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ORIGINAL ARTICLE

Replicator dynamics under perturbations and time delays

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Abstract Along with the appearance of new optimization and control problems, novel paradigms emerge. A large number of them are based on behavioral ecology, where population dynamics play an important role. One of the most known models of population dynamics is the replicator equation, whose applications in optimization and control have increased in recent years. This fact motivates the study of the replicator dynamics' properties that are related to the implementation of this method for solving optimization and control problems. This paper addresses implementation issues of the replicator equation in engineering problems. We show by means of the Lyapunov theory that the replicator dynamics model is robust under perturbations that make the state to leave the simplex (among other reasons, this phenomenon can emerge due to numerical errors of the solver employed to obtain the replicator dynamic's response). A refinement of these results is obtained by introducing a novel robust dynamical sys-

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tem inspired by the replicator equation that allows to control and optimize plants under arbitrary initial conditions on the positive orthant. Finally, we characterize stability bounds of the replicator dynamics model in problems that involve N strategies that are subject to time delays. We illustrate our results via simulations.

Keywords Replicator dynamics · Evolutionary game theory · Singular perturbation · Time-delay systems

1 Introduction

Evolutionary game theory was introduced in [1] based on the idea that if we look at the interactions among individuals as a game, then the better strategies will eventually dominate in the population. The core of evolutionary game theory lies in the concept of the evolutionarily stable strategy (ESS), which is a strategy that cannot be invaded by any mutant. Therefore, if the players have adopted an ESS, bad mutations cannot overtake the population. Another notion that is closely related to the previous one is the so-called evolutionarily stable state [2,3], which refers to a population state that is capable to reject disturbances. Both ideas can be associated with the natural selection process, i.e., the dynamic mechanism by which only the best strategies survive.

One important issue related to the previous concepts is that players involved in a strategic game will eventually choose strategies that produce a better-than-average payoff. This behavior is modeled by a first-order differential equation called the frequency dynamics or replicator dynamics. The underlying principle behind the replicator model is based on the fact that the rate at which the percentage of the population using a certain strategy changes depends on how much greater (or less) the payoff of this strategy is compared to the payoff of the other strategies. The replicator equation was introduced in 1978 by Taylor and Jonker [4]. Since then, its properties have been extensively studied in the literature, e.g., we refer the reader to [5–8] for a comprehensive review, and to [9–13] for recent results.

Most of the early work on the replicator dynamics model is devoted to analyze its applications in biology and economics. Nevertheless, its usefulness is not restricted to those areas. Recently, some authors have shown the potential that the replicator equation has to optimize and control a variety of systems. For instance, in [14], the authors use replicator dynamics to regulate the temperature inside a building. They show that this technique performs optimally under different scenarios. A similar method is developed in [15] to provide adequate levels of service in water distribution systems. The key idea in [15] is to equalize and maximize the volume of water allocated in the distribution tanks. Other applications include: combinatorial optimization [16], access control [17,18] and bandwidth allocation [19] in communication networks, economic dispatch of distributed generators [20], adaptive control [21], energy saving in lighting systems [22], and so forth.

The use of methods based on the replicator equation in optimization and control poses new research challenges related to these fields. An important one involves the robustness analysis in the face of: (i) disturbances on the state and (ii) time delays. From the best of our knowledge, topic (i) has been largely unexplored in the literature, and only some articles have considered it as a secondary problem. We highlight the work conducted in [14,15], where the authors prove, by means of passivity theory, the emergence of asymptotically stable equilibrium points when replicator dynamics-based controllers are applied. Furthermore, they show by means of simulations that the closed-loop system response is capable to reject certain kind of disturbances. Nonetheless, they do not provide a thorough analysis on the robustness of the method. On the other hand, although there exist some articles that deal with topic (ii), most of the results reported in them are only valid for two strategy models. For instance, the authors in [23] derive stability conditions under time-delayed 2×2 matrix games. These conditions are extended in [17] by considering asymmetric delays, and a similar study is performed in [24] for a discrete-time version of the replicator dynamics. The drawback of the existing results is that they cannot be applied in the analysis of complex problems, since the number of strategies is generally related to the number of decision variables or subsystems involved in optimization and control schemes.

This paper addresses some of the problems referred above. Specifically, we investigate the stability properties of the replicator dynamics under certain class of perturbations and time delays. First, we derive some mild conditions that guarantee the emergence of an attractive set (i.e., the $(N-1)$-simplex), which is robust in the face of state disturbances. This result provides some guidelines on the design of fitness functions to diminish the dependence of the replicator equation's solution on errors induced by numerical solvers. Then, we propose a modified version of the classic replicator dynamics model that exhibits a two time-scale behavior and achieves fitness-independent convergence to the simplex. We use singular perturbation analysis to demonstrate that this new model has some appealing properties related to the robustness under disturbances on states and fitness functions. In addition, the proposed model exhibits the same optimality characteristics as the classic one. Hence, it is suitable to be applied as a control method. Finally, we derive necessary and sufficient conditions for delay-dependent stability of the replicator equation with N strategies under a certain class of fitness functions. We characterize the time-delay stability margin, and provide a direct connection between its value, the population size, and the fitness function's parameters. Since our results are not limited to 2×2 games, they can be applied to characterize the robustness of the replicator dynamics in large-scale problems subject to delays.

The remainder of this paper is organized as follows: Sect. 2 introduces the replicator dynamics and describes the use of this model in optimization and control problems. In Sect. 3, we study the behavior of replicator dynamics under state disturbances. Section 4 provides a robustness analysis of the replicator equation in the face of time delays. Finally, a discussion and some concluding remarks are drawn in Sects. 5 and 6, respectively.

1.1 Notation

We use $\mathbb{R}^N_+$ to denote the set of vectors in $\mathbb{R}^N$, whose entries are non-negative. Similarly, $\mathbb{R}^N_{++}$ represents the set of vectors with strictly positive components. Furthermore, Π denotes the product operator. On the other hand, $\mathrm{supp}(\xi)$ stands for the support of the vector $\xi = [\xi_1, \ldots, \xi_N]^\top$, which corresponds to the set of indices of nonzero vector's entries. Mathematically, the support of ξ is defined as follows: $\text{supp}(\xi) = \{i \in \{1, \ldots, N\} : \xi_i \neq 0\}$.

2 Replicator dynamics model

2.1 Preliminaries

Let us consider a population of size X (i.e., its individuals form a continuum of mass X) that is interacting in a game, where each individual can choose between N pure strategies. Let $x_i \in \mathbb{R}_+$ and $x = [x_1, \ldots, x_N]^\top$, be, respectively, the portion of population that has selected the ith strategy and the distribution of the whole population among the strategies. Thus, $\sum_{i=1}^{N} x_i = X$. The replicator dynamics model describes the evolution of x based on the difference between the payoff received by the individuals playing a certain strategy and the average payoff of the entire population. Mathematically, this model is given by the following differential equation [4]:

$$\dot{x}_i = x_i \left(f_i(x) - \bar{f}(x) \right), \quad \text{for all } i = 1, \ldots, N, \tag{1}$$

where the fitness function $f_i : \mathbb{R}_+^N \mapsto \mathbb{R}$ is the payoff obtained by the individuals playing the ith strategy, and $\bar{f}(x) := \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x)$ corresponds to the average fitness of the population. Throughout this paper, it is assumed that the fitness functions are Lipschitz continuous. We notice that the growth rate $\dot{x}_i$ is positive if and only if the ith fitness is greater than the average, i.e., the most successful populations increase, while the less successful ones decrease. This behavior is closely related to the natural selection process, in which the fittest individuals have better potential to survive and reproduce.

In previous work (e.g., [5,7]), it is shown that the population's size remains invariant under the replicator equation; indeed, it has been proven that x evolves in the following simplex under appropriate initialization:

$$\Delta_X = \left\{ x \in \mathbb{R}_+^N : \sum_{i=1}^{N} x_i = X \right\}. \tag{2}$$

We formalize this property in the next lemma (adapted from [5, pp. 74]).

Lemma 1 *If $x(0) \in \Delta_X$, then $x(t) \in \Delta_X$, for all $t \geq 0$, under Eq.* (1).

The above characteristic is crucial in many applications of the replicator dynamics model, especially in those related to optimization and control, where the invariance of the simplex Δ_X guarantees the satisfaction of a given constraint (see [14,20]).

2.2 Game-theoretic properties

The replicator equation exhibits appealing properties, some of which can be found in [5–7]. Next, we present certain results that will be useful in subsequent sections.

Let us begin by introducing the definition of Nash equilibrium and evolutionarily stable state. These two concepts are related to the output of a game when each individual in the population tries to maximize its own profit (i.e., players are rational).

Definition 1 *Nash equilibria* (adapted from [5,7,25]): $x^* \in \Delta_X$ is a Nash Equilibrium if each used strategy entails the maximum benefit for the players who chose it. Equivalently, the set of Nash Equilibria is given by $NE = \{x^* \in \Delta_X : x_i^* > 0 \Rightarrow f_i(x^*) \geq f_j(x^*), \forall i, j = 1, \ldots, N\}$.

Definition 2 *Evolutionarily stable state* (adapted from [5–7]): $x^* \in \Delta_X$ is an evolutionarily stable state if $\sum_{i=1}^N x_i^* f_i(x) > \sum_{i=1}^N x_i f_i(x)$, for all $x \neq x^*$ in a neighborhood of x^*.

While a Nash equilibrium is a population state in which individuals do not have incentives to unilaterally change their strategy, an evolutionarily stable state is a refinement of the previous concept. In fact, it can be shown that if a population has adopted an evolutionarily stable state x^*, any invader group, whose distribution is close to x^*, cannot obtain a better average payoff. As a result, the invaders are extinguished.

Given the fact that populations using worse strategies than the average decrease according to Eq. (1), it is natural to expect that evolutionarily stable states are locally attractive under replicator dynamics. This notion is formalized in the next theorem.

Theorem 1 (Adapted from [5–7]) *If x^* is an evolutionarily stable state, then x^* is asymptotically stable under Eq. (1).*

Proof Let us define the following Lyapunov function

$$V(x) = -\sum_{i=1}^N x_i^* \ln\left(\frac{x_i}{x_i^*}\right), \tag{3}$$

where we have adopted the convention $0\ln(0) = 0\ln(\infty) = 0$ (positive definiteness of $V(x)$ is proved in [5, Lemma 3.1]). The derivative of $V(x)$ along the trajectories of the system described in Eq. (1) is given by

$$\begin{aligned}\dot{V}(x) &= -\sum_{i=1}^N x_i^* (\dot{x}_i / x_i) \\ &= -\sum_{i=1}^N x_i^* \left(f_i(x) - \frac{1}{X}\sum_{j=1}^N x_j f_j(x)\right) \\ &= -\sum_{i=1}^N x_i^* f_i(x) + \sum_{i=1}^N x_i f_i(x).\end{aligned}$$

Since x^* is an evolutionarily stable state, $\dot{V}(x)$ is negative definite in a neighborhood of x^*. Hence, x^* is asymptotically stable. □

The above theorem shows that if the game has an evolutionarily stable state, then it is attractive under the replicator equation. A particular case that guarantees the existence of that state is when the fitness functions are strictly decreasing, as it is proven in the following corollary.

