MA-412

Complex Analysis

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§1. Preliminaries

Region: An open, connected subset of $\mathbb{C}$ is called domain or region.

$$D(a,r) = \{z \in \mathbb{C} |\ |z-a| < r\}$$

$$\{z \in \mathbb{C} |\ r < |z-a| < R\} \rightarrow \text{annulus}$$

Exponential function: $exp(z) : \mathbb{C} \rightarrow \mathbb{C}$

$$e^z = e^x(cos(y) + isin(y))$$

$$e^{z_1+z_2} = e^{z_1}.e^{z_2}, 0 \notin Rng(exp(z))$$

$$|e^{ix}| = 1 \ \ (\forall x \in \mathbb{R})$$

$$e^z = 1 \text{ iff } z = 2n\pi i,\ n \in \mathbb{Z}$$

$$e^{z_1} = e^{z_2} \text{ iff } z_1 = z_2 + 2n\pi i$$

Argument: $arg(z) : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$

$$arg(z) = \theta \text{ (Angle made wrt positive real axis)}$$

$$arg(z) \text{ for } z \in \mathbb{C} \setminus \{0\} \text{ is multivalued function}$$

$$arg(z) : \mathbb{C} \setminus \{0\} \rightarrow (\alpha, \alpha + 2\pi] \text{ or } [\alpha, \alpha + 2\pi) \text{ is well-defined } (\forall \alpha \in \mathbb{R})$$

Principal value: $Arg(z) = \theta \ (-\pi < \theta \leq \pi)$

$$arg(z) = \{Arg(z) + 2n\pi : n \in \mathbb{Z}\}$$

Complex Log: For $z \in \mathbb{C} \setminus \{0\}$, define:

$$log(z) = log(|z|) + iarg(z)$$

Notice that log is multivalued

Whenever $arg(z)$ is well-defined (i.e $Rng(arg(z)) = [\alpha, \alpha + 2\pi)$ or $(\alpha, \alpha + 2\pi]$)

$\Rightarrow log(z)$ is well defined

$$Log(z) = log(|z|) + i.Arg(z) \text{ (principal log)}$$
$$log(z) = Log(z) + 2n\pi i$$

Lemma 1.1: For $z \in \mathbb{C} \setminus \{0\}$, the values of $log(z)$ are the complex numbers ω such that $e^{\omega} = z$

Proof: Do it on your own!

Limit: $f : E \subset \mathbb{C} \to \mathbb{C}$

$$\lim_{z \to z_0} f(z) = l$$
$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$$
$$z \in (B(z, z_0) \setminus \{0\}) \cap E \Rightarrow f(z) \in B(l, \epsilon)$$

Proposition 1.1 (Sequential criteria for limit):

A function $f : E \subset \mathbb{C} \to \mathbb{C}$ has a limit point l as $z \to z_0$ iff $f(z_n) \to l$ for every sequence $\{z_n\} \subset E \setminus \{z_0\}$ with $z_n \to z_0$ as $n \to \infty$

Theorem 1.1: If $f : E \subset \mathbb{C} \to \mathbb{C}$ has a limit at z_0, then f is bounded near z_0

Continuity: $f : E \subset \mathbb{C} \to \mathbb{C}$

f is continuous at $z_0 \in E$ if for every basic neighbourhood V of $f(z_0)$, there is a basic neighbourhood U of z_0 such that $f(U) \subset V$. Let $V = B(f(z_0), \epsilon)$ & $U = B(z_0, \delta)$,

$$z \in B(z_0, \delta) \Rightarrow f(z) \in B(f(z_0), \epsilon)$$
$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$
$$\text{Then, } z \in B(z_0, \delta) \setminus \{z_0\} \Rightarrow f(z) \in B(f(z_0), \epsilon)$$
$$\lim_{z \to z_0} f(z) = f(z_0)$$

