2.2 Repeat Problem 2.1, but now derive a fourth-order accurate central-difference approximation for $\frac{\partial u}{\partial x}$.

SOLUTION: To derive a fourth-order accurate central-difference approximation for $\frac{\partial u}{\partial x}$ using Taylor's series expansion, we can expand $u(x \pm \Delta x)$ and $u(x \pm 2\Delta$ in Taylor series around x up to the third order:

$$u(x+2\Delta x) = u(x) + 2\Delta x\frac{\partial u}{\partial x}(x) + 2\Delta x^2\frac{\partial^2 u}{\partial x^2}(x) + \frac{4}{3}\Delta x^3\frac{\partial^3 u}{\partial x^3}(x) + O(\Delta x^4)$$

$$u(x+\Delta x) = u(x) + \Delta x\frac{\partial u}{\partial x}(x) + \frac{1}{2}\Delta x^2\frac{\partial^2 u}{\partial x^2}(x) + \frac{1}{6}\Delta x^3\frac{\partial^3 u}{\partial x^3}(x) + O(\Delta x^4)$$

$$u(x-\Delta x) = u(x) - \Delta x\frac{\partial u}{\partial x}(x) + \frac{1}{2}\Delta x^2\frac{\partial^2 u}{\partial x^2}(x) - \frac{1}{6}\Delta x^3\frac{\partial^3 u}{\partial x^3}(x) + O(\Delta x^4)$$

$$u(x-2\Delta x) = u(x) - 2\Delta x\frac{\partial u}{\partial x}(x) + 2\Delta x^2\frac{\partial^2 u}{\partial x^2}(x) - \frac{4}{3}\Delta x^3\frac{\partial^3 u}{\partial x^3}(x) + O(\Delta x^4)$$

First, we note that

$$-u(x+2\Delta x) + 8u(x+\Delta x) = 7u(x) + 6\Delta x\frac{\partial u}{\partial x}(x) + 2\Delta x^2\frac{\partial^2 u}{\partial x^2}(x) + O(\Delta x^4),$$

$$-8u(x-\Delta x) + u(x-2\Delta x) = -7u(x) + 6\Delta x\frac{\partial u}{\partial x}(x) - 2\Delta x^2\frac{\partial^2 u}{\partial x^2}(x) + O(\Delta x^4).$$

Adding these two equations yields

$$u(x-2\Delta x) - 8u(x-\Delta x) + 8u(x+\Delta x) - u(x+2\Delta x) = 12\Delta x\frac{\partial u}{\partial x}(x) + O(\Delta x^4).$$

Solving this equation for $\frac{\partial u}{\partial x}(x)$, we have

$$\frac{\partial u}{\partial x} = \frac{u(x-2\Delta x) - 8u(x-\Delta x) + 8u(x+\Delta x) - u(x+2\Delta x)}{12\Delta x} + O(\Delta x^3).$$