Tutorial-7

Om Swostik

15.9(ii):

Asssume for contradiction there exists a FO sentence F without $=$ such that F doesn't satisfy any model with > 2 elements.

Let m be a model over $\{a, b\}$ such that $m \models F$. (with any signature)

Construct a model $m^{'}$ (over the same signature) such that $D_{m'} = \{a, b, c\}$ and $m^{'}(x) = m(x)$ $(\forall x \in FVars)$

And $m^{'}(c) = a$, $m^{'}$ is identity over $\{a, b\}$.

If $f \in R$, $f_{m'}(l_1, \ldots, l_n)$ is true if $f_m(m^{'}(l_1), \ldots, m^{'}(l_n))$ holds. $(l_i \in \{a, b, c\})$

For $g \in F$, $g_{m'}(l_1, \ldots, l_n) = g_m(m^{'}(l_1), \ldots, m^{'}(l_n))$ $(l_i \in \{a, b, c\})$

(Basically replace all occurences of c with a)

Observe that, since we don't have $=$, $m^{'} \models F$.

Contradiction. (since $|D_{m'}| = 3$)

16.6:

This is a tricky problem. It is quite tempting to try $\forall$-elim n times and substitute the appropriate terms (wrt σ)

However, this approach fails as the following example shows:

Consider the formula $\forall x_1, x_2, x_3\ \ x_1 = x_3$ and let $\sigma = [x_1 \to x_2, x_2 \to x_1]$

If we try to apply $\forall$-elim starting from x_1 and substituting each term (according to σ), we get the formula $x_1 = x_3$.

However, $F\sigma =$ $(x_2 = x_3)$ (As x_2 doesn't occur in F)

A possible solution to this issue is the following:

We use extra variables $y_1, y_2 \ldots y_n$ (y_i are fresh variables not in F and distinct from x_i)

If σ maps x_i to t_i, we replace all occurences of x_i in t_i with y_i to create a new term $t_i^{'}$

We apply $\forall$-elim on the x_i's and replace with the corresponding terms $t_i^{'}$

We obtain a formula $F^{'}$ which has some y_i's occuring in it (we want to replace these y_i's with x_i's to obtain $F\sigma$)

Now apply $\forall$-intro for y_i's in order, starting from y_1 (Why can we do this?)

Now, apply $\forall$-elim again (on the the universally quantified y_i's) and replace each term with the corresponding x_i to get $F\sigma$ (Check that $\forall$-elim can be applied in this context)

With this, we get a proof of atmost $2n$ steps

16.8:

We will simulate $\forall$-elim using $\exists$-def and other proof rules

t refers to any arbitrary term

1. $\Sigma \vdash \forall x F(x)$ (Premise)
2. $\Sigma \vdash \neg\exists x(\neg F(x))$ ($\exists$-def-1)
3. $\Sigma \cup \{\neg F(t)\} \vdash \neg F(t)$ (Associativity)
4. $\Sigma \cup \{\neg F(t)\} \vdash \exists x(\neg F(x))$ ($\exists$-intro-3)
5. $\Sigma \cup \{\neg F(t)\} \vdash \neg\exists x(\neg F(x))$ (Monotonicity-2)
6. $\Sigma \vdash \neg\neg F(t)$ (By-Contra-4-5)
7. $\Sigma \vdash F(t)$ (Double-neg-elim-6)

17.4.1:

1. $\phi \vdash R(s_1, s_2) \vee \neg R(s_1, s_2)$ (Tautology)
2. $\phi \vdash \exists w(R(s_1, s_2) \vee \neg R(w, s_2))$ ($\exists$-intro-1)
3. $\phi \vdash \forall z \exists w(R(s_1, s_2) \vee \neg R(w, z))$ ($\forall$-intro-2)