ELEN30011 EDM Task

- Xiufu SUN 1372750
- Wenyang SUN 1354302

1.1

$\boldsymbol{\nabla} \cdot \mathbf{F}$ (div $\mathbf{F}$) is a scalar, $\boldsymbol{\nabla} \times \mathbf{F}$ is a vector field.

Explaination: $\Downarrow$

1.2

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field with

$$\mathbf{F}(x, y, z) = F_x(x, y, z)\hat{\mathbf{x}} + F_y(x, y, z)\hat{\mathbf{y}} + F_z(x, y, z)\hat{\mathbf{z}}$$

The divergence of $\mathbf{F}$ (div $\mathbf{F}$) is

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{F} &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (F_x, F_y, F_z) \\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \end{aligned}$$

$\boldsymbol{\nabla} \cdot \mathbf{F}$ is a scalar.

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{F} &= (\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}}) \times (F_x(x, y, z)\hat{\mathbf{x}} + F_y(x, y, z)\hat{\mathbf{y}} + F_z(x, y, z)\hat{\mathbf{z}}) \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\hat{\mathbf{x}} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\hat{\mathbf{y}} + (\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x})\hat{\mathbf{z}} \end{aligned}$$

$\boldsymbol{\nabla} \times \mathbf{F}$ is a vector field.

1.3

(a)

$$grad f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

(b)

$$grad f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = 1\hat{\mathbf{x}} + z\hat{\mathbf{y}} + y\hat{\mathbf{z}}$$

(c)

$$grad f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = x\hat{\mathbf{x}} + (y + \frac{1}{2}z^2 siny)\hat{\mathbf{y}} - zcosy\hat{\mathbf{z}}$$

(d)

$$grad f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = \frac{-x}{x^2+y^2+z^2}\hat{\mathbf{x}} + \frac{-y}{x^2+y^2+z^2}\hat{\mathbf{y}} + \frac{-z}{x^2+y^2+z^2}\hat{\mathbf{z}}$$

1.4

(a)

$$div\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$$

$$curl\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 0 \end{vmatrix} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

(b)

$$div\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = -1 + 1 = 0$$

$$curl\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & 0 & z \end{vmatrix} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

(c)

$$div\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$$

$$curl\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & -x \end{vmatrix} = [1-(-1)]\hat{\mathbf{y}} = 2\hat{\mathbf{y}}$$

(d)

$$div\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y} + \frac{\partial^2 f}{\partial z^2}$$

$$\begin{aligned} curl\mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= (\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y})\hat{\mathbf{x}} + (\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z})\hat{\mathbf{y}} + (\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y})\hat{\mathbf{z}} \\ &= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \end{aligned}$$