Corollary 1 *Assume that f_i only depends on the population that has chosen the ith strategy, i.e., x_i. If $f_i(x_i)$ is a strictly decreasing function, for all $i = 1, \ldots, N$, then the population game has a unique Nash equilibrium x^*, which is asymptotically stable under the replicator dynamics given in Eq. (1). Furthermore, the region of attraction of x^* is $\mathcal{Q}_{x^*} = \{x \in \Delta_X : supp(x^*) \subset supp(x)\}$.*

Proof First, let us prove that the Nash equilibrium is unique. To do this, we proceed by contradiction. Assume that $\left\{x^{(1)}, x^{(2)}\right\} \in NE$, and $x^{(1)} \neq x^{(2)}$. Without loss of generality, let $x_i^{(1)} > x_i^{(2)}$, where $x_i^{(1)}$ and $x_i^{(2)}$ are the ith entries of vectors $x^{(1)}$ and $x^{(2)}$, respectively. Since f_i is strictly decreasing and $\left\{x^{(1)}, x^{(2)}\right\} \in NE$, $\bar{f}(x^{(1)}) < \bar{f}(x^{(2)})$. Let $j \neq i$. If $x_j^{(1)} > 0$, then $f_j(x_j^{(1)}) = \bar{f}(x^{(1)}) < \bar{f}(x^{(2)})$. Thus, $x_j^{(2)} < x_j^{(1)}$. Moreover, if $x_j^{(1)} = 0$, then $f_j(x_j^{(1)}) < f_i(x_i^{(1)}) < \bar{f}(x^{(2)})$. Hence, $x_j^{(2)} = 0$. In conclusion, $x_j^{(1)} \geq x_j^{(2)}$, for all $j = 1, \ldots, N, j \neq i$. Therefore, $\sum_{k=1}^{N} x_k^{(1)} > \sum_{k=1}^{N} x_k^{(2)}$, which is a contradiction due to simplex invariance.

To prove that the Nash equilibrium x^* is asymptotically stable under (1), it is sufficient to show that x^* is an evolutionarily stable state (c.f. Theorem 1). To do that, we define the quantity $\psi(x) = \sum_{i=1}^{N} (x_i^* - x_i) f_i(x_i)$. Let $\hat{f}_i(x_i) = f_i(x_i) - f_i(x_i^*)$. Thus

$$\begin{aligned}\psi(x) &= \sum_{i=1}^{N} (x_i^* - x_i) \left(\hat{f}_i(x_i) + f_i(x_i^*)\right) \\ &= \sum_{i=1}^{N} (x_i^* - x_i) \hat{f}_i(x_i) + \sum_{i=1}^{N} (x_i^* - x_i) f_i(x_i^*).\end{aligned}$$

We notice that $x^* \in \Delta_X$ since $x^* \in NE$. For the same reason, $x_i^* > 0$ if $f_i(x_i^*) = \bar{f}(x^*)$, and $x_i^* = 0$ if $f_i(x_i^*) < \bar{f}(x^*)$. Without loss of generality, we assume that, at equilibrium, M fitness functions are equal ($M \leq N$), i.e., $f_i(x_i^*) = \bar{f}(x^*)$ for $i = 1, \ldots, M$ (if the fitness functions are sorted in descending order). Under this assumption, $\psi(x)$ can be expressed as follows:

$$\psi(x) = \sum_{i=1}^{N} (x_i^* - x_i) \hat{f}_i(x_i) - \left(\bar{f}(x^*) \sum_{i=1}^{M} (x_i - x_i^*) + \sum_{i=M+1}^{N} (x_i - x_i^*) f_i(x_i^*) \right).$$

Given the fact that $f_i(x_i^*) < \bar{f}(x^*)$ and $x_i - x_i^* = x_i \geq 0$ for $i = M+1, \ldots, N$, we have

$$\psi(x) \geq \sum_{i=1}^{N} (x_i^* - x_i)\hat{f}_i(x_i) - \left(\bar{f}(x^*) \sum_{i=1}^{M} (x_i - x_i^*) + \sum_{i=M+1}^{N} (x_i - x_i^*)\bar{f}(x^*) \right)$$

$$= \sum_{i=1}^{N} (x_i^* - x_i)\hat{f}_i(x_i) - \bar{f}(x^*) \left(\underbrace{\sum_{i=1}^{N} x_i}_{X} - \underbrace{\sum_{i=1}^{N} x_i^*}_{X} \right)$$

$$= \sum_{i=1}^{N} (x_i^* - x_i)\hat{f}_i(x_i).$$

Moreover, since $\hat{f}_i(x_i)$ is strictly decreasing and zero in x_i^*, we have that $\hat{f}_i(x_i) > 0$ if $x_i < x_i^*$, and $\hat{f}_i(x_i) < 0$ if $x_i > x_i^*$. Therefore, $\psi(x)$ is positive definite, i.e., $\psi(x) = \sum_{i=1}^{N} (x_i^* - x_i) f_i(x_i) > 0$, for all $x \neq x^*$. Thus, x^* is an evolutionarily stable state.

To show that $\mathcal{Q}_{x^*}$ is the region of attraction of x^*, notice that the derivative of the Lyapunov function given in Eq. (3) (which is used in Theorem 1 to show asymptotic stability of evolutionarily stable states) is equal to $-\psi(x)$. Thus, this derivative is negative definite in Δ_X. However, the cited Lyapunov function is only valid in $\mathcal{Q}_{x^*}$. Therefore, $\mathcal{Q}_{x^*}$ is a candidate set to be the region of attraction of x^*. To complete the proof, consider the following argument: for each $x(0) \notin \mathcal{Q}_{x^*}$, there exists some $i = 1, \ldots, N$, such that $x_i(0) = 0$ and $x_i^* \neq 0$. Under this scenario, $x_i(t) = 0$, for all $t \geq 0$, following the replicator dynamics given in Eq. (1). Hence, $x(t)$ does not converge to x^*. In conclusion, the set $\mathcal{Q}_{x^*}$ completely characterizes the region of attraction of x^*. □

Remark 1 An alternative proof of Corollary 1 can be derived from the more general result in [7, Theorem 7.1.2]. That proof does not rely on the concept of evoultionarily stable state, but exploits the properties of potential games.

Theorem 1 and Corollary 1 provide conditions that guarantee the convergence of the replicator equation's solution to a desired output of a population game. Next, we present two results that link the replicator dynamics model with the solution of optimization and control problems.

2.3 Replicator dynamics applied to optimization and control

Before introducing the optimality properties of the replicator dynamics, let us define the interior of the simplex Δ_X as

$$\text{int}\Delta_X := \left\{ x \in \mathbb{R}_{++}^N : \sum_{i=1}^{N} x_i = X \right\}.$$

From a population games perspective, the interior of Δ_X is the set of possible population states in which all the strategies are being played.

Once defined intΔ_X, the following theorem (adapted from [6,26]) clarifies the optimality characteristics of the replicator equation.

Theorem 2 Shahshahani gradient system*: Let $\dot{x}_i = \frac{\partial V}{\partial x_i}$, for $i = 1, \ldots, N$, be a Euclidean gradient system on $\mathbb{R}^N$ with $V(x)$ as its potential function. If $f_i(x) := \frac{\partial V}{\partial x_i}$ in* (1)*, the replicator equation is a gradient system on* $\mathrm{int}\Delta_X := \{x \in \mathbb{R}^N_{++} : \sum_{i=1}^{N} x_i = X\}$ *with the same potential function and the Shahshahani metric.*

Thus, if the fitness functions are selected in a proper way, i.e., as the partial derivative of an objective function $V(x)$, the replicator dynamics model becomes a gradient ascent method on the interior of the simplex Δ_X following a particular metric (we notice that the Shahshahani metric referred in Theorem 2 is characterized by the inner product $\langle \xi, \eta \rangle_x = \sum_{i=1}^{N} \frac{\xi_i, \eta_i}{x_i}$, where ξ, η belong to the tangent space to intΔ_X at x. We refer the reader to [6] for further details). This fact has been exploited in recent work, such as in [20,22,27], where the authors seek to maximize certain criteria subject to the simplex constraint, which represent a resource restriction in most of the problems. A particular situation is when the objective function $V(x)$ is strictly concave. In this case, the inequality in Definition 2 is satisfied at the maximum x^* by choosing $f_i(x) = \frac{\partial V}{\partial x}$. Therefore, x^* is also an evolutionarily stable state, and the solution of the replicator equation will converge to it. On the other hand, it is worth noting that the application of the replicator dynamics model is not limited to static optimization. Indeed, this model has been used to design time-varying adaptive controllers based on a extremum seeking architecture (see [19,21]).

As it was noticed in the introduction, last years have also witnessed the emergence of novel control techniques based on replicator dynamics (several applications can be found in [14,15]). The core of these techniques relies on the following property [9,28].

Proposition 1 Passivity: *The replicator dynamics system with input $u = [f_1, \ldots, f_N]^\top$ and output $y = [(x_1 - x_1^*), \ldots, (x_N - x_N^*)]^\top$ is passive.*

Proof Using the Lyapunov function given in Eq. (3) as the system storage function, we have that $\dot{V}(x) = \sum_{i=1}^{N} (x_i - x_i^*) f_i = u^\top y$. This implies that the stored energy is equal to the power supplied into the system along the time. Hence, the system is passive and lossless [29]. $\square$

Passivity is a key feature, because it allows the interconnection of a controller based on replicator dynamics with other passive systems avoiding instability issues. The general control scheme is shown in Fig. 1, where Σ_P represents the controlled plant, and Σ_{RD} is the replicator dynamics model. Notice that, in this case, the fitness functions depend on the output of a dynamical system (the controlled system) instead of the population's state (we refer the reader to [30] for a biological point of view of this dependence, and to [9] for a population dynamics perspective). As in the static case, the selection of fitness functions is crucial to achieve the control goals. In [14],

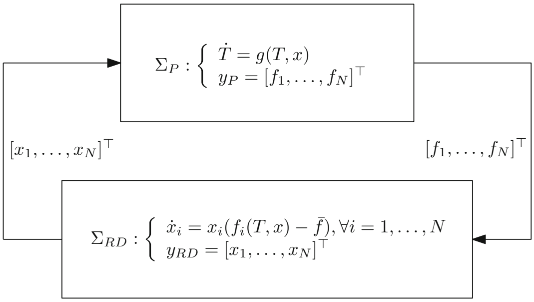

Fig. 1 Standard configuration of a plant controlled via replicator dynamics

the authors outline some guidelines for a proper design of fitness functions. However, this is still an open problem.

The application of the replicator equation in optimization and control poses new challenges related to these fields. In the next sections, we provide our main results regarding the robustness of the replicator dynamics model under a class of perturbations and time delays.

3 Robustness of replicator dynamics in optimization and control applications

As it was stated earlier, the simplex Δ_X plays an important role in formulating problems that involve constrained optimization and control. This situation is due to the fact that Δ_X is closely related to certain restrictions. For instance, if we have N zones, and x_i represents the resource allocated to the ith zone, then $x \in \Delta_X$ implies that the resource allocated to all the zones must be equal to X (c.f., Eq. (2)). In this case, Δ_X is associated with the preservation of a resource. Similarly, if we have a set of N controllers acting on a plant, and x_i is the energy provided by the ith controller, then Δ_X is related to a limitation of the total energy that can be applied to the plant. In this regard, an interesting property of the replicator dynamics model is given by Lemma 1, which ensures that the simplex Δ_X is invariant, i.e., the replicator equation produces solutions that remain in Δ_X provided that the initial condition belongs to Δ_X. However, Lemma 1 does not give us insights on the replicator dynamics' behavior when $x(t_o) \notin \Delta_X$, for some $t_o \geq 0$. The problem is that this scenario is common when the replicator dynamics model is applied as an optimization or control method, since external disturbances, such as noise and bounded perturbations, can act on the state x (e.g., solvers used in the implementation stage induce state perturbations due to numerical approximations). The following example illustrates the implications of having a state that is out of Δ_X.

