Chapter 4

Vector Spaces

4.1 The Theory of linear equations

Definition 36. The *null space* of a matrix A is the set of all vectors $\boldsymbol{x}$ that satisfy $A\boldsymbol{x} = \mathbf{0}$. We denote it as $\text{nullsp}(A)$.

Example 4.1.1. Find the null space of A.

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

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Example 4.1.2. Identify geometrically the null space of $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

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Theorem 4.1.3. *If A is an $m \times n$ matrix and $m < n$, then A has a nontrivial null space, i.e. there exists a vector $\boldsymbol{x} \neq \mathbf{0}$ such that $A\boldsymbol{x} = \mathbf{0}$.*

Example 4.1.4. Let $A = \begin{bmatrix} 1 & -3 & -2 & 4 \\ 2 & 0 & 2 & 2 \\ 0 & 4 & 4 & -4 \end{bmatrix}$. Find the null space of A.

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Definition 37. The *column space*(*row*) of a matrix A is the set of all linear combinations of the columns (row) of A. It is denoted as $\operatorname{colsp}(A)$ $(\operatorname{rowsp}(A))$.

Since A is $m \times n$, the columns of A are vectors in $\mathbb{R}^m$, so column space of A is a collection of vectors in $\mathbb{R}^m$, where as the null space of A consists of vectors in $\mathbb{R}^n$.

Example 4.1.5. Find the column space of A.

1. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

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Example 4.1.6. Suppose $A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$. The columns of this matrix are nonparallel vectors in $\mathbb{R}^2$, so they span the xy-plane. The column space of the matrix is also $\mathbb{R}^2$.

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Example 4.1.7. Find the column space of $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ and describe geometrically.

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Example 4.1.8. Find the column space of A in Example 4.1.4.

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Theorem 4.1.9. *A vector $\boldsymbol{b}$ is in the column space of A if and only if the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.*

Theorem 4.1.10. *The column space of an $m \times n$ matrix A is the set of all $A\boldsymbol{x}$ as $\boldsymbol{x}$ ranges over all vectors in $\mathbb{R}^n$.*

Example 4.1.11. Is $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ in the column space of $A = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 3 & 5 & 4 \\ 2 & 1 & -3 & -1 \end{bmatrix}$? What is the column space of A?

To find the column space of A, we consider

$$\begin{bmatrix} 1 & 0 & -2 & -1 \\ -1 & 3 & 5 & 4 \\ 2 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

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Definition 38. The *rank* of a matrix A (denoted as rank(A) is the number of columns with a leading entry in its r.e.f.

Example 4.1.12. $\begin{bmatrix} 4 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ has rank 2. Show that $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ has rank 1.

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Example 4.1.13. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 12 & 11 & 10 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. What type of solutions does $A\boldsymbol{x} = \boldsymbol{b}$ have? Find the rank of A.

■

Example 4.1.14. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. What type of solutions does $A\boldsymbol{x} = \boldsymbol{b}$ have? Find the rank of A.

■

Theorem 4.1.15. *Let A be an $m \times n$ matrix.*

1. *Let* $\operatorname{rank}(A) = m$. *Then, for any vector* $\boldsymbol{b}$ *in* $\mathbb{R}^m$, $A\boldsymbol{x} = \boldsymbol{b}$ *has at least one solution.*

2. *Let* $\operatorname{rank}(A) = n$. *Then, for any vector* $\boldsymbol{b}$ *in* $\mathbb{R}^m$, $A\boldsymbol{x} = \boldsymbol{b}$ *has at most one solution.*

3. *If $m = n$, the following statements are equivalent:*

   (a) $\operatorname{rank}(A) = n$.
   (b) For any vector $\boldsymbol{b}$ *in* $\mathbb{R}^m$, $A\boldsymbol{x} = \boldsymbol{b}$ *has a unique solution.*
   (c) A is invertible.
   (d) $\det(A) \neq 0$.

4.2 Vector spaces and subspaces

Definition 39 (Vector Space Axioms)**.** A real vector space consists of

(a) a set V of objects (called *vectors*);

(b) an operation of *vector addition* $\boldsymbol{u} + \boldsymbol{v}$ between any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in V; and

(c) an operation of *scalar multiplication* $c\boldsymbol{u}$ between any real number c and any vector $\boldsymbol{u}$ in V.

