Solutions to Jech's Set Theory, The Third Millenium Edition, Revised and Expanded

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Contents

Part I

Basic Set Theory

Chapter 1

Axioms of Set Theory

1.1 Exercise 1

Verify (1.1).

Solution. If $a = c$ and $b = d$, then $(a, b) = (c, d)$ as a consequence of an axiom of first-order logic with equality, namely, that equals may be substituted for equals in a formula.

Conversely, suppose that $(a, b) = (c, d)$. Then

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}.$$

If $a = b$, then

$$\{\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\}.$$

Therefore $\{a\} = \{c\}$ and $\{a\} = \{c, d\}$. Hence, $a = c = d$, from which it follows that $a = c$ and $b = d$. If $a \neq b$, then $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$. Therefore $a = c$, and from this it follows that $\{a, b\} = \{a, d\}$. Hence $b = d$. $\square$

1.2 Exercise 2

There is no set X such that $P(X) \subset X$.

Solution. Suppose there exists a set X such that $P(X) \subset X$. Let $Y = \{x : x \in X \text{ and } x \notin x\}$. Clearly, $Y \subset X$, hence $Y \in P(X)$ and therefore $Y \in X$. However $Y \in Y$ if and only if $Y \notin Y$. We have therefore reached a contradiction and conclude that no such set X exists. $\square$

Let

$$\boldsymbol{N} = \bigcap\{X : X \text{ is inductive}\}.$$

$\boldsymbol{N}$ is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0, 1\}, \quad 3 = \{0, 1, 2\}, \quad \ldots .$$

If $n \in \boldsymbol{N}$, let $n + 1 = n \cup \{n\}$. Let us define $<$ (on $\boldsymbol{N}$) by $n < m$ if and only if $n \in m$.

A set T is *transitive* if $x \in T$ implies $x \subset T$.

1.3 Exercise 3

If X is inductive, then the set $\{x \in X : x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive, and for each n, $n = \{m \in \boldsymbol{N} : m < n\}$.

Solution. Let X be inductive. Let $Y = \{x \in X : x \subset X\}$. Since X is inductive, $\emptyset \in X$. Since $\emptyset \subset X$, $\emptyset \in Y$. Let $x \in Y$, then $x \in X$ and $x \subset X$. Since X is inductive, $x \cup \{x\} \in X$. If $y \in x \cup \{x\}$, then $y \in x$ or $y = x$. If $y \in x$, then since $x \subset X$, we have $y \in X$. If $y = x$, then clearly $y \in X$. Hence $x \cup \{x\} \subset X$. Thus, it follows that $x \cup \{x\} \in Y$ and therefore Y is inductive.

Since $\boldsymbol{N}$ is inductive, the set $M = \{n \in \boldsymbol{N} : n \subset \boldsymbol{N}\}$ is inductive. Clearly, $M \subset \boldsymbol{N}$, and since M is inductive, $\boldsymbol{N} \subset M$, and therefore $M = \boldsymbol{N}$. From this it follows that for every $n \in \boldsymbol{N}$, $n \subset \boldsymbol{N}$. Hence, $\boldsymbol{N}$ is transitive. If $n \in \boldsymbol{N}$, then $n \subset \boldsymbol{N}$. Hence if $m \in n$, then $m \in \boldsymbol{N}$, and by definition, $m < n$. Therefore $n \subset \{m \in \boldsymbol{N} : m < n\}$. Conversely, if $k \in \{m \in \boldsymbol{N} : m < n\}$, then $k < n$ and consequently $k \in n$. It follows that $\{m \in \boldsymbol{N} : m < n\} \subset n$ and therefore that $n = \{m \in \boldsymbol{N} : m < n\}$. □

1.4 Exercise 4

If X is inductive, then the set $\{x \in X : x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

Solution. Let X be inductive. Let $Y = \{x \in X : x \text{ is transitive}\}$. Since X is inductive, $\emptyset \in X$. Since $\emptyset$ is transitive (vacuously), $\emptyset \in Y$. Let $x \in Y$, then $x \in X$ and x is transitive. Since X is inductive, $x \cup \{x\} \in X$. If $y \in x \cup \{x\}$, then $y \in x$ or $y = x$. If $y \in x$, then since x is transitive, $y \subset x$ and therefore $y \subset x \cup \{x\}$. If $y = x$, then clearly $y \subset x \cup \{x\}$. Hence $x \cup \{x\}$ is transitive, from which it follows that $x \cup \{x\} \in Y$. Therefore Y is inductive.

Since $\boldsymbol{N}$ is inductive, the set $M = \{n \in \boldsymbol{N} : n \text{ is transitive}\}$ is inductive. Clearly, $M \subset \boldsymbol{N}$, and since M is inductive, $\boldsymbol{N} \subset M$, and therefore $M = \boldsymbol{N}$. From this it follows that for every $n \in \boldsymbol{N}$, n is transitive. □

1.5 Exercise 5

If X is inductive, then the set $\{x \in X : x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in \boldsymbol{N}$.