Example 1 Effect of state disturbances: Consider the problem described in [28], where there are N zones that need to reach the maximum uniform temperature subject to a power limitation. The temperature T_i of the ith zone is described by means of the following differential equation:

$$\dot{T}_i = k_{i,1}(T_a - T_i) + k_{i,2}x_i,$$

where T_a is the ambient temperature, $x_i > 0$ is the power applied to the zone, and $k_{i,1}, k_{i,2} > 0$ are system's parameters. In addition, the total power supplied to all the zones is constrained to a certain value X, i.e., $\sum_{i=1}^{N} x_i = X$. To achieve the control goal, we apply replicator dynamics following the scheme depicted in Fig. 1. The fitness used is given by $f_i = \gamma(B - T_i)$, where $\gamma > 0$ and $B \geq 0$ are tuning parameters. This fitness is employed, since, at equilibrium of the replicator dynamics given in Eq. (1), $f_i^* = f_j^*$, for all $i, j = 1, \ldots, N$, which implies that the temperature in the zones reach the same value. Besides, if $x(0) \in \Delta_X$, then $x^* \in \Delta_X$. Therefore, the temperature is maximized due to the fact that all the power available is used in steady state. We simulate the controlled system in Matlab® with the following conditions: $N = 4$, $k_{i,1} = k_{i,2} = 0.001$, for all $i = 1, \ldots, N$, $T_a = 20$ °C, $X = 10$ KW, $\gamma = 0.001$, and $B = 0$. Figure 2 shows the corresponding results. Notice that initially, the controller based on replicator dynamics performs in a proper way, i.e., power constraint is satisfied (dashed line in Fig. 2b) and more power is allocated to the cooler zones. Nonetheless, at around $t = 30$ min, the numerical solver introduces a disturbance that makes x to leave the simplex Δ_X. After this disturbance, the resource constraint is not satisfied anymore. Indeed, the total amount of power allocated to the different zones increases with time, and the closed-loop system response diverges.

The previous example evidences that a bad design of the replicator dynamics-based controller produces a non-robust performance. In fact, as we will show later, this adverse effect is due to a bad choice of the fitness function.

One of the key problems in several applications is that the state of the controller does not satisfy the simplex constraint at certain times. To analyze the behavior of the replicator dynamics model when $x \notin \Delta_X$, we need a stronger result than the one given in Lemma 1.

Proposition 2 Attractivity of the simplex: *Assume that, $x(0) \in \mathbb{R}_+^N$ and $x_i(0) > 0$, for some $i = 1, \ldots, N$. If $f_i > 0$, for all $i = 1, \ldots, N$ and for all $t \geq 0$, then the simplex Δ_X is attractive under Eq.* (1)*, i.e., $x^* \in \Delta_X$ in steady state.*

Proof First, notice that if $x_i = 0$, then $\dot{x}_i = 0$ according to Eq. (1). Therefore, the non-negative orthant $\mathbb{R}_+^N$ is positively invariant under replicator dynamics. In fact, [5, pp. 74] shows that if $x_i(0) > 0$, then $x_i(t) > 0$, for all $t \geq 0$. Hence, since $x(0) \in \left(\mathbb{R}_+^N - \{0\}\right)$ by assumption, $x(t) \in \left(\mathbb{R}_+^N - \{0\}\right)$, for all $t \geq 0$.

Let $V(x) = \frac{1}{2}\left\| X - \sum_{i=1}^{N} x_i \right\|^2$ be a measure related to the distance between the state x and the simplex Δ_X. The derivative of $V(x)$ along the trajectories of the system described in (1) is given by

$$
\begin{aligned}
\dot{V}(x) &= -\left(X - \sum_{i=1}^{N} x_i\right)\sum_{i=1}^{N} \dot{x}_i \\
&= -\left(X - \sum_{i=1}^{N} x_i\right)\left[\sum_{i=1}^{N}\left(x_i\left(f_i - \frac{1}{X}\sum_{j=1}^{N} x_j f_j\right)\right)\right]
\end{aligned}
$$

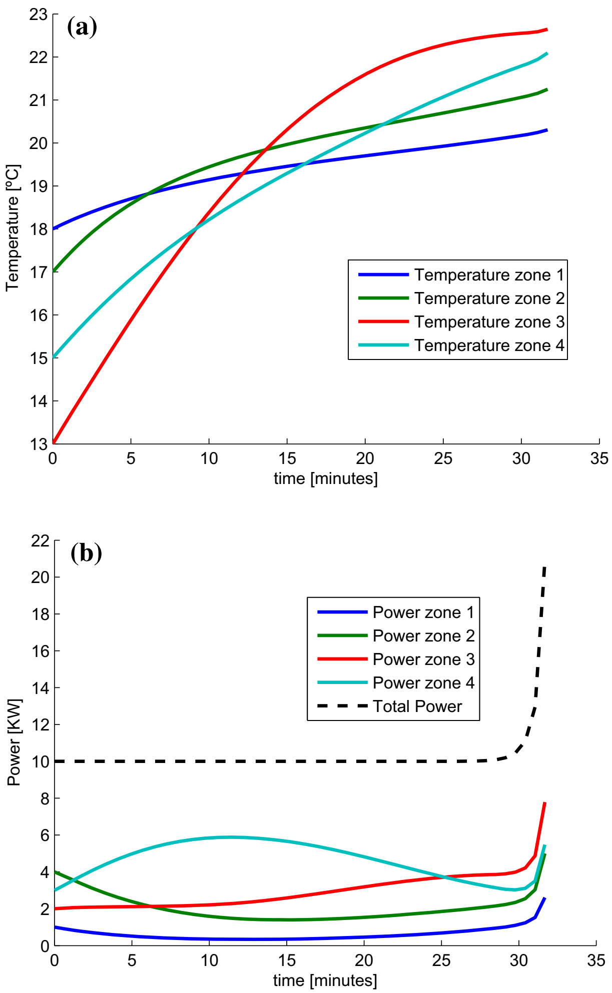

Fig. 2 Effect of state disturbances under replicator dynamics. **a** Evolution of the temperature in each zone. **b** Power supplied to each zone and total power used by the controller (*dashed line*)

$$
\begin{aligned}
&= -\left(X - \sum_{i=1}^{N} x_i\right)\left[\sum_{i=1}^{N} x_i f_i - \frac{1}{X}\sum_{i=1}^{N} x_i \sum_{j=1}^{N} x_j f_j\right] \\
&= -\left(X - \sum_{i=1}^{N} x_i\right)\sum_{i=1}^{N} x_i f_i \left(1 - \frac{1}{X}\sum_{i=1}^{N} x_i\right) \\
&= -\frac{1}{X}\underbrace{\sum_{i=1}^{N} x_i f_i}_{>0}\left(X - \sum_{i=1}^{N} x_i\right)^2
\end{aligned}
$$

Since $\dot{V}(x)$ is negative for all points x satisfying $\sum_{i=1}^{N} x_i \neq X$, we have that the quantity $V(x) = \frac{1}{2}\left\| X - \sum_{i=1}^{N} x_i \right\|^2$ decreases in time, i.e., x converges to the simplex Δ_X. □

If the conditions of Proposition 2 are satisfied, the trajectories that leave Δ_X will eventually converge back to it, i.e., the simplex constraint is asymptotically guaranteed if the fitness functions are always positive and the state x is non-negative.

Remark 2 To ensure positivity of the fitness functions, we can add an auxiliary positive function $\Psi(x)$ (e.g., a large positive constant) to all f_i. It can be verified that this modification does not affect either the equilibrium points nor the behavior of Eq. (1) on Δ_X. In addition, although non-negativity of the state x is guaranteed in noise-free scenarios if $x(0) \in \mathbb{R}_+^N$ (c.f., Lemma 1), this constraint can be unsatisfied in the face of state perturbations. However, non-negativity of x can be easily ensured by setting the value of x_i to zero if x_i becomes negative due to a perturbation.

The previous analysis allows us to correct the problems found in the simulations of Example 1. Let us consider the same scenario described in that example. However, in this case, we choose B, such that $f_i = B - T_i$ is always positive, e.g., $B = 100$. Results using this new parameter are shown in Fig. 3. Notice that, under the new conditions, the closed-loop-system response converges to the desired solution, i.e., all the zones reach the maximum uniform temperature despite disturbances induced by the numerical solver.

The next section goes deeper into the analysis of the replicator equation in the face of state disturbances.

3.1 Generalized replicator dynamics model

We continue considering the case when the replicator dynamics equation is used to optimize and/or control systems in the simplex Δ_X. As it was shown before, in most applications the states of the replicator dynamics cannot be guaranteed to stay in Δ_X for all time. This is mostly due to external disturbances, such as noise and bounded perturbations acting on the state x. To leverage the positive fitness assumption of Proposition 2, we analyze the behavior of the replicator dynamics when the state is outside of the simplex. This analysis is made by considering a generalized replicator equation, whose behavior depends on a positive parameter δ, which can be tuned to achieve a desired performance.

Let us consider the following nonlinear autonomous dynamical system:

$$\dot{x}_i = x_i \left[\left(\frac{\mathbb{1}^\top x}{X} \right) f_i(x) + \frac{1}{\delta} - \frac{1}{X} \sum_{j=1}^{N} \left(f_j(x) + \frac{1}{\delta} \right) x_j \right], \quad \forall \; i = 1, \ldots, N, \quad (4)$$

where $\mathbb{1}$ is the column vector in $\mathbb{R}^N$, whose entries are all 1. Equation (4) will be seen as a "robustified" version of Eq. (1). We proceed to study the stability properties of this system.

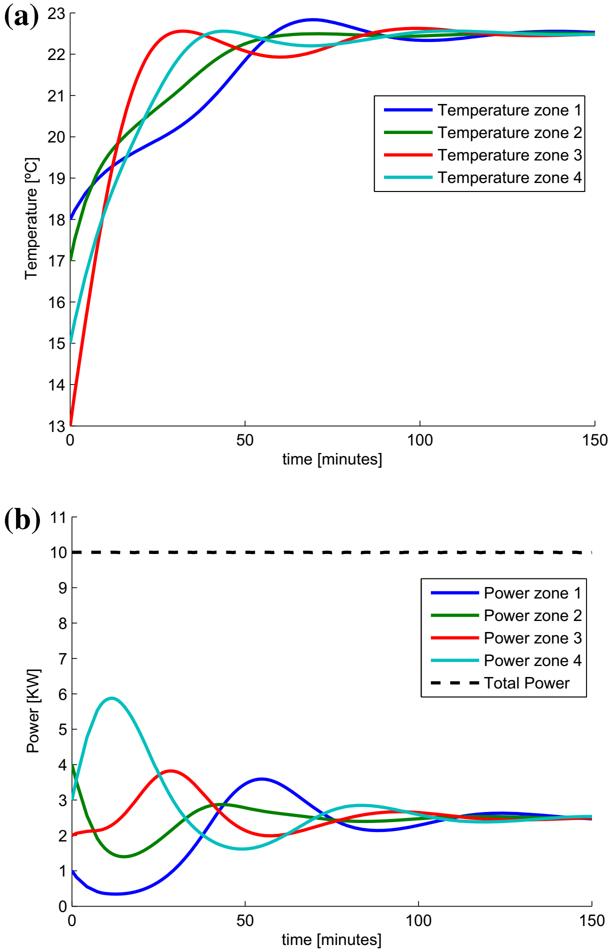

Fig. 3 Mitigation of the effect of state disturbances under replicator dynamics. **a** Evolution of the temperature in each zone. **b** Power supplied to each zone and total power used by the controller (*dashed line*)

3.2 Stability analysis

We now make the following assumption on the vector of fitness functions.

