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MODELING AND ANALYSIS OF A HARVESTING FISHERY MODEL IN A TWO-PATCH ENVIRONMENT

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In this paper, a harvesting fishery model in a two-patch environment: one free-fishing zone and the other one reserved zone where fishing is strictly prohibited, is proposed and analyzed. The existence of possible biological steady states, along with their local stability, instability and global stability is discussed. The existence of bioeconomic equilibrium is derived. An optimal harvesting policy is also given by applying pontryagin's maximum principle.

Keywords: Stage-structure; global stability; migration rates; optimal harvesting policy.

Mathematics Subject Classification 2000: 92D25, 34D20

1. Introduction

Regulation of exploitation to marine fishery resource has become a problem of major concern in recent years in view of the growing human needs for more food and energy, which has led to the depletion of some fish species. One potential solution to these problems is the creation of marine reserves where fishing and other extractive activities are prohibited. Marine reserves cannot only protect species inside the reserve area but they can also increase fish abundance in adjacent areas. Thus, marine reserves can offer a sustainable development policy for exploiting of the marine resources. In order to exploit reasonable fishery resource in marine ecosystems, various dynamic models have been proposed and analyzed in recent years [1–8]. In [1, 2], Clark has provided us with some techniques and methods for bioeconomic exploitation of fishery resource. Based on the work of Clark, Chaudhuri [3] proposes a model to study the combined harvesting of two competing fish species and show that the open-access fishery may have an optimal bioeconomic equilibrium. In [4], Dew presents a discrete time, age structured, single species fishery model with dynamic total allowable catch (TAC) to exploit reasonable fishery resource in the absence of fishery population data. In [5], Ardito *et al.* consider a

Gordon–Schaefer model with periodic coefficients to investigate the effects of seasonal fluctuations on an open access fishery problem. In [6], Chen and Hsui consider harvest tax adopted by a governmental agency as administering fishery policies for the objective of a sustainable development. In [7], Kar *et al.* takes the fishing effort as a dynamic variable and tax as a control instrument to investigate the problem of selective harvesting in a hybrid type of prey predator model. Recently, Fan and Wang [8] investigate the exploitation of a single-fish population model and show the system has a unique globally asymptotically stable positive periodic solution. In addition to these, we refer the reader to [9] for a general reference.

On the other hand, the description of the stage structure of the population in the life history is also an interesting problem in population dynamics. In the natural world, there are many species whose individual members have a life history that takes them through two stages, immature and mature. In particular, some mammalian populations and amphibious animals usually exhibit two stages. Stage-structured ecological models have received much attention in recent years [10–19]. However, in [14, 15], they always assume that populations migrate at the same diffusion rates in two patches. Moreover, they consider only the populations have logistically growing resource in one patch. The present paper deals with the model for stage-structured fishery resource in an aquatic habitat that consists of two zones: one free fishing zone and the other a reserve zone where fishing is prohibited. Whether in the reserved zones or in free fishing zone, fish population grows logistically with intrinsic growth rate and migrates at the different diffusion rates, which seems to be more realistic. The main aim of this paper is to examine the combined effects of stage-structure for fish species in the open region, dispersal and harvesting on the stability of fishery resource with stage structure and to find the optimal harvesting policy using the Pontryagin's maximum principle [20].

This organization of this paper is as follows. In Sec. 2, the basic model is given. In Sec. 3, boundedness of the system is obtained. The existence of nonnegative equilibria and their stability behavior have been discussed in Sec. 4. The existence of bioeconomic equilibrium and optimal harvesting policy are also discussed in Secs. 5 and 6, respectively. Finally, we conclude with numerical example and brief remarks.

2. The Model Formulation

In this section, we consider a fishery habitat in an aquatic ecosystem, where there is a two-patch environment: one free fishing zone and the other a reserved zone, where no fishing is permitted. Fish population in free fishing zone consists of immature fish and mature fish. The fish population in a reserved zone grows logistically with intrinsic rate s and carrying capacity k. It is assumed that the mature fish population in the free fishing zone and fish population in the reserved zone can migrate randomly. However, the immature fish species cannot disperse between a two-patch environment due to its weak strength. And only the mature fish is harvested in one free fishing area while the immature fish is not harvested due to its lower commercial profits. Keep these in view, the model is described as follows:

