MATH 161: Some Practice Final Problems

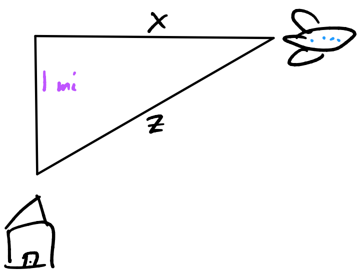
Here are problems that cover the last two weeks of our class.

Remember the final is cumulative; you should look at Practice Midterm 1+2 and Midterm 1+2 as well.

---

1. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.

(1)



x: distance from plane to straight above the station

z: distance between plane and radar station

functions of time!

(2) Given:

$$z = 2\text{ mi}, \quad \frac{dx}{dt} = 500\text{ mi}/h$$

Need:

$$\frac{dz}{dt}$$

(3) $x^2 + 1^2 = z^2$ Pythagorean Thm

(4) $\frac{d}{dt}[x^2] + \frac{d}{dt}[1^2] = \frac{d}{dt}[z^2]$

$$2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$$

missing from (2)! Find what x is.

$$\frac{dz}{dt} = \frac{2x}{2z}\frac{dx}{dt} = \frac{x}{z}\cdot\frac{dx}{dt}$$

(5) Now plug in z.

$$1^2 + x^2 = 2^2$$

$$x^2 = 4-1$$

$$x^2 = 3$$

$$\sqrt{x^2} = \pm\sqrt{3} \longrightarrow x = \sqrt{3}$$

(6) $\frac{dz}{dt} = \frac{x}{z}\cdot\frac{dx}{dt}$

$$= \frac{\sqrt{3}\,\cancel{\text{mi}}}{\cancel{2}\,\cancel{\text{mi}}}\cdot\frac{\overset{250}{\cancel{500}}\ \text{mi}}{h}$$

$$= \boxed{250\sqrt{3}\ \text{mi}/h}$$

2. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = 300$

$$\boxed{f'(x) = 0}$$

(b) $f(x) = 5x^4 - x^2 + 3x$

$$f'(x) = 5 \cdot 4x^3 - 2x + 3 \cdot 1$$

$$= \boxed{20x^3 - 2x + 3}$$

(c) $f(x) = \dfrac{\sin^2(x)}{x^2}$

quotient

chain. outside x^2 inside $\sin(x)$

$$f'(x) = \frac{x^2 \cdot \frac{d}{dx}\left[\sin^2(x)\right] - \sin^2(x) \frac{d}{dx}[x^2]}{(x^2)^2}$$

$$= \frac{x^2 \cdot 2\sin(x) \cdot \frac{d}{dx}[\sin(x)] - \sin^2(x) \cdot 2x}{x^4}$$

$$= \frac{2x^2 \sin(x)\cos(x) - 2x\sin^2(x)}{x^4}$$

$$= \frac{2\cancel{x}\sin(x)\left(x\cos(x) - \sin(x)\right)}{\cancel{x^4}\ x^3}$$

$$= \boxed{\frac{2\sin(x)\left(x\cos(x) - \sin(x)\right)}{x^3}}$$

product

(d) $g(x) = \underbrace{x^2}_{L}\underbrace{\cos(x^2)}_{R}$

Chain

out $\cos(x)$
in x^2

$$g'(x) = \cos(x^2)\cdot\frac{d}{dx}[x^2] - x^2\cdot\frac{d}{dx}[\cos(x^2)]$$

$$= 2x\cos(x^2) - x^2\cdot(-\sin(x^2))\cdot\frac{d}{dx}[x^2]$$

$$= 2x\cos(x^2) + 2x^3\sin(x^2)$$

$$= \boxed{2x\left(\cos(x^2) + x^2\sin(x^2)\right)}$$

(e) $f(x) = \left(\frac{x^2-1}{x^2+3}\right)^4$ chain

out	x^4
in	$\frac{x^2-1}{x^2+3}$

quotient

$$f'(x) = 4\left(\frac{x^2-1}{x^2+3}\right)^3\cdot\frac{d}{dx}\left[\frac{x^2-1}{x^2+3}\right]$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3\cdot\frac{(x^2+3)\cdot\frac{d}{dx}[x^2-1] - (x^2-1)\cdot\frac{d}{dx}[x^2+3]}{(x^2+3)^2}$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3\cdot\frac{(x^2+3)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+3)^2}$$

