Solutions to Jech's Set Theory, The Third Millenium Edition, Revised and Expanded

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Contents

Part I

Basic Set Theory

Chapter 1

Axioms of Set Theory

1.1 Exercise 1

Verify (1.1).

Solution. If $a = c$ and $b = d$, then $(a, b) = (c, d)$ as a consequence of an axiom of first-order logic with equality, namely, that equals may be substituted for equals in a formula.

Conversely, suppose that $(a, b) = (c, d)$. Then

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}.$$

If $a = b$, then

$$\{\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\}.$$

Therefore $\{a\} = \{c\}$ and $\{a\} = \{c, d\}$. Hence, $a = c = d$, from which it follows that $a = c$ and $b = d$. If $a \neq b$, then $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$. Therefore $a = c$, and from this it follows that $\{a, b\} = \{a, d\}$. Hence $b = d$. $\square$

1.2 Exercise 2

There is no set X such that $P(X) \subset X$.

Solution. Suppose there exists a set X such that $P(X) \subset X$. Let $Y = \{x : x \in X \text{ and } x \notin x\}$. Clearly, $Y \subset X$, hence $Y \in P(X)$ and therefore $Y \in X$. However $Y \in Y$ if and only if $Y \notin Y$. We have therefore reached a contradiction and conclude that no such set X exists. $\square$

Let

$$\boldsymbol{N} = \bigcap \{X \colon X \text{ is inductive}\}.$$

$\boldsymbol{N}$ is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0,1\}, \quad 3 = \{0,1,2\}, \quad \ldots .$$

If $n \in \boldsymbol{N}$, let $n+1 = n \cup \{n\}$. Let us define $<$ (on $\boldsymbol{N}$) by $n < m$ if and only if $n \in m$.

A set T is *transitive* if $x \in T$ implies $x \subset T$.

1.3 Exercise 3

If X is inductive, then the set $\{x \in X : x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive, and for each n, $n = \{m \in \boldsymbol{N} : m < n\}$.

Solution. Let X be inductive. Let $Y = \{x \in X : x \subset X\}$. Since X is inductive, $\emptyset \in X$. Since $\emptyset \subset X$, $\emptyset \in Y$. Let $x \in Y$, then $x \in X$ and $x \subset X$. Since X is inductive, $x \cup \{x\} \in X$. If $y \in x \cup \{x\}$, then $y \in x$ or $y = x$. If $y \in x$, then since $x \subset X$, we have $y \in X$. If $y = x$, then clearly $y \in X$. Hence $x \cup \{x\} \subset X$. Thus, it follows that $x \cup \{x\} \in Y$ and therefore Y is inductive.

Since $\boldsymbol{N}$ is inductive, the set $M = \{n \in \boldsymbol{N} : n \subset \boldsymbol{N}\}$ is inductive. Clearly, $M \subset \boldsymbol{N}$, and since M is inductive, $\boldsymbol{N} \subset M$, and therefore $M = \boldsymbol{N}$. From this it follows that for every $n \in \boldsymbol{N}$, $n \subset \boldsymbol{N}$. Hence, $\boldsymbol{N}$ is transitive. If $n \in \boldsymbol{N}$, then $n \subset \boldsymbol{N}$. Hence if $m \in n$, then $m \in \boldsymbol{N}$, and by definition, $m < n$. Therefore $n \subset \{m \in \boldsymbol{N} : m < n\}$. Conversely, if $k \in \{m \in \boldsymbol{N} : m < n\}$, then $k < n$ and consequently $k \in n$. It follows that $\{m \in \boldsymbol{N} : m < n\} \subset n$ and therefore that $n = \{m \in \boldsymbol{N} : m < n\}$. □

1.4 Exercise 4

If X is inductive, then the set $\{x \in X : x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

Solution. Let X be inductive. Let $Y = \{x \in X : x \text{ is transitive}\}$. Since X is inductive, $\emptyset \in X$. Since $\emptyset$ is transitive (vacuously), $\emptyset \in Y$. Let $x \in Y$, then $x \in X$ and x is transitive. Since X is inductive, $x \cup \{x\} \in X$. If $y \in x \cup \{x\}$, then $y \in x$ or $y = x$. If $y \in x$, then since x is transitive, $y \subset x$ and therefore $y \subset x \cup \{x\}$. If $y = x$, then clearly $y \subset x \cup \{x\}$. Hence $x \cup \{x\}$ is transitive, from which it follows that $x \cup \{x\} \in Y$. Therefore Y is inductive.

Since $\boldsymbol{N}$ is inductive, the set $M = \{n \in \boldsymbol{N} : n \text{ is transitive}\}$ is inductive. Clearly, $M \subset \boldsymbol{N}$, and since M is inductive, $\boldsymbol{N} \subset M$, and therefore $M = \boldsymbol{N}$. From this it follows that for every $n \in \boldsymbol{N}$, n is transitive. $\square$

1.5 Exercise 5

If X is inductive, then the set $\{x \in X : x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n + 1$ for each $n \in \boldsymbol{N}$.

Solution. Let X be inductive. Let $Y = \{x \in X : x \text{ is transitive and } x \notin x\}$. Since X is inductive, $\emptyset \in X$. Since $\emptyset$ is transitive and $\emptyset \notin \emptyset$, $\emptyset \in Y$. Let $x \in Y$, then $x \in X$, x is transitive, and $x \notin x$. Since X is inductive, $x \cup \{x\} \in X$. If $y \in x \cup \{x\}$, then $y \in x$ or $y = x$. If $y \in x$, then since x is transitive, $y \subset x$ and therefore $y \subset x \cup \{x\}$. If $y = x$, then clearly $y \subset x \cup \{x\}$. Hence $x \cup \{x\}$ is transitive. Suppose that $x \cup \{x\} \in x \cup \{x\}$. Then $x \cup \{x\} \in x$ or $x \cup \{x\} = x$. If $x \cup \{x\} \in x$, then since x is transitive, we have $x \cup \{x\} \subset x$. Therefore, since $x \in x \cup \{x\}$, we have $x \in x$, which is a contradiction. If $x \cup \{x\} = x$, then since $x \in x \cup \{x\}$, we again have $x \in x$. Thus in either case we have reached a contradiction, and conclude that $x \cup \{x\} \notin x \cup \{x\}$. Therefore $x \cup \{x\} \in Y$. Hence Y is inductive.

Since $\boldsymbol{N}$ is inductive, the set $M = \{n \in \boldsymbol{N} : n \text{ is transitive and } n \notin n\}$ is inductive. Clearly, $M \subset \boldsymbol{N}$, and since M is inductive, $\boldsymbol{N} \subset M$, and therefore $M = \boldsymbol{N}$. Therefore $n \notin n$ for every $n \in \boldsymbol{N}$. Since $n \in n \cup \{n\}$, but $n \notin n$, it follows that $n \neq n + 1$. $\square$