

MA-412

Complex Analysis

Om Swostik

<https://github.com/swostikom20/>

Spring Semester 2022-2023

§1. Preliminaries

Region: An open, connected subset of \mathbb{C} is called domain or region.

$$D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$$

$$\{z \in \mathbb{C} \mid r < |z - a| < R\} \rightarrow \text{annulus}$$

Exponential function: $\exp(z) : \mathbb{C} \rightarrow \mathbb{C}$

$$e^z = e^x(\cos(y) + i\sin(y))$$

$$e^{z_1+z_2} = e^{z_1}.e^{z_2}, 0 \notin \text{Rng}(\exp(z))$$

$$|e^{ix}| = 1 \quad (\forall x \in \mathbb{R})$$

$$e^z = 1 \text{ iff } z = 2n\pi i, n \in \mathbb{Z}$$

$$e^{z_1} = e^{z_2} \text{ iff } z_1 = z_2 + 2n\pi i$$

Argument: $\arg(z) : \mathbb{C} \rightarrow \mathbb{C}$

$$\arg(z) = \theta \quad (\text{Angle made wrt positive real axis})$$

$$\arg(z) \text{ for } z \in \mathbb{C} \setminus \{0\} \text{ is multivalued function}$$

$$\arg(z) : \mathbb{C} \setminus \{0\} \rightarrow (\alpha, \alpha + 2\pi] \text{ or } [\alpha, \alpha + 2\pi) \text{ is well-defined } (\forall \alpha \in \mathbb{R})$$

Principal value: $\text{Arg}(z) = \theta \quad (-\pi < \theta \leq \pi)$

$$\arg(z) = \{\text{Arg}(z) + 2n\pi : n \in \mathbb{Z}\}$$

Complex Log: For $z \in \mathbb{C} \setminus \{0\}$, define:

$$\log(z) = \log(|z|) + i\arg(z)$$

Notice that \log is multivalued

Whenever $\arg(z)$ is well-defined (i.e $\text{Rng}(\arg(z)) = [\alpha, \alpha + 2\pi)$ or $(\alpha, \alpha + 2\pi]$)
 $\Rightarrow \log(z)$ is well defined

$$\text{Log}(z) = \log(|z|) + i.\text{Arg}(z) \text{ (principal log)}$$

$$\log(z) = \text{Log}(z) + 2n\pi i$$

Lemma 1.1: For $z \in \mathbb{C} \setminus \{0\}$, the values of $\log(z)$ are the complex numbers ω such that $e^\omega = z$

Proof: Do it on your own!

Limit: $f : E \subset \mathbb{C} \rightarrow \mathbb{C}$

$$\lim_{z \rightarrow z_0} f(z) = l$$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$$

$$z \in (B(z, z_0) \setminus \{0\}) \cap E \Rightarrow f(z) \in B(l, \epsilon)$$

Proposition 1.1 (Sequential criteria for limit):

A function $f : E \subset \mathbb{C} \rightarrow \mathbb{C}$ has a limit point l as $z \rightarrow z_0$ iff $f(z_n) \rightarrow l$ for every sequence $\{z_n\} \subset E \setminus \{z_0\}$ with $z_n \rightarrow z_0$ as $n \rightarrow \infty$

Theorem 1.1: If $f : E \subset \mathbb{C} \rightarrow \mathbb{C}$ has a limit at z_0 , then f is bounded near z_0

Continuity: $f : E \subset \mathbb{C} \rightarrow \mathbb{C}$

f is continuous at $z_0 \in E$ if for every basic neighbourhood V of $f(z_0)$, there is a basic neighbourhood U of z_0 such that $f(U) \subset V$. Let $V = B(f(z_0), \epsilon)$ & $U = B(z_0, \delta)$,

$$z \in B(z_0, \delta) \Rightarrow f(z) \in B(f(z_0), \epsilon)$$

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

$$\text{Then, } z \in B(z_0, \delta) \setminus \{z_0\} \Rightarrow f(z) \in B(f(z_0), \epsilon)$$

