# MATH 141: Midterm 2

Name: key

Directions:

- Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
- **Remember to simplify each expression.**
- If you do not know how to solve a problem, try your best and/or explain in English what you would do.
- Good luck!

| Problem | Score | Points |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| | | **50** |

1. Suppose $f(x) = 3x^2 - x$.

(a) What does the expression $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ represent?

The derivative of $f(x)$, which represents the slope of the tangent line at the same x-coordinates.

(b) Find

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for the given function $f(x)$. You must use this limit definition to receive credit.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\overbrace{3(x+h)^2 - (x+h)}^{f(x+h)} - \overbrace{(3x^2 - x)}^{f(x)}}{h}$$

$(A+B)^2$ dist law

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - x - h - 3x^2 + x}{h}$$

dist

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 - x - h - 3x^2 + x}{h}$$

$$= \lim_{h \to 0} \frac{6xh + h^2 - h}{h}$$

GCF

$$= \lim_{h \to 0} \frac{\cancel{h} \cdot (6x + h - 1)}{\cancel{h}}$$

$$\lim_{h \to 0} [6x + h - 1]$$

$$= 6x + 0 - 1$$

$$= \boxed{6x - 1}$$

(c) Find the equation of the tangent line of $f(x)$ at the point $(1, 1)$.

The equation is $y - 1 = f'(1)\,(x - 1)$

$$y - 1 = (6 \cdot 1 - 1)(x - 1)$$

$$y - 1 = 5(x - 1)$$

$$y = 5x - 5 + 1$$

$$\boxed{y = 5x - 4}$$

2. Short answer questions:

   (a) If a function $f(x)$ is differentiable at $x = a$, must it be continuous at $x = a$ as well? If not, draw a graph of a function that is differentiable but not continuous at $x = a$.

   Yes, it must be continuous.

   (b) True or false:

   $$f(x) = x^2 \cdot \sin x$$

   is continuous on $\mathbb{R}$.

   True because the domain of $x^2$ is $\mathbb{R}$, the domain of $\sin(x)$ is $\mathbb{R}$, so the domain of $x^2 \cdot \sin(x)$ is $\mathbb{R} \cap \mathbb{R} = \mathbb{R}$. Because this function is continuous on its' domain, the conclusion follows.

   (c) Given $f(x) = x^2$, find an equation of the normal line at $(1, 1)$.

   $f'(x) = 2x$

   Normal line has equation $y - 1 = -\frac{1}{f'(1)}(x-1)$

   $$y - 1 = -\frac{1}{2 \cdot 1}(x-1)$$

   $$y - 1 = -\frac{1}{2}(x-1)$$

   $$y = 1 - \frac{1}{2}x + \frac{1}{2}$$

   $$\boxed{y = -\frac{1}{2}x + \frac{3}{2}}$$

3. Answer the following:

(a) Given a function $f(x)$, if

$$\lim_{x\to a} f(x) = \frac{0}{0}$$

what global factor do you need to manifest in the numerator and denominator and why?

the factor $x-a$ needs to be created in order to use fraction law 5 to cancel, removing the 0's from the numerator and denominator.

(b) Find

$$\lim_{t\to 0}\left[\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right]$$

needs factor of $\sqrt{1+t}$

Hint: Subtract to get one fraction first.

$$\lim_{t\to 0}\left[\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\cdot\frac{\sqrt{1+t}}{\sqrt{1+t}}\right] \underset{\text{law \#1}}{\overset{\text{frac}}{=}} \lim_{t\to 0}\left[\frac{1}{t\sqrt{1+t}} - \frac{\sqrt{1+t}}{t\sqrt{1+t}}\right]$$

rationalize numerator to create a factor of $t$ in the numerator!

$$\underset{\text{law \#3}}{\overset{\text{frac}}{=}} \lim_{t\to 0}\left[\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\cdot\frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}\right]$$

$$\underset{\text{frac law \#1}}{\overset{A^2-B^2,}{=}} \lim_{t\to 0}\left[\frac{1^2-(\sqrt{1+t})^2}{t\sqrt{1+t}\,(1+\sqrt{1+t})}\right]$$

$$= \lim_{t\to 0}\left[\frac{1-(1+t)}{t\sqrt{1+t}\,(1+\sqrt{1+t})}\right]$$

do not forget parenthesis.

