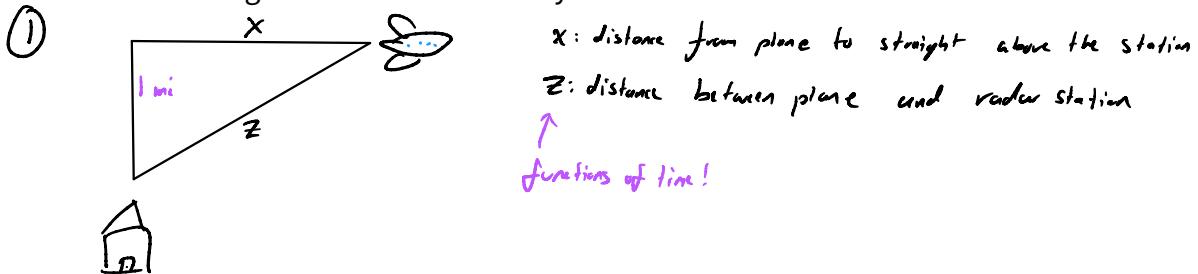


MATH 161: Some Practice Final Problems

Here are problems that cover the last two weeks of our class.

Remember the final is cumulative; you should look at Practice Midterm 1+2 and Midterm 1+2 as well.

1. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.



② Given:
 $z = 2 \text{ mi}$, $\frac{dx}{dt} = 500 \text{ mi/h}$

Need:

$$\frac{dz}{dt}$$

③ $x^2 + 1^2 = z^2$ Pythagorean Thm

④ $\frac{d}{dt}[x^2] + \frac{d}{dt}[1^2] = \frac{d}{dt}[z^2]$

$$2x \frac{dx}{dt} + 0 = 2z \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{2x}{2z} \frac{dx}{dt} = \frac{x}{z} \cdot \frac{dx}{dt}$$

missing from ②! Find what x is.

⑤ Now plug in z .

$$1^2 + x^2 = 2^2$$

$$x^2 = 4 - 1$$

$$x^2 = 3$$

$$\sqrt{x^2} = \pm \sqrt{3}$$

$$x = \sqrt{3}$$

⑥ $\frac{dz}{dt} = \frac{x}{z} \cdot \frac{dx}{dt}$

$$= \frac{\sqrt{3}}{2} \frac{\text{mi}}{\text{mi}} \cdot 500 \frac{\text{mi}}{\text{h}}$$

$$= 250\sqrt{3} \text{ mi/h}$$

2. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = 300$

$$\boxed{f'(x) = 0}$$

(b) $f(x) = 5x^4 - x^2 + 3x$

$$f'(x) = 5 \cdot 4x^3 - 2x + 3 \cdot 1$$

$$= \boxed{20x^3 - 2x + 3}$$

(c) $f(x) = \frac{\sin^2(x)}{x^2}$

quotient
 ↘ chain. outside x^2
 inside $\sin(x)$

$$f'(x) = \frac{x^2 \cdot \frac{d}{dx} [\sin^2(x)] - \sin^2(x) \frac{d}{dx} [x^2]}{(x^2)^2}$$

$$= \frac{x^2 \cdot 2\sin(x) \cdot \frac{d}{dx} [\sin(x)] - \sin^2(x) \cdot 2x}{x^4}$$

$$= \frac{2x^2 \sin(x) \cos(x) - 2x \sin^2(x)}{x^4}$$

$$= \frac{2x \sin(x) (x \cos(x) - \sin(x))}{x^4}$$

$$= \boxed{\frac{2 \sin(x) (x \cos(x) - \sin(x))}{x^3}}$$

product

(d) $g(x) = \overbrace{x^2}^L \overbrace{\cos(x^2)}^R$

Chain
↓
out $\cos(x)$
in x^2

$$g'(x) = \cos(x^2) \cdot \frac{d}{dx}[x^2] + x^2 \cdot \frac{d}{dx}[\cos(x^2)]$$