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

Prove: Let $f : G \rightarrow \mathbb{C}$ such that $f = u + iv$ (where $u, v : G \rightarrow \mathbb{R}$)

f is continuous at a point z_0 iff u, v are continuous at z_0

§2. Branch of Log and Power function

$$\text{Arg} : \mathbb{C} \setminus \{0\} \rightarrow (-\pi, \pi]$$

$$\text{Log}(z) = \log(|z|) + i.\text{Arg}(z) \text{ (defined on } \mathbb{C} \setminus \{0\})$$

$$\text{Log}(z) \text{ is continuous on } \mathbb{C} \setminus (-\infty, 0] \text{ as } \text{Arg}(z) \text{ is continuous on } \mathbb{C} \setminus (-\infty, 0]$$

§§2.1. Branch of Log

Let $G \subset \mathbb{C}$ be a region. A continuous function $f : G \rightarrow \mathbb{C}$ is called a branch of logarithm in G if:

$$e^{f(z)} = z \quad (\forall z \in G) (\Rightarrow 0 \notin G)$$

$$e^{\text{Log}(z)} = z \quad (\forall z \in \mathbb{C} \setminus \{0\})$$

But $\text{Log}(z)$ isn't continuous on $\mathbb{C} \setminus \{0\}$

Therefore, $\text{Log}(z)$ is a branch of log in $\mathbb{C} \setminus (-\infty, 0]$

$$f_k = \text{Log}(z) + 2k\pi i, \quad k \in \mathbb{Z}$$

$$e^{f_k(z)} = z \quad (\text{Each } f_k \text{ is a branch of log in } G)$$

Theorem 2.1: Let $f : G \rightarrow \mathbb{C}$ be a branch of log. Then $g : G \rightarrow \mathbb{C}$ is a branch of log iff

$$g(z) = f(z) + 2k\pi i \text{ (for some } k \in \mathbb{Z})$$

Proof: (\Rightarrow) if f is a branch of log, then so is g

(\Leftarrow) Let $g : G \rightarrow \mathbb{C}$ be a branch of log.

Then, (i) g is continuous

$$(ii) \quad e^{g(z)} = z, \quad \forall z \in G$$

Since $f : G \rightarrow \mathbb{C}$ is a branch of log, we have,

(i) f is continuous

$$(ii) \quad e^{f(z)} = z, \quad \forall z \in G$$

$$e^{g(z)} = e^z = e^{f(z)} \quad (\forall z \in G)$$

$$\Rightarrow g(z) = f(z) + 2k(z)\pi i \quad (k \text{ depends on } z)$$

$$k : G \subset \mathbb{C} \rightarrow \mathbb{Z}$$

$$k(z) = \frac{1}{2\pi i}(g(z) - f(z)) \quad (k \text{ is continuous})$$

Since G is connected, $\text{Im}(k)$ is connected (subset of \mathbb{Z})

$\Rightarrow k(z)$ is constant

Hence, the claim follows.

Unit disk in \mathbb{C} : $D = \{z \in \mathbb{C} : |z| < 1\}$

$0 \in D$, hence D cannot be a branch of \log .

$D \setminus \{0\}$ isn't a branch of \log . (Why?)

$\text{Log}(z) : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$

The half-line is called a branch-cut for any member of $\{\text{Log}(z) + 2k\pi i : k \in \mathbb{Z}\}$

Question: How do you make a branch-cut to define a branch of $\text{Log}(z + i - 1)$?

§§2.2. Power functions

Let $\alpha \in \mathbb{C}$. We define z^α to be the multi-valued function:

$$\begin{aligned} z^\alpha &= e^{\alpha \log(z)} = e^{\alpha(\log(|z|) + i \arg(z))} \\ &= e^{\alpha(\text{Log}(z) + 2k\pi i)} \\ &= e^{\alpha \text{Log}(z)} \cdot e^{2\pi i k \alpha} \quad (z \neq 0, k \in \mathbb{Z}) \end{aligned}$$

Let $\alpha = n \in \mathbb{N}$. Then $z^\alpha = e^{n \text{Log}(z)}$ which is single-valued.