$$\overset{\text{dist}}{=} \lim_{t\to 0}\frac{1-1-t}{t\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$= \lim_{t\to 0}\frac{-\cancel{t}}{\cancel{t}\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$\lim_{t\to 0}$ says to cancel out $t-0=t$ from part (a).

$$= \lim_{t\to 0}\frac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$\overset{\text{continuity}}{=} \frac{-1}{\sqrt{1+0}\,(1+\sqrt{1+0})}$$

$$= \frac{-1}{\sqrt{1}\,(1+\sqrt{1})} = \frac{-1}{1\cdot(1+1)} = \boxed{-\frac{1}{2}}$$

(c) Find

$$\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$ ← deal with compound fraction first.

LCD of $\frac{1}{x+h}$ and $\frac{1}{x}$ is $x(x+h)$.

$$= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)}$$

frac law 1

$$= \lim_{h\to 0}\frac{\left(\frac{1}{x+h}-\frac{1}{x}\right)\cdot x(x+h)}{h\cdot x\cdot(x+h)}$$

dist

$$= \lim_{h\to 0}\frac{\frac{1}{\cancel{x+h}}\cdot x\cancel{(x+h)}-\frac{1}{\cancel{x}}\cdot\cancel{x}(x+h)}{hx(x+h)}$$

frac law 5

$$= \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}$$

dist law

$$= \lim_{h\to 0}\frac{x-x-h}{hx(x+h)}$$

$$= \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}x(x+h)}$$

frac law 5

$$= \lim_{h\to 0}\frac{-1}{x(x+h)}$$

continuity

$$= \frac{-1}{x(x+0)}$$

$$= \boxed{-\frac{1}{x^2}}$$

4. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = 300$

$$f'(x) = \frac{d}{dx} 300 = \boxed{0}$$

(b) $f(x) = 5x^4 - x^2 + 3x$

$$f'(x) = 5 \cdot \frac{d}{dx} x^4 - \frac{d}{dx} x^2 + 3 \frac{d}{dx} x$$

power rule
$$= 5 \cdot 4x^{4-1} - 2x^{2-1} + 3x^{1-1}$$

$$= \boxed{20x^3 - 2x + 3}$$

(c) $f(x) = \dfrac{\sin^2(x)}{x^2}$ ← high ← low

Quotient Rule.

$$f'(x) = \frac{\underbrace{x^2}_{\text{low}} \cdot \underbrace{\frac{d}{dx}[\sin^2(x)]}_{\text{D high}} - \underbrace{\sin^2(x)}_{\text{high}} \cdot \underbrace{\frac{d}{dx} x^2}_{\text{D low}}}{(x^2)^2}$$

chain rule, power rule, LoE #3
$$= \frac{x^2 \cdot 2\sin(x) \cdot \frac{d}{dx}\sin(x) - \sin^2(x) \cdot 2x}{x^4}$$

$$= \frac{\overbrace{2x^2 \sin(x)\cos(x)}^{\text{term}} - \overbrace{2x\sin^2(x)}^{\text{term}}}{x^4}$$

GCF
$$= \frac{2 \cdot \cancel{x} \cdot \sin(x)\,(x\cos(x) - \sin(x))}{\cancel{x^4}\; x^3}$$

frac law #5
$$= \boxed{\frac{2\sin(x)\,(x\cos(x) - \sin(x))}{x^3}}$$

Product Rule

(d) $g(x) = \overbrace{x^2}^{\text{left}} \cdot \overbrace{\cos(x^2)}^{\text{right}}$

$$g'(x) = \overbrace{x^2}^{\text{left}} \cdot \overbrace{\frac{d}{dx}\left[\cos(x^2)\right]}^{(\text{right})'} + \overbrace{\cos(x^2)}^{\text{right}} \cdot \overbrace{\frac{d}{dx}\left[x^2\right]}^{(\text{left})'}$$

chain rule, power rule
$$= x^2 \cdot \left(-\sin(x^2)\right) \cdot \frac{d}{dx}\left[x^2\right] + \cos(x^2) \cdot 2x$$

power
$$= -x^2 \sin(x^2) \cdot 2x + 2x\cos(x^2)$$

$$= \overbrace{-2x^3 \sin(x^2)}^{\text{term}} + \overbrace{2x\cos(x^2)}^{\text{term}}$$

GCF
$$= \boxed{-2x\left(x^2 \sin(x^2) + \cos(x^2)\right)}$$

(e) $f(x) = \left(\dfrac{x^2 - 1}{x^2 + 3}\right)^4$ ← chain rule.

$$f'(x) = \frac{d}{dx}\left[\left(\frac{x^2-1}{x^2+3}\right)^4\right]$$

quotient rule.

