Spectral Methods: Advanced

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Minimum Variance Spectrum Estimation

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- Let $x(n)$ be zero-mean, WSS process with power spectrum $P_x(e^{j\omega})$
- Let $g_i(n)$ be an ideal bandpass filter with bandwidth Δ and center frequency ω_i

$$|G(e^{j\omega})| = \begin{cases} 1 & ; |\omega - \omega_i| < \Delta/2 \\ 0 & ; \text{ otherwise.} \end{cases}$$

Minimum Variance Spectrum Estimation

- If $x(n)$ is filtered with $g_i(n)$ to obtain $y_i(n)$ then the power spectrum

$$P_{\mathrm{i}}(e^{j\omega}) = P_{\mathrm{x}}(e^{j\omega})|G_i(e^{j\omega_i})|^2$$

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and power in $y_i(n)$ is

$$E\{|y_i(n)|^2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{\mathrm{i}}(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{\mathrm{x}}(e^{j\omega})|G_i(e^{j\omega_i})|^2 d\omega$$

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- If Δ is small and $P_x(e^{j\omega})$ is approximately constant over this range
$$P_{\mathrm{i}}(e^{j\omega}) \approx P_{\mathrm{x}}(e^{j\omega})\frac{\Delta}{2\pi}$$

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$$P_{\text{i}}(e^{j\omega}) \approx P_{\text{x}}(e^{j\omega})\frac{\Delta}{2\pi}$$

- PSD of $x(n)$ at $\omega = \omega_i$ can be estimated from $y_i(n)$ as

$$P_{\text{x}}(e^{j\omega_i}) = \frac{E\{|y_i(n)|^2\}}{}$$

Minimum Variance Spectrum Estimation

- Periodogram can also be thought of as outputs from identical filters centered at varying ω_is

  However, leakage through sidelobes of the filter leads to distortion in the PSD

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- Better approach to use data adaptive filters
  - Optimum in some sense
  - Reduce out-of-band signal power

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- The minimum variance approach does this

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- Let $g_i(n)$ be a complex-valued FIR bandpass filter of order p
- Let $\mathbf{g}_i$ be the vector of filter coefficients

$$\mathbf{g}_i = \begin{bmatrix} g_i(0) & g_i(1) & \cdots & g_i(p) \end{bmatrix}^T$$

  and $\mathbf{e}_i$ be the vector of complex exponentials

$$\mathbf{e}_i = \begin{bmatrix} 1 & e^{j\omega_i} & \cdots & e^{jp\omega_i} \end{bmatrix}.$$

- $G_i(e^{j\omega})$ will be constrained to have a gain of one at $\omega = \omega_i$

$$G_i(e^{j\omega}) = \sum_{n=0}^{p} g_i(n) e^{jn\omega_i} = 1$$

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- Constraint can be written in vector form as

$$\mathbf{g}_i^H \mathbf{e}_i = \mathbf{e}_i^h \mathbf{g}_i = 1.$$

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- Power in $y_i(n)$ can be expressed as

$$E\left\{|y_i(n)|^2\right\} = \sum_{l=0}^{N-1}\sum_{m=0}^{N-1} g_i(l) r_x(m-l) g_i^*(m) = \mathbf{g}_i^H \mathbf{R}_x \mathbf{g}_i$$

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- Our objective is to solve the constrained optimization problem

$$\min_{\mathbf{g}_i} \mathbf{g}_i^H \mathbf{R}_x \mathbf{g}_i \quad \text{subject to} \quad \mathbf{e}_i^H \mathbf{g}_i = 1.$$

Minimum Variance Spectrum Estimation

- Rewriting using Lagrange multiplier

$$Q(\mathbf{g}_i, \lambda) = \mathbf{g}_i^H \mathbf{R}_x \mathbf{g}_i + \lambda(1 - \mathbf{g}_i^H \mathbf{e}_i).$$

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- Setting the gradient of $Q(\mathbf{g}_i, \lambda)$ w.r.t $\mathbf{g}_i^*$ equal to zero

$$\begin{aligned}\nabla_{\mathbf{g}_i^*} Q(\mathbf{g}_i, \lambda) &= \mathbf{R}_x \mathbf{g}_i - \lambda \mathbf{e}_i = 0 \\ &\implies \mathbf{g}_i = \lambda \mathbf{R}_x^{-1} \mathbf{e}_i.\end{aligned}$$