§3. Differentiability and Power series

§§3.1. Differentiability

Let $\Omega \subset \mathbb{C}$ and $f : \Omega \rightarrow \mathbb{C}$. Then f is said to be differentiable at $z_0 \in \Omega$ if,

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists

Example: $f(z) = \bar{z}$ isn't differentiable at any point in \mathbb{C} .

Cauchy-Riemann equations: Let $f : \Omega \rightarrow \mathbb{C}$ be differentiable at $z_0 = x_0 + iy_0$. Then,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Further,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$$

Note that the last equation only has partial derivatives wrt x .

Proof (Sketch): In the differential, approach the origin once through the real axis and then the imaginary axis to get some equations

Holomorphic/Analytic: A function is said to be **holomorphic** or **analytic** at z_0 if it is differentiable in a neighbourhood of z_0 .

If a function is analytic at a point, it is called a regular point for the function

Entire functions: If a function f is analytic at every point in $\mathbb{C} \Rightarrow f$ is entire.

Example:

1. $f(z) \rightarrow$ polynomial in \mathbb{C}
2. $f(z) = \frac{1}{z}$ (differentiable at every point on $\mathbb{C} \setminus \{0\}$)

Proposition 3.1: If f and g are differentiable at $z_0 \in \Omega \subset \mathbb{C}$, then so are:

(i) $f + g$, (ii) fg , (iii) f/g ($g(z_0) \neq 0$)

Proof: Trivial

Chain Rule: $f : \Omega \rightarrow U$, $g : U \rightarrow \mathbb{C}$ are holomorphic at z_0 and $f(z_0)$ respectively. Then

$g \circ f : \Omega \rightarrow \mathbb{C}$ is holomorphic at z_0 and $(g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0)$

Proof: Standard (found in any introductory complex analysis text)

§§3.2. Power series

A series of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ ($a_n, z, z_0 \in \mathbb{C}$; z_0 is center)

Theorem 3.1(Radius of convergence): Given a power series $\sum_{n=0}^{\infty} a_n(z)^n$,

$\exists R \in [0, \infty) \cup \{\infty\}$ such that:

1. the series converges absolutely ($\forall z \in B(0, R)$; the disk of convergence)
2. diverges $\forall z$ such that $|z| > R$
- 3.

$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$$

(Convention: $\frac{1}{0} = +\infty$, $\frac{1}{\infty} = 0$)

Proof: Can be found in any standard text (Try Conway!)

Theorem 3.2: The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function (in its disk of convergence)

Derivative of f is obtained by differentiation of each term:

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

f' has same disk of convergence

Proof: Found in any standard text

Remark: A power series is infinitely differentiable (analytic) in its disk of convergence

$$\Rightarrow f(z) = \sum a_n z^n = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

Proposition 3.2: Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be domains and let $f : \Omega_1 \rightarrow \mathbb{C}$, $g : \Omega_2 \rightarrow \mathbb{C}$ be continuous function such that $f(\Omega_1) \subset \Omega_2$ and that $g(f(z)) = z$ ($\forall z \in \Omega_1$). If g is differentiable on Ω_2 and if $g'(w) \neq 0$ ($\forall w \in \Omega_2$), then f is differentiable and $f'(z) = \frac{1}{g'(f(z))}$ ($\forall z \in \Omega_1$)

Proof (Sketch): Use the differentiability of g and the continuity of f to get the result

Theorem 3.3: A branch of logarithm is analytic and its derivative is $\frac{1}{z}$.

