# MA-412
# Complex Analysis

Om Swostik
https://github.com/swostikom20/

Last updated: March 4, 2023

# Contents

# Chapter 0

# Preliminaries

**Definition 0.1** (**Region**). An open, connected subset of $\mathbb{C}$ is called domain or region.

$$D(a,r) = \{z \in \mathbb{C} | \, |z-a| < r\}$$
$$\{z \in \mathbb{C} | \, r < |z-a| < R\} \rightarrow \text{annulus}$$

**Definition 0.2** (**Exponential function**). $exp(z) : \mathbb{C} \rightarrow \mathbb{C}$

$$e^z = e^x(cos(y) + isin(y))$$
$$e^{z_1+z_2} = e^{z_1}.e^{z_2}, 0 \notin Rng(exp(z))$$
$$|e^{ix}| = 1 \ \ (\forall x \in \mathbb{R})$$
$$e^z = 1 \ \text{ iff } \ z = 2n\pi i, \ n \in \mathbb{Z}$$
$$e^{z_1} = e^{z_2} \ \text{ iff } \ z_1 = z_2 + 2n\pi i$$

**Definition 0.3** (**Argument**). $arg(z) : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$

$arg(z) = \theta$ (Angle made wrt positive real axis)
$arg(z)$ for $z \in \mathbb{C} \setminus \{0\}$ is multivalued function
$arg(z) : \mathbb{C} \setminus \{0\} \rightarrow (\alpha, \alpha + 2\pi]$ or $[\alpha, \alpha + 2\pi)$ is well-defined $(\forall \alpha \in \mathbb{R})$

**Definition 0.4 (Principal value).** $Arg(z) = \theta \ (-\pi < \theta \leq \pi)$
$arg(z) = \{Arg(z) + 2n\pi : n \in \mathbb{Z}\}$

**Definition 0.5 (Complex Log).** For $z \in \mathbb{C} \setminus \{0\}$, define:

$$log(z) = log(|z|) + iarg(z)$$

Notice that $log$ is multivalued
Whenever $arg(z)$ is well-defined (i.e $Rng(arg(z)) = [\alpha, \alpha + 2\pi)$ or $(\alpha, \alpha + 2\pi]$)
$\Rightarrow log(z)$ is well defined

$$\begin{aligned} &Log(z) = log(|z|) + i.Arg(z) \ \text{(principal log)} \\ &log(z) = Log(z) + 2n\pi i \end{aligned}$$

**Lemma 0.6.** For $z \in \mathbb{C} \setminus \{0\}$, the values of $log(z)$ are the complex numbers $\omega$ such that $e^{\omega} = z$

*Proof.* Do it on your own! □

**Definition 0.7 (Limit).** $f : E \subseteq \mathbb{C} \to \mathbb{C}$

$$\begin{aligned} &\lim_{z \to z_0} f(z) = l \\ &\forall \epsilon > 0, \exists \delta > 0 \ \text{such that} \ 0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon \\ &z \in (B(z, z_0) \setminus \{0\}) \cap E \Rightarrow f(z) \in B(l, \epsilon) \end{aligned}$$

**Proposition 0.8 (Sequential criteria for limit).** A function $f : E \subseteq \mathbb{C} \to \mathbb{C}$ has a limit point $l$ as $z \to z_0$ iff $f(z_n) \to l$ for every sequence $\{z_n\} \subset E \setminus \{z_0\}$ with $z_n \to z_0$ as $n \to \infty$

**Theorem 0.9.** If $f : E \subset \mathbb{C} \to \mathbb{C}$ has a limit at $z_0$, then $f$ is bounded near $z_0$

**Definition 0.10** (**Continuity**)**.** $f : E \subseteq \mathbb{C} \to \mathbb{C}$
$f$ is continuous at $z_0 \in E$ if for every basic neighbourhood $V$ of $f(z_0)$, there is a basic neighbourhood $U$ of $z_0$ such that $f(U) \subset V$. Let $V = B(f(z_0), \epsilon)$ & $U = B(z_0, \delta)$,

$$z \in B(z_0, \delta) \Rightarrow f(z) \in B(f(z_0), \epsilon)$$
$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$
$$\text{Then, } z \in B(z_0, \delta) \setminus \{z_0\} \Rightarrow f(z) \in B(f(z_0), \epsilon)$$
$$\lim_{z \to z_0} f(z) = f(z_0)$$

**Prove**: Let $f : G \to \mathbb{C}$ such that $f = u + iv$ (where $u, v : G \to \mathbb{R}$)
$f$ is continuous at a point $z_0$ iff $u, v$ are continuous at $z_0$

# Chapter 1

# Branch of Log and Power function

$Arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$
$Log(z) = log(|z|) + i.Arg(z)$ (defined on $\mathbb{C} \setminus \{0\}$)
$Log(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$ as $Arg(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$

## §1.1. Branch of Log

**Definition 1.1** (Branch of Log)**.** Let $G \subset \mathbb{C}$ be a region. A continuous function $f : G \to \mathbb{C}$ is called a branch of logarithm in $G$ if:

$$e^{f(z)} = z \ (\forall z \in G)(\Rightarrow 0 \notin G)$$

$$e^{Log(z)} = z \ (\forall z \in \mathbb{C} \setminus \{0\})$$

But $Log(z)$ isn't continuous on $\mathbb{C} \setminus \{0\}$
Therefore, $Log(z)$ is a branch of log in $\mathbb{C} \setminus (-\infty, 0]$

$$f_k = Log(z) + 2k\pi i, \ k \in \mathbb{Z}$$
$$e^{f_k(z)} = z \quad \text{(Each } f_k \text{ is a branch of log in G)}$$

**Theorem 1.2.** Let $f : G \to \mathbb{C}$ be a branch of log. Then $g : G \to \mathbb{C}$ is a branch of log iff $g(z) = f(z) + 2k\pi i$ (for some $k \in \mathbb{Z}$)

*Proof.* ($\Rightarrow$) if $f$ is a branch of log, then so is $g$
($\Leftarrow$) Let $g : G \to \mathbb{C}$ be a branch of log.
Then, (i) $g$ is continuous
(ii) $e^{g(z)} = z, \forall z \in G$
Since $f : G \to \mathbb{C}$ is a branch of log, we have,
(i) $f$ is continuous
(ii)$e^{f(z)} = z, \forall z \in G$
$e^{g(z)} = z = e^{f(z)}$ $(\forall z \in G)$
$\Rightarrow g(z) = f(z) + 2k(z)\pi i$ ($k$ depends on $z$)
$k : G \subset \mathbb{C} \to \mathbb{Z}$
$k(z) = \frac{1}{2\pi i}(g(z) - f(z))$ ($k$ is continuous)
Since $G$ is connected, $Img(k)$ is connected (subset of $\mathbb{Z}$)
$\Rightarrow k(z)$ is constant
Hence, the claim follows. $\square$

**Definition 1.3** **(Unit disk in $\mathbb{C}$).** $D = \{z \in \mathbb{C} : |z| < 1\}$

$0 \in D$, hence $D$ cannot be a branch of log.
$D \setminus \{0\}$ isn't a branch of log. (Why?)
$Log(z) : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$
The half-line is called a branch-cut for any member of $\{Log(z) + 2k\pi i : k \in \mathbb{Z}\}$
**Question**: How do you make a branch-cut to define a branch of $Log(z + i - 1)$?