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- λ can be obtained by setting $\nabla_\lambda Q(\mathbf{g}_i, \lambda)$ equal to zero

$$1 - \mathbf{g}_i^H \mathbf{e}_i = 0 \implies \lambda = \frac{1}{\mathbf{e}_i^H \mathbf{R}_x^{-1} \mathbf{e}_i}.$$

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- Solution is similar to the one obtained in min-norm method

$$\mathbf{g}_i = \frac{\mathbf{R}_x^{-1} \mathbf{e}_i}{\mathbf{e}_i^H \mathbf{R}_x^{-1} \mathbf{e}_i}.$$

Minimum Variance Spectrum Estimation

- The optimum filter for estimating the power in $x(n)$ at frequency ω is
$$\mathbf{g} = \frac{\mathbf{R}_x^{-1}\mathbf{e}}{\mathbf{e}^H\mathbf{R}_x^{-1}\mathbf{e}}.$$

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- The corresponding minimum power estimate is

$$\hat{\sigma}_x^2(\omega) = E\{|y_i(n)|^2\} = \mathbf{g}^H\mathbf{R}_x\mathbf{g} = \frac{1}{\mathbf{e}^H\mathbf{R}_x^{-1}\mathbf{e}}$$

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- Next objective is to obtain the power spectrum
- To do this we need to obtain the bandwidth of the filter Δ

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$$\hat{P}(e^{j\omega}) = \sigma_x^2 \tag{1}$$

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$$E\{|y_i(n)|^2\} = \frac{\sigma_x^2}{p+1}$$

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- As seen earlier

$$\hat{P}_x(e^{j\omega_i}) = \frac{E\{|y_i(n)|^2\}}{\Delta/2\pi}$$

  and for white noise we have

$$\hat{P}_x(e^{j\omega_i}) = \frac{\sigma_x^2}{p+1}\frac{2\pi}{\Delta} \tag{2}$$

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- From (1) and (2), we have the required bandwidth

$$\Delta = 2\pi/p+1$$

Minimum Variance Spectrum Estimation

- Hence, the general PSD estimate using minimum variance method is

$$
\begin{aligned}
\hat{P}_{\mathrm{MV}}(e^{j\omega}) = \frac{E\{|y_i(n)|^2\}}{\Delta/2\pi} &= (p+1)E\{|y_i(n)|^2\} \\
&= \frac{p+1}{\mathbf{e}^H \mathbf{R}_x^{-1} \mathbf{e}}.
\end{aligned}
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- Using estimated $\hat{\mathbf{R}}_x$, we can obtain $\hat{P}_{\mathrm{MV}}(e^{j\omega})$

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- Using estimated $\hat{\mathbf{R}}_x$, we can obtain $\hat{P}_{\mathrm{MV}}(e^{j\omega})$
- The order of filter p is related to data length M, i.e., $p \leq M$
- Compare with BT using Bartlett window

$$\hat{P}_{\mathrm{BT}}(e^{j\omega}) = \frac{1}{M+1} \sum_{k=-M}^{M} (M+1-|k|)\, \hat{r}_x(k) e^{-jk\omega}$$
$$= \frac{1}{M+1} \mathbf{e}^H \mathbf{R}_x \mathbf{e}.$$

Array Processing

Spatial spectral estimation

- Distribution of energy in space - signal source position in space
- Estimate of source parameters - azimuth, elevation, and range
- Near field or far-field sources

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Assumptions (in our case)

- Far-field sources
- Narrowband sources
- Source and sensor aarry in same plane - only azimuth is of interest
- Signal impinging on the array is in the form of a narrowband plan wave

Array Processing

- Array model: Uniform linear array (ULA)
- Output of sensor k

$$y_k(t) = h_k(t) * x(t - \tau_k) + e_k(t)$$

where

$x(t)$ is signal waveform at a reference source,
$h_k(t)$ known impulse response
τ_k time for source signal to reach sensor k

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- Equivalently estimation of time-delay

Array Processing

- Signal of centre frequency ω_c output from sensor,

$$y_k(t) = H_k(\omega_c)e^{-j\omega_c\tau_k}s(t) + e_k(t)$$

where $s(t) = Ae^{j\phi t}$ and $H_k(\omega_c)$ response of each sensor

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- Array transfer function (direction vector)

$$\mathbf{a}(\theta) = [H_1(\omega_c)e^{-j\omega_c\tau_1} \cdots H_M(\omega_c)e^{-j\omega_c\tau_m}]^T$$