$$\begin{aligned}\dot{x}_1(t) &= \alpha x_2(t) - r x_1(t) - \beta x_1(t),\\ \dot{x}_2(t) &= \beta x_1(t) - \eta x_2^2(t) + \sigma_2 y(t) - \sigma_1 x_2(t) - qEx_2(t),\\ \dot{y}(t) &= sy(t)\left(1 - \frac{y(t)}{k}\right) + \sigma_1 x_2(t) - \sigma_2 y(t),\end{aligned} \tag{2.1}$$

where $x_1(t)$ and $x_2(t)$ are the biomass densities of the immature fish species and mature fish species at time t in the free fishing zone, respectively. $y(t)$ is he biomass densities of the fish species at time t in the reserved zone. α is the birth rate of the immature fish species. r and η represent the death rate of the immature and mature fish species, respectively. β is immature stage's transformation rate into maturation. E is the total effort applied for harvesting the mature fish population and q is the catchability coefficient of the mature fish population in the unreserved zone. σ_1 and σ_2 are the migration rates from the free fishing zone to the reserved zone and from the reserved zone to the free fishing zone, respectively. All the parameters are assumed to be positive. Here, we observe that if there is no diffusion migration of fish population from the free zone to the reserved zone (i.e. $\sigma_1 = 0$) and $s - \sigma_2 < 0$, then $\dot{y}(t) < 0$. Hence, in this paper, we always assume that $s - \sigma_2 > 0$.

3. Boundedness of the System (2.1)

Let

$$\begin{aligned} D &= \{(x_1, x_2, y) \in R^3 : x_1 \geq 0, x_2 \geq 0, y \geq 0\},\\ \operatorname{Int} D &= \{(x_1, x_2, y) \in R^3 : x_1 > 0, x_2 > 0, y > 0\}.\end{aligned}$$

Considering the biological significance, we investigate system (2.1) in the region $\operatorname{Int} D$.

Theorem 3.1. $\operatorname{Int} D$ *is positive invariant for system* (2.1).

Proof. It is easy to show that the plane of $x_1 = 0, x_2 = 0, y = 0$ are not the trajectories of system (2.1). Since

$$\dot{x}_1(t)|_{x_1=0} = \alpha x_2 > 0, \quad \dot{y}(t)|_{y=0} = \sigma_1 x_2 > 0.$$

It follows from the above expression and the second equation of (2.1) that $\dot{x}_2(t)|_{x_2=0} > 0$. Hence, each trajectory starting from the $\operatorname{Int} D$ cannot cross through the boundary of $\operatorname{Int} D$, that is, each trajectory cannot cross through x_1x_2-plane, x_1y-plane, x_2y-plane. Hence, $\operatorname{Int} D$ is positive invariant for system (2.1).

This completes the proof of Theorem 3.1. □

Theorem 3.2. *All the solutions of system* (2.1) *which start in* $\operatorname{Int} D$ *are uniformly bounded.*

Proof. Let

$$V_1(t) = x_1(t) + x_2(t) + y(t). \tag{3.1}$$

Calculating the derivative of (3.1) along the system (2.1), we obtained

$$\frac{dV_1(t)}{dt} = \alpha x_2(t) - rx_1(t) - \eta x_2^2 + sy\left(1 - \frac{y}{k}\right).$$

It follows from the above equation that

$$\begin{aligned}\frac{dV_1(t)}{dt} + rV_1(t) &= \frac{(\alpha+r)^2}{4\eta} + \frac{k(r+s)^2}{4s} - \eta\left(x_2 - \frac{\alpha+r}{2\eta}\right)^2 - \frac{s}{k}\left(y - \frac{k(r+s)}{2s}\right)^2 \\ &\le \frac{(\alpha+r)^2}{4\eta} + \frac{k(r+s)^2}{4s} \stackrel{\text{def}}{=} \Lambda. \end{aligned} \tag{3.2}$$

By the theory of differential inequality, we have

$$0 \le V_1(t) \le \frac{\Lambda}{r}(1 - e^{-rt}) + V_1(0)e^{-rt}. \tag{3.3}$$

Let $t \to \infty$, it yields from (3.3) that $V_1(t) < \frac{\Lambda}{r}$. So, we derive that all the solutions of system (2.1) that start in $\operatorname{Int} D$ are confined to the region W, where

$$W = \left\{(x_1, x_2, y) \in \operatorname{Int} D : V_1(t) = \frac{\Lambda}{r} + \epsilon,\ \text{for any } \epsilon > 0\right\}.$$