## §1.2. Power functions

**Definition 1.4** (Power function)**.** Let $\alpha \in \mathbb{C}$. We define $z^\alpha$ to be the multi-valued function:

$$\begin{aligned} z^\alpha = e^{\alpha log(z)} &= e^{\alpha(log(|z|) + iarg(z))} \\ &= e^{\alpha(Log(z) + 2k\pi i)} \\ &= e^{\alpha Log(z)}.e^{2\pi i k\alpha} \quad (z \neq 0, k \in \mathbb{Z}) \end{aligned}$$

Let $\alpha = n \in \mathbb{N}$. Then $z^\alpha = e^{nLog(z)}$ which is single-valued.

# Chapter 2

# Differentiability and Power series

## §2.1. Differentiability

**Definition 2.1.** Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Then $f$ is said to be differentiable at $z_0 \in \Omega$ if,
$f^{'}(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists

**Example 2.2.** $f(z) = \bar{z}$ isn't differentiable at any point in $\mathbb{C}$.

**Proposition 2.3. Cauchy-Riemann equations**: Let $f : \Omega \to C$ be differentiable at $z_0 = x_0 + iy_0$. Then,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Further,

$$f^{'}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

Note that the last equation only has partial derivatives wrt $x$.

*Proof.* In the differential, approach the origin once through the real axis and and then the imaginary axis to get some equations ☐

**Definition 2.4** (**Holomorphic/Analytic Function**). A function is said to be **holomorphic** or **analytic** at $z_0$ if it is differentiable in a neighbourhood of $z_0$. If a function is analytic at a point, it is called a *regular* point for the function

**Definition 2.5** (**Entire function**). If a function $f$ is analytic at every point in $\mathbb{C} \Rightarrow f$ is entire.

**Example 2.6.**
1. $f(z) \to$ polynomial in $\mathbb{C}$
2. $f(z) = \frac{1}{z} \to$ not an entire function (differentiable at every point on $\mathbb{C} \setminus \{0\}$)

**Proposition 2.7.** If $f$ and $g$ are differentiable at $z_0 \in \Omega \subset \mathbb{C}$, then so are: (i)$f + g$, (ii)$fg$, (iii)$f/g$ $(g(z_0) \neq 0)$

*Proof.* Trivial ☐

**Theorem 2.8** (**Chain Rule**). $f : \Omega \to U$, $g : U \to \mathbb{C}$ are holomorphic at $z_0$ and $f(z_0)$ respectively.
Then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic at $z_0$ and $(g \circ f)'(z_0) = g'(f(z_0)).f'(z_0)$

*Proof.* Let $h = (g \circ f)$

$$\begin{aligned} h'(z_0) &= \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \\ &= \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} . \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{f(z) \to f(z_0)} \frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} . \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = g'(f(z_0))\, f'(z_0) \quad \text{(Using continuity of } f) \end{aligned}$$

☐

## §2.2. Power series

**Definition 2.9** (Power series)**.** A series of the form $\sum_0^\infty a_n(z-z_0)^n$ ($a_n, z, z_0 \in \mathbb{C}$; $z_0$ is center)

**Theorem 2.10** (**Radius of convergence**)**.** Given a power series $\sum_{n=0}^\infty a_n(z)^n$, $\exists R \in [0,\infty) \cup \{\infty\}$ such that:

1. the series converges absolutely ($\forall z \in B(0,R)$; the disk of convergence)
2. diverges $\forall z$ such that $|z| > R$
3. $$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$$

(Convention: $\frac{1}{0} = +\infty$ , $\frac{1}{\infty} = 0$)

*Proof.* Theorem 2.5 in Stein-Shakarchi Complex Analysis □

**Theorem 2.11.** The power series $f(z) = \sum_{n=0}^\infty a_n z^n$ defines a holomorphic function (in its disk of convergence)
Derivative of $f$ is obtained by differentiation of each term:
$f'(z) = \sum_{n=0}^\infty n a_n z^{n-1}$
$f'$ has same disk of convergence

*Proof.* Theorem 2.6 in Stein-Shakarchi Complex Analysis □

**Remark 2.12.** A power series is infinitely differentiable (analytic) in its disk of convergence

$\Rightarrow$ $$f(z) = \sum a_n z^n = \sum_{k=0}^\infty \frac{f^k(0)}{k!} z^k$$

**Proposition 2.13.** Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be domains and let $f : \Omega_1 \to \mathbb{C}$, $g : \Omega_2 \to \mathbb{C}$ be continuous function such that $f(\Omega_1) \subseteq \Omega_2$ and that $g(f(z)) = z$ ($\forall z \in \Omega_1$). If

$g$ is differentiable on $\Omega_2$ and if $g^{'}(w) \neq 0$ $(\forall w \in \Omega_2)$, then $f$ is differentiable and $f^{'}(z) = \frac{1}{g^{'}(f(z))}$ $(\forall z \in \Omega_1)$

*Proof.* Use the differentiability of $g$ and the continuity of $f$ to get the result □

**Theorem 2.14.** A branch of logarithm is analytic and its derivative is $\frac{1}{z}$.

*Proof.* Let $f$ be a branch of log. Define $g(z) = e^z$. Then $g(f(z)) = z$. $(e^{f(z)} = z)$ □

**Lemma 2.15.** Let $f : \Omega \to \mathbb{C}$. Then $f$ is differentiable at $z_0 \in \Omega$ iff $\exists a \in \mathbb{C}$ such that
$f(z_0 + h) - f(z_0) = ah + h\psi(h)$ $(\lim_{h\to 0} \psi(h) = 0)$

*Proof.* Use the definition of differentiability of a multivariable function □

**Theorem 2.16.** Let $\Omega \subset \mathbb{C}$ be a domain and let $f = u + iv$ is a function from $\Omega$ to $\mathbb{C}$.
Then $f$ is analytic at $z_0 \in \Omega$ iff the partial derivatives exist, are continuous and satisfy Cauchy-Riemann equations.

*Proof.* $\Rightarrow$ If $f$ is analytic at $z_0$, then partial derivatives are continuous and satisfy the Cauchy-Riemann equations (follows from theorem 3.2)
$\Leftarrow$ Let $\tilde{f} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $\tilde{f} = (u, v)$
$\tilde{f}$ is differentiable at $z_0 = (x_0, y_0)$
(to complete) (Refer to Theorem 2.4 in stein-shakarchi) □

**Definition 2.17.** For domain $\Omega$, $H(\Omega)$ is the collection of all analytic functions from $\Omega \to \mathbb{C}$

**Proposition 2.18.** Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ such that $f \in H(\Omega)$. If $D$ is a disk with center at $z_0 \in \Omega$ and if $\bar{D} \subseteq \Omega$, then $f$ has a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \ \ (\forall z \in D)$$

$$\text{where } a_n = \frac{f^{(n)}(z_0)}{n!}, \ (n \in \mathbb{N})$$

*Proof.* Differentiate the power series repeatedly (Use Theorem 3.2) □

**Example 2.19.**

1. $\sum_{n=1}^{\infty} nz^n < \infty$ (converges $\forall z \in B(0,1)$; doesn't converge for $|z|= 1$)
2. $\sum_{n=1}^{\infty} \frac{z^n}{n^2} < \infty$ (converges $\forall z \in B(0,1)$; converges for $|z|= 1$)
3. $\sum_{n=1}^{\infty} \frac{z^n}{n} < \infty$ (converges for $|z|< 1$; diverges otherwise)
4. $f(z) = e^z \Rightarrow f^{'}(z) = e^z$ (by definition of $e^z$)

   Consider $g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$

   $g(z)$ is an entire function (Why?)