Solution. Let X be inductive. Let $Y = \{x \in X : x \text{ is transitive and } x \notin x\}$. Since X is inductive, $\emptyset \in X$. Since $\emptyset$ is transitive and $\emptyset \notin \emptyset$, $\emptyset \in Y$. Let $x \in Y$, then $x \in X$, x is transitive, and $x \notin x$. Since X is inductive, $x \cup \{x\} \in X$. If $y \in x \cup \{x\}$, then $y \in x$ or $y = x$. If $y \in x$, then since x is transitive, $y \subset x$ and therefore $y \subset x \cup \{x\}$. If $y = x$, then clearly $y \subset x \cup \{x\}$. Hence $x \cup \{x\}$ is transitive. Suppose that $x \cup \{x\} \in x \cup \{x\}$. Then $x \cup \{x\} \in x$ or $x \cup \{x\} = x$. If $x \cup \{x\} \in x$, then since x is transitive, we have $x \cup \{x\} \subset x$. Therefore, since $x \in x \cup \{x\}$, we have $x \in x$, which is a contradiction. If $x \cup \{x\} = x$, then since $x \in x \cup \{x\}$, we again have $x \in x$. Thus in either case we have reached a contradiction, and conclude that $x \cup \{x\} \notin x \cup \{x\}$. Therefore $x \cup \{x\} \in Y$. Hence Y is inductive.

Since $\boldsymbol{N}$ is inductive, the set $M = \{n \in \boldsymbol{N} : n \text{ is transitive and } n \notin n\}$ is inductive. Clearly, $M \subset \boldsymbol{N}$, and since M is inductive, $\boldsymbol{N} \subset M$, and therefore $M = \boldsymbol{N}$. Therefore $n \notin n$ for every $n \in \boldsymbol{N}$. Since $n \in n \cup \{n\}$, but $n \notin n$, it follows that $n \neq n + 1$. □

1.6 Exercise 6

If X is inductive, then the set $\{x \in X : x \text{ is transitive and every nonempty } z \subset x \text{ has an } \in\text{-minimal element}\}$ is inductive (t is $\in$-*minimal* in z if there is no $s \in z$ such that $s \in t$).

Solution. Let X be inductive. Let $Y = \{x \in X : x \text{ is transitive and every nonempty } z \subset x \text{ has an } \in\text{-minimal element}\}$. Since X is inductive, $\emptyset \in X$. Since $\emptyset$ is transitive and since $\emptyset$ has no nonempty subsets, $\emptyset \in Y$. Let $x \in Y$, then $x \in X$, x is transitive, and every nonempty subset of x has an $\in$-minimal element. Since X is inductive, $x \cup \{x\} \in X$. If $y \in x \cup \{x\}$, then $y \in x$ or $y = x$. If $y \in x$, then since x is transitive, $y \subset x$ and therefore $y \subset x \cup \{x\}$. If $y = x$, then clearly $y \subset x \cup \{x\}$. Hence $x \cup \{x\}$ is transitive. Let z be a nonempty subset of $x \cup \{x\}$. If $z - \{x\}$ is empty, then clearly $z = \{x\}$, hence z has an $\in$-minimal element, namely, x. If $z - \{x\}$ is nonempty, then since $z - \{x\} \subset x$, it follows that $z - \{x\}$ has an $\in$-minimal element. Let t be an $\in$-minimal element of $z - \{x\}$. Then t is also an $\in$-minimal element of z. To see this, suppose that $x \in t$. Since x is transitive, $t \in x$ implies $t \subset x$. Hence, $x \in t$ implies $x \in x$. This means that x has no $\in$-minimal element, which is a contradiction. Therefore, $x \cup \{x\} \in Y$. Hence Y is inductive. $\square$

1.7 Exercise 7

Every nonempty $X \subset \boldsymbol{N}$ has an $\in$-minimal element. [Pick $n \in X$ and look at $X \cap n$.]

Solution. Let $X \subset \boldsymbol{N}$ be nonempty. Since X is nonempty, let $n \in X$. If $n \cap X = \emptyset$, then $m \in n$ implies $m \notin X$. Hence n is an $\in$-minimal element of X. If $n \cap X$ is nonempty, then since $n \cap X$ is a nonempty subset of n, by Exercise 1.6, $n \cap X$ has an $\in$-minimal element. Let t be an $\in$-minimal element of $n \cap X$. Suppose that $s \in X$ such that $s \in t$. Since, n is transitive, $t \in n$ implies $t \subset n$, and therefore $s \in n$. This contradicts the minimality of t. Therefore, no such s exists and we conclude that t is an $\in$-minimal element of X. $\square$

1.8 Exercise 8

If X is inductive then so is $\{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$. Hence each $n \neq 0$ is $m + 1$ for some m.

Solution. Let X be inductive. Let $Y = \{x \in X : x = \emptyset \text{ or } x = y \cup \{y\} \text{ for some } y\}$. Clearly, $\emptyset \in Y$. Let $x \in Y$, then $x \in X$. Since X is inductive, $x \cup \{x\} \in X$, and it follows that $x \cup \{x\} \in Y$.

Hence $\boldsymbol{N} \subset Y$. Therefore, either $n = \emptyset = 0$ or $n = m \cup \{m\}$ for some m. If the latter, then $m \in n$ and therefore $m \in \boldsymbol{N}$. Hence $n = m + 1$. $\square$

1.9 Exercise 9

Induction. Let A be a subset of $\boldsymbol{N}$ such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \boldsymbol{N}$.

