EOSC 511
SAM STEVENS

Lab 2 - Problem Error

a) Backwards difference formula = $T'(t_i) \simeq \dfrac{T_i - T_{i-1}}{\Delta t}$

Expand the solution to make it continuous and then write as Taylor series:

$$T(t_{i-1}) = T(t_i - \Delta t) = T(t_i) - (\Delta t)\,T'(t_i) - \frac{1}{2}(\Delta t)^2\,T''(t_i) - O(\Delta t^3)$$

Calculate the local truncation error as follows:

$$\frac{T_i - T_{i-1}}{\Delta t} - T'(t_i)$$

Substitute Taylor series into error equation and simplify:

$$\frac{\cancel{T(t_i)} - \left(\cancel{T(t_i)} - (\Delta t)\,T'(t_i) + \frac{1}{2}(\Delta t)^2\,T''(t_i) - O(\Delta t^3)\right)}{\Delta t} - T'(t_i)$$

$$= \frac{\Delta t\, T_i - \frac{1}{2}\Delta t^2\, T_i'' + O(\Delta t^3)}{\Delta t} - T_i'$$

$$= \cancel{T_i' - T_i'} - \frac{1}{2}\Delta t\, T_i'' + O(\Delta t^2)$$

AS Δt is assumed to be small, the next term in the series (Δt^2) will be relatively smaller and the backward Euler method will be first order accurate.

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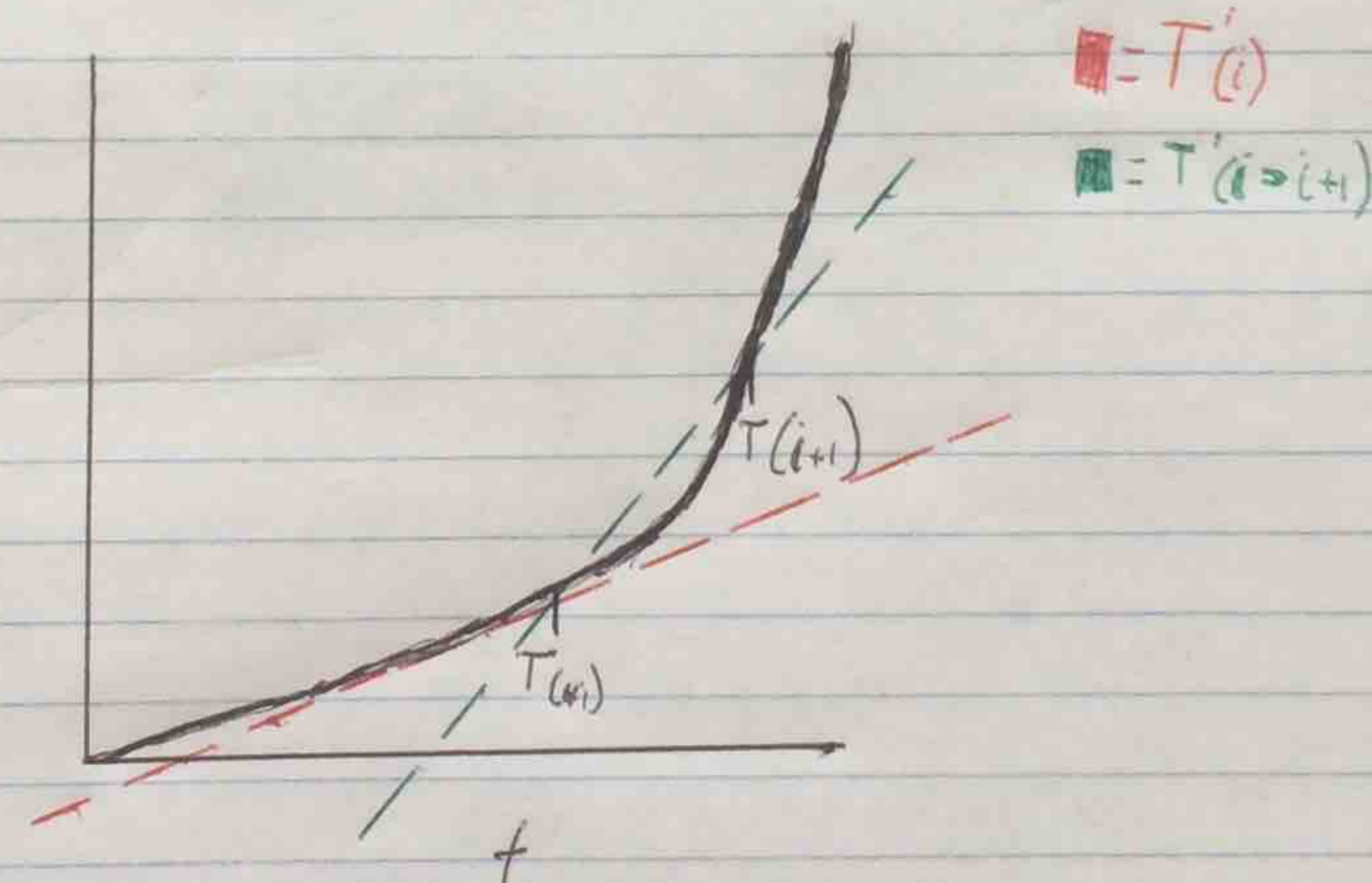
b)

