MATH 118: Midterm 2

Name: Key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
		60

1. Short answer questions:

   (a) Given three functions

$$f(x) = \sqrt{x-2}, \qquad g(x) = \frac{2x}{3x+1}, \qquad h(x) = x^4$$

   find the composition $g \circ f \circ h$.

$$(g \circ f \circ h)(x) = g(f(h(x))) = g(f(x^4)) = g(\sqrt{x^4-2}) = \boxed{\frac{2\sqrt{x^4-2}}{3\sqrt{x^4-2}+1}}$$

   (b) Given a polynomial $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)$, should $f(4.5)$ be positive or negative and why?

$$f(4.5) = (4.5-1)\cdot(4.5-2)\cdot(4.5-3)\cdot(4.5-4)\cdot(4.5-5)$$

$$= - \cdot - \cdot - \cdot - \cdot +$$

$$= \boxed{+}$$

   (c) Find a degree four polynomial with zeros $i\sqrt{3}$ and $5i$.

$$P(x) = (x - i\sqrt{3})(x + i\sqrt{3})(x - 5i)(x + 5i)$$

   (d) Given a base function $f(x) = \sqrt{x}$ and two transformed functions

$$g(x) = \sqrt{x-2} \qquad\qquad h(x) = \sqrt{\frac{1}{2}x - 2}$$

   do both $g(x)$ and $h(x)$ have the same horizontal shift from $f(x)$? If not, state both of the horizontal shifts of $g(x)$ and $h(x)$.

No. $h(x) = \sqrt{\frac{1}{2}(x-4)}$

$g(x)$ is shifted 2 units to the right from $f(x)$

$h(x)$ is shifted 4 units to the right from $f(x)$

2. Suppose $f(x) = x^2 - x$.

(a) What is the domain of $f(x)$?

$\mathbb{R}$ or $(-\infty, \infty)$

(b) Find a complete factorization of $f(x)$.

$$f(x) = x^2 - x = \boxed{x \cdot (x-1)}$$

(c) Calculate and **fully expand + simplify** the expression $\dfrac{f(x+h) - f(x)}{h}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h}$$

expand dist law

$$= \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h}$$

$$= \frac{2xh + h^2 - h}{h}$$

GCF

$$= \frac{\cancel{h} \cdot (2x + h - 1)}{\cancel{h}}$$

frac #5

$$= \boxed{2x + h - 1}$$

3. Given the polynomial $P(x) = x^4 + 4x^3 + 5x^2 + 4x + 4$:

(a) What is the average rate of change of $P(x)$ on $[0, 1]$?

$$\frac{P(1) - P(0)}{1 - 0} = \frac{1 + 4 + 5 + 4 + 4 - (0 + 0 + 0 + 0 + 4)}{1}$$

$$= \frac{18 - 4}{1} = \boxed{14}$$

(b) Is $P(x)$ even, odd, or neither? Full credit requires using the definition of even/odd.

$$P(-x) = (-x)^4 + 4(-x)^3 + 5(-x)^2 + 4(-x) + 4$$

$$= x^4 \underbrace{- 4x^3}_{} + 5x^2 \underbrace{- 4x}_{} + 4$$

only two terms flipped sign

no way for $P(-x)$ to equal $P(x)$ or $-P(x)$. $\boxed{\text{neither}}$

(c) $x = -2$ is a zero of multiplicity two for $P(x)$. Use this information to find a complete factorization of $P(x)$.

$x = -2$ is multiplicity 2 means $(x - (-2))^2 = (x+2)^2$ is a factor

expanding $(x+2)^2$ gives $x^2 + 4x + 4$. Long divide:

$$\begin{array}{r|l} & x^2 + 1 \\ \hline x^2 + 4x + 4 & x^4 + 4x^3 + 5x^2 + 4x + 4 \\ & -(x^4 + 4x^3 + 4x^2) \\ \hline & 0 + 0 + x^2 + 4x + 4 \\ & x^2 + 4x + 4 \\ \hline & \underbrace{0}_{R(x)} \end{array}$$