Solution. Suppose that $A \neq \boldsymbol{N}$. Then $\boldsymbol{N} - A$ is a nonempty subset of $\boldsymbol{N}$. Therefore, by Exercise 1.7, $\boldsymbol{N} - A$ has an $\in$-minimal element. Let t be an $\in$-minimal element of $\boldsymbol{N} - A$. Since $0 \in A$, we have $t \neq 0$. Therefore, by Exercise 1.8, $t = s + 1$ for some $s \in \boldsymbol{N}$. Suppose that $s \notin A$. Then $s \in \boldsymbol{N} - A$. However, since $s \in t$, this contradicts the minimality of t in $\boldsymbol{N} - A$. Hence $s \in A$, from which it follows that $t = s + 1 \in A$. But this is another contradiction, and we conclude that $A = \boldsymbol{N}$. $\square$

1.10 Exercise 10

Each $n \in \boldsymbol{N}$ is T-finite.

Solution. Let $A = \{n \in \boldsymbol{N} : n \text{ is T-finite}\}$. Clearly 0 is T-finite, since if X is a nonempty subset of $P(0) = \{0\}$, then $X = \{0\}$, where obviously 0 is $\subset$-maximal in X. Let $n \in A$ and X a nonempty subset of $P(n+1)$. Consider the set $Y = \{y : y = x - \{n\} \text{ for some } x \in X\}$. Clearly $Y \subset P(n)$ and Y is nonempty because X is. Therefore let u be a $\subset$-maximal element of Y. If $u \cup \{n\} \notin X$, then u is a $\subset$-maximal element of X. To see this, suppose that $v \in X$ with $u \subset v$. Then $v - \{n\} \in Y$, and since $n \notin u$, we have $u \subset v - \{n\}$, which contradicts the maximality of u. On the other hand, if $u \cup \{n\} \in X$, then $u \cup \{n\}$ is a $\subset$-maximal element of X. In this case, we suppose that $v \in X$ with $u \cup \{n\} \subset v$. Then $u = (u \cup \{n\}) - \{n\} \subset v - \{n\}$. Since $v - \{n\} \in Y$, this contradicts the maximality of u. Hence X has a $\subset$-maximal element. Therefore $n + 1 \in A$. By Exercise 1.9, $A = \boldsymbol{N}$, and therefore every $n \in \boldsymbol{N}$ is T-finite. $\square$

1.11 Exercise 11

$\boldsymbol{N}$ is T-infinite; the set $\boldsymbol{N} \subset P(\boldsymbol{N})$ has no $\subset$-maximal element.

Solution. Suppose that $\boldsymbol{N}$ has a $\subset$-maximal element k. Since $k + 1 = k \cup \{k\} \in \boldsymbol{N}$ and since $k \subset k + 1$, we have reached a contradiction. Hence $\boldsymbol{N}$ has no $\subset$-maximal element. $\square$

1.12 Exercise 12

Every finite set is T-finite.

Solution. Let X be a finite set. Then there exists an $n \in \boldsymbol{N}$ such that there exists a one-to-one mapping, f, of X onto n, i.e., $f : X \to n$. Let $A \subset P(X)$ be nonempty. Thus, for each $x \in A$, we have $f(x) \subset n$. Let $M = \{y : y = f(x) \text{ for some } x \in A\}$. Then M is a nonempty collection of subsets of n, and since n is T-finite, M has a $\subset$-maximal element k. The claim is that $u = f_{-1}(k)$ is $\subset$-maximal in X. Indeed, let $v \in A$ with $u \subset v$. Since f is one-to-one, we therefore have $f(u) \subset f(v)$. However, this contradicts the maximality of $f(u)$. Hence every nonempty collection of subsets of X has a $\subset$-maximal element, and hence X is T-finite. $\square$

1.13 Exercise 13

Every infinite set is T-infinite. [If S is infinite, consider $X = \{u \subset S : u \text{ is finite}\}$.]

Solution. Let S be infinite and $X = \{u \subset S : u \text{ is finite}\}$. Since $\emptyset \subset S$ and $\emptyset$ is finite, we have $\emptyset \in X$ and hence X is nonempty. Suppose that X has a $\subset$-maximal element, v. Then $S - v$ is nonempty, hence, let $x \in S - v$. It follows that $v \subset v \cup \{x\}$. However, $v \cup \{x\}$ is finite since v is finite. Therefore, we have reached a contradiction and conclude that X has no $\subset$-maximal element, i.e., X is T-infinite. $\square$

1.14 Exercise 14

The Separation Axioms follow from the Replacement Schema. [Given ϕ, let $F = \{(x, x) : \phi(x)\}$. Then $\{x \in X : \phi(x)\} = F(X)$ for every X.]