This completes the proof of Theorem 3.2. □

4. Existence of Nonnegative Equilibria and Their Stability Analysis

System (2.1) has two possible nonnegative equilibria $E_0(0,0,0)$ and $E^*(x_1^*, x_2^*, y^*)$. Obviously, $E_0(0,0,0)$ always exists. x_1^*, x_2^*, y^* are the positive solutions of the following algebraic equations:

$$\begin{cases} \alpha x_2 - (r+\beta)x_1 = 0, \\ \beta x_1 - \eta x_2^2 + \sigma_2 y - \sigma_1 x_2 - qEx_2 = 0, \\ sy(1 - \frac{y}{k}) + \sigma_1 x_2 - \sigma_2 y = 0. \end{cases} \tag{4.1}$$

For the above Eq. (4.1), eliminating x_1 and y, we get a cubic equation in x_2 as

$$ax_2^3 + bx_2^2 + cx_2 + d = 0, \tag{4.2}$$

where

$$\begin{aligned}
a &= \frac{s\eta^2}{k\sigma_2^2} > 0,\\
b &= \frac{2s\eta}{k\sigma_2^2}\left(\sigma_1 + qE - \frac{\alpha\beta}{r+\beta}\right),\\
c &= \frac{s}{k\sigma_2^2}\left(\sigma_1 + qE - \frac{\alpha\beta}{r+\beta}\right)^2 - \frac{s-\sigma_2}{\sigma_2}\eta,\\
d &= \frac{s-\sigma_2}{\sigma_2}\left(\frac{\alpha\beta}{r+\beta} - \sigma_1 - qE\right) - \sigma_1.
\end{aligned} \tag{4.3}$$

Equation (4.2) has a unique positive solution $x_2 = x_2^*$ if the following inequalities hold

$$\sigma_1 < \frac{\alpha\beta}{r+\beta} - qE < \frac{s\sigma_1}{s-\sigma_2}, \quad \frac{s}{k\sigma_2^2}\left(\sigma_1 + qE - \frac{\alpha\beta}{r+\beta}\right)^2 < \frac{s-\sigma_2}{\sigma_2}\eta. \tag{4.4}$$

For $x_2 = x_2^*$, we get $x_1^* = \frac{\alpha}{r+\beta}x_2^*$ and $y^* = \frac{1}{\sigma_2}\left(\eta x_2^{*2} + \left(\sigma_1 + qE - \frac{\alpha\beta}{r+\beta}\right)x_2^*\right)$.

Noting that y^* is positive, we must have

$$\eta x_2^* > \frac{\alpha\beta}{r+\beta} - \sigma_1 - qE > 0. \tag{4.5}$$

In the following, we firstly consider the local stability of the nonnegative equilibria. The variational matrix of the system (2.1) is

$$J(x_1, x_2, y) = \begin{bmatrix} -r-\beta & \alpha & 0 \\ \beta & -2\eta x_2 - \sigma_1 - qE & \sigma_2 \\ 0 & \sigma_1 & s - \sigma_2 - \frac{2s}{k}y \end{bmatrix}.$$

The characteristic equation of $J(E_0)$ is

$$f(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{4.6}$$

where

$$\begin{aligned}
a_1 &= \sigma_1 + \sigma_2 - s + r + \beta + qE,\\
a_2 &= (r+\beta)(\sigma_1 + \sigma_2 + qE - s) - \sigma_1 s + qE(\sigma_2 - s) - \alpha\beta,\\
a_3 &= (qE(r+\beta) - \alpha\beta)(\sigma_2 - s) - \sigma_1 s(r+\beta).
\end{aligned}$$

From condition (4.4), we have $f(0) = a_3 < 0$ and $f(+\infty) = +\infty$. Therefore, the characteristic equation (4.6) of $J(E_0)$ has at least one positive root. So, $E_0(0,0,0)$ is unstable.

The characteristic equation of $J(E^*)$ is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{4.7}$$

where

$$b_1 = \frac{\alpha x_2^*}{x_1^*} + \eta x_2^* + \beta\frac{x_1^*}{x_2^*} + \sigma_2\frac{y^*}{x_2^*} + \frac{s}{k}y^* + \sigma_1\frac{x_2^*}{y^*} > 0,$$

$$b_2 = \alpha\eta\frac{{x_2^*}^2}{x_1^*} + \alpha\sigma_2\frac{y^*}{x_1^*} + \left(\frac{s}{k}y^* + \sigma_1\frac{x_2^*}{y^*}\right)\left(\frac{\alpha x_2^*}{x_1^*} + \eta x_2^* + \beta\frac{x_1^*}{x_2^*}\right) + \frac{s\sigma_2{y^*}^2}{kx_2^*},$$

$$b_3 = \left(\frac{s}{k}y^* + \sigma_1\frac{x_2^*}{y^*}\right)\frac{\alpha\eta{x_2^*}^2}{x_1^*} + \frac{\alpha s\sigma_2{y^*}^2}{kx_1^*} > 0.$$

Noting that $b_1 > 0,\ b_3 > 0$, after a little algebraic calculation, it is easy to verify that $b_1b_2 - b_3 > 0$. By the Routh–Hurwitz criterion, it follows that all the roots of Eq. (4.7) have negative real parts. Hence, $E^*(x_1^*, x_2^*, y^*)$ is locally asymptotically stable.

