Tutorial solutions (Part-II)

Om Swostik

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1 Tutorial-1

1:

Given $\phi = \forall x \exists y\, R(x, y) \;\wedge\; \exists y \forall x\, \neg R(x, y)$.

Take the model to be the set of naturals $\mathbb{N}$ with $<$ relation.

Then, $m \models \phi$ (Why?)

(**Hint**: $\mathbb{N}$ is an well-ordered set (i.e has a minimum) and isn't bounded above)

2:

$\varphi_B(x, y) = \exists z(P(z, x) \wedge P(z, y)) \wedge \neg F(x)$

$\varphi_A(x, y) = \exists z(P(z, y) \wedge \varphi_S(x, z)) \qquad (\varphi_S(x, y) = \exists z(P(z, x) \wedge P(z, y)) \wedge F(x)$, x is sister of y)

$\varphi_C(x, y) = \exists z(\varphi_A(z, x) \wedge P(z, y))$

$\varphi_O(x) = \forall z, y(P(z, y) \wedge P(z, x) \Rightarrow (x = y))$

The spousal relationship cannot be defined (Why?)

3:

$Zero(x) = \; +(x, x) = x$

$One(x) = \forall y(\times(x, y) = y)$

$Two(x) = \exists z((+(z, z) = x) \wedge One(z))$

$Even(x) = \exists z, y((\times(z, y) = x) \wedge Two(y))$

$Odd(x) = \neg Even(x)$

$Prime(x) = \neg One(x) \wedge (\neg \exists w, y((\times(w, y) = x) \wedge (\neg One(w) \wedge \neg One(y))))$

Goldbach conjecture in FO:

$\forall x(\neg One(x) \wedge \neg Two(x) \wedge Even(x) \Rightarrow \exists z, w(Prime(z) \wedge Prime(w) \wedge +(z, w) = x))$

4:

Encoding associativity of $+$: $\forall x, y, z(+(x, +(y, z)) = +(+(x, y), z))$

Encoding the right identity as 0: $\forall x(+(x, 0) = x)$

Encoding right inverse: $\forall x \exists y(+(x, y) = 0)$

Encoding A(4): $\forall x, y, z(+(x, z) = +(y, z) \Rightarrow x = y)$

Here we have used the signature $\tau = (0, +)$.

5:

(i) Consider the set of integers $\mathbb{Z}$ with the induced relation $+_Z$ referring to the usual addition in $\mathbb{Z}$. The constant 0_Z refers to 0 in $\mathbb{Z}$. Observe that addition is associative and admits both left and right inverses. Also 0 is a identity for addition. We can conclude the τ-structure $\mathbb{Z}$ satisfies ψ.

(ii) Consider the set $\mathbb{N}_0$ of whole numbers and the corresponding induced relation being addition and the constant being 0 (in $\mathbb{N}_0$). This τ-structure doesn't satisfy ψ as φ_3 fails to be true (non-zero elements in $\mathbb{N}_0$ don't have inverses).

(iii) Consider the set of all $n \times n$ invertible matrices with complex values, $GL_n(\mathbb{C})$. Let the induced binary operation be matrix multiplication and let the constant 0 map to the identity $n \times n$ matrix.

It's clear that the τ-structure $GL_n(\mathbb{C})$ satisfies ψ, however, it doesn't satisfy $\forall x, y(+(x, y) = +(y, x))$ (Why?).

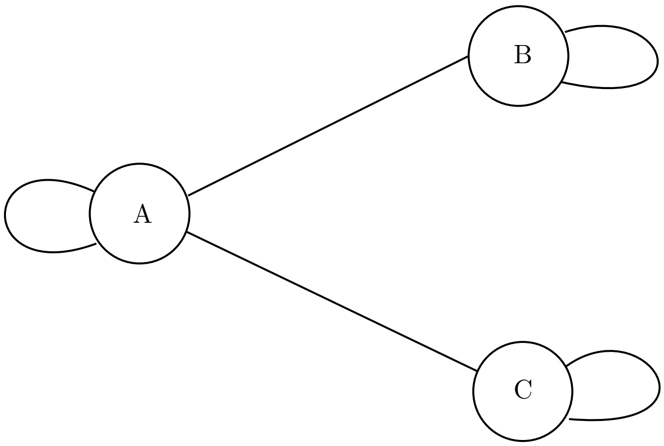
(iv) As before, consider the set $\mathbb{N}_0$ of whole numbers with the usual addition. This satisfies $\varphi_1 \wedge \varphi_2$ but doesn't satisfy φ_3.

Consider the set of non-negative reals $\mathbb{R}_{\geq 0}$, with the binary operation defined as $+(a, b) = |a - b|$ and the constant mapping to 0. Show that this structure satisfies $\varphi_2 \wedge \varphi_3$ but fails to satisfy φ_1.

Consider $\mathbb{Z}$ with the usual addition and the constant 0 mapping to 1 (in $\mathbb{Z}$). This satisfies $\varphi_1 \wedge \varphi_3$ but fails to satisfy φ_2.

We can conclude that ψ isn't equivalent to any of $\varphi_1 \wedge \varphi_2$, $\varphi_2 \wedge \varphi_3$ or $\varphi_1 \wedge \varphi_3$.

7:



Consider the undirected graph $\mathcal{G}$ above (with loops). This (with its natural edge relation) satisfies the second formula but not the first.

8:

$\exists^{\geq n}x(x = x) \ \wedge \ \neg\exists^{\geq n+1}x(x = x)$ is true for all models whose universe has exactly n elements.

Let $\varphi = \exists x_1, x_2 \ldots x_n(\wedge_{i \neq j}(x_i \neq x_j))$.

$\varphi \equiv \exists^{\geq n}x(x = x)$ (Why?)

9:

Using counting quantifiers, we can write,

$\varphi = \exists^{\geq n}x(x = x) \ \wedge \ \neg\exists^{\geq m+1}x(x = x)$

φ evaluates to true only over models with atleast n and atmost m elements.