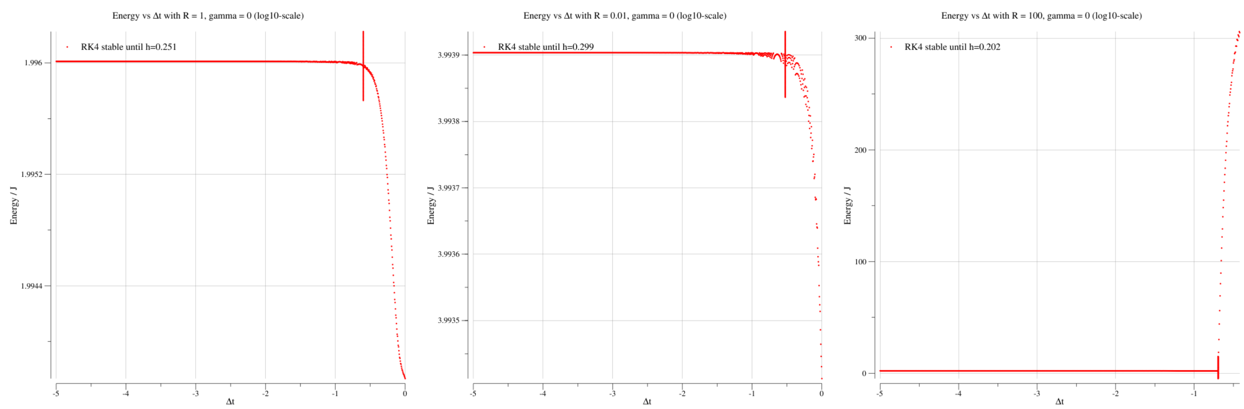
(a) System with $\gamma = 0$ (b) System with$\gamma = 0.2$

Figure 1: Single pendulum: Total energy against Δt, with $t_f = 40$

3.2 Double Pendulum

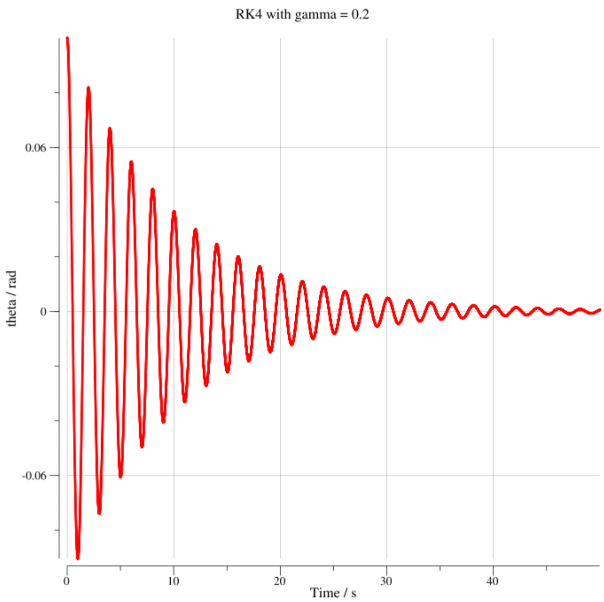
We used the Runge-Kutta method of order 4 to solve the equation of motion for a double pendulum. For $\gamma = 0$, we see a plot of total energy, E against time step Δt (figure 2) for three different mass ratios: $R = 1,\ 0.01,\ 100$. We note that E deviates suddenly at a certain point, where we can locate a Δt limit to ensure stability. We see that the maximum stable Δt reduces as R increases. This means that the higher R is, it is more stable for a lower Δt to be used. This can also be rephrased to say that the system is more unstable as R increases. Disregarding R, the cases are stable for values of Δt below the determined limits, indicating the usefulness of the Runge-Kutta 4 method for varying initial conditions.

When considering the change in θ and ϕ over time for $\gamma = 1$, as shown in figure 3, we can see that the angles exhibit behaviour observed in the single pendulum case. For $R = 1$, the angles are out of phase but both generally follow the pattern of decay shown for single pendulum (figure 3a). For $R = 0.01$ where the top pendulum is much more heavy, we see that the top pendulum moves like a single pendulum, while the bottom pendulum moves with much higher amplitude, seemingly driven by the top pendulum. For $R = 100$, where the bottom pendulum has the greatest effect, we see that the pendulums are aligned in a straight line like a single pendulum. This is expected as