Prove: Let $f : G \to \mathbb{C}$ such that $f = u + iv$ (where $u, v : G \to \mathbb{R}$)

f is continuous at a point z_0 iff u, v are continuous at z_0

§2. Branch of Log and Power function

$Arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$

$Log(z) = log(|z|) + i.Arg(z)$ (defined on $\mathbb{C} \setminus \{0\}$)

$Log(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$ as $Arg(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$

§§2.1. Branch of Log

Let $G \subset \mathbb{C}$ be a region. A continuous function $f : G \to \mathbb{C}$ is called a branch of logarithm in G if:

$$e^{f(z)} = z \ \ (\forall z \in G)(\Rightarrow 0 \notin G)$$

$$e^{Log(z)} = z \ \ (\forall z \in \mathbb{C} \setminus \{0\})$$

But $Log(z)$ isn't continuous on $\mathbb{C} \setminus \{0\}$

Therefore, $Log(z)$ is a branch of log in $\mathbb{C} \setminus (-\infty, 0]$

$$f_k = Log(z) + 2k\pi i, \ k \in \mathbb{Z}$$

$$e^{f_k(z)} = z \ \text{(Each } f_k \text{ is a branch of log in G)}$$

Theorem 2.1: Let $f : G \to \mathbb{C}$ be a branch of log. Then $g : G \to \mathbb{C}$ is a branch of log iff

$g(z) = f(z) + 2k\pi i$ (for some $k \in \mathbb{Z}$)

Proof: $(\Rightarrow)$ if f is a branch of log, then so is g

$(\Leftarrow)$ Let $g : G \to \mathbb{C}$ be a branch of log.

Then, (i) g is continuous

(ii) $e^{g(z)} = z$, $\forall z \in G$

Since $f : G \to \mathbb{C}$ is a branch of log, we have,

(i) f is continuous

(ii)$e^{f(z)} = z$, $\forall z \in G$

$e^{g(z)} = z = e^{f(z)}$ $(\forall z \in G)$

$\Rightarrow g(z) = f(z) + 2k(z)\pi i$ (k depends on z)

$k : G \subset \mathbb{C} \to \mathbb{Z}$

$k(z) = \frac{1}{2\pi i}(g(z) - f(z))$ (k is continuous)

Since G is connected, $Img(k)$ is connected (subset of $\mathbb{Z}$)

$\Rightarrow k(z)$ is constant

Hence, the claim follows.

Unit disk in $\mathbb{C}$: $D = \{z \in \mathbb{C} : |z| < 1\}$

$0 \in D$, hence D cannot be a branch of log.

$D \setminus \{0\}$ isn't a branch of log. (Why?)

$Log(z) : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$

The half-line is called a branch-cut for any member of $\{Log(z) + 2k\pi i : k \in \mathbb{Z}\}$

Question: How do you make a branch-cut to define a branch of $Log(z + i - 1)$?

§§2.2. Power functions

Let $\alpha \in \mathbb{C}$. We define z^α to be the multi-valued function:

$$\begin{aligned} z^\alpha = e^{\alpha log(z)} &= e^{\alpha(log(|z|)+iarg(z))} \\ &= e^{\alpha(Log(z)+2k\pi i)} \\ &= e^{\alpha Log(z)}.e^{2\pi ik\alpha} \ \ (z \neq 0, k \in \mathbb{Z}) \end{aligned}$$

Let $\alpha = n \in \mathbb{N}$. Then $z^\alpha = e^{nLog(z)}$ which is single-valued.

§3. Differentiability and Power series

§§3.1. Differentiability

Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Then f is said to be differentiable at $z_0 \in \Omega$ if,

$$f^{'}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists

Example: $f(z) = \bar{z}$ isn't differentiable at any point in $\mathbb{C}$.

Cauchy-Riemann equations: Let $f : \Omega \to C$ be differentiable at $z_0 = x_0 + iy_0$. Then,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Further,

$$f^{'}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

Note that the last equation only has partial derivatives wrt x.