   Note that $f^{(k)}(0) = 1 \Rightarrow a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!} \Rightarrow f(z) = g(z)$

# Chapter 3

# Complex integration

**Definition 3.1 (Path).** A **path** or a **curve** is a continuous function, $\gamma : [a, b] \to \mathbb{C}$ ($Rng(\gamma) \subset \mathbb{C}$)

$\gamma(a)$: initial point of path; $\gamma(b)$: endpoint of path
$[a, b]$: parameter interval

**Definition 3.2.** $\gamma$ is said to be:

1. closed if $\gamma(a) = \gamma(b)$
2. smooth or $C^1$ if $\gamma$ is differentiable and $\gamma'$ is continuous
3. simple if $\gamma$ is one-one
4. simple closed if $\gamma(a) = \gamma(b)$ and $\gamma$ is one-one on $(a, b)$
5. piecewise smooth if there are finitely many points $s_0, s_1 \ldots s_n \in [a.b]$ with $a = s_0 < s_1 < s_2 \cdots < s_n = b$ such that the restriction of $\gamma$ to each $(s_i, s_{i+1})$ is smooth.

$-\gamma$ or $\gamma^{-1}$ is defined by $\gamma^{-1}(t) = \gamma(a + b - t)$
$\phi : [0, 1] \to [a, b]$ defined as: $\phi(t) = a + (b - a)t$ (one-one and differentiable)

## §3.1. Line integral

$f : [a, b] \to \mathbb{C}$ : continuous
$f = u + iv$, where $u, v : [a, b] \to \mathbb{R}$

Define $\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$
**Properties**:

1. $\int_a^b c.f(t)dt = c.\int_a^b f(t)dt$
2. $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$

**Length of a smooth curve**: Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve.
$L(\gamma) = \int_a^b |\gamma'(t)|dt$
$= \int_a^b \sqrt{\gamma_1'(t)^2 + \gamma_2'(t)^2}dt \quad (\gamma(t) = \gamma_1(t) + i.\gamma_2(t))$
If $\gamma : [a, b] \to C$ is piecewise smooth then $L(\gamma)$ is the sum of the length of its smooth parts.

**Definition 3.3** (**Orientation**). A curve $\gamma$ is *positively* oriented if traversed in anti-clockwise direction else is *negatively* oriented.

**Example 3.4.**

1. $\gamma(t) = re^{it}$, $(t \in [0, 2\pi])$ ($r > 0$: simple, smooth curve); $L(\gamma) = \int_0^{2\pi} |ire^{it}|dt = r(2\pi)$
2. $\gamma(t) = e^{it}$, $(t \in [0, 4\pi])$: closed, smooth, traverses the unit circle twice in the positive direction

**Definition 3.5** (**integration over paths**). $\gamma[a, b] \to \mathbb{C}$ is a smooth curve and $f : \gamma \to \mathbb{C}$: continuous
$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b g(t)dt$
$(g(t) = f(\gamma(t))\gamma'(t)$ where $g : [a, b] \to \mathbb{C})$

Let $[a_1, b_1]$ be any closed interval. Then $\exists \phi : [a_1, b_1] \to [a, b]$ (one-one,differentiable and $\phi(a_1) = a; \phi(a_2) = b$)
$\phi[a_1, b_1] \to \mathbb{C}$: smooth
$\int_{a_1}^{b_1} f(\gamma_1(t)).\gamma_1'(t)dt \quad (= \int_{\gamma_1} f(z)dz)$
$= \int_{a_1}^{b_1} f(\gamma(\phi(t))).\phi'(t)dt$
$= \int_\gamma f(\gamma(s))\gamma'(s)ds = \int_\gamma f(z)dz \quad (\phi(t) = s)$
If $\gamma$ is piecewise smooth, the integral can be split into the sum of its smooth components:
if $\gamma = \gamma_1 + \gamma_2 \cdots + \gamma_n$, then $\int_\gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f$.
Note that $\gamma_i's$ are smooth.

**Proposition 3.6.** If $f$ and $g$ are continuous on a smooth curve $\gamma$, then

1. $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$
2. $\int_{\gamma^-} f = - \int_\gamma f$
3. $|\int_\gamma f(z)dz| \leq \|f\|_{\infty,\gamma} L(\gamma) \quad (\|f\|_{\infty,\gamma} = sup_{z\in\{\gamma\}}|f(z)|)$

*Proof.*

$$\begin{aligned}|\int_\gamma f| = |\int_a^b f(\gamma(t)).\gamma^{'}(t)dt| &\leq \int_\gamma |f(\gamma(t)).\gamma^{'}(t)|dt \\ &\leq \|f\|_{\infty,\gamma} \int_a^b |\gamma^{'}(t)|dt \quad (L(\gamma) = \int_a^b |\gamma^{'}(t)|dt)\end{aligned}$$

□

**Examples**:
(i)Let $\gamma$ be the arc of a circle of radius 3 ($|z|= 3$) from 3 to $3i$.
Show that:

$$|\int_\gamma \frac{z+4}{z^3-1}dz| \leq \frac{21\pi}{52}$$

(ii) $\gamma : |z|= 2$ (traverse curve in positive direction)
Prove:

$$|\int_\gamma \frac{e^z dz}{z^2+1}| \leq \frac{4\pi e^2}{3}$$

**Theorem 3.7 (Fundamental theorem of calculus).** If $f : [a, b] \to \mathbb{R}$ has a primitive F, then $\int_a^b f(x)dx = F(b) - F(a) \quad (F^{'}(x) = f(x), \forall x \in [a, b])$
**For complex case**: Suppose $G \in \mathbb{C}$ be a domain. If a continuous function $f : G \to \mathbb{C}$ has a primitive $F$ on $G$ and if $\gamma$ is a smooth curve in G with initial and terminal points $\omega_1$ and $\omega_2$ respectively, then:
$\int_\gamma f = F(\omega_1) - F(\omega_2)$

*Proof.* Let $[a, b] \in \mathbb{R}$ be a parameter interval for $\gamma$ and $\gamma(a) = \omega_1$; $\gamma(b) = \omega_2$
Given $F^{'}(z) = f(z)$ $(\forall z \in G)$

$$\int_{\gamma} f = \int_a^b f(\gamma(t)).\gamma^{'}(t)dt = \int_a^b F^{'}(\gamma(t))\gamma^{'}(t)dt$$

$$= \int_a^b (F \circ \gamma)^{'}(t)dt = F \circ \gamma(b) - F \circ \gamma(a) = F(\omega_2) - F(\omega_1)$$

□

**Corollary 3.8.** If $\gamma$ is a closed curve (smooth), then
$\int_{\gamma} f = 0$

*Proof.* Follows from FTC □

**Corollary 3.9.** If $f \in H(\Omega)$ for a region $\Omega \in \mathbb{C}$ and if $f^{'} = 0$ on $\Omega$, then $f$ is a constant function.