1.5

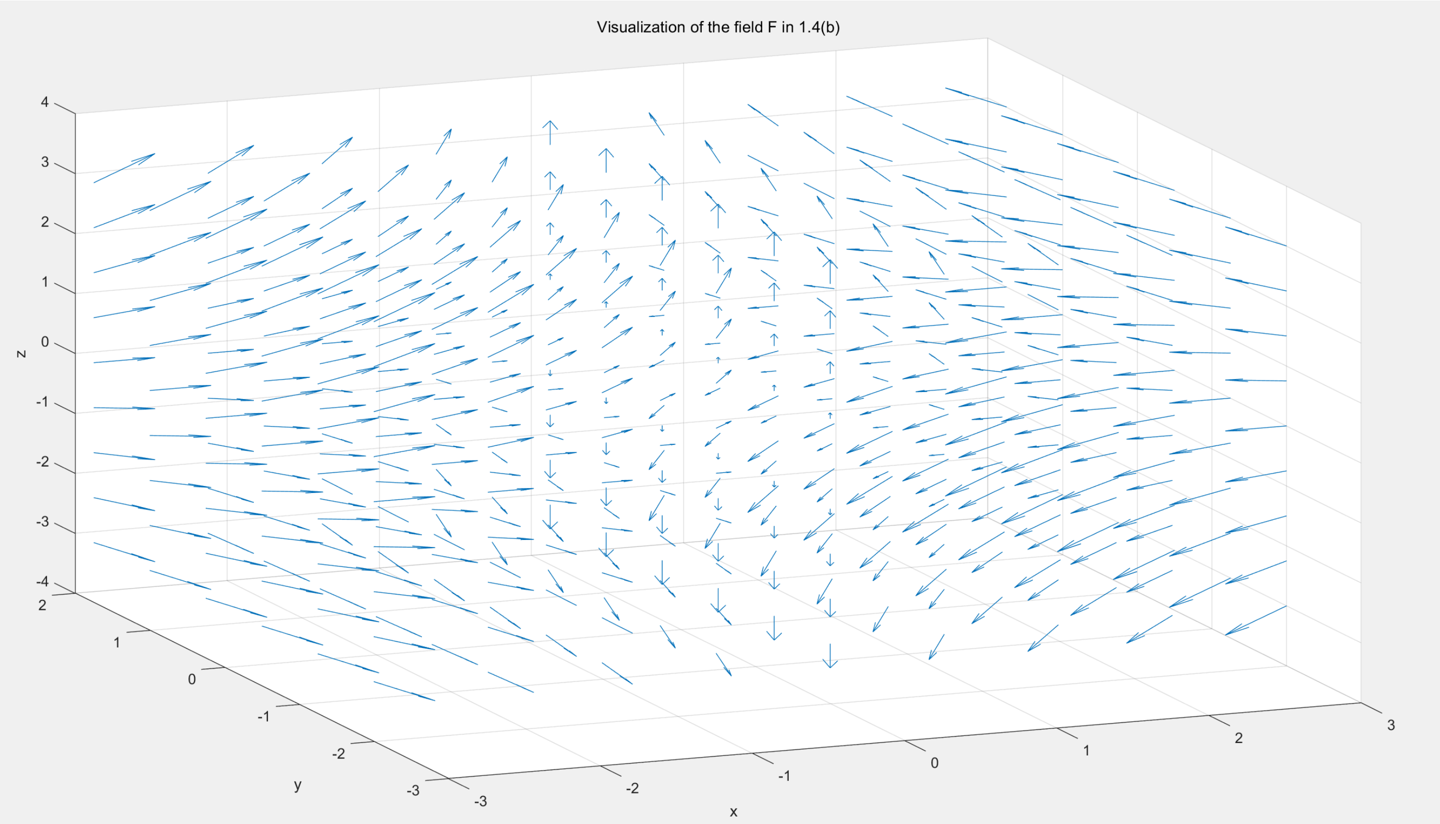
```
close all
clear
clc

x = -3:.75:3;
y = -2:.75:2;
z = -3:.75:3;

[X, Y, Z] = meshgrid(x, y, z);

FX = -X;
FY = 0.*Y;
FZ = Z;

figure(1);
quiver3(X,Y,Z,FX,FY,FZ)
xlabel("x");
ylabel("y");
zlabel("z");
title("Visualization of the field F in 1.4(b)")
```