Forward Euler Method	Backward Euler Method
$T(t_i) + \Delta t\, T'(t_i) + O(\Delta t^2)$	$-\frac{1}{2}\Delta t\, T'' + O(\Delta t^2)$

The forward Euler method involves the first derivative of $T(t_i)$. If we assume the function is a concave curve upwards (like in our example), the $T'(t_i)$ will be an underestimate of the derivate that is actually representative of the entire timestep.

The backward Euler method involves the second derivative of $T(t_i)$. [redacted] If the curve is concave up then $T''(t_i)$ will overestimate the second derivative that is representative of the entire timestep.



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Lab 2 - Problem Backwards Euler

$$T'_{(t_i)} = \frac{T_i - T_{i-1}}{\Delta t}$$

We can use a test equation to determine the stability of a scheme:

$$\frac{dz}{dt} = \lambda z \quad \text{so} \quad \frac{z_i - z_{i-1}}{\Delta t} = \lambda z$$

$$\Rightarrow -z_{i-1} = \lambda \Delta t \, z_i - z_i$$

$$\Rightarrow z_{i-1} = z(1 - \lambda \Delta t)$$

$$\Rightarrow z_i = \frac{z_{i-1}}{1 - \lambda \Delta t}$$

Apply this formula iteratively:

$$\left(\frac{1}{1 - \lambda \Delta t}\right)^2 z_{i-2}$$

$$\cdots$$

$$\Rightarrow z_i = \left(\frac{1}{1 - \lambda \Delta t}\right)^{i+1} z_0.$$

$\therefore$ If the first term of the equation is >1, the solution would blow up.

So if $\left| \frac{1}{1 - \lambda \Delta t} \right| < 1$

then $$|1 - \lambda \Delta t| < 1$$

$$|-\lambda \Delta t| < 0$$

Δt is more than 0, so we know that $\lambda < 0$ and real.

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Lab 2 - Taylor Problem

a) Centred difference formula =

$$T'(t_i) = \frac{T_{i+1} - T_{i-1}}{2\Delta t}$$

Do Taylor expansions:

make continuous

$$T_{i+1} = T(t_i + \Delta t) = T(t_i) + \Delta t \, T_i' + \frac{1}{2!}\Delta t^3 T_i'' + \frac{1}{3!}\Delta t^3 T'''(t_i) + O(t^4)$$

$$T_{i-1} = T(t_i - \Delta t) = T(t_i) - \Delta t \, T_i' + \frac{1}{2!}\Delta t^3 T_i'' - \frac{1}{3!}\Delta t^3 T'''(t_i) + O(t^4)$$

Due to the opposite signs on these terms, they cancel out.

So to simplify:

$$\frac{T_{i+1} + T_{i-1}}{2\Delta t} = \frac{\left(T(t_i) + \frac{\cancel{1}}{\cancel{2}}\Delta t^{\cancel{3}} T_i'' + O(t^4)\right) + \left(T(t_i) + \frac{\cancel{1}}{\cancel{2}}\Delta t^{\cancel{3}} T_i'' + O(t^4)\right)}{\cancel{2\Delta t}}$$

$$T_{i+1} + T_{i-1} = 2T(t_i) + \Delta t^2 T_i'' + O(\Delta t^4)$$

$$T_{i+1} + T_{i-1} - 2T_i = \Delta t^2 T_i'' + O(\Delta t^4)$$

$$\rightarrow \frac{T_{i+1} + T_{i-1} - 2}{\Delta t^2} = T_i'' + O(\Delta t^2)$$