dist law

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3\cdot\frac{2x^3+6x-2x^3+2x}{(x^2+3)^2}$$

$$= 4\,\frac{(x^2-1)^3}{(x^2+3)^3}\cdot\frac{8x}{(x^2+3)^2}$$

LoE 5

$$= \boxed{\frac{32x(x^2-1)^3}{(x^2+3)^5}}$$

frac law 1, then LoE 1

3. The following three equations are in implicit form. Find $\frac{dy}{dx}$.

(a) $3x^2 + 2y = 2x^4 + 3y^2$

$$3\frac{d}{dx}[x^2] + 2\frac{d}{dx}[y] = 2\frac{d}{dx}[x^4] + 3\frac{d}{dx}[y^2]$$

$$3\cdot 2x + \underline{2\cdot\frac{dy}{dx}} = 2\cdot 4x^3 + \underline{3\cdot 2y\cdot\frac{dy}{dx}}$$

$$6x - 8x^3 = 6y\frac{dy}{dx} - 2\frac{dy}{dx}$$

$$6x - 8x^3 = \frac{dy}{dx}(6y - 2)$$

$$\frac{dy}{dx} = \frac{6x - 8x^3}{6y - 2} = \frac{\cancel{2}(3x - 4x^3)}{\cancel{2}(3y - 1)} = \boxed{\frac{3x - 4x^3}{3y - 1}}$$

(b) $x^2 - 2xy + y^2 = 5$

product rule

$$\frac{d}{dx}[x^2] - 2\frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = \frac{d}{dx}[5]$$

$$2x - 2\left(x\frac{d}{dx}[y] + y\cdot\frac{d}{dx}[x]\right) + 2y\frac{dy}{dx} = 0$$

$$2x \underline{- 2x\frac{dy}{dx}} - 2y \underline{+ 2y\frac{dy}{dx}} = 0$$

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 2x$$

$$\frac{dy}{dx} = \frac{\cancel{(2y - 2x)}}{\cancel{(2y - 2x)}}$$

$$\boxed{\frac{dy}{dx} = 1}$$

out cos(x)

chain

in $x \cdot y$

(c) $\cos(xy) = 1 + \sin y$

$$\frac{d}{dx}\left[\cos(xy)\right] = \frac{d}{dx}[1] + \frac{d}{dx}\left[\sin(y)\right]$$

term with y!

product rule

$$-\sin(xy) \cdot \frac{d}{dx}[xy] = 0 + \cos(y) \cdot \frac{dy}{dx}$$

$$-\sin(xy) \cdot \left(x \cdot \frac{d}{dx}[y] + y \cdot \frac{d}{dx}[x]\right) = \cos(y)\frac{dy}{dx}$$

$$-\sin(xy)\, x \frac{dy}{dx} - \sin(xy)\, y = \cos(y)\frac{dy}{dx}$$

$$-\sin(xy)\, y = \cos(y)\frac{dy}{dx} + x\sin(xy)\frac{dy}{dx}$$

$$-\sin(xy)\, y = \frac{dy}{dx}\left(\cos(y) + x\sin(xy)\right)$$

$$\boxed{\frac{dy}{dx} = \frac{-y\sin(xy)}{\cos(y) + x\sin(xy)}}$$

4. Find the absolute minimum and maximum value, if any, of

$$f(x) = \frac{1}{8}x^2 - 4\sqrt{x} \qquad [0, 9]$$

We can use the Closed Interval Method. EVT applies here.

