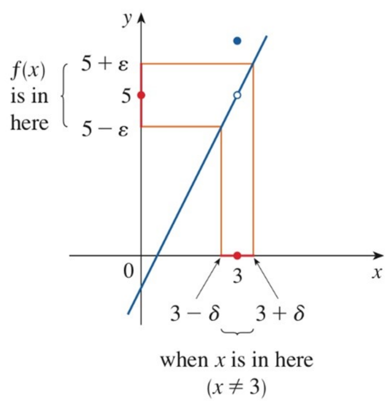
Section 2.4 The Precise Definition of a Limit

1. Recall: The Intuitive Definition of a Limit

   When we see the function "$\lim_{x \to 3} f(x) = 5$", we read it "as when x approaches 3, the limit of $f(x)$ equals to 5". By reading this function, we get to know the intuitive definition of this function is when x is close to 3, the value of $f(x)$ will be closer and closer to 5 but note that the final value of $f(x)$ will only be sufficiently close to 5 but not exactly 5.

2. The Precise Definition of a Limit (Graphically)

   Now, in order to introduce the precise definition of a limit, let's look at the following graph.

   In the graph, it's clear to see that when x is in the range of $3 \pm \delta$ and approaching 3, $f(x)$ must then be in the range of $5 \pm \varepsilon$ and approaching 5. Therefore, we now introduce two new notations, δ and ε, to explain how close exactly x is regarding to 3 and $f(x)$ is regarding to 5. In the next topic, we'll be using these two notations to further prove the precise definition of a limit numerically, this proving method is called "the Epsilon-Delta proof".

3. Epsilon-Delta Definition of Limit (The Epsilon-Delta proof)

   After understanding the graph, we now extend the Epsilon-Delta definition to any limit function, let's say $\lim_{x \to c} f(x) = L$. In this case, we must discuss two "approaches" on x-axis and y-axis, respectively.

   For the approach on y-axis, we can set y_1 as the value of y, and we can define the distance of y_1 and L is ε. ($|y_1 - L| < \varepsilon$)

And the approach on x-axis, we let x_1 as the value of x, and the distance of x_1 and c is defined as δ. $(0 < |x_1 - c| < \delta)$

We can notice the relationship between ε and δ, "ε approaching is accomplished by δ approaching". To express the relationship with math language:

$$Def.\ "\lim_{x\to\infty} f(x) = L" \ \Leftrightarrow\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0,$$
$$s.t.\ \forall\ x\ (\,x \in X\,),\ if\ 0 < |x - c| < \delta,\ then\ \ |f(x) - L| < \varepsilon$$

4. Examples

1. Prove $\lim_{x\to 1}\ 3x + 2 = 5$
   Given $\varepsilon > 0, choose\ \delta =$________ > 0
   If $0 < |x - 1| < \delta$
   $\Rightarrow |(3x + 2) - 5| = |$______$| = 3\,|$______$| <$ ______
   Since ε is arbitrary, $\lim_{x\to 1}\ 3x + 2 = 5\ (Q.E.D.)$

2. Prove $\lim_{x\to 2}\ x^2 + 5 = 9$
   Given $\varepsilon > 0, choose\ \delta = min\ (\ 1,$ ____$) > 0.$
   If $0 < |x - 2| < \delta$
   $|x - 2| < \delta \le 1 \Rightarrow$ __ $< x <$ __ $\Rightarrow$ __ $<$ ______ $<$ __ $\Rightarrow |x + 2| < 5$
   $\Rightarrow |(x^2 + 5) - 9| = |x^2 - 4| = |$____$| \cdot |$____$| < |$____$| \cdot \delta <$ ____
   Since ε is arbitrary, $\lim_{x\to 2}\ x^2 + 5 = 9\ (Q.E.D.)$

3. Prove $\lim_{x\to 3}\ \frac{1}{x-1} = \frac{1}{2}$
   Given $\varepsilon > 0,\ choose\ \delta = min\ (\ 1,$ __$) > 0$
   If $0 < |x - 3| < \delta$
   $|x - 3| < \delta \le 1 \Rightarrow$ __ $< x <$ __ $\Rightarrow$ __ $< x - 1 <$ __ $\Rightarrow$ __ $< \frac{1}{x-1} <$ __
   $\Rightarrow \left|\frac{1}{x-1} - \frac{1}{2}\right| = \frac{|\quad\ |}{|\quad|} = \frac{|\quad|}{|\quad|} < \frac{\quad}{|\quad|} \cdot \delta < \frac{\quad}{\quad} \cdot \delta \le \frac{1}{2} \cdot 2\varepsilon = \varepsilon$
   Since ε is arbitrary, $\lim_{x\to 3} \frac{1}{x-1} = \frac{1}{2}\ (Q.E.D)$

5. Exercises

1. Prove $\lim\limits_{x \to -2} (-2x + 1) = 5$

2. Prove $\lim\limits_{x \to -2} (x^2 - 1) = 3$

3. Prove $\lim\limits_{x \to 2} \frac{1}{x} = \frac{1}{2}$