Solution. Let p and X be sets and let $\phi(u, p)$ be a formula. Let $F = \{(x, x) : \phi(x, p)\}$. Clearly, $\{x \in X : \phi(x, p)\} = F(X)$. Therefore, by Replacement, $\{x \in X : \phi(x, p)\}$ is a set. $\square$

1.15 Exercise 15

Instead of Union, Power Set, and Replacement Axioms consider the following weaker versions:

$$\forall X \exists Y \bigcup X \subset Y, \qquad \text{i.e., } \forall X \exists Y \, (\forall x \in X)(\forall u \in x) u \in Y, \tag{1.8}$$

$$\forall X \exists Y \; P(X) \subset Y, \qquad \text{i.e., } \forall X \exists Y \, \forall u (u \subset X \to u \subset Y), \tag{1.9}$$

$$\text{If a class } F \text{ is a function, then } \forall X \exists Y \; F(X) \subset Y. \tag{1.10}$$

Then axioms 1.4, 1.5, and 1.7 can be proved from (1.8), (1.9), and (1.10), using the Separation Schema (1.3).

Solution. Using (1.8), let X and Y be such that $(\forall x \in X)(\forall u \in x) \; u \in Y$. Let $Z = \{u \in Y : (\exists x \in X) \; u \in x\}$. Then by the Separation Schema, Z is a set, and $Z = \bigcup X$.

Similarly, using (1.9), consider $Z = \{u \in Y : u \subset X\}$ And using (1.10), consider $Z = \{u \in Y : (\exists x \in X) \; \phi(x, u, p)\}$. $\square$

Chapter 2

Ordinal Numbers

2.1 Exercise 1

The relation "$(P,<)$ is isomorphic to $(Q,<)$" is an equivalence relation (on the class of all partially ordered sets).

Solution. Let $(P,<)$ be a partially ordered set. Let $\mathrm{id}_P : P \to P$ be the identity function, i.e., $\mathrm{id}_P(p) = p$ for every $p \in P$. Clearly, id_P is a one-to-one function of P onto itself. Also, id_P is obviously order-preserving, and since $\mathrm{id}_P = \mathrm{id}_P^{-1}$, it follows that id_P^{-1} is order-preserving. Therefore, id_P is an automorphism of $(P,>)$. Hence, $(P,>)$ is isomorphic to itself.

Let $P(,<)$ be isomorphic to $Q(,<)$ and let $f : P \to Q$ be an isomorphism. Then $f^{-1} : Q \to P$ is one-to-one, onto, and order-preserving. f^{-1} is therefore an isomorphism, and hence, $(Q,<)$ is isomorphic to $(P,<)$.

Let $(P,<)$ be isomorphic to $(Q,<)$ and $(Q,<)$ be isomorphic to $(R,<)$. Then there exist isomorphisms $f : P \to Q$ and $g : Q \to R$. Therefore the composition $g \circ f$ is one-to-one and onto since f and g are. Furthermore, $g \circ f$ is order-preserving, since $x > y$ implies $f(x) > f(y)$ which further implies $g(f(x)) > g(f(y))$. The inverse composition, $f^{-1} \circ g^{-1}$ is also order-preserving. Hence $g \circ f$ is an isomorphism and thus $(P,<)$ is isomorphic to $(R,<)$. □

2.2 Exercise 2

α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$, for every β.

Solution. Let α be a limit ordinal. If $\alpha = 0$, then there is no β such that $\beta < \alpha$. On the other hand, if $\alpha \neq 0$, then let β be an ordinal such that $\beta < \alpha \leq \beta + 1$. Let $x \in \beta + 1$. Then, either $x \in \beta$, whereupon $x \in \alpha$, or $x = \beta$, where again we have $x \in \alpha$. Hence, $x \in \beta + 1$ implies $x \in \alpha$, or, $\beta + 1 \subset \alpha$. If $\beta + 1 = \alpha$, then α is a successor ordinal and therefore not a limit ordinal, which is a contradiction. If $\beta+1 \neq \alpha$, then by Lemma 2.11(iii), we have $\beta + 1 < \alpha$, which is also a contradiction and we therefore conclude that $\beta < \alpha$ implies $\beta + 1 < \alpha$

Conversely, suppose that α is not a limit ordinal, i.e., α is a successor ordinal. Then $\alpha = \beta+1$ for some β. Clearly, $\beta < \alpha$. However, $\beta+1 \not< \alpha$. $\square$

2.3 Exercise 3

If a set X is inductive, then $X \cap \mathit{Ord}$ is inductive. The set $\boldsymbol{N} = \bigcap\{X : X \text{ is inductive}\}$ is the least limit ordinal $\neq 0$.

Solution. Let X be an inductive set. Then, by the Separation Schema, we have that $X \cap \mathit{Ord}$ is a set. Since X is inductive, $\emptyset \in X$. Clearly, $0 = \emptyset \in \mathit{Ord}$. Hence $\emptyset \in X \cap \mathit{Ord}$. Let $x \in X \cap \mathit{Ord}$. Since X is inductive, $x+1 = x \cap \{x\} \in X$. Using (2.5), we have $x + 1 \in \mathit{Ord}$. Therefore $x + 1 \in X \cap \mathit{Ord}$. Hence, $X \cap \mathit{Ord}$ is inductive.

