MATH 161: Midterm 2

Name: key

Directions:

* Show your thought process (commonly said as "show your work") when solving each problem for full credit.
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
		60

1. Draw **one single graph** of a function which satisfies the following:

   (a) $f(0) = 1$

   (b) $f(2) = 1$

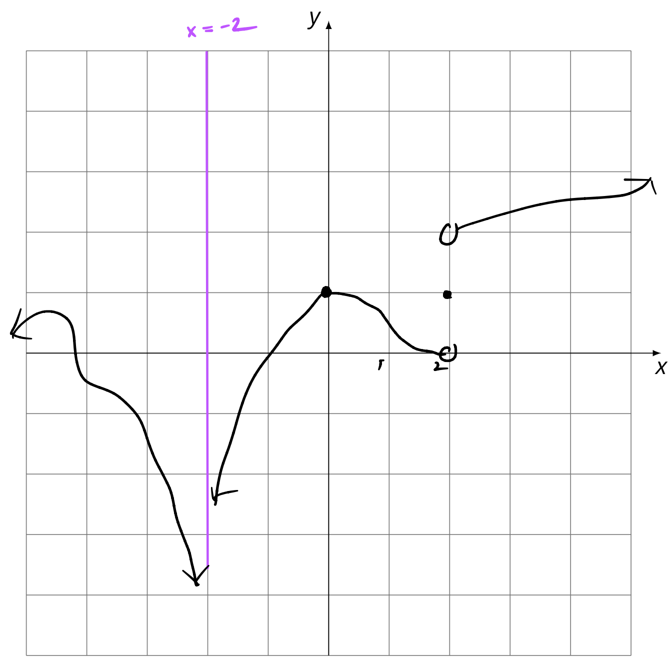
   (c) $\lim_{x \to 0} f(x) = 1$

   (d) $\lim_{x \to 2^-} f(x) = 0$

   (e) $\lim_{x \to 2^+} f(x) = 2$

   (f) $\lim_{x \to -2} f(x) = -\infty$

Answers may vary.



2. Consider this limit:

$$\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h}$$

(a) Try using Limit Laws to find the limit. What ends up happening?

$$\lim_{h\to 0}\frac{\frac{1}{3+h}-\frac{1}{3}}{h} \overset{\substack{\text{limit law}\\ 5\\ 1,2}}{=} \frac{\frac{\lim_{h\to 0} 1}{\lim_{h\to 0} 3+\lim_{h\to 0} h}-\lim_{h\to 0}\left(\frac{1}{3}\right)}{\lim_{h\to 0} h}$$

$$\overset{\substack{\text{limit law}\\ 7,8}}{=} \frac{\frac{1}{3+0}-\frac{1}{3}}{0}$$

$$= \frac{0}{0} \leftarrow$$ You end up with an indeterminate form of type $\frac{0}{0}$.

(b) Now find the actual limit.

The $\lim_{h\to 0}$ says you're looking to create a global factor of $h-0=\boxed{h}$ in the numerator. So, simplify the compound fraction.

$$\lim_{h\to 0}\frac{\frac{1}{(3+h)}-\frac{1}{3}}{h} = \lim_{h\to 0}\frac{\frac{3}{3}\frac{1}{(3+h)}-\frac{1}{3}\cdot\frac{3+h}{3+h}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{3}{3(3+h)}-\frac{3+h}{3(3+h)}}{h}$$ frac law 1

$$= \lim_{h\to 0}\frac{\frac{3-(3+h)}{3(3+h)}}{h}$$ frac law 3

$$= \lim_{h\to 0}\frac{\frac{3-3-h}{3(3+h)}}{h}$$ dist law

$$= \lim_{h\to 0}\frac{\frac{-h}{3(3+h)}}{h}$$

$$= \lim_{h\to 0}\frac{-h}{3(3+h)}\cdot\frac{1}{h}$$ frac law 2

$$= \lim_{h\to 0}\frac{-\cancel{h}}{3\cdot\cancel{h}\cdot(3+h)}$$ frac law 5

$$= \lim_{h\to 0}\frac{-1}{3(3+h)}$$ now use limit laws

$$= \frac{-1}{3(3+0)}$$

$$= \boxed{-\frac{1}{9}}$$

3

3. Use **the mathematical definition of continuity** to prove the function

$$f(x) = \begin{cases} x(x-1) & x < 1 \\ 0 & x = 1 \\ \sqrt{x-1} & x > 1 \end{cases}$$

is continuous at the number $x = 1$.