The operations must satisfy the following properties:

(V1) Closure under the addition: For all $\boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} + \boldsymbol{v} \in V$.

(V2) Closure under the scalar multiplication: For all $\boldsymbol{u} \in V$ and $c \in \mathbb{R}, c\boldsymbol{u} \in V$.

(V3) Existence of zero vector: There exists a *zero vector* $\mathbf{0} \in V$ such that

$$\boldsymbol{u} + \mathbf{0} = \boldsymbol{u} = \mathbf{0} + \boldsymbol{u}$$

for all $\boldsymbol{u} \in V$.

(V4) Existence of negatives: For each $\boldsymbol{u} \in V$, there exists $\boldsymbol{v} \in V$ such that

$$\boldsymbol{u} + \boldsymbol{v} = \mathbf{0} = \boldsymbol{v} + \boldsymbol{u}.$$

($\boldsymbol{v}$ is called the *negative* of $\boldsymbol{u}$ and denoted by $-\boldsymbol{u}$.)

(V5) Standard operation axioms: For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in \mathbb{R}$,

(i) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$;

(ii) $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$;

(iii) $a(b\boldsymbol{u}) = (ab)\boldsymbol{u}$;

(iv) $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$;

(v) $(a + b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$;

(vi) $1\boldsymbol{u} = \boldsymbol{u}$.

Example 4.2.1.

1. $\mathbb{R}^n$ is a vector space under the standard operations of addition and scalar multiplication defined in Chapter 1.

2. The set of all 2×2 matrices is a vector space under the addition and scalar multiplication defined in Chapter 2.

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}.$$

3. Let $V = \mathbb{R}^2$ and define addition and scalar multiplication operations as follows: If $\boldsymbol{u} = [u_1, u_2]^T$ and $\boldsymbol{v} = [v_1, v_2]^T$, then define

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

and if $k \in \mathbb{R}$, then define

$$k\boldsymbol{u} = \begin{bmatrix} ku_1 \\ 0 \end{bmatrix}.$$

Is V a vector space under the stated operations?

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Definition 40. A subset W of a vector space V is called a *subspace* of V if W is itself a vector space under the addition and scalar multiplication defined on V. We denote W is a subspace of V by $W \leq V$.

Theorem 4.2.2. *If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.*

(a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in W, then $\boldsymbol{u}+\boldsymbol{v} \in W$.

(b) If k is any scalar and $\boldsymbol{u}$ is any vector in W, then $k\boldsymbol{u}$ is in W.

Example 4.2.3. Determine whether W is a subspace of V for each of the following.

1. $V = \mathbb{R}^3$ and $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x + y - z = 0 \right\}$.

2. $V = \mathbb{M}_2$ and W is the set of all 2×2 diagonal matrices.

3. $V = \mathbb{R}^2$ and $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x = 1 \text{ and } y \in \mathbb{R} \right\}$.

4. $V = \mathbb{R}^3$ and $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x < y < z \right\}$.

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Definition 41. Let V be a vector space. If $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}$ is a subset of V, then the subspace

$$W = \{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_n\boldsymbol{u}_n \mid c_1, c_2, \cdots, c_n \in \mathbb{R}\}$$

of V is called the *space spanned* by S (or by $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n$) and we write

$$W = \text{span}(S) \text{ or } W = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}.$$

We also say that S spans W.

Note S is a subset of W.

Example 4.2.4.

1. The standard basis vectors of $\mathbb{R}^n$ span $\mathbb{R}^n$ i.e.
$$\operatorname{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n\} = \mathbb{R}^n.$$

2. Let $V = \mathbb{M}_2$ and $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Every element in $\operatorname{span}(S)$ is of the form

3. The span of a single vector $\boldsymbol{v}$ is $\{a\boldsymbol{v} |\, a \in \mathbb{R}\}$ is just the set of all scalar multiples of $\boldsymbol{v}$, a line passes through the origin.

4. The plane with equation $2x - y + 3z = 0$ is spanned by $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. In fact any two nonparallel vectors in this plane span it.

