OPT201 Project

Solving am otimization problem using the SQP algorithm

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1 Introduction

In this project, we will try to solve numerically the optimization problem (P_{26}) :

$$\inf_{\substack{x=(x_1,\dots,x_5)^T\in\mathbf{R}^5\\ \sum_{i=1}^5 x_i^2=10\\ x_2x_3=5x_4x_5\\ x_1^3+x_2^3=-1}} \left\{ \mathrm{e}^{\prod_{i=1}^n x_i} - \frac{1}{2}(x_1^3+x_2^3-1)^2 \right\} \tag{1}$$

We can rewrite the problem : where : $f(x) = \left\{ \mathrm{e}^{\prod_{i=1}^n x_i} - \frac{1}{2}(x_1^3+x_2^3-1)^2 \right\}$ and $g(x) = \begin{pmatrix} \sum_{i=1}^5 x_i^2 - 10 \\ x_2x_3 - 5x_4x_5 \\ x_1^3 + x_2^3 + 1 \end{pmatrix}$.

The algorithm that was chosen in this project is : SQP algorithm.

2 SQP algorithm

2.1 Description of the algorithm

2.2 Steps of the algorithm

The steps of the SQP algorithm are :

1. Let $x_0 \in \mathbf{R}^5$ and $\lambda_0 \in \mathbf{R}^3$, set $k=0$
2. If (x_k, λ_k) is a KKT-pair of (PCE), then stop
3. If not, then get (a_k, ξ_k) solving (QS_k) :

$$\begin{pmatrix} H_xL(x_k,\lambda_k) & Jg(x_k)^T \\ Jg(x_k) & 0 \end{pmatrix} \begin{pmatrix} a_k \\ \xi_k \end{pmatrix} = -\begin{pmatrix} \nabla f(x_k) \\ g(x_k) \end{pmatrix}$$

4. Let $x_{k+1} = x_k + a_k$, $\lambda_{k+1} = \xi_k$ and $k = k+1$
5. Go to step 2

3 Results

We applied the SQP Algorithm to 3 different initial values :

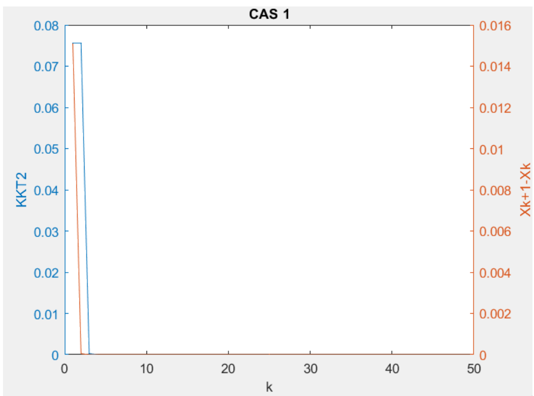
$$\begin{cases} x^0 = (1.71, 1.59, 1.82, 0.763, 0.763)^T \\ \tilde{x}^0 = (1.9, 1.82, 2.02, 0.9, 0.9)^T \\ \hat{x}^0 = (1, 0, 3, 0, 0)^T \end{cases}$$

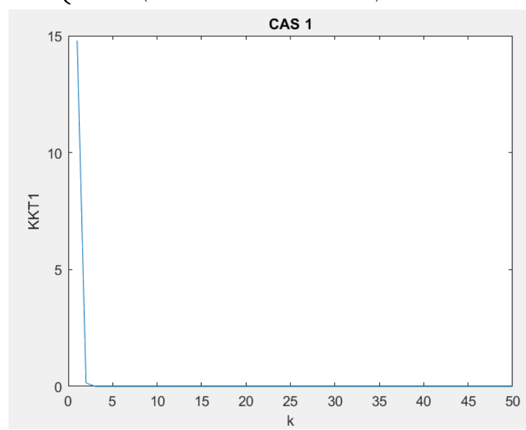
The initial value λ was the same for the three cases which is : $\lambda_0 = (1,1,1)^T$.
The maximum number of iterations is $N = 50$. The algorithm is implemented to stop if the optimal solution is obtained. However, this was never the case in any of the x_0 that were suggested. The epsilon considered in the resolution of this problem is : $\epsilon = 10^{-3}$.

