OrbitKepler

Finding Kepler orbit in Newtonian dynamics

Sangwook Ryu

I present `OrbitKepler` class, which calculates orbit of a test object under the influence of point mass. If the test object is bound to the gravitational field, its trajectory forms an elliptic orbit. This is demonstrated by solving equations of motion by means of Runge-Kutta method with initial condition. One can begin with the Newton's second law.

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{\mathbf{F}}{m} = -\frac{GM}{r^2}\frac{\mathbf{x}}{r} = -\frac{GM}{r^2}\hat{\mathbf{e}}_r \tag{1}$$

where G is the gravitational constant and M is mass responsible for the gravitational force. Due to the fact that we have a central force field, it will be convenient to work in spherical polar coordinate.

$$\begin{aligned}
\mathbf{x} &= (x,\, y,\, z) \\
&= r\,\hat{\mathbf{e}}_r = r\,(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta) && (2) \\
\frac{d\mathbf{x}}{dt} &= \frac{dr}{dt}\,\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + r\sin\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi && (3) \\
\frac{d^2\mathbf{x}}{dt^2} &= \frac{d^2r}{dt^2}\hat{\mathbf{e}}_r + \frac{dr}{dt}\left(\frac{d\theta}{dt}\,\hat{\mathbf{e}}_\theta + \sin\theta\frac{d\phi}{dt}\,\hat{\mathbf{e}}_\phi\right) \\
&\quad + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\hat{\mathbf{e}}_\theta + r\frac{d\theta}{dt}\left(\cos\theta\frac{d\phi}{dt}\hat{\mathbf{e}}_\phi - \frac{d\theta}{dt}\hat{\mathbf{e}}_r\right) \\
&\quad + \frac{d}{dt}\left(r\sin\theta\frac{d\phi}{dt}\right)\hat{\mathbf{e}}_\phi - r\sin\theta\left(\frac{d\phi}{dt}\right)^2(\sin\theta\,\hat{\mathbf{e}}_r + \cos\theta\,\hat{\mathbf{e}}_\theta) && (4) \\
&= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]\hat{\mathbf{e}}_r \\
&\quad + \frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right]\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right)\hat{\mathbf{e}}_\phi && (5)
\end{aligned}$$

Given that $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$ forms orthonormal basis, by considering equation (1) and (5), one has the following equations of motion for r, θ and ϕ.

$$\begin{aligned}
-\frac{GM}{r^2} &= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2 && (6) \\
0 &= \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 && (7) \\
0 &= \frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right) \qquad \to \qquad r^2\sin^2\theta\frac{d\phi}{dt} = l = \text{constant}\,. && (8)
\end{aligned}$$

One can simply the problem by defining the coordinate system such that an orbit lies on the xy-plane. It corresponds to set $\theta = \pi/2$ and $d\theta/dt = 0$ to get the following differential equations.

$$\frac{d^2r}{dt^2} - \frac{l^2}{r^3} = -\frac{GM}{r^2} \qquad \text{and} \qquad \frac{d\phi}{dt} = \frac{l}{r^2} \tag{9}$$

where l is angular momentum per unit mass of the test object. One can multiply dr/dt to the radial equation and integrate over time to get the following equation for energy conservation.

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{l^2}{2\,r^2} - \frac{GM}{r} = \mathcal{E} \tag{10}$$

where $\mathcal{E}$ is the total (kinetic + potential) energy per unit mass of the test object. In the case of a bound orbit i.e., $\mathcal{E} < 0$, the conservation equation (10) has the following form.

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &= -2\,\mathcal{E}\,\frac{(r - r_{\rm min})(r_{\rm max} - r)}{r^2} \qquad (11) \\ r_{\rm max,min} &= a\,(1 \pm e) \qquad (12) \\ \mathcal{E} &= -\frac{GM}{2\,a} \qquad (13) \\ l &= \sqrt{GMa\,(1 - e^2)} \qquad (14) \end{aligned}$$

where a and e are the semi-major axis and eccentricity, respectively. We have the following solution for r, ϕ and t as functions of a parametric variable u.

$$\begin{aligned} t\,(u) &= \sqrt{\frac{a^3}{GM}}(u - e\,\sin u) \qquad (15) \\ r\,(u) &= a\,(1 - e\,\cos u) \qquad (16) \\ \phi\,(u) &= \sqrt{1 - e^2}\int_0^u \frac{du'}{1 - e\,\cos u'} \qquad (17) \end{aligned}$$

One can see that r and ϕ are periodic functions in u such that

$$r\,(u + 2\pi) = r\,(u) \quad \text{and} \quad \phi\,(u + 2\pi) = \phi\,(u) + 2\pi \qquad (18)$$

and, therefore, period T of the orbit becomes

$$T = t\,(u + 2\pi) - t\,(u) = 2\pi\sqrt{\frac{a^3}{GM}}\,. \qquad (19)$$

To numerically solve the equations of motion, one can define the following dimensionless quantities.

$$\hat{t} = \sqrt{\frac{GM}{a^3}}\,t, \quad \hat{r} = \frac{r}{a} \quad \text{and} \quad \hat{l} = \frac{l}{\sqrt{GMa}} \qquad (20)$$