Assumption 1 The vector of fitness functions $f = [f_1, \ldots, f_N]^\top$ is such that the classic replicator dynamics given in Eq. (1) has an equilibrium point $x^* \in \Delta_X$ that is globally (with respect to $\mathcal{Q}_{x^*}$) asymptotically stable. Moreover, $f_i : \mathbb{R}^N \mapsto \mathbb{R}$ is continuously differentiable, for all $i = 1, \ldots, N$.

Notice that, according to Corollary 1 and Theorem 2, Assumption 1 is satisfied if, for instance, the fitness functions are strictly decreasing, or if the vector f corresponds to the gradient of a strictly concave potential function V. Assumption 1 also holds if x^* is an evolutionarily stable state with basin of attraction equal to $\text{int}\Delta_X$.

Lemma 2 *If $x \in \Delta_X$, Eq. (4) is equivalent to the standard replicator dynamics (1).*

Proof By expanding the dynamics (4), and using the fact that $\mathbb{1}^\top x = X$, we obtain

$$\begin{aligned}
\dot{x}_i &= x_i \left[f_i(x) + \frac{1}{\delta} - \frac{1}{X} \sum_{j=1}^{N} \left(f_j(x) + \frac{1}{\delta} \right) x_j \right], \quad \forall\, i = 1, \ldots, N. \\
&= x_i \left[f_i(x) + \frac{1}{\delta} - \frac{1}{X} \sum_{j=1}^{N} f_j(x) x_j - \frac{1}{X} \sum_{j=1}^{N} \frac{x_j}{\delta} \right], \quad \forall\, i = 1, \ldots, N. \\
&= x_i \left[f_i(x) + \frac{1}{\delta} - \frac{1}{X} \sum_{j=1}^{N} f_j(x) x_j - \frac{1}{\delta} \right], \quad \forall\, i = 1, \ldots, N. \\
&= x_i \left[f_i(x) - \frac{1}{X} \sum_{j=1}^{N} f_j(x) x_j \right], \quad \forall\, i = 1, \ldots, N. \\
&= x_i \left[f_i(x) - \bar{f}(x) \right], \quad \forall\, i = 1, \ldots, N,
\end{aligned}$$

where the average fitness $\bar{f}(x)$ is given by $\bar{f}(x) = \frac{1}{X} \sum_{j=1}^{N} f_j(x) x_j$. □

When we restrict our attention to the simplex Δ_X, Lemma 2 shows that the generalized system proposed in Eq. (4) corresponds to the standard replicator dynamics. However, this is not the case when $x \notin \Delta_X$. To characterize Eq. (4) when x does not belong to Δ_X we first review the notion of semiglobal-practical asymptotic stability [31].

Definition 3 For a Lipchitz continuous dynamical system represented by $\dot{x} = g(x, \delta)$, where $\delta \in \mathbb{R}_{++}$ is a small positive parameter, the origin $x = 0$ is said to be semiglobally practically asymptotically stable as $\delta \mapsto 0^+$, if for every pair $(M, \epsilon) \in \mathbb{R}^2_{++}$, there exists a $\delta^* \in \mathbb{R}_{++}$ and $T \in \mathbb{R}_{++}$ such that for each $\delta \in (0, \delta^*)$ any solution $x(t)$ starting in the ball of radius M is at time T in the ball of radius ϵ, and never leaves this ball.

To analyze the system given in Eq. (4), we will show that it is possible to model this system as a singularly perturbed system [29], where the constant δ acts as a small perturbation.

To start, we introduce the auxiliary variable $z = \frac{\mathbb{1}^\top x - X}{\delta}$, and we obtain that

$$\mathbb{1}^\top x = \delta z + X, \tag{5}$$

Replacing (5) in (4), we have that

$$\dot{x}_i = x_i \left[\left(\frac{\delta z + X}{X} \right) f_i(x) + \frac{1}{\delta} - \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x) - \frac{1}{X} \sum_{j=1}^{N} \frac{x_j}{\delta} \right]. \qquad (6)$$

This can be rewritten as

$$\begin{aligned}
\dot{x}_i =& x_i \left[\left(\frac{\delta z}{X} + 1 \right) f_i(x) - \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x) \right] + \frac{x_i}{\delta} - \frac{x_i}{X} \sum_{j=1}^{N} \frac{x_j}{\delta} \\
=& x_i \left[\left(\frac{\delta z}{X} + 1 \right) f_i(x) - \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x) \right] + \frac{x_i}{\delta} - \frac{x_i}{X} \left(\frac{\delta z + X}{\delta} \right) \\
=& x_i \left[\left(\frac{\delta z}{X} + 1 \right) f_i(x) - \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x) \right] + \frac{x_i}{\delta} - \frac{x_i z}{X} - \frac{x_i}{\delta}.
\end{aligned}$$

Therefore

$$\dot{x}_i = x_i \left[\left(\frac{\delta z}{X} + 1 \right) f_i(x) - \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x) \right] - \frac{x_i z}{X}. \qquad (7)$$

In the same way, we have that $\dot{z}$ is given by

$$\dot{z} = \frac{\mathbb{1}^\top \dot{x}}{\delta} = \frac{1}{\delta} \sum_{i=1}^{N} \dot{x}_i. \qquad (8)$$

Replacing (7) in (8) we obtain that

$$\begin{aligned}
\dot{z} =& \frac{1}{\delta} \sum_{i=1}^{N} \left(x_i \left[\left(\frac{\delta z}{X} + 1 \right) f_i(x) - \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x) \right] - \frac{x_i z}{X} \right) \\
=& \frac{1}{\delta} \left(\left(\frac{\delta z}{X} + 1 \right) \sum_{i=1}^{N} x_i f_i(x) - \frac{\sum_{i=1}^{N} x_i}{X} \sum_{j=1}^{N} x_j f_j(x) - \frac{z \sum_{i=1}^{N} x_i}{X} \right),
\end{aligned}$$

and since $\sum_{i=1}^{N} x_i / X = \left(\frac{\delta z}{X} + 1 \right)$, we have that

$$\dot{z} = -\frac{1}{\delta} \frac{z}{X} \sum_{i=1}^{N} x_i,$$

and hence the complete extended system is given by

$$\dot{x}_i = x_i \left[\left(\frac{\delta z}{X} + 1 \right) f_i(x) - \frac{1}{X} \sum_{j=1}^{N} x_j f_j(x) \right] - \frac{x_i z}{X} \tag{9a}$$

$$\delta \dot{z} = - \frac{z}{X} \sum_{i=1}^{N} x_i. \tag{9b}$$

System (9) is in standard singular perturbation form [29], where the small perturbation δ induces a separation of time scales between the evolution of the state x and the evolution of the state z. We use this singular perturbation form of the original system (4) to provide the following result regarding the robustness of the generalized replicator dynamics.

Theorem 3 *Suppose that Assumption 1 holds, and consider the singularly perturbed system (9). Then, the equilibrium point x^* is semi-globally (with respect to $\mathcal{B}_{x^*}$) practically asymptotically stable as $\delta \to 0^+$, where $\mathcal{B}_{x^*} = \{x \in \mathbb{R}_+^N : supp(x^*) \subset supp(x)\}$.*

Proof Let us study the behavior of the system in (9). For sufficiently small values of δ, the dynamics (9b) move in a faster time-scale than the dynamics in (9a). To see this, consider the introduction of the new time variable $\theta = \frac{t - t_0}{\delta}$. In the new time-scale θ, we obtain that

$$\frac{\partial x_i}{\partial \theta} = \delta x_i \left[\left(\frac{\delta z}{X} + 1 \right) f_i(x) - \frac{1}{X} \sum_{i=1}^{N} x_i f_i(x) \right] - \frac{x_i z}{X} \delta \tag{10a}$$

$$\frac{\partial z(\theta)}{\partial \theta} = - \frac{z}{X} \sum_{i=1}^{N} x_i(t). \tag{10b}$$

Since $t = t_0 + \delta\theta$ and $x(t) = x(t_0 + \delta\theta)$, notice that for small values of δ, the state x will change slowly. Now, at the limit case when $\delta = 0$, the state x is frozen at $x(t_0)$, and hence, the auxiliary state z will evolve according to

$$\frac{\partial z(\theta)}{\partial \theta} = - \frac{z}{X} \sum_{i=1}^{N} x_i(t_0). \tag{11}$$

Thus, if $\sum_{i=1}^{N} x_i(t_0) > 0$, the equilibrium point $z^* = 0$, or equivalently the set $\sum_{i=1}^{N} x_i^* = X$, will be globally asymptotically stable (indeed exponentially stable). Notice that this condition is satisfied in $\mathbb{R}_+^N - \{0\}$, and hence, the simplex Δ_X will be attractive if $x(t_0) \in \mathbb{R}_+^N$ and at least one component $x_i(t_0)$ is positive, where $i = 1, \ldots, N$. Now, to analyze the slow or "reduced" dynamics, consider the dynamics (10a) evaluated at the equilibrium of (10b), i.e., when $z = 0$. In this case, (10a) reduces to the classic replicator dynamics (1), whose equilibrium point x^* is asymptotically stable under Assumption 1. Since the origin of the boundary layer system

(11) and the reduced system (1) are both globally asymptotically stable (with respect to $\mathcal{B}_{x^*} \subset \mathbb{R}_+^N - \{0\}$), and $\mathbb{R}_+^N$ and $\mathbb{R}_{++}^N$ are both positively invariant for the original system (4), we have that the original system (4) is semi-globally (with respect to $\mathcal{B}_{x^*}$) practically asymptotically stable as $\delta \to 0^+$ [32, Theorem 4]. □

Remark 3 In contrast to Proposition 2, the convergence to Δ_X, which is modeled by the dynamics $\dot{z}$, is independent of the vector of fitness functions f as $\delta \to 0^+$. Thus, if f is selected such that the reduced model, i.e., the standard replicator equation, is a Shahshahani gradient, convergence to the optimal point $x^* \in \Delta_X$ of the potential function V is not affected by other possible maximizers of V located outside of Δ_X. This property is achieved by the introduction of the term $(\frac{\mathbb{1}^\top x}{X})$ that multiplies the individual fitness functions in (4).

