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(PROJECT-3)

Q6. Maths based question.

1. Given:

Least squares solution:

$u = (A^TA)^{-1}A^Tb$ for solving the overdetermined system $Au = b$.

Let $G = A^TA$ be the 2×2 structure ~~matrix~~ Tensor at the window

The Lucas-Kanade method assumes the optical flow vector $u = [u, v]^T$ is constant within a local window W. The error fn to be minimized is the sum of squared residuals of the optical flow constraint equation for all pixels p_i, in the window:

$$E(u,v) = \sum_{p_i \in W} \left(I_x(p_i)u + I_y(p_i)v + I_t(p_i)\right)^2$$

Rewriting this in matrix form as minimizing the norm $\|Au - b\|^2$, where

$$A = \begin{bmatrix} I_x(p_1) & I_y(p_1) \\ \vdots & \vdots \\ I_x(p_n) & I_y(p_n) \end{bmatrix}, \quad u = \begin{bmatrix} u \\ v \end{bmatrix}, \quad b = \begin{bmatrix} -I_t(p_1) \\ \vdots \\ -I_t(p_n) \end{bmatrix}$$

To find minimum, we take derivative w.r.t u and set it to zero:

$$\nabla E = 2A^T(Au - b) = 0$$

$$\therefore A^TAu = A^Tb.$$

Here $G = A^TA$ is the structure Tensor, a 2×2 matrix

$$G = \begin{bmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{bmatrix}$$

- The linear system $Gu = A^T b$ has a unique solution for u if and only if G is non singular (i.e invertible)
- G is invertible if and only if it's determinant is non-zero (or equivalently it has full rank, meaning it's two eigen values λ_1, λ_2 are both strictly positive)
- Hence concluded. if G is not invertible (e.g in a flat region where gradients are zero, or along a single edge where gradients are linearly dependant), the system is under constraints (the aperture problem), and no unique solution exists. Therefore a reliable and unique solution exists if and only if G is invertible.

2. Second moment Matrix and edges.

Let the region W contain an ideal, straight edge oriented such the normal direction to edge is $n = [\cos\theta, \sin\theta]^T$ and the tangent direction is $t = [-\sin\theta, \cos\theta]^T$

In an ideal edge:

1. The gradient ∇I at any point is either zero or points in the direction of the normal n.
2. Thus, for any pixel i, $\nabla I_i = k_i n$ where k_i is the magnitude of the gradient.
3. There is no gradient variation along the edge direction t (i.e. $\nabla I \cdot t = 0$)

The Second Moment matrix M is defined as:

$$M = \sum_i (\nabla I_i)(\nabla I_i)^T$$

Sub $\nabla I_i = k_i n$:

$$\therefore M = \sum_i (k_i n)(k_i n)^T = \left(\sum k_i\right)^2 n n^T$$

lets test the eigenvalues by multiplying M by the tangent vector t:

$$Mt = \cancel{\left[\left(\sum_i k_i\right)^2 n n^T\right.} \left[\left(\sum_i k_i^2\right) n n^T\right] t$$

Since n and t are orthogonal $(n^T t = 0)$:

$$\therefore Mt = \left(\sum_i k_i^2\right) n (0) = 0 = 0 \cdot t$$

This shows that ~~eigenv~~ t is an eigenvector of M with a corresponding eigenvalue of 0 (The other eigenvalue corresponds to eigenvector n and is equal to $\sum_i k_i^2 > 0$)

Hence concluded, for an ideal straight edge, the rank M is 1, implying one eigenvalue is strictly positive and the other must be 0.

3. Horn Schunk Smoothness term.

The smoothness energy term $\mathcal{E}_s$ penalizes variations in the flow field (u, v). In the continuous domain it is:

$$\mathcal{E}_s = \iint (\|\nabla u\|^2 + \|\nabla v\|^2)\, dx\, dy$$

$$= \iint \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) dx\, dy$$

To minimize this numerically, we discretize the derivatives on a pixel grid. The squared magnitude of the gradient at pixel (i,j) can be approximated using finite differences between neighbours:

$$\mathcal{E}_s \approx \sum_{i,j} \left[(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + (v_{i+1,j} - v_{i,j})^2 + (v_{i,j+1} - v_{i,j})^2\right]$$

To find minimum, we take partial derivative of $\mathcal{E}_s$ with single flow component $u_{i,j}$ and set it to zero:

$$\frac{\partial E_s}{\partial u_{i,j}} = 2(u_{i,j} - u_{i-1,j}) + 2(u_{i,j} - u_{i+1,j})$$

$$+ 2(u_{i,j} - u_{i,j-1}) + 2(u_{i,j} - u_{i,j+1}) = 0$$

Dividing by 2 and grouping terms:

$$4u_{i,j} - (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) = 0$$

$$\therefore 4u_{i,j} = \sum_{(k,l)\in N_{i,j}} u_{k,l}$$

$$\therefore u_{i,j} = \frac{1}{4} \sum_{(k,l)\in N_{i,j}} u_{k,l}$$

Let the local average be $\bar{u}_{i,j} = \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})$

Then the minimization condition implies:

$$u_{i,j} - \bar{u}_{i,j} = 0$$

Hence concluded that minimizing the squared Frobenius norm of the gradient (smoothness term) implies that the Laplacian of the flow field is zero ($\nabla^2 u = 0$) or equivalently in the discrete case, that the flow vector at each pixel equals the average of it's neighbours.