Proof (Sketch): Let f be a branch of log. Define $g(z) = e^z$. Then $g(f(z)) = z$. ($e^{f(z)} = z$)

Lemma 3.1: Let $f : \Omega \rightarrow \mathbb{C}$. Then f is differentiable at $z_0 \in \Omega$ iff $\exists a \in \mathbb{C}$ such that

$$f(z_0 + h) - f(z_0) = ah + h\psi(h) \quad (\lim_{h \rightarrow 0} \psi(h) = 0)$$

Proof: Use the definition of differentiability of a multivariable function

Theorem 3.4: Let $\Omega \subset \mathbb{C}$ be a domain and let $f = u + iv$ is a function from Ω to \mathbb{C} . Then f is analytic at $z_0 \in \Omega$ iff the partial derivatives exist, are continuous and satisfy Cauchy-Riemann equations.

Proof (To complete): \Rightarrow If f is analytic at z_0 , then partial derivatives are continuous and satisfy the Cauchy-Riemann equations (follows from theorem 3.2)

\Leftarrow Let $\tilde{f} : \Omega \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined as $\tilde{f} = (u, v)$

\tilde{f} is differentiable at $z_0 = (x_0, y_0)$

Definition: For domain Ω , $H(\Omega)$ is the collection of all analytic functions from $\Omega \rightarrow \mathbb{C}$

Proposition 3.3: Let $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$ such that $f \in H(\Omega)$. If D is a disk with center at $z_0 \in \Omega$ and if $\bar{D} \subseteq \Omega$, then f has a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (\forall z \in D)$$

$$\text{where } a_n = \frac{f^{(n)}(z_0)}{n!}, \quad (n \in \mathbb{N})$$

Proof (Sketch): Differentiate the power series repeatedly (Use Theorem 3.2)

Examples:

$$1. \sum_{n=1}^{\infty} nz^n < \infty \quad (\text{converges } \forall z \in B(0, 1); \text{ doesn't converge for } |z| = 1)$$

$$2. \sum_{n=1}^{\infty} \frac{z^n}{n^2} < \infty \quad (\text{converges } \forall z \in B(0, 1); \text{ converges for } |z| = 1)$$

$$3. \sum_{n=1}^{\infty} \frac{z^n}{n} < \infty \quad (\text{converges for } |z| < 1; \text{ diverges otherwise})$$

$$4. f(z) = e^z \Rightarrow f'(z) = e^z \quad (\text{by definition of } e^z)$$

$$\text{Consider } g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$g(z)$ is an entire function (Why?)

$$\text{Note that } f^{(k)}(0) = 1 \Rightarrow a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \Rightarrow f(z) = g(z)$$

§4. Complex integration

A **path** or a **curve** is a continuous function, $\gamma : [a, b] \rightarrow \mathbb{C}$ ($\text{Rng}(\gamma) \subset \mathbb{C}$)

$\gamma(a)$: initial point of path; $\gamma(b)$: endpoint of path

$[a, b]$: parameter interval

γ is said to be:

1. closed if $\gamma(a) = \gamma(b)$
2. smooth or C^1 if γ is differentiable and γ' is continuous
3. simple if γ is one-one
4. simple closed if $\gamma(a) = \gamma(b)$ and γ is one-one on (a, b)
5. piecewise smooth if there are finitely many points $s_0, s_1 \dots s_n \in [a, b]$ with $a = s_0 < s_1 < s_2 < \dots < s_n = b$ such that the restriction of γ to each (s_i, s_{i+1}) is smooth.

$-\gamma$ or γ^{-1} is defined by $\gamma^{-1}(t) = \gamma(a + b - t)$

$\phi : [0, 1] \rightarrow [a, b]$ defined as: $\phi(t) = a + (b - a)t$ (one-one and differentiable)

§§4.1. Line integral

$f : [a, b] \rightarrow \mathbb{C}$: continuous

$f = u + iv$, where $u, v : [a, b] \rightarrow \mathbb{R}$

Define $\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$

Properties:

1. $\int_a^b c.f(t)dt = c. \int_a^b f(t)dt$
2. $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$

Length of a smooth curve: Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a smooth curve.

$$L(\gamma) = \int_a^b |\gamma'(t)|dt = \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}dt \quad (\gamma(t) = \gamma_1(t) + i.\gamma_2(t))$$

If $\gamma : [a, b] \rightarrow C$ is piecewise smooth then $L(\gamma)$ is the sum of the length of its smooth parts.

