MATH 141: Midterm 2

Name: key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* **Remember to simplify each expression.**
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
		50

1. Suppose $f(x) = 3x^2 - x$.

(a) What does the expression $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ represent?

The derivative of $f(x)$, or the slope of the tangent line.

(b) Find

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for the given function $f(x)$. You must use this limit definition to receive credit.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - (x+h) - (3x^2 - x)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - x - h - 3x^2 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x + 3h - 1$$

$$= 6x + 0 - 1$$

$$= \boxed{6x - 1}$$

(c) Find the equation of the tangent line of $f(x)$ at the point $(1, 2)$.

$f'(1) = 6 \cdot 1 - 1 = 5$

$y - 2 = 5 \cdot (x - 1)$

$y = 5x - 5 + 2$

$\boxed{y = 5x - 3}$

2. Short answer questions:

   (a) If a function $f(x)$ is differentiable at $x = a$, must it be continuous at $x = a$ as well? If not, draw a graph of a function that is differentiable but not continuous at $x = a$.

   yes.

   (b) True or false:

   $$f(x) = x^2 \cdot \sin x$$

   is continuous on $\mathbb{R}$.

   True, domain of x^2 is $\mathbb{R}$, $\sin(x)$ is $\mathbb{R}$,

   Product still has domain $\mathbb{R}$.

   (c) Given $f(x) = x^2$, find an equation of the normal line at $(1, 1)$.

   $f'(x) = 2x$

   Slope of normal at $(1,1)$ is $-\frac{1}{f'(1)} = -\frac{1}{2}$

   $$y - 1 = -\frac{1}{2}(x-1)$$

   $$y = -\frac{1}{2}x + \frac{1}{2} + 1$$

   $$\boxed{y = -\frac{1}{2}x + \frac{3}{2}}$$

3. Answer the following:

(a) Given a function $f(x)$, if

$$\boxed{\lim_{x\to a}} f(x) = \frac{0}{0}$$

what global factor do you need to manifest in the numerator and denominator and why?

$(x-a)$

$\frac{0}{0}$ if try lim laws.

(b) Find

$$\lim_{x\to 0}\left[\frac{1}{x\sqrt{1+x}} - \frac{1}{x}\right]$$

needs factor of $\sqrt{1+x}$

Hint: Subtract to get one fraction first.

$$\lim_{x\to 0}\left[\frac{1}{x\sqrt{1+x}} - \frac{1}{x}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\right] = \lim_{x\to 0}\left[\frac{1}{x\sqrt{1+x}} - \frac{\sqrt{1+x}}{x\sqrt{1+x}}\right]$$

rationalize to create glob. factor of x.

$$= \lim_{x\to 0}\frac{1-\sqrt{1+x}}{x\sqrt{1+x}}\cdot\frac{1+\sqrt{1+x}}{1+\sqrt{1+x}}$$

$$= \lim_{x\to 0}\frac{1^2-(\sqrt{1+x})^2}{x\sqrt{1+x}\,(1+\sqrt{1+x})}$$

$$= \lim_{x\to 0}\frac{1-(1+x)}{x\sqrt{1+x}\,(1+\sqrt{1+x})}$$

$$= \lim_{x\to 0}\frac{-\cancel{x}}{\cancel{x}\sqrt{1+x}\,(1+\sqrt{1+x})}$$

$$= \lim_{x\to 0}\frac{-1}{\sqrt{1+x}\,(1+\sqrt{1+x})}$$

$$= \frac{-1}{\sqrt{1+0}\,(1+\sqrt{1+0})}$$

continuity

$$= \boxed{-\frac{1}{2}}$$

$\frac{0}{0}$ if try lim laws.

(c) Find

$$\lim_{h\to 0} \frac{\frac{1}{(x+h+1)} - \frac{1}{(x+1)}}{h}$$

compound fraction.

$$\lim_{h\to 0} \frac{\frac{(x+1)}{(x+1)} \cdot \frac{1}{(x+h+1)} - \frac{1}{(x+1)} \cdot \frac{(x+h+1)}{(x+h+1)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}}{h}$$

free law 1 then 3

$$= \lim_{h\to 0} \frac{\frac{x+1-x-h-1}{(x+1)(x+h+1)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{-h}{(x+1)(x+h+1)}}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(x+1)(x+h+1)}$$

!! free law 2 then 1

$$= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)}$$

$$= \frac{-1}{(x+1)(x+0+1)}$$

continuity

$$= \boxed{-\frac{1}{(x+1)^2}}$$

4. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = 300$

$$\boxed{f'(x) = 0}$$

(b) $f(x) = 5x^4 - x^2 + 3x$

$$f'(x) = 5 \cdot 4x^3 - 2x + 3 \cdot 1$$

$$= \boxed{20x^3 - 2x + 3}$$

(c) $f(x) = \dfrac{\sin^2(x)}{x^2}$ quotient

chain. outside x^2 inside $\sin(x)$

$$f'(x) = \frac{x^2 \cdot \frac{d}{dx}\left[\sin^2(x)\right] - \sin^2(x)\frac{d}{dx}[x^2]}{(x^2)^2}$$