Proof (Sketch): In the differential, approach the origin once through the real axis and and then the imaginary axis to get some equations

Holomorphic/Analytic: A function is said to be **holomorphic** or **analytic** at z_0 if it is differentiable in a neighbourhood of z_0.

If a function is analytic at a point, it is called a regular point for the function

Entire functions: If a function f is analytic at every point in $\mathbb{C} \Rightarrow f$ is entire.

Example:

1. $f(z) \to$ polynomial in $\mathbb{C}$
2. $f(z) = \frac{1}{z}$ (differentiable at every point on $\mathbb{C} \setminus \{0\}$)

Proposition 3.1: If f and g are differentiable at $z_0 \in \Omega \subset \mathbb{C}$, then so are:

(i)$f + g$, (ii)fg, (iii)f/g $(g(z_0) \neq 0)$

Proof: Trivial

Chain Rule: $f : \Omega \to U$, $g : U \to \mathbb{C}$ are holomorphic at z_0 and $f(z_0)$ respectively. Then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic at z_0 and $(g \circ f)^{'}(z_0) = g^{'}(f(z_0)).f^{'}(z_0)$

Proof: Standard (found in any introductory complex analysis text)

§§3.2. Power series

A series of the form $\sum_0^\infty a_n(z - z_0)^n$ ($a_n, z, z_0 \in \mathbb{C}$; z_0 is center)

Theorem 3.1(Radius of convergence): Given a power series $\sum_{n=0}^\infty a_n(z)^n$, $\exists R \in [0, \infty) \cup \{\infty\}$ such that:

1. the series converges absolutely ($\forall z \in B(0, R)$; the disk of convergence)
2. diverges $\forall z$ such that $|z| > R$
3. $$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$$

(Convention: $\frac{1}{0} = +\infty$, $\frac{1}{\infty} = 0$)

Proof: Can be found in any standard text (Try Conway!)

Theorem 3.2: The power series $f(z) = \sum_{n=0}^\infty a_n z^n$ defines a holomorphic function (in its disk of convergence)

Derivative of f is obtained by differentiation of each term:

$f^{'}(z) = \sum_{n=0}^\infty n a_n z^{n-1}$

$f^{'}$ has same disk of convergence

Proof: Found in any standard text

Remark: A power series is infinitely differentiable (analytic) in its disk of convergence

$$\Rightarrow \qquad f(z) = \sum a_n z^n = \sum_{k=0}^\infty \frac{f^k(0)}{k!} z^k$$

Proposition 3.2: Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be domains and let $f : \Omega_1 \to \mathbb{C}$, $g : \Omega_2 \to \mathbb{C}$ be continuous function such that $f(\Omega_1) \subset \Omega_2$ and that $g(f(z)) = z$ ($\forall z \in \Omega_1$). If g is differentiable on Ω_2 and if $g^{'}(w) \neq 0$ ($\forall w \in \Omega_2$), then f is differentiable and $f^{'}(z) = \frac{1}{g^{'}(f(z))}$ ($\forall z \in \Omega_1$)

Proof (Sketch): Use the differentiability of g and the continuity of f to get the result

Theorem 3.3: A branch of logarithm is analytic and its derivative is $\frac{1}{z}$.

Proof (Sketch): Let f be a branch of log. Define $g(z) = e^z$. Then $g(f(z)) = z$. ($e^{f(z)} = z$)

Lemma 3.1: Let $f : \Omega \to \mathbb{C}$. Then f is differentiable at $z_0 \in \Omega$ iff $\exists a \in \mathbb{C}$ such that

$f(z_0 + h) - f(z_0) = ah + h\psi(h)$ ($\lim_{h\to 0} \psi(h) = 0$)

Proof: Use the definition of differentiability of a multivariable function

Theorem 3.4: Let $\Omega \subset \mathbb{C}$ be a domain and let $f = u + iv$ is a function from Ω to $\mathbb{C}$. Then f is analytic at $z_0 \in \Omega$ iff the partial derivatives exist, are continuous and satisfy Cauchy-Riemann equations.