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  where $\mathbf{y}(t) = [y_1(t) \cdots y_M(t)]^T$ and $e(t) = [e_1(t) \cdots e_M(t)]^T$
- $H_k(\omega_c)$ depends on array configuration. Considering first sensor as reference and identical sensors

$$\mathbf{a}(\theta) = [1\, e^{-j\omega_c\tau_1} \cdots e^{-j\omega_c\tau_{M-1}}]^T$$

Array Processing

- Consider ULA with *M* sensors
- Wave propagation direction θ with respect to center of array
- Planar wave experiences delay τ between adjacent sensors

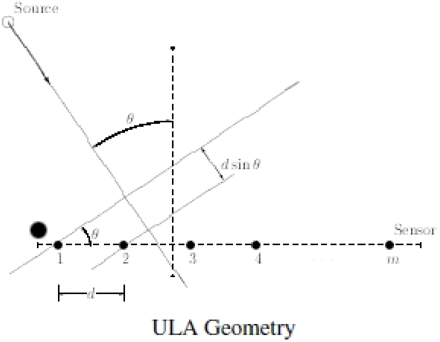

Figure : Uniform Linear Array [Stoica and Moses]

Array Processing

- Assume sensor 1 as reference

$$\tau_m = (m-1)\frac{d\sin\theta}{c}$$

where

  - d array separation
  - c speed of propagation in the medium

- Corresponding direction vector

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 & e^{-j\omega_c \frac{d\sin\theta}{c}} & \cdots & e^{-j(M-1)\omega_c \frac{d\sin\theta}{c}} \end{bmatrix}^T$$

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- Spatial frequency

$$f_s = f_c \frac{d \sin\theta}{c} = \frac{d \sin\theta}{\lambda}$$

- Using $\omega_s = 2\pi f_s$ (or wavenumber k),

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 & e^{-j\omega_s} & \cdots & e^{-j(M-1)\omega_s} \end{bmatrix}^T$$

or

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 & e^{-jk} & \cdots & e^{-j(M-1)k} \end{bmatrix}^T$$

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- To avoid spatial aliasing $|\omega_s| < \pi$

$$\implies |f_s| < \frac{1}{2} \implies d|\sin \theta| < \frac{\lambda}{2} \implies d < \frac{\lambda}{2}$$

- Array spacing d constrained by signal wavelength λ

Array Processing: Beamforming

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Array Processing: Beamforming

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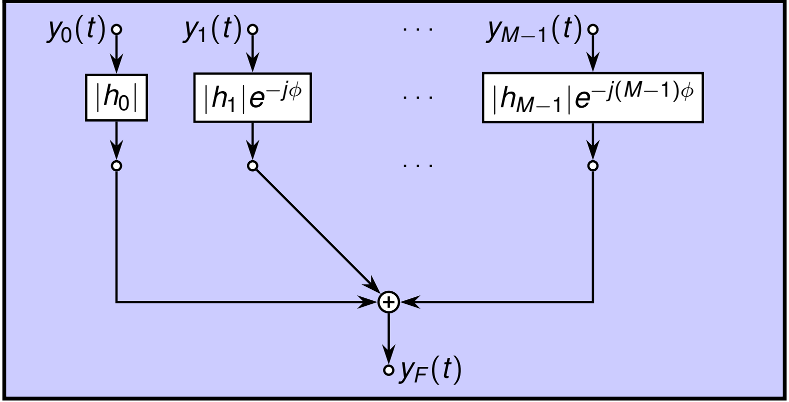
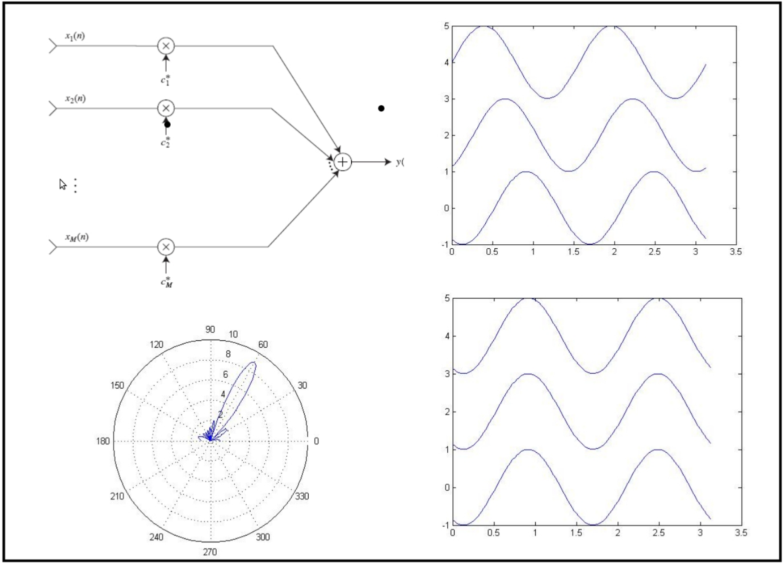
- Array weights are adjusted to compensate for propagation time of waves arriving from direction θ_0
- Set phase of array $\phi = k_0$ - “steering the beam”
  - to produce large output from the array for waves arriving from direction θ_0
  - other directions will result in smaller outputs

