MATH 141: Midterm 2

Name: key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* **Remember to simplify each expression.**
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
		80

conceptual understanding

execution

common pitfalls

1. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = 300$

$$f'(x) = \frac{d}{dx} 300 = \boxed{0}$$

(b) $f(x) = 5x^4 - x^2 + 3x$

$$f'(x) = 5 \cdot \frac{d}{dx} x^4 - \frac{d}{dx} x^2 + 3 \frac{d}{dx} x$$

$$= 5 \cdot 4x^{4-1} - 2x^{2-1} + 3x^{1-1}$$ power rule

$$= \boxed{20x^3 - 2x + 3}$$

(c) $f(x) = \frac{\sin^2(x)}{x^2}$ ← high ← low

Quotient Rule.

$$f'(x) = \frac{\underbrace{x^2}_{\text{low}} \cdot \underbrace{\frac{d}{dx}[\sin^2(x)]}_{\text{D high}} - \underbrace{\sin^2(x)}_{\text{high}} \cdot \underbrace{\frac{d}{dx}[x^2]}_{\text{D low}}}{(x^2)^2}$$

$$= \frac{x^2 \cdot 2\sin(x) \cdot \frac{d}{dx}\sin(x) - \sin^2(x) \cdot 2x}{x^4}$$ chain rule, power rule, LoE #3

$$= \frac{\overbrace{2x^2 \sin(x)\cos(x)}^{\text{term}} - \overbrace{2x\sin^2(x)}^{\text{term}}}{x^4}$$

$$= \frac{2 \cdot \cancel{x} \cdot \sin(x)\,(x\cos(x) - \sin(x))}{\cancel{x^4}\; x^3}$$ GCF

$$= \boxed{\frac{2\sin(x)\,(x\cos(x) - \sin(x))}{x^3}}$$ x is a global factor.

left · right Product Rule

(d) $g(x) = \underbrace{x^2}_{\text{left}} \underbrace{\cos(x^2)}_{\text{right}}$

$$g'(x) = \underbrace{x^2}_{\text{left}} \cdot \underbrace{\frac{d}{dx}[\cos(x^2)]}_{(\text{right})'} + \underbrace{\cos(x^2)}_{\text{right}} \cdot \underbrace{\frac{d}{dx}[x^2]}_{(\text{left})'}$$

$$= x^2 \cdot (-\sin(x^2)) \cdot \frac{d}{dx}[x^2] + \cos(x^2) \cdot 2x$$ chain + power

$$= -x^2 \sin(x^2) \cdot 2x + 2x\cos(x^2)$$ power

$$= \underbrace{-2x^3 \sin(x^2)}_{\text{term}} + \underbrace{2x\cos(x^2)}_{\text{term}}$$

$$= \boxed{-2x\left(x^2\sin(x^2) - \cos(x^2)\right)}$$

(e) $f(x) = \left(\dfrac{x^2-1}{x^2+3}\right)^4$ ← chain rule.

$$f'(x) = \frac{d}{dx}\left[\left(\frac{x^2-1}{x^2+3}\right)^4\right]$$

quotient rule.

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{d}{dx}\left[\frac{x^2-1}{x^2+3}\right]$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{(x^2+3)\frac{d}{dx}[x^2-1] - (x^2-1)\frac{d}{dx}[x^2+3]}{(x^2+3)^2}$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{(x^2+3)\cdot 2x - (x^2-1)\,2x}{(x^2+3)^2}$$

both − and 2x need to be distributed.

dist

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{2x^3+6x - 2x^3 + 2x}{(x^2+3)^2}$$

$$= 4\underbrace{\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{8x}{(x^2+3)^2}}_{\text{combine w/ LoE}}$$

LoE 5

$$= 4\,\frac{(x^2-1)^3}{(x^2+3)^3} \cdot \frac{8x}{(x^2+3)^2}$$

frac law #1 and LoE #1

$$= \boxed{\frac{32x(x^2-1)^3}{(x^2+3)^5}}$$

2. The following three equations are in implicit form. Find $\frac{dy}{dx}$.

(a) $3x^2 + 2y = 2x^4 + 3y^2$

$$3\frac{d}{dx}[x^2] + 2\frac{d}{dx}[y] = 2\frac{d}{dx}[x^4] + 3\frac{d}{dx}[y^2]$$

$$3\cdot 2x + \underline{2\cdot\frac{dy}{dx}} = 2\cdot 4x^3 + \underline{3\cdot 2y\cdot\frac{dy}{dx}}$$

$$6x - 8x^3 = 6y\frac{dy}{dx} - 2\frac{dy}{dx}$$

$$6x - 8x^3 = \frac{dy}{dx}(6y - 2)$$

$$\frac{dy}{dx} = \frac{6x - 8x^3}{6y - 2} = \frac{\cancel{2}(3x - 4x^3)}{\cancel{2}(3y - 1)} = \boxed{\frac{3x - 4x^3}{3y - 1}}$$