① Find crit #'s.

$$f(x) = \frac{1}{8}x^2 - 4x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{4}x - 2x^{-\frac{1}{2}}$$

$$= \frac{1}{4}x - \frac{2}{\sqrt{x}}$$

$$= \frac{x}{4}\cdot\frac{\sqrt{x}}{\sqrt{x}} - \frac{2}{\sqrt{x}}\cdot\frac{4}{4}$$

$$= \frac{x^{\frac{3}{2}}}{4\sqrt{x}} - \frac{8}{4\sqrt{x}}$$

$$= \frac{x^{\frac{3}{2}} - 8}{4\sqrt{x}}$$ ← one fraction!

Remember, numerator is for $f'(x) = 0$, denom is for $f'(x)$ DNE.

ⓐ Solve $f'(x) = 0$

$$4\sqrt{x}\cdot\frac{x^{\frac{3}{2}} - 8}{4\sqrt{x}} = 0\cdot 4\sqrt{x}$$

$$x^{\frac{3}{2}} - 8 = 0$$

$$\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = (8)^{\frac{2}{3}}$$

$$x = 8^{\frac{2}{3}} = \sqrt[3]{8^2}$$

$$= \sqrt[3]{(2^3)^2}$$

$$= \sqrt[3]{2^6} = \boxed{4}$$

ⓑ find when $f'(x)$ DNE. Set denom of $f'(x) = 0$.

$$4\sqrt{x} = 0$$

$$\sqrt{x} = 0$$

$$x = 0^2 = 0$$

② Plug in crit #'s $x = 0, x = 4$ and endpoints.

$$f(0) = \frac{1}{8}\cdot 0^2 - 4\sqrt{0} = 0$$ ← largest

$$f(4) = \frac{1}{8}\cdot 4^2 - 4\sqrt{4} = 2 - 8 = -6$$ ← Smallest

$$f(9) = \frac{1}{8}\cdot 9^2 - 4\sqrt{9} = \frac{81}{8} - 12\cdot\frac{8}{8}$$

$$= \frac{81}{8} - \frac{96}{8}$$

$$= -\frac{15}{8}$$

$$= -1\frac{7}{8}$$

∴, absolute maximum of $f(0) = 0$

absolute minimum of $f(4) = -6$

$$\frac{1}{e^{-x}} \overset{\text{def}}{=} \frac{1}{\frac{1}{e^x}} = 1 \cdot \frac{e^x}{1} = e^x$$

chain

out	$x^{\frac{3}{2}}$
in	$x^2 + e^x$

5. Find the derivative of the following functions:

(a) $f(x) = \left(x^2 + \frac{1}{e^{-x}}\right)^{3/2} = \left(x^2 + e^x\right)^{\frac{3}{2}}$

$$f'(x) = \frac{3}{2}\left(x^2 + e^x\right)^{\frac{1}{2}} \cdot \frac{d}{dx}\left[x^2 + e^x\right]$$

$$= \boxed{\frac{3}{2}\sqrt{x^2 + e^x} \cdot \left(2x + e^x\right)}$$

chain

out	x^3
in	$\ln(x)$

(b) $f(x) = (\ln(x))^3$

$$f'(x) = 3\left(\ln(x)\right)^2 \cdot \frac{d}{dx}\left[\ln(x)\right]$$

$$= 3\left(\ln(x)\right)^2 \cdot \frac{1}{x}$$

$$= \boxed{\frac{3\left(\ln(x)\right)^2}{x}}$$

(c) $f(x) = x^{\sin x}$ log diff b/c no laws can be used.

$$y = x^{\sin x}$$

$$\ln y = \ln\left(x^{\sin(x)}\right)$$

now implicitly differentiate

$$\ln y = \sin(x) \cdot \ln(x)$$

prod rule

$$\frac{d}{dx}\left[\ln(y)\right] = \frac{d}{dx}\left[\overset{L}{\overline{\sin(x)}} \cdot \overset{R}{\overline{\ln(x)}}\right]$$

$$\frac{1}{y} \cdot y' = \ln(x) \cdot \frac{d}{dx}\left[\sin(x)\right] + \sin(x) \cdot \frac{d}{dx}\left[\ln(x)\right]$$

$$\frac{y'}{y} = \ln(x)\cos(x) + \frac{\sin(x)}{x}$$

$$y' = y\left(\ln(x)\cos(x) + \frac{\sin(x)}{x}\right) = \boxed{x^{\sin(x)}\left(\ln(x)\cos(x) + \frac{\sin(x)}{x}\right)}$$