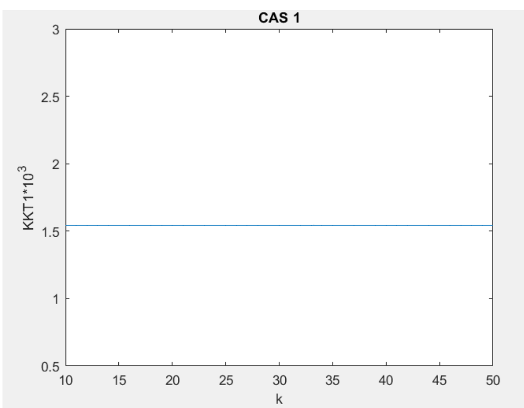
Remark :
In each case, we drew three graphs : the first which shows the evolution of $\|x_{k+1} - x_k\|$ to check if the algorithm converges, but also the evolution of each condition of the KKT system : $\begin{cases} \nabla f(x) + Jg(x)^T\lambda = 0 \\ g(x) = 0 \end{cases}$ by representing their norm to see to what extent the solution obtained respects with the KKT conditions. The third figure is only a zoom on the second figure. The objective is to see if the solution respects the contraints (KKT2) and the if it minimize f (KKT1), to know how suitable the solution obtained is.

3.1 Result for x^0

The optimal solution of (P_{26}) according to this case is : $\begin{cases} \bar{x} = (-1.721; 1.600; 1.820; -0.763; -0.763)^T \\ \bar{\lambda} = (0.040, -0.038, 0.005)^T \end{cases}$



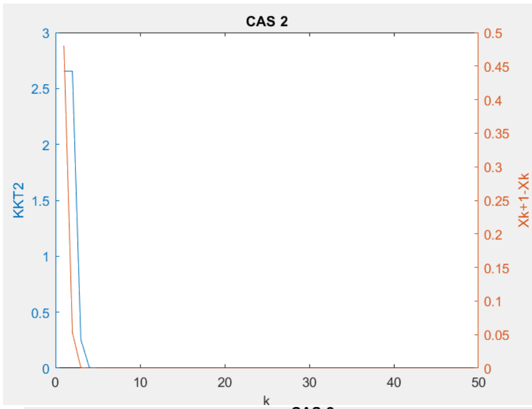


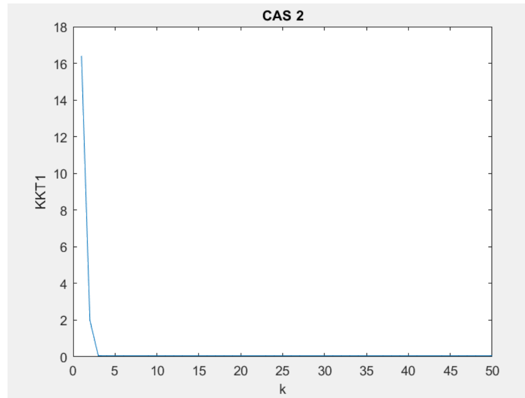


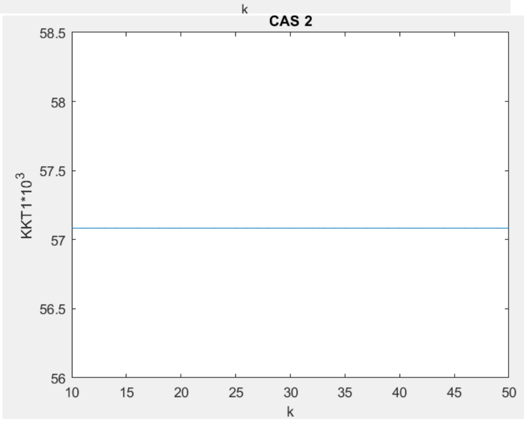
We notice that $\|x_{k+1} - x_k\|$ tends to 0. This shows that the algorithm converges to a given value . It remains to be seen whether this value verifies the conditions of KKT or not. From the second graph, we note that g tends towards 0. One condition of KKT is therefore verified. Figure 2 and especially figure 3 shows that $\nabla f(x) + Jg(x)^T\lambda$ stabilizes at a value close to zero, wich is : $1, 5.10^{-3}$. We may consider in this case that $\nabla f(x) + Jg(x)^T\lambda$ is converting to zero and consequently $\bar{x}$ is a local optimal solution to the problem.

