MATH 119: Midterm 2

Name: key

Directions:

- Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
- If you do not know how to solve a problem, try your best and/or explain in English what you would do.
- Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
		120

conceptual understanding

execution

common pitfalls

1. Prove the identity

$$\frac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$$

Know these. Recognize them.

① $(A + B)^2 = A^2 + 2AB + B^2$

② $(A - B)^2 = A^2 - 2AB + B^2$

③ $A^2 - B^2 = (A - B)(A + B)$

$$LHS = \frac{\overbrace{(\sin x + \cos x)^2}^{(A + B)^2}}{\underbrace{\sin^2 x - \cos^2 x}_{A^2 - B^2}} = \frac{(\sin x + \cos x)\cancel{(\sin x + \cos x)}}{\underbrace{(\sin x - \cos x)\cancel{(\sin x + \cos x)}}_{(A - B)(A + B)}}$$

factor denom!

$$= \frac{(\sin x + \cos x)}{(\sin x - \cos x)} \cdot \frac{(\sin x - \cos x)}{(\sin x - \cos x)}$$

why? RHS missing it!

$$= \frac{(\sin x + \cos x)(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

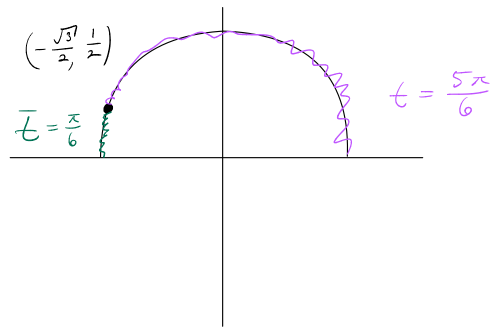
$$= \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2} = RHS$$

$\blacksquare$

2. Evaluate the following:

(a) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

$\cos(x)$ 1-1 on $[0, \pi]$



(b) $\cos 112.5°$

On next page.

(c) $\sin 75° = \sin(30° + 45°)$

$30° \cdot \frac{\pi}{180} = \frac{\pi}{6}$ rad

$45° \cdot \frac{\pi}{180} = \frac{\pi}{4}$ rad.

$= \sin(30°)\cos(45°) + \cos(30°)\sin(45°)$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$

$= \boxed{\frac{\sqrt{6} + \sqrt{2}}{4}}$

2b. $\cos 112.5°$

when I see .5, I suspect 112.5 is half of the desired angle.

$$\cos(112.5°) = \cos\left(\frac{1}{2}\cdot 225°\right)$$

because x coordinate of 112.5° is in II, negative.

$$= -\sqrt{\frac{1+\cos 225°}{2}}$$

half angle

$$= -\sqrt{\frac{1+\cos(180°+45°)}{2}}$$

prepare for addition identity

$$= -\sqrt{\frac{1+\cos(180°)\cos(45°)-\sin(180°)\sin(45°)}{2}}$$

$$= -\sqrt{\frac{1+1\cdot\frac{\sqrt{2}}{2}-0\cdot\frac{\sqrt{2}}{2}}{2}}$$

compound frac.

$$= -\sqrt{\frac{\frac{2}{2}+\frac{\sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2+\sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2+\sqrt{2}}}{\sqrt{4}} = \boxed{-\frac{\sqrt{2+\sqrt{2}}}{2}}$$

3. Solve the equation for θ. Check your work if necessary.

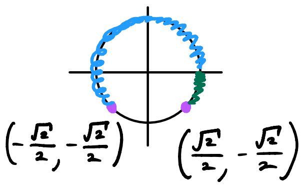
(a) $\sqrt{2}\sin\theta + 1 = 0$

isolate $\sin\theta$.

$$\sqrt{2}\sin\theta = -1$$

$$\sin\theta = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

① in one period.



$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ $\quad \frac{7\pi}{4}$

$$\theta = \frac{5\pi}{4}, \quad \theta = \frac{7\pi}{4}$$

② account for periodicity

period of $\sin\theta$ is 2π long.

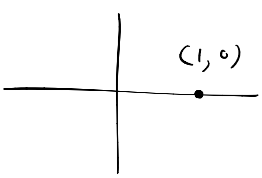
$$\boxed{\begin{aligned} \theta &= \frac{5\pi}{4} + 2k\pi \\ \theta &= \frac{7\pi}{4} + 2k\pi, \end{aligned} \quad k \in \mathbb{Z}}$$

(b) $\sqrt{2}\tan\theta\sin\theta - \tan\theta = 0$

$$\underbrace{\tan\theta}_{\text{factor}} \cdot \underbrace{\left(\sqrt{2}\sin\theta - 1\right)}_{\text{factor}} = 0$$

$$\tan\theta = 0$$

① Solutions in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$$\theta = 0$$

② account for periodicity.