3.3 Approximate convergence point in $\boldsymbol{\Delta_X}$

The previous section states that if $x(0) \notin \Delta_X$, then the state x will converge to Δ_X under the generalized replicator dynamics given in Eq. (4) for δ sufficiently small. As a matter of fact, it is possible to calculate the point in Δ_X, where the trajectories x will converge as $\delta \to 0^+$. For this purpose, consider the expansion of the system (4):

$$\begin{aligned}
\dot{x}_i &= x_i \left[\left(\frac{\mathbb{1}^\top x}{X} \right) f_i(x) + \frac{1}{\delta} - \frac{1}{X} \sum_{j=1}^N \left(f_j(x) + \frac{1}{\delta} \right) x_j \right], \\
&= x_i \left[\left(\frac{\mathbb{1}^\top x}{X} \right) f_i(x) + \frac{1}{\delta} - \frac{1}{X} \sum_{j=1}^N f_j(x) x_j - \frac{1}{\delta X} \sum_{j=1}^N x_j \right], \\
&= x_i \left(\frac{\mathbb{1}^\top x}{X} \right) f_i(x) + \frac{x_i}{\delta} - \frac{x_i}{X} \sum_{j=1}^N f_j(x) x_j - \frac{x_i}{\delta X} \sum_{j=1}^N x_j, \quad \forall\, i = 1, \ldots, N.
\end{aligned}$$

Rearranging the terms we have that

$$\dot{x}_i = \left[x_i \left(\frac{\mathbb{1}^\top x}{X} \right) f_i(x) - \frac{x_i}{X} \sum_{j=1}^N f_j(x) x_j \right] + \frac{1}{\delta} \left[x_i - \frac{x_i}{X} \sum_{j=1}^N x_j \right], \quad \forall\, i = 1, \ldots, N. \tag{12}$$

For δ sufficiently small, the terms inside the right brackets will dominate the terms of the left brackets, and system (12) can be approximated by

$$\dot{x}_i \approx \frac{1}{\delta} \left(x_i - \frac{x_i}{X} \sum_{j=1}^N x_j \right), \quad \forall\, i = 1, \ldots, N.$$

Thus,

$$\dot{x}_i \approx -\frac{x_i}{\delta X}\left(\mathbb{1}^\top x - X\right), \quad \forall\, i = 1, \ldots, N. \tag{13}$$

Hence, when $\mathbb{1}^\top x \neq X$ and δ is sufficiently small, the trajectory of x is approximately given by the solution of (13), where x_i flows towards the origin if $\mathbb{1}^\top x - X > 0$, or away from the origin if $\mathbb{1}^\top x - X < 0$, in both cases with the same rate for all $i = 1, \ldots, N$. Since the simplex Δ_X is asymptotically stable in the positive orthant, the point $x^\Delta \in \Delta_X$, where the trajectories will converge when they are initialized at $x(0) \notin \Delta_X$ will be given by the intersection of Δ_X and a vector that starts at $x(0)$ and points towards the origin if $\mathbb{1}^\top x(0) - X > 0$, or away from the origin if $\mathbb{1}^\top x(0) - X < 0$. Therefore, $x^\Delta = [x_1^\Delta, \ldots, x_N^\Delta]^\top$ is given by

$$x_i^\Delta = \frac{x_i(0)}{\mathbb{1}^\top x(0)} X, \quad \forall\, i = 1, \ldots, N. \tag{14}$$

Notice that the error in the approximation of x_i^Δ given by (14) will converge to zero as $\delta \to 0^+$. Moreover, the singular perturbation analysis from Sect. 3.2 also tells us that the convergence to $x^\Delta \in \Delta_X$ will occur in a faster time scale compared to the convergence to $x^* \in \Delta_X$, i.e., the state x will first hit a neighborhood of x^Δ before starting to flow towards x^*.

Remark 4 In the case that an additional fitness $f_{N+1} = 0$ is added to the system, the states $x_1, \ldots, x_N$ will converge to the equilibrium point in the set $\{x \in \mathbb{R}^N_{++} : \sum_{i=1}^N x_i \leq X\}$. Under this scenario, the additional state x_{N+1} associated with f_{N+1} acts as a slack variable that can be used to perform optimization under inequality constraints. This approach has been used in [14,22,33]) to solve optimal resource allocation problems in engineering systems.

3.4 Numerical examples

We illustrate the foregoing results by means of the following examples.

Example 2 Attractivity of the simplex I: Consider the case when the fitness functions are selected as $f_1 = -2(x_1 - 0.6)$, $f_2 = -2(x_2 - 0.4)$, and the parameter δ is defined as $\delta = 0.005$. In this case, the point $x^* = [0.6,\ 0.4]^\top$ is asymptotically stable in $\mathcal{Q}_{x^*}$ for the reduced model of system (10). Figure 4a shows three different trajectories initialized at the points $x_a(0) = [0.2, 1.4]^\top$, $x_b(0) = [0.18, 0.2]^\top$, and $x_c(0) = [1.6,\ 0.2]^\top$. The trajectories converge in a faster time scale to a neighborhood of the points x_a^Δ, x_b^Δ, and x_c^Δ, respectively, located in the simplex and given by Eq. (14). After this, the trajectories converge slowly to a neighborhood of the point x^*. Figure 4b shows the different behaviors of the system's trajectories from two different initial conditions and different values of the parameter δ. As it can be observed, the smaller the value of δ, the closer the trajectories approximate the predicted behavior of the singularly perturbed system (9b)–(9a).

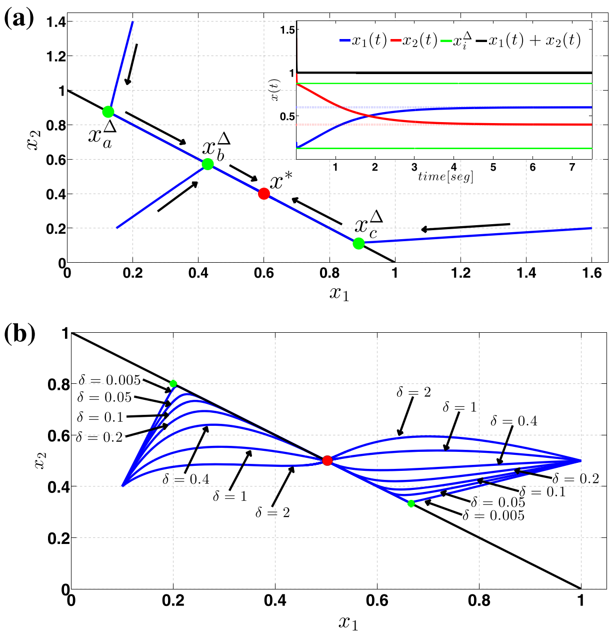

Fig. 4 Attractivity of the simplex in $\mathbb{R}^2_+$ under the dynamics (4). **a** Convergence to x^{Δ} under different initial conditions $x(0)$. **b** Trajectories of (4) for different values of δ

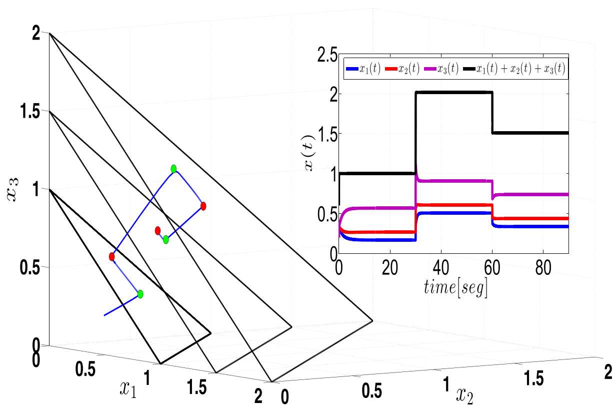

Fig. 5 Convergence of the dynamics (4) for different simplexes

Example 3 Attractivity of the simplex II: Figure 5 shows the trajectories of the system (4) when $x \in \mathbb{R}^3$, and the fitness functions are selected as $f_1 = -2(x_1 - 0.9)$, $f_2 = -2(x_2 - 1)$, $f_3 = -2(x_3 - 1.3)$, and the parameter δ is defined as $\delta = 0.02$. In this case, we vary the size of the simplex obtaining convergence to the point x^{Δ}, represented by a green circle, and subsequently to the point x^*, represented with a red circle that is associated with the respective simplex.

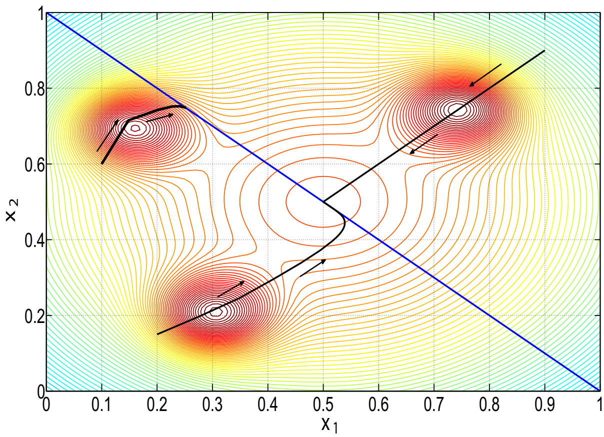

Fig. 6 Convergence of the dynamics (4) in presence of local maxima located outside the set Δ_X

Example 4 Optimization in Δ_X with local maxima in $\mathbb{R}_{++}\backslash\Delta_X$: We now consider the implications of Remark 3. In this case we consider a potential function given by

$$V(x) = \sum_{j=1}^{4} B_j \exp\left(-\frac{(x_1 - x_{1j}^*)^2 + (x_2 - x_{2j}^*)^2}{c_j}\right),$$

where the parameters B_j, c_j, x_{1j}^*, and x_{2j}^* correspond to the jth entries of the vectors $B = [3,\ 15,\ 3,\ 3]$, $c = [0.01,\ 1.5,\ 0.01,\ 0.01]$, $x_1^* = [0.65,\ 0.5,\ 0.35,\ 0.8]$, and $x_2^* = [0.9,\ 0.5,\ 0.35,\ 0.7]$, respectively. Furthermore, the fitness functions are given by $f_1(x) := \frac{\mathrm{d}V(x)}{\mathrm{d}x_1}$ and $f_2(x) := \frac{\mathrm{d}V(x)}{\mathrm{d}x_2}$. We notice that the underlying game has an evolutionarily stable state at $x^* = [0.5,\ 0.5]^\top$. Figure 6 shows three trajectories converging to the unique maximizer of V in Δ_X, and passing through other maximals of V located outside of Δ_X. This behavior is consistent with the model given by Eqs. (9b)–(9a), where the dynamics of z are independent of f.

4 Time-delayed replicator equation

In the classic replicator dynamics model given in Eq. (1), the population growth rate of individuals, who are playing the ith strategy at time t, depends on the fitness perceived by each of the players at that time. In some cases, it is more realistic to consider that the population evolves taking into account the fitness perceived by the individuals in the past. According to [24], this kind of lag in the fitness functions is called social-type delay. Its implications in the behavior of the players have been studied in [17,18,23,24, 34] for games with two strategies. To provide a tool to analyze the stability properties of large-scale systems based on replicator dynamics that are subject to time delays, we are interested in extending some of the existing results to games with N strategies (we notice that the number of strategies is generally related to the scale of the system (see

[14,15,20])). The social-type delay models several scenarios in engineering problems. For instance, the time taken by embedded computers to process the information or the delay associated with measurement devices due to their sensor response and transport times.