From the above discussion, we can summarize the following conclusion.

Theorem 4.1. *System* (2.1) *has always a nonnegative equilibrium* $E_0(0,0,0)$. *If* $\frac{\alpha\beta}{r+\beta} - qE < \frac{s\sigma_1}{s-\sigma_2}$, $E_0(0,0,0)$ *is unstable. If* (4.4) *and* (4.5) *hold, then the positive equilibrium* $E^*(x_1^*, x_2^*, y^*)$ *exists, which is locally asymptotically stable.*

Remark. We claim that $E_0(0,0,0)$ is always unstable for $s - \sigma_2 > 0$. In fact, from the above discussion, we have obtained that if $\frac{\alpha\beta}{r+\beta} - qE < \frac{s\sigma_1}{s-\sigma_2}$, $E_0(0,0,0)$ is unstable. If $s > \sigma_2$, then from the third equation of system (2.1), we obtain

$$\dot{y}(t) \geq (s-\sigma_2)y - \frac{s}{k}y^2 = y(s-\sigma_2)\left(1 - \frac{s}{(s-\sigma_2)k}y\right). \tag{4.8}$$

Consider the auxiliary equation

$$\dot{u} = u(s-\sigma_2)\left(1 - \frac{s}{(s-\sigma_2)k}u\right). \tag{4.9}$$

It follows from (4.8) that equilibrium $u = 0$ is always unstable for $s > \sigma_2$. Therefore, from (4.7) and by a standard comparative argument, it is easy to see that the fish species y in the reserved zone never tend to extinction. Integrating the biological significance of system (2.1), it is reasonable that $E_0(0,0,0)$ should be unstable for $s > \sigma_2$. In addition, we can give an example to explain that if $\frac{\alpha\beta}{r+\beta} - qE \geq \frac{s\sigma_1}{s-\sigma_2}$, Eq. (4.6) has at least one positive root.

Example. Let $s = 0.8, k = 100, \alpha = 1, r = 0.01, \eta = 0.03, \beta = 0.8, \sigma_1 = 0.21, \sigma_2 = 0.2, q = 0.01$ and $E = 30$. From the above the values of parameters with satisfying $\frac{\alpha\beta}{r+\beta} - qE > \frac{s\sigma_1}{s-\sigma_2}$, we obtain that the characteristic equation $f(\lambda) = 0$ of $J(E_0)$ has two positive real roots: $\lambda_1 = 0.188810, \lambda_2 = 0.666216$.

Now we shall investigate the global stability of the interior equilibrium $E^*(x_1^*, x_2^*, y^*)$ of system (2.1).

Theorem 4.2. *The positive equilibrium* $E^*(x_1^*, x_2^*, y^*)$ *of system* (2.1) *is globally asymptotically stable.*

Proof. Let us consider the following Liapunov function:

$$V(t) = \sum_{i=1}^{2} \lambda_i \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*} \right) + \lambda_3 \left(y - y^* - y^* \ln \frac{y}{y^*} \right), \tag{4.10}$$

where $\lambda_i,\ i = 1, 2, 3$ are positive constants, to be chosen suitably later on.

Calculating the derivative $V(t)$ along each solution of system (2.1), we have

$$\begin{aligned} \frac{dV(t)}{dt} &= \lambda_1 \frac{x_1 - x_1^*}{x_1} \frac{dx_1}{dt} + \lambda_2 \frac{x_2 - x_2^*}{x_2} \frac{dx_2}{dt} + \lambda_3 \frac{y - y^*}{y} \frac{dy}{dt} \\ &= \lambda_1 \alpha \frac{x_1 - x_1^*}{x_1} \left(x_2 - \frac{x_2^*}{x_1^*} x_1 \right) + \lambda_2 \frac{x_2 - x_2^*}{x_2} \left[\frac{\beta}{x_2^*} (x_1 x_2^* - x_1^* x_2) - \eta x_2 (x_2 - x_2^*) \right. \\ &\quad \left. + \frac{\sigma_2}{x_2^*} (y x_2^* - x_2 y^*) \right] + \lambda_3 \frac{y - y^*}{y} \left[\frac{\sigma_1}{y^*} (x_2 y^* - x_2^* y) - \frac{s}{k} y (y - y^*) \right]. \end{aligned}$$

