

MATH 119: Midterm 2

Name: bey

Directions:

- * Show your thought process (commonly said as "show your work") when solving each problem for full credit.
- * If you do not know how to solve a problem, try your best and/or explain in English what you would do.
- * Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10

120

— Conceptual understanding
— Execution
— Common pitfalls

1. Prove the identity

$$\frac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$$

Know these. Recognize them.

- ① $(A + B)^2 = A^2 + 2AB + B^2$
- ② $(A - B)^2 = A^2 - 2AB + B^2$
- ③ $A^2 - B^2 = (A - B)(A + B)$

$$LHS = \frac{(A + B)^2}{\frac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x}} = \frac{(\sin x + \cos x)(\sin x + \cos x)}{(\sin x - \cos x)(\sin x + \cos x)}$$

factor \uparrow

denom!

$$= \frac{(\sin x + \cos x)}{(\sin x - \cos x)} \cdot \frac{(\sin x - \cos x)}{(\sin x - \cos x)}$$

$$= \frac{(\sin x + \cos x)(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

Why?

RHS

missing it!

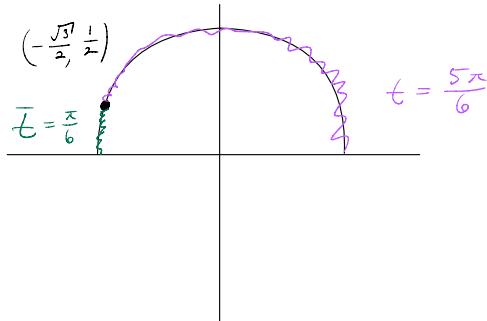
$$= \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2} = RHS$$



2. Evaluate the following:

$$(a) \cos^{-1} \left(-\frac{\sqrt{3}}{2} \right) = \frac{5\pi}{6}$$

$\cos(x)$ 1-1 on $[0, \pi]$



$$(b) \cos 112.5^\circ$$

On next page.

$$(c) \sin 75^\circ = \sin (30^\circ + 45^\circ)$$

$$30^\circ \cdot \frac{\pi}{180} = \frac{\pi}{6} \text{ rad}$$

$$45^\circ \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ rad.}$$

$$= \sin(30^\circ) \cos(45^\circ) + \cos(30^\circ) \sin(45^\circ)$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$= \boxed{\frac{\sqrt{6} + \sqrt{2}}{4}}$$

$$2b. \cos 112.5^\circ$$

When I see .5, I suspect 112.5° is half of the desired angle.

$$\cos(112.5^\circ) = \cos\left(\frac{1}{2} \cdot 225^\circ\right)$$

$$= -\sqrt{\frac{1 + \cos 225^\circ}{2}}$$

because x coordinate
of 112.5° is
in II, negative.

half angle

$$= -\sqrt{\frac{1 + \cos(180^\circ + 45^\circ)}{2}}$$

prepare for addition
identity

$$= -\sqrt{\frac{1 + \cos(180^\circ) \cos(45^\circ) - \sin(180^\circ) \sin(45^\circ)}{2}}$$

$$= -\sqrt{\frac{1 - 1 \cdot \frac{\sqrt{2}}{2} - 0 \cdot \frac{\sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{\frac{2}{2} - \frac{\sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{\frac{2 - \sqrt{2}}{2}}{2}}$$

$$= -\sqrt{\frac{2 - \sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2 - \sqrt{2}}}{\sqrt{4}} = -\frac{\sqrt{2 - \sqrt{2}}}{2}$$

Compound
frac.

3. Solve the equation for θ . Check your work if necessary.

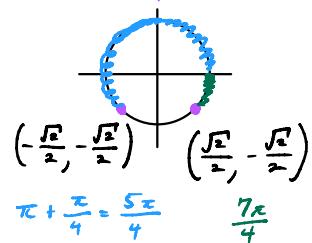
$$(a) \sqrt{2} \sin \theta + 1 = 0$$

isolate $\sin \theta$.

$$\sqrt{2} \sin \theta = -1$$

$$\sin \theta = -\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

① in one period.



$$\theta = \frac{5\pi}{4}, \theta = \frac{7\pi}{4}$$

$$(b) \sqrt{2} \tan \theta \sin \theta - \tan \theta = 0$$

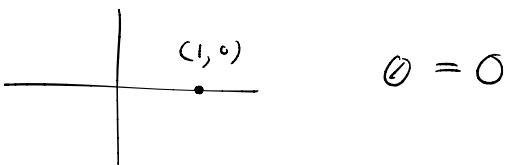
② account for periodicity

period of $\sin \theta$ is 2π long.

$$\boxed{\begin{aligned} \theta &= \frac{5\pi}{4} + 2k\pi & k \in \mathbb{Z} \\ \theta &= \frac{7\pi}{4} + 2k\pi \end{aligned}}$$

$$\tan \theta = 0$$

① Solutions in $(-\frac{\pi}{2}, \frac{\pi}{2})$



$$\theta = 0$$

② account for periodicity.

