EE 779 Advanced Topics in Signal Processing
Assignment 1
Assigned: 22/08/11, Due: 30/08/11
Indian Institute of Technology Bombay

Note

- Most of these problems are from Chapter 2 in [1]. A scanned copy of these questions is placed in moodle.
- Submit the starred ($*$) problems and **all** simulations.
- For the simulation problems, you can use the Matlab functions provided by S & M [1] or Hayes [2]. A copy of the files is in moodle.

Problems

1. A continuous-time signal $x_a(t)$ is bandlimited to 5 kHz, i.e., has a spectrum $X_a(f)$ that is zero for $|f| > 5$ kHz. Only 10 seconds of the signal has been recorded and is available for processing. We would like to estimate the power spectrum of $x_a(t)$ using the available data in a radix-2 FFT algorithm, and it is required that the estimate have a resolution of at least 10 Hz. Suppose that we use Bartlett's method of periodogram averaging.

   (a) If the data is sampled at Nyquist rate, what is the minimum section length that you may use to get the desired resolution ?

   (b) Using the minimum section length determined in part (a), with 10 seconds of data, how many sections are available for averaging ?

   (c) How does your choice of the sampling rate affect the resolution and variance of your estimate ? Are there any benefits to sampling above the Nyquist rate ?

2. Whenever the signal mean is unknown, a natural modification of the unbiased estimator of the autocorrelation function (ACF) is,

$$\tilde{r}_x(k) = \frac{1}{N-k} \sum_{n=k}^{N-1} \left(x(n) - \overline{x}\right)\left(x(n-k) - \overline{x}\right), \quad k = 0, \ldots, N-1. \tag{1}$$

   and the biased estimator is,

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=k}^{N-1} \left(x(n) - \overline{x}\right)\left(x(n-k) - \overline{x}\right), \quad k = 0, \ldots, N-1. \tag{2}$$

   where $\overline{x}$ is the sample mean

$$\overline{x} = \frac{1}{N} \sum_{n=0}^{N-1} x(n).$$

   Show that in the unknown mean case, the usual names, "unbiased" and "biased" sample covariance associated with the above might no longer be appropriate. Indeed, both these estimators could be biased; furthermore, $\hat{r}_x(k)$ could be less biased than $\tilde{r}_x(k)$. To simplify calculations, assume that $x(n)$ is white noise.

3. [$*$]Show that the following definitions of the periodogram are equivalent.

$$P_{\text{per}}(e^{j\omega}) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)(e^{-j\omega n}) \right|^2. \tag{3}$$

$$P_{\text{per}}(e^{j\omega}) = \sum_{k=-(N-1)}^{N-1} \hat{r}_x(k) e^{-j\omega k}, \tag{4}$$

where $\hat{r}_x(k)$ is the biased autocorrelation estimate (assuming zero-mean $x(n)$).

4. [$*$]Let $x(n), n = 0, \ldots, N-1$, be a real Gaussian process with zero mean and autocorrelation $r_x(k)$. The autocorrelation can be estimated using

$$\hat{r}_x(k) = \alpha(k) \sum_{n=k}^{N-1} \left(x(n) - \overline{x}\right)\left(x(n-k) - \overline{x}\right), \quad k = 0, \ldots, N-1, \tag{5}$$

where

$$\alpha(k) = \begin{cases} \frac{1}{N-k}, & \text{for unbiased ACF estimate} \\ \frac{1}{N}, & \text{for biased ACF estimate.} \end{cases}$$

Using the moment factoring theorem for real Guassian RVs (which is different from that for complex Gaussian RVs mentioned in class),

$$E\{abcd\} = E\{ab\}E\{cd\} + E\{ac\}E\{bd\} + E\{ad\}E\{bc\} - 2E\{a\}E\{b\}E\{c\}E\{d\}, \tag{6}$$

show that

$$Var\{\hat{r}_x(k)\} = \alpha^2(k) \sum_{m=-(N-k-1)}^{N-k-1} (N-k-|m|)\left(r_x^2(m) + r_x(m+k)r_x(m-k)\right).$$

Simulations

1. Problem C2.23 in S & M [1].

Reference

1. Petre Stoica and Randolph Moses, "Spectral analysis of signals", Prentice Hall, 2005. (Indian edition available)

2. Monson H. Hayes, "Statistical signal processing and modeling", Wiley India Pvt. Ltd., 2002. (Indian edition available)