Lab 1

Magical Romance

Lab Objective: *We present an application of optimal control to teenage romance. Specifically, we will apply LQR to determine the optimal amount of magical effort Ron Weasley must utilize to realize his goal of a happily ever after with Hermione Granger.*

The Linear Love Affair

In order to arouse your interest in the classification of of linear systems, we are going to modify a famous problem from Strogatz (Linear Systems p.g. 138). Nerds will be familiar with the love story between Ron Weasley and Hermione Granger in the popular Harry Potter series written by J.K. Rowling. Ron is madly in love with Hermione, but in our version of the story, Hermione is not entirely sure of her feelings towards Ron. The more that Ron shows that he loves her, the more Hermione wants to apparate away from him. In classic female fashion, when Ron gets discouraged and backs off, Hermione starts to find him more and more attractive. Ron on the other hand, seems to follow what Hermione feels, when she is attracted to him he reciprocates, but when she starts to back off his love starts to grow colder.

Let $R(t)$ be Ron's love/hate for Hermione at time t, and $H(t)$ be Hermione's love/hate for Ron.

Positive values for R, H signify love, while negative values signify hate.

Now consider the general linear system,

$$\begin{aligned} \dot{R}(t) &= aR(t) + bH(t) \\ \dot{H}(t) &= cR(t) + dH(t) \end{aligned} \tag{1.1}$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles. For example, if $a > 0$ and $b > 0$, the gossip around the Gryffindor Common Room would label Ron as an "eager beaver". If $a < 0$ and $b > 0$, he would be much more cautious about his love.

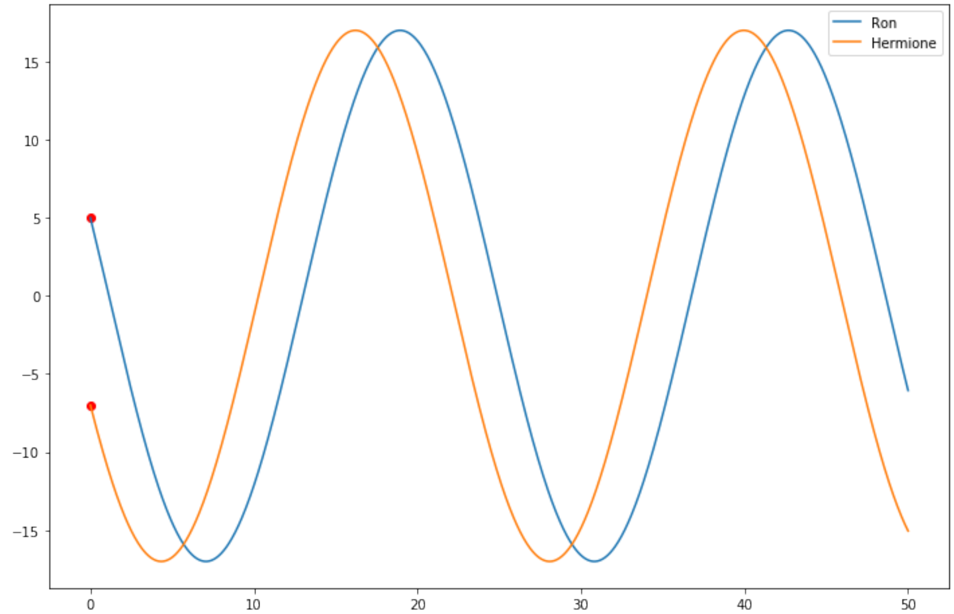

Figure 1.1: Ron and Hermione's love for each other over time.

Problem 1. 1. Using equation 1.1 and `scipy.integrate.odeint`, solve for $H(t)$ and $R(t)$ with the parameters below.

2. Recreate Figure 1.1 showing Ron and Hermione's feelings for each other over time.

3. Recreate Figure 1.2 plotting Ron's feelings against Hermione's feelings. What do you notice about their relationship?

```
a = -0.3
b = 0.4
c = -0.4
d = 0.3
r0 = 5
h0 = -7
tf = 50
```

Chocolate Frogs

As you can see in the graphs above, Ron and Hermione are caught in a never ending loop of falling in and out of love. One day, Ron realizes this and decides that he must do something about it. He loves Hermione and wants to end up with her. He knows she has a deep passionate burning love for chocolate frogs. He concocts a plan to keep her in love by regularly gifting her chocolate frogs.

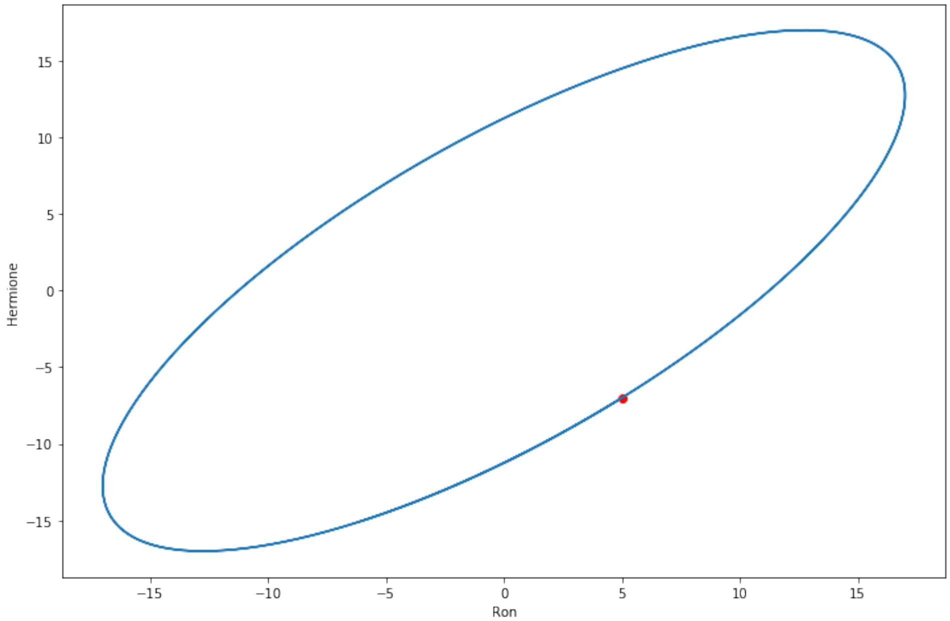

Figure 1.2: Ron's love plotted against Hermione's love.

