First we prove that

$$Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}} \tag{1}$$

For $n = 0$,

$$Fib\,(0) = 0 = \frac{\phi^0 - \psi^0}{\sqrt{5}}$$

For $n = 1$,

$$Fib\,(1) = 1 = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} = \frac{\phi^1 - \psi^1}{\sqrt{5}}$$

Assume that (1) holds for $n = k$ and $n = k + 1$, $k \in N$.

For $n = k + 2$, we have

$$\begin{aligned} Fib\,(k+2) &= Fib(k) + Fib(k+1) \\ &= \frac{\phi^k - \psi^k}{\sqrt{5}} + \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} \\ &= \frac{\phi^k\,(\phi+1) - \psi^n\,(\psi+1)}{\sqrt{5}} \qquad (2) \end{aligned}$$

Also, ϕ and ψ satisfy the equations:

$$\phi^2 = \phi + 1 \tag{3}$$

$$\psi^2 = \psi + 1 \tag{4}$$

Substitute (3) and (4) into (2), we have

$$\begin{aligned} Fib(k+2) &= \frac{\phi^k\left(\phi^2\right) - \psi^n\left(\psi^2\right)}{\sqrt{5}} \\ &= \frac{\phi^{n+2} - \psi^{n+2}}{\sqrt{5}} \end{aligned}$$

Thus, (1) holds for all $n \in N$.

Now we have

$$\left|Fib(n) - \frac{\phi^n}{\sqrt{5}}\right| = \left|\frac{\phi^n - \psi^n}{\sqrt{5}} - \frac{\phi^n}{\sqrt{5}}\right| = \left|\frac{\psi^n}{\sqrt{5}}\right| = \frac{\left(\frac{\sqrt{5}-1}{2}\right)^n}{\sqrt{5}} \leq \frac{1}{\sqrt{5}} < \frac{1}{2}$$

Thus, $Fib(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$.

Q.E.D.