Complex notes

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Definition: An open, connected subset of $\mathbb{C}$ is called domain or region.

$D(a, r) = \{z \in \mathbb{C} | \; |z - a| < r\}$

$\{z \in \mathbb{C} | \; r < |z - a| < R\}$→annulus

Exponential function: $exp(z) : \mathbb{C} \to \mathbb{C}$

$e^z = e^x(cos(y) + isin(y))$

$e^{z_1+z_2} = e^{z_1}.e^{z_2}$, $0 \notin Rng(exp(z))$

$|e^{ix}| = 1$ $(\forall x \in \mathbb{R})$

$e^z = 1$ iff $z = 2n\pi i$,$n \in \mathbb{Z}$

$e^{z_1} = e^{z_2}$ iff $z_1 = z_2 + 2n\pi i$

Argument: $\theta = arg(z)$ (Angle made wrt positive real axis)

$arg(z)$ for $z \in \mathbb{C} \setminus \{0\}$ is multivalued function.

$arg(z) : \mathbb{C} \setminus \{0\} \to (\alpha, \alpha + 2\pi]$ or $[\alpha, \alpha + 2\pi)$ is well-defined $(\forall \alpha \in \mathbb{R})$

Principal value:$Arg(z) = \theta$ $(-\pi < \theta \leq \pi)$

$arg(z) = \{Arg(z) + 2n\pi : n \in \mathbb{Z}\}$

Complex Log: For $z \in \mathbb{C} \setminus \{0\}$, define $log(z) = log(|z|) + arg(z)$ (multivalued)

Whenever $arg(z)$ is well-defined (i.e $R(arg(z)) = [\alpha, \alpha + 2\pi)$ or $(\alpha, \alpha + 2\pi]$)

$\Rightarrow log(z)$ is well defined

$Log(z) = log(|z|) + i.Arg(z)$: principal log $(log(z) = Log(z) + i2n\pi)$

Lemma: For $z \in \mathbb{C} \setminus \{0\}$, the values of log(z) are the complex numbers ω such that $e^\omega = z$

Proof: Do it on your own!

Limit: $f : E \subset \mathbb{C} \to \mathbb{C}$

$\lim_{z \to z_0} f(z) = l$

$\forall \epsilon > 0, \exists \delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$

$z \in (B(z, z_0) \setminus \{0\}) \cap E \Rightarrow f(z) \in B(l, \epsilon)$

Proposition: A function $f : E \subset \mathbb{C} \to \mathbb{C}$ has a limit point l as $z \to z_0$ iff $f(z_n) \to l$ for every sequence $\{z_n\} \subset E \setminus \{z_0\}$ with $z_n \to z_0$ as $n \to \infty$

Theorem: If $f : E \subset \mathbb{C} \to \mathbb{C}$ has a limit at z_0, then f is bounded near z_0

Continuity: $f : E \subset \mathbb{C} \to \mathbb{C}$

f is continuous at $z_0 \in E$ if for every basic neighbourhood V of $f(z_0)$, there is a basic neighbourhood U of z_0 such that $f(U) \subset V$

Let $V = B(f(z_0), \epsilon)$, $U = B(z_0, \delta)$

$z \in B(z_0, \delta) \Rightarrow f(z) \in B(f(z_0), \epsilon)$

$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$

Then, $z \in B(z_0, \delta) \setminus \{z_0\} \Rightarrow f(z) \in B(f(z_0), \epsilon)$

$\lim_{z \to z_0} f(z) = f(z_0)$

$Arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$

$Log(z) = log(|z|) + i.Arg(z)$ (defined on $\mathbb{C} \setminus \{0\}$)

$Log(z)$ is continuous on $C \setminus (-\infty, 0]$ as $Arg(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$

Definition: Let $G \subset \mathbb{C}$ be a region. A continuous function $f : G \to \mathbb{C}$ is called a branch of logarithm in G if:

$e^{f(z)} = z$ $(\forall z \in G)$ $(\Rightarrow 0 \notin G)$

$e^{Log(z)} = z$ $(\forall z \in \mathbb{C} \setminus \{0\})$

But $Log(z)$ isn't continuous on $\mathbb{C} \setminus \{0\}$

Therefore, $Log(z)$ is a branch of log in $\mathbb{C} \setminus (-\infty, 0]$

$f_k = Log(z) + 2k\pi i$, $k \in \mathbb{Z}$

$e^{f_k(z)} = z$ (Each f_k is a branch of log in G)

Theorem: Let $f : G \to \mathbb{C}$ be a branch of log. Then $g : G \to \mathbb{C}$ is a branch of log iff $g(z) = f(z) + 2k\pi i$ (for some $k \in \mathbb{Z}$)

Proof: ($\Rightarrow$) if f is a branch of log, then so is g

($\Leftarrow$) Let $g : G \to \mathbb{C}$ be a branch of log.

Then, (i) g is continuous

(ii) $e^{g(z)} = z$, $\forall z \in G$

Since $f : G \to \mathbb{C}$ is a branch of log, we have,

(i) f is continuous

(ii)$e^{f(z)} = z$, $\forall z \in G$

$e^{g(z)} = e^z = e^{f(z)}$ $(\forall z \in G)$

$\Rightarrow g(z) = f(z) + 2k(z)\pi i$ (k depends on z)

$k : G \subset \mathbb{C} \to \mathbb{Z}$

$k(z) = \frac{1}{2\pi i}(g(z) - f(z))$ (k is continuous)

Since G is connected, $Img(k)$ is connected (subset of $\mathbb{Z}$)

$\Rightarrow k(z)$ is constant

Hence, the claim follows.

Unit disk in $\mathbb{C}$: $D = \{z \in \mathbb{C} : |z| < 1\}$

$0 \in D$, hence D cannot be a branch of log.

$D \setminus \{0\}$ isn't a branch of log. (Why?)

$Log(z) : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$

The half-line is called a branch-cut for any member of $\{Log(z) + 2k\pi i : k \in \mathbb{Z}\}$

Q: How do you make a branch-cut to define a branch of $Log(z + i - 1)$?

Power functions: Let $\alpha \in \mathbb{C}$. We define z^α to be the multi-valued function:

$z^\alpha = e^{\alpha log(z)}$, $z \neq 0$

$z^\alpha = e^{\alpha log(z)} = e^{\alpha(log(|z|)+arg(z))} = e^{\alpha(Log(z)+2k\pi i)}$ $(k \in \mathbb{Z})$

$= e^{\alpha Log(z)}.e^{2\pi i k\alpha}$

Let $\alpha = n \in \mathbb{N}$. Then $z^\alpha = e^{nLog(z)}$ which is single-valued.

Definition: Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. Then f is said to be differentiable at $z_0 \in \Omega$ if,

$f^{'}(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists

Definition: A function is said to be holomorphic at z_0 if it is differentiable in a neighbourhood of z_0

Entire functions: If a function f is analytic at every point in $\mathbb{C} \Rightarrow f$ is entire.

Example: $f(z) \to$ polynomial in $\mathbb{C}$

$f(z) = \frac{1}{z}$ (differentiable at every point on $\mathbb{C} \setminus \{0\}$)

Proposition: If f and g are differentiable at $z_0 \in \Omega \subset \mathbb{C}$, then so are:

(i)$f + g$, (ii)fg, (iii)f/g $(g(z_0) \neq 0)$

Chain Rule: $f : \Omega \to U$, $g : U \to \mathbb{C}$ are holomorphic at z_0 and $f(z_0)$ respectively. Then $g \circ f : \Omega \to \mathbb{C}$ is holomorphic at z_0 and $(g \circ f)^{'}(z_0) = g^{'}(f(z_0)).f^{'}(z_0)$

Example: $f(z) = \bar{z}$ isn't differentiable at any point in $\mathbb{C}$.