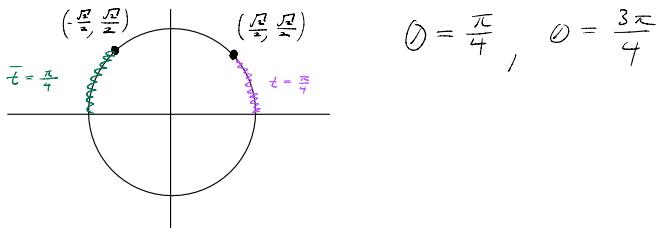
$$\boxed{\begin{aligned} \theta &= 0 + k\pi \\ &= k\pi \quad k \in \mathbb{Z} \end{aligned}}$$

$$\sqrt{2} \sin \theta - 1 = 0$$

$$\sqrt{2} \sin \theta = 1$$

$$\sin \theta = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

① Solutions on $(0, 2\pi)$



$$\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}$$

② Account for periodicity

$$\boxed{\begin{aligned} \theta &= \frac{\pi}{4} + 2k\pi & k \in \mathbb{Z} \\ \theta &= \frac{3\pi}{4} + 2k\pi \end{aligned}}$$

4. Prove these identities algebraically:

$$(a) \frac{\sin \theta}{\tan \theta} = \cos \theta$$

$$LHS = \frac{\sin \theta}{\tan \theta} = \frac{\sin \theta}{\frac{\sin \theta}{\cos \theta}}$$

$$= \frac{\sin \theta \cdot \cos \theta}{\sin \theta}$$

$$= \frac{\sin \theta \cos \theta}{\sin \theta}$$

$$= \cos \theta = RHS \quad \blacksquare$$

$$(b) \frac{\cos x}{\sec x} + \frac{\sin x}{\csc x} = 1$$

$$LHS = \frac{\cos x}{\sec x} + \frac{\sin x}{\csc x} \stackrel{\text{trig functions}}{=} \frac{\cos x}{\frac{1}{\cos x}} + \frac{\sin x}{\frac{1}{\sin x}} \stackrel{\frac{1}{\frac{1}{b}} = 1 \cdot \frac{c}{b}}{=} \cos x \cdot \cos x + \sin x \cdot \sin x$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

Ch 5 identity

$$= RHS \quad \blacksquare$$

$$(c) \cos^4 x - \sin^4 x = \cos 2x$$

$$LHS = \cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$$

$$\text{Let } A = \cos^2 x, B = \sin^2 x$$

$$\text{Then } \cos^4 x - \sin^4 x \stackrel{\text{LoE}}{=} (\cos^2 x)^2 - (\sin^2 x)^2$$

$$= A^2 - B^2$$

$$\stackrel{\text{Special product}}{=} (A - B)(A + B)$$

$$= [(\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)]$$

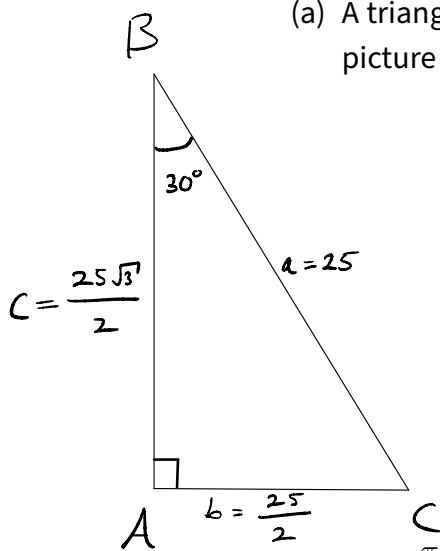
$$= \cos 2x$$

$$= RHS \quad \blacksquare$$

$$30^\circ \cdot \frac{\pi}{180} = \frac{\pi}{6}$$

5. Answer the following:

- (a) A triangle ABC has $\angle A = 90^\circ$, $\angle B = 30^\circ$ and $a = 25$. Solve the triangle and draw a picture of it.



