Linear Algebra

1 Solving a system of linear Equation.

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30$$

find the value of x, y, z.

2 Matrix operation.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $C = A \times B$

NORMAL OR GAUSSIAN DISTRIBUTON.

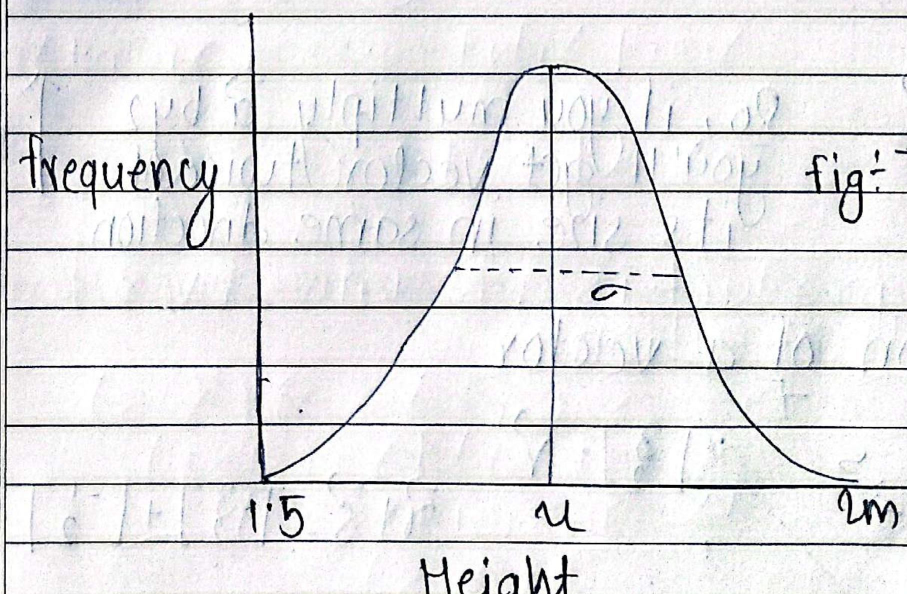
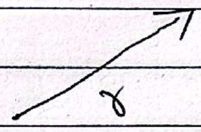


fig: The distribution of Height with population.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-u)^2}{2\sigma^2}\right\}$$

Vector

We can think of vector just as a list.



In computer we think of a vector as a list of attribute of an object.

Suppose for a house, attributes are :

120 sq m
2 bedroom
1 bathroom
$ 150,000

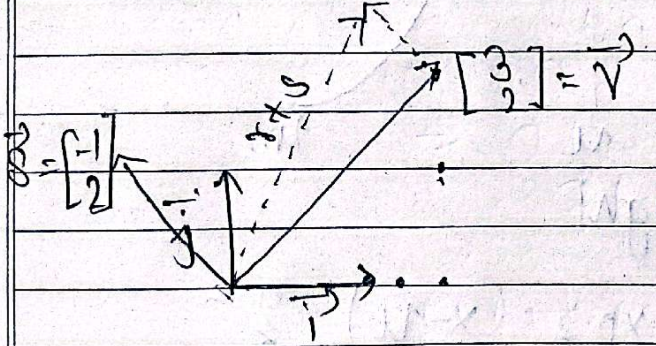
We generalize the vectors like:

$$\begin{bmatrix} 120 \\ 2 \\ 1 \\ 150,000 \end{bmatrix}$$

- Multiplying a vector by a scaler

$\uparrow \vec{a}$ So, if you multiply $\vec{a}$ by 2 you'll get vector twice of its size in same direction $\uparrow 2\vec{a}$

- Addition of a vector



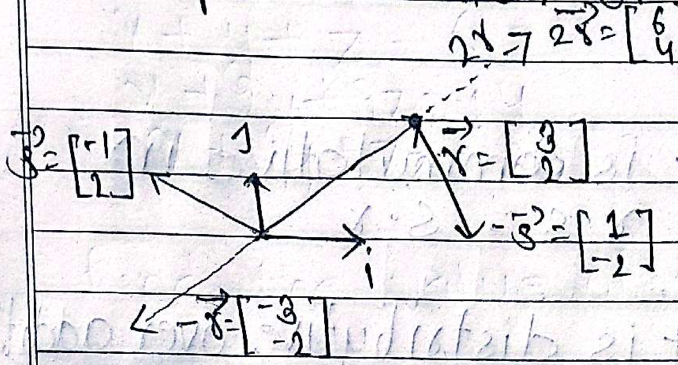
$$\vec{r} + \vec{s} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