Ron decides to use his skills from arithmancy (magic math) and set up an optimal control problem. His control will be to give Hermione chocolate frogs and other favors. We can add this control variable to the evolution equations as follows:

$$\begin{aligned} \dot{R}(t) &= aR(t) + bH(t) \\ \dot{H}(t) &= cR(t) + dH(t) + kU(t) \end{aligned} \tag{1.2}$$

Ron wants to maximize the attention he gets from Hermione while not scaring her off by being too aggressive. Therefore he chooses to minimize the cost functional

$$J[u] = \int_0^{t_f} qH^2 + rU^2 dt \tag{1.3}$$

Problem 2. Solve the optimal control problem as defined by equations 1.2 and 1.3 with the following initial conditions. Recreate Figure 1.3. What relationship is there between the control and Hermione's feelings for Ron?

```
a = 1.5
b = 1.2
c = 1.9
d = 1
e = 1
q = 2
r = 0.1
tf = 6
```

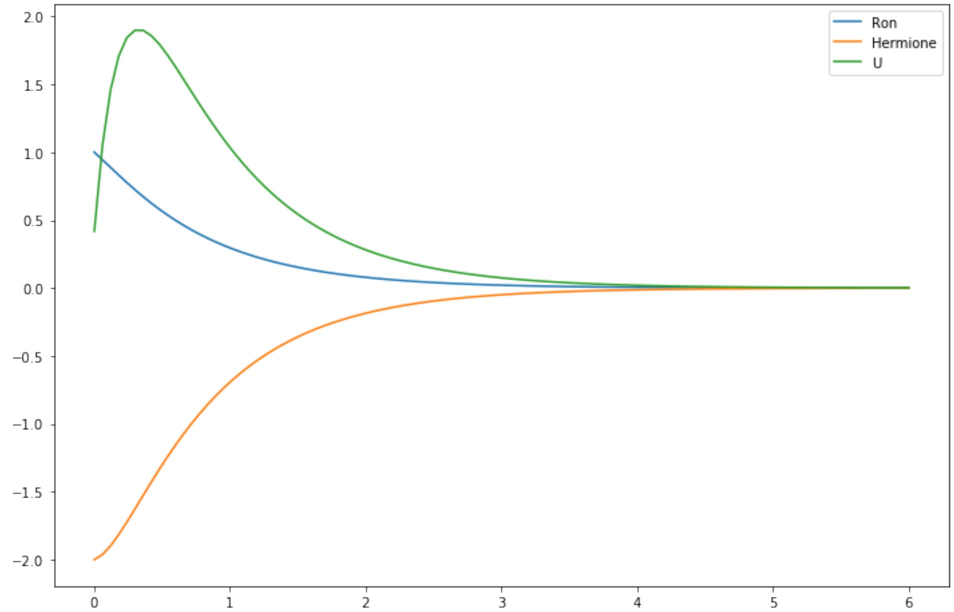

Figure 1.3: Ron and Hermione's Love with Control

```
x0 = np.array([1,-2])
```

Here Comes the Competition

It's been a few years, and all seems to be moving along quite swimmingly with Ron and Hermione. Ron knows just how many chocolate frogs to buy for Hermione on each day of the week to promote the maximum devotion without making her sick. The grass is green and the Pygmy Puffs are squeaking and all is right in the world.

However, unbeknownst to these two oblivious lovebirds, Harry has recently begun puberty, and has realized two great Truths: one, he is inveterately single; and two, Hermione is surpassingly hot. Consumed by jealousy each time he sees her with Ron, he settles on a plan.

At breakfast one day, he awaits the arrival of the mail owls. While the room is in general commotion and everyone is distracted, he surreptitiously spikes Hermione's butterbeer with a love potion, which he purchased for a pricey sum at Zonko's. "Faint heart never won fair lady," he murmurs to himself. "If I want to succeed in this game, it's time I take control!"

Harry wants to do all he can to win Hermione's love, but love potions are expensive. His massive inherited fortune presents no real constraint, but he wants to manage it judiciously. Help him figure out just what steps to take to minimize his expenses while maximizing Hermione's love for him!

To model these complex social dynamics, we split Hermione's love into two variables: one describing her love for Ron, and one for Harry. Note she could end up hating both, but she cannot love both equally.

Hermione's love for Harry evolves according to her love for Ron, Harry's love for her, Ron's love for her, and Harry's control.

$$\dot{G}_P = aG_W + bP + cW + dU \tag{1.4}$$

Her love for Ron evolves according to her love for Harry, Harry's love for her, and Ron's love for her.

$$\dot{G}_W = eG_P + fP + gW \tag{1.5}$$

Ron's and Harry's love for Hermione vary as functions of their own love and of Hermione's love for them.

$$\begin{aligned} \dot{P} &= hP + iG_P \\ \dot{W} &= jW + kG_W \end{aligned} \tag{1.6}$$

Problem 3. Using the evolution equations (1.4,1.5,1.6), minimize this TEMPORARY cost functional $\int_0^\infty (rU^2 - qG_P^2)dt$ and plot the state equations over time.