$$= \frac{x^2 \cdot 2\sin(x) \cdot \frac{d}{dx}[\sin(x)] - \sin^2(x) \cdot 2x}{x^4}$$

$$= \frac{2x^2 \sin(x)\cos(x) - 2x\sin^2(x)}{x^4}$$

$$= \frac{2\cancel{x}\sin(x)\left(x\cos(x) - \sin(x)\right)}{\cancel{x^4}\ x^3}$$

$$= \boxed{\frac{2\sin(x)\left(x\cos(x) - \sin(x)\right)}{x^3}}$$

product

(d) $g(x) = \underbrace{x^2}_{L}\,\underbrace{\cos(x^2)}_{R}$

chain

out $\cos(x)$

in x^2

$$g'(x) = \cos(x^2)\cdot\frac{d}{dx}[x^2] - x^2\cdot\frac{d}{dx}[\cos(x^2)]$$

$$= 2x\cos(x^2) - x^2\cdot(-\sin(x^2))\cdot\frac{d}{dx}[x^2]$$

$$= 2x\cos(x^2) + 2x^3\sin(x^2)$$

$$= \boxed{2x\left(\cos(x^2) + x^2\sin(x^2)\right)}$$

(e) $f(x) = \left(\frac{x^2-1}{x^2+3}\right)^4$ chain

out	x^4
in	$\frac{x^2-1}{x^2+3}$

quotient

$$f'(x) = 4\left(\frac{x^2-1}{x^2+3}\right)^3\cdot\frac{d}{dx}\left[\frac{x^2-1}{x^2+3}\right]$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3\cdot\frac{(x^2+3)\cdot\frac{d}{dx}[x^2-1] - (x^2-1)\cdot\frac{d}{dx}[x^2+3]}{(x^2+3)^2}$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3\frac{(x^2+3)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+3)^2}$$

dist law

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3\cdot\frac{2x^3+6x-2x^3+2x}{(x^2+3)^2}$$

$$= 4\,\frac{(x^2-1)^3}{(x^2+3)^3}\cdot\frac{8x}{(x^2+3)^2}$$

LoE 5

$$= \boxed{\frac{32x(x^2-1)^3}{(x^2+3)^5}}$$

frac law 1, then LoE 1

5. The following three equations are in implicit form. Find $\frac{dy}{dx}$.

(a) $3x^2 + 2y = 2x^4 + 3y^2$

$$3\frac{d}{dx}[x^2] + 2\frac{d}{dx}[y] = 2\frac{d}{dx}[x^4] + 3\frac{d}{dx}[y^2]$$

$$3 \cdot 2x + \underline{2 \cdot \frac{dy}{dx}} = 2 \cdot 4x^3 + \underline{3 \cdot 2y \cdot \frac{dy}{dx}}$$

$$6x - 8x^3 = 6y\frac{dy}{dx} - 2\frac{dy}{dx}$$

$$6x - 8x^3 = \frac{dy}{dx}(6y - 2)$$

$$\frac{dy}{dx} = \frac{6x - 8x^3}{6y - 2} = \frac{\cancel{2}(3x - 4x^3)}{\cancel{2}(3y - 1)} = \boxed{\frac{3x - 4x^3}{3y - 1}}$$

(b) $x^2 - 2xy + y^2 = 5$

product rule

$$\frac{d}{dx}[x^2] - 2\frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = \frac{d}{dx}[5]$$

$$2x - 2\left(x\frac{d}{dx}[y] + y \cdot \frac{d}{dx}[x]\right) + 2y\frac{dy}{dx} = 0$$

$$2x \underline{- 2x\frac{dy}{dx}} - 2y \underline{+ 2y\frac{dy}{dx}} = 0$$

$$2y\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - 2x$$

$$\frac{dy}{dx}(2y - 2x) = 2y - 2x$$

$$\frac{dy}{dx} = \frac{\cancel{(2y - 2x)}}{\cancel{(2y - 2x)}}$$

$$\boxed{\frac{dy}{dx} = 1}$$

out $\cos(x)$

chain in $x \cdot y$

(c) $\cos(xy) = 1 + \sin y$

$$\frac{d}{dx}\left[\cos(xy)\right] = \frac{d}{dx}[1] + \frac{d}{dx}\left[\sin(y)\right]$$

term with y!

product rule

$$-\sin(xy) \cdot \frac{d}{dx}[xy] = 0 + \cos(y) \cdot \frac{dy}{dx}$$

$$-\sin(xy) \cdot \left(x \cdot \frac{d}{dx}[y] + y \cdot \frac{d}{dx}[x]\right) = \cos(y) \frac{dy}{dx}$$

$$-\sin(xy)\, x \frac{dy}{dx} - \sin(xy)\, y = \cos(y) \frac{dy}{dx}$$

$$-\sin(xy)\, y = \cos(y) \frac{dy}{dx} + x \sin(xy) \frac{dy}{dx}$$

$$-\sin(xy)\, y = \frac{dy}{dx}\left(\cos(y) + x \sin(xy)\right)$$

$$\boxed{\frac{dy}{dx} = \frac{-y \sin(xy)}{\cos(y) + x \sin(xy)}}$$