- vector addition is associative

$$\vec{r} + \vec{s} = \vec{s} + \vec{r}$$

- Vector substraction is the addition of negative representation of a vector



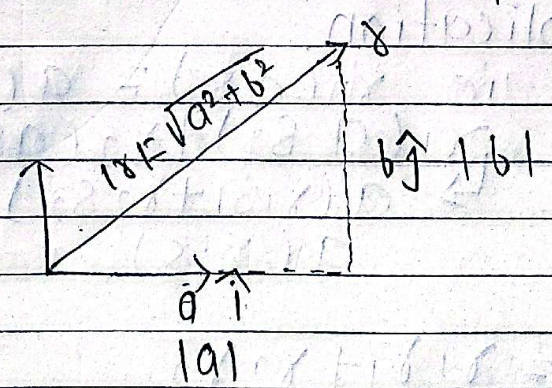
$$\vec{r} - \vec{s} = \begin{bmatrix} 3+1 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\vec{r} + (-\vec{r}) = \begin{bmatrix} 3-3 \\ 2-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- As we know from the attributes of house

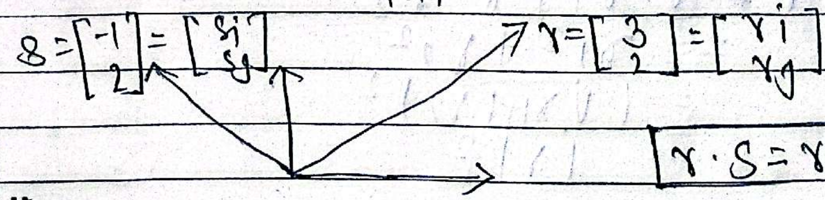
$$⇧ = \begin{bmatrix} 120 \\ 2 \\ 1 \\ 150 \end{bmatrix} \qquad ⇧ ⇧ = \begin{bmatrix} 120 \\ 2 \\ 1 \\ 150 \end{bmatrix} \times 2 = \begin{bmatrix} 240 \\ 4 \\ 2 \\ 300 \end{bmatrix}$$

- Moduls and the dot product of vector



$$r = a\hat{i} + b\hat{j} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$|r| = \sqrt{a^2 + b^2}$$



$$\boxed{r \cdot s = r_i s_i + r_j s_j}$$

$r \cdot s = r_i s_i + r_j s_j$
$= 3 \times (-1) + 2 \times 2$
$= 1$

- Dot product is commutative in nature ie $r \cdot s = s \cdot r$

- Dot product is distributive over addition.
$$r \cdot (s+t) = r \cdot s + r \cdot t$$

proof ÷

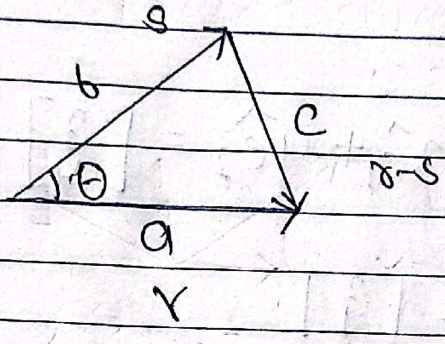
$$r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \quad s = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \quad t = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}$$

$$r \cdot (s+t) = r_1(s_1+t_1) + r_2(s_2+t_2) + \dots + r_n(s_n+t_n)$$
$$= r_1 s_1 + r_1 t_1 + \dots + r_n s_n + r_n t_n$$
$$= r \cdot s + r \cdot t$$

- Dot product is associative over scaler multiplication
ie $r \cdot (as) = a(r \cdot s)$
$r_1(a s_1) + r_2(a s_2)$
$\rightarrow a(r_1 s_1 + r_2 s_2)$
$a(r \cdot s)$

$r \cdot r = r_1 r_1 + r_2 r_2$
$= r_1^2 + r_2^2$
$= (\sqrt{r_1 + r_2})^2$
$= |r|^2$

- The cosine rule & the dot product.



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$|r-s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos\theta$$

Taking LHS $= |r-s|^2$

$= (r-s)\cdot(r-s)$

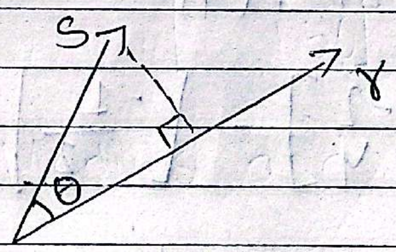
$= r\cdot r - r\cdot s - s\cdot r + s\cdot s$