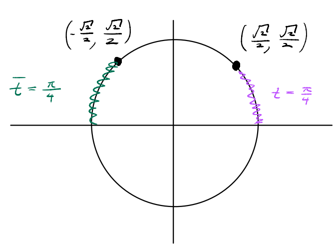
$$\boxed{\begin{aligned} \theta &= 0 + k\pi \\ &= k\pi \quad k \in \mathbb{Z} \end{aligned}}$$

$$\sqrt{2}\sin\theta - 1 = 0$$

$$\sqrt{2}\sin\theta = 1$$

$$\sin\theta = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

① Solutions on $(0, 2\pi)$



$$\theta = \frac{\pi}{4}, \quad \theta = \frac{3\pi}{4}$$

② Account for periodicity

$$\boxed{\begin{aligned} \theta &= \frac{\pi}{4} + 2k\pi \\ \theta &= \frac{3\pi}{4} + 2k\pi \end{aligned} \quad k \in \mathbb{Z}}$$

4. Prove these identities algebraically:

(a) $\dfrac{\sin\theta}{\tan\theta} = \cos\theta$

$$LHS = \frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\sin\theta}{1} \cdot \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\cancel{\sin\theta}\cos\theta}{\cancel{\sin\theta}}$$

$$= \cos\theta = RHS \quad \blacksquare$$

$$\frac{a}{\frac{b}{c}} = a \cdot \frac{c}{b}$$

(b) $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

$$\frac{1}{\frac{b}{c}} = 1 \cdot \frac{c}{b}$$

$$LHS = \frac{\cos x}{\sec x} + \frac{\sin x}{\csc x} \overset{\text{trig function definition}}{=} \frac{\cos x}{\frac{1}{\cos x}} + \frac{\sin x}{\frac{1}{\sin x}} = \cos x \cdot \cos x + \sin x \cdot \sin x$$

$$= \cos^2 x + \sin^2 x$$

$$= 1 \qquad \text{ch 5 identity}$$

$$= RHS \quad \blacksquare$$

(c) $\cos^4 x - \sin^4 x = \cos 2x$

$$LHS = \cos^4 x - \sin^4 x = \underbrace{(\cos^2 x - \sin^2 x)}_{\cos(2x)}\underbrace{(\cos^2 x + \sin^2 x)}_{1}$$

$$= \cos(2x) \cdot 1$$

$$= \cos 2x$$

$$= RHS \quad \blacksquare$$

Let $A = \cos^2 x$, $B = \sin^2 x$

Then $\cos^4 x - \sin^4 x \overset{\text{LoE}}{=} (\cos^2 x)^2 - (\sin^2 x)^2$

$$= A^2 - B^2$$

$$= (A-B)(A+B)$$

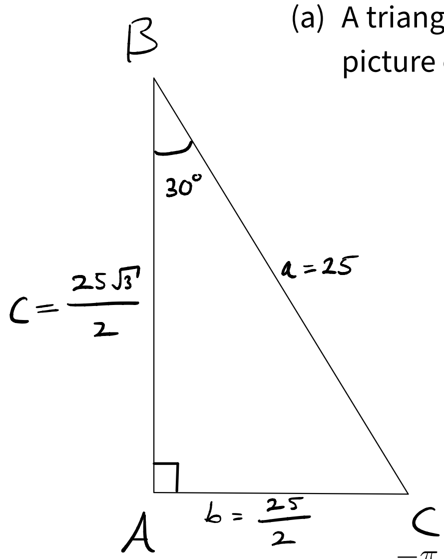
Special product

$$= \boxed{(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)}$$

$30^\circ \cdot \frac{\pi}{180} = \frac{\pi}{6}$

5. Answer the following:

   (a) A triangle ABC has $\angle A = 90^\circ$, $\angle B = 30^\circ$ and $A = 25$. Solve the triangle and draw a picture of it.