Array Processing: ULA

Uniform linear array (ULA) with M elements



Array Processing: Beamforming



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$$y_F(t) = S(\phi)e^{j2\pi f_c t} = \sum_{n=0}^{M-1} y_n e^{j2\pi f_c t} e^{-jn\phi}$$

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- This implies that the complex amplitude of the output y_n is

$$Y(\phi) = \sum_{n=0}^{M-1} y_n e^{-jn\phi}$$

- Fourier transform of spatially sampled signal y_n, where ϕ plays the role of digital frequency

Array Processing: Beamforming

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- Signal received at sensor 0

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  where A is the complex amplitude
- Signal received at sensor n

$$y_n(t) = Ae^{j2\pi f_c t + nk}$$

  where $k = \frac{2\pi f_c d \sin\theta}{c}$ is the wavenumber
- Complex amplitude of signal from θ_0 (k_0) at n-th sensor is

$$y_n = Ae^{jnk_0}$$

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- Signal received at sensor n

$$y_n(t) = Ae^{j2\pi f_c t + nk}$$

  where $k = \frac{2\pi f_c d \sin\theta}{c}$ is the wavenumber
- Complex amplitude of signal from θ_0 (k_0) at n-th sensor is

$$y_n = Ae^{jnk_0}$$

- Corresponding array output

$$Y(k; k_0) = \sum_{n=0}^{M-1} Ae^{jnk_0} e^{-jnk}$$

Array Processing: Beamforming

- Array output is referred to as the beampattern

$$|Y(k;k_0)| = |\sum_{n=0}^{M-1} Ae^{jn(k-k_0)}| = |A|\left|\frac{\sin M(k-k_0)/2}{\sin(k-k_0)/2}\right|$$

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  - good response to signals arriving from a chosen direction θ_0
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- Non-uniform weights $|h_n|$ can be used to suppress side-lobes at the expense of widening the main lobe
- This approach is referred to as conventional beamforming

Beamforming: Analogy with Temporal Filtering

Temporal FIR

- With generic input $u(t)$

$$y_F(t) = \sum_{k=0}^{M-1} h_k u(t-k) \triangleq \mathbf{h}^H \mathbf{y}(t)$$

where

$$\mathbf{h} = \begin{bmatrix} h_0 & h_1 & \cdots & h_{M-1} \end{bmatrix}^H \quad \mathbf{y}(t) = \begin{bmatrix} u(t) & \cdots & u(t-M+1) \end{bmatrix}^T$$

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- With complex exponential input $u(t) = e^{j\omega t}$

$$y_F(t) = [\mathbf{h}^H a(\omega)] u(t)$$

  where

$$\mathbf{a}(\omega) = \begin{bmatrix} 1 & e^{-j\omega} & \cdots & e^{-j(M-1)\omega} \end{bmatrix}^T$$

  - large $\mathbf{h}^H \mathbf{a}(\omega)$ enhances or passes frequency component ω
  - small $\mathbf{h}^H \mathbf{a}(\omega)$ attenuates frequency component ω

Beamforming: Analogy with Temporal Filtering

Spatial filter

- With input $s(t)$

  $$y_F(t) = \mathbf{h}^H \mathbf{y}(t) = [\mathbf{h}^H \mathbf{a}(\theta)] s(t)$$

  where

  $$\mathbf{a}(\theta) = \begin{bmatrix} 1 & e^{-j\omega\tau_2(\theta)} & \cdots & e^{-j\omega\tau_M(\theta)} \end{bmatrix}^T$$

  - large $\mathbf{h}^H \mathbf{a}(\theta)$ enhances or passes signals arriving at an angle θ
  - small $\mathbf{h}^H \mathbf{a}(\theta)$ attenuates signals arriving at an angle θ
- Find $\mathbf{h}$ such that it
  - Passes undistorted signal for a given θ
  - Attenuates all other signals from DOAs other than θ