$$= 2x \cos(x^2) + x^2 \cdot (-\sin(x^2)) \cdot \frac{d}{dx}[x^2]$$

$$= 2x \cos(x^2) - 2x^3 \sin(x^2)$$

$$= \boxed{2x (\cos(x^2) - x^2 \sin(x^2))}$$

(e) $f(x) = \left(\frac{x^2-1}{x^2+3}\right)^4$ chain

out x^4
in $\frac{x^2-1}{x^2+3}$

quotient

$$f'(x) = 4 \left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{d}{dx} \left[\frac{x^2-1}{x^2+3}\right]$$

$$= 4 \left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{(x^2+3) \cdot \frac{d}{dx}[x^2-1] - (x^2-1) \cdot \frac{d}{dx}[x^2+3]}{(x^2+3)^2}$$

$$= 4 \left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{(x^2+3) \cdot 2x - (x^2-1) \cdot 2x}{(x^2+3)^2}$$

$$= 4 \left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{2x^3 + 6x - 2x^3 + 2x}{(x^2+3)^2}$$

$$= 4 \frac{(x^2-1)^3}{(x^2+3)^3} \cdot \frac{8x}{(x^2+3)^2}$$

LoE 5

$$= \boxed{\frac{32x(x^2-1)^3}{(x^2+3)^5}}$$

frac law 1, thru LoE 1

3. The following three equations are in implicit form. Find $\frac{dy}{dx}$.

$$(a) 3x^2 + 2y = 2x^4 + 3y^2$$

$$3 \frac{d}{dx}[x^2] + 2 \frac{d}{dx}[y] = 2 \frac{d}{dx}[x^4] + 3 \frac{d}{dx}[y^2]$$

$$3 \cdot 2x + 2 \cdot \frac{dy}{dx} = 2 \cdot 4x^3 + 3 \cdot 2y \cdot \frac{dy}{dx}$$

$$6x - 8x^3 = 6y \frac{dy}{dx} - 2 \frac{dy}{dx}$$

$$6x - 8x^3 = \frac{dy}{dx}(6y - 2)$$

$$\frac{dy}{dx} = \frac{6x - 8x^3}{6y - 2} = \frac{2(3x - 4x^3)}{2(3y - 1)} = \boxed{\frac{3x - 4x^3}{3y - 1}}$$

$$(b) x^2 - 2xy + y^2 = 5$$

$$\frac{d}{dx}[x^2] - 2 \frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = \frac{d}{dx}[5]$$

$$2x - 2 \left(x \frac{d}{dx}[y] + y \cdot \frac{d}{dx}[x] \right) + 2y \frac{dy}{dx} = 0$$

$$2x - 2x \frac{dy}{dx} - 2y + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 2x$$

$$\frac{dy}{dx} = \frac{(2y - 2x)}{(2y - 2x)}$$

$$\boxed{\frac{dy}{dx} = 1}$$

out $\cos(x)$
chain in $x \cdot y$

(c) $\cos(xy) = 1 + \sin y$

$$\frac{d}{dx} [\cos(xy)] = \frac{d}{dx}[1] + \frac{d}{dx} [\sin(y)]$$

term with y !

$$-\sin(xy) \cdot \frac{d}{dx}[xy] = 0 + \cos(y) \cdot \frac{dy}{dx}$$

product rule

$$-\sin(xy) \cdot \left(x \cdot \frac{d}{dx}[y] + y \cdot \frac{d}{dx}[x] \right) = \cos(y) \frac{dy}{dx}$$

$$\underbrace{-\sin(xy) \cdot \frac{dy}{dx}}_{- \sin(xy)y} - \sin(xy)y = \cos(y) \frac{dy}{dx}$$

$$-\sin(xy)y = \cos(y) \frac{dy}{dx} + x \sin(xy) \frac{dy}{dx}$$

$$-\sin(xy)y = \frac{dy}{dx} \left(\cos(y) + x \sin(xy) \right)$$

$$\frac{dy}{dx} = \frac{-y \sin(xy)}{\cos(y) + x \sin(xy)}$$

4. Find the absolute minimum and maximum value, if any, of

$$f(x) = \frac{1}{8}x^2 - 4\sqrt{x} \quad [0, 9]$$

We can use the Closed Interval Method. EVT applies here.