Cauchy-Riemann equations: Let $f : \Omega \to C$ be differentiable at $z_0 = x_0 + iy_0$. Then,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$$

Further,

$$f^{'}(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

Note that the last equation only has partial derivatives wrt x.

Power series: A series of the form $\sum_0^\infty a_n(z-z_0)^n$ ($a_n, z, z_0 \in \mathbb{C}$; z_0 is center)

Theorem(Radius of convergence): Given a power series $\sum_0^\infty a_n(z)^n$,

$\exists R \in [0,\infty) \cup \{\infty\}$ such that:

(i) the series converges ($\forall z \in B(0,R)$; the disk of convergence)

(ii) diverges $\forall z$ such that $|z| > R$

Convention: if $\frac{1}{0} = +\infty$; $\frac{1}{\infty} = 0$, then $R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}}$

Proof: Can be found in any standard text (Try Conway!)

$e^z = \sum \frac{z^n}{n!}$ ($R = +\infty$ i.e an entire function)

Theorem: The power series $f(z) = \sum_{n=0}^\infty a_n z^n$ defines a holomorphic function (in its disk of convergence)

Derivative of f is obtained by differentiation of each term:

$f^{'}(z) = \sum_{n=0}^\infty n a_n z^{n-1}$

$f^{'}$ has same disk of convergence

Proof: Found in any standard text

Remark: A power series is infinitely differentiable and analytic in its disk of convergence

$f(z) = \sum a_n z^n = \sum_{k=0}^\infty \frac{f^k(0)}{k!} z^k$

Examples: (i) $\sum_{n=1}^\infty n z^n < \infty$ ($\forall z \in B(0,1)$; doesn't converge for $|z| = 1$)

(ii) $\sum_{n=1}^\infty \frac{z^n}{n^2} < \infty$ ($\forall z \in B(0,1)$; converges for $|z| = 1$)

(iii) $\sum_{n=1}^\infty \frac{z^n}{n} < \infty$ (converges for $|z| < 1$; diverges otherwise)

Proposition: Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be domains and let $f : \Omega_1 \to \mathbb{C}$, $g : \Omega_2 \to \mathbb{C}$ be continuous function such that $f(\Omega_1) \subset \Omega_2$ and that $g(f(z)) = z$ ($\forall z \in \Omega_1$). If g is differentiable on Ω_2 and if $g^{'}(w) \neq 0$ ($\forall w \in \Omega_2$), then f is differentiable and $f^{'}(z) = \frac{1}{g^{'}(f(z))}$ ($\forall z \in \Omega_1$)

Theorem: A branch of logarithm is analytic and its derivative is $\frac{1}{z}$.

Proof: Let f be a branch of log. Define $g(z) = e^z$. Then $g(f(z)) = z$. ($e^{f(z)} = z$)

Lemma: Let $f : \Omega \to \mathbb{C}$. Then f is differentiable at $z_0 \in \Omega$ iff $\exists a \in \mathbb{C}$ such that

$f(z_0 + h) - f(z_0) = ah + h\psi(h)$ ($\lim_{h\to 0} \psi(h) = 0$)

Theorem: Let $\Omega \subset \mathbb{C}$ be a domain and let $f = u + iv$ is a function from Ω to $\mathbb{C}$. Then f is analytic at $z_0 \in \Omega$ iff the partial derivatives exist, are continuous and satisfy Cauchy-Riemann equations.