Let $\lambda_1 = 1,\ \lambda_2 = \frac{\alpha x_2^*}{\beta x_1^*},\ \lambda = \frac{\alpha \sigma_2 y^*}{\beta \sigma_1 x_1^*}$. After a little algebraic calculation and Theorem 3.1, we have

$$\begin{aligned} \frac{dV(t)}{dt} &= -\frac{\alpha (x_1 x_2^* - x_1^* x_2)^2}{x_1 x_2 x_1^*} - \frac{\alpha x_2^*}{x_1^*} \eta (x_2 - x_2^*)^2 - \frac{\alpha \sigma_2 (x_2^* y - x_2 y^*)^2}{\beta x_2 y x_1^*} \\ &\quad - \frac{s \alpha \sigma_2 y^*}{k \beta \sigma_1 x_1^*} (y - y^*)^2 \\ &\leq 0. \end{aligned}$$

Set $D_1 = \{(x_1, x_2, y) \in \operatorname{Int} D : \frac{dV(t)}{dt} = 0\} = \{(x_1, x_2, y) \in \operatorname{Int} D : x_1 = x_1^*, x_2 = x_2^*, y = y^*\} = E^*$. According to Liapunov's theorem on stability [21], it follows that the positive equilibrium E^* is globally asymptotically stable with respect to all solutions initiating in $\operatorname{Int} D$. This completes the proof of Theorem 4.2. □

5. Bioeconomic Equilibrium

In the previous section, we have discussed the dynamical behavior of biological equilibria, which are given by $\dot{x}_1 = \dot{x}_2 = \dot{y} = 0$. In this section, we mainly consider the bieconomic equilibrium, which is to be achieved when the total revenue obtained by selling the harvested biomass equals the total cost utilized in harvesting it. Let c be the fishing cost per unit effort for mature fish in the open-access zone and p be the constant price per unit biomass of the landed mature fish in the free fishing zone. Therefore, the net economic rent at any time is given by

$$\pi(x_1, x_2, y, E, t) = (pqx_2 - c)E. \tag{5.1}$$

The bioeconomic equilibrium is $[x_{1\infty}, x_{2\infty}, y_\infty, E_\infty]$, where $x_{1\infty}, x_{2\infty}, y_\infty, E_\infty$ are the positive solutions of $\dot{x}_1 = \dot{x}_2 = \dot{y} = \pi = 0$. Noticing that if $c > pqx_2$, i.e. the cost is greater than the revenue obtained from it, then the fishing in the open-access will be stopped. So, for the existence of bioeconomic equilibrium, it is natural to assume that $c < pqx_2$. Thus, we have

$$x_{1\infty} = \frac{\alpha c}{(r+\beta)pq}, \quad y_\infty = \frac{k}{2s}\left[(s-\sigma_2) + \left\{(s-\sigma_2)^2 + \frac{4s\sigma_1 c}{pqk}\right\}^{1/2}\right],$$

$$x_{2\infty} = \frac{c}{pq}, \qquad E_\infty = \frac{\alpha\beta}{q(r+\beta)} - \frac{\sigma_1}{q} - \eta\frac{c}{pq^2} + \frac{\sigma_2 p y_\infty}{c}.$$

Noting that $E_\infty > 0$ if

$$\frac{\alpha\beta}{q(r+\beta)} + \frac{\sigma_2 p y_\infty}{c} > \frac{\sigma_1}{q} + \eta\frac{c}{pq^2}. \tag{5.2}$$

Thus, the bioeconomic equilibrium $(x_{1\infty}, x_{2\infty}, y_\infty, E_\infty)$ exists if inequality (5.2) holds. Obviously, if $E > E_\infty$, then the total cost utilized in harvesting the mature fish population in the open-access zone would exceed the total revenues obtained from the fishery industry. At this time, some fishermen would be in loss, and naturally they might withdraw their participation from the fishery. Hence, $E > E_\infty$ cannot be maintained indefinitely. On the other hand, if the $E < E_\infty$, the fishery is more profitable. At this time, the fishery would attract more and more fishermen in the open-access zone. This will eventually have an increasing effect on the harvesting effort. Hence, $E < E_\infty$ also cannot be maintained indefinitely.