① Find crit #'s.

$$\begin{aligned} f(x) &= \frac{1}{8}x^2 - 4x^{\frac{1}{2}} \\ f'(x) &= \frac{1}{4}x - 2x^{-\frac{1}{2}} \\ &= \frac{1}{4}x - \frac{2}{\sqrt{x}} \\ &= \frac{x}{4}\cdot\frac{\sqrt{x}}{\sqrt{x}} - \frac{2}{\sqrt{x}}\cdot\frac{4}{4} \\ &= \frac{x^{\frac{3}{2}}}{4\sqrt{x}} - \frac{8}{4\sqrt{x}} \\ &= \frac{x^{\frac{3}{2}} - 8}{4\sqrt{x}} \end{aligned}$$

one fraction!
Remember, numerator
is for $f'(x) = 0$,
denom is for $f'(x)$ DNE.

(a) Solve $f'(x) = 0$

$$\begin{aligned} 4\sqrt{x} \cdot \frac{x^{\frac{1}{2}} - 8}{4\sqrt{x}} &= 0 \cdot 4\sqrt{x} \\ x^{\frac{1}{2}} - 8 &= 0 \\ (x^{\frac{1}{2}})^{\frac{2}{3}} &= (8)^{\frac{2}{3}} \\ x = 8^{\frac{2}{3}} &= \sqrt[3]{8^2} \\ &= \sqrt[3]{(2^3)^2} \\ &= \sqrt[3]{26} = \boxed{4} \end{aligned}$$

(b) find where $f'(x)$ DNE. Set denom of $f'(x) = 0$.

$$\begin{aligned} 4\sqrt{x} &= 0 \\ \sqrt{x} &= 0 \\ x = 0^2 &= \circled{0} \end{aligned}$$

② Plug in crit #'s $x=0, x=4$

and endpoints.

$$f(0) = \frac{1}{8} \cdot 0^2 - 4\sqrt{0} = 0 \quad \leftarrow \text{largest}$$

$$f(4) = \frac{1}{8} \cdot 4^2 - 4\sqrt{4} = 2 - 8 = -6 \quad \leftarrow \text{smallest}$$

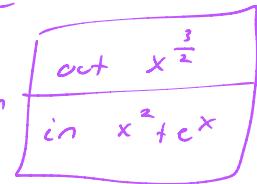
$$\begin{aligned} f(9) &= \frac{1}{8} \cdot 9^2 - 4\sqrt{9} = \frac{81}{8} - 12 \cdot \frac{8}{8} \\ &= \frac{81}{8} - \frac{96}{8} \\ &= -\frac{15}{8} \\ &= -1\frac{7}{8} \end{aligned}$$

∴ absolute maximum of $f(0) = 0$
absolute minimum of $f(4) = -6$

$$\frac{1}{e^{-x}} \stackrel{\text{def}}{=} \frac{1}{\frac{1}{e^x}} = 1 \cdot \frac{e^x}{1} = e^x$$

5. Find the derivative of the following functions:

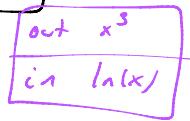
$$(a) f(x) = \left(x^2 + \frac{1}{e^{-x}} \right)^{3/2} = \left(x^2 + e^x \right)^{\frac{3}{2}}$$



$$f'(x) = \frac{3}{2} \left(x^2 + e^x \right)^{\frac{1}{2}} \cdot \frac{d}{dx} [x^2 + e^x]$$

$$= \boxed{\frac{3}{2} \sqrt{x^2 + e^x} \cdot (2x + e^x)}$$

$$(b) f(x) = (\ln(x))^3$$



$$f'(x) = 3 (\ln(x))^2 \cdot \frac{d}{dx} [\ln(x)]$$

$$= 3 (\ln(x))^2 \cdot \frac{1}{x}$$

$$= \boxed{\frac{3 (\ln(x))^2}{x}}$$

$$(c) f(x) = x^{\sin x} \quad \text{log diff b/c no laws can be used.}$$

$$y = x^{\sin x}$$

$$\ln y = \ln (x^{\sin x}) \quad \text{now implicitly differentiate}$$

$$\ln y = \sin(x) \cdot \ln(x)$$

$$\frac{d}{dx} [\ln(y)] = \frac{d}{dx} \left[\overset{L}{\sin(x)} \cdot \overset{R}{\ln(x)} \right] \quad \text{prod rule}$$