*Proof.* Fix a point $\omega_0 \in \Omega$. It suffices to show that $f(\omega) = f(\omega_0), \forall \omega \in \Omega$ □

## §3.2. Simple Closed Curve

**Theorem 3.10 (Jordan-curve theorem).** Every simple closed curve in $\mathbb{C}$ divides the complex plane into two regions. One of these regions is bounded and the other is unbounded. The bounded region is called the interior of the curve.

**Example 3.11.** $G = \mathbb{C} \setminus \{0\}$
$f(z) = \frac{1}{z}$ on $G$, $\gamma : |z| = 1$, $\gamma(t) = e^{it}$, $(t \in [0, 2\pi])$

$$\int_{\gamma} f = \int_0^{2\pi} f(\gamma(t)).\gamma^{'}(t)dt = \int_0^{2\pi} \frac{i.e^{it}}{e^{it}} dt = 2\pi i \neq 0$$

**Definition 3.12** (**Winding number or index of a closed curve**)**.** Let $\gamma$ be a closed curve on $\mathbb{C}$ and let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. The winding number of $\gamma$ about $\alpha$ or the index of $\gamma$ with respect to $\alpha$ is denoted by, $\eta(\gamma;\alpha)/Ind_\gamma(\alpha)$ defined by:

$$\eta(\gamma;\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha}$$

**Example 3.13.** $\gamma : [0, 6\pi] \to \mathbb{C}$
$\gamma(t) = a + re^{it}$

$$\eta(\gamma;\alpha) = \frac{1}{2\pi i}\int_\gamma \frac{\gamma'(t)}{\gamma(t)-a} = \frac{1}{2\pi i}\int_\gamma \frac{1}{a+re^{it}-a}.ire^{it}dt = 3$$

**Theorem 3.14.** Let $\gamma$ be a smooth, closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then $\eta(\gamma;\alpha) \in \mathbb{Z}$.

*Proof.* $\phi : [0,1] \to \mathbb{C}$
$\phi = \dfrac{\gamma'(s)}{\gamma(s)-\alpha}$ and $g : [0,1] \to \mathbb{C}$, $g(t) = \displaystyle\int_0^t \phi(s)ds$
$g(0) = 0$ and $g(1) = \displaystyle\int_0^1 \phi(s)ds = \int_0^1 \frac{\gamma'(s)}{\gamma-\alpha}ds = \int_\gamma \frac{dz}{z-\alpha}$
**Claim**: $g'(t) = \phi(t)$
Proof: To show that $\displaystyle\lim_{h\to 0}\frac{g(t+h)-g(t)}{h} - \phi(t) = 0$
$\dfrac{g(t+h)-g(t)}{h} - \phi(t) = \dfrac{1}{h}\displaystyle\int_t^{t+h}[\phi(t+h)-\phi(t)]ds$ (for $h > 0$, similar for $h < 0$)
Since $\phi$ is uniformly continuous on $[0,1]$, $\forall\epsilon > 0$, $\exists\delta > 0$ such that
$|s-t| < \delta \Rightarrow |\phi(s)-\phi(t)| < \epsilon$
If $h < \delta$, then
$|\dfrac{1}{h}\displaystyle\int_t^{t+h}[\phi(t+h)-\phi(t)]ds| \leq \frac{1}{h}\int_t^{t+h}|\phi(t+h)-\phi(t)|ds < \epsilon$
Same thing holds if $h < 0$
Therefore, $h < \delta \Rightarrow |\dfrac{g(t+h)-g(t)}{h} - \phi(t)| < \epsilon$
Hence, $g' = \phi$
Set $h(t) = e^{-g(t)}(\gamma(t)-\alpha)$

We have, $h^{'}(t) = e^{-g(t)}\gamma^{'}(t) - e^{-g(t)}(\gamma(t)-\alpha)g^{'}(t) = 0$
Hence, $h(t)$ is a constant function.

$$\begin{aligned} e^{-g(0)}(\gamma(0)-\alpha) &= e^{-g(1)}\ (\gamma(1)-\alpha) \\ \Rightarrow e^{-g(0)} &= e^{-g(1)} = 1 \ (\text{As } \gamma \text{ is a closed curve}) \end{aligned}$$

hence $g(1) = 2k\pi i$ (for $k \in \mathbb{Z}$)
Therefore, $\displaystyle\int_\gamma \frac{dz}{z-\alpha} = 2k\pi i \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha} = k \in \mathbb{Z}$ □

**Remark 3.15.** The theorem is true if $\gamma$ is a closed contour. (Prove it!)
(A contour is a piecewise smooth curve)

**Theorem 3.16.** Let $\gamma$ be a closed contour and let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then,
(a) the function $f_\gamma : \mathbb{C} \setminus \{\gamma\} \to \mathbb{Z}$ is continuous. $(\alpha \to \eta(\gamma;\alpha))$
(b) $f$ is constant on every component of $\mathbb{C} \setminus \{\gamma\}$

*Proof.* (a) Let $\alpha_0 \in \mathbb{C} \setminus \{\gamma\}$. Then the function $g : t \to |\alpha_0 - \gamma(t)|$ is continuous.
$g$ attains its infimum, say $s = \inf\limits_{t\in[0,1]} g(t)$
If $\alpha$ is very close to $\alpha_0$, then $|\alpha - \gamma(t)| \geq \frac{s}{2}$. Then,

$$\begin{aligned} &|\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}| = \frac{|\alpha-\alpha_0|}{|z-\alpha||z-\alpha_0|} \leq \frac{2}{s^2}|\alpha-\alpha_0| \ \ (z \in \gamma) \\ &|f_\gamma(\alpha) - f_\gamma(\alpha_0)| \leq \frac{1}{2\pi i}\int_\gamma |\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}|dz \\ &\qquad\qquad \leq \frac{2}{s^2}|\alpha-\alpha_0|\frac{1}{2\pi i}L(\gamma) = M(\alpha-\alpha_0) \ \ (\text{Lipschitz continuous} \Rightarrow \text{continuous}) \end{aligned}$$

(b) Let $V$ be a component, then $f(V)$ is connected in $\mathbb{Z} \Rightarrow f(V)$ is a constant $\in \mathbb{Z}$ □

**Proposition 3.17.** Let $\gamma$ be a closed contour in $\mathbb{C}$. Then $\eta(\gamma;\alpha) = 0$ $\forall \alpha$ in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$

*Proof.* Since $\gamma$ is closed and bounded, $\{\gamma\} \subseteq \bar{B}(0;R)$ for some $R > 0$.
Let $\alpha \in \mathbb{C} \setminus \bar{B}(0;R)$

$|z-\alpha| \geq |\alpha| - |z| \geq |\alpha - R|$
$|\eta(\gamma;\alpha)| = \frac{1}{2\pi} | \int_\gamma \frac{dz}{z-\alpha} | \leq \frac{1}{2\pi} \int_\gamma \frac{dz}{|z-\alpha|} \leq \frac{1}{2\pi} \frac{1}{|\alpha| - R} L(\gamma)$
One can find a large enough $|\alpha|$ to make $\eta(\gamma;\alpha) < 1$
Hence, $\eta(\gamma;\alpha) = 0$, when $|\alpha|$ is sufficiently large
Since, $\eta(\gamma;\alpha)$ is constant within a component, $\eta(\gamma,\beta) = 0$ ($\forall \beta$ in unbounded component)

□

**Proposition 3.18.** Let $\gamma$ be a closed contour consisting of curves $\gamma_1, \ldots \gamma_n$.Then, $\eta(\gamma;\alpha) = \eta(\gamma_1;\alpha) + \cdots + \eta(\gamma_n;\alpha)$ (Prove!)