Orientation: A curve γ is *positively* oriented if traversed in anti-clockwise direction else is *negatively* oriented.

Examples:

1. $\gamma(t) = re^{it}$, ($t \in [0, 2\pi]$) ($r > 0$: simple, smooth curve);

$$L(\gamma) = \int_0^{2\pi} |ire^{it}| dt = r(2\pi)$$

2. $\gamma(t) = e^{it}$, ($t \in [0, 4\pi]$): closed, smooth, traverses the unit circle twice in the positive direction

Integration over paths: $\gamma[a, b] \rightarrow \mathbb{C}$ is a smooth curve and $f : \gamma \rightarrow \mathbb{C}$: continuous

Definiton: $\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt = \int_a^b g(t) dt$

($g(t) = f(\gamma(t)) \gamma'(t)$ where $g : [a, b] \rightarrow \mathbb{C}$)

Let $[a_1, b_1]$ be any closed interval. Then $\exists \phi : [a_1, b_1] \rightarrow [a, b]$ (one-one, differentiable and

$\phi(a_1) = a; \phi(a_2) = b$)

$\phi[a_1, b_1] \rightarrow \mathbb{C}$: smooth

$$\int_{a_1}^{b_1} f(\gamma_1(t)) \cdot \gamma_1'(t) dt \quad (= \int_{\gamma_1} f(z) dz)$$

$$= \int_{a_1}^{b_1} f(\gamma(\phi(t))) \cdot \phi'(t) dt = \int_{\gamma} f(\gamma(s)) \gamma'(s) ds = \int_{\gamma} f(z) dz \quad (\phi(t) = s)$$

If γ is piecewise smooth, the integral can be split into the sum of its smooth components:

if $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$, then $\int_{\gamma} f = \int_{\gamma_1} f + \dots + \int_{\gamma_n} f$.

Note that γ_i' s are smooth.

Proposition: If f and g are continuous on a smooth curve γ , then

1. $\int_{\gamma} \alpha f + \beta g = \alpha \int_{\gamma} f + \beta \int_{\gamma} g$
2. $\int_{\gamma^-} f = - \int_{\gamma} f$
3. $|\int_{\gamma} f(z) dz| \leq \|f\|_{\infty, \gamma} L(\gamma) \quad (\|f\|_{\infty, \gamma} = \sup_{z \in \{\gamma\}} |f(z)|)$

$$|\int_{\gamma} f| = |\int_a^b f(\gamma(t)) \cdot \gamma'(t) dt| \leq \int_{\gamma} |f(\gamma(t)) \cdot \gamma'(t)| dt \leq \|f\|_{\infty, \gamma} \int_a^b |\gamma'(t)| dt \quad (L(\gamma) = \int_a^b |\gamma'(t)| dt)$$

Examples:

(i) Let γ be the arc of a circle of radius 3 ($|z| = 3$) from 3 to $3i$.

Show that:

$$|\int_{\gamma} \frac{z+4}{z^3-1} dz| \leq \frac{21\pi}{52}$$

(ii) $\gamma : |z| = 2$ (traverse curve in positive direction)

Prove:

$$|\int_{\gamma} \frac{e^z dz}{z^2+1}| \leq \frac{4\pi e^2}{3}$$

Fundamental theorem of calculus:

If $f : [a, b] \rightarrow \mathbb{R}$ has a primitive F , then $\int_a^b f(x)dx = F(b) - F(a)$ ($F'(x) = f(x), \forall x \in [a, b]$)

For complex case: Suppose $G \subset \mathbb{C}$ be a domain. If a continuous function $f : G \rightarrow \mathbb{C}$ has a primitive F on G and if γ is a smooth curve in G with initial and terminal points ω_1 and ω_2 respectively, then:

$$\int_{\gamma} f = F(\omega_1) - F(\omega_2)$$

Proof: Let $[a, b] \subset \mathbb{R}$ be a parameter interval for γ and $\gamma(a) = \omega_1; \gamma(b) = \omega_2$

