# CS-228
# Logic

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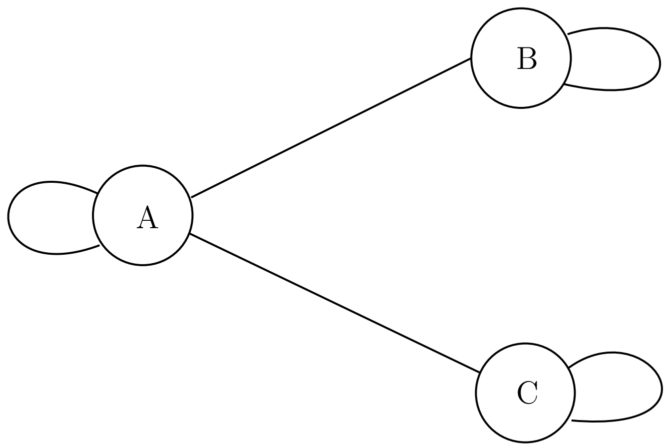
# Contents

# Tutorial-1

**1**: Given $\phi = \forall x \exists y \, R(x, y) \ \wedge \ \exists y \forall x \, \neg R(x, y)$. Take the model $m$ to be the set of naturals $\mathbb{N}$ with $<$ relation. Then, $m \models \phi$ (Why?)
(**Hint**: $\mathbb{N}$ is an well-ordered set (i.e has a minimum) and isn't bounded above)
**2**: $\varphi_B(x, y) = \exists z(P(z, x) \wedge P(z, y)) \wedge \neg F(x)$
$\varphi_A(x, y) = \exists z(P(z, y) \wedge \varphi_S(x, z))$ $\quad$ ($\varphi_S(x, y) = \exists z(P(z, x) \wedge P(z, y)) \wedge F(x)$, $x$ is sister of $y$)
$\varphi_C(x, y) = \exists z(\varphi_A(z, x) \wedge P(z, y))$
$\varphi_O(x) = \forall z, y(P(z, y) \wedge P(z, x) \Rightarrow (x = y))$
The spousal relationship cannot be defined (Why?)
**3**: $Zero(x) = \ +(x, x) = x$
$One(x) = \forall y(\times(x, y) = y)$
$Two(x) = \exists z((+(z, z) = x) \wedge One(z))$
$Even(x) = \exists z, y((\times(z, y) = x) \wedge Two(y))$
$Odd(x) = \neg Even(x)$
$Prime(x) = \neg One(x) \wedge (\neg \exists w, y((\times(w, y) = x) \wedge (\neg One(w) \wedge \neg One(y))))$
Goldbach conjecture in FO: $\forall x(\neg One(x) \wedge \neg Two(x) \wedge Even(x) \Rightarrow \exists z, w(Prime(z) \wedge Prime(w) \wedge +(z, w) = x))$
**4**: Encoding associativity of $+$: $\forall x, y, z(+(x, +(y, z)) = +(+(x, y), z))$
Encoding the right identity as 0: $\forall x(+(x, 0) = x)$
Encoding right inverse: $\forall x \exists y(+(x, y) = 0)$
Encoding A(4): $\forall x, y, z(+(x, z) = +(y, z) \Rightarrow x = y)$
Here we have used the signature $\tau = (0, +)$.
**5**: **(i)** Consider the set of integers $\mathbb{Z}$ with the induced relation $+_Z$ referring to the usual addition in $\mathbb{Z}$. The constant $0_Z$ refers to 0 in $\mathbb{Z}$. Observe that addition is associative and admits both left and right inverses. Also $0_{\mathbb{Z}}$ is a identity for addition. We can conclude the $\tau$-structure $\mathbb{Z}$ satisfies $\psi$.
**(ii)** Consider the set $\mathbb{N}_0$ of whole numbers and the corresponding induced relation being addition and the constant being 0 (in $\mathbb{N}_0$). This $\tau$-structure doesn't satisfy $\psi$ as $\varphi_3$ fails to be true (non-zero elements in $\mathbb{N}_0$ don't have inverses).
**(iii)** Consider the set of all $n \times n$ invertible matrices with complex values, $GL_n(\mathbb{C})$.