$= |r|^2 - 2s\cdot r + |s|^2$

$= |r|^2 + |s|^2 - 2|s||r|\cos\theta$

$$\boxed{r\cdot s = |r||s|\cos\theta}$$

- Projection



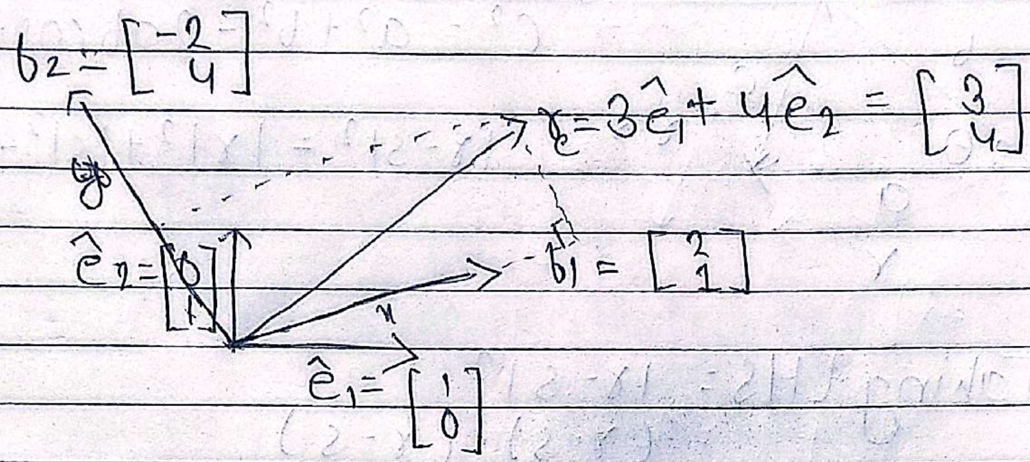
projection of $\vec{a}$ on $\vec{b}$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ (scaler)

vector projection of $\vec{a}$ on $\vec{b}$

$= \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right) \times \vec{b}$

Coordinate system of vector space:



$$x = \frac{r_e \cdot b_1}{|b_1|^2} \times b_1 = \frac{3 \times 2 + 4 \times 1}{2^2 + 1^2} b_1 = \frac{10}{5} b_1$$

$$= 2 \times b_1 = 2 \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$y = \frac{r_e \cdot b_2}{|b_2|^2} \times b_2 = \frac{3 \times -2 + 4 \times 4}{(-2)^2 + (4)^2} \times b_2$$

$$= \frac{1}{2} b_2 = \frac{1}{2} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

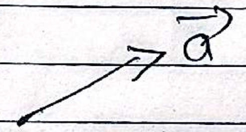
$\hat{i}$ and $\hat{j}$ are the basis vectors of the xy coordinate system.

We could have chosen a different basis vector and got a new coordinate system.

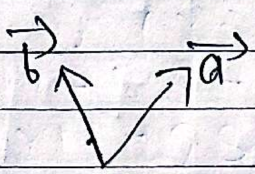
You can reach must of the points in the plane with $a\vec{v} + b\vec{w}$

The "span" of $\vec{v}$ and $\vec{w}$ is the combine set of all their linear Combination : $a\vec{v} + b\vec{w}$

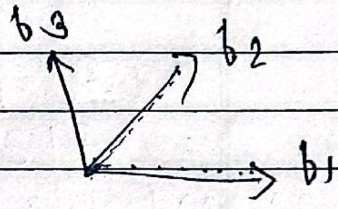
Basis is a set of n vector that :
i) are not linear combinations of each other (linearly independent)
ii) Span the space : The space is then n-dimension.



1-D



2-D



You can move anywhere in the line

You can move anywhere in the 2-D space

You can move freely in the 3D space if $b_3 \neq a_1 b_1 + a_2 b_2$

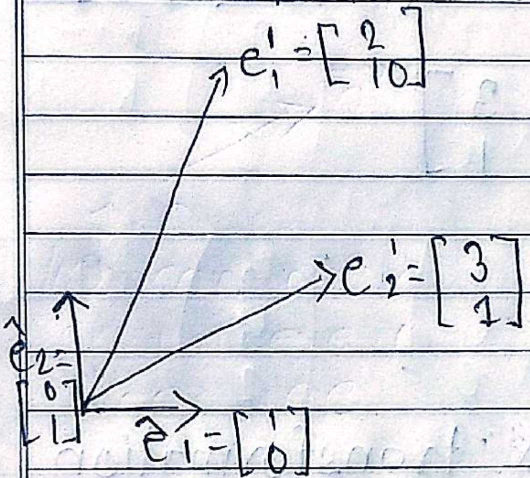
Matrices

$2a + 3b = 8$
$10a + 1b = 13$

converting in matrix form:

$$\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

How matrix transforms space:



$$\begin{pmatrix} 2 & 3 \\ 1 & 10 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 10 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Some rules:

$A \quad r = r'$
$A \quad (nr) = nr'$
$A \quad (r + s) = Ar + As$
$A \quad (n\hat{e}_1 + n\hat{e}_2) = nA\hat{e}_1 + mA\hat{e}_2$
$= ne_1' + me_2'$

$$\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 32 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \times \left(3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 3\left(\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2\left(\begin{bmatrix} 2 & 3 \\ 30 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$\begin{bmatrix}2&3\\10&1\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}12\\32\end{bmatrix}$$

$$=\begin{bmatrix}2&3\\10&1\end{bmatrix}\left(3\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$=3\left(\begin{bmatrix}2&3\\10&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)+2\left(\begin{bmatrix}2&3\\10&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right)$$

