Section 6, 8 Application of Integration

After learning the techniques of integration, we now look at how these techniques can be applied in calculation of area and volume.

1. Area Between Curves

(1) Integration with respect to x.

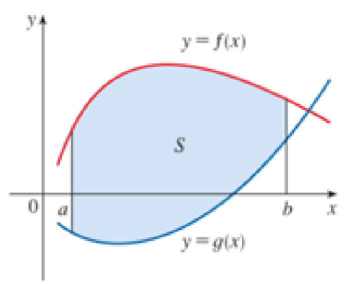
$$\text{If } f(x) \geq g(x) \text{ and } x \in [a,b], \text{ then } A = \int_a^b [f(x) - g(x)]dx$$

It's important to notice that we always use the upper function to minus lower function when we integrate with respect to x-axis, the graph below clearly demonstrate how this method works.

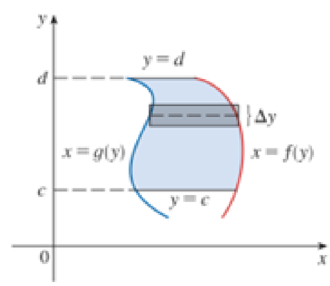
(2) Integration with respect to y.

$$\text{If } f(y) \geq g(y) \text{ and } y \in [c,d], \text{ then } A = \int_c^d [f(y) - g(y)]dy$$

Similar with the idea above, when integrating with respect to y, we normally use the function on the right side to minus the function on the left side



Integration with respect to x.



Integration with respect to y.

If the interval of the integration is not provided, try to find the intercept of f and g.

Example 1

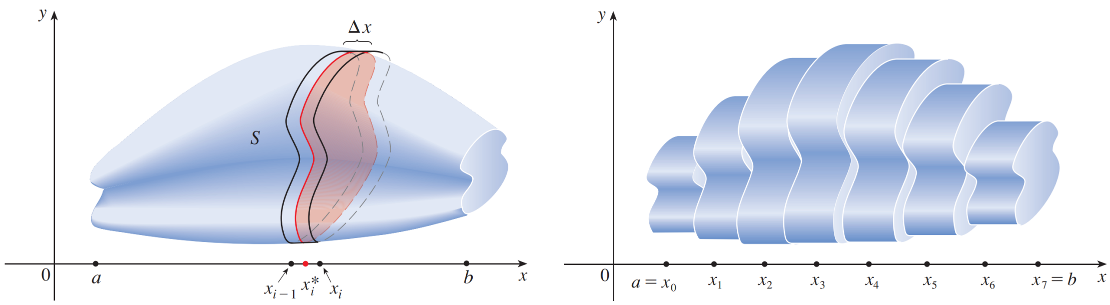
The line $y = mx$ cuts the region bounded above by the curve $y = x - x^2$ and below by the x-axis into two parts.Then, the areas of the two parts are equal when m is?

Exercise 1

Find the values of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576.

2. Volume

Calculating volume can be considered as accumulating small layers of cross-sectional areas until a certain value of height or thickness is reached, the following two pictures demonstrates this idea clearly.



Therefore, the volume can be expressed as following eqaution:

$$V = \int_a^b A(x)\,dx$$

Next, we'll be introducing some common methods to calculate volume using integration, note that similarly, integration of volume can also be done in either x or y direction, depending on the axis of rotation.

(1) Disk Method

Used when calculating volume of one single variable function rotating a specific line.

Rotate around $y = k$:

$$V = \pi \int_a^b (radius)^2\,dx = \pi \int_a^b [f(x) - k]^2\,dx$$

Rotate around $x = k$:

$$V = \pi \int_a^b (radius)^2\,dy = \pi \int_a^b [f(y) - k]^2\,dy$$

Example 1

Evaluate the volume enclosed by the curves, $x = 2\sqrt{y}$, $x = 0$, $y = 9$, about $x = 0$.

Exercise 1

The integral $\int_0^{\frac{\pi}{2}} \pi sin^2 * (x)\,dx$ represents the volume of a solid, describe the solid.

(2) Washer Method

Used when calculating volume between two single variable functions, we choose the area enclosed by two curves $f(x)$ and $g(x)$, then rotate around a specific line.

