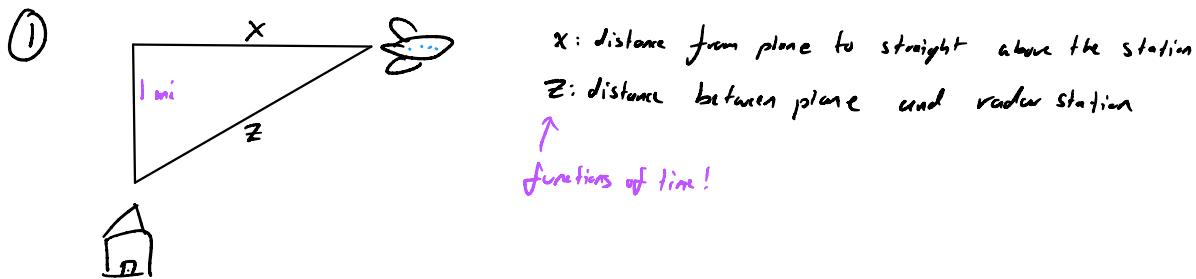


MATH 141: Some Practice Final Problems Key

Here are problems that cover the last two weeks of our class.

Remember the final is cumulative; you should look at Practice Midterm 1+2 and Midterm 1+2 as well.

1. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 miles away from the station.



(2) Given:
 $z = 2 \text{ mi}$, $\frac{dx}{dt} = 500 \text{ mi/h}$

Need:

$$\frac{dz}{dt}$$

(3) $x^2 + 1^2 = z^2$ Pythagorean Thm

(4) $\frac{d}{dt}[x^2] + \frac{d}{dt}[1^2] = \frac{d}{dt}[z^2]$

$$2x \frac{dx}{dt} + 0 = 2z \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{2x}{2z} \frac{dx}{dt} = \frac{x}{z} \cdot \frac{dx}{dt}$$

missing from (2)! Find what x is.

(5) Now plug in z .

$$1^2 + x^2 = 2^2$$

$$x^2 = 4 - 1$$

$$x^2 = 3$$

$$\sqrt{x^2} = \pm \sqrt{3} \quad \rightarrow \quad x = \sqrt{3}$$

(6) $\frac{dz}{dt} = \frac{x}{z} \cdot \frac{dx}{dt}$

$$= \frac{\sqrt{3} \text{ mi}}{2 \text{ mi}} \cdot 500 \frac{\text{mi}}{\text{h}}$$

$$= 250\sqrt{3} \text{ mi/h}$$

2. Short answer questions:

- (a) What are the two methods for finding local minimums and maximums called? State the weaknesses of one of the methods.

① First derivative test.

② Second derivative test; cannot determine local extrema in the case where $f''(c)$ DNE (the pointy corner extrema)

- (b) What is the method for finding absolute minimums and maximums called?

Closed interval method.

- (c) What is the strength of **Part 2** of the Fundamental Theorem of Calculus?

When evaluating integrals, instead of taking an infinite limit of a sum, you only need to take the antiderivative of the function you are integrating.

- (d) Find three particular antiderivatives of the function $f(x) = \sin(x)$.

$$\boxed{\begin{aligned} F_1(x) &= -\cos(x) + \pi \\ F_2(x) &= -\cos(x) + 3 \\ F_3(x) &= -\cos(x) - 1 \end{aligned}}$$

$$F_1'(x) = -(-\sin(x)) + 0 = \sin(x)$$

$$\text{because } F_2'(x) = -(-\sin(x)) + 0 = \sin(x)$$

$$F_3'(x) = -(-\sin(x)) + 0 = \sin(x)$$

3. Suppose $f(x) = \frac{x}{x^2 + 1}$.

(a) Find all intervals on which $f(x)$ is increasing and decreasing.

① crit #'s

$$f'(x) = \frac{(x^2 + 1) \cdot \frac{d}{dx}[x] - x \cdot \frac{d}{dx}[x^2 + 1]}{(x^2 + 1)^2}$$

Quotient Rule

$$= \frac{x^2 + 1 - x \cdot 2x}{(x^2 + 1)^2}$$

$$\textcircled{a} \quad f'(x) = 0$$

$$\frac{-x^2 + 1}{(x^2 + 1)^2} = 0$$

$$-x^2 + 1 = 0$$

$$1 = x^2$$

$$\boxed{x = \pm 1}$$

$$= \frac{-x^2 + 1}{(x^2 + 1)^2}$$

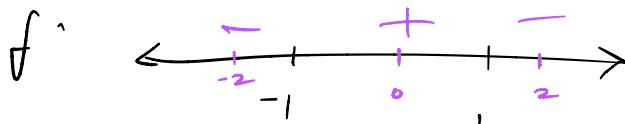
$$x^2 = 1$$

$$\textcircled{b} \quad f'(x) \text{ DNE when } (x^2 + 1)^2 = 0$$

but $x^2 + 1 > 0$ so not applicable.