6. Optimal Harvesting Policy

In this section, our objective is to select a harvesting strategy that maximizes the present value J of a continuous time-stream of revenues given by

$$J = \int_0^\infty e^{-\delta t}(pqx_2 - c)E(t)dt, \tag{6.1}$$

where δ denotes the instantaneous annual rate of discount. We intend to maximize (6.1) subject to the state equation (2.1) with the control constraints $0 \le E \le E_{\max}$ by invoking Pontryagin's maximum principle [20]. The Hamiltonian of the above control problem is given by

$$\begin{aligned} H = {} & e^{-\delta t}(pqx_2 - c)E + \lambda_1(\alpha x_2 - rx_1 - \beta x_1) + \lambda_2(\beta x_1 - \eta x_2^2 + \sigma_2 y - \sigma_1 x_2 - qEx_2) \\ & + \lambda_3\left[sy\left(1 - \frac{y}{k}\right) - \sigma_2 y + \sigma_1 x_2\right], \end{aligned} \tag{6.2}$$

where λ_i $(i = 1,2,3)$ are the adjoint variables and $\theta(t) = e^{-\delta t}(pqx_2 - c) - \lambda_2 qEx_2$ is called the switching function (see Clark [2]). Because H is linear in the control variable E, the optimal control will be a combination of bang-bang control and singular control. The optimal control must satisfy the following conditions:

$$E = E_{\max}, \quad \text{when } \theta(t) > 0, \text{ i.e. } \quad \text{when } \lambda_2(t)e^{\delta t} < p - \frac{c}{qx_2}\,, \tag{6.3}$$

$$E = 0, \quad \text{when } \theta(t) < 0, \text{ i.e. } \quad \text{when } \lambda_2(t)e^{\delta t} > p - \frac{c}{qx_2}\,. \tag{6.4}$$

$\lambda_2(t)e^{\delta t}$ is called usually the shadow price and $p - \frac{c}{qx_2}$ is the net economic revenue on a unit harvest (see Clark [2]). This shows that whether $E = E_{\max}$ or zero depends on the shadows price, which is less than or greater than the net economic revenue on a unit harvest. If $\theta(t) = 0$, i.e. when the shadow price equals the net economic revenue on a unit harvest, then the Hamiltonian H is independent of the control variable $E(t)$, i.e. $\frac{\partial H}{\partial E} = 0$. It is the necessary condition for the singular control $E^*(t)$ to be optimal over the control set $0 < E^* < E_{\max}$. Thus, the optimal harvesting policy is

$$E(t) = \begin{cases} E_{\max}, & \theta(t) > 0, \\ 0, & \theta(t) < 0, \\ E^*, & \theta(t) = 0. \end{cases} \tag{6.5}$$

In fact, if $\theta(t) = 0$, it follows that

$$\lambda_2 q x_2 = e^{-\delta t}(pqx_2 - c) = e^{-\delta t}\frac{\partial \pi}{\partial E}. \tag{6.6}$$

This implies that the user's cost of harvest per unit of effort equals the discounted value of the future marginal profit of the effort at the steady-state level. Now we are in the position to find the path of singular control by applying the pontragin's maximum principle and to find eventually the optimal equilibrium solution to the problem. Thus, we shall consider the following adjoint equations:

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x_1} = -[-(r+\beta)\lambda_1 + \lambda_2\beta], \tag{6.7}$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial x_2} = -[e^{-\delta t}pqE + \alpha\lambda_1 - (2\eta x_2 + qE)\lambda_2 + \lambda_3\sigma_1], \tag{6.8}$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial y} = -\left[\lambda_2\sigma_2 + \lambda_3\left(s - \sigma_2 - \frac{2sy}{k}\right)\right]. \tag{6.9}$$

Since Eq. (6.7) can be written as $\frac{d\lambda_1}{dt} - (r+\beta)\lambda_1 = -A_1 e^{-\delta t}$, where $A_1 = (p - \frac{c}{qx_2^*})\beta$, whose solution is given by

$$\lambda_1(t) = \frac{A_1}{r+\beta+\delta}e^{-\delta t}. \tag{6.10}$$

From (6.9), we have $\frac{d\lambda_3}{dt} - A_2\lambda_3 = -A_3 e^{-\delta t}$, whose solution is given by

$$\lambda_3(t) = \frac{A_3}{A_2+\delta}e^{-\delta t}, \tag{6.11}$$

where $A_2 = \frac{2sy^*}{k} + \sigma_2 - s$, $A_3 = \big(p - \frac{c}{qx_2^*}\big)\sigma_2$.

