Section 2.5 Continuity, I.V.T. and Limits at Infinity

1. Continuity

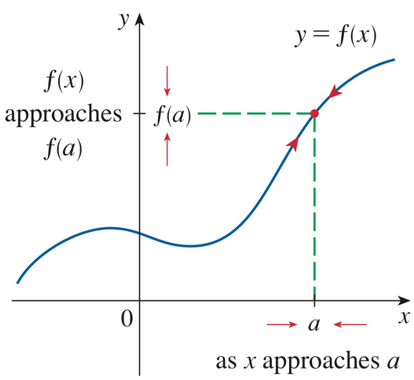
   (1) Recall: Limits

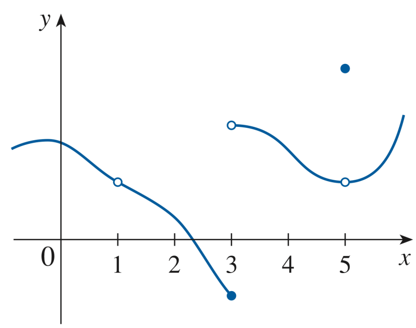
   We have learned how to find the limits of the function at specific point, while doing so, you might find that some of the limits are not equal to the definition of the function at the same points. So we are going to introduce the concept of ”Continuity”.

   (2) Definition

   Continuity means the function is defined at $(a, f(a))$ ($a \in Domain\ of\ f(x)$), and the limits approach a exist and is equal to $f(a)$. We can enumerate 3 requirements of the definition:

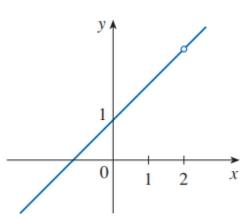
   i. $f(a)$ is defined ii. $\lim_{x\to a} f(x)$ exists iii. $\lim_{x\to a} = f(a)$

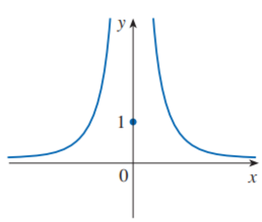


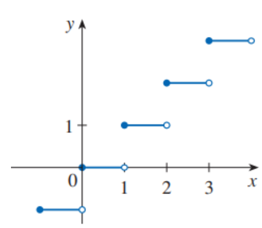


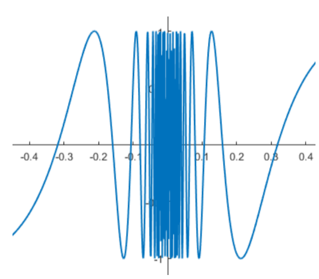
   (3) Type of Discontinuity

   i. Hole ii. Infinity iii. Break iv. Oscillation









2. Intermediate Value Theorem (I.V.T.)

$$\textit{Assume } f(x) \textit{ is continuous on } [a, b],$$

$$\exists\ c\ \in [a, b],\ s.t.\ f(c) = k,\ f(a) < k < (b)$$

3. Limits at Infinity

General form:

$$\lim_{x\to\infty} \frac{a_m x^m + a_{m-1}x^{m-1} + ... + a_0 x^0}{b_n x^n + b_{n-1}x^{n-1} + ... + b_0 x^0} = \begin{cases} \infty(-\infty),\ m > n \\ \dfrac{a_m}{b_n},\ m = n \\ 0,\ m < n \end{cases}$$

Expansion speed: $n^n > n! > a^n > n^a > ln(n)$

4. Asymptotes

(1) Horizontal Asymptotes (H.A.)

H.A. $\Rightarrow$ Occurs when x approaches $\pm\infty$.

So let x approaches $\pm\infty$ can find whether the function has horizontal asymptotes.

Ex. Find the asymptotes of $f(x) = e^{-x}$

$$\lim_{x\to\infty} e^x = \infty$$
$$\lim_{x\to-\infty} e^x = 0$$

(2) Vertical Asymptotes (V.A.)

V.A. $\Rightarrow$ Occurs at $f(x)$ approaches $\pm\infty$.

So find the points where $f(x)$ approaches $\pm\infty$, that is the position of vertical asymptotes.

Ex. Find the asymtote of $f(x) = \frac{1}{x}$

$f(x)$ approaches ∞ as x approaches 0^+, and approaches $-\infty$ as x approaches 0^-

(3) Slant asymptotes (S.A.)

Methods to check the slant asymptotes of a function is as following:

Method 1: Rationalization

Ex. Find the asymtote of $f(x) = \frac{3x^2+2x+1}{x+3}$

$$f(x) = \frac{3x^2 + 2x + 1}{x + 3} = (3x - 7) + \frac{22}{x + 3}$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} [(3x + 7) + \frac{22}{x + 3}]$$

Method 2: Derivative

5. Examples and Exercises

(1) Continuity

Example:

Show that the following function is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} x^3 sin(\dfrac{1}{x}), & if\ x \neq 0 \\ \\ 0, & if\ x = 0 \end{cases}$$

Exercise:

Where is the following function continuous?

$$f(x) = \begin{cases} tan(\dfrac{\pi}{4})x, & if\ \ |x| < 1 \\ \\ x, & if\ \ |x| \geqslant 1 \end{cases}$$

(2) Intermediate Value Theorem

Prove that the equation $\frac{x-1}{x^2+2} = \frac{3-x}{x+1}$ is solvable.

(3) Infinite Limit

i. $\lim\limits_{x\to\infty} cos^{-1}(\sqrt{x^2+x}-x)$

ii. $\lim\limits_{x\to\infty} \frac{4x^5+3x^2+2}{7x^5+3}$

iii. $\lim\limits_{x\to\infty} \frac{2\cdot 5^x-3\cdot 2^x}{7\cdot 5^x+2\cdot 4^x}$

iv. $\lim\limits_{x\to\infty} \frac{5^x}{x!}$

v. $\lim\limits_{x\to\infty} x^{\frac{1}{x!}}$

i. $\lim\limits_{x\to\infty} (\sqrt{x^2+ax}-\sqrt{x^2+bx})$

ii. $\lim\limits_{x\to\infty} \frac{2(1/5)^x-3(1/3)^x}{7(2/3)^x+2(1/5)^x}$

iii. $\lim\limits_{x\to\infty} \frac{\sqrt{9x^2+1}}{x+4}$

iv. $\lim\limits_{x\to\infty} \frac{x^{2000}(\ln x)^2}{e^x}$

v. $\lim\limits_{x\to\infty} \sqrt[x]{x}$

(4) Asymptotes

Find the horizontal and vertical asymptotes for the function, $f(x) = tan^{-1}(\frac{x-1}{x+1})$

Find all asymptotes of the following function $f(x) = \frac{x^3+4}{x^2}$

(5) Curve sketching

Plot the graph satisfying the following conditions,

$$\begin{cases} \lim\limits_{x \to \pm\infty} = 0 \\ \lim\limits_{x \to 3^-} = -\infty \\ \lim\limits_{x \to 3^+} = \infty \end{cases}$$