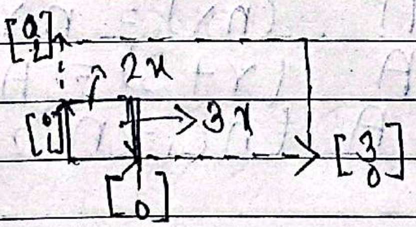
$$=3\begin{bmatrix}2\\10\end{bmatrix}+2\begin{bmatrix}3\\1\end{bmatrix}$$

$$=\begin{bmatrix}12\\32\end{bmatrix}$$

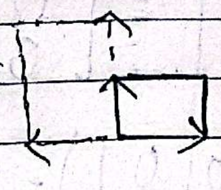
- Types of matrix transformation

A matrix is just composed of basis vector in the space ie $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ (Identity matrix)

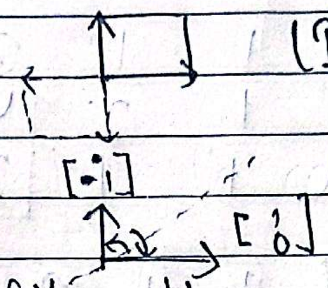
$$\begin{bmatrix}3&0\\0&1\end{bmatrix}=$$



$$\begin{bmatrix}-1&0\\0&2\end{bmatrix}=$$



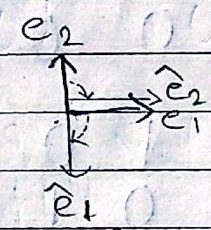
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \quad \text{(Inversion)}$$



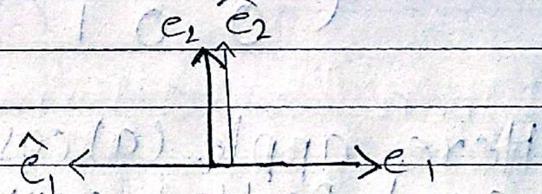
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \quad \text{(flips the axis)}$$

Composition or Combination of Matrix Transformation

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



$$A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$A_1 \cdot A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 \cdot A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

- Matrix Multiplication is not commutative.

working with $AA^{-1} = I$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 15 \\ 21 \\ 13 \end{pmatrix}$$

Elimination
$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1_{(+)} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \\ -2_{(+)} \end{pmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Back Substitution
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 2 \end{pmatrix} \begin{matrix} R_1 - 3R_3 \\ R_2 - R_3 \\ \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} \begin{matrix} R_1 - R_2 \\ \\ \end{matrix}$$

Here apple (a) cost Rs 5, banana (b) cost Rs 4 & carrot (c) cost Rs 2

- Going from Gaussian Elimination to find the Inverse Matrix.

$$A A^{-1} = I$$

lets say $A B = I$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ (-) \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & \underset{(+)}{-1} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ +1 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -2 & 0 & 3 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boxed{\begin{pmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}}$$

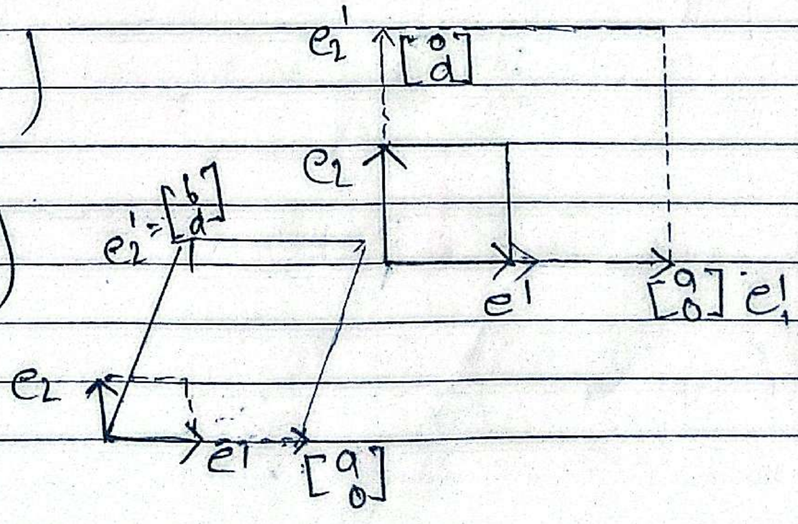
- Computationally this is one of the very effective way to solve these sort of problems.

This is the required A^{-1}/ B

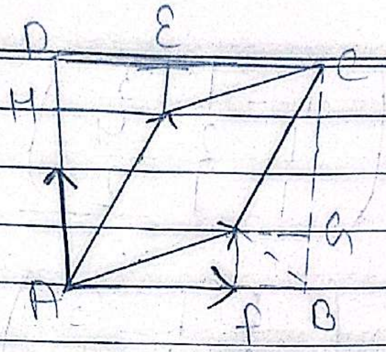
- Determinant and Inverse.