Proof(To complete): $\Rightarrow$ If f is analytic at z_0, then partial derivatives are continuous and satisfy the Cauchy-Riemann equations (follows from theorem 3.2)

$\Leftarrow$ Let $\tilde{f} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $\tilde{f} = (u, v)$

$\tilde{f}$ is differentiable at $z_0 = (x_0, y_0)$

Definition: For domain Ω, $H(\Omega)$ is the collection of all analytic functions from $\Omega \to \mathbb{C}$

Proposition 3.3: Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ such that $f \in H(\Omega)$. If D is a disk with center at $z_0 \in \Omega$ and if $\bar{D} \subseteq \Omega$, then f has a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \ \ (\forall z \in D)$$

$$\text{where } a_n = \frac{f^{(n)}(z_0)}{n!}, \ (n \in \mathbb{N})$$

Proof (Sketch): Differentiate the power series repeatedly (Use Theorem 3.2)

Examples:

1. $\displaystyle\sum_{n=1}^{\infty} nz^n < \infty$ (converges $\forall z \in B(0, 1)$; doesn't converge for $|z| = 1$)

2. $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2} < \infty$ (converges $\forall z \in B(0, 1)$; converges for $|z| = 1$)

3. $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n} < \infty$ (converges for $|z| < 1$; diverges otherwise)

4. $f(z) = e^z \Rightarrow f^{'}(z) = e^z$ (by definition of e^z)
   Consider $g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n!}$
   $g(z)$ is an entire function (Why?)
   Note that $f^{(k)}(0) = 1 \Rightarrow a_k = \dfrac{f^{(k)}(0)}{k!} = \dfrac{1}{k!} \Rightarrow f(z) = g(z)$

§4. Complex integration

A **path** or a **curve** is a continuous function, $\gamma : [a, b] \to \mathbb{C}$ $(Rng(\gamma) \subset \mathbb{C})$

$\gamma(a)$: initial point of path; $\gamma(b)$: endpoint of path

$[a, b]$: parameter interval

γ is said to be:

1. closed if $\gamma(a) = \gamma(b)$
2. smooth or C^1 if γ is differentiable and $\gamma^{'}$ is continuous
3. simple if γ is one-one
4. simple closed if $\gamma(a) = \gamma(b)$ and γ is one-one on (a, b)
5. piecewise smooth if there are finitely many points $s_0, s_1 \ldots s_n \in [a.b]$ with $a = s_0 < s_1 < s_2 \cdots < s_n = b$ such that the restriction of γ to each (s_i, s_{i+1}) is smooth.

$-\gamma$ or γ^{-1} is defined by $\gamma^{-1}(t) = \gamma(a + b - t)$

$\phi : [0, 1] \to [a, b]$ defined as: $\phi(t) = a + (b - a)t$ (one-one and differentiable)

§§4.1. Line integral

$f : [a, b] \to \mathbb{C}$: continuous

$f = u + iv$, where $u, v : [a, b] \to \mathbb{R}$

Define $\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$

Properties:

1. $\int_a^b c.f(t)dt = c. \int_a^b f(t)dt$
2. $| \int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$

Length of a smooth curve: Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve.

$L(\gamma) = \int_a^b |\gamma^{'}(t)|dt = \int_a^b \sqrt{\gamma_1^{'}(t)^2 + \gamma_2^{'}(t)^2}dt \quad (\gamma(t) = \gamma_1(t) + i.\gamma_2(t))$

If $\gamma : [a, b] \to C$ is piecewise smooth then $L(\gamma)$ is the sum of the length of its smooth parts.

Orientation: A curve γ is *positively* oriented if traversed in anti-clockwise direction else is *negatively* oriented.