(b) Find all local minimums and maximums.

② Sign diagram of f'



factor

$$f'(x) = \frac{-x^2 + 1}{(x^2 + 1)^2} = \frac{-(x^2 - 1)}{(x^2 + 1)^2} = \frac{-(x-1)(x+1)}{(x^2 + 1)^2}$$

$$f'(-2) = \frac{-(-2-1)(-2+1)}{(-2^2 + 1)^2} = \frac{-(-1)(-3)}{(-3)^2} = \frac{3}{9} = +$$

$$f'(0) = \frac{-(0-1)(0+1)}{(0^2 + 1)^2} = \frac{-1 \cdot 1}{1^2} = -$$

$$f'(2) = \frac{-(2-1)(2+1)}{(2^2 + 1)^2} = \frac{-1 \cdot 3}{3^2} = -\frac{1}{3}$$

Increasing on
 $(-1, 1)$

Decreasing on

$$(-\infty, -1) \cup (1, \infty)$$

$$\text{Local minimum of } f(-1) = \frac{-1}{(-1)^2 + 1} = -\frac{1}{2}$$

$$\text{Local maximum of } f(1) = \frac{1}{1^2 + 1} = \frac{1}{2}$$

4. Evaluate the following expressions. If applicable, you are allowed to use Fundamental Theorem of Calculus.

(a) $\sum_{i=1}^5 \frac{f(i)}{i}$ given that $f(x) = x^2$

$$\begin{aligned}\sum_{i=1}^5 \frac{f(i)}{i} &= \frac{f(1)}{1} + \frac{f(2)}{2} + \frac{f(3)}{3} + \frac{f(4)}{4} + \frac{f(5)}{5} \\ &= 1 + \frac{2^2}{2} + \frac{3^2}{3} + \frac{4^2}{4} + \frac{5^2}{5} \\ &= 1 + 2 + 3 + 4 + 5 = \boxed{15}\end{aligned}$$

(b) $\int_1^5 x dx = \frac{1}{2} x^2 \Big|_1^5 = \frac{1}{2} \cdot 5^2 - \frac{1}{2} \cdot 1^2 = \frac{25}{2} - \frac{1}{2} = \frac{24}{2} = \boxed{12}$

(c) $\lim_{x \rightarrow \infty} \frac{x+1}{\sqrt{x^4 - 2}}$

largest power is x^4

$$\begin{aligned}&= \lim_{x \rightarrow \infty} \frac{\frac{x+1}{x^4}}{\frac{\sqrt{x^4 - 2}}{x^4}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{\sqrt{\frac{x^4 - 2}{x^4}}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{\sqrt{1 - \frac{2}{x^4}}} \\ &= \frac{\lim_{x \rightarrow \infty} \frac{1}{x} + \lim_{x \rightarrow \infty} \frac{1}{x^2}}{\sqrt{\lim_{x \rightarrow \infty} 1 - 2 \cdot \lim_{x \rightarrow \infty} \frac{1}{x^4}}} = \frac{0 + 0}{\sqrt{1 - 2 \cdot 0}} = \boxed{0}\end{aligned}$$

(d) $\lim_{x \rightarrow \infty} (x^5 - x^3) = \lim_{x \rightarrow \infty} x^2 (x^3 - 1) = \infty \cdot \infty = \boxed{\infty}$

$$(e) \int_{-1}^1 (3x^2 + 4x + 4) dx = 3 \int_{-1}^1 x^2 dx + 4 \int_{-1}^1 x dx + \int_{-1}^1 4 dx$$

$$= 3 \cdot \frac{1}{3} x^3 \Big|_{-1}^1 + 4 \cdot \frac{1}{2} x^2 \Big|_{-1}^1 + 4x \Big|_{-1}^1$$

$$= 1^3 - (-1)^3 + 2 \cdot (1^2 - (-1)^2) + 4(1 - (-1))$$

$$(f) \int_{\pi/2}^{\pi} \cos(x) dx = F(\pi) - F(\pi/2) = 1 + 1 + 2(1-1) + 4(1+1) = 2 + 8 = \boxed{10}$$