Goal: find, b, c, $\angle C$

For $\angle C$:

$$\angle A + \angle B + \angle C = 180^\circ$$

$$90^\circ + 30^\circ + \angle C = 180^\circ$$

$$\boxed{\angle C = 60^\circ}$$

$$\text{For } b: \sin(30^\circ) = \frac{b}{25}$$

$$b = 25 \sin\left(\frac{\pi}{6}\right)$$

$$= 25 \cdot \frac{1}{2}$$

$$= \boxed{\frac{25}{2}}$$

$$\text{For } c: \cos(30^\circ) = \frac{c}{25}$$

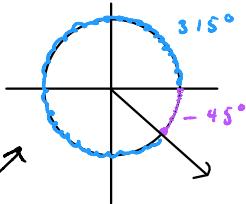
$$c = 25 \cos(30^\circ)$$

$$= 25 \cos\left(\frac{\pi}{6}\right) = 25 \cdot \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{25\sqrt{3}}{2}}$$

- (b) Are $\frac{-\pi}{4}$ rad and 315° coterminal? Show with calculations.

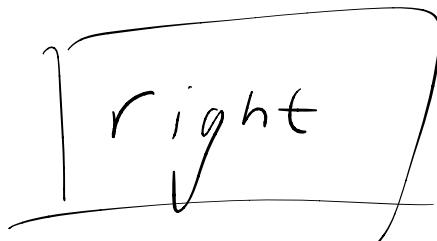
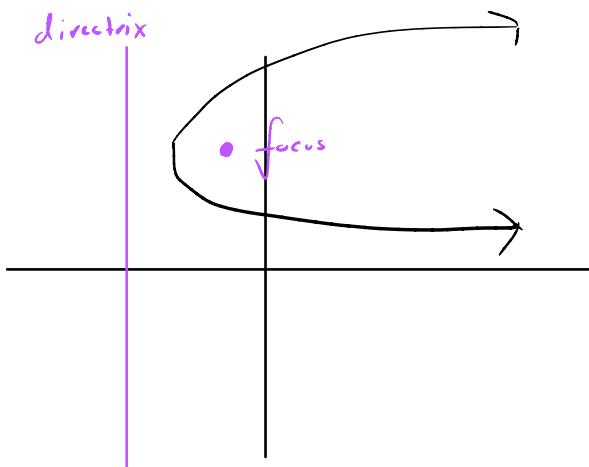
$$-\frac{\pi}{4} \cdot \frac{180^\circ}{\pi} = -45^\circ$$



A full circle is highlighted since $315^\circ + 45^\circ = 360^\circ$

The terminal sides therefore lie on top of each other
so they are coterminal.

- (c) If the directrix of a parabola is a vertical line and the focus is to the right of the directrix, which way does the parabola open?



6. Answer the following:

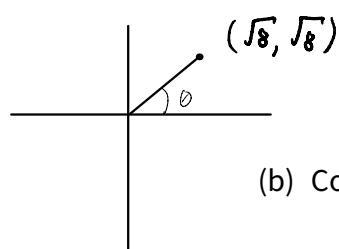
→ find r and θ

- (a) Convert $(\sqrt{8}, \sqrt{8})$ into polar coordinates.

$$\begin{matrix} \uparrow & \uparrow \\ x & y \\ 2 & \end{matrix}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{8})^2 + (\sqrt{8})^2} = \sqrt{8+8} = \sqrt{16} = 4$$

$$\tan \theta = \frac{\sqrt{8}}{\sqrt{8}} = 1 \leftarrow$$



looking here $\theta = \frac{\pi}{4}$, only way to point in quadrant I.

$$\boxed{(4, \frac{\pi}{4})}$$

- (b) Convert $r = \frac{1}{1+\sin\theta}$ into rectangular form.

use this r to make a $r\sin\theta$. Missing r !

$$(1+\sin\theta) \cdot r = \frac{1}{1+\sin\theta} \cdot (1+\sin\theta)$$

$$r + r\sin\theta = 1 \quad \text{dist law}$$

$$r + y = 1$$

$$\begin{aligned} r &= 1-y && \text{isolate } r \\ r^2 &= (1-y)^2 && \text{create } r^2 \\ x^2 + y^2 &= (1-y)^2 && \text{in order to use } r^2 = x^2 + y^2 \end{aligned}$$

- (c) Convert $r = 6\cos\theta$ into rectangular form.

both missing r to use formulas. multiply!

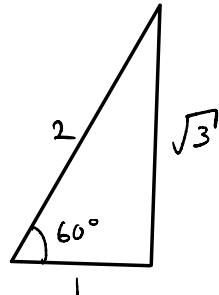
$$r \cdot r = 6 \cos\theta \cdot r$$

$$r^2 = 6r\cos\theta$$

$$\boxed{x^2 + y^2 = 6x}$$

7. Simplify the following trigonometric expressions:

$$(a) \sin 20^\circ \cos 40^\circ + \cos 20^\circ \sin 40^\circ = \sin(20^\circ + 40^\circ)$$



$$\begin{aligned} &= \sin(60^\circ) \\ &= \boxed{\frac{\sqrt{3}}{2}} \end{aligned}$$

generates 2 forms.