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Theorem 4.2.5. *The span of any set of vectors is a subspace.*

Proof.

□

Theorem 4.2.6.

1. *The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.*

2. *The row space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$ with the vectors are row vectors.*

Definition 42. If A and B are $m \times n$ matrices, we say that B is *row-equivalent* to A if B can be obtained from A by a finite sequence of elementary row operations.

Theorem 4.2.7. *Let A and B be row equivalent matrices. Then the row space of A is the row space of B i.e. elementary row operations preserve the row space of a matrix.*

Example 4.2.8. Suppose

$$A = \begin{bmatrix} 1 & 3 & 1 & -1 & 1 \\ -1 & -1 & 3 & 1 & 3 \\ 1 & 4 & 3 & -1 & 3 \end{bmatrix}.$$

Reduce matrix A to r.e.f. and hence verify that the row space of r.e.f.(A) is the row space of A.

■

4.3 Linear independence

Recall that

1. $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ are *linearly independent* if and only if the equation

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n = \mathbf{0}$$

   has only trivial solution i.e. $c_1 = 0, c_2 = 0, \cdots, c_n = 0$;

2. $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ are linearly dependent if and only if the equation

$$c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n = \mathbf{0}$$

   has nontrivial solutions, i.e. there exist real numbers $c_1, c_2, \cdots, c_n$, not all c_i's are zero, such that $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \cdots + c_n\boldsymbol{x}_n = \mathbf{0}$.

Example 4.3.1. Determine whether

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

are linearly independent.

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Example 4.3.2. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are as in Example 4.3.1 and $\boldsymbol{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Are $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ linearly independent?

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Theorem 4.3.3. *If $n > m$, any n vectors in $\mathbb{R}^m$ are linearly dependent.*

Theorem 4.3.4. *Vectors are linearly dependent if and only if one of them is a linear combination of the others.*

Proof.

□

Example 4.3.5. Note that $\begin{bmatrix}0\\3\\4\end{bmatrix} = 3\begin{bmatrix}1\\1\\2\end{bmatrix} - \begin{bmatrix}3\\0\\2\end{bmatrix}$. Then, $\begin{bmatrix}0\\3\\4\end{bmatrix}$, $\begin{bmatrix}1\\1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\0\\2\end{bmatrix}$ are linearly dependent.

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Example 4.3.6. The matrix

$$U = \begin{bmatrix}3 & -1 & 4 & 1 & 2 & 8\\0 & -2 & 0 & 5 & 7 & 5\\0 & 0 & 0 & 8 & 3 & -2\\0 & 0 & 0 & 0 & 0 & 9\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

is in r.e.f. The leading entries, $3, -2, 8, 0$ are in column $1, 2, 4, 6$. The column $1, 2, 4, 6$ are linearly independent.

■

Theorem 4.3.7. *Let U be a matrix in r.e.f.*

1. *The columns that contain leading entries of U are linearly independent.*
2. *The nonzero rows of U are linearly independent.*

Theorem 4.3.8. *If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are linearly independent vectors in vectors space V, then any vector in their span can be expressed in exactly one way as a linear combination of these vectors; the coefficient are unique.*

4.4 Basis and dimension, change of basis

Theorem 4.4.1. *Any linearly independent set of vectors in $\mathbb{R}^m$ contains at most m vectors. Any spanning set for $\mathbb{R}^m$ contains at least m vectors.*

Example 4.4.2. Let $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \boldsymbol{u}_3 = \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix}, \boldsymbol{u}_4 = \begin{bmatrix} 13 \\ 14 \\ 15 \\ 16 \end{bmatrix}$. Do $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4$ span $\mathbb{R}^4$?

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Definition 43. Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ be a subset of a vector space V. Then S is called a *basis* for V if

1. S is linearly independent and
2. S spans V.

Remark 4.4.3.

1. A basis for a vector space V contains the smallest possible number of vectors that can span V.
2. In general, a vector space has many different bases.

Example 4.4.4.

1. Standard basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$.
2. Show that $\left\{[1,1]^T, [1,-1]^T\right\}$ is a basis for $\mathbb{R}^2$.