**Theorem 3.19** (**Cauchy-Goursat theorem**)**.** Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f \in H(\Omega)$. Then for any closed contour $\gamma$ lying in the interior of $\Omega$,
$\int_\gamma f(z)dz = 0$

*Proof.* Step-I (Goursat's theorem):
When $\gamma = T$, a triangle
Let $T^{(0)} = T$
Let $diam(T^{(0)}) = d^{(0)}$ and $peri(T^{(0)}) = p^{(0)}$

$$\begin{aligned}
&\int_{T^{(0)}} f(z)dz \\
&= \int_{T^{(1)}} f(z)dz + \int_{T^{(2)}} f(z)dz + \int_{T^{(3)}} f(z)dz + \int_{T^{(4)}} f(z)dz \\
&|\int_{T^{(0)}} f(z)dz| \;\leq\; 4|\int_{T^{(j)}} f(z)dz| \qquad\qquad \text{(for some } j \in \{1,2,3,4\})
\end{aligned}$$

Call this $T^{(j)}$ to be $T^{(1)}$ (suppose)

$$diam(T^{(1)}) = \frac{1}{2} diam(T^{(0)})$$
$$d^{(1)} = \frac{d^{(0)}}{2} \text{ and } p^{(1)} = \frac{p^{(0)}}{2}$$

Do the same process with $T^{(1)}$ to get,

$$|\int_{T^{(1)}} f(z)dz| \;\leq\; 4|\int_{T^{(2)}} f(z)dz|$$

Continuing,

$$
\begin{aligned}
&|\int_{T^{(0)}} f(z)dz| \leq 4^n |\int_{T^{(n)}} f(z)dz| \\
&d^{(n)} = \frac{d^{(0)}}{2^n} \text{ and } p^{(n)} = \frac{p^{(0)}}{2^n} \\
&\triangle_n = T^{(n)} \cup Int(T^{(n)}) \text{ (Int refers to interior of triangle)} \\
&\triangle_0 \supseteq \triangle_1 \supseteq \ldots \triangle_n \supseteq \ldots \text{ (nested compact sets)}
\end{aligned}
$$

$d^{(n)}$ tends to 0
Therefore, exists ! $z_0 \in \bigcap_{i=0}^{\infty} \triangle_n$
$f$ is holomorphic at $z_0$

$$
\begin{aligned}
&f(z_0 + h) - f(z_0) = hf'(z_0) + h\psi(h) \ (\lim_{h\to 0} \psi(h) = 0) \\
&\text{So, } f(z) - f(z_0) = (z - z_0)f'(z_0) + (z - z_0)\psi_1(h) \text{ where } \lim_{z\to z_0} \psi_1(z) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \int_T f(z)dz \\
&= \int_T f(z_0)dz + \int_T (z - z_0)f'(z_0)dz + \int_T (z - z_0)\psi_1(z)dz = \int_T (z - z_0)\psi_1(z)dz
\end{aligned}
$$

Then,

$$\psi_1(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$$

Let $\sup_{z\in T^{(n)}} |\psi_1(z)| = E_n$ ($E_n \to 0$ as $n \to \infty$)

$$
\begin{aligned}
&|\int_{T^{(n)}} f(z)dz| \\
&= |\int_{T^{(n)}} (z - z_0)\psi_1(z)dz| \leq \int_{T^{(n)}} |z - z_0||\psi_1(z)|dz \\
&\leq d^{(n)} E_n p^{(n)} = \frac{d^{(0)}p^{(0)}}{4^n} E_n \\
&|\int_{T^{(0)}} f(z)dz| \leq 4^n |\int_{T^{(n)}} f(z)dz| \leq d^{(0)}p^{(0)} E_n \ (\forall n)
\end{aligned}
$$

Take limit on both sides as $n \to \infty \Rightarrow |\int_{T^{(0)}} f(z)dz| = 0$

□

# Chapter 4

# Cauchy Goursat Theorem

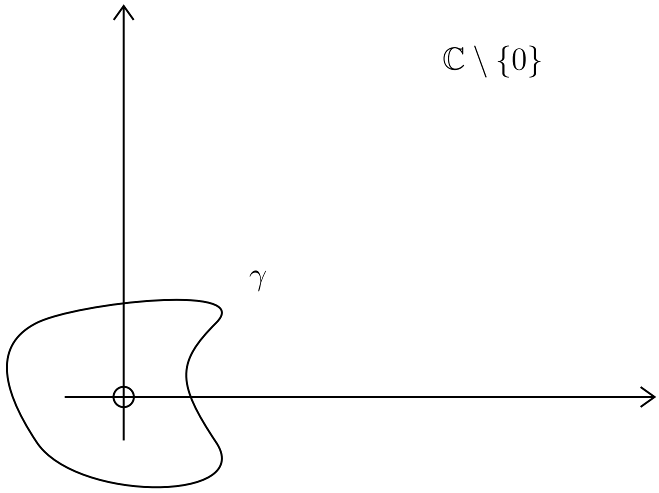
The Cauchy-Goursat Theorem, which as the name of the chapter suggests, is central to this chapter. So we begin by first stating the theorem itself.

**Theorem 4.1** (**Cauchy-Goursat**)**.** If a function $f$ is analytic at all points interior to and on a simple-closed contour $C$ then $\int_C f(z)\, dz = 0$.

This theorem is rather hard to prove, so we will be building tools to deal with it for the remainder of the chapter.
Before moving ahead, we would like to define what a *simply connected* region is.

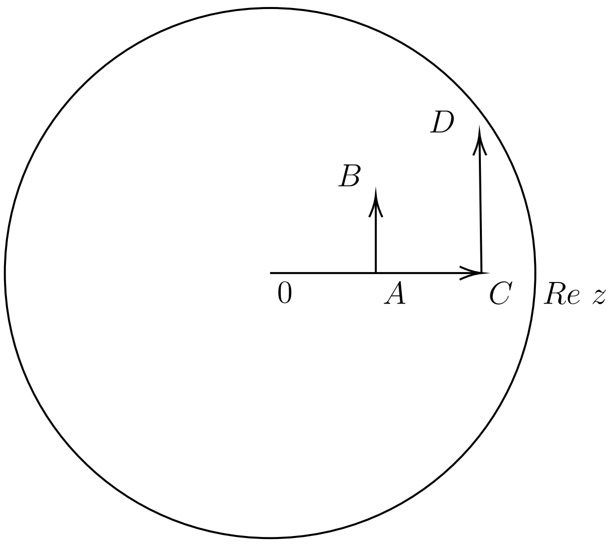
**Definition 4.2** (**Simply-connected Region**)**.** A domain $D$ is said to be *simply connected* if every simple closed contour within it is encloses points of $D$ only.
A domain $D$ is said to be *multiply connected* if it is not simply connected.