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



To find the area visualize all the empty spaces as a bunch of rectangle and triangle and substracting the empty spaces from the rectangle ABCD

$$|\text{Area}| = ad - bc$$

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

When the determinant are zero it means than the basis vector aren't independent that means the inverse doesn't exists

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Date:
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Eintein summation convention

$$\begin{array}{ccc} A & B & AB \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} & \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = & \begin{pmatrix} & \\ & \end{pmatrix} \end{array}$$

rows columns

$$(ab)_{23} = a_{21} b_{13} + a_{22} b_{23} + \cdots + a_{2n} b_{n3}$$

$$(ab)_{ik} = \sum_j a_{ij} b_{jk} = a_{ij} b_{jk}$$

$$AB = C$$

$$C_{ik} = a_{ij} b_{jk}$$

As long as you got the same number of j you can multiply the matrix

$$\begin{array}{c}3\\ 2\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\end{array} \begin{array}{c}4\\ 3\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}\end{array} = \begin{array}{c}4\\ 2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}\end{array}$$

The connection between dot product and projection.

Projection and dot product are symmetric. Projection is the dot product. A matrix multiplication of a with a vector is the projection of that vector onto the vector composing the matrix.

Changing the basis

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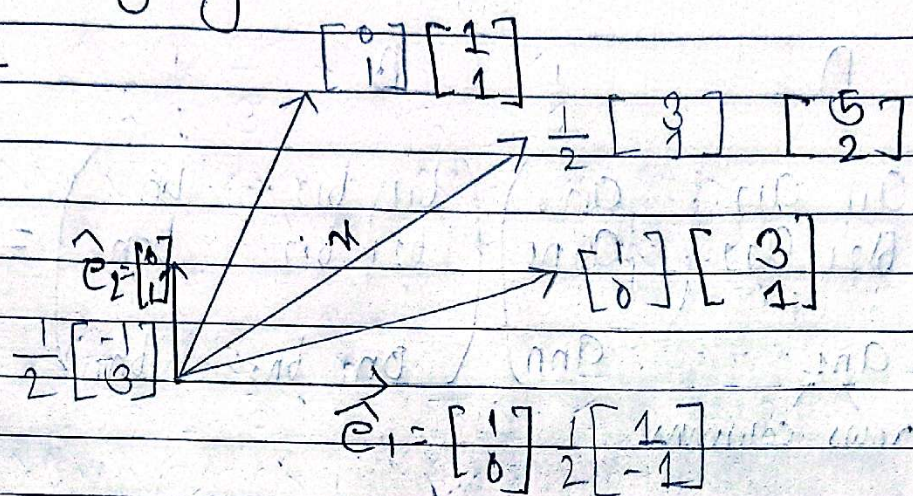


Bear's basic vector is $\begin{bmatrix}3\\1\end{bmatrix}$ & $\begin{bmatrix}1\\1\end{bmatrix}$ in my frame $\begin{bmatrix}3&1\\1&1\end{bmatrix}$

$$\begin{bmatrix}3&1\\1&1\end{bmatrix}\begin{bmatrix}3/2\\1/2\end{bmatrix}=\begin{bmatrix}5\\2\end{bmatrix}$$

Bear's basis in my world | Bear's vector | my vector

" B then $B^{-1}=\frac{1}{2}\begin{bmatrix}1&-1\\-1&3\end{bmatrix}$

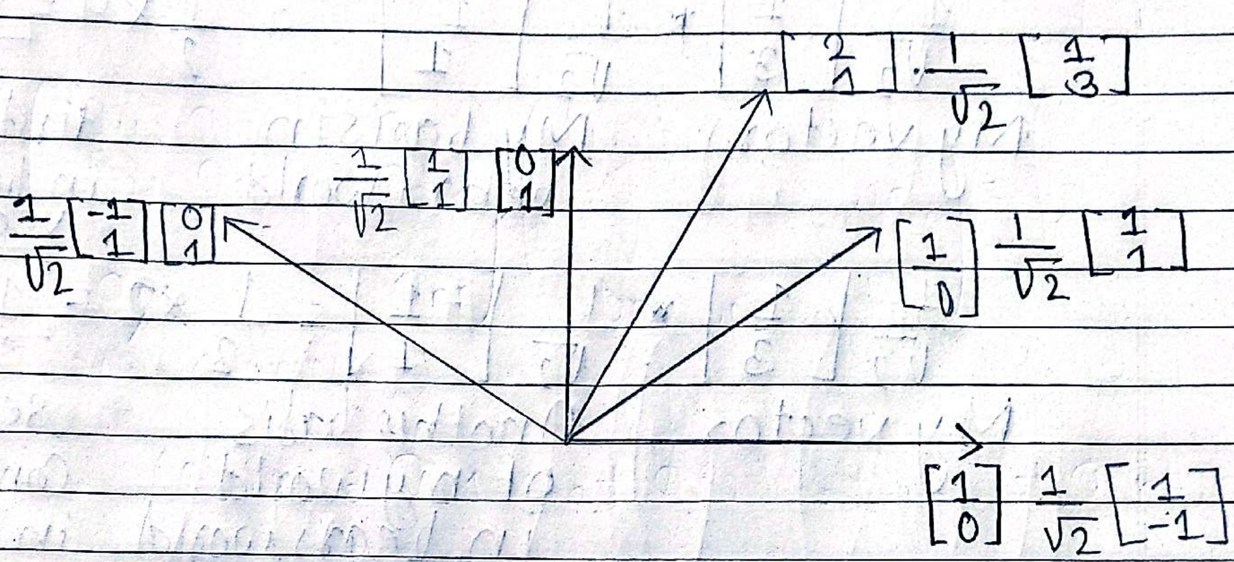
$$=\frac{1}{2}\begin{bmatrix}1&-1\\-1&3\end{bmatrix}$$

my basis in bears world.

$$\frac{1}{2}\begin{bmatrix}1&-1\\-1&3\end{bmatrix}\times\begin{bmatrix}5\\2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}3\\2\end{bmatrix}$$

my basis in bear's world | Bears vector

Pen = My world pencil = bear's world.



