Vector, Geometry of Space and Coorodinate

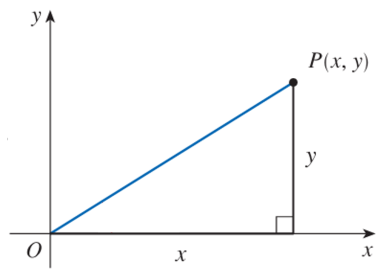
"You had studied in high school, right?" (Prof. Duc-Thang Vo, 2023)

1. Coordinate

   (1) Cartesian Coordinate:

   There are 2/3 mutually perpendicular directions in the 2D/3D Cartesian Coordinate.

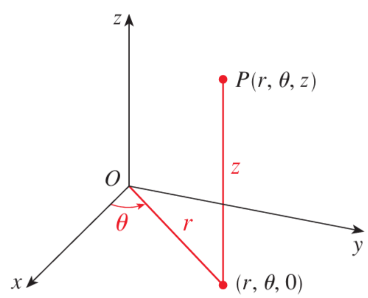
   We will represent point P as $P(x,\ y,\ z)$.



   (2) Polar (Cylindrical) Coordinate:

   There are 1 length and 1 angle in the 2D Polar Coordinate. Adding a Z-axis will turn into Cylindrical coordinates.

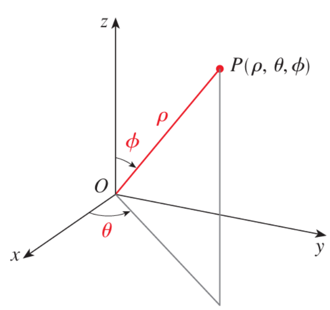
   We will represent point P as $P(r,\ \theta,\ z)$.



   (3) Spherical Coordinate:

   There are 1 length and 2 angles in the Spherical Coordinate.

   We will represent point P as $P(\rho,\ \theta,\ \phi)$.



2. Vector and Vector Algebra

   (1) Scalar and Vector

      i. Scarlar: Only has magnitude.

      ii. Vector: Has both magnitude and direction

   (2) Representation of Vector:

      We usually use the bold font "$\mathbf{v}$" or superscript arrow "$\overrightarrow{v}$" to represent the vector v.

      i. Tuple notation: $\mathbf{v} =< v_1,\ v_2,\ \dots,\ v_n >$, and n is finite integer.

      ii. Matrix notation: $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, and n is finite integer.

   (3) Basic Operations of Vector

$$\mathbf{v} =< v_1,\ v_2,\ \dots,\ v_n >,\ \mathbf{u} =< u_1,\ u_2,\ \dots,\ u_n >$$

      i. Addition and Subtraction:

$$\mathbf{v} + \mathbf{u} =< v_1 + u_1,\ v_2 + u_1,\ \dots,\ v_n + u_n >$$

$$\mathbf{v} - \mathbf{u} =< v_1 - u_1,\ v_2 - u_1,\ \dots,\ v_n - u_n >$$

      ii. Scarlar Multiplication:

$$k\mathbf{v} =< kv_1,\ kv_2,\ \dots,\ kv_n >$$

      iii. Inner Product: (Scalar)

      Geometry Meaning: $v \cdot u$ = (v's Projection Length on u)*(u's Length)

$$\mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + \cdots + v_nu_n$$

      or we can use row vectors to calculate:

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{v}^T\mathbf{u}$$

$$\mathbf{v}^T\mathbf{u} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1u_1 + v_2u_2 + \cdots + v_nu_n \end{bmatrix} = (Scalar)$$

iv. Cross Product: (Vector)

$$\text{Geometry Meaning: } \begin{cases} \mathbf{v} \times \mathbf{u} \perp \mathbf{v} \text{ and } \mathbf{u} \text{ (Perpendicular or Orthogonal)} \\ |\mathbf{v} \times \mathbf{u}| = \text{(Area of Parallelogram expand by}\mathbf{v}\text{and}\mathbf{u}\text{.)} \end{cases}$$

$$\mathbf{v} \times \mathbf{u} \ = \begin{vmatrix} \hat{x_1} & \hat{x_2} & \dots & \hat{x_n} \\ v_1 & v_2 & \dots & v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \dots & u_n \end{vmatrix} \ = \ (Vector)$$

v. Triple Product: (Volume $\rightarrow$ Scalar)

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

vi. Outer Product: (Matrix)

$$\mathbf{v} \otimes \mathbf{u} = \mathbf{v}\mathbf{u}^T$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} = \begin{bmatrix} v_1u_1 & v_1u_2 & \dots & v_1u_n \\ v_2u_1 & v_2u_2 & \dots & v_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ v_nu_1 & v_nu_2 & \dots & v_nu_n \end{bmatrix}$$

(4) Geometry of Space

i. Equation of Plane in $\mathbb{R}^3$.

If the normal vector of the plane is $< a,\ b,\ c >$, and the plane passes through point $P(x_0,\ y_0,\ z_0)$, then the equation of the plane is

$$E: \ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or we can use the intercept form

$$\frac{x}{x_i} + \frac{y}{y_i} + \frac{z}{z_i} = 1$$

$x_i,\ y_i,\ z_i$ are the intercepts of the plane with each axis.

ii. Equation of Line in $\mathbb{R}^3$. If the direction vector of the line is $< a,\ b,\ c >$, and the line passes through point $P(x_0,\ y_0,\ z_0)$, then the equation of the line is

$$L: \ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \ (abc \neq 0)$$

or we can use the parametric form

$$L: \ \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases} \ (t \in \mathbb{R})$$

Example 1

Let $\mathbf{a} = (1, 2, 1),\ \mathbf{b} = (1, 0, -1)$, and $\mathbf{c}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

If the volume spanned by $\mathbf{a},\ \mathbf{b}$ and $\mathbf{c}$ is 12, find $\mathbf{c}$.

Example 2

Let $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, and $\mathbf{d} = \mathbf{a} \times \mathbf{c}$, calculate $\mathbf{b} \cdot \mathbf{d}$

Example 3

Let $A(2, 0, 2)$ and $B(1, 2, 2)$ be two points in the 3D space, find the point P on the plane $E:\ x + 2y + 2z + 3 = 0$ such that the total length of $\overline{AP} + \overline{BP}$ is minimal.

Example 4

Find the distance between L_1 and L_2.

$$L_1 : \begin{cases} x = t - 1 \\ y = 2t \\ z = -t + 3 \end{cases} \qquad L_2 : \begin{cases} x = 2t - 1 \\ y = 3t \\ z = -3t \end{cases}$$