Examples:

1. $\gamma(t) = re^{it}$, $(t \in [0, 2\pi])$ $(r > 0$: simple, smooth curve);

   $L(\gamma) = \int_0^{2\pi} |ire^{it}|dt = r(2\pi)$

2. $\gamma(t) = e^{it}$, $(t \in [0, 4\pi])$: closed, smooth, traverses the unit circle twice in the positive direction

Integration over paths: $\gamma[a, b] \to \mathbb{C}$ is a smooth curve and $f : \gamma \to \mathbb{C}$: continuous

Definiton: $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma^{'}(t)dt = \int_a^b g(t)dt$

$(g(t) = f(\gamma(t))\gamma^{'}(t)$ where $g : [a, b] \to \mathbb{C})$

Let $[a_1, b_1]$ be any closed interval. Then $\exists \phi : [a_1, b_1] \to [a, b]$ (one-one,differentiable and $\phi(a_1) = a; \phi(a_2) = b)$

$\phi[a_1, b_1] \to \mathbb{C}$: smooth

$\int_{a_1}^{b_1} f(\gamma_1(t)).\gamma_1^{'}(t)dt \quad (= \int_{\gamma_1} f(z)dz)$

$= \int_{a_1}^{b_1} f(\gamma(\phi(t))).\phi^{'}(t)dt = \int_\gamma f(\gamma(s))\gamma^{'}(s)ds = \int_\gamma f(z)dz \quad (\phi(t) = s)$

If γ is piecewise smooth, the integral can be split into the sum of its smooth components: if $\gamma = \gamma_1 + \gamma_2 \cdots + \gamma_n$, then $\int_\gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f$.

Note that $\gamma_i's$ are smooth.

Proposition: If f and g are continuous on a smooth curve γ, then

1. $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$
2. $\int_{\gamma^-} f = -\int_\gamma f$
3. $|\int_\gamma f(z)dz| \leq \|f\|_{\infty,\gamma} L(\gamma) \quad (\|f\|_{\infty,\gamma} = sup_{z\in\{\gamma\}}|f(z)|)$

$|\int_\gamma f| = |\int_a^b f(\gamma(t)).\gamma^{'}(t)dt| \leq \int_\gamma |f(\gamma(t)).\gamma^{'}(t)|dt \leq \|f\|_{\infty,\gamma} \int_a^b |\gamma^{'}(t)|dt \quad (L(\gamma) = \int_a^b |\gamma^{'}(t)|dt)$

Examples:

(i)Let γ be the arc of a circle of radius 3 ($|z| = 3$) from 3 to $3i$.

Show that:

$$|\int_\gamma \frac{z+4}{z^3-1}dz| \leq \frac{21\pi}{52}$$

(ii) $\gamma : |z| = 2$ (traverse curve in positive direction)

Prove:

$$|\int_\gamma \frac{e^z dz}{z^2+1}| \leq \frac{4\pi e^2}{3}$$

Fundamental theorem of calculus:

If $f:[a,b]\to\mathbb{R}$ has a primitive F, then $\int_a^b f(x)dx = F(b)-F(a) \quad (F^{'}(x)=f(x),\forall x\in[a,b])$

For complex case: Suppose $G\in\mathbb{C}$ be a domain. If a continuous function $f:G\to\mathbb{C}$ has a primitive F on G and if γ is a smooth curve in G with initial and terminal points ω_1 and ω_2 respectively, then:

$\int_\gamma f = F(\omega_1)-F(\omega_2)$

Proof: Let $[a,b]\in\mathbb{R}$ be a parameter interval for γ and $\gamma(a)=\omega_1$; $\gamma(b)=\omega_2$

Given $F^{'}(z)=f(z)$ $(\forall z\in G)$

$$\int_\gamma f = \int_a^b f(\gamma(t)).\gamma^{'}(t)dt = \int_a^b F^{'}(\gamma(t))\gamma^{'}(t)dt$$

$$= \int_a^b (F\circ\gamma)^{'}(t)dt = F\circ\gamma(b) - F\circ\gamma(a) = F(\omega_2)-F(\omega_1)$$

Corollary-1: If γ is a closed curve (smooth), then

$\displaystyle\int_\gamma f = 0$

Proof: Follows from FTC

Corollary-2: If $f\in H(\Omega)$ for a region $\Omega\in\mathbb{C}$ and if $f^{'}=0$ on Ω, then f is a constant function.