(b) $x^2 - 2xy + y^2 = 5$

product rule.

$$\frac{d}{dx}[x^2] - 2\frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = \frac{d}{dx}[5]$$

many of you will forget that -2 is multiplied into ≥ 2 terms b/c the product rule generates two terms.

don't be that person!

$$2x - 2\left(x\frac{d}{dx}[y] + y\frac{d}{dx}[x]\right) + 2y\cdot y' = 0$$

$$2x - 2(\underline{xy'} + y) + \underline{2y\cdot y'} = 0$$

isolate these terms.

$$2x - 2xy' - 2y + 2yy' = 0$$

$$2yy' - 2xy' = 2y - 2x$$

$$y'(2y - 2x) = 2y - 2x$$

$$\to y' = \frac{2y - 2x}{2y - 2x}$$

$$\boxed{y' = 1}$$

(c) $\cos(xy) = 1 + \sin y$

chain rule!

$$\frac{d}{dx}\left[\cos(xy)\right] = \frac{d}{dx}[1] + \frac{d}{dx}\left[\sin(y)\right]$$

product rule!

$$-\sin(xy)\cdot\frac{d}{dx}[xy] = 0 + \cos(y)\cdot y'$$

Commonly forgot.

$$-\sin(xy)\cdot\left(x\cdot\frac{d}{dx}[y] + y\cdot\frac{d}{dx}[x]\right) = \cos(y)y'$$

$$-\sin(xy)\cdot\left(x\underline{y'} + y\right) = \underline{\cos(y)y'}$$

$$-\sin(xy)\cdot xy' - \sin(xy)y = \cos(y)y'$$

collect terms w/ y'

$$-\sin(xy)y = \cos(y)y' + \sin(xy)\cdot xy'$$

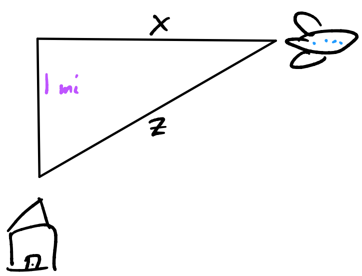
GCF

$$-\sin(xy)y = y'\left(\cos(y) + \sin(xy)x\right)$$

divide

$$\boxed{y' = \frac{-\sin(xy)y}{\cos(y) + x\sin(xy)}}$$

3. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.

① 

x: distance from plane to straight above the station

z: distance between plane and radar station

these are functions of time.

②

Given: $z = 2$ mi $\quad \frac{dx}{dt} = 500$ mi/h $\quad$ Need: $\frac{dz}{dt}$

③ $x^2 + 1^2 = z^2$ Pythagorean Thm

④ $\frac{d}{dt}[x^2] + \frac{d}{dt}[1^2] = \frac{d}{dt}[z^2]$

$$2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$$

missing from ②! Find x in ⑤.

$$\frac{dz}{dt} = \frac{2x}{2z}\frac{dx}{dt} = \frac{\boxed{x}}{z}\cdot\frac{dx}{dt}$$

⑤ Now plug in $z = 2$ from ② into ③.

$$1^2 + x^2 = 2^2$$

$$x^2 = 4 - 1$$

$$x^2 = 3$$

$$\sqrt{x^2} = \pm\sqrt{3}$$

$$x = \sqrt{3}$$ distance isn't negative here.

⑥ $$\frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

$$= \frac{\sqrt{3}\,\cancel{mi}}{\cancel{2}\,\cancel{mi}}\cdot \cancel{500}^{250}\frac{mi}{h}$$

$$= \boxed{250\sqrt{3}\ \frac{mi}{h}}$$

4. Short answer questions:

(a) What are the two methods for finding local minimums and maximums called? Compare the strength of both methods.

First and Second derivative test.