Proof: Conway

A **path** or a **curve** is a continuous function, $\gamma : [a, b] \to \mathbb{C}$ ($Rng(\gamma) \subset \mathbb{C}$)

$\gamma(a)$: initial point of path; $\gamma(b)$: endpoint of path

$[a, b]$: parameter interval

γ is said to be:

1. closed if $\gamma(a) = \gamma(b)$
2. smooth or C^1 if γ is differentiable and $\gamma^{'}$ is continuous
3. simple if γ is one-one
4. simple closed if $\gamma(a) = \gamma(b)$ and γ is one-one on (a, b)
5. piecewise smooth if there are finitely many points $s_0, s_1 \dots s_n \in [a.b]$ with $a = s_0 < s_1 < s_2 \dots < s_n = b$ such that the restriction of γ to each (s_i, s_{i+1}) is smooth.

$-\gamma$ or γ^{-1} is defined by $\gamma^{-1}(t) = \gamma(a + b - t)$

$\phi : [0, 1] \to [a, b]$ defined as: $\phi(t) = a + (b - a)t$ (one-one and differentiable)

Line integral: $f : [a, b] \to \mathbb{C}$: continuous

$f = u + iv$, where $u, v : [a, b] \to \mathbb{R}$

Define $\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$

Properties:

1. $\int_a^b c.f(t)dt = c. \int_a^b f(t)dt$
2. $| \int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$

Length of a smooth curve: Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve.

$L(\gamma) = \int_a^b |\gamma^{'}(t)|dt = \int_a^b \sqrt{\gamma_1^{'}(t)^2 + \gamma_2^{'}(t)^2}dt$ ($\gamma(t) = \gamma_1(t) + i.\gamma_2(t)$)

If $\gamma : [a, b] \to C$ is piecewise smooth then $L(\gamma)$ is the sum of the length of its smooth parts.

Definition(orientation):A curve γ is positively oriented if traversed in anti-clockwise direction else is negatively oriented.

Examples:

1. $\gamma(t) = r.e^{it}$, ($t \in [0, 2\pi]$) ($r > 0$: simple, smooth curve);

   $L(\gamma) = \int_0^{2\pi} |i.r.e^{it}|dt = r.(2\pi)$
2. $\gamma(t) = e^{it}$, ($t \in [0, 4\pi]$): closed, smooth, traverses the unit circle twice in the positive direction

Integration over paths: $\gamma[a,b] \to \mathbb{C}$ is a smooth curve and $f : \gamma \to \mathbb{C}$: continuous

Definiton: $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma^{'}(t)dt = \int_a^b g(t)dt$

$(g(t) = f(\gamma(t))\gamma^{'}(t)$ where $g : [a,b] \to \mathbb{C})$

Let $[a_1, b_1]$ be any closed interval. Then $\exists \phi : [a_1, b_1] \to [a, b]$ (one-one,differentiable and

$\phi(a_1) = a; \phi(a_2) = b)$

$\phi[a_1, b_1] \to \mathbb{C}$: smooth

$\int_{a_1}^{b_1} f(\gamma_1(t)).\gamma_1^{'}(t)dt$ $(= \int_{\gamma_1} f(z)dz)$

$= \int_{a_1}^{b_1} f(\gamma(\phi(t))).\phi^{'}(t)dt = \int_\gamma f(\gamma(s))\gamma^{'}(s)ds = \int_\gamma f(z)dz$ $(\phi(t) = s)$

If γ is piecewise smooth, the integral can be split into the sum of its smooth components:

if $\gamma = \gamma_1 + \gamma_2 \cdots + \gamma_n$, then $\int_\gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f$.

Note that $\gamma_i^{\prime}s$ are smooth.

Proposition: If f and g are continuous on a smooth curve γ, then

1. $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$
2. $\int_{\gamma^-} f = -\int_\gamma f$
3. $|\int_\gamma f(z)dz| \leq \|f\|_{\infty,\gamma} L(\gamma)$ $(\|f\|_{\infty,\gamma} = sup_{z\in\{\gamma\}}|f(z)|)$

$|\int_\gamma f| = |\int_a^b f(\gamma(t)).\gamma^{'}(t)dt| \leq \int_\gamma |f(\gamma(t)).\gamma^{'}(t)|dt \leq \|f\|_{\infty,\gamma} \int_a^b |\gamma^{'}(t)|dt$ $(L(\gamma) = \int_a^b |\gamma^{'}(t)|dt)$

Examples:

(i)Let γ be the arc of a circle of radius 3 $(|z| = 3)$ from 3 to $3i$.