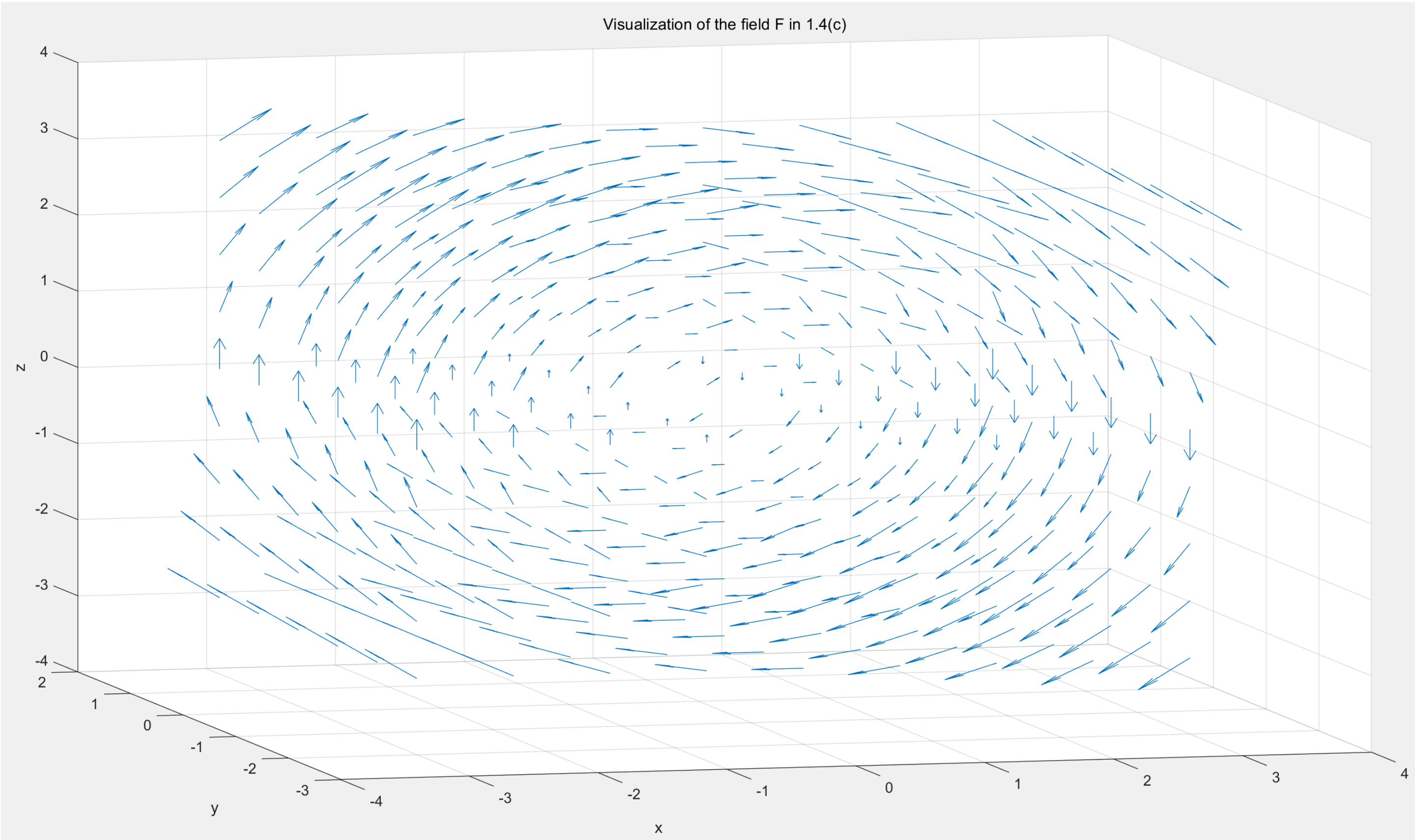

```
close all
clear
clc

x = -3:.75:3;
y = -2:.75:2;
z = -3:.75:3;

[X, Y, Z] = meshgrid(x, y, z);

FX = Z;
FY = 0.*Y;
FZ = -X;

figure(1);
quiver3(X,Y,Z,FX,FY,FZ)
xlabel("x");
ylabel("y");
zlabel("z");
title("Visualization of the field F in 1.4(c)")
```



2.1

2.1.1

$\mathbf{l_1}$ is a straight line.

$$\mathbf{l_1}(s) = (1-s)\mathbf{P_1} + s\mathbf{P_2}$$

$\mathbf{l_2}$ is circular arc of radius 1.

Let $x = cos(\frac{\pi s}{4})$, $y = sin(\frac{\pi s}{4})$.

$$x^2 + y^2 = cos^2(\frac{\pi s}{4}) + sin^2(\frac{\pi s}{4}) = 1$$

In cartesian coordinates, $x^2 + y^2 = 1$ is describing a circle with a radius of 1.

Plus, since $s \in [0,1]$, $x \in [\frac{\sqrt{2}}{2}, 1]$, $y \in [0, \frac{\sqrt{2}}{2}]$.

When s increases, $\mathbf{l_2}$ moves from $(1,0)$ to $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ along this circle, anticlockwise.