$\boxed{2^{nd} \text{ vs. } 1^{st}:}$

Pro:

① faster on polynomials.

Con:

① 2^{nd} can't detect $f'(x)$ DNE critical #'s, or pointy peak/valleys.

② may be inconclusive.

(b) Under which conditions are both the absolute minimum and maximum of $f(x)$ guaranteed to exist?

Two conditions:

① $f(x)$ continuous on

② closed interval $[a, b]$

a.k.a. EVT.

(c) What is the method for finding absolute minimums and maximums called? How do you use it?

Closed Interval Method.

① Check EVT satisfied.

② Find crit #'s, plug into $f(x)$

③ plug in endpoints.

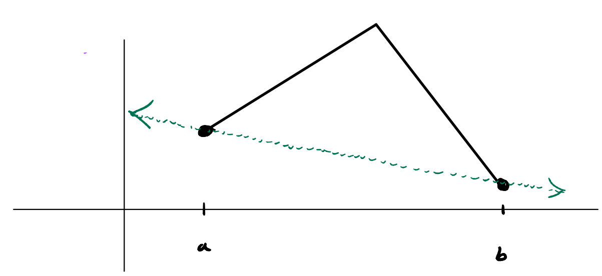
④ largest # is absolute max. smallest # is absolute min.

(d) Suppose $f(x)$ is continuous on $[a, b]$. Sketch a graph of $f(x)$ which shows there does not necessarily need to be a c where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

MVT problem. Notice $f(x)$ isn't required to be differentiable on (a, b).

So draw a graph where slopes of tangent is never the slope of secant line through endpoints. Ex:



5. Suppose $f(x) = \dfrac{x}{x^2+1}$.

(a) Find all intervals on which $f(x)$ is increasing and decreasing.

(b) Find all local minimums and maximums.

1 crit #'s

$$f'(x) = \frac{(x^2+1)\cdot \frac{d}{dx}[x] - x\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$$

$$= \frac{x^2+1 - x\cdot 2x}{(x^2+1)^2}$$

$$= \frac{-x^2+1}{(x^2+1)^2}$$

a solve $f'(x)=0$

$$(x^2+1)^2\,\frac{-x^2+1}{(x^2+1)^2} = 0\,(x^2+1)^2$$

$$-x^2+1=0$$

$$x^2 = 1$$

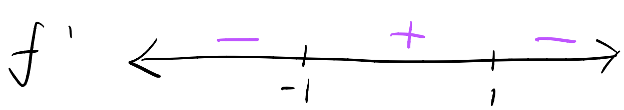
$$x = \pm\sqrt{1}$$

$$x = \pm 1$$

b $f'(x)$ DNE when $(x^2+1)^2 = 0$

but $(x^2+1)^2 > 0$ so N/A

2 Sign diagram of f'



Factor $f'(x)$: $f'(x) = \dfrac{-x^2+1}{(x^2+1)^2} = \dfrac{-(x^2-1)}{(x^2+1)^2} = \dfrac{-(x-1)(x+1)}{(x^2+1)^2}$

$$f'(-2) = \frac{-(-2-1)(-2+1)}{((-2)^2+1)^2} = \frac{-\cdot-\cdot-}{+} = -$$

denom squared. always +

$$f'(0) = \frac{-(0-1)(0+1)}{+} = \frac{-\cdot-\cdot+}{+} = +$$

$$f'(2) = \frac{-(2-1)(2+1)}{+} = \frac{-\cdot+\cdot+}{+} = -$$

∴, $f(x)$ is increasing on $(-1,1)$

decreasing on $(-\infty,-1)\cup(1,\infty)$

Local maximum of

$$f(-1) = \frac{-1}{(-1)^2+1} = -\frac{1}{2}$$

Local minimum of

$$f(1) = \frac{1}{1^2+1} = \frac{1}{2}$$

6. Determine the intervals of concavity of

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$

① Find potential inflection points.

$$f'(x) = 6x^2 - 18x + 12$$

$$f''(x) = 12x - 18 = 6(2x - 3)$$

ⓐ Solve $f''(x) = 0$

$$6(2x-3) = 0$$

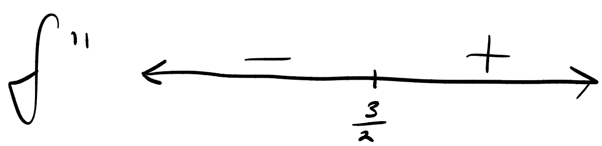
$$2x - 3 = 0$$

$$x = \frac{3}{2}$$

ⓑ find when $f''(x)$ DNE

not applicable, $f''(x)$ has domain $\mathbb{R}$.