Show that:

$$|\int_\gamma \frac{z+4}{z^3-1}dz| \leq \frac{21\pi}{52}$$

(ii) $\gamma : |z| = 2$ (traverse curve in positive direction)

Prove:

$$|\int_\gamma \frac{e^z dz}{z^2+1}| \leq \frac{4\pi e^2}{3}$$

Fundamental theorem of calculus:

If $f : [a,b] \to \mathbb{R}$ has a primitive F, then $\int_a^b f(x)dx = F(b) - F(a)$ $(F^{'}(x) = f(x), \forall x \in [a,b])$

Definition: Suppose $G \in \mathbb{C}$ be a domain. If a continuous function $f : G \to \mathbb{C}$ has a primitive F on G and if γ is a smooth curve in G with initial and terminal points ω_1 and ω_2 respectively, then:

$\int_\gamma f = F(\omega_1) - F(\omega_2)$

Proof: Let $[a,b] \in \mathbb{R}$ be a parameter interval for γ and $\gamma(a) = \omega_1$; $\gamma(b) = \omega_2$

Given $F^{'}(z) = f(z)$ $(\forall z \in G)$

$$\int_{\gamma} f = \int_a^b f(\gamma(t)).\gamma^{'}(t)dt = \int_a^b F^{'}(\gamma(t))\gamma^{'}(t)dt$$

$$= \int_a^b (F \circ \gamma)^{'}(t)dt = F \circ \gamma(b) - F \circ \gamma(a) = F(\omega_2) - F(\omega_1)$$

Corollary-1: If γ is a closed curve (smooth), then

$\int_{\gamma} f = 0$ (Proof follows from FTC)

Corollary-2: If $f \in H(\Omega)$ for a region $\Omega \in \mathbb{C}$ and if $f^{'} = 0$ on Ω, then f is a constant function.

Proof: Fix a point $\omega_0 \in \Omega$. It suffices to show that $f(\omega) = f(\omega_0), \forall \omega \in \Omega$

Simple Closed Curve:

Jordan-curve theorem: Every simple closed curve in $\mathbb{C}$ divides the complex plane into two regions. One of these regions is bounded and the other is unbounded. The bounded region is called the interior of the curve.

Example: $G = \mathbb{C} \setminus \{0\}$

$f(z) = \frac{1}{z}$ on G, $\gamma : |z| = 1$, $\gamma(t) = e^{it}$, $(t \in [0, 2\pi])$

$$\int_{\gamma} f = \int_0^{2\pi} f(\gamma(t)).\gamma^{'}(t)dt = \int_0^{2\pi} \frac{i.e^{it}}{e^{it}} dt = 2\pi i \neq 0$$

Winding number or index of a closed curve: Letγ be a closed curve on $\mathbb{C}$ and let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. The winding number of γ about α or the index of γ with respect to α is denoted by: $\eta(\gamma; \alpha)/Ind_{\gamma}(\alpha)$ defined by:

$$\eta(\gamma; \alpha) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - \alpha}$$

Example: $\gamma : [0, 6\pi] \to \mathbb{C}$

$\gamma(t) = a + re^{it}$

$$\eta(\gamma; \alpha) = \frac{1}{2\pi i} \int_{\gamma} \frac{\gamma^{'}(t)}{\gamma(t) - a} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{a + re^{it} - a} .ire^{it} dt = 3$$

Theorem: Let γ be a smooth, closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \{\gamma\}$. Then $\eta(\gamma; \alpha) \in \mathbb{Z}$.

Proof: To be done