① Show $\lim_{x \to 1} f(x)$ exists.

limit laws

we have: $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x-1} = \sqrt{\lim_{x \to 1^+} x - \lim_{x \to 1^+} 1} = \sqrt{1-1} = 0$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} [x(x-1)] = \left[\lim_{x \to 1^-} x\right]\left[\lim_{x \to 1^-} x - \lim_{x \to 1^-} 1\right]$

limit laws

$= 1 \cdot (1-1)$

$= 0$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$, we conclude $\lim_{x \to 1} f(x)$ exists and is equal to 0.

② Show $f(1)$ is defined.

$f(1) = 0$ ✓

③ Show $\lim_{x \to 1} f(x) = f(1)$

from parts ① and ②, $\lim_{x \to 1} f(x) = 0$ and $f(1) = 0$.

∴, this condition is satisfied.

By the definition of continuity $f(x)$ is continuous at $x = 1$.

4. Suppose $f(x) = \sqrt{x}$.

(a) What the expression $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ represent?

The derivative of $f(x)$, which represents the slope of the tangent line at the same x-coordinates.

(b) Find the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for the given function $f(x)$. You must use this limit definition to receive credit.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$A^2 - B^2$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

frac law 5

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

continuity

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

(c) Find the equation of the tangent line of $f(x) = \sqrt{x}$ at the point $(1, 1)$.

Tangent line of $f(x)$ at $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a)$$

Given $(1, 1)$, we have

a $f(a)$

$$y - 1 = f'(1)(x - 1)$$

$$y - 1 = \frac{1}{2\sqrt{1}}(x - 1)$$

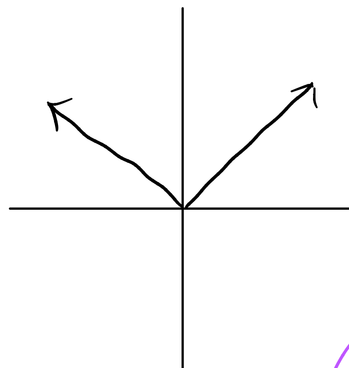
$$y = 1 + \frac{1}{2}x - \frac{1}{2}$$

$$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

5. Short answer questions:

(a) If a function $f(x)$ is continuous at $x = a$, must it be differentiable at $x = a$?
If not, draw a graph of a function that is continuous but not differentiable.

No. The function $f(x) = |x|$ has the following graph.



$f(x)$ is continuous on $\mathbb{R}$ but differentiable on $(-\infty, 0) \cup (0, \infty)$.

(b) True or False:

$$f(x) = \sin(x) + \frac{x}{x+1}$$

is continuous on $\mathbb{R}$.

False. Finding continuity is just finding domain.

$\frac{x}{x+1}$ has domain $(-\infty, -1) \cup (-1, \infty)$ because you cannot divide by 0.

$\therefore$, $f(x)$ is continuous on $(-\infty, -1) \cup (-1, \infty)$.

(c) Given $f(x) = x$, find $f'(x)$.

$f'(x) = 0$

$\boxed{f''(x) = 0}$

6. Find the following derivatives. You are allowed to use the Differentiation Rules.

(a) $f(x) = \pi^2$

constant!

$$f'(x) = \frac{d}{dx}\left[\pi^2\right] = \boxed{0}$$

(b) $f(\theta) = \theta^3 - e^\theta + 2\sin\theta$

$$f'(\theta) = \frac{d}{d\theta}\left[\theta^3 - e^\theta + 2\sin\theta\right]$$

$$= \frac{d}{d\theta}[\theta^3] - \frac{d}{d\theta}[e^\theta] + 2\frac{d}{d\theta}[\sin\theta]$$

power rule

$$= \boxed{3\theta^2 - e^\theta + 2\cos\theta}$$

(c) $g(x) = (x-1)(x+1)$

$$= x^2 - 1$$

$$g'(x) = \frac{d}{dx}\left[x^2 - 1\right]$$

$$= \frac{d}{dx}[x^2] - \frac{d}{dx}[1]$$

$$= 2x - 0$$

$$= \boxed{2x}$$