By Exercise 1.3, $\boldsymbol{N}$ is transitive. By Exercise 1.7, $(\boldsymbol{N}, \in)$ is well-founded. Since $X \cap \mathit{Ord}$ is inductive, we have $\boldsymbol{N} \subset X \cap \mathit{Ord} \subset \mathit{Ord}$. Therefore, by Lemma 2.11(iv), it follows that $\boldsymbol{N}$ is linearly ordered by $\in$. Hence, $\boldsymbol{N}$ is transitive and well-ordered by $\in$, that is, $\boldsymbol{N}$ is an ordinal. Since $\emptyset \in \boldsymbol{N}$, we have $\boldsymbol{N} \neq 0$. Let $n \in \boldsymbol{N}$ be a nonzero ordinal. Then, by Exercise 1.8, there exists $m \in \boldsymbol{N}$ such that $n = m + 1$. Thus, n is a successor ordinal. Therefore, by Exercise 2.2, $\boldsymbol{N}$ is a limit ordinal, in fact, it is the least nonzero limit ordinal. $\square$

2.4 Exercise 4

(Without the Axiom of Infinity). Let ω = least limit $\alpha \neq 0$ if it exists, $\omega = \mathit{Ord}$ otherwise. Prove that the following statements are equivalent:

1. There exists an inductive set.
2. There exists an infinite set.

3. ω is a set.

   [For (ii) $\to$ (iii), apply Replacement to the set of all finite subsets of X.]

Solution. (i) $\to$ (ii). Suppose that there exists an inductive set. Let $\boldsymbol{N} = \bigcap\{X : X \text{ is inductive}\}$. By Exercise 1.11, $\boldsymbol{N}$ is T-infinite. Hence, by Exercises 1.12 and 1.13, $\boldsymbol{N}$ is infinite.

(ii) $\to$ (iii). Suppose that there exists an infinite set, X, and that $\omega = Ord$. Let $Y = \{y \in P(Y) : y \text{ is finite}\}$. If $y \in Y$, then y is finite; therefore there exists a one-to-one mapping, f, of y onto some $n \in \omega$. By Replacement, we have $f(Y)$ is a set. Clearly $\emptyset \in Y$ and $f(\emptyset) = 0$. Thus, $f(Y)$ is nonempty, since $0 \in f(Y)$. Suppose that $n \in f(Y)$. Then there exists $y \in Y$ such that $f(y) = n$. Since X is infinite, it follows that $X - y$ is nonempty. Hence, let $z \in X - y$. Therefore, $y \cup \{z\} \in Y$ and thus $f(y \cup \{z\}) = n + 1$. It follows that $n + 1 \in f(Y)$. Therefore, by induction, $f(Y) = \omega$, and we have ω is a set

(iii) $\to$ (i). Let ω be a set. Then $0 < \omega$, and for every $n \in \omega$, $n + 1 \in \omega$ since ω is a limit ordinal. Hence ω is inductive. □

2.5 Exercise 5

If W is a well-ordered set, then there exists no sequence $\langle a_n : n \in \boldsymbol{N}\rangle$ in W such that $a_0 > a_1 > a_2 > \ldots$.

Solution. Suppose that there exists such a sequence. Let $X = \{a_n : n \in \boldsymbol{N}\}$. Then X is a nonempty subset of W. Therefore, let x be the least element of X. Let n be the least $n \in \boldsymbol{N}$ such that $a_n = x$. Hence, $a_{n+1} \geq a_n$. We have therefore reached a contradiction, and conclude that no such sequence exists. □

2.6 Exercise 6

There are arbitrarily large limit ordinals; i.e., $\forall\alpha\ \exists\beta > \alpha$ (β is a limit.)

[Consider $\lim_{n\to\omega} \alpha_n$, where $\alpha_{n+1} = \alpha_n + 1$.]

Solution. Suppose that there exists a largest limit ordinal, and let α_0 be this ordinal. For every $n < \omega$, let $\alpha_{n+1} = \alpha_n + 1$. Since $\langle \alpha_n : n < \omega\rangle$ is clearly a nondecreasing sequence of ordinals, we may define the limit $\lim_{n\to\omega} \alpha_n = \sup\{\alpha_n : n < \omega\}$. By (2.4), $\sup\{\alpha_n : n < \omega\}$ is an ordinal, hence, let $\alpha_\omega = \sup\{\alpha_n : n < \omega\}$. If α_ω is a successor, then there exists an ordinal $\xi < \alpha_\omega$ such that $\xi + 1 = \alpha_\omega$.. However, if $\xi < \alpha_\omega$, then $\xi \in \alpha_\omega$, and it follows that $\xi + 1 \in \alpha_\omega$. We have therefore reached a contradiction and conclude that α_ω is a limit ordinal. Since $\alpha_0 < \alpha_\omega$, we see that α_0 is not the largest limit ordinal, which is yet another contradiction. Hence, we find that there indeed arbitrarily large limit ordinals. $\square$

2.7 Exercise 7

Every normal sequence $\langle \gamma_\alpha : \alpha \in \mathit{Ord} \rangle$ has arbitrarily large *fixed points*, i.e., α such that $\gamma_\alpha = \alpha$.

[Let $\alpha_{n+1} = \gamma_{\alpha_n}$, and $\alpha = \lim_{n\to\omega} \alpha_n$.]