A simple example for simply connected region would be $\mathbb{C}$ itself. Any curve $\{\gamma\} \in \mathbb{C}$ can only contain points of $\mathbb{C}$ itself.
Consider the space $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, it is multiply connected.




It is apparent that the curve contains $0 \notin \mathbb{C} \setminus \{0\}$

Let $\gamma_z :=$ OAB.
Define $F(z) = \int_{\gamma_z} f(z)\, dz$
Then, we have,




Since $f$ is continuous at $z$,
$f\left(w\right) = f\left(z\right) + \phi\left(\omega\right), \omega \in$ BD
where $\lim_{w \to z} \phi(\omega) = 0$
Therefore,

$$\omega \to z, f(\omega) \to f\left(z\right)$$

Since $g(\omega) = \omega$ is a primitive for 1,
$\int_{\mathrm{BD}} d\omega = h$
Therefore, (to complete proof)

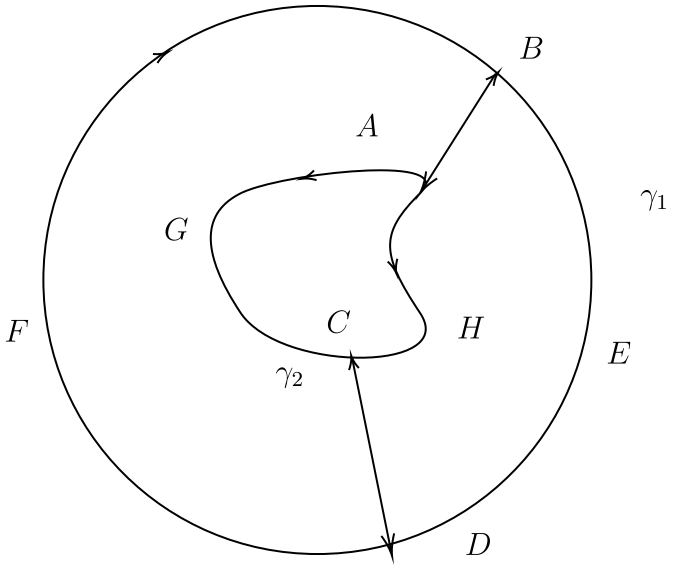
**Remark 4.3.**

1. The above theorem holds if $f \in H(D)$ where $D \subset \mathbb{C}$ is any disk.
2. Let $\Omega \subset \mathbb{C}$ be a disk in $\mathbb{C}$ and if $f \in H(\Omega)$ then,
   $\int_\gamma f(z)\, dz = 0$
   for any closed contour $\gamma \subset \Omega$
3. If $A \subset \mathbb{C}$ and $f$ is holomorphic on $A$ if there is an open set $A \subset U$ such that $f \in H(U)$
4. If $\gamma$ is an closed contour then $f$ is said to be analytic on and inside $\gamma$ if $f \in H(U)$

**Definition 4.4** (Simply-connected)**.** A domain $\Omega \subset \mathbb{C}$ is said to be *simply connected* if for every simple closed curve $\gamma$ lying in $\Omega$, $Int(\gamma) \subset \Omega$

Let $f \in H(\Omega), \Omega \subset \mathbb{C}$ is simply connected and let $\alpha, \beta \in \Omega$.
Then the integral of $f$ along any contour joining $\alpha$ and $\beta$ is same. Let $\gamma_1, \gamma_2$ be two distinct contours joining $\alpha, \beta$. Then, consider the path $\gamma = \gamma_1 + -(\gamma_2)$. Then, we have that
$\int_\gamma f(z)\, dz = 0$ (By Cauchy-Goursat theorem)
Let $\gamma_1$ and $\gamma_2$ be two simple closed contours with same orientations such that $\{\gamma_2\} \subset Int(\gamma_1)$. If a function is holomorphic in the closed contour region bounded by $\gamma_1$ and $\gamma_2$, then,

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$




**Proposition 4.5** (**Cauchy's Integral Formula**)**.** Let $\gamma$ be a simple closed contour, and let $z \in Int\,(\gamma)$. If $f$ is analytic on and inside $\gamma$, then,

$$f\,(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f\,(z)}{z - z_0}\, dz.$$

*Proof.* Witout loss of generality, let $\gamma$ be positively orineted. Since $f$ is continuous at $z_0$, $\forall \epsilon > 0, \exists \delta > 0$ such that,

$$|z - z_0| < \delta \implies |f\,(z) - f\,(z_0)\,| < \epsilon.$$

Set $\gamma_1 := |z - z_0| = r$, where $0 < r < \delta$, (Simple Circle, positively oriented)

The function $\frac{f(z)}{z - z_0}$ is analytic in the closed annular region bounded by $\gamma$ andd $\gamma_1$. Then, by the previous lemma, we have,

$$\int_\gamma \frac{f\,(z)}{z - z_0}\, dz = \int_{\gamma_1} \frac{f\,(z)}{z - z_0}\, dz$$

Now, we can write the corresponding integral as follows:

$$\int_{\gamma_1} \frac{f(z)}{z - z_0}\, dz = \int_{\gamma_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz + \int_{\gamma_1} \frac{f(z_0)}{z - z_0}\, dz$$

Now,

$$|\int_{\gamma_1} \frac{f(z) - f(z_0)}{z - z_0}\, dz| \leq \frac{\epsilon}{r}(2\pi r)$$

Hence the first term vanishes as $\epsilon > 0$

Now the second term is:

$$\begin{aligned}\int_{\gamma_1} \frac{f(z_0)}{z - z_0}\, dz &= f(z_0) \int_{\gamma_1} \frac{1}{z - z_0}\, dz \\ &= f(z_0)\, 2\pi i\end{aligned}.$$

Hence, we have that

$$\begin{aligned}\frac{1}{2\pi i} \int_{\gamma} f(z)\, dz &= \int_{\gamma_1} f(z)\, dz \\ &= \frac{f(z_0)}{2\pi i} 2\pi i = f(z_0)\end{aligned}.$$

□

**Proposition 4.6 (Cauchy's Integral Formula for derivatives).** Let $\gamma$ be a simple closed contour and let $z_0 \in Int(\gamma)$. If $f$ is holomorphic on and inside $\gamma$, then $f$ is infinitely differentiable at any point in $Int(\gamma)$ and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

*Proof.* For $n = 0$, the conclusion is true (by Cauchy's Integral formula)
Suppose it holds for $k = n - 1$. Then,

$$f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^n}dz$$
$$\lim_{h\to 0}\frac{f^{n-1}(z_0+h) - f^{n-1}(z_0)}{h}$$
$$\lim_{h\to 0}\frac{(n-1)!}{2\pi i}\int_\gamma f(z).\frac{1}{h}[\frac{1}{(z-z_0-h)^n} - \frac{1}{(z-z_0^n)}]dz$$
$$\lim_{h\to 0}\frac{(n-1)!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0-h)(z-z_0)}[A^{n-1}+A^{n-2}B+\cdots+B^{n-1}] \text{ (where)}$$