3. Is $S = \left\{[1,1,1,1]^T, [0,0,1,2]^T, [-1,0,0,1]^T\right\}$ a basis for $\mathbb{R}^4$?

4. Find a basis of the plane $2x - y + z = 0$.

$\blacksquare$

Definition 44. A nonzero vector space V is called *finite dimensional* if it has a basis with finite number of vectors. If V does not have a finite basis, then V is called *infinite dimensional*.

Definition 45. The *dimension* of a finite dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, the dimension of the zero vector space is defined to be zero.

Example 4.4.5.

1. $\dim(\mathbb{R}^n) = n$.

2. Find a basis for and determine the dimension of the subspace

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x, y, z \in \mathbb{R} \text{ and } y = z \right\}$$

of $\mathbb{R}^3$.

3. Find a basis for and determine the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \text{ and } b + c = 0 \right\}$$

of $\mathbb{M}_2$.

■

Example 4.4.6. Find the dimension of the row space and the null space of

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -3 & -6 & 4 & -13 & -17 \\ 4 & 8 & -6 & 25 & 29 \\ -1 & -2 & 0 & 10 & 6 \end{bmatrix}.$$

■

Definition 46. The *nullity* of a matrix A is the dimension of its null space.

Theorem 4.4.7. *For any $m \times n$ matrix A, $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$, the number of columns of A.*

Example 4.4.8. Verify Theorem 4.4.7:

1. $A = \begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

2. $B = \begin{bmatrix} -3 & 4 & 5 & 8 & 1 \\ 1 & 0 & -2 & 1 & -1 \\ -2 & 4 & 3 & 9 & 0 \\ 4 & -4 & -7 & -7 & 3 \end{bmatrix}$

■

Definition 47. Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}$ be a basis for a vector space V and $\boldsymbol{v}$ be a vector in V. If

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_n\boldsymbol{u}_n,$$

then the coefficients $c_1, c_2, \cdots, c_n$ are called the *coordinates* of $\boldsymbol{v}$ relative to the basis S. The vector

$$(\boldsymbol{v})_S = [c_1, c_2, \cdots, c_n]^T$$

in $\mathbb{R}^n$ is called the *coordinate vector* of $\boldsymbol{v}$ relative to the basis S.

Example 4.4.9. Let $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $\mathcal{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, where $\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find $(\boldsymbol{x})_{\mathcal{C}}$, given $(\boldsymbol{x})_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

■

Definition 48. Let $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}$ and $\mathcal{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ be bases for a vector space V. Then $n \times n$ matrix whose columns are the coordinate vectors $(\boldsymbol{u}_1)_{\mathcal{C}}, (\boldsymbol{u}_2)_{\mathcal{C}}, \cdots, (\boldsymbol{u}_n)_{\mathcal{C}}$ of the vectors in $\mathcal{B}$ with respect to $\mathcal{C}$ is denoted by $P_{\mathcal{C} \leftarrow \mathcal{B}}$ and is called the *change-of-basis matrix* from $\mathcal{B}$ to $\mathcal{C}$.

Theorem 4.4.10. *Let* $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\}$ *and* $\mathcal{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ *be bases for a vector space* V *and let* $P_{\mathcal{C}\leftarrow\mathcal{B}}$ *be the change-of-basis matrix from* $\mathcal{B}$ *to* $\mathcal{C}$*. Then*

(a) $P_{\mathcal{C}\leftarrow\mathcal{B}}(\boldsymbol{x})_{\mathcal{B}} = (\boldsymbol{x})_{\mathcal{C}}$ *for all* $\boldsymbol{x}$ *in* V*.*

(b) $P_{\mathcal{C}\leftarrow\mathcal{B}}$ *is the unique matrix* P *with the property that* $P(\boldsymbol{x})_{\mathcal{B}} = (\boldsymbol{x})_{\mathcal{C}}$ *for all* $\boldsymbol{x}$ *in* V*.*

(c) $P_{\mathcal{C}\leftarrow\mathcal{B}}$ *is invertible and* $(P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = P_{\mathcal{B}\leftarrow\mathcal{C}}$*.*

Example 4.4.11. Let $\mathcal{B} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are standard basis vectors in 3-space and let

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find the coordinate vector of $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ with respect to $\mathcal{C}$.