Proof: Fix a point $\omega_0\in\Omega$. It suffices to show that $f(\omega)=f(\omega_0),\forall\omega\in\Omega$

§§4.2. Simple Closed Curve

Jordan-curve theorem: Every simple closed curve in $\mathbb{C}$ divides the complex plane into two regions. One of these regions is bounded and the other is unbounded. The bounded region is called the interior of the curve.

Example: $G=\mathbb{C}\setminus\{0\}$

$f(z)=\frac{1}{z}$ on G, $\gamma:|z|=1$, $\gamma(t)=e^{it}$, $(t\in[0,2\pi])$

$$\int_\gamma f = \int_0^{2\pi} f(\gamma(t)).\gamma^{'}(t)dt = \int_0^{2\pi}\frac{i.e^{it}}{e^{it}}dt = 2\pi i\neq 0$$

Winding number or index of a closed curve: Let γ be a closed curve on $\mathbb{C}$ and let $\alpha\in\mathbb{C}\setminus\{\gamma\}$. The winding number of γ about α or the index of γ with respect to α is denoted by, $\eta(\gamma;\alpha)/Ind_\gamma(\alpha)$ defined by:

$$\eta(\gamma;\alpha) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-\alpha}$$

Example: $\gamma : [0, 6\pi] \to \mathbb{C}$

$\gamma(t) = a + re^{it}$

$$\eta(\gamma;\alpha) = \frac{1}{2\pi i}\int_{\gamma}\frac{\gamma^{'}(t)}{\gamma(t)-a} = \frac{1}{2\pi i}\int_{\gamma}\frac{1}{a+re^{it}-a}.ire^{it}dt = 3$$

Theorem 4.1: Let γ be a smooth, closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then $\eta(\gamma;\alpha) \in \mathbb{Z}$.

Proof: $\phi : [0,1] \to \mathbb{C}$

$\phi = \dfrac{\gamma^{'}(s)}{\gamma(s)-\alpha}$ and $g : [0,1] \to \mathbb{C}$, $g(t) = \displaystyle\int_0^t \phi(s)ds$

$g(0) = 0$ and $g(1) = \displaystyle\int_0^1 \phi(s)ds = \int_0^1 \frac{\gamma^{'}(s)}{\gamma-\alpha}ds = \int_{\gamma}\frac{dz}{z-\alpha}$

Claim: $g^{'}(t) = \phi(t)$

Proof: To show that $\displaystyle\lim_{h\to 0}\frac{g(t+h)-g(t)}{h} - \phi(t) = 0$

$\dfrac{g(t+h)-g(t)}{h} - \phi(t) = \dfrac{1}{h}\displaystyle\int_t^{t+h}[\phi(t+h)-\phi(t)]ds$ (for $h > 0$, similar for $h < 0$)

Since ϕ is uniformly continuous on $[0,1]$, $\forall\epsilon > 0$, $\exists\delta > 0$ such that

$|s-t| < \delta \Rightarrow |\phi(s)-\phi(t)| < \epsilon$

If $h < \delta$, then

$|\dfrac{1}{h}\displaystyle\int_t^{t+h}[\phi(t+h)-\phi(t)]ds| \leq \frac{1}{h}\int_t^{t+h}|\phi(t+h)-\phi(t)|ds < \epsilon$

Same thing holds if $h < 0$

Therefore, $h < \delta \Rightarrow |\dfrac{g(t+h)-g(t)}{h} - \phi(t)| < \epsilon$