6. Find the 4th degree Taylor polynomial of

$$f(x) = \cos(x)$$

at $a = 0$.

$$f(x) = \cos(x) \longrightarrow f(0) = \cos(0) = 1$$

$$f'(x) = -\sin(x) \longrightarrow f'(0) = -\sin(0) = 0$$

$$f''(x) = -\cos(x) \longrightarrow f''(0) = -\cos(0) = -1$$

$$f'''(x) = \sin(x) \longrightarrow f'''(0) = \sin(0) = 0$$

$$f^{(4)}(x) = \cos(x) \longrightarrow f^{(4)}(0) = \cos(0) = 1$$

So we have:

$$T_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$$

$$= f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(x)}{24}x^4$$

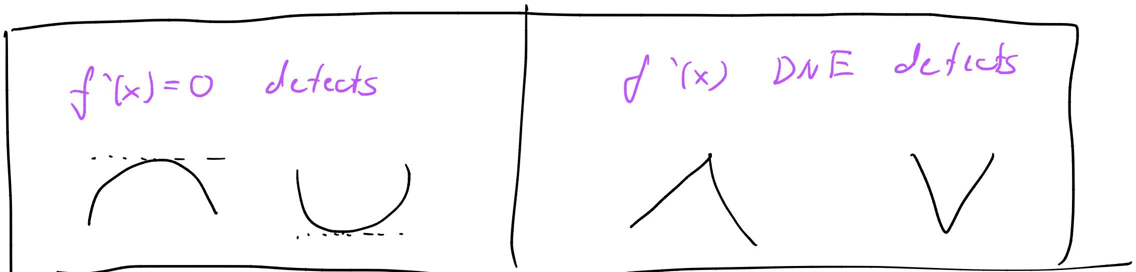
$$= 1 + 0 \cdot x - \frac{1}{2}x^2 + \frac{0}{6}x^3 + \frac{1}{24}x^4$$

$$= \boxed{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4}$$

7. Short answer questions:

(a) What do critical numbers tell us when finding absolute minima/maxima?

Potential locations (x-values) of local minima/maxima, which could themselves be absolute minima/maxima. Recall:



(b) Suppose $f(x)$ is continuous on $\mathbb{R}$ and you find $f(1) = -1$ and $f(2) = -3$. Must $f(1)$ be an absolute maximum? Explain your answer with at least one reason.

No. EVT requires continuous on a closed interval $\mathbb{R}$ to guarantee existence of absolute maxima.

$\mathbb{R}$ is not a closed interval because the endpoints are not included.

(c) What is the 100th degree Taylor polynomial of $f(x) = x$? forgot to add at $a = 0$.

$f(x) = x \longrightarrow f(0) = 0$

$f'(x) = 1 \longrightarrow f'(0) = 1$

$f''(x) = 0 \longrightarrow f''(0) = 0$

$f'''(x) = 0 \longrightarrow f'''(0) = 0$

$\vdots$

$f^{(99)}(x) = 0 \longrightarrow f^{(99)}(0) = 0$

$f^{(100)}(x) = 0 \longrightarrow f^{(100)}(0) = 0$

So only first degree Taylor exists.

$$T_{100}(x) = T_1(x)$$
$$= f(0) + f'(0)(x-0)$$
$$= 0 + 1(x-0)$$
$$= x$$

$$\boxed{T_{100}(x) = x}$$

Look! $T_{100}(x) = f(x)$ This is because $f(x)$ is a line. The best linear approximation of a line is the line itself.