Then, equation (9) can be rewritten as

$$\frac{d^2\hat{r}}{d\hat{t}^2} = \frac{\hat{l}^2}{\hat{r}^3} - \frac{1}{\hat{r}^2} \qquad \text{and} \qquad \frac{d\phi}{d\hat{t}} = \frac{\hat{l}}{\hat{r}^2}\,. \qquad (21)$$

In `OrbitKepler` class, one solves a set of three first-order ordinary differential equations for `y[1]` $= \hat{r}$, `y[2]` $= d\hat{r}/d\hat{t}$ and `y[3]` $= \phi$ as functions of $\hat{t}$. For a given eccentricity, $\hat{l} = \sqrt{1 - e^2}$ and the initial condition is

$$\hat{r}\,(\hat{t} = 0) = 1 - e, \quad \text{and} \quad \frac{d\hat{r}}{d\hat{t}}\,(\hat{t} = 0) = 0\,. \qquad (22)$$

Kepler orbits calculated in `OrbitKepler` class with different eccentricities are shown in Figure 1.

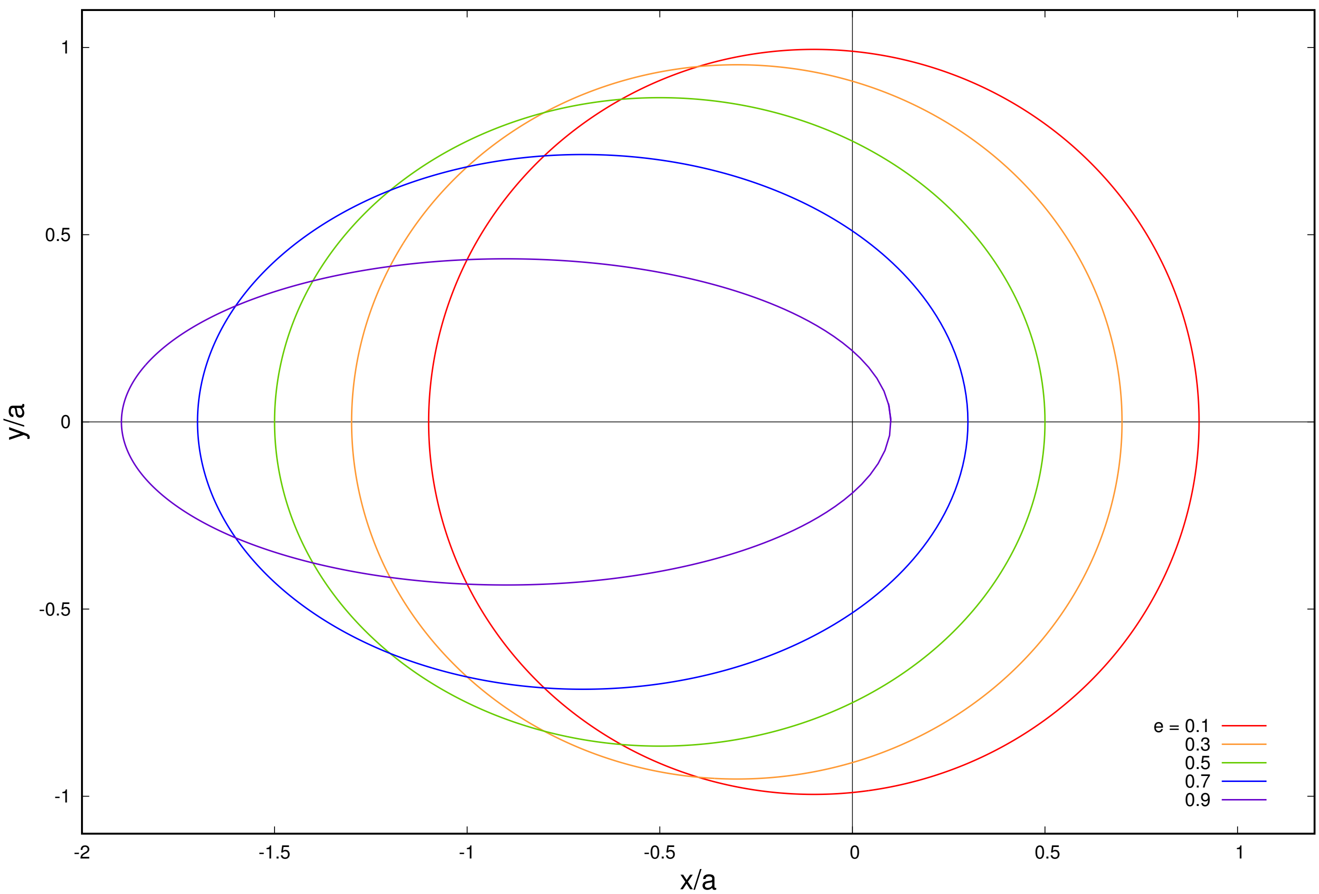

Figure 1: Kepler orbits with different eccentricities on the xy-plane