From (6.8), we get $\frac{d\lambda_2}{dt} - B_1\lambda_2 = -B_2e^{-\delta t}$, whose solution is given

$$\lambda_2(t) = \frac{B_2}{B_1+\delta}e^{-\delta t}, \tag{6.12}$$

where $B_1 = 2\eta x_2^* + qE$, $B_2 = pqE + \frac{\sigma_1 A_3}{A_2+\delta} + \frac{A_1\alpha}{r+\beta+\delta}$. From (6.6) and (6.12), we obtain a singular path

$$p - \frac{c}{qx_2^*} = \frac{B_2}{B_1+\delta}. \tag{6.13}$$

Noting that $y^* = \frac{1}{2s}[k(s-\sigma_2) + \{k^2(s-\sigma_2)^2 + 4sk\sigma_1 x_2^*\}^{\frac{1}{2}}]$ and from (4.1), we can rewrite A_2, B_1, B_2 as

$$\begin{aligned} A_2 &= [(s-\sigma_2)^2 + 4s\sigma_1 x_2^*/k]^{\frac{1}{2}}, \\ B_1 &= 2\eta x_2^* + qE, \\ B_2 &= pqE + \frac{\sigma_1 A_3}{A_2+\delta} + \frac{A_1\alpha}{r+\beta+\delta}. \end{aligned}$$

Let $x_2^* = x_{2\delta}$ be a solution (if it exists) of Eq. (6.13), using this value $x_{2\delta}$, we get the optimal equilibrium solution $(x_{1\delta}, x_{2\delta}, y_\delta, E_\delta)$, where

$$\begin{aligned} x_{1\delta} &= \frac{\alpha}{r+\beta}x_{2\delta}, \quad & y_\delta &= \frac{1}{2s}[k(s-\sigma_2) + \{k^2(s-\sigma_2)^2 + 4sk\sigma_1 x_{2\delta}\}^{\frac{1}{2}}], \\ x_2^* &= x_{2\delta}, & E_\delta &= \frac{1}{qx_{2\delta}}[\beta x_{1\delta} - \eta x_{2\delta}^2 + \sigma_2 y_\delta - \sigma_1 x_{2\delta}]. \end{aligned} \tag{6.14}$$

Here, $E_\delta > 0$ if $\beta x_{1\delta} + \sigma_2 y_\delta > \eta x_{2\delta}^2 + \sigma_1 x_{2\delta}$.

From (6.10)–(6.12), we observe that $\lambda_i(t)e^{\delta t} (i = 1, 2, 3)$ are independent of time t in an optimum equilibrium. They satisfy the transversality condition at ∞, i.e. they remain bounded as $t \to \infty$. From Eq. (6.13), we observed that

$$pqx_2^* - c = \frac{B_2 q x_2^*}{B_1+\delta} \to 0, \quad \text{as } \delta \to \infty. \tag{6.15}$$

Thus, the net economic revenue $\pi(x_{1\infty}, x_{2\infty}, y_\infty, E, t) = 0$. This implies that the net economic revenue leading to zero if the discount rate tends to infinity. Therefore, the fishery would remain closed.

Numerical example. Let $\alpha = 2$, $r = 0.2$, $\beta = 0.7$, $\eta = 0.03$, $\sigma_1 = 0.51$, $\sigma_2 = 0.5$, $s = 1.0$, $k = 10$, $q = 0.01$, $p = 8$, $\delta = 1$, $c = 0.5$ and $E = 100$ in appropriate units. These parameter values satisfy inequality (4.4) and (4.5), hence the nontrivial equilibrium point exists. From (4.1), we have $E^*(32.484, 14.618, 2.000)$ as the nontrivial equilibrium point. From the above values of parameters and (6.14), we derive that the optimal bioeconomic equilibrium $(28.636, 12.88637, 10.983)$ and the corresponding optimal harvesting efforts are $E_\delta = 7.760$.