Hence, $g^{'} = \phi$

Set $h(t) = e^{-g(t)}(\gamma(t)-\alpha)$

We have, $h^{'}(t) = e^{-g(t)}\gamma^{'}(t) - e^{-g(t)}(\gamma(t)-\alpha)g^{'}(t) = 0$

Hence, $h(t)$ is a constant function.

$$e^{-g(0)}(\gamma(0)-\alpha) = e^{-g(1)}\ (\gamma(1)-\alpha)$$

$$\Rightarrow e^{-g(0)} = e^{-g(1)} = 1 \text{ (As } \gamma \text{ is a closed curve)}$$

hence $g(1) = 2k\pi i$ (for $k \in \mathbb{Z}$)

Therefore, $\displaystyle\int_\gamma \frac{dz}{z-\alpha} = 2k\pi i \Rightarrow \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\alpha} = k \in \mathbb{Z}$

Remark: The theorem is true if γ is a closed contour. (Prove it!)

(A contour is a piecewise smooth curve)

Theorem 4.2: Let γ be a closed contour and let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then,

(a) the function $f_\gamma : \mathbb{C} \setminus \{\gamma\} \to \mathbb{Z}$ is continuous. ($\alpha \to \eta(\gamma;\alpha)$)

(b) f is constant on every component of $\mathbb{C} \setminus \{\gamma\}$

Proof: (a) Let $\alpha_0 \in \mathbb{C} \setminus \{\gamma\}$. Then the function $g : t \to |\alpha_0 - \gamma(t)|$ is continuous.

g attains its infimum, say $s = \inf\limits_{t\in[0,1]} g(t)$

If α is very close to α_0, then $|\alpha - \gamma(t)| \geq \dfrac{s}{2}$. Then,

$$\begin{aligned}
&|\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}| = \frac{|\alpha-\alpha_0|}{|z-\alpha||z-\alpha_0|} \leq \frac{2}{s^2}|\alpha-\alpha_0| \;\; (z \in \gamma) \\
&|f_\gamma(\alpha) - f_\gamma(\alpha_0)| \leq \frac{1}{2\pi i}\int_\gamma |\frac{1}{z-\alpha} - \frac{1}{z-\alpha_0}|dz \\
&\qquad\qquad\qquad\quad \leq \frac{2}{s^2}|\alpha-\alpha_0|\frac{1}{2\pi i}\,L(\gamma) = M(\alpha-\alpha_0) \;\; \text{(Lipschitz continuous} \Rightarrow \text{continuous)}
\end{aligned}$$

(b) Let V be a component, then $f(V)$ is connected in $\mathbb{Z} \Rightarrow f(V)$ is a constant $\in \mathbb{Z}$

Proposition 4.1: Let γ be a closed contour in $\mathbb{C}$. Then $\eta(\gamma;\alpha) = 0$ $\forall\alpha$ in the unbounded component of $\mathbb{C} \setminus \{\gamma\}$

Proof: Since γ is closed and bounded, $\{\gamma\} \subseteq \bar{B}(0;R)$ for some $R > 0$.

Let $\alpha \in \mathbb{C} \setminus \bar{B}(0;R)$

$|z-\alpha| \geq |\alpha| - |z| \geq |\alpha - R|$

$|\eta(\gamma;\alpha)| = \dfrac{1}{2\pi}|\displaystyle\int_\gamma \frac{dz}{z-\alpha}| \leq \frac{1}{2\pi}\int_\gamma \frac{dz}{|z-\alpha|} \leq \frac{1}{2\pi}\frac{1}{|\alpha|-R}\,L(\gamma)$

One can find a large enough $|\alpha|$ to make $\eta(\gamma;\alpha) < 1$

Hence, $\eta(\gamma;\alpha) = 0$, when $|\alpha|$ is sufficiently large

Since, $\eta(\gamma;\alpha)$ is constant within a component, $\eta(\gamma,\beta) = 0$ ($\forall\beta$ in unbounded component)

Proposition 4.2: Let γ be a closed contour consisting of curves $\gamma_1, \ldots \gamma_n$.Then,

$\eta(\gamma;\alpha) = \eta(\gamma_1;\alpha) + \cdots + \eta(\gamma_n;\alpha)$ (Prove!)