Goal: find, b, c, $\angle C$

For $\angle C$:

$\angle A + \angle B + \angle C = 180^\circ$

$90^\circ + 30^\circ + \angle C = 180^\circ$

$\boxed{\angle C = 60^\circ}$

For b: $\sin(30^\circ) = \frac{b}{25}$

$b = 25 \sin\left(\frac{\pi}{6}\right)$

$= 25 \cdot \frac{1}{2}$

$= \boxed{\frac{25}{2}}$

For c: $\cos(30^\circ) = \frac{c}{25}$

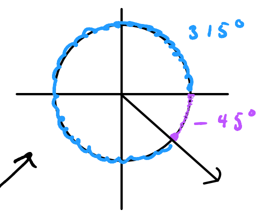
$c = 25 \cos(30^\circ)$

$= 25 \cos\left(\frac{\pi}{6}\right) = 25 \cdot \frac{\sqrt{3}}{2}$

$= \boxed{\frac{25\sqrt{3}}{2}}$

   (b) Are $\frac{-\pi}{4}$ rad and 315° coterminal? Show with calculations.

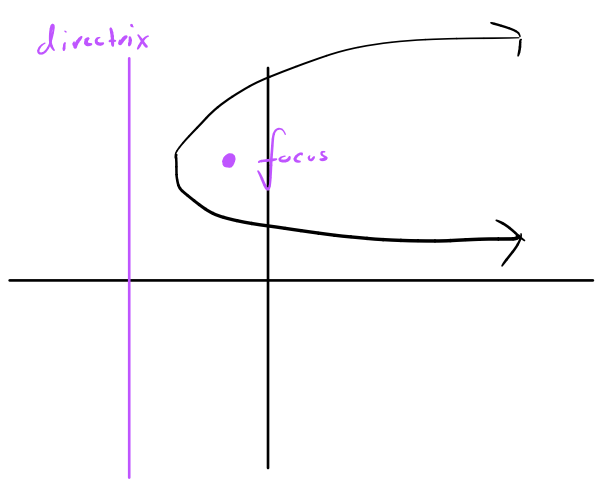
$-\frac{\cancel{\pi}}{\cancel{4}} \cdot \frac{\overset{45^\circ}{\cancel{180^\circ}}}{\cancel{\pi}} = -45^\circ$



A full circle is highlighted since $315^\circ + 45^\circ = 360^\circ$

$\boxed{\text{The terminal sides therefore lie on top of each other so they are coterminal.}}$

   (c) If the directrix of a parabola is a vertical line and the focus is to the right of the directrix, which way does the parabola open?



$\boxed{\text{right}}$

6. Answer the following: find r and θ

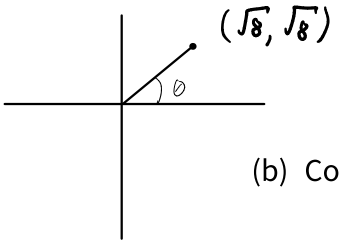
(a) Convert $(\sqrt{8}, \sqrt{8})$ into polar coordinates.

x y

$$r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{8})^2 + (\sqrt{8})^2} = \sqrt{8+8} = \sqrt{16} = 4$$

$$\tan\theta = \frac{\sqrt{8}}{\sqrt{8}} = 1$$

$$\boxed{\left(4, \frac{\pi}{4}\right)}$$



looking here $\theta = \frac{\pi}{4}$, only way to point in quadrant 1.

(b) Convert $r = \dfrac{1}{1+\sin\theta}$ into rectangular form.

use this r to make a $r\sin\theta$. Missing r!

$$(1+\sin\theta)\cdot r = \frac{1}{\cancel{1+\sin\theta}} \cdot \cancel{(1+\sin\theta)}$$

$$r + r\sin\theta = 1 \quad \text{dist law}$$

$$r + y = 1$$

$$r = 1 - y \quad \text{isolate } r$$

$$r^2 = (1-y)^2 \quad \text{create } r^2$$

$$\boxed{x^2 + y^2 = (1-y)^2} \quad \text{in order to use } r^2 = x^2 + y^2$$

(c) Convert $r = 6\cos\theta$ into rectangular form.

both missing r to use formulas. multiply!

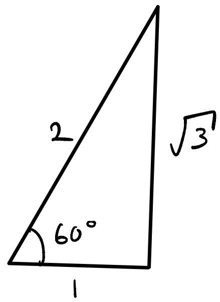
$$r \cdot r = 6\cos(\theta)\cdot r$$

$$r^2 = 6r\cos(\theta)$$

$$\boxed{x^2 + y^2 = 6x}$$

7. Simplify the following trigonometric expressions:

(a) $\sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ$ $= \sin(20^\circ + 40^\circ)$



$$= \sin(60^\circ)$$

$$= \boxed{\frac{\sqrt{3}}{2}}$$

generates 2 terms

subtracting!