By division algorithm $x^4 + 4x^3 + 5x^2 + 4x + 4 = \underbrace{(x^2 + 4x + 4)}_{D(x)} \underbrace{(x^2 + 1)}_{Q(x)} + \underbrace{0}_{R(x)}$

$$= (x+2)^2 (x^2 + 1)$$

Factor $x^2 + 1$ using quadratic formula: $a = 1, b = 0, c = 1$

$$x = \frac{-0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{\pm \sqrt{-4}}{2} = \frac{\pm i\sqrt{4}}{2} = \frac{\pm \cancel{2} i}{\cancel{2}} \overset{\text{frac}}{\underset{\#5}{=}} \pm i$$

so i and $-i$ are zeros of $x^2 + 1$, meaning $(x - i)(x - (-i)) = (x - i)(x + i) = x^2 + 1$

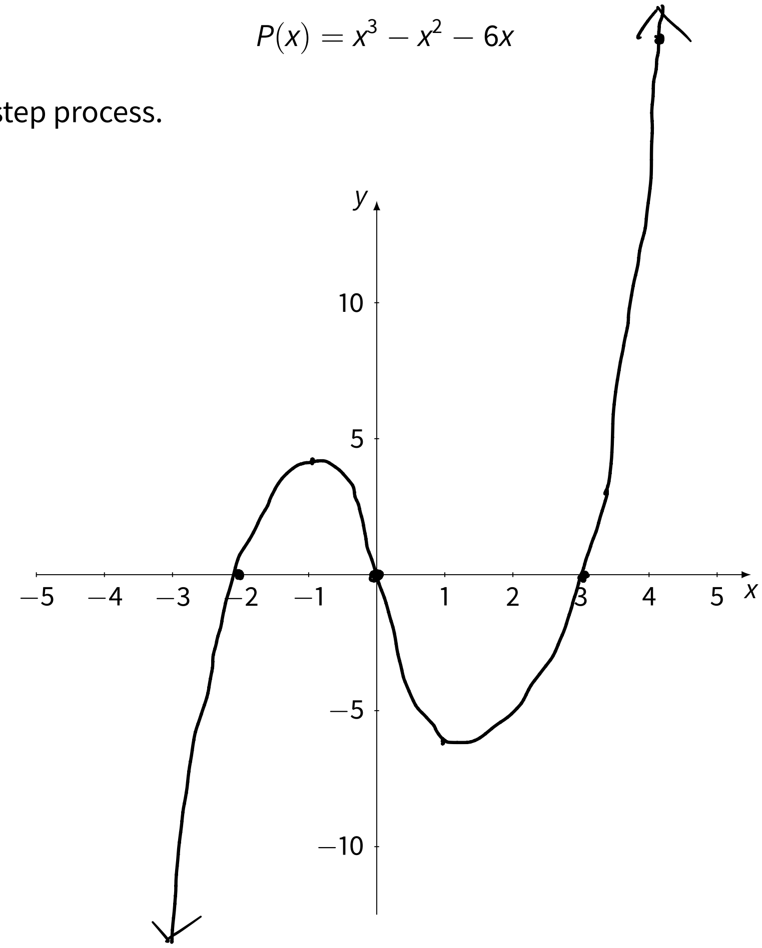
$\therefore$ a complete factorization is $\boxed{(x+2)^2 (x - i)(x + i)}$

4

4. Sketch an accurate graph of the polynomial

$$P(x) = x^3 - x^2 - 6x$$

using the four step process.