B

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Bears basis vector in my frame — Bear's vector — In my world

for doing reverse transformation we need

$$B^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

My basis vector in bears frame — My vector — Bear's vector.

If the new basis vector are orthognal the we can solve it using projection.

Projections

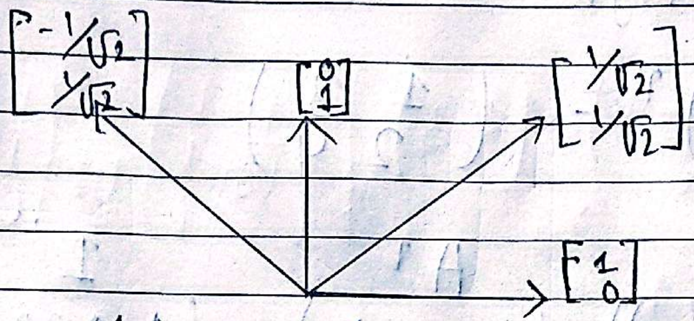
$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2} \times 4 = \boxed{2}$$

My vector — My basis in bear's world — first comp in bear's vector

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} \cdot \frac{1}{2} \times 2 = \boxed{1}$$

My vector — Another basis of my world in bear's world — sec Component in bear vector.

- Transformation of vector that's already in changed basis



{In My world}

$$R\circlearrowleft 45^\circ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

{In Bears world}
with basis $= \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ for our world

$$\underbrace{\underbrace{\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}}_{B^{-1}} \underbrace{\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{R\circlearrowleft 45^\circ} \underbrace{\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}}_{B} \cdot \begin{pmatrix} x \\ y \end{pmatrix}}_{\text{vector in my frame}}}_{\text{vector in Bears frame}}$$

$$B^{-1} R B = \underbrace{RB}$$

This gives us the rotation in Bear's world.

$$RB = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}$$

$$B^{-1}RB = \frac{1}{2\sqrt{2}}\begin{pmatrix} -2 & -2 \\ 10 & 6 \end{pmatrix}$$

- Orthogonal Matrices.

$$A^T_{ij} = A_{ji}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \begin{pmatrix} a_1 \end{pmatrix} \begin{pmatrix} a_2 \end{pmatrix} \begin{pmatrix} a_3 \end{pmatrix} \cdots \begin{pmatrix} a_n \end{pmatrix} \end{pmatrix}}_{A_{n \times n}} \times \underbrace{\begin{pmatrix} (\; a_1 \;) \\ (\; a_2 \;) \\ \vdots \\ (\; a_n \;) \end{pmatrix}}_{A^T} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{I}$$

$$\therefore A \times A^T = I$$

$$a_i \times a_j = 0 \quad i \neq j$$
$$= 1 \quad i = j$$

- How to construct orthonormal basis.

The Gram-Schmidt Process

let's say I have a set of linearly Independent vector $\vec{V_1}, \vec{V_2}, \ldots \vec{V_k}$ that are the basis for V

V_1 — $e_1 = \dfrac{V_1}{|V_1|} =$

V_2 — $e_2 = (V_2 \cdot e_1) \times \dfrac{e_1}{|e_1|} + U_2$

$$U_2 = V_2 - (V_2 \cdot e_1) \cdot e_1$$

$$\frac{U_2}{|U_2|} = e_2$$

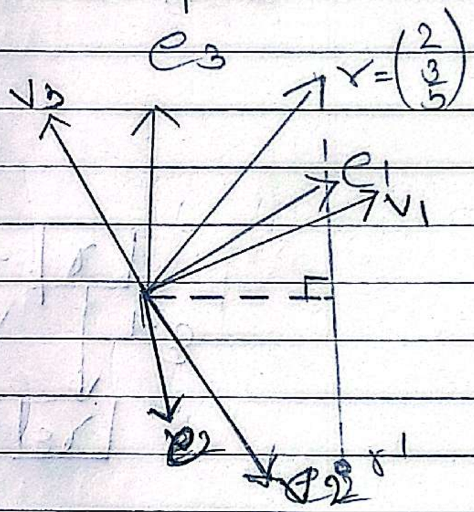
V_3 isn't in the plane defined by V_1 & V_2.