The mathematical model of the replicator dynamics for a game with N strategies, considering the social-type time delay can be expressed as

$$\dot{x}_i(t) = x_i(t)\left(f_i(x(t-\tau)) - \frac{1}{X}\sum_{j=1}^{N} x_j(t) f_j(x(t-\tau))\right) \text{ for all } i = 1, \ldots, N,$$
$$x(t) = x_0(t), \text{ for all } t \in [-\tau, 0], \tag{15}$$

where $\tau \geq 0$ is the associated delay, and $x_0(t) \in \Delta_X$ is the preshape function (equivalent to the initial conditions for the non-delayed case). It can be verified that Lemma 1 and Proposition 2 also hold for the delayed case. This implies that $\Delta_X = \left\{x \in \mathbb{R}_+^N : \sum_{i=1}^N x_i = X\right\}$ remains invariant and attractive under Eq. (15).

In our analysis of the time-delayed replicator dynamics given in Eq. (15), we consider monotonically decreasing fitness functions that linearly depend on the population playing each strategy, that is

$$f_i(x(t-\tau)) = -c_i x_i(t-\tau) + b, \quad \text{for all } i = 1, \ldots, N, \tag{16}$$

where $c_i > 0$ for $i = 1, \ldots, N$, and $b > 0$ is a constant guaranteeing that $f_i(x)$ is positive, for all $x \in \Delta_X$ (e.g., $b = \sum_{i=1}^N c_i$, or $b > \max_i\{c_i\}$). The constant b is important to ensure the robustness of the system, given the fact that, according to Proposition 2, the simplex Δ_X is attractive if the fitness functions are always positive (notice that b acts as $\Psi(x)$ in Remark 2). On the other hand, this kind of fitness function is common in several applications given the fact that it is related to the derivative of a quadratic utility function (see [20]).

The time-delayed replicator dynamics system stated in Eq. (15), using the fitness function given in Eq. (16), has a unique equilibrium point $x^* = [x_1^*, \ldots, x_N^*]^\top$ in the interior of the simplex Δ_X, where all the fitness functions are equal (i.e., $f_i(x^*) = f^*$, for all $i = 1, \ldots, N$). Explicitly, x^* is given by

$$x_i^* = \frac{\prod_{j=1, j\neq i}^N c_j}{\sum_{k=1}^N \prod_{j=1, j\neq k}^N c_j} X, \quad \text{for all } i = 1, \ldots, N. \tag{17}$$

It can be verified that $\sum_{i=1}^N x_i^* = X$, where $x_i^* > 0$.

4.1 Stability analysis

According to Proposition 1, x^* is asymptotically stable for the non-delayed replicator equation [when $\tau = 0$ in Eq. (15)] given the fact that the fitness functions under consideration are monotonically decreasing. However, this is not necessarily true for the delayed case. For instance, it has been shown that, if a sufficiently large time delay is introduced in the fitness functions, the replicator dynamics system becomes unstable for games with two strategies (see [23]). In this section, we derive some delay-dependent stability conditions for the equilibrium point given in Eq. (17) of the replicator dynamics system presented in Eq. (15). For this purpose, it is useful to linearize the system around x^*. The linearized version of the delayed replicator equation can be expressed as

$$\dot{\zeta}(t) = \mathbf{A_0}\zeta(t) + \mathbf{A_1}\zeta(t-\tau), \tag{18}$$

where ζ are the error coordinates, i.e., $\zeta = x - x^*$; $\mathbf{A_0} \in \mathbb{R}^{N\times N}$, and $\mathbf{A_1} \in \mathbb{R}^{N\times N}$ are respectively, the Jacobian matrix, and the delayed Jacobian matrix of the replicator dynamics system evaluated at equilibrium. The stability of this system is determined by the roots of its characteristic equation, which is given by

$$\det\left(\mathbf{I}s - \mathbf{A_0} - \mathbf{A_1}e^{-s\tau}\right) = 0, \tag{19}$$

where $s \in \mathbb{C}$, $\mathbf{I}$ is the $N \times N$ identity matrix, and the exponential component corresponds to the Laplace transform of the delayed term in Eq. (18). It is well known that the system given in Eq. (18) is stable if and only if all of the roots of its characteristic equation have negative real part. Calculation of these roots can be problematic due to the presence of the quasi-polynomial on the left-hand side of the expression given in Eq. (19). However, we can exploit the structure of the matrices $\mathbf{A_0}$ and $\mathbf{A_1}$ to simplify the analysis. To do that, first we introduce the following lemma.

Lemma 3 *Assume that $\tau \in \mathbb{R}$ and $s \in \mathbb{C}$. Furthermore, let $\mathbf{A_0}$ and $\mathbf{A_1}$ be, respectively, the Jacobian matrix, and the delayed Jacobian matrix of the replicator dynamics system given in Eq.* (15)*, evaluated at the equilibrium point stated in Eq.* (17)*. The characteristic polynomial of the matrix $\mathbf{A_0} + \mathbf{A_1}e^{-s\tau}$ is given by $p(\lambda) = \left(\lambda + (b - f^*)e^{-s\tau}\right)^{N-1}(\lambda + f^*)$, where $f^* = -c_i x_i^* + b$.*

Proof We need to prove that the eigenvalues of the matrix $\mathbf{A_0} + \mathbf{A_1}e^{-s\tau}$ are $\lambda_1 = -(b - f^*)e^{-s\tau}$ with multiplicity equal to $N-1$ and $\lambda_2 = -f^*$ with multiplicity equal to 1.

According to [35], if the matrix $\mathbf{M}' = \lambda_k\mathbf{I} - \mathbf{A}_0 - \mathbf{A}_1e^{-s\tau}$ is singular, then λ_k is an eigenvalue of the matrix $\mathbf{A}_0 + \mathbf{A}_1e^{-s\tau}$. Furthermore, the multiplicity of λ_k is equal to the nullity of $\mathbf{M}'$. It can be shown that the element of the row i, column j of the matrix $\mathbf{M}'$ is given by

$$m_{i,j} = \begin{cases} \lambda_k - f_i(x^*) + \dfrac{1}{X}\displaystyle\sum_{l=1}^{N} x_l^* f_l(x^*) + \dfrac{1}{X}x_i^* f_i(x^*) + c_i x_i^*\left(1 - \dfrac{x_i^*}{X}\right)e^{-s\tau} & \text{if } i = j \\ \dfrac{1}{X}x_i^* f_j(x^*) - \dfrac{1}{X}c_j x_j^* x_i^* e^{-s\tau} & \text{if } i \neq j. \end{cases}$$

Using the fact that $\sum_{l=1}^{N} x_l^* = X$, $f_i(x^*) = f^*$, and $c_i x_i^* = b - f^*$, for all $i = 1, \ldots, N$, we obtain

$$m_{i,j} = \begin{cases} \lambda_k + \frac{1}{X} x_i^* f^* - \frac{1}{X}(b - f^*)x_i^* e^{-s\tau} + (b - f^*)e^{-s\tau} & \text{if } i = j \\ \frac{1}{X} x_i^* f^* - \frac{1}{X}(b - f^*)x_i^* e^{-s\tau} & \text{if } i \neq j. \end{cases} \tag{20}$$

We have then

1. For the case $\lambda_1 = -(b - f^*)e^{-s\tau}$: Replacing $\lambda_k = -(b - f^*)e^{-s\tau}$ in Eq. (20), we have that $m_{i,j} = \frac{1}{X} x_i^* f^* - \frac{1}{X}(b - f^*)x_i^* e^{-s\tau}$ for all $i = 1, \ldots, N$, and all $j = 1, \ldots, N$. In other words, the matrix $\mathbf{M}'$ is singular, because its columns are equal. Furthermore, $\text{null}(\mathbf{M}') = N - 1$. Therefore, the multiplicity of $\lambda_1 = -(b - f^*)e^{-s\tau}$ is equal to $N - 1$.
2. For the case $\lambda_2 = -f^*$: according to [35], we have that the trace of a matrix is equal to the sum of its eigenvalues (counting their multiplicities), that is

$$\begin{aligned} \text{tr}(\mathbf{A}_0 + \mathbf{A}_1 e^{-s\tau}) &= \lambda_2 + \sum_{i=1}^{N-1} \left(-(b - f^*)e^{-s\tau}\right) \\ &= \lambda_2 + (1 - N)(b - f^*)e^{-s\tau}. \end{aligned} \tag{21}$$

On the other hand, we can calculate $\text{tr}(\mathbf{A}_0 + \mathbf{A}_1 e^{-s\tau})$ using Eq. (20), as follows:

$$\begin{aligned} \text{tr}(\mathbf{A}_0 + \mathbf{A}_1 e^{-s\tau}) &= \sum_{i=1}^{N} \left(-\frac{x_i^*}{X} f^* + (b - f^*)\frac{x_i^*}{X} e^{-s\tau} - (b - f^*)e^{-s\tau} \right) \\ &= -f^* + (b - f^*)e^{-s\tau} - N(b - f^*)e^{-s\tau} \\ &= -f^* + (1 - N)(b - f^*)e^{-s\tau}. \end{aligned} \tag{22}$$

Using Eqs. (21) and (22), we can conclude that $\lambda_2 = -f^*$ is an eigenvalue of the matrix $\mathbf{A}_0 + \mathbf{A}_1 e^{-s\tau}$, and its multiplicity is equal to 1.

□

Usefulness of Lemma 3 lies in the relationship between the characteristic polynomial of a matrix and its characteristic equation. This relationship allows us to provide our first result regarding the location of the roots of Eq. (19).

Theorem 4 *For a given τ, the roots of Eq.* (19)*, which corresponds to the characteristic equation of the linearized version of the time-delayed replicator dynamics system around x^*, are located at $s = -f^*$ (with multiplicity equal to 1), and at $s = \frac{1}{\tau} W_k \left(-(b - f^*)\tau\right)$ (with multiplicity equal to $N - 1$), for $k = 0, \pm 1, \pm 2, \ldots$, where W_k is the* kth *branch of the Lambert W function.*

Proof It is known that $p(s) = \det\left(\mathbf{I}s - \mathbf{A_0} - \mathbf{A_1} e^{-s\tau}\right)$, where $p(s)$ is the characteristic polynomial of the matrix $\mathbf{A_0} + \mathbf{A_1} e^{-s\tau}$ (see, e.g., [35]). Therefore, using Lemma 3, we have that

$$\det\left(\mathbf{I}s - \mathbf{A_0} - \mathbf{A_1} e^{-s\tau}\right) = \left(s + (b - f^*)e^{-s\tau}\right)^{N-1} (s + f^*).$$

Hence, the roots of Eq. (19) are given by $s = -f^*$, and the solutions of $\left(s + (b - f^*)e^{-s\tau}\right)^{N-1} = 0$. Rewriting the last expression, we have $s\tau e^{s\tau} = -(b - f^*)\tau$, and solving for s, we obtain $s = \frac{1}{\tau} W_k\left(-(b - f^*)\tau\right)$, for $k = 0, \pm 1, \pm 2, \ldots$, where W_k is the kth branch of the Lambert W function. □

According to Theorem 4, if $\tau = 0$, the characteristic equation of our system has one root at $-f^*$ and $N-1$ roots at $-(b - f^*)$. Therefore, in this case, the equilibrium point x^* is asymptotically stable, since both $b - f^*$ and f^* are greater than zero (we notice that the constant b ensures $f^* > 0$). If a small delay is introduced in the system, i.e., $\tau \to 0^+$, an infinite number of roots appear corresponding to the branches of the Lambert W function evaluated at $-(b - f^*)\tau$. These roots have real part that tends to $-\infty$ as τ tends to zero (for a detailed description of the Lambert W function, we refer the reader to [36,37]). Hence, for a small time delay, the equilibrium point of the replicator dynamics system remains asymptotically stable. However, if the delay is large enough, some solutions of the equation $s = \frac{1}{\tau} W_k\left(-(b - f^*)\tau\right)$ lie on the right-hand side of the complex plane, and the system becomes unstable.