① zeros: factor $P(x) = x^3 - x^2 - 6x$

$P(x) = x^3 - x^2 - 6x$

new x

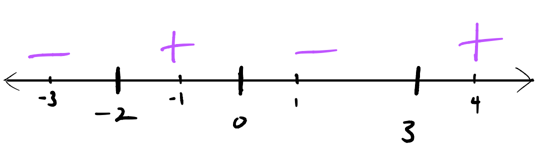
$= x(x^2 - x - 6)$

1 −3
1 2

$= x(x-3)(x+2)$

zeros are $x = 0, 3, -2$

② Diagram / IVT



$P(-3) = (-3)\cdot(-3-3)\cdot(-3+2) = (-3)(-6)(-1)$
$= -18$

$P(-1) = (-1)(-1-3)(-1+2) = (-1)(-4)(1) = 4$

$P(1) = 1\cdot(1-3)(1+2) = 1\cdot(-2)\cdot 3 = -6$

$P(4) = 4\cdot(4-3)\cdot(4+2) = 4\cdot 1\cdot 6 = 24$

③ End behavior: leading term x^3

$y \to \infty$ as $x \to \infty$

$y \to -\infty$ as $x \to -\infty$

④ graph

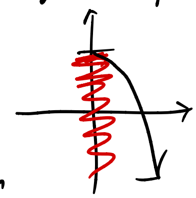
x	−3	−2	−1	0	1	3	4
P(x)	−18	0	4	0	−6	0	24

5. Given the function

x^2 reflected around x-axis, shifted up 4 gives

$$f(x) = 4 - x^2, \; x \geq 0$$

domain restriction



range is $y \leq 4$

(a) Calculate the inverse f^{-1} algebraically.

① 1-1 from picture on right

② $y = 4 - x^2, \quad y \leq 4$

③ $x^2 = 4 - y, \quad y \leq 4$

$x = \pm\sqrt{4-y}, \quad y \leq 4$

choose $x = +\sqrt{4-y}$ since $x \geq 0$ and square root outputs positive numbers.

$x = \sqrt{4-y}, \quad y \leq 4$

④ $f^{-1}(x) = y = \sqrt{4-x}, \quad x \leq 4$

(b) Use the Inverse Function Property to verify your result of f^{-1} is actually the inverse of $f(x)$.

$$\begin{aligned}
(f \circ f^{-1})(x) &= f\left(f^{-1}(x)\right) \\
&= f\left(\sqrt{4-x}\right) \\
&= 4 - \left(\sqrt{4-x}\right)^2 \\
&= 4 - (4 - x) \\
&= 4 - 4 + x \\
&= x \checkmark
\end{aligned}$$

6. Given

$$F(x) = \sqrt{x-1} \qquad G(x) = -(x^2 - 1)$$

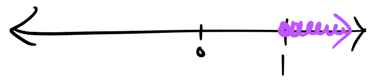
(a) Find the domain of $F(x)$.

$F(x) = \sqrt{x-1}$

problems : $x - 1 < 0$

(remove from $\mathbb{R}$) $x < 1$

what's left



$\boxed{[1, \infty)}$

(b) Decompose $F(x)$ into two functions f and g where $f \circ g = F$. You are not allowed to choose $f(x) = x$ or $g(x) = x$.

$\boxed{f(x) = \sqrt{x}, \; g(x) = x - 1}$ why:

$(f \circ g)(x) = f(g(x)) = f(x-1) = \sqrt{x-1} = F(x)$ ✓

(c) Calculate $(F \circ G)(0)$.

$(F \circ G)(0) = F(G(0))$

$= F(-(0^2 - 1))$

$= F(-(-1))$

$= F(1)$

$= \sqrt{1-1}$

$= \sqrt{0}$

$= \boxed{0}$

(d) Find the function $F \circ G$ and explain why the domain of this function is the single number $x = 0$.

$(F \circ G)(x) = F(G(x))$

$= F(-(x^2 - 1))$

$= F(-x^2 + 1)$

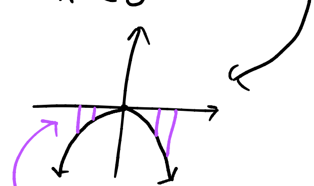
$= \sqrt{-x^2 + 1 - 1}$

$= \boxed{\sqrt{-x^2}}$

so $F \circ G = \sqrt{-x^2}$

problems (remove from $\mathbb{R}$) $-x^2 < 0$

use graph of $-x^2$ and check when $-x^2 < 0$



every height of $-x^2$ is less than 0 except at 0. so we remove all real #'s except for 0 itself.