2.1.2

Fomular (8) shows that,

$$\mathbf{E}(x,y,z) = -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$$

For $\mathbf{l_1}$, $x = 1 - s + \frac{s}{\sqrt{2}}$, $y = \frac{s}{\sqrt{2}}$

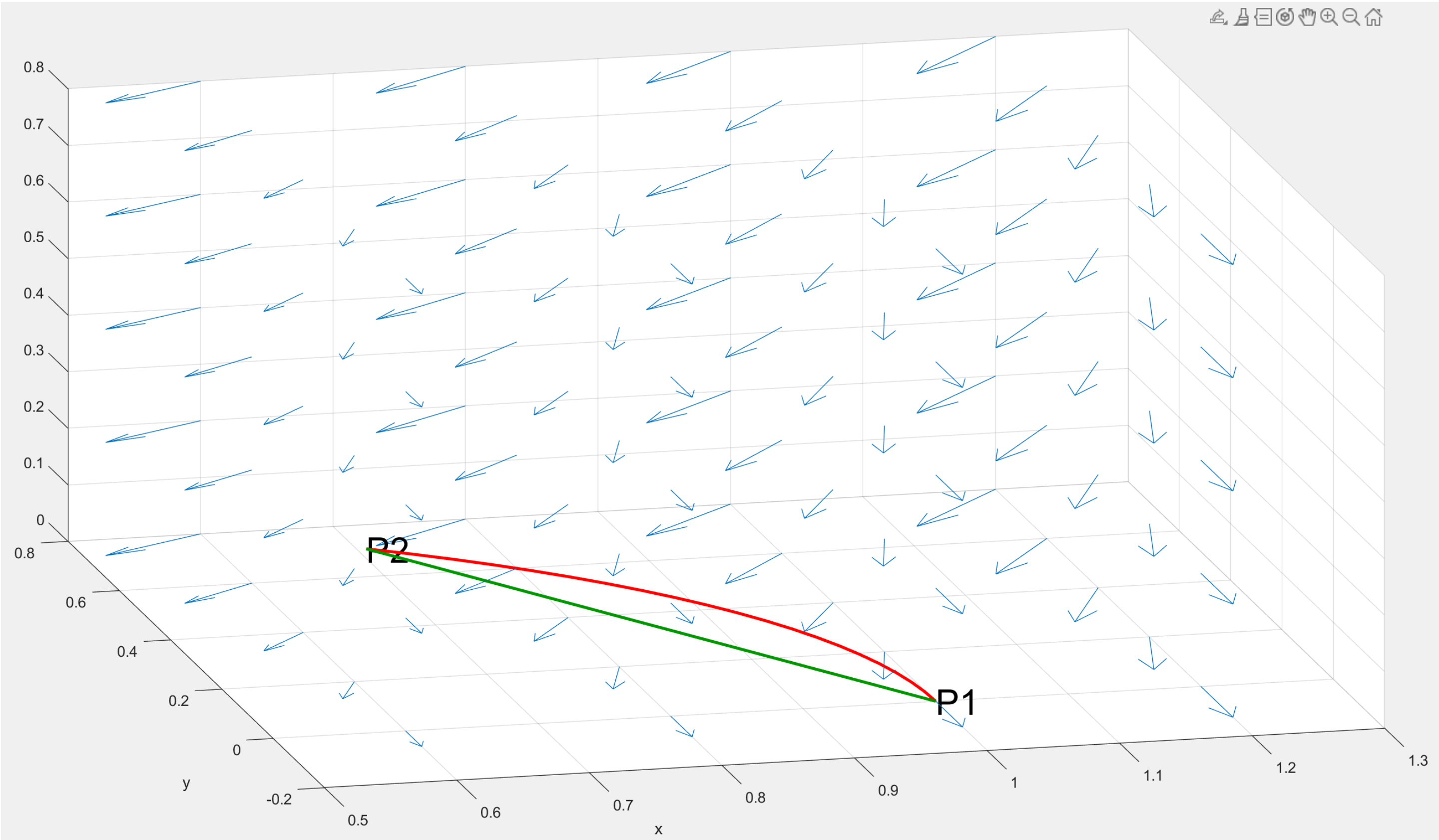
$$\mathbf{E}(\mathbf{l_1}(s)) = -\frac{s}{\sqrt{2}}\hat{\mathbf{x}} - (1 - s + \frac{s}{\sqrt{2}})\hat{\mathbf{y}}$$