7. Concluding Remarks

In this paper, a mathematical model with stage structure and harvesting has been proposed and investigated to study the dynamics of fishery resource in a two-patch environment. Our result Theorem 4.1 shows that the positive equilibrium of the system under consideration is globally asymptotically stable if it exists. The above result implies that if the mature fish populations are harvested in an open-access fishery zone while fishing is not allowed in a reserved region, then fish species in the two-patch environment settle down to their respective equilibrium levels, whose magnitudes depend upon the birth rate and the death rate of immature fish species, immature stage's transformation rate into maturation, the death rate of mature fish species in an open-access fishery zone, the intrinsic growth rate of fish species, carrying capacities in a reserved zone and diffusion migration coefficients between two zones. Though the different diffusion rates σ_1 and σ_2 do not influence directly the stability of the nonnegative equilibria, from (4.4) and (4.5), it is easy to see that they have an effect on existence of the unique positive equilibrium. Our results also imply that even after continuous harvesting of mature fish population in unreserved area, fish population may be sustained at an appropriate equilibrium level.

The optimal harvesting policy has been discussed by applying pontryagin's maximum principle. We have found the harvesting strategy that maximizes the present value of a continuous time-stream of revenues. From (6.15), it has also been noticed that the economic rent decreases with the discount rate increasing and even may tend to zero if the discount rate tends to infinity. Thus, we drew the conclusion that high interest rate will cause high inflation rate. In this paper, it has been assumed that the harvesting effort E is constant, but the assumption may be not practical. A more reasonable assumption should be that the harvesting effort is regarded as an impulse function. We shall leave this problem to further investigation.

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References

[1] C. W. Clark, *Bioeconomic Modelling and Fisheries Management* (John Wiley and Sons, New York, 1985).
[2] C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, 2nd edn. (John Wiley and Sons, New York, 1990).
[3] K. S. Chaudhuri, A bioeconomic model of harvesting a multispecies fishery, *Eco. Model* **32** (1986) 267–279.
[4] I. M. Dew, Theoretical model of a new fishery under a simple quota management system, *Eco. Model* **143** (2001) 59–70.

[5] A. Ardito, P. Ricciardi and A. Schiaffino, The effects of seasonal fluctuations on an open access fishery problem, *J. Math. Biol.* **31** (1993) 879–890.
[6] C. C. Chen and C. Y. Hsui, Fishery policy when considering the future opportunity of harvesting, *Math. Biosci.* **207** (2007) 138–160.
[7] M. Fan and K. Wang, Optimal harvesting policy for single population with periodic co-efficients, *Math. Biosci.* **152** (1998) 165–177.
[8] T. K. Kar, U. K. Pahart and K. S. Chaudhuri, Management of a prey-predator fishery based on continuous fishing effort, *J. Biol. Syst.* **12**(3) (2004) 301–313.
[9] L. Chen, X. Song and Z. Lu, *Mathematical Models and Methods in Ecology* (Sichuan Science and Technology, Chendu, China, 2003) (in Chinese).
[10] W. G. Aiello and H. I. Freedman, A time delay model of single-species growth with stage structure, *Math. Biosci.* **101** (1990) 139–153.
[11] S. A. Gourley and Y. Kuang, A stage structured predator-prey model and its dependence on through-stage delay and death rate, *J. Math. Biol.* **49** (2004) 188–200.
[12] W. Wang, Y. Takeuchi, Y. Saito and S. Nakaoka, Prey-predator system with parental care for predators, *J. Theor. Biol.* **241** (2006) 451–458.
[13] S. Liu, L. Chen and R. Agarwal, Recent progress on stage-structured population dynamics, *Math. Comput. Model.* **36** (2002) 1319–1360.
[14] J. Cui and L. Chen, The effect of dispersal on population growth with stage structure, *Comput. Math. Appl.* **39**(1/2) (2000) 91–102.
[15] Z. Zhang and X. Zheng, A periodic stage-structure model, *Appl. Math. Lett.* **16** (2003) 1053–1061.
[16] R. Xu, M. A. J. Chaplain and F. A. Davidson, Global stability of a Lotka-Volterra type predator-prey model with stage structure and time delay, *Appl. Math. Comput.* **159** (2004) 863–880.
[17] L. Cai and X. Song, Permanence and stability of a predator-prey system with stage structure for predator, *J. Comput. Appl. Math.* **201** (2007) 356–366.
[18] S. Liu and E. Beretta, A stage-structured predator-prey model of Beddington-DeAngelis type, *SIAM J. Appl. Math.* **66** (2006) 1101–1129.
[19] R. Xu, M. A. J. Chaplain and F. A. Davidson, Persistence and global stability of a ratio-dependent predator-prey model with stage structure, *Appl. Math. Comput.* **158** (2004) 729–744.
[20] L. S. Pontryagin, V. G. Boltyonskii, R. V. Gamkrelidre and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes* (Wiley, NewYork, 1962).
[21] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications* (Academic Press, New York, London, 1961).