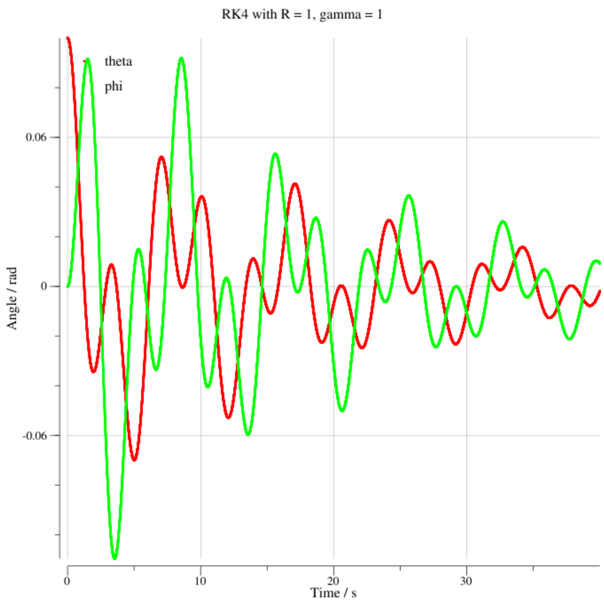


(a) System with $R = 1$ (b) System with $R = 0.01$ (c) System with $R = 100$

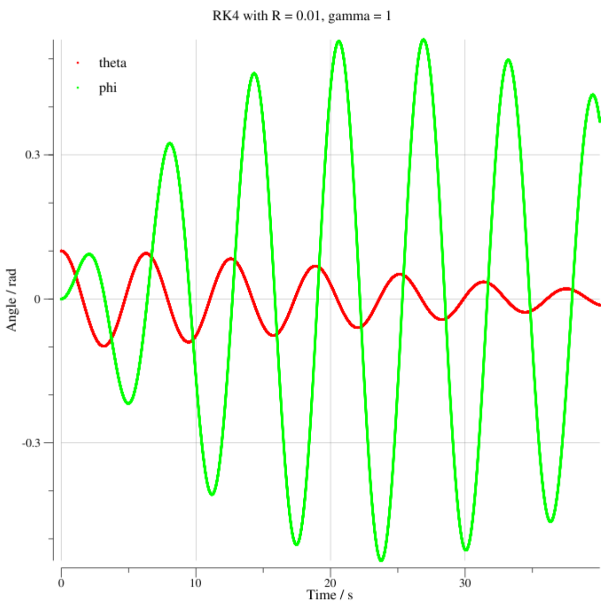
Figure 2: Total energy against Δt, with $t_f = 20$, $\gamma = 0$



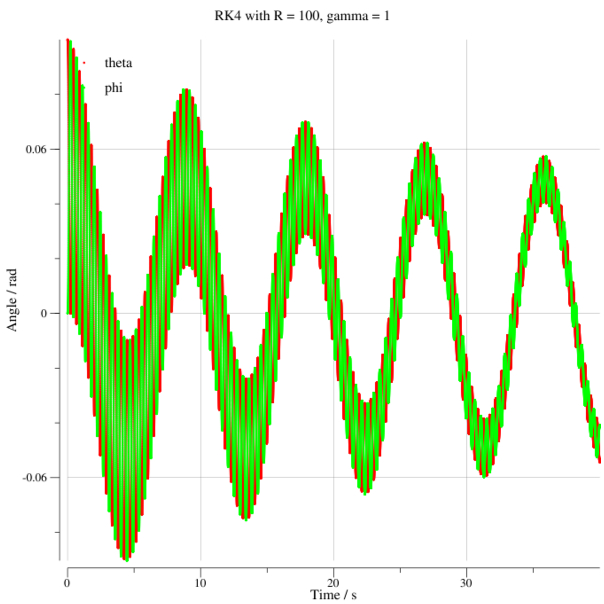
(a) Singe pendulum system with $\gamma = 0.2, t_f = 50$



(b) System with $R = 1$



(c) System with $R = 0.01$



(d) System with $R = 100$

Figure 3: Double pendulum plots of θ and ϕ against t, with $t_f = 40$, $\gamma = 1$

the tension due to M and the relatively low tension due to m would reduce variations in θ.

4 Conclusion

We assessed the stability of Euler's method, leapfrog method, and Runge-Kutta method of order 4 for the simple single pendulum system. We find that the methods are generally stable for very low Δt values, with Euler becoming unstable for lower Δt, leapfrog coming next, and RK4 being most stable. We also note that with damping, the leapfrog method's errors converge towards Euler's method as the first term becomes more prominent.

For double pendulum, we note that RK4 becomes unstable for $\Delta t > 0.2$, and that the stable maximum Δt reduces with increasing R. We notice that the method is still stable for low Δt as in the single pendulum case. To improve the evaluations, it would be beneficial to use a higher order RK4 method, as well as lower Δt values. It is worth mentioning however, that for Δt below a limit for Δt found for stable evaluations, errors vary negligibly. In real world situations therefore, a Δt limit should be found by inspection of trial evaluations over Δt to ensure stability of the RK4 method.

Approximate word count: 1450 words