$$U_3 = V_3 - (V_3 \cdot e_1)e_1 - (V_3 \cdot e_2)e_2$$

$$\frac{U_3}{|U_3|} = e_3$$

Example: reflecting in a plane

$$\underset{V_1}{\begin{pmatrix}1\\1\\1\end{pmatrix}} \quad \underset{V_2}{\begin{pmatrix}2\\0\\1\end{pmatrix}} \quad \underset{V_3}{\begin{pmatrix}3\\1\\-1\end{pmatrix}}$$



$$e_1 = \frac{V_1}{|V_1|} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$U_2 = V_2 - (V_2 \cdot e_1)e_1 \qquad e_2 = \frac{U_2}{|U_2|}$$

$$= \begin{pmatrix}2\\0\\1\end{pmatrix} - \left(\begin{pmatrix}2\\0\\1\end{pmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right) \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$= \begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{1}{\cancel{3}} \times \cancel{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$U_3 = V_3 - (V_3 \cdot e_1)e_1 - (V_3 \cdot e_2)e_2 = \begin{bmatrix}1\\1\\-2\end{bmatrix} \qquad e_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

$V_1 = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad V_2 = \begin{pmatrix}2\\0\\1\end{pmatrix} \quad V_3 = \begin{pmatrix}3\\1\\-1\end{pmatrix}$

$\mathcal{E} = \begin{bmatrix} \begin{pmatrix}e_1\end{pmatrix} & \begin{pmatrix}e_2\end{pmatrix} & \begin{pmatrix}e_3\end{pmatrix}\end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} & \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\2\end{pmatrix}\end{bmatrix}$

$$\begin{array}{cc} T_{\mathcal{E}} = & \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \\ & \begin{matrix} e_1 & e_2 & e_3\end{matrix}\end{array}$$

$$\begin{tikzcd} \boxed{r} \arrow[r, "\text{Hard}"] \arrow[d, "\mathcal{E}^{-1}"'] \arrow[r, bend right] & \boxed{r'} \\ \boxed{r_{\mathcal{E}}} \arrow[r, "T_{\mathcal{E}}"'] & \boxed{r'_{\mathcal{E}}} \arrow[u, "\mathcal{E}"'] \end{tikzcd}$$

$\mathcal{E} T_{\mathcal{E}} \mathcal{E}^{-1} r = r'$

We Here know that $\mathcal{E}^T = \mathcal{E}^{-1}$

$r' = \mathcal{E} T_{\mathcal{E}} \mathcal{E}^{-1} r$

$T_{\mathcal{E}} \cdot \mathcal{E}^{-1} = T_{\mathcal{E}} \cdot \mathcal{E}^T = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \begin{pmatrix} 1/\sqrt{3}\,(1\ 1\ 1) \\ 1/\sqrt{2}\,(1\ {-1}\ 0) \\ 1/\sqrt{6}\,(1\ 1\ 2)\end{pmatrix}$

$= \begin{pmatrix} 1/\sqrt{3}\,(1\ {-1}\ 1) \\ 1/\sqrt{2}\,(1\ {-1}\ 0) \\ 1/\sqrt{6}\,(1\ 1\ 2)\end{pmatrix}$

$$\mathcal{E}^T\mathcal{E}^T = \begin{pmatrix} \frac{1}{3}+\frac{1}{2}-\frac{1}{6} & \frac{1}{3}-\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+0+\frac{3}{6} \\ \frac{1}{3}-\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+\frac{1}{2}-\frac{1}{6} & \frac{1}{3}+0+\frac{1}{6} \\ \frac{1}{3}+0+\frac{2}{6} & \frac{1}{3}+0+\frac{2}{6} & \frac{1}{3}+0-\frac{4}{6} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix} = T$$

$$r' = T \cdot r = T\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 11 \\ 14 \\ 5 \end{pmatrix}$$

Process

- Given a subspace W of R^n, we should like to find an orthogonal or orthonormal basis for W.

- The Gram-Schmidt process allows up to start with any basis for W and construct a new basis that is orthogonal (or orthonormal, as desired).

let $\{x_1, x_2, \cdots, x_p\}$ be a basis for W. Then we construct new vectors $v_1, v_2, \cdots, v_p$ as follows:

- $v_1 = x_1$
- $v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$ projection of $\vec{x_2}$ onto $\vec{v_1}$
- $v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$

$\cdots\cdots\cdots\cdots$

- $v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$

$\therefore$ The vector $\{v_1, v_2, \cdots, v_p\}$ form an orthogonal basis for W.

Idea of Proof:

- v_2 is the component of x_2 that is orthogonal to v_1.
- v_3 is the component of x_3 that is orthogonal to both v_1 and v_2.
- etc

Orthogonal Basis.

- If an orthonormal basis is desired, then we first use Gram-Schmidt to find an orthogonal basis.
- Then we normalize each vector to obtain an orthonormal basis.

Example:

Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, and $x_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}$

and let $W = \text{Span}\{x_1, x_2, x_3\}$

find an ortho~~gona~~normal basis for W.