$$\frac{1}{y} \cdot y' = \ln(x) \cdot \frac{d}{dx} [\sin(x)] + \sin(x) \cdot \frac{d}{dx} [\ln(x)]$$

$$\frac{y'}{y} = \ln(x) \cos(x) + \frac{\sin(x)}{x}$$

$$y' = y \left(\ln(x) \cos(x) + \frac{\sin(x)}{x} \right) = \boxed{x^{\sin(x)} \left(\ln(x) \cos(x) + \frac{\sin(x)}{x} \right)}$$

6. Find the 4th degree Taylor polynomial of

$$f(x) = \cos(x)$$

at $a = 0$.

$$f(x) = \cos(x) \longrightarrow f(0) = \cos(0) = 1$$

$$f'(x) = -\sin(x) \longrightarrow f'(0) = -\sin(0) = 0$$

$$f''(x) = -\cos(x) \longrightarrow f''(0) = -\cos(0) = -1$$

$$f'''(x) = \sin(x) \longrightarrow f'''(0) = \sin(0) = 0$$

$$f^{(4)}(x) = \cos(x) \longrightarrow f^{(4)}(0) = \cos(0) = 1$$

So we have:

$$T_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$$

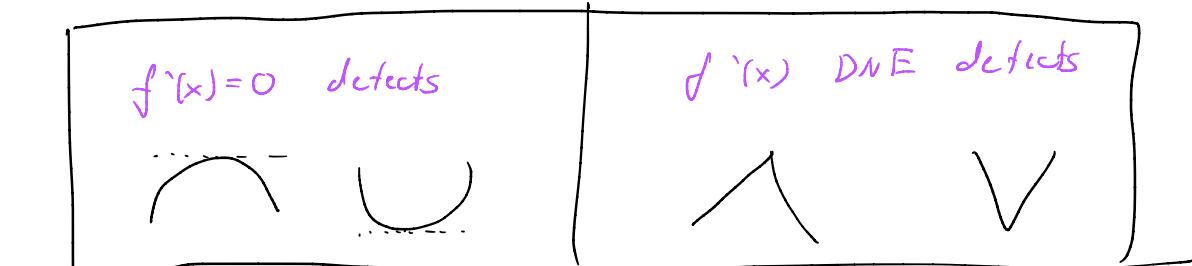
$$= 1 + 0 \cdot x - \frac{1}{2}x^2 + \frac{0}{6}x^3 + \frac{1}{24}x^4$$

$$= \boxed{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4}$$

7. Short answer questions:

- (a) What do critical numbers tell us when finding absolute minima/maxima?

Potential locations (x -values) of local minima/maxima, which could themselves be absolute minima/maxima. Recall:



- (b) Suppose $f(x)$ is continuous on \mathbb{R} and you find $f(1) = -1$ and $f(2) = -3$. Must $f(1)$ be an absolute maximum? Explain your answer with at least one reason.

No. EVT requires continuous on a closed interval \mathbb{R} to guarantee existence of absolute maxima.

\mathbb{R} is not a closed interval because the endpoints are not included.

- (c) What is the 100th degree Taylor polynomial of $f(x) = x$?

$$f(x) = x \rightarrow f(0) = 0$$

$$f'(x) = 1 \rightarrow f'(0) = 1$$

$$f''(x) = 0 \rightarrow f''(0) = 0$$

$$f'''(x) = 0 \rightarrow f'''(0) = 0$$

$$f^{(99)}(x) = 0 \rightarrow f^{(99)}(0) = 0$$

$$f^{(100)}(x) = 0 \rightarrow f^{(100)}(0) = 0$$

So only first degree Taylor exists.

$$T_{100}(x) = T_1(x)$$

$$= f(0) + f'(0)(x-0)$$

$$= 0 + 1(x-0)$$

$$= x$$

$$T_{100}(x) = x$$

Look! $T_{100}(x) = f(x)$

This is because $f(x)$ is a line. The best

linear approximation of

a line is the line itself.