Given $F'(z) = f(z)$ ($\forall z \in G$)

$$\begin{aligned} \int_{\gamma} f &= \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt = \int_a^b F'(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b (F \circ \gamma)'(t) dt = F \circ \gamma(b) - F \circ \gamma(a) = F(\omega_2) - F(\omega_1) \end{aligned}$$

Corollary-1: If γ is a closed curve (smooth), then

$$\int_{\gamma} f = 0$$

Proof: Follows from FTC

Corollary-2: If $f \in H(\Omega)$ for a region $\Omega \subset \mathbb{C}$ and if $f' = 0$ on Ω , then f is a constant function.

Proof: Fix a point $\omega_0 \in \Omega$. It suffices to show that $f(\omega) = f(\omega_0), \forall \omega \in \Omega$

§§4.2. Simple Closed Curve

Jordan-curve theorem: Every simple closed curve in \mathbb{C} divides the complex plane into two regions. One of these regions is bounded and the other is unbounded. The bounded region is called the interior of the curve.

Example: $G = \mathbb{C} \setminus \{0\}$

$f(z) = \frac{1}{z}$ on G , $\gamma : |z| = 1, \gamma(t) = e^{it}, (t \in [0, 2\pi])$

$$\int_{\gamma} f = \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t) dt = \int_0^{2\pi} \frac{i \cdot e^{it}}{e^{it}} dt = 2\pi i \neq 0$$

Winding number or index of a closed curve: Let γ be a closed curve on \mathbb{C} and let

$\alpha \in \mathbb{C} \setminus \{\gamma\}$. The winding number of γ about α or the index of γ with respect to α is denoted by, $\eta(\gamma; \alpha) / \text{Ind}_{\gamma}(\alpha)$ defined by:

$$\eta(\gamma; \alpha) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - \alpha}$$

Example: $\gamma : [0, 6\pi] \rightarrow \mathbb{C}$

$$\gamma(t) = a + re^{it}$$

$$\eta(\gamma; \alpha) = \frac{1}{2\pi i} \int_{\gamma} \frac{\gamma'(t)}{\gamma(t) - a} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{a + re^{it} - a} \cdot ire^{it} dt = 3$$

Theorem 4.1: Let γ be a smooth, closed curve in \mathbb{C} . Let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then $\eta(\gamma; \alpha) \in \mathbb{Z}$.

Proof: $\phi : [0, 1] \rightarrow \mathbb{C}$

$$\phi = \frac{\gamma'(s)}{\gamma(s) - \alpha} \text{ and } g : [0, 1] \rightarrow \mathbb{C}, g(t) = \int_0^t \phi(s) ds$$

$$g(0) = 0 \text{ and } g(1) = \int_0^1 \phi(s) ds = \int_0^1 \frac{\gamma'(s)}{\gamma(s) - \alpha} ds = \int_{\gamma} \frac{dz}{z - \alpha}$$

Claim: $g'(t) = \phi(t)$

Proof: To show that $\lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h} - \phi(t) = 0$

$$\frac{g(t+h) - g(t)}{h} - \phi(t) = \frac{1}{h} \int_t^{t+h} [\phi(t+h) - \phi(t)] ds \quad (\text{for } h > 0, \text{ similar for } h < 0)$$

Since ϕ is uniformly continuous on $[0, 1]$, $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|s - t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \epsilon$$

If $h < \delta$, then

$$\left| \frac{1}{h} \int_t^{t+h} [\phi(t+h) - \phi(t)] ds \right| \leq \frac{1}{h} \int_t^{t+h} |\phi(t+h) - \phi(t)| ds < \epsilon$$

Same thing holds if $h < 0$

$$\text{Therefore, } h < \delta \Rightarrow \left| \frac{g(t+h) - g(t)}{h} - \phi(t) \right| < \epsilon$$