Solution. First, since it is not done in the text, we state explicitly that Ord is well-ordered as a class. That is, if C is any nonempty class of ordinals, then C has a least element. By virtue of remark (2.3) following Lemma 2.11, $\bigcap C$ is the least element of C.

Given this declaration, it is necessary to verify the following extension of Lemma 2.4. If $\langle \gamma_\alpha : \alpha \in \mathit{Ord} \rangle$ is an increasing sequence of ordinals, then $\gamma_\alpha \geq \alpha$ for every $\alpha \in \mathit{Ord}$. To see this, suppose that the class $X = \{\gamma_\alpha : \gamma_\alpha < \alpha\}$ is nonempty, and let β be the least element of X. Then $\gamma_\beta < \beta$, and since $\langle \gamma_\alpha : \alpha \in \mathit{Ord} \rangle$ is an increasing sequence, we have $\gamma_{\gamma_\beta} < \gamma_\beta$. But this contradicts the fact that β is the least ordinal such that $\gamma_\alpha < \alpha$. We therefore conclude that X is empty, and therefore that $\gamma_\alpha \geq \alpha$ for every $\alpha \in \mathit{Ord}$.

Next, if $\langle \gamma_\alpha : \alpha \in \mathit{Ord} \rangle$ is a normal sequence and X is any nonempty set of ordinals, then $\gamma_{\sup X} = \sup\{\gamma_\alpha : \alpha \in X\}$. For this, let $\beta = \sup X$. By remark (2.4) following Lemma 2.11, β is an ordinal. If $\beta = 0$, then $X = \{0\}$, and clearly $\gamma_0 = \sup\{\gamma_0\}$. If β is a successor ordinal, then $\beta = \delta + 1$ for some ordinal δ. Since $\beta = \sup X$, we have $\beta \geq \xi$ for every $\xi \in X$. Suppose, in addition, that $\delta \geq \xi$ for every $\xi \in X$. Then δ is an upper bound of X, and since $\delta < \beta$, we have a contradiction of the fact that $\beta = \sup X$. We therefore conclude that there exists $\epsilon \in X$ such that $\delta < \epsilon$. Since $\beta = \sup X$, we have $\beta \geq \epsilon$. Since $\beta = \inf\{\zeta : \zeta > \delta\}$, we have $\epsilon \geq \beta$. Therefore $\beta = \epsilon$, and it follows that $\beta \in X$. Since $\langle \gamma_\alpha \rangle$ is increasing, it follows that $\gamma_\beta \geq \gamma_\xi$ for every

$\xi \in X$. Since $\gamma_\beta \in \{\gamma_\alpha : \alpha \in X\}$, it follows that $\gamma_\beta = \sup\{\gamma_\alpha : \alpha \in X\}$. If β is a nonzero limit ordinal, then let δ be an ordinal such that $\delta < \beta$. Since β is a limit and since $\beta = \sup X$, there exists $\epsilon \in X$ such that $\delta < \epsilon$. Since $\langle \gamma_\alpha \rangle$ is increasing, we have $\gamma_\delta < \gamma_\epsilon$. Hence, $\gamma_\delta < \sup\{\gamma_\alpha : \alpha \in X\}$, which yields, since $\langle \gamma_\alpha \rangle$ is continuous, $\gamma_\beta = \sup\{\gamma_\delta : \delta < \beta\} \leq \sup\{\gamma_\alpha : \alpha \in X\}$. However, since $\langle \gamma_\alpha \rangle$ increasing and $\beta = \sup X$, we have $\sup\{\gamma_\alpha : \alpha \in X\} \leq \gamma_\beta$. Therefore, $\gamma_\beta = \sup\{\gamma_\alpha : \alpha \in X\}$, as desired.

We are now in a position to prove the main result. We define a sequence, $\langle \alpha_n : n < \omega \rangle$ recursively, as follows: Let $\alpha \in \mathit{Ord}$ and let $\alpha_0 = \alpha$, and $\alpha_{n+1} = \gamma_{\alpha_n}$. Let $\beta = \sup\{\alpha_n : n < \omega\}$. Thus, $\beta > \alpha$. Moreover, $\gamma_\beta = \gamma_{\sup\{\alpha_n : n < \omega\}} = \sup\{\gamma_{\alpha_n} : n < \omega\} = \sup\{\alpha_{n+1} : n < \omega\} = \beta$.

We may then find an even larger fixed point by taking $\alpha = \gamma_\beta$ and repeating the process. Hence $\langle \gamma_\alpha \rangle$ has arbitrarily large fixed points. $\square$

2.8 Exercise 8

For all α, β, and γ,

(i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$,

(ii) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$,

(iii) $\left(\alpha^\beta\right)^\gamma = \alpha^{\beta \cdot \gamma}$.

Solution. In each case, we proceed by transfinite induction on γ.