(to complete, refer to corollary 4.2 of stein-shakarchi for proof) ☐

**Proposition 4.7** (**Cauchy's estimate**). Let $f$ be analytic on and inside a simple circle $\gamma : |z - z_0| = r$ and let $|f(z)| \leq M \ \forall z \in \{\gamma\}$. Then,

$$|f^{(n)}(z_0)| \leq \frac{n!\, M}{r^n}$$

*Proof.*

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\int_\gamma |\frac{f(z)}{(z-z_0)}|dz$$
$$\frac{n!}{2\pi}.\frac{M}{r^{n+1}}2\pi r = \frac{n!\, M}{r^n}$$

(to complete, refer to corollary 4.3 of stein-shakarchi for proof) ☐

**Theorem 4.8** (**Liouville's theorem**). A function which is entire and bounded is a constant function.

*Proof.* Let $f$ be an entire function with $|f(z)| \leq M \ \forall z \in \mathbb{C}$
$|f'(z_0)| \leq \frac{M}{R} < \epsilon$, for given $\epsilon > 0$ when $R$ is sufficiently large.
Therefore, $f'(z_0) = 0$ ☐

**Theorem 4.9.** Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f \in H(\Omega)$. If $D$ is a disk centered at $z_0 \in \Omega$ such that $\bar{D} \subseteq \Omega$, then $f$ has a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \forall z \in D$$
$$\text{where } \ a_n = \frac{f^{(n)}(z_0)}{n!}, \ \forall n = 0, 1, 2, \ldots$$

*Proof.* Fix $z \in D$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{w - z} dw \ \ \gamma = \partial D$$
$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} . \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

$\exists \in (0, 1)$ such that

$$|\frac{z - z_0}{w - z_0}| < r < 1$$

(Refer to Theorem 4.4 of stein-shakarchi) □

**Questions**:

1. Prove that every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$
   (A continuous function is bounded inside a disk)

2. Every monic polynomial $p(z)$ with degree $n$ ($\geq 1$) has precisely $n$ roots in $\mathbb{C}$. If the roots are $z_1, z_2 \ldots z_n \in \mathbb{C}$, then $p(z) = (z - z_1) \ldots (z - z_n)$.

# Chapter 5

# NotAssigned

**Morera's theorem**:
Suppose $f$ is continuous complex-valued function on a domain $\Omega \subseteq \mathbb{C}$ such that $\int_T f(z)dz = 0$ for all triangles $T \subseteq \Omega$
Then, $f \in H(\Omega)$
Proof: Let $D \subseteq \Omega$ be any arbitrary disk. Then $\int_T f(z)dz = 0$, $\forall T \subseteq D$. Following the proof of Cauchy-Goursat Theorem for a disk, we have that,
$f$ has a primitive in $D$.
$F^{'}(z) = f(z)$, $\forall z \in D$
Thus $f \in H(D)$.
Since $\Omega$ is a union of such disks, we can conclude. **Theorem**: Let $\Omega \subseteq \mathbb{C}$ be a region and let $f_n \in H(\Omega)$, $\forall n \in \mathbb{N}$. If $f_n$ converges uniformly to $f$ on every compact subset of $\Omega$, then $f \in H(\Omega)$
Also, $f_n^{'}$ converges to $f^{'}$ uniformly on all compact subsets of $\Omega$.
Proof: Let $D \subseteq \Omega$ be a disk such that $\bar{D} \subseteq \Omega$. Then,
$\int_T f_n(z)dz = 0$ for all triangle $T \subseteq D$ (by Cauchy Goursat theorem)
Since $f_n \to f$ uniformly,
$\lim_{n\to\infty} \int_T f_n(z)dz = \int_T \lim_{n\to\infty} f_n(z)dz = \int_T f(z)dz$
$\Rightarrow \int_T f(z)dz = 0$
$\Rightarrow f \in H(D)$ (by Morera's theorem)
To show $f_n^{'} \to f$ uniformly, consider $g_n^{'} = f_n^{'} - f$ and use Cauchy's theorem for $g_n^{'}$.
**The Identity Theorem**: Let $\Omega \subseteq \mathbb{C}$ be a region and let $f \in H(\Omega)$. If $\omega_n$ is a sequence of distinct points on $\Omega$ such that $f(\omega_n) = 0$, $\forall n \in \mathbb{N}$ and if $w_n$ has a limit point in $\Omega$, then $f = 0$ on $\Omega$
Proof: Let $z_0 \in \Omega$ be a limit point of $\omega_n$. Let $D \subseteq \Omega$ be a disk with centre at $z_0$. Since $f \in H(\Omega)$, $f$ has a power series in $D$.
$f = \sum_0^\infty a_n(z - z_0)^n$

$f(z_0) = a_0 = 0$ (by the continuity of $f$, $\omega_n \to z_0$ and $f(\omega_n) = 0$)
$f(z) = \sum_1^\infty a_n(z - z_0)^n$
Let $f \neq 0$ on $D$.
Then $\exists n_0 \in \mathbb{N}$ such that $a_{n_0} \neq 0$
Let $m \in \mathbb{N}$ be the least positive integer such that $a_m \neq 0$. Then,
$f(z) = a_m(z - z_0)^m[1 + \frac{a_{m+1}}{a_m}(z - z_0) + \dots] = a_m(z - z_0)^m[1 + g(z - z_0)]$ (where $g(z - z_0)$ goes to 0 as $z \to z_0$)
Since $\omega_n$ is a sequence of distinct points, $\exists k \in \mathbb{N}$ such that $|g(\omega_k - z_0)| < 1$ and $\omega_k \neq z_0$
Therefore, $g(\omega_k - z_0) \neq -1$
Now, $f(\omega_k) = a_m(\omega_k - z_0)^m[1 + g(\omega_k - z_0)]$
$\Rightarrow LHS = 0$ and $RHS \neq 0$, contradiction.
We conclude $f = 0$ on $D$.
Let $Z(f) = \{z \in \Omega : f(z) = 0\} \subseteq \Omega$
Let $G$ be the interior of $Z(f)$.
Then, $G \neq \phi$ as $D \subseteq G$. We will show $G$ is closed.
Let $y_0$ be a limit point of $G$. Then there is a sequence of distinct points $\{y_n\} \subseteq G$ such that $\lim_{n\to\infty} y_n = y_0$
$\{y_n\} \subseteq G \subseteq Z(f) \Rightarrow y_0 \in Z(f) \subseteq \Omega$
Since $\Omega$ is open, $y_0 \in \Omega$ is an interior point of $\Omega$ and thus $\exists$ disk $D(y_0; r) \subseteq \Omega$.
Then there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\} \subseteq D(y_0; r)$ and $y_{n_k} \to y_0$.
Since $f(y_{n_k}) = 0$, $\forall n_k$ by the previous part $f = 0$ on $D(y_0; r)$
Then, $D(y_0; r) \subseteq G$ and thus $y_0$ is an interior point of $G$. So $y_0 \in G$ and $G$ is closed.
$G \subseteq \Omega$ (both open and closed)
$G = \Omega = Z(f) \Rightarrow f = 0$ on $\Omega$
**Corollary**: Let $f$ and $g$ belong to $H(\Omega)$, $\Omega$ is a domain and let $f(z) = g(z)$, $\forall z \in \{\omega_n\} \subseteq \Omega$, where $\{\omega_n\}$ is a sequence of distinct points having a limit point in $\Omega$.
Then, $f = g$ on $\Omega$
**Theorem**: Let $\Omega$ be a domain and let $f \in H(\Omega)$ not be identically zero. Then corresponding to every zero $\alpha$ of $f$, there is a unique positive integer $m$ and a unique $g \in H(\Omega)$ such that
$f(z) = (z - \alpha)^m g(z)$, where $g(z) \neq 0$ in a neighbourhood of $\alpha$.
Proof: Since $f \neq 0$ on $\Omega$, there is a disk $D$ centred at $\alpha$ such that $D \subseteq \Omega$ and $f(z_0) \neq 0$, $\forall z \in D \setminus \{\alpha\}$
$f(z) = \sum_1^\infty a_n(z - z_0)^n$, $z \in D$ $(a_0 = f(\alpha) = 0)$
Since $f \neq 0$ on $D$, there is a positive integer $m$ such that $a_m \neq 0$ and $a_0 = a_1 = \dots = a_{m-1} = 0$
Then, $f(z) = (z - \alpha)^m[a_m + a_{m+1}(z - \alpha) + \dots]$