② Sign diagram of f''



$$\boxed{\begin{array}{l} \text{Concave up on } \left(\frac{3}{2}, \infty\right) \\ \text{Concave down on } \left(-\infty, \frac{3}{2}\right) \end{array}}$$

$$f''(0) = 6 \cdot (2 \cdot 0 - 3) = -$$

$$f''(2) = 6 \cdot (2 \cdot 2 - 3) = +$$

7. Sketch a possible graph of a function which satisfies the following:

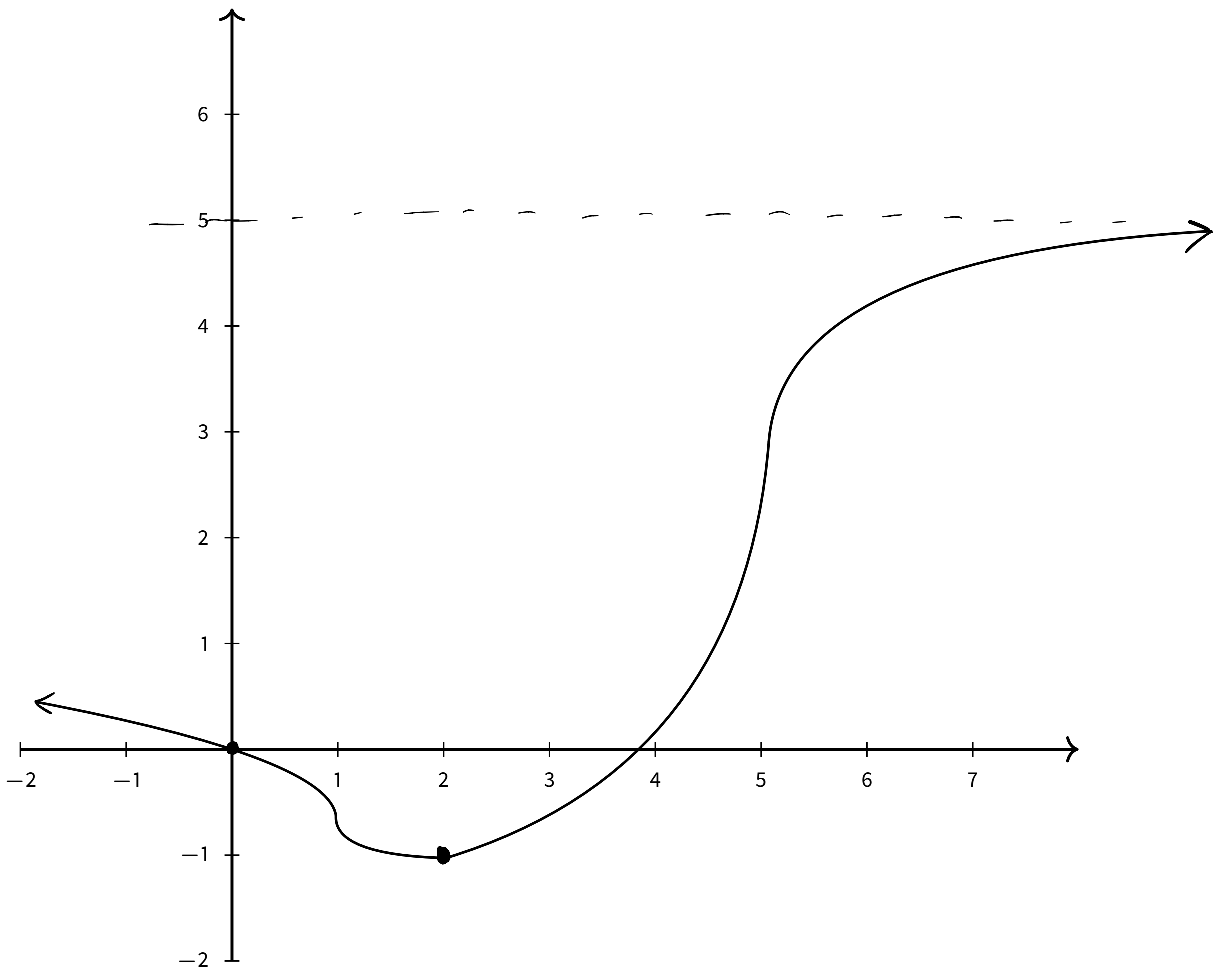
   (a) $f(0) = 0$, ~~[illegible]~~, $f'(2) = 0$ $\quad f(2) = -1$