We are interested in finding the critical time delay for which the equilibrium point of the system changes from stable to unstable. This delay is called delay margin and it is denoted by $\bar{\tau}$ [38]. To find $\bar{\tau}$ is not practical to calculate all the roots of the characteristic equation for each value of τ using the result in Theorem 4, because the number of roots is infinite for $\tau \neq 0$. However, according to [38], the solutions of the quasi-polynomial shown in Eq. (19) have two important properties that can be exploited: (i) for a given delay, the number of roots, counting their multiplicities, that are on the imaginary axis is finite and (ii) the roots are continuous with respect to the delay, i.e., the system becomes unstable/stable only if at least one root crosses the imaginary axis (the stability boundary) for some delay. Therefore, to compute $\bar{\tau}$, it is preferable to find the time delay for which the system has a certain number of roots on the imaginary axis, i.e., calculating the delay that produces characteristic roots of the form $s = \pm j\omega_c$, where $\omega_c \in \mathbb{R}$, and $j = \sqrt{-1}$ [39]. The next corollary provides some stability properties of the time-delayed replicator dynamics system based on the idea described above.

Corollary 2 *The linearized version of the delayed replicator equation given in Eq.* (18) *is*

- (i) *asymptotically stable for any* $\tau \in [0, \bar{\tau})$, *where* $\bar{\tau} = \frac{\pi \sum_{i=1}^{N} \prod_{j=1, j\neq i}^{N} c_j}{2X \prod_{i=1}^{N} c_i}$.
- (ii) *critically stable for* $\tau = \bar{\tau}$, *i.e., the characteristic equation of the system has roots at* $s = \pm j\omega_c$, *where* $\omega_c = \frac{\prod_{i=1}^{N} c_i}{\sum_{i=1}^{N} \prod_{j=1, j\neq i}^{N} c_j} X$.
- (iii) *unstable for* $\tau > \bar{\tau}$.

Proof – (i) Let us note that, if $\xi \in [0, \frac{\pi}{2})$, the real part of $W_k(-\xi)$ is strictly negative, for all $k = 0, \pm 1, \pm 2, \ldots$. Moreover, given the fact that $b - f^* = \frac{\prod_{i=1}^{N} c_i}{\sum_{i=1}^{N} \prod_{j=1, j\neq i}^{N} c_j} X$, then $(b - f^*)\tau \in [0, \frac{\pi}{2})$ if $\tau \in \left[0, \frac{\pi \sum_{i=1}^{N} \prod_{j=1, j\neq i}^{N} c_j}{2X \prod_{i=1}^{N} c_i}\right)$. Hence, according to Theorem 4, for any $\tau \in [0, \bar{\tau})$ all the characteristic roots of Eq. (19) have strictly negative real part, and the system is asymptotically stable.

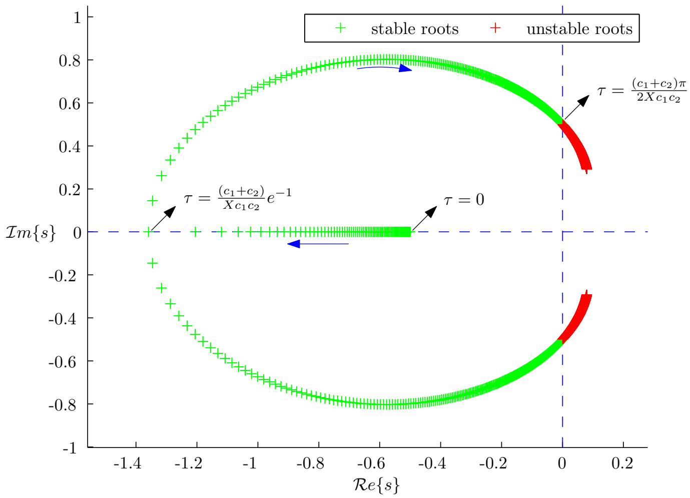

Fig. 7 Location of the rightmost characteristic roots for a linearized delayed replicator system with 2 strategies

– (ii) If $\tau = \bar{\tau}$, then $-(b - f^*)\tau = \frac{\pi}{2}$. Thus, using Theorem 4, we can determine that the characteristic roots of Eq. (19) with non negative real part are located at $s = \frac{1}{\tau} W_0(\frac{\pi}{2}) = j \frac{\prod_{i=1}^N c_i}{\sum_{i=1}^N \prod_{j=1, j\neq i}^N c_j} X = j\omega_c$, and at $s = \frac{1}{\tau} W_{-1}(\frac{\pi}{2}) = -j \frac{\prod_{i=1}^N c_i}{\sum_{i=1}^N \prod_{j=1, j\neq i}^N c_j} X = -j\omega_c$.

– (iii) If $\tau \in (\bar{\tau}, +\infty)$, then $(b - f^*)\tau \in \left(\frac{\pi}{2}, +\infty\right)$. In that interval, we have that $\mathcal{R}e\{W_0\,(-(b - f)\tau)\} > 0$. Hence, according to Theorem 4, at least one root of Eq. (19) has real part greater than zero, and the system is unstable. □

The result on delay-dependent stability pointed out in Corollary 2, reveals the existence of a single bifurcation at $\tau = \bar{\tau}$, for which, the equilibrium point of the system given in Eq. (15), changes its stability properties (from asymptotically stable for $\tau < \bar{\tau}$ to unstable for $\tau > \bar{\tau}$). This result is consistent with the one presented in [23], where the authors show that the rest point of the replicator equation is asymptotically stable if $\tau < \frac{(c_1+c_2)\pi}{2c_1c_2}$, and unstable if $\tau > \frac{(c_1+c_2)\pi}{2c_1c_2}$, for games with two strategies ($N = 2$), payoff matrix of the form diag($[-c_1, -c_2]$), and population size $X = 1$.

Figure 7 shows the location of the rightmost characteristic roots (the ones with larger real part) for a linearized time-delayed replicator system with $N = 2$ strategies and different values of $\tau \in [0, 2\bar{\tau}]$. In this case, we use the fitness function described in Eq. (16), with $c_1 = c_2 = 1$, and $b = 2$. Moreover, we take $X = 1$. The figure is obtained by applying Theorem 4, taking into account that we only need to calculate the branches of the Lambert W function with larger real part, in this case, they correspond to W_0 and W_{-1}. It is worth noting that we obtain the same result using DDE-BIFTOOL [40], which is a Matlab® package for bifurcation analysis of delay differential equations.

As it was aforementioned, the rightmost characteristic roots are important, because they define the stability of the system. In the case shown in Fig. 7, these roots cross the imaginary axis at $\tau = \frac{(c_1+c_2)\pi}{2Xc_1c_2}$, making the system unstable. Moreover, in the region of stability (depicted in green), which corresponds to $\tau \in \left[0, \frac{(c_1+c_2)\pi}{2Xc_1c_2}\right)$, the rightmost characteristic roots are related to the convergence rate to the equilibrium point. The closer the roots to the imaginary axis, the slower the system response. In this regard, it can be noticed that the response of the replicator dynamics is faster when a small delay is introduced in the fitness function. For the case shown in Fig. 7, the roots are farther from the imaginary axis when $\tau = \frac{c_1+c_2}{Xc_1c_2}e^{-1}$.

Remark 5 More generally, the convergence of the replicator equation's solution is faster when the rightmost characteristic roots are located at $-(b - f^*)e$, which corresponds to the solution of $s = (b - f^*)eW_0(-e^{-1})$, and $s = (b - f^*)eW_{-1}(-e^{-1})$, i.e., when $\tau = (b - f^*)^{-1}e^{-1}$. The above is valid only if $-(b - f^*)e \geq -f^*$, e.g., if $b = \sum_{i=1}^{N} c_i$.

4.2 Numerical examples

To illustrate the result proposed in Sect. 4.1, we simulate the time response of the delayed replicator dynamics described in Eq. (15) for a game with five strategies ($N = 5$), using the fitness functions given in Eq. (16), where $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_4 = 4$, $c_5 = 5$, and $b = \sum_{i=1}^{N} c_i = 12$. Furthermore, the population size is $X = 1$. In this case, the equilibrium point stated in Eq. (17) is $x_1^* = 0.4380$, $x_2^* = 0.2190$, $x_3^* = 0.1460$, $x_4^* = 0.1095$, $x_5^* = 0.0876$, and the delay margin is $\bar{\tau} = 3.5867$. The simulations results are shown in Fig. 8 for different values of τ. According to this figure, for the simulation conditions described above, the equilibrium point of the system is asymptotically stable if $\tau < \bar{\tau}$ (Fig. 8a, b), and unstable if $\tau > \bar{\tau}$ (Fig. 8c, d). Moreover, the system becomes more oscillatory when the delay is increased. The oscillations can be explained if we consider that the growth rate of each population, i.e., $\dot{x}$ in Eq. (15), depends on the fitness perceived by the players in the past. For instance, it is possible that $\dot{x}_i(t) > 0$ although $f_i(t) - \sum_{j=1}^{N} x_j(t) f_j(t) < 0$, given that $\dot{x}_i$ depends on that difference, but evaluated τ units of time before, i.e., $f_i(t-\tau) - \sum_{j=1}^{N} x_j(t) f_j(t - \tau)$.

The results stated in Sect. 4.1 are local, since they are obtained using the linearized version of the delayed replicator dynamics. The following simulation scenario gives us insights on the region where the aforementioned results hold for the original nonlinear replicator dynamics: consider a population of size $X = 1$, $N = 4$ available strategies, and fitness functions defined as $f_i = -x_i$, for all $i = 1, \ldots, 4$. Figure 9 shows the trajectories of the population state under the delayed replicator dynamics given in Eq. (15) for different initial conditions inside the simplex Δ_X. Notice that, although some initial conditions are far from the equilibrium point (placed in the center of the simplex), the conditions stated in Corollary 2 remains valid, i.e., the equilibrium point is asymptotically stable for $\tau < \bar{\tau}$ (Fig. 9a), stable for $\tau = \bar{\tau}$ (Fig. 9b), and unstable for $\tau > \bar{\tau}$ (Fig. 9c). Thus, simulations suggest that the region, where Corollary 2 is applicable for the nonlinear replicator dynamics, is not very restrictive. A similar conclusion is drawn in [41] for the delayed logistic equation, which is closely related to the replicator dynamics. On the other hand, it is worth noting that the trajectories of