(i) If $\gamma = 0$ then

$$\begin{aligned} \alpha \cdot (\beta + 0) &= \alpha \cdot \beta \\ &= \alpha \cdot \beta + 0 \\ &= \alpha \cdot \beta + \alpha \cdot 0 \end{aligned}$$

If $\gamma + 1$ is a successor ordinal then

$$\begin{aligned} \alpha \cdot (\beta + (\gamma + 1)) &= \alpha \cdot ((\beta + \gamma) + 1) \\ &= \alpha \cdot (\beta + \gamma) + \alpha \\ &= (\alpha \cdot \beta + \alpha \cdot \gamma) + \alpha \\ &= \alpha \cdot \beta + (\alpha \cdot \gamma + \alpha) \\ &= \alpha \cdot \beta + \alpha \cdot (\gamma + 1) \end{aligned}$$

If γ is a nonzero limit ordinal then

$$\begin{aligned}\alpha\cdot(\beta+\gamma) &= \alpha\cdot\lim_{\xi\to\gamma}(\beta+\xi)\\ &= \lim_{\xi\to\gamma}\alpha\cdot(\beta+\xi)\\ &= \lim_{\xi\to\gamma}\alpha\cdot\beta+\alpha\cdot\xi\\ &= \alpha\cdot\beta+\lim_{\xi\to\gamma}\alpha\cdot\xi\\ &= \alpha\cdot\beta+\alpha\cdot\gamma\end{aligned}$$

(ii) If $\gamma = 0$ then

$$\begin{aligned}\alpha^{\beta+0} &= \alpha^{\beta}\\ &= \alpha^{\beta}\cdot 1\\ &= \alpha^{\beta}\cdot\alpha^{0}\end{aligned}$$

If $\gamma+1$ is a successor ordinal then

$$\begin{aligned}\alpha^{\beta+(\gamma+1)} &= \alpha^{(\beta+\gamma)+1}\\ &= \alpha^{\beta+\gamma}\cdot\alpha\\ &= (\alpha^{\beta}\cdot\alpha^{\gamma})\cdot\alpha\\ &= \alpha^{\beta}\cdot(\alpha^{\gamma}\cdot\alpha)\\ &= \alpha^{\beta}\cdot\alpha^{\gamma+1}\end{aligned}$$

If γ is a nonzero limit ordinal then

$$\begin{aligned}\alpha^{\beta+\gamma} &= \alpha^{\beta+\lim_{\xi\to\gamma}\xi}\\ &= \alpha^{\lim_{\xi\to\gamma}(\beta+\xi)}\\ &= \lim_{\xi\to\gamma}\alpha^{\beta+\xi}\\ &= \lim_{\xi\to\gamma}\left(\alpha^{\beta}\cdot\alpha^{\xi}\right)\\ &= \alpha^{\beta}\cdot\lim_{\xi\to\gamma}\alpha^{\xi}\\ &= \alpha^{\beta}\cdot\alpha^{\lim_{\xi\to\gamma}\xi}\\ &= \alpha^{\beta}\cdot\alpha^{\gamma}\end{aligned}$$

(iii) If $\gamma = 0$ then

$$\begin{aligned}\left(\alpha^\beta\right)^0 &= 1\\ &= \alpha^0\\ &= \alpha^{\beta\cdot 0}\end{aligned}$$

If $\gamma + 1$ is a successor ordinal then

$$\begin{aligned}\left(\alpha^\beta\right)^{\gamma+1} &= \left(\alpha^\beta\right)^\gamma \cdot \alpha^\beta\\ &= \alpha^{\beta\cdot\gamma}\cdot\alpha^\beta\\ &= \alpha^{\beta\cdot(\gamma+1)}\end{aligned}$$

If γ is a nonzero limit ordinal then

$$\begin{aligned}\left(\alpha^\beta\right)^\gamma &= \left(\alpha^\beta\right)^{\lim_{\xi\to\gamma}\xi}\\ &= \lim_{\xi\to\gamma}\left(\alpha^\beta\right)^\xi\\ &= \lim_{\xi\to\gamma}\alpha^{\beta\cdot\xi}\\ &= \alpha^{\lim_{\xi\to\gamma}\beta\cdot\xi}\\ &= \alpha^{\beta\cdot\lim_{\xi\to\gamma}\xi}\\ &= \alpha^{\beta\cdot\gamma}\end{aligned}$$

□

2.9 Exercise 9

(i) Show that $(\omega+1)\cdot 2 \neq \omega\cdot 2 + 1\cdot 2$.

(ii) Show that $(\omega\cdot 2)^2 \neq \omega^2\cdot 2^2$.

Solution. (i)

$$\begin{aligned}(\omega+1)\cdot 2 &= (\omega+1)+(\omega+1)\\ &= (\omega+(1+\omega))+1\\ &= (\omega+\omega)+1\\ &= \omega\cdot 2+1\\ &< \omega\cdot 2+2\\ &= \omega\cdot 2+1\cdot 2\end{aligned}$$

(ii)

$$
\begin{aligned}
(\omega \cdot 2)^2 &= (\omega \cdot 2) \cdot (\omega \cdot 2) \\
&= (\omega \cdot (2 \cdot \omega)) \cdot 2 \\
&= (\omega \cdot \omega) \cdot 2 \\
&= \omega^2 \cdot 2 \\
&< \omega^2 \cdot 4 \\
&= \omega^2 \cdot 2^2
\end{aligned}
$$

□

2.10 Exercise 10

If $\alpha < \beta$ then $\alpha + \gamma \leq \beta + \gamma$, $\alpha \cdot \gamma \leq \beta \cdot \gamma$, and $\alpha^\gamma \leq \beta^\gamma$.