Hence, $g' = \phi$

$$\text{Set } h(t) = e^{-g(t)}(\gamma(t) - \alpha)$$

$$\text{We have, } h'(t) = e^{-g(t)}\gamma'(t) - e^{-g(t)}(\gamma(t) - \alpha)g'(t) = 0$$

Hence, $h(t)$ is a constant function.

$$\begin{aligned} e^{-g(0)}(\gamma(0) - \alpha) &= e^{-g(1)}(\gamma(1) - \alpha) \\ \Rightarrow e^{-g(0)} &= e^{-g(1)} = 1 \quad (\text{As } \gamma \text{ is a closed curve}) \end{aligned}$$

hence $g(1) = 2k\pi i$ (for $k \in \mathbb{Z}$)

Therefore, $\int_{\gamma} \frac{dz}{z-\alpha} = 2k\pi i \Rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-\alpha} = k \in \mathbb{Z}$

Remark: The theorem is true if γ is a closed contour. (Prove it!)

(A contour is a piecewise smooth curve)

Theorem 4.2: Let γ be a closed contour and let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then,

(a) the function $f_{\gamma} : \mathbb{C} \setminus \{\gamma\} \rightarrow \mathbb{Z}$ is continuous. ($\alpha \rightarrow \eta(\gamma; \alpha)$)

(b) f is constant on every component of $\mathbb{C} \setminus \{\gamma\}$

Proof: (a) Let $\alpha_0 \in \mathbb{C} \setminus \{\gamma\}$. Then the function $g : t \rightarrow |\alpha_0 - \gamma(t)|$ is continuous.

g attains its infimum, say $s = \inf_{t \in [0,1]} g(t)$

If α is very close to α_0 , then $|\alpha - \gamma(t)| \geq \frac{s}{2}$. Then,

$$\begin{aligned} \left| \frac{1}{z-\alpha} - \frac{1}{z-\alpha_0} \right| &= \frac{|\alpha - \alpha_0|}{|z-\alpha||z-\alpha_0|} \leq \frac{2}{s^2} |\alpha - \alpha_0| \quad (z \in \gamma) \\ |f_{\gamma}(\alpha) - f_{\gamma}(\alpha_0)| &\leq \frac{1}{2\pi i} \int_{\gamma} \left| \frac{1}{z-\alpha} - \frac{1}{z-\alpha_0} \right| dz \\ &\leq \frac{2}{s^2} |\alpha - \alpha_0| \frac{1}{2\pi i} L(\gamma) = M(\alpha - \alpha_0) \quad (\text{Lipschitz continuous} \Rightarrow \text{continuous}) \end{aligned}$$

(b) Let V be a component, then $f(V)$ is connected in $\mathbb{Z} \Rightarrow f(V)$ is a constant $\in \mathbb{Z}$

Proposition 4.1: Let γ be a closed contour in \mathbb{C} . Then $\eta(\gamma; \alpha) = 0 \forall \alpha$ in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$

Proof: Since γ is closed and bounded, $\{\gamma\} \subseteq \bar{B}(0; R)$ for some $R > 0$.

Let $\alpha \in \mathbb{C} \setminus \bar{B}(0; R)$

$$\begin{aligned} |z - \alpha| &\geq |\alpha| - |z| \geq |\alpha| - R \\ |\eta(\gamma; \alpha)| &= \frac{1}{2\pi} \left| \int_{\gamma} \frac{dz}{z-\alpha} \right| \leq \frac{1}{2\pi} \int_{\gamma} \frac{dz}{|z-\alpha|} \leq \frac{1}{2\pi} \frac{1}{|\alpha| - R} L(\gamma) \end{aligned}$$

One can find a large enough $|\alpha|$ to make $\eta(\gamma; \alpha) < 1$

Hence, $\eta(\gamma; \alpha) = 0$, when $|\alpha|$ is sufficiently large

Since, $\eta(\gamma; \alpha)$ is constant within a component, $\eta(\gamma; \beta) = 0$ ($\forall \beta$ in unbounded component)