Cauchy-Goursat theorem: Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f \in H(\Omega)$. Then for any closed contour γ lying in the interior of Ω,

$\int_\gamma f(z)dz = 0$

Proof: Step-I (Goursat's theorem):

When $\gamma = T$, a triangle

Let $T^{(0)} = T$

Let $diam(T^{(0)}) = d^{(0)}$ and $peri(T^{(0)}) = p^{(0)}$

$\int_{T^{(0)}} f(z)dz = \int_{T^{(1)}} f(z)dz + \int_{T^{(2)}} f(z)dz + \int_{T^{(3)}} f(z)dz + \int_{T^{(4)}} f(z)dz$

$|\int_{T^{(0)}} f(z)dz| \leq 4|\int_{T^{(j)}} f(z)dz|$ (for some $j \in \{1, 2, 3, 4\}$)

Call this $T^{(j)}$ to be $T^{(1)}$ (suppose)

$diam(T^{(1)}) = \frac{1}{2} diam(T^{(0)})$

$d^{(1)} = \frac{d^{(0)}}{2}$ and $p^{(1)} = \frac{p^{(0)}}{2}$

Do the same process with $T^{(1)}$ to get $T^{(2)} \Rightarrow |\int_{T^{(1)}} f(z)dz| \leq 4|\int_{T^{(2)}} f(z)dz|$

Continuing, $|\int_{T^{(0)}} f(z)dz| \leq 4^n |\int_{T^{(n)}} f(z)dz|$

$d^{(n)} = \frac{d^{(0)}}{2^n}$ and $p^{(n)} = \frac{p^{(0)}}{2^n}$

$\triangle_n = T^{(n)} \cup Int(T^{(n)})$ (Int refers to interior of triangle)

$\triangle_0 \supseteq \triangle_1 \supseteq \ldots \triangle_n \supseteq \ldots$ (nested compact sets)

$d^{(n)}$ tends to 0

Therefore, $\exists! z_0 \in \bigcap\limits_{n=0}^{\infty} \triangle_n$

f is holomorphic at z_0

$f(z_0 + h) - f(z_0) = hf^{'}(z_0) + h\psi(h)$ ($\lim_{h \to 0} \psi(h) = 0$)

So, $f(z) - f(z_0) = (z - z_0)f^{'}(z_0) + (z - z_0)\psi_1(h)$ where $\lim_{z \to z_0} \psi_1(z) = 0$

$\Rightarrow \int_T f(z)dz = \int_T f(z_0)dz + \int_T (z - z_0)f^{'}(z_0)dz + \int_T (z - z_0)\psi_1(z)dz = \int_T (z - z_0)\psi_1(z)dz$

Then, $\psi_1(z) = \frac{f(z) - f(z_0)}{z - z_0} - f^{'}(z_0)$

Let $\sup_{z \in T^{(n)}} |\psi_1(z)| = E_n$ ($E_n \to 0$ as$\to \infty$)

$|\int_{T^{(n)}} f(z)dz| = |\int_{T^{(n)}} (z - z_0)\psi_1(z)dz| \leq \int_{T^{(n)}} |z - z_0||\psi_1(z)|dz$

$\leq d^{(n)} E_n p^{(n)} = \frac{d^{(0)} p^{(0)}}{4^n} E_n$

$|\int_{T^{(0)}} f(z)dz| \leq 4^n |\int_{T^{(n)}} f(z)dz| \leq d^{(0)} p^{(0)} E_n$ ($\forall n$)

Take limit on both sides as $n \to \infty \Rightarrow |\int_{T^{(0)}} f(z)dz| = 0$