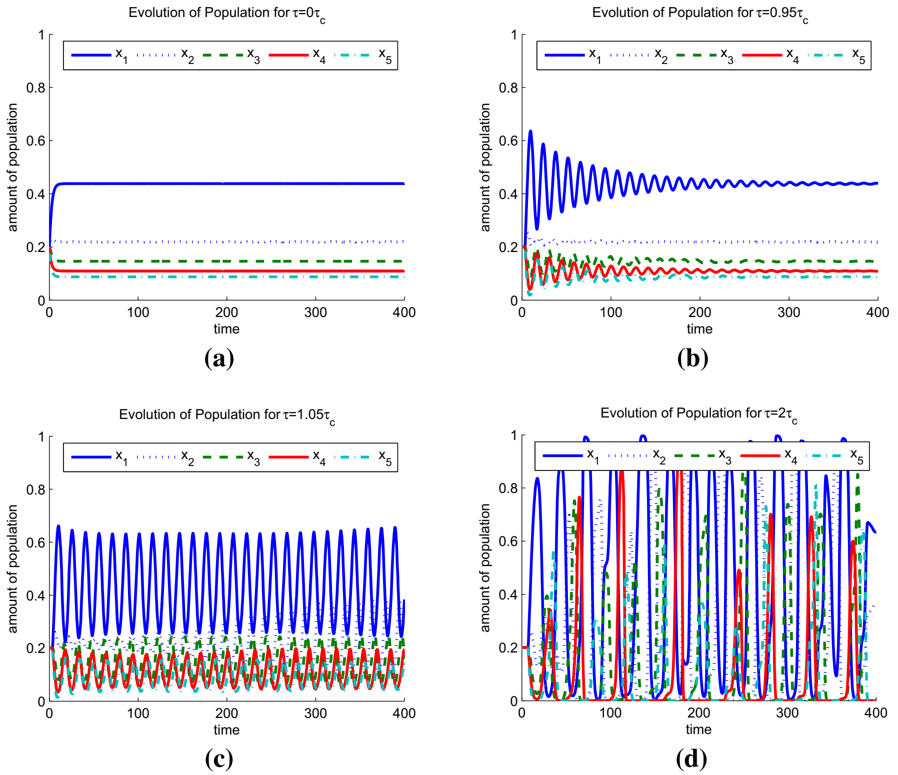

Fig. 8 Evolution of the population under the delayed replicator dynamics for **a** $\tau = 0$, **b** $\tau = 0.95\bar{\tau}$, **c** $\tau = 1.05\bar{\tau}$, and **d** $\tau = 2\bar{\tau}$

the system belong to the simplex Δ_X, for all $t \geq 0$, even when the delay is arbitrarily increased.

Finally, to show the accelerating effect induced by small time delays, which was pointed out in the last part of Sect. 4.1, we also simulate a delayed replicator system with two strategies ($N = 2$), $X = 1$, and fitness functions given by $f_1 = -x_1 + 2$, and $f_2 = -x_2 + b$. Figure 10 shows, in solid line, the result for the non-delayed case, and in dashed line, the response when a small delay is introduced. For the delayed case, we have chosen $\tau = 2e^{-1}$, which is the delay that produces characteristic roots with more negative real part. It can be noticed that the response of the delayed system is faster and free of oscillations.

5 Discussion

Typically, the replicator dynamics model is subject to disturbances when it is used to solve optimization and control problems. This fact is due to several factors. For instance, inaccuracy of the numerical solvers used in the implementation stage. If additionally, we let the fitness functions depend on the output of a physical system,

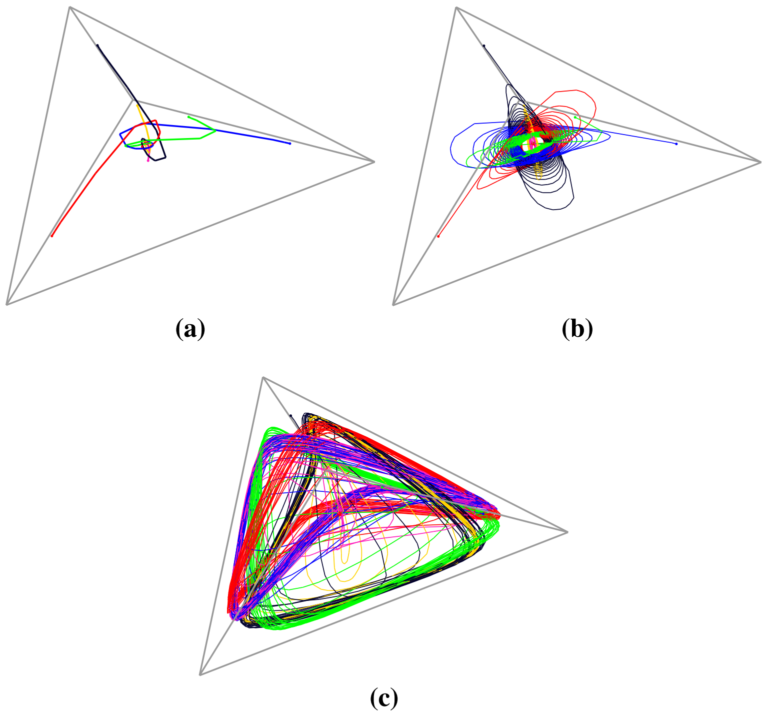

Fig. 9 Trajectories of the population state under the delayed replicator dynamics for different initial conditions and **a** $\tau = 0.5\bar{\tau}$, **b** $\tau = \bar{\tau}$, **c** $\tau = 1.3\bar{\tau}$

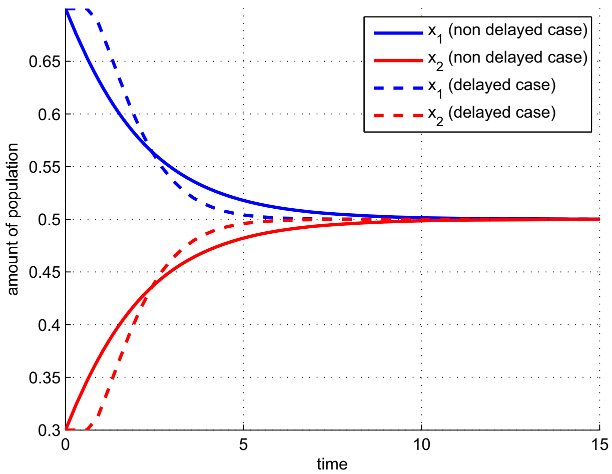

Fig. 10 Time response of a replicator dynamics system in a game with two strategies, for $\tau = 0$ (*solid line*), and $\tau = 2e^{-1}$ (*dashed line*)

other kind of perturbations arises as a result of sensors' noise, processing delays, influence of external signals, etc. The study presented in this paper shows that the replicator equation performs properly in non-ideal scenarios.

In this regard, Proposition 2 gives conditions on the fitness functions that guarantee the robustness of the simplex Δ_X in the face of state disturbances. This is a desirable property, since Δ_X is generally related to a constraint that must be satisfied, e.g., the power limitation of a heating system [14], or the amount of generation required to supply the load of a micro-grid [20].

In Sect. 3.1, we relax the assumptions stated in Proposition 2 by modifying the classic replicator dynamics model. We prove that the new robustified model, which is given in Eq. (4), preserves the optimality characteristics of the classic one, while its robustness is significantly improved. Indeed, the proposed dynamics are not only asymptotically stable in the simplex Δ_X (as is the case of the classic replicator equation) but in $\mathbb{R}^N_{++}$. Besides, when optimizing a potential function V, the proposed dynamics allows us to pass through local maxima located outside Δ_X. Summarizing, we have introduced a novel system that allows us to control and optimize plants subject to arbitrary initial conditions on the positive orthant. Therefore, the introduced system is capable to deal with disturbances (e.g., bounded impulse disturbances due to imperfect computation) that make the system state to leave the simplex Δ_X.

On the other hand, robustness of the replicator equation under social-type delays is characterized in Corollary 2 for a special class of fitness function [i.e., the one stated in Eq. (16)]. We analyze social-type delays, because they are common in several scenarios. For instance, they can be employed to model the time taken by individuals for processing the information received from their environment (i.e., there is a lag in the fitness functions). Moreover, they have been considered in engineering problems, such as the one described in [17]. It should be noticed that our results assume the same delay for all fitness functions. Even though this assumption is somewhat restrictive, it is satisfied in several problems; for instance, in situations where the elements that process the information (e.g., persons, animals, computers, etc.) have similar characteristics. Indeed, the same consideration has been taken into account in other works, such as [23,24].

In [42], the authors analyze the stability of the replicator equation from a networked control systems perspective. They prove delay-independent stability for a particular class of delay that models the effect of using non-ideal communication channels. However, for the social-type-delay studied in this manuscript, Corollary 2 shows the existence of a marginal delay that makes the replicator dynamics unstable. Specifically, the delay margin is given by $\bar{\tau} = \frac{\pi \sum_{i=1}^{N} \prod_{j=1, j\neq i}^{N} c_j}{2X \prod_{i=1}^{N} c_i} = \frac{\pi}{2(b-f^*)}$, which reveals a connection with the steady state value of the fitness functions. If we scale all the fitness functions by a positive parameter γ, it can be easily verified that its equilibrium points and stability properties do not change. In this case, the parameter γ can be adjusted to provide a better robustness under time delays since $\bar{\tau} = \frac{\pi}{2\gamma(b-f^*)}$, i.e., the smaller γ, the larger the delay margin. The drawback of the above procedure is that it reduces the convergence rate to the equilibrium. Thus, there exists a tradeoff between delay tolerance and settling time. Following a similar analysis, we have noticed that the same tradeoff occurs by modifying the population size.

Although in most of the cases is preferable to avoid time delays, there are some scenarios in which they provide desirable effects. For instance, authors in [39] use them for stabilization purposes. In this regard, Remark 5 shows an interesting consequence of delays in the replicator dynamics model, which is then illustrated in Fig. 10, i.e., time delays can accelerate the convergence of the replicator equation's solution. This effect is due to the fact that small delays move the characteristic roots far from the imaginary axis (on the left-hand side), as it is shown in Fig. 7. From another perspective, acceleration can be explained if we take into consideration that the induced delay gives more time to the individuals involved in the game to process the available information. Thus, they can take better decisions. Nevertheless, if the delay is large, it causes loss of coordination and hence instability.

Finally, the major advantage of Corollary 2 is that it allows the analysis of large-scale systems based on replicator dynamics in the face of social-type delays. As yet, the results reported in the literature have dealt only with two-strategies games [17,23,24,34], which implied a strong limitation to analyze more complex cases. For instance, those related to engineering applications, where the number of strategies is generally large (see [14,20,22]). A drawback of the method developed in this paper is its flexibility, i.e., to apply the proposed analysis, we require that the delayed fitness functions of the addressed problem have the form given in Eq. (16), i.e., they must be affine and decreasing functions. Nonetheless, these assumptions are not very severe. While the first one can be relaxed using linearization around the equilibrium point, the second one is common in several problems, among which we can mention: dispatch of distributed generators [20], lighting control [22], and habitat selection [30].

6 Conclusions

Applications of the replicator dynamics model have extended from biology and economics to engineering problems. This is mainly because of two facts. First, gradient characteristics of the replicator dynamics model coupled with its invariance properties make it suitable to be applied as a constrained optimization method. In addition, since the replicator dynamics system is passive, this can be used to control a large number of dynamical systems without creating stability problems.

We have addressed implementation issues of the replicator equation in engineering applications. We have shown that this equation is robust in the face of state disturbances assuming that the vector of fitness functions evolve in $\mathbb{R}^N_{++}$. To relax this assumption, we have built on the replicator dynamics and proposed a robust model that exhibits a multi-time-scale behavior. Singular perturbation theory is used to prove that the proposed model forces the trajectories to converge to the simplex Δ_X, and once in Δ_X, the model preserves the same optimality characteristics of the replicator equation. Furthermore, we have characterized the stability margin of the replicator dynamics in the face of time delays. Necessary and sufficient conditions for delay-dependent stability are provided and extended to N-strategies models. Finally, we have shown that small delays can be used to accelerate the convergence of the replicator equation's solution.

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