Let's say we rotate around y-axis, then the volume will be:

$$V = \pi \int_a^b R^2(x) - r^2(x)\, dx$$

If we rotate around $y = k$, then the volume will be:

$$V = \pi \int_a^b [R(x) - k]^2 - [r(x) - k]^2\, dx$$

Example 1

The base of solid S is a circular disk with radius r. Parallel cross-sections prependecular to the base are squares. Find the volume of S.

Exercise 1

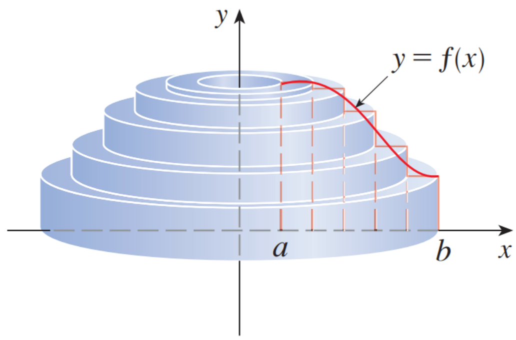
Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere.

(3) Cylindrical Shell Method

Cylindrical Methods is dividing the solid bodies into several cylinders, and calculate the volume of the solid bodies by sum up the volume of cylinders.

We can list the general equation for this method:

$$V = \int_{r_1}^{r_2} 2\pi r \cdot h(r) \cdot dr$$



Find the volume generated by rotating the region bounded by the given curves about the specified axis.

Example 1

$x = 2y^2$, $y \geq 0$, $x = 2$; about $y = 2$.

Exercise 1

$x = (y-1)^2$, $x - y = 1$; about $x = -1$.

3. Arc Length

(1) The Arc Length of the Formula

The arc length formula is used to calculate the arc length of a specific function which has a definite range of x.

$$\text{If } f' \text{ is continuous on } [a,b], \text{ and the curve } y = f(x),\ a \leq x \geq b$$

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2} dx$$

(2) The Arc Length Function

Obviously, the arc length function will be used when calculating the arc length of a specific function which doesn't have a definite range of x, in other words, it measures the arc length starting from a specific point to any other point on the curve.

$$\text{If } f' \text{ is continuous on } [a,b], \text{ and the curve } y = f(x),\ a \leq x \leq b$$

$$\text{Arc Length} = s(x) = \int_a^x \sqrt{1+[f'(x)]^2} dx$$

Since the function above is an integrating function, it can therefore be transformed into a differential equation by the F.T.C., which is:

$$\frac{d}{dx} s(x) = \sqrt{1+[f'(x)]^2}$$

$$ds = \sqrt{1+(\frac{dy}{dx})^2}\, dx \text{ or } ds = \sqrt{1+(\frac{dx}{dy})^2}\, dy \text{ or } ds = \sqrt{(dx)^2+(dy)^2}$$

Example 1

Find the length of the curve $y = \int_1^x \sqrt{t^3-1}\, dt$, $1 \leq x \leq 4$

Exercise 1

Find the arc length function of the curve $y = sin^{-1}x + \sqrt{1-x^2}$ with starting point $(0,1)$.

4. Area of a Surface of Revolution

If f has continuous derivative and $y = f(x),\ a \le x \le b\ \equiv\ x = g(y),\ c \le x \le d$.

(1) Rotate $y = f(x)$ around $y = c$.

$$\begin{aligned} S &= \int_a^b 2\pi r(x)\, ds \\ &= \int_a^b 2\pi |f(x) - c| \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_c^d 2\pi |y - c| \sqrt{1 + [g'(y)]^2}\, dy \end{aligned}$$

(2) Rotate $x = g(y)$ around $x = c$.

$$\begin{aligned} S &= \int_a^b 2\pi r(x)\, ds \\ &= \int_c^d 2\pi |g(y) - c| \sqrt{1 + [g'(y)]^2}\, dy \\ &= \int_a^b 2\pi |x - c| \sqrt{1 + [f'(x)]^2}\, dx \end{aligned}$$

Example 1

The curve $y = \sqrt{r^2 - x^2},\ -r \le x \le r$, is an arc of the circle $x^2 + y^2 = r^2$. Find the area of the surface by rotating this arc about the x-axis.

Exercise 1

Find the area of the surface generated by revolving the curve $x = \frac{e^y + e^{-y}}{2}$, where $0 \le y \le ln(2)$, about the y-axis.