Let $g(z) = a_m + a_{m+1}(z-\alpha) + \dots \ (z \in D)$
$g \in H(D)$ Let us define $g$ on $\Omega$ by: $g(z) = \frac{f(z)}{(z-\alpha)^m}$ (for $z \neq \alpha$)
$g(z) = a_m$ (for $z = \alpha$)
We can conclude $g \in H(\Omega)$
Let $f(z) = (z-\alpha)^m g(z) = (z-\alpha)^p h(z)$
If $p > m$, then $g(z) = (z-\alpha)^{p-m} h(z)$ At $\alpha$, $LHS = 0$ and $RHS \neq 0$
We can conclude uniqueness (complete proof)
**Lemma**: Let $E$ be the annular region determined by two closed contours $\gamma_1$ and $\gamma_2$ with $\gamma_2 \subseteq Int(\gamma_1)$. If $f$ is analytic on $\bar{E}$, then for any point $z_0 \in E$:
$f(z_0) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(z)}{z-z_0}$ ($\gamma_1$ and $\gamma_2$ have positive orientation)
**Theorem(Laurent)**: Let $f$ be analytic on the closure of an annular region $E = \{z \in \mathbb{C} : r < |z-z_0| < R\}$. Then,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

where, $a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$ and $b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{-n+1}}dz$ $(\forall n)$
Here, $\gamma : |z-z_0| = \rho$ , $r \leq \rho R$
Proof: Let $\alpha \in E$. Then by previous lemma,

$$f(\alpha) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-\alpha}dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(z)}{z-\alpha}$$

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-\alpha}dz &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{(z-z_0)-(\alpha-z_0)}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{(z-z_0)[1-\frac{\alpha-z_0}{z-z_0}]}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{(z-z_0)(1-\frac{1}{1-t})}dz \ \ (t = \frac{\alpha-z_0}{z-z_0}) \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-z_0}(1+t+t^2\dots)dz \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z-z_0}t^n dz \\
&= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)(\alpha-z_0)}{(z-z_0)^{n+1}}dz \\
&= \sum_{n=0}^{\infty} a_n(\alpha-z_0)^n
\end{aligned}$$

**Singularity**: A point $z_0 \in \mathbb{C}$ is called a point of singularity of a function $f$ if $f$ is not analytic at $z_0$.
Singularity can either be *isolated* or *non-isolated.* An *isolated* singularity can either be *removable* or *non-removable.*
**Definitions**: A singularity $z_0 \in \mathbb{C}$ is said to be isolated singularity of $f$ if there exists a neighbourhood $N(z_0)$ of $z_0$ such that $f$ is analytic on $N(z_0) \setminus \{z_0\}$
A singularity which is not isolated is called a non-isolated singularity.

**Definition 5.1** (Removable and non-removable)**.** (i) A function $f$ has a removable singularity at $z_0$ if $z_0$ is an isolated singularity of $f$ and $f$ can be redefined at $z_0$ so that $f$ becomes analytic at $z_0$ i.e if
$\lim_{z\to z_0} f(z) = l \in \mathbb{C}$ (ii) If $f_1$ and $f_2$ are analytic at $z_0$ with $f_1(z_0) \neq 0$ and $z_0$ is a zero of order $m$ then $f = \frac{f_1}{f_2}$ is said to have a *pole* of order $m$ at $z_0$. A pole of order 1 is called a *simple* pole.
(iii) An isolated singularity which is neither removable nor a pole is called a *essential* singularity.

**Examples**: (i) Consider the function $f(z) = \frac{1-cos(z)}{z^2}$. At $z = 0$, the function doesn't have a pole (since the numerator vanishes) but the point is a isolated removable singularity (Why?)
(ii) $f(z) = e^{\frac{1}{z}}$, $z = 0$ is an esential singularity. (Why?)
**Theorem**: Let $z_0 \in \mathbb{C}$ be an isolated singularity of $f$. Then $z_0$ is a removable singularity iff $\lim_{z\to z_0}(z - z_0)f(z) = 0$
Proof: Forward direction is trivial.
For the converse, let $\lim_{z\to z_0}(z - z_0)f(z) = 0$.
We show that, $\lim_{z\to z_0} f(z)$ exists.
$f$ is analytic in a deleted neighbourhood $N(z_0) \setminus \{z_0\}$
Define $g$ on $N(z_0)$ by:
$g(z) = (z - z_0)f(z)$ if $z \neq z_0$ otherwise $g(z) = 0$
$g$ is analytic in $N(z_0) \setminus \{z_0\}$. Since $f$ is analytic in $N(z_0) \setminus \{z_0\}$.
Also, $\lim_{z\to z_0} g(z) = 0$ (exists)
Therefore, $g \in H(N(z_0))$
Since $z_0$ is a zero of $g$, there is a unique $m \in \mathbb{N}$ and a unique $\phi \in H(N(z_0))$ such that
$g(z) = (z - z_0)^m \phi(z)$ ($\phi(z) \neq 0$ in $N(z_0)$)
$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{g(z)}{z-z_0} = \lim_{z\to z_0} \frac{(z-z_0)^m \phi(z)}{(z-z_0)} = \lim_{z\to z_0}(z - z_0)^{m-1}\phi(z)$
**Theorem**: Let $z_0$ be an isolated singularity of $f$ and let
$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$

Then $z_0$ is a removable singularity iff $b_n = 0$ $(\forall n \in \mathbb{N})$ Proof: $z_0$ is a removable singularity of $f$
$\Leftrightarrow \lim_{z \to z_0} (z - z_0) f(z) = 0$
$\Leftrightarrow$