Let the induced binary operation be matrix multiplication and let the constant 0 map to the identity $n \times n$ matrix.
It's clear that the $\tau$-structure $GL_n(\mathbb{C})$ satisfies $\psi$, however, it doesn't satisfy $\forall x, y(+(x, y) = +(y, x))$ (Why?).
**(iv)** As before, consider the set $\mathbb{N}_0$ of whole numbers with the usual addition. This satisfies $\varphi_1 \wedge \varphi_2$ but doesn't satisfy $\varphi_3$.
Consider the set of non-negative reals $\mathbb{R}_{\geq 0}$, with the binary operation defined as $+(a, b) = |a - b|$ and the constant mapping to 0. Show that this structure satisfies $\varphi_2 \wedge \varphi_3$ but fails to satisfy $\varphi_1$.
Consider $\mathbb{Z}$ with the usual addition and the constant 0 mapping to 1 (in $\mathbb{Z}$). This satisfies $\varphi_1 \wedge \varphi_3$ but fails to satisfy $\varphi_2$.
We can conclude that $\psi$ isn't equivalent to any of $\varphi_1 \wedge \varphi_2$, $\varphi_2 \wedge \varphi_3$ or $\varphi_1 \wedge \varphi_3$.

**7**:




Consider the undirected graph $\mathcal{G}$ above (with loops). This (with its natural edge relation) satisfies the second formula but not the first.
**8**: $\exists^{\geq n} x(x = x) \wedge \neg\exists^{\geq n+1} x(x = x)$ is true for all models whose universe has exactly $n$ elements.
Let $\varphi = \exists x_1, x_2 \ldots x_n(\wedge_{i \neq j}(x_i \neq x_j))$.
$\varphi \equiv \exists^{\geq n} x(x = x)$ (Why?)
**9**: Using counting quantifiers, we can write,
$\varphi = \exists^{\geq n} x(x = x) \ \wedge \ \neg\exists^{\geq m+1} x(x = x)$
$\varphi$ evaluates to true only over models with atleast $n$ and atmost $m$ elements.

# Tutorial-2

**1**(a) Any word $w$ over $\{a, b\}$ having equal number of occurences of $ab$ and $ba$ must start and end with the same character. (Why?)
$Q_a(first) = \exists x(Q_a(x) \wedge \forall y(x \leq y))$ (a is the first character in $w$)
$Q_a(last) = \exists x(Q_a(x) \wedge \forall y(y \leq x))$ (a is the last character in $w$)
Now, let $\varphi = (Q_a(first) \wedge Q_a(last)) \vee (Q_b(first) \wedge Q_b(last))$
$\varphi$ is only satisfied by words which have equal number of $ab$ and $ba$.
(b) $\exists x(Q_{\#}(x) \wedge \forall y(x < y \Rightarrow Q_b(y)) \wedge \forall z(z < x \Rightarrow Q_a(z)))$
(c) $\forall x, y(S(x, y) \wedge Q_b(x) \Rightarrow Q_b(y))$ (Note that word with all $a$'s and empty word also need to satisfy)
(d) $Q_0(second) = \exists x(\forall y(x \leq y) \wedge \forall y(S(x, y) \Rightarrow Q_0(y)))$
$Q_0(second - last) = \exists x(\forall y(y \leq x) \wedge \forall z(S(z, x) \Rightarrow Q_0(z)))$
Now, let $\varphi = Q_0(second) \wedge Q_0(second - last)$
(e) $\Sigma = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\}$

Now, for the top row to be bigger than bottom row, there must exist a $\begin{pmatrix}1\\0\end{pmatrix}$ somewhere and everything before this must only consist of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}0\\0\end{pmatrix}$.
$\exists x(Q_{\{1\,0\}}(x) \wedge \forall y(y < x \Rightarrow Q_{\{0\,0\}}(y) \vee Q_{\{1\,1\}}(y)))$
**2** FO $\subseteq$ Regular languages