Subtracting!

four applications of addition/subtraction formula

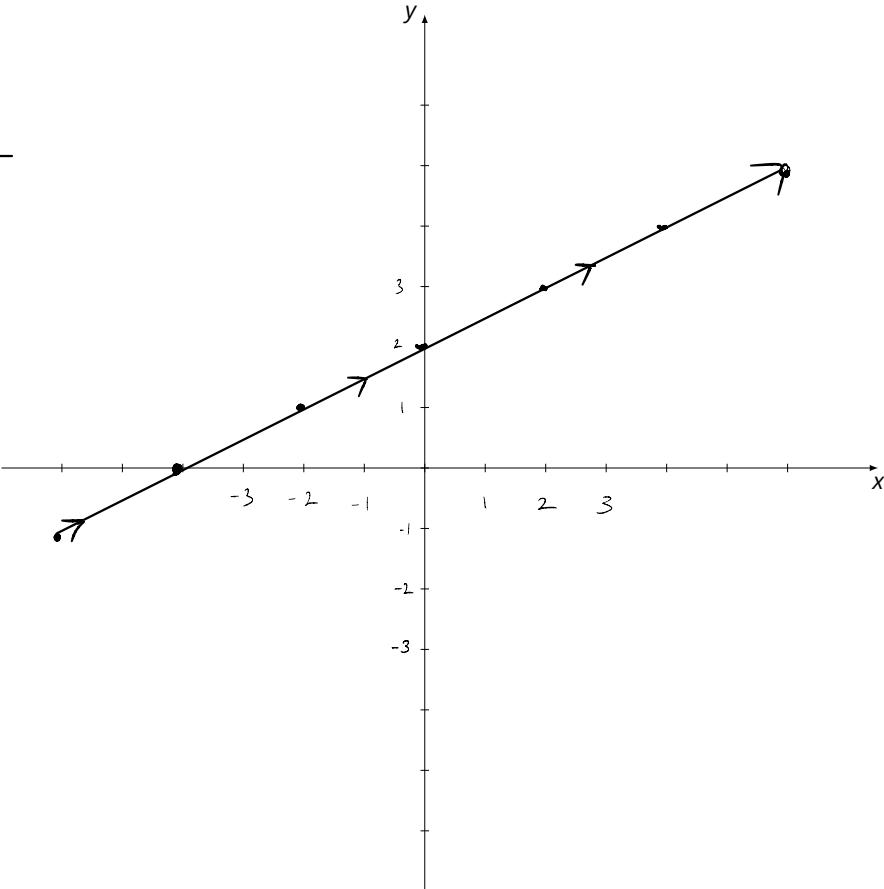
$$\begin{aligned} (b) \frac{\sin(x+y) - \sin(x-y)}{\cos(x+y) + \cos(x-y)} &= \frac{\sin(x)\cos(y) + \cos(x)\sin(y) - (\sin(x)\cos(y) - \cos(x)\sin(y))}{\cos(x)\cos(y) - \sin(x)\sin(y) + \cos(x)\cos(y) + \sin(x)\sin(y)} \\ &\stackrel{\text{dist}}{=} \frac{\cancel{\sin(x)\cos(y)} + \cos(x)\sin(y) - \cancel{\sin(x)\cos(y)} + \cos(x)\sin(y)}{2\cos(x)\cos(y)} \\ &= \frac{2 \cdot \cancel{\cos(x)\sin(y)}}{2 \cdot \cancel{\cos(x)\cos(y)}} \\ \frac{1}{5} &= \frac{\sin(y)}{\cos(y)} \\ &= \boxed{\tan(y)} \end{aligned}$$

8. Here is a pair of parametric equations

$$x = 2t \quad y = t + 2$$

(a) Sketch the curve represented by the equations.

t	x	y
-3	-6	-1
-2	-4	0
-1	-2	1
0	0	2
1	2	3
2	4	4
3	6	5



(b) Find a rectangular coordinate equation for the curve by eliminating the parameter.

$$y = t + 2 \xrightarrow{\text{Solve for } t} t = y - 2 \xrightarrow{\text{Substitute}} x = 2(y - 2)$$

$$\begin{aligned} & \xrightarrow{\text{Solve for } y} x = 2y - 4 \\ & \text{if you can. } 2y = x + 4 \rightarrow y = \frac{x + 4}{2} = \frac{x}{2} + \frac{4}{2} = \frac{1}{2}x + 2 \end{aligned}$$

$$y = \frac{1}{2}x + 2$$