Example:

- $v_1 = x_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$
- $v_2 = x_2 - \dfrac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \left(\dfrac{-2}{6}\right) \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1 \\ 2/3 \\ 2/3 \end{bmatrix}$
- $v_3 = x_3 - \dfrac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \dfrac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 4/7 \\ 8/7 \\ -4/7 \\ -4/7 \end{bmatrix}$

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$$V_1 = \frac{v_1}{|v_1|} = \frac{v_1}{\sqrt{6}} = \begin{bmatrix} 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$V_2 = \frac{v_2}{|v_2|} = \frac{v_2}{\sqrt{7/3}} = \begin{bmatrix} -2/\sqrt{21} \\ \sqrt{3/7} \\ 2/\sqrt{21} \\ 2/\sqrt{21} \end{bmatrix}$$

$$V_3 = \frac{v_3}{|v_3|} = \frac{v_3}{4/\sqrt{7}} = \begin{bmatrix} 1/\sqrt{7} \\ 2/\sqrt{7} \\ -1/\sqrt{7} \\ -1/\sqrt{7} \end{bmatrix}$$

Using Mathematica

- Since this process is tedious we use technology to perform it.
- In Mathematica, we use the Orthogonalize command to produce orthogonal basis.

Orthogonalize [{{1,0,2,-1}, {-1,1,0,1}, {2,0,0,-2}}]

Output: $\left\{\left\{\frac{1}{\sqrt{6}}, 0, \sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{6}}\right\}, \left\{-\frac{2}{\sqrt{21}}, \sqrt{\frac{3}{7}}, \frac{2}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right\},\right.$

$\left.\left\{\frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}}, -\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right\}\right\}$

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Eigenproblems

Eigen vectors are those which lie along the same span both before and after apply a linear transformation into the space.

Eigen value as the amount each of those vectors have stretched during the process.

Special cases:-

- for uniform scaling any vector will be an Eigen vector.
- for rotation there are no Eigen vector but for 180° rotation the three Eigon vectors are still lying on the same span as before but pointing in opposite direction. All vector for this transformation are Eigen vector and they all have Eigen value -1.
- for Horizontal scaling and shearing in 2D the x axis vector is obviously Eigen but other Eigen vector are difficult to observe.
- 3D rotation ÷ Eigen vector = axis of rotation

Mathematics of Eigen vector (x)

$$Ax = \lambda x$$

A → It must be $n \times n$ matrix

x → It also must be a n dimensional matrix

$$\underbrace{(A - \lambda I)}\, x = 0$$

$$\det(A - \lambda I) = 0 \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

Eigen for Vertical scaling

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(2-\lambda) = 0$$

$$(A - \lambda I)x = 0$$

$$@\ \lambda = 1 \quad \begin{pmatrix} 1-1 & 0 \\ 0 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = 0$$

@ $\lambda = 2$ $\begin{pmatrix} 1-2 & 0 \\ 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_1 \\ 0 \end{pmatrix} = 0$$

@ $\lambda = 1$ $x = \begin{pmatrix} t \\ 0 \end{pmatrix}$ @ $\lambda = 2$ $x = \begin{pmatrix} 0 \\ t \end{pmatrix}$

Eg :
Rotation by 90° anticlockwise

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\det \begin{pmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{pmatrix}$

$$f(\lambda) = \lambda^2 + 1 = 0$$

Calculating Eigen.

$$T = \begin{pmatrix} 0.9 & 0.8 \\ -1 & 0.35 \end{pmatrix} \qquad v_0 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$V_1 = T V_0$$

$$V_n = T^n V_0$$

For a diagonal matrix

$$T^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$$

$$C = \begin{pmatrix} x_1 & x_2 & x_3 \\ | & | & | \\ | & | & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$T = CDC^{-1}$$

$$T^2 = CDC^{-1}CDC^{-1} = CDDC^{-1}$$
$$= CD^2C^{-1}$$

$$T^n = CD^nC^{-1}$$

Eigen basis diagonalization :

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

Show that the following matrix is diagonalizable.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{pmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - 6 = 0$$

$$\lambda^2 - 7x + 6 = 0$$

$$\underline{(\lambda-6)}\ \underline{(\lambda-1)} = 0$$

$$\lambda = 6 \quad \lambda = 1$$

Eigen vector for $\lambda_1 = 1$

$(A - \lambda I)(x) = 0$

$@\lambda=1 \begin{pmatrix} 4-1 & 2 \\ 3 & 3-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$(R_2 - R_1)$

$\begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$3x_1 + 2x_2 = 0$

for $x_2 = 3k$

$3x_1 = -6k$

$x_1 = -2k$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2k \\ 3k \end{pmatrix}$

$= k' \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Eigen vec $= \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

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ⓐ $\lambda = 6$ $\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$-2x_1 + 2x_2 = 0$$

$$(x_1 = x_2 = k)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -3 & 6 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 30 \end{bmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

1) $\frac{1}{2} tr \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2} = \frac{\lambda_1 + \lambda_2}{2} = m$

2) $\det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc = \lambda_1 \lambda_2 = p$

3) $\lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}$

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