GeodesicSphere

Finding shortest path on a 2D sphere

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I present `GeodesicSphere` class, which calculates the shortest path between two points — origin (i) and destination (f) — on a 2-dimensional sphere with unit radius. Each point on a sphere can be specified by latitude ϕ and longitude λ. Distance ds between two infinitesimally close points becomes

$$ds^2 = d\phi^2 + \cos^2\phi\, d\lambda^2 \tag{1}$$

where $d\phi$ and $d\lambda$ are difference in latitude and longitude, respectively. The task on hand is to find a trajectory which minimizes

$$S_{\mathrm{i}\to\mathrm{f}}[\phi,\, \lambda] = \int_{\mathrm{i}}^{\mathrm{f}} \sqrt{d\phi^2 + \cos^2\phi\, d\lambda^2} \tag{2}$$

The equation (2) can be written in terms of a parametric variable ξ, such that a trajectory can be specified by $\phi\,(\xi)$ and $\lambda\,(\xi)$ as functions in ξ with boundary conditions.

$$u\, S_{\mathrm{i}\to\mathrm{f}}[\phi,\, \lambda] \;=\; \int_0^u d\xi \left[\left(\frac{d\phi}{d\xi}\right)^2 + \cos^2\phi \left(\frac{d\lambda}{d\xi}\right)^2\right]^{1/2} \tag{3}$$

$$\text{where} \quad \begin{array}{l} \phi\,(\xi = 0,\, 1) = \phi_{\mathrm{i,f}} \\ \lambda\,(\xi = 0,\, 1) = \lambda_{\mathrm{i,f}} \end{array} . \tag{4}$$

Let us consider small variations in ϕ and λ and see how much deviation in S we have.

$$\begin{aligned} \delta S_{\mathrm{i}\to\mathrm{f}} &= S_{\mathrm{i}\to\mathrm{f}}[\phi + \delta\phi, \lambda + \delta\lambda] - S_{\mathrm{i}\to\mathrm{f}}[\phi, \lambda] && (5) \\ &= -\frac{1}{S_{\mathrm{i}\to\mathrm{f}}[\phi,\, \lambda]} \int_0^1 d\xi \left\{ \left[\frac{d^2\phi}{d\xi^2} + \cos\phi\, \sin\phi \left(\frac{d\lambda}{d\xi}\right)^2\right] \delta\phi\,(\xi) + \frac{d}{d\xi}\left(\cos^2\phi\, \frac{d\lambda}{d\xi}\right) \delta\lambda\,(\xi) \right\} && (6) \end{aligned}$$

If a trajectory is the shortest path, we have $\delta S_{\mathrm{i}\to\mathrm{f}} = 0$ for any arbitrary infinitesimal $\delta\phi$ and $\delta\lambda$. Equation (6) implies that $\phi\,(\xi)$ and $\lambda\,(\xi)$ meet the following set of differential equations, which is also called *geodesic equation*.

$$\begin{aligned} \frac{d\phi}{d\xi} &= \dot{\phi} && (7) \\ \frac{d\lambda}{d\xi} &= \dot{\lambda} && (8) \\ \frac{d\dot{\phi}}{d\xi} &= -\cos\phi\, \sin\phi\, \dot{\lambda}^2 && (9) \\ \frac{d\dot{\lambda}}{d\xi} &= 2\tan\phi\, \dot{\phi}\, \dot{\lambda} && (10) \end{aligned}$$

`GeodesicSphere` class implements relaxation methods[1] to solve the geodesic equation with boundary conditions. There are aforementioned 4 functions of interest — `y[1]` $= \phi$, `y[2]` $= \lambda$, `y[3]` $= \dot{\phi}$ and `y[4]` $= \dot{\lambda}$, and two boundary conditions (4) at each of boundaries (origin and destination).

[1] The mathematical algorithm is described in *Numerical Recipes in C* (2nd edition).