$$\int_{\pi/2}^{\pi} \cos(x) dx = \sin(x) \Big|_{\pi/2}^{\pi} = \sin(\pi) - \sin\left(\frac{\pi}{2}\right) = 0 - 1 = \boxed{-1}$$

$$(g) \lim_{x \rightarrow \infty} \frac{x^2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^3}}{\frac{x^3 + 1}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1 + \frac{1}{x^3}} = \frac{\lim_{x \rightarrow \infty} \frac{1}{x}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{1}{x^3}}$$

longest power

$$= \frac{0}{1 + 0} = \boxed{0}$$

$$(h) \int_1^4 \frac{1}{\sqrt{x}} dx = \int_1^4 x^{-\frac{1}{2}} dx = -\frac{1}{\frac{1}{2}} x^{\frac{1}{2}} \Big|_1^4$$

$$= 2\sqrt{x} \Big|_1^4$$

$$= 2 \cdot \sqrt{4} - 2 \cdot \sqrt{1}$$

$$= 2 \cdot 2 - 2 = \boxed{2}$$

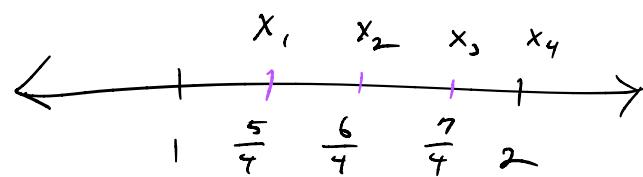
5. Suppose $f(x) = x^2$. Approximate the area underneath the curve on the interval $[1, 2]$ using four rectangles and right endpoints.

Only set up the sum; do not compute it.

$$n = 4$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$$\begin{matrix} \uparrow \\ a \end{matrix} \quad \begin{matrix} \uparrow \\ b \end{matrix}$$



$$\begin{aligned}
 A &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
 &= f\left(\frac{5}{4}\right) \cdot \frac{1}{4} + f\left(\frac{6}{4}\right) \cdot \frac{1}{4} + f\left(\frac{7}{4}\right) \cdot \frac{1}{4} + f(2) \cdot \frac{1}{4} \\
 &= \boxed{\frac{1}{4} \left(\left(\frac{5}{4}\right)^2 + \left(\frac{6}{4}\right)^2 + \left(\frac{7}{4}\right)^2 + 2^2 \right)}
 \end{aligned}$$

6. Consider the functions

$$g(x) = \int_0^x t^2 dt \quad h(x) = \int_0^x \sin(t^3) dt$$

(a) What is the geometric meaning of the number $g(5)$?

The area between the curve and the x -axis
of the function $f(x) = x^2$ on the interval $[0, 5]$

(b) What is the geometric meaning of the number $h(3)$?

The area between the curve and the x -axis
of the function $f(x) = \sin(x^3)$ on the interval $[0, 3]$

(c) Find the derivative with respect to x of $g(x)$.

$$\frac{d}{dx} g(x) = \frac{d}{dx} \int_0^x t^2 dt = x^2$$

by FTC 1

(d) Evaluate the following expression:

Differentiation
formula (4)
Section 2-3

$$\begin{aligned} \frac{d}{dx} [g(x) + h(x)] &= \frac{d}{dx} [g(x)] + \frac{d}{dx} [h(x)] \\ &= \frac{d}{dx} \int_0^x t^2 dt + \frac{d}{dx} \int_0^x \sin(t^3) dt \\ &= \boxed{x^2 + \sin(x^3)} \end{aligned}$$

7. Determine the intervals of concavity of

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$

① Find inflection points

$$f'(x) = 6x^2 - 18x + 12$$

$$f''(x) = 12x - 18 = 6(2x - 3)$$

a) solve $f''(x) = 0$

$$6(2x - 3) = 0$$

$$2x - 3 = 0$$

$$x = \frac{3}{2}$$

② Sign diagram of f''

$$\begin{array}{c} f'' \\ \hline - + \\ \frac{3}{2} \end{array}$$

not applicable, $f''(x)$ has domain \mathbb{R} .

(concave up on $(\frac{3}{2}, \infty)$)

(concave down on $(-\infty, \frac{3}{2})$)

$$f''(0) = 6 \cdot (2 \cdot 0 - 3) = -$$

$$f''(2) = 6 \cdot (2 \cdot 2 - 3) = +$$