MV notes

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Cantor set($\mathscr{C}$):

$C_0 = [0,1]$

$C_1 = [0,1] - (\frac{1}{3}, \frac{2}{3})$ and so on (Basically, keep removing the middle one third)

$\mathscr{C} = \bigcap\limits_{i=0}^{\infty} \mathcal{C}_i$

(I) $\mathscr{C} \neq \phi$

(II) $\mathscr{C}$ is closed (hence, compact)

(III) $\mathscr{C}$ has length 0

(IV) $\mathscr{C}$ is totally disconnected

Exercise: Can you characterize the elements of $\mathscr{C}$?

Integration:

<u>**In one variable**</u>:

Definition: A partition P of $[a,b] \subset \mathbb{R}$ is a collection $t_0 = a \leq t_1 \cdots \leq t_k = b$

Rectangle in $\mathbb{R}^n$: $[a_1,b_1] \times [a_2,b_2] \cdots \times [a_n,b_n] = A$

If P_i is a partition of $[a_i,b_i]$, $P = (P_1, P_2, \ldots P_n)$ is a partition of A.

$f : A \to \mathbb{R}$ is a bounded function. (P is a partition of A)

If S = rectangle in P, $m_S(f) = \inf_{x \in S} f(x)$, $M_S(f) = \sup_{x \in S} f(x)$

$L(f,P) = \sum_{S \in P} m_S(f)\nu(S)$

$U(f,P) = \sum_{S \in P} M_S(f)\nu(S)$ ($\nu(S)$ = volume of S)

$L(f,P) \leq U(f,P)$

If P and P' are partitions, $P^{'}$ is said to be an refinement of P if any subrectangle $S^{'}$ in $P^{'}$ is contained in contained in some subrectangle S in P.

If $P = (P_1, P_2 \ldots P_n)$ and $Q = (Q_1, Q_2 \ldots Q_n)$, then

$\mathcal{T} = (P_1 \cup Q_1, \ldots, P_n \cup Q_n)$ is a refinement of both P and Q.

Proposition: If $P^{'}$ is a refinement of P, then $L(f,P) \leq L(f,P^{'}) \leq U(f,P^{'}) \leq U(f,P)$.

If $S^{'} \in P^{'}$, then $S^{'} \subseteq S$ for some $S \in P$, $m_{S^{'}}(f) \geq m_S(f)$, $M_{S^{'}} \leq M_S(f)$

$S = \bigcup_{i=1}^{m} S_i^{'}, \; S_i^{'} \in P^{'}$

$\sum_{i=1}^{m} m_{S'}(f).\nu(S^{'}) \geq \sum_{i=1}^{m} m_S(f).\nu(S^{'}) = m_S(f) \sum_{i=1}^{m} \nu(S_i^{'}) = m_S(f)\nu(S)$

Summing up, $L(f, P^{'}) \geq L(f, P)$

Similarly, $U(f, P^{'}) \leq U(f, P)$

Corollary: If P and $P^{'}$ are any two partitions: $L(f, P^{'}) \leq U(f, P)$

Take $Q =$ common refinement of P, $P^{'}$.

Then, $L(f, P^{'}) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$

$\sup_P L(f, P) \leq \inf_{P'} U(f, P^{'})$

$f : A \to \mathbb{R}$: a bounded function is said to be integrable if $\sup_P L(f, P) = \inf_P U(f, P)$

Value is denoted as $\int_A f$

Propositon: f is integrable $\Leftrightarrow \forall \epsilon > 0$, $\exists$ a partition P such that, $U(f, P) - L(f, P) < \epsilon$

Theorem: Let $f : A \to \mathbb{R}$ be a bounded function on a closed rectangle A. Then f is integrable on A iff the set $B = \{x : f \text{ is discontinuous at } x\}$ has measure 0.

Definition: A set $S \subseteq \mathbb{R}^n$ is said to have measure 0 if given $\epsilon > 0$ there exists a sequence of closed rectangles $Q_1, Q_2 \ldots Q_n$, such that $S \subseteq \bigcup_{i=0}^{\infty} Q_i$ and $\sum_{i=1}^{\infty} \nu(Q_i) < \epsilon$

Exercise:

(I) If $\{A_i\}_1^{\infty}$ is a countable collection of measure zero sets,then $\bigcup_{i=0}^{\infty} A_i$ has measure zero.

(II)$\mathscr{C}$ has measure zero

Definition: A set $A \subseteq \mathbb{R}^n$ is said to have content 0, if given $\epsilon > 0$, there exists a finite collection of closed rectangles $Q_1, Q_2 \ldots Q_n$ such that $\sum_{i=1}^{n} \nu(Q_i) < \epsilon$

content 0 $\Rightarrow$ measure 0

Theorem: measure 0 $\Rightarrow$ content 0 if A is compact

Proposition: If $a < b$, then $S = [a, b]$ doesn't have measure zero.

Proof(Sketch): Let $S \subseteq \bigcup_{i=0}^{\infty} Q_i$

$\sum \nu(Q_i) \geq (b - a)$