Section 1-4 Midterm Review

1. **Function**

   (1) Exponential Function ($e \cong 2.7183,\ f(x) = e^x,\ f'(0) = e^0 = 1$)

   i. $b^{x+y} = b^x b^y$ ii. $b^{x-y} = b^x b^y$ iii. $(b^x)^y = b^{xy}$ iv. $(ab)^x = a^x b^x$

   (2) Logarithmic Function ($ln(x) \equiv log_e(x)$.) **Caution:** Should be written as ln not In. **(Lowercase LN!)**

   i. $\ln(xy) = \ln x + \ln y$ ii. $\ln(\frac{x}{y}) = \ln x - \ln y$ iii. $\ln(x^r) = r \ln x$ iv. $\ln(e^x) = x$

   (3) Inverse Function ($f(x) = y \Leftrightarrow f^{-1}(y) = x$)

   Step 1. Write $y = f(x)$.

   Step 2. Solve the equation of x in terms of y, then we can get $f^{-1}(y) = x$

   Step 3. Express the f^{-1} as a function of x. (As Required)

   (4) Inverse Trigonometric Function

$y = sin^{-1}(x) \Rightarrow$	Domain: $-1 \le x \le 1,$	Range: $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
$y = cos^{-1}(x) \Rightarrow$	Domain: $-1 \le x \le 1,$	Range: $0 \le y \le \pi$
$y = tan^{-1}(x) \Rightarrow$	Domain: $x \in \mathbb{R},$	Range: $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

2. **Limit**

   (1) One-side Limit and Existence

   If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, then we can say $\lim_{x \to a} f(x)$ exist.

   (2) Asymptotes

   i. Vertical Asymptotes: Find the x in the domain of $f(x)$ such that y approach $\pm\infty$.

   ii. Horizontal Asymptotes: Check if $f(x)$ approach constant as x approaches $\pm\infty$.

   iii. Slant Asymptotes: Factorize the fraction and check if $f(x)$ approach a specific linear function $g(x)$ as x approach $\pm\infty$.

   (3) Calculation

   i. Direct Substitutions: The most intuitive way, just substitute x into the approaching value.

   ii. Fractional Reduction: Factorize or Reduce the fraction then use the substitution.

   iii. Absolute & Peicewise Function:
   E.g.: Guassian Floor or Ceiling Function, Heaviside Unit Step Function.
   Discuss different interval using one-side limit.

   iv. Squeeze Theorem:
   If $g(x) \le f(x) \le h(x)$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$
   Hint: Think of the Squeeze theorem whlie dealing with the limit of trigonometric Function.

v. L'Hospital Rule: Using to deal with the limit of undeterminate form.

$$\text{Suppose} \begin{cases} \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \ or \\ \lim_{x\to a} \frac{f(x)}{g(x)} = \pm\frac{\infty}{\infty} \end{cases} \Rightarrow \begin{matrix} \lim_{x\to a} \frac{f(x)}{g(x)} \ (\text{"}\frac{0}{0}\text{"}) & \overset{L.H.}{=} & \lim_{x\to a} \frac{f'(x)}{g'(x)} \\ \lim_{x\to a} \frac{f(x)}{g(x)} \ (\text{"}\frac{\infty}{\infty}\text{"}) & \overset{L.H.}{=} & \lim_{x\to a} \frac{f'(x)}{g'(x)} \end{matrix}$$

Type of undeterminate form:

(1) Fraction: $\frac{0}{0}$, $\frac{\infty}{\infty}$ (Use L'Hospital Rule directly)

(2) Product: $0 \cdot \infty$ (Move 0 or ∞ to the denomenator to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

(3) Power: 0^0, 1^∞, ∞^0 (Try to use Natural Log (ln) to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

(4) Subtraction: $\infty - \infty$ (Reduce or Factorize the Square root or Fraction to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$)

3. **Continuity**

(1) Definition

i. $\lim_{x\to a} f(x)$ exist.

ii. $f(a)$ is defined.

iii. $\lim_{x\to a} f(x) = f(a)$

(2) Type of Discontinuity

i. Hole

ii. Infinity

iii. Break

iv. Oscillation

4. **Derivative**

(1) Definition

If $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$, then $f'(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

(2) Common Derivative

i. $\frac{d}{dx}\ (Constant) = 0$

ii. $\frac{d}{dx}\ (x^n) = nx^{n-1}$

iii. $\frac{d}{dx}\ (e^x) = e^x$

iv. $\frac{d}{dx}\ (\log_n x) = \frac{1}{x \ln n}$

v. $\frac{d}{dx}\ (\ln x) = \frac{1}{x}$

vi. $\frac{d}{dx}\ (sinx) = cosx$

vii. $\frac{d}{dx}\ (cosx) = -sinx$

viii. $\frac{d}{dx}\ (tanx) = sec^2x$

ix. $\frac{d}{dx}\ (secx) = secx \cdot tanx$

(3) Product Rule and Quotient Rule

i. Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$

ii. Quotient Rule: $\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

(4) Chain Rule: Treat the function as a composite function and differentiate layer by layer.

$\frac{d}{dx}\ f(g(h(x))) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$

(5) Differentiation in graphics

i. First derivatie $f'(x) \Rightarrow$ Slope of tangent $\equiv$ Rate of change. ($= 0 \Rightarrow$ Critical Point)

ii. First derivatie $f'(x) \Rightarrow$ Concavity of the function. ($= 0 \Rightarrow$ Inflection Point)

(6) Implicit Differentiation

Process: Derivative the equation, rearrange the equation into the form $y'(x) = \cdots$

(7) Mean Value Theorem

If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there exist c, s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(8) Application

i. Rate of Change: Tangent Slope, $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

ii. Linear Approximation: $f(x) \cong f(a) + f'(a)(x - a)$

iii. Related Error: $\frac{\Delta y}{y} \cong \frac{dy}{y}$

iv. Maximum, Minimum and Optimization

$$\begin{cases} \text{Local Maximum (L.M.)} \\ \text{Local Minimum (L.m.)} \end{cases} \Rightarrow \text{Occur while } f'(x) = 0 \text{ or D.N.E.}$$

$$\begin{cases} \text{Absolute Maximum (A.M.)} \\ \text{Absolute Minimum (A.m.)} \end{cases} \Rightarrow \text{Occur at Local } \begin{pmatrix} \text{Maximum} \\ \text{Minimum} \end{pmatrix} \text{ or End points}$$

Procedure of Optimization:

[1] Comprehend the question

[2] List the equation

[3] Find the A.M. or A.m. (Check 1^{st} Derivative and the End Points.)