Solution. In each case, we proceed by induction on γ. First we consider sums. If $\gamma = 0$, we have $\alpha + 0 = \alpha < \beta = \beta + 0$. If $\gamma + 1$ is a successor ordinal, then $\alpha + \gamma \leq \beta + \gamma < (\beta + \gamma) + 1$, therefore $\alpha + (\gamma + 1) = (\alpha + \gamma) + 1 \leq (\beta + \gamma) + 1 = \beta + (\gamma + 1)$. Finally, if γ is a nonzero limit ordinal, then for every $\xi < \gamma$, we have $\alpha + \xi \leq \beta + \xi$. From this it follows that $\lim_{\xi \to \gamma} \alpha + \xi \leq \lim_{\xi \to \gamma} \beta + \xi$, or, equivalently, $\alpha + \gamma \leq \beta + \gamma$.

Next we consider products. If $\gamma = 0$, then $\alpha \cdot 0 = 0 = \beta \cdot 0$. If $\gamma + 1$ is a successor ordinal, then since $\alpha < \beta$, by virtue of Lemma 2.25(i) we have $\alpha \cdot (\gamma + 1) = \alpha \cdot \gamma + \alpha < \alpha \cdot \gamma + \beta$ and by the preceding result for sums, $\alpha \cdot \gamma + \beta \leq \beta \cdot \gamma + \beta = \beta \cdot (\gamma + 1)$. Therefore, $\alpha \cdot (\gamma + 1) \leq \beta \cdot (\gamma + 1)$. Finally, if γ is a nonzero limit ordinal, then for every $\xi < \gamma$, we have $\alpha \cdot \xi \leq \beta \cdot \xi$. From this it follows that $\lim_{\xi \to \gamma} \alpha \cdot \xi \leq \lim_{\xi \to \gamma} \beta \cdot \xi$, or, equivalently, $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

Lastly we consider exponentials. If $\gamma = 0$, then $\alpha^0 = 1 = \beta^0$. If $\gamma + 1$ is a successor ordinal, then since $\alpha < \beta$, by Lemma 2.25(iii), we have $\alpha^{\gamma+1} = \alpha^\gamma \cdot \alpha < \alpha^\gamma \cdot \beta$, and by the preceding result for products, $\alpha^\gamma \cdot \beta \leq \beta^\gamma \cdot \beta = \beta^{\gamma+1}$. Therefore, $\alpha^{\gamma+1} \leq \beta^{\gamma+1}$. Finally, if γ is a nonzero limit ordinal, then for every $\xi < \gamma$, we have $\alpha^\xi \leq \beta^\xi$. From this it follows that $\lim_{\xi \to \gamma} \alpha^\xi \leq \lim_{\xi \to \gamma} \beta^\xi$, or, equivalently, $\alpha^\gamma \leq \beta^\gamma$. □

2.11 Exercise 11

Find α, β, γ such that

(i) $\alpha < \beta$ and $\alpha + \gamma = \beta + \gamma$,

(ii) $\alpha < \beta$ and $\alpha \cdot \gamma = \beta \cdot \gamma$,

(iii) $\alpha < \beta$ and $\alpha^\gamma = \beta^\gamma$.

Solution. (i) $0 + \omega = 1 + \omega = \omega$.

(ii) $1 \cdot \omega = 2 \cdot \omega = \omega$.

(iii) $2^\omega = 3^\omega = \omega$.

□

2.12 Exercise 12

Let $\varepsilon_0 = \lim_{n\to\omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$ for all n. Show that ε_0 is the least ordinal ε such that $\omega^\varepsilon = \varepsilon$.

Solution. Let $\beta_n = \omega^{\alpha_n}$ for all $n < \omega$, that is, $\beta_n = \alpha_{n+1}$ for all n. Then $\lim_{n\to\omega} \beta_n = \lim_{n\to\omega} \alpha_{n+1} = \lim_{n\to\omega} \alpha_n = \varepsilon_0$. Additionally, $\lim_{n\to\omega} \beta_n = \lim_{n\to\omega} \omega^{\alpha_n} = \omega^{\lim_{n\to\omega} \alpha_n} = \omega^{\varepsilon_0}$. Hence, $\varepsilon_0 = \omega^{\varepsilon_0}$.

Suppose that there exists an ordinal $\varepsilon < \varepsilon_0$ such that $\varepsilon = \omega^\varepsilon$. Since $n < \omega^n$ for all $n < \omega$, it follows that $\omega \leq \varepsilon$. Furthermore, by Lemma 2.25(v), $\omega < \omega^\omega$. Therefore $\omega < \varepsilon$. Let n be the least natural number such that $\varepsilon < \alpha_n$. Since $n > 0$, there exists a natural number m such that $m + 1 = n$. Hence, $\alpha_n = \omega^{\alpha_m}$. Since $\alpha_m \leq \varepsilon$, we have $\alpha_n = \omega^{\alpha_m} \leq \omega^\varepsilon = \varepsilon$, which is a contradiction. □