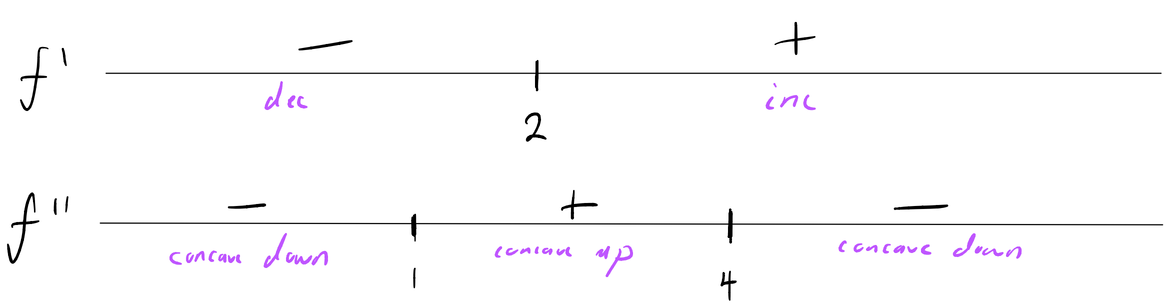
   (b) $f'(x) < 0$ when $0 < x < 2$ and $f'(x) > 0$ when $x > 2$

   (c) $f''(x) < 0$ when $0 < x < 1$ and $x > 4$

   (d) $f''(x) > 0$ when $1 < x < 4$

   (e) $\lim_{x \to \infty} f(x) = 5$

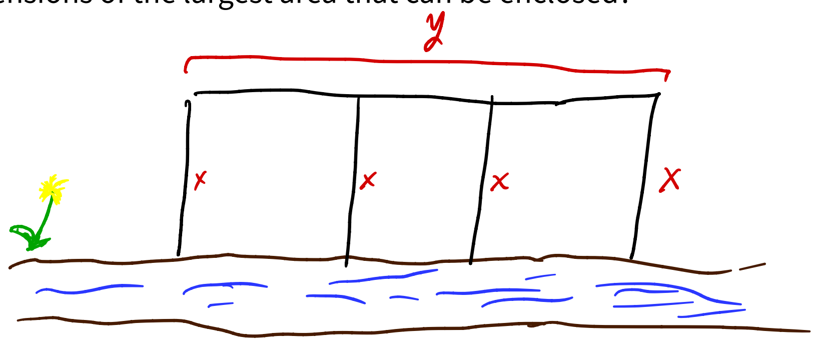




8. Suppose someone owns 4000 meters of fencing. They wish to create a rectangular piece of grazing land where one side is along a river. This means no fence is needed for that side. Moreover, they wish to subdivide the rectangle into three separate sections with two pieces of fence, both of which are parallel to the sides not along the river.

What are the dimensions of the largest area that can be enclosed?

① Diagram



② Maximize $A = x \cdot y$ ← middle two pieces don't contribute to total area.

③ eliminate y: $4x + y = 4000$

$y = 4000 - 4x$ Substitute into ②

so maximize $A = xy = \underbrace{x(4000-4x)}_{f(x)}$, $x \in [0, 1000]$

Use CI Method b/c EVT applies.

$f(x) = x(4000-4x) = 4000x - 4x^2$

$f'(x) = 4000 - 8x$

$f'(x) = 0$

$4000 - 8x = 0$

$x = 500$

$f(500) = 500 \cdot (4000 - 4 \cdot 500)$

$= 500 \cdot 2000$

also, $f(0) = 0 \cdot (4000 - 4 \cdot 0) = 0$

$f(1000) = 1000 \cdot (4000 - 4 \cdot 1000)$

$= 0$

$\therefore$ absolute max when $x = 500$.

Need dimensions, solve for y.

From ③: $y = 4000 - 4x$

$= 4000 - 4 \cdot 500$

$= 2000$

$\boxed{\text{Maximum dimensions is } x = 500 \text{ meters, } y = 2000 \text{ meters}}$