Beamforming

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- Power of the spatially filtered signal is

$$E\left\{|y_F(t)|^2\right\} = \mathbf{h}^H\mathbf{R}\mathbf{h}$$

where

$$\mathbf{R} = E\left\{\mathbf{y}(t)\mathbf{y}(t)^T\right\} \text{ with } \mathbf{y}(t) = s(t)\mathbf{a}(\theta)$$

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- The signal power $\mathbf{h}^H\mathbf{R}\mathbf{h}$ should peak at source DOA and pass a signal at a given θ undistorted

$$\mathbf{h}^H\mathbf{a}(\theta) = 1,$$

and $a(\theta)$ will be normalized

$$\mathbf{a}(\theta)^H\mathbf{a}(\theta) = M.$$

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- Assume spatially white noise as input, i.e., $\mathbf{R} = \mathbf{I}$

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- Objective is to design **h** such that

$$\min_{\mathbf{h}} \mathbf{h}^H\mathbf{h} \quad \text{subject to} \quad \mathbf{h}^H\mathbf{a}(\theta) = 1.$$

- Note: Here we are minimizing the output power independent of the data

Beamforming

- Rewriting using Lagrange multiplier

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- Solving this we have

$$\mathbf{h} = \frac{\mathbf{a}(\theta)}{\mathbf{a}^H(\theta)\mathbf{a}(\theta)} = \frac{\mathbf{a}(\theta)}{M}.$$

- The array steering vector corresponding to a particular θ is the filter

Beamforming

- Corresponding output power

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- Note that $\hat{\mathbf{R}}$ depends on incoming data (at some $\theta = \theta_0$)
- DOA estimates are given by location of the n highest peaks in

$$\mathbf{a}^H(\theta)\hat{\mathbf{R}}\mathbf{a}(\theta)$$

  assuming n sources

Beamforming: DOA Resolution

- DOA resolution limit

$$\Delta\theta \approx \frac{\text{wavelength}}{\text{array "length"}} = \frac{\lambda}{(M-1)d}$$

- This can be obtained from (for ULA)

$$\Delta\omega_s = \frac{2\pi}{M} \Leftrightarrow \Delta f_s = \frac{1}{M}$$

- Resolution depends on the signal direction relative to the array center
  - Broadside - DOAs about $\theta = 0$
  - End-fire - DOAs about $\theta = \pm\frac{\pi}{2}$

Capon's Beamforming

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- Note: Compare to conventional beamforming that we discussed earlier
    - Minimizing output power in a data-dependent way
    - Attenuates signals impinging on the array at an angle $\theta_0 \neq \theta$

Capon's Beamforming

- Corresponding solution and the optimum filter is

$$\mathbf{h} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta)}{\mathbf{a}^{H}(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)}.$$

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$$E\{|y_F(t)|^2\} = \mathbf{h}^H\mathbf{R}\mathbf{h} = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)}$$

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- Referred to as minimum variance as mean square value or output power is the variance if we assume zero-mean output signal

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$$\mathbf{y}(t) = \mathbf{a}(\theta)s(t) + e_n(t)$$

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- Using Matrix inversion lemma

$$(\mathbf{C} + \mu\mathbf{x}\mathbf{y}^H)^{-1} = \mathbf{C}^{-1}(\mathbf{I} - \frac{\mathbf{x}\mathbf{y}^H}{\mu^{-1} + \mathbf{y}^H\mathbf{C}^{-1}\mathbf{x}}\mathbf{C}^{-1})$$

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- We have

$$\mathbf{R}^{-1} = \frac{1}{\sigma_n^2}(\mathbf{I} - \frac{A^2}{MA^2 + \sigma_n^2}\mathbf{a}(\theta)\mathbf{a}^H(\theta))$$

Capon's Beamforming

- Output power in a beamformer designed for θ_d,

$$
\begin{aligned}
P_{\text{MVDR}} &= [\mathbf{a}^H(\theta_d)\mathbf{R}^{-1}\mathbf{a}(\theta_d)]^{-1} \\
&= \sigma_n^2[M - \frac{A^2}{MA^2 + \sigma_n^2}|\mathbf{a}^H(\theta_d)\mathbf{a}(\theta)|^2]^{-1}
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- Power is maximum when $\theta = \theta_d$,

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P_{\text{MVDR}} = A^2 + \frac{\sigma_n^2}{M}
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- Power is maximum when $\theta = \theta_d$,

$$P_{\mathrm{MVDR}} = A^2 + \frac{\sigma_n^2}{M}$$

- Power is minimum for other propagating directions $\theta \neq \theta_d$

$$P_{\mathrm{MVDR}} = \frac{\sigma_n^2}{M}$$

Thanks for your attention