3.2 Result for $\tilde{x}^0$

In the second case, the solution is : $\begin{cases} \bar{x} = (-1.599; 1.457; 2.020; -0.654; -0.900)^T \\ \bar{\lambda} = (0.045, -0.042, 0.007)^T \end{cases}$



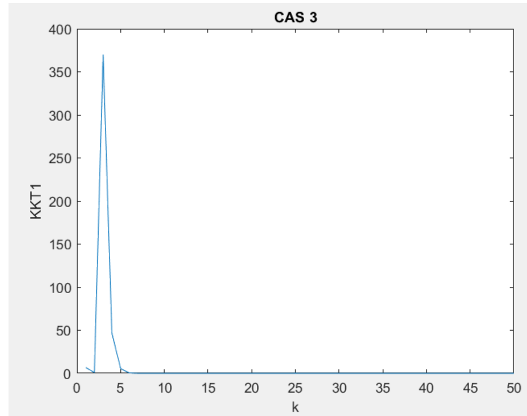


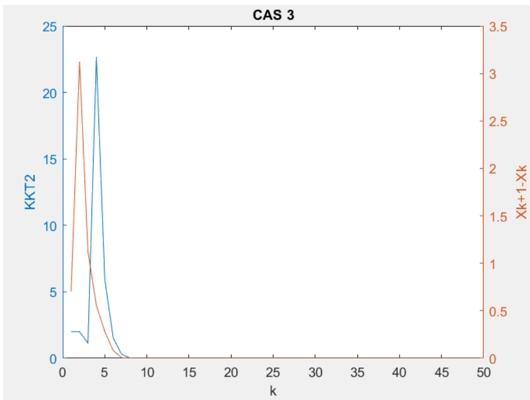


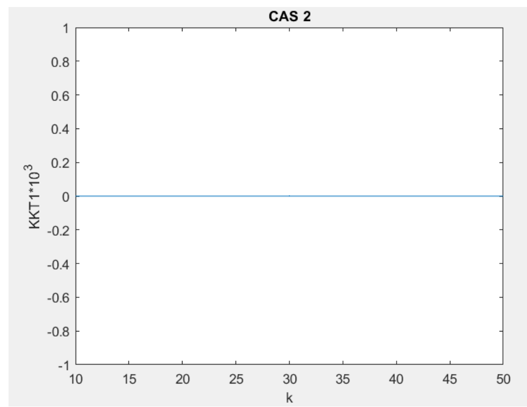
Note that this case is very similar to the previous case. $\|x_{k+1} - x_k\|$ tends to 0.This shows that the algorithm converges to a given value . From the second graph, we note that g tends towards 0. One condition of KKT is therefore verified. Figure 2 and especially figure 3 shows that $\nabla f(x) + Jg(x)^T\lambda$ stabilizes at a value close to zero, but higher then the previous case, wich is : $5.70^{1}0-2$. We may consider in this case that $\nabla f(x) + Jg(x)^T\lambda$ is converting to zero and consequently $\bar{x}$ is a local optimal solution to the problem.

3.3 Result for $\hat{x}^0$

In this case, we got : $\begin{cases} \bar{x} = (-1, 0, 3, 0, 0)^T \\ \bar{\lambda} = (0, 0, 0)^T \end{cases}$







Like the previous cases, $\|x_{k+1} - x_k\|$ tends and so the algorithm converges to a given value $\bar{x}$. The difference this time is that $\nabla f(x) + Jg(x)^T \lambda$ is converging exactly to 0. We can clearly observe on figure 3 that $\nabla f(x) + Jg(x)^T \lambda$ is stabilizing at 0. $\bar{x}$ in this case should be very close, compared to the previous cases, to the real optimal solution.