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{d}{dx}\left[\frac{x^2-1}{x^2+3}\right]$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{(x^2+3)\frac{d}{dx}\left[x^2-1\right] - (x^2-1)\frac{d}{dx}\left[x^2+3\right]}{(x^2+3)^2}$$

both − and 2x need to be distributed.

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{(x^2+3) \cdot 2x - (x^2-1)\,2x}{(x^2+3)^2}$$

dist
$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{2x^3 + 6x - 2x^3 + 2x}{(x^2+3)^2}$$

$$= 4\left(\frac{x^2-1}{x^2+3}\right)^3 \cdot \frac{8x}{(x^2+3)^2} = \boxed{\frac{32x\,(x^2-1)^3}{(x^2+3)^5}}$$

combine w/ LoE

from low #1 and LoE #1

LoE 5
$$= 4\,\frac{(x^2-1)^3}{(x^2+3)^3} \cdot \frac{8x}{(x^2+3)^2}$$

5. The following three equations are in implicit form. Find $\frac{dy}{dx}$.

(a) $\sqrt{x} + \sqrt{y} = 1$

$$\frac{d}{dx}\sqrt{x} + \frac{d}{dx}\sqrt{y} = \frac{d}{dx}[1]$$

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}} \cdot \underbrace{y'}_{\text{isolate}} = 0$$

$$2\sqrt{y} \cdot \frac{1}{2\sqrt{y}} y' = -\frac{1}{2\sqrt{x}} \cdot 2\sqrt{y}$$

$$y' = -\frac{\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}}$$

$$\boxed{y' = -\frac{\sqrt{y}}{\sqrt{x}}}$$

(b) $x^2 - 2xy + y^2 = 5$

product rule.

$$\frac{d}{dx}[x^2] - 2\frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = \frac{d}{dx}[5]$$

many of you forgot that $-2$ is multiplied into $\geq 2$ terms b/c the product rule generates two terms.

$$2x - 2\left(x\frac{d}{dx}[y] + y\frac{d}{dx}[x]\right) + 2y \cdot y' = 0$$

$$2x - 2(\underline{xy'} + y) + \underline{2y \cdot y'} = 0$$

isolate these terms.

$$2x - 2xy' - 2y + 2yy' = 0$$

$$2yy' - 2xy' = 2y - 2x$$

$$y'(2y - 2x) = 2y - 2x$$

$$y' = \frac{2y - 2x}{2y - 2x}$$

$$\boxed{y' = 1}$$

Chain rule!

(c) $\cos(xy) = 1 + \sin y$

$$\frac{d}{dx}\left[\cos(xy)\right] = \frac{d}{dx}[1] + \frac{d}{dx}\left[\sin(y)\right]$$

product rule!

$$-\sin(xy)\cdot\frac{d}{dx}[xy] = 0 + \cos(y)\cdot y'$$

many of you forgot parenthesas again.

$$-\sin(xy)\cdot\left(x\cdot\frac{d}{dx}[y] + y\cdot\frac{d}{dx}[x]\right) = \cos(y)y'$$

$$-\sin(xy)\cdot\left(xy' + y\right) = \cos(y)y'$$

$$-\sin(xy)\cdot xy' - \sin(xy)y = \cos(y)y'$$

collect terms w/ $y'$

$$-\sin(xy)y = \cos(y)y' + \sin(xy)\cdot xy'$$

GCF

$$-\sin(xy)y = y'\left(\cos(y) + \sin(xy)x\right)$$

divide

$$\boxed{y' = \frac{-\sin(xy)y}{\cos(y) + x\sin(xy)}}$$