(b) $$\frac{\sin(x+y) - \sin(x-y)}{\cos(x+y) + \cos(x-y)} = \frac{\sin(x)\cos(y) + \cos(x)\sin(y) - \left(\sin(x)\cos(y) - \cos(x)\sin(y)\right)}{\cos(x)\cos(y) - \cancel{\sin(x)\sin(y)} + \cos(x)\cos(y) + \cancel{\sin(x)\sin(y)}}$$

four applications of addition/subtraction formula

$$\overset{\text{dist}}{=} \frac{\cancel{\sin(x)\cos(y)} + \cos(x)\sin(y) - \cancel{\sin(x)\cos(y)} + \cos(x)\sin(y)}{2\cos(x)\cos(y)}$$

$$= \frac{\cancel{2}\cdot\cancel{\cos(x)}\cdot\sin(y)}{\cancel{2}\cdot\cancel{\cos(x)}\cdot\cos(y)}$$

frac law 5

$$= \frac{\sin(y)}{\cos(y)}$$

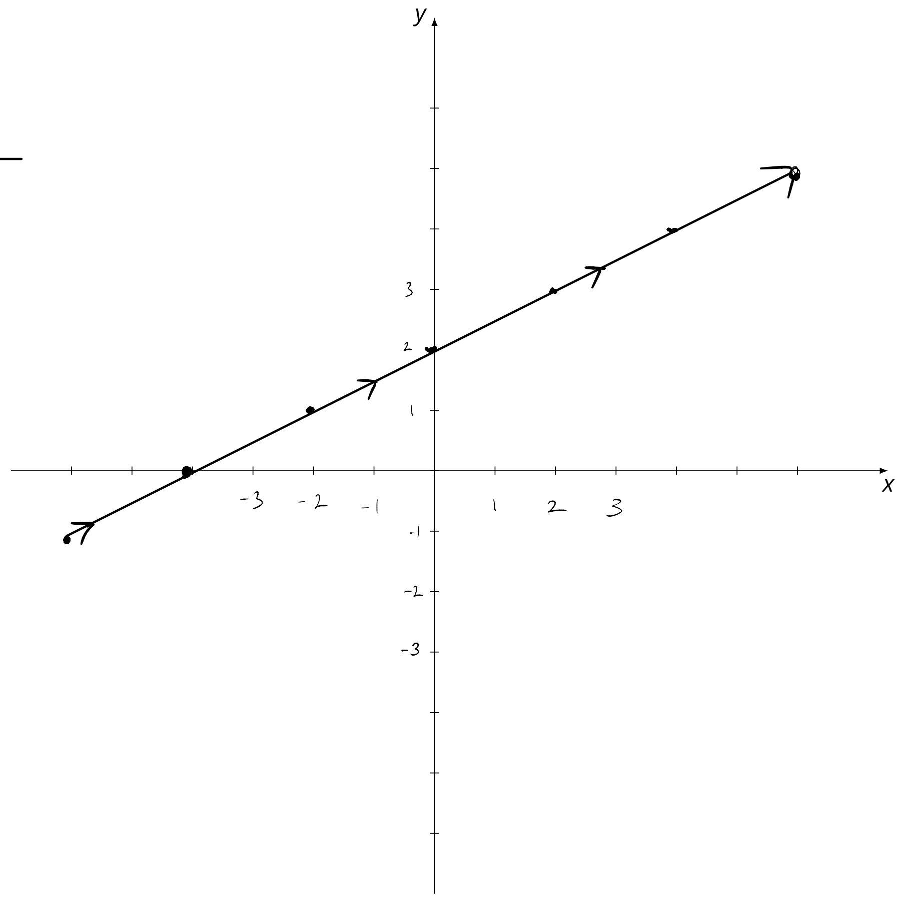
$$= \boxed{\tan(y)}$$

8. Here is a pair of parametric equations

$$x = 2t \qquad y = t + 2$$

(a) Sketch the curve represented by the equations.

t	x	y
-3	-6	-1
-2	-4	0
-1	-2	1
0	0	2
1	2	3
2	4	4
3	6	5



(b) Find a rectangular coordinate equation for the curve by eliminating the parameter.

$$y = t + 2 \xrightarrow[t]{\text{solve for}} t = y - 2 \xrightarrow{\text{substitute}} x = 2(y-2)$$

$$\xrightarrow[\text{for } y]{\text{solve}} \quad x = 2y - 4$$

if you can.

$$2y = x + 4 \longrightarrow y = \frac{x+4}{2} = \frac{x}{2} + \frac{4}{2} = \frac{1}{2}x + 2$$

$$\boxed{y = \frac{1}{2}x + 2}$$