Proposition 4.2: Let γ be a closed contour consisting of curves $\gamma_1, \dots, \gamma_n$. Then,

$$\eta(\gamma; \alpha) = \eta(\gamma_1; \alpha) + \dots + \eta(\gamma_n; \alpha) \quad (\text{Prove!})$$

Cauchy-Goursat theorem: Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f \in H(\Omega)$. Then for any closed contour γ lying in the interior of Ω ,

$$\int_{\gamma} f(z) dz = 0$$

Proof: Step-I (Goursat's theorem):

When $\gamma = T$, a triangle

Let $T^{(0)} = T$

Let $diam(T^{(0)}) = d^{(0)}$ and $peri(T^{(0)}) = p^{(0)}$

$$\int_{T^{(0)}} f(z)dz = \int_{T^{(1)}} f(z)dz + \int_{T^{(2)}} f(z)dz + \int_{T^{(3)}} f(z)dz + \int_{T^{(4)}} f(z)dz$$

$$|\int_{T^{(0)}} f(z)dz| \leq 4|\int_{T^{(j)}} f(z)dz| \text{ (for some } j \in \{1, 2, 3, 4\})$$

Call this $T^{(j)}$ to be $T^{(1)}$ (suppose)

$$diam(T^{(1)}) = \frac{1}{2}diam(T^{(0)})$$

$$d^{(1)} = \frac{d^{(0)}}{2} \text{ and } p^{(1)} = \frac{p^{(0)}}{2}$$

Do the same process with $T^{(1)}$ to get $T^{(2)} \Rightarrow |\int_{T^{(1)}} f(z)dz| \leq 4|\int_{T^{(2)}} f(z)dz|$

Continuing, $|\int_{T^{(0)}} f(z)dz| \leq 4^n |\int_{T^{(n)}} f(z)dz|$

$$d^{(n)} = \frac{d^{(0)}}{2^n} \text{ and } p^{(n)} = \frac{p^{(0)}}{2^n}$$

$$\Delta_n = T^{(n)} \cup Int(T^{(n)}) \text{ (Int refers to interior of triangle)}$$

$$\Delta_0 \supseteq \Delta_1 \supseteq \dots \Delta_n \supseteq \dots \text{ (nested compact sets)}$$

$d^{(n)}$ tends to 0

Therefore, $\exists! z_0 \in \bigcap_{n=0}^{\infty} \Delta_n$

f is holomorphic at z_0

$$f(z_0 + h) - f(z_0) = hf'(z_0) + h\psi(h) \text{ (}\lim_{h \rightarrow 0} \psi(h) = 0\text{)}$$

So, $f(z) - f(z_0) = (z - z_0)f'(z_0) + (z - z_0)\psi_1(h)$ where $\lim_{z \rightarrow z_0} \psi_1(z) = 0$

$$\Rightarrow \int_T f(z)dz = \int_T f(z_0)dz + \int_T (z - z_0)f'(z_0)dz + \int_T (z - z_0)\psi_1(z)dz = \int_T (z - z_0)\psi_1(z)dz$$

$$\text{Then, } \psi_1(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

Let $\sup_{z \in T^{(n)}} |\psi_1(z)| = E_n$ ($E_n \rightarrow 0$ as $n \rightarrow \infty$)

$$|\int_{T^{(n)}} f(z)dz| = |\int_{T^{(n)}} (z - z_0)\psi_1(z)dz| \leq \int_{T^{(n)}} |z - z_0| |\psi_1(z)| dz$$

$$\leq d^{(n)} E_n p^{(n)} = \frac{d^{(0)} p^{(0)}}{4^n} E_n$$

$$|\int_{T^{(0)}} f(z)dz| \leq 4^n |\int_{T^{(n)}} f(z)dz| \leq d^{(0)} p^{(0)} E_n \text{ (}\forall n\text{)}$$

Take limit on both sides as $n \rightarrow \infty \Rightarrow |\int_{T^{(0)}} f(z)dz| = 0$