For $\mathbf{l_2}$, $x = cos(\frac{\pi s}{4})$, $y = sin(\frac{\pi s}{4})$

$$\mathbf{E}(\mathbf{l_2}(s)) = -sin(\frac{\pi s}{4})\hat{\mathbf{x}} - cos(\frac{\pi s}{4})\hat{\mathbf{y}}$$

2.1.3

```
x = 0.6:.2:1.2; y = 0:.2:.8; z = 0:.2:.8;
[xx,yy,zz] = meshgrid(x,y,z);
Exx =-yy; Eyy =-xx; Ezz = 0*xx;
figure(2);
quiver3(xx,yy,zz,Exx,Eyy,Ezz);
grid on; hold on;
xhat = [1;0;0]; yhat = [0;1;0];
s = 0:.05:1;
L1 = (1- s).*xhat + s.*(xhat + yhat)/sqrt(2);
L2 = cos(pi .*s ./4).*xhat + sin(pi .*s ./4).*yhat;
plot3(L1(1,:),L1(2,:),L1(3,:),'Color',[0 .6 0],'LineWidth',2);
plot3(L2(1,:),L2(2,:),L2(3,:),'Color',[1 0 0],'LineWidth',2);
xlabel("x")
ylabel("y")
text(1,0,0,"P1","FontSize",24)
text(1/sqrt(2), 1/sqrt(2), 0, "P2", "FontSize",24)
```



2.1.4

There is no singularity here, the result is not related to the path.

$$\begin{aligned}
V_{l_1} &= -\int_{l_1} \mathbf{E} \cdot dl_1 \\
&= -\int_{l_1} \mathbf{E}(\mathbf{l_1}(\mathbf{s})) \cdot \frac{d\mathbf{l_1}}{ds}(s)ds \\
&= -\int_{s} (-\frac{s}{\sqrt{2}}, -(1 - s + \frac{s}{\sqrt{2}}))(-1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})ds \\
&= -\int_{0}^{1} \sqrt{2}s - s - \frac{\sqrt{2}}{2}ds \\
&= \frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
V_{l_2} &= -\int_{l_2} \mathbf{E} \cdot dl_2 \\
&= -\int_{l_2} \mathbf{E}(\mathbf{l_2}(\mathbf{s})) \cdot \frac{d\mathbf{l_2}}{ds}(s)ds \\
&= -\int_s (-sin(\frac{\pi s}{4}), -cos(\frac{\pi s}{4}))(-\frac{\pi}{4}sin(\frac{\pi s}{4}), \frac{\pi}{4}cos(\frac{\pi s}{4}))ds \\
&= \int_0^1 cos^2(\frac{\pi s}{4}) - sin^2(\frac{\pi s}{4})ds \\
&= \frac{\pi}{4}\int_0^1 cos(\frac{\pi s}{2})ds \\
&= \frac{\pi}{4} \times \frac{2}{\pi} sin(\frac{\pi s}{2})|_{s=0}^{s=1} \\
&= \frac{1}{2}
\end{aligned}
$$

- Force:

$$\mathbf{F} = q\mathbf{E}$$

- Work:

$$W = -\int_l \mathbf{F}dl = -q\int_l \mathbf{E}dl$$

- Voltage

$$V = \frac{W}{q} = -\int_l \mathbf{E}dl$$

2.1.5

Since $\mathbf{E}(x, y, z) = -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$ (shown in Fomular 8)

This field is static,

$$-\nabla U = E$$

$$
\begin{aligned}
\frac{\partial U}{\partial x} = y \\
\frac{\partial U}{\partial y} = x
\end{aligned}
$$

Hence,

$$U = xy + c$$

where c is constant.

$\mathbf{P_1} = (1, 0), \mathbf{P_2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$U(\mathbf{P_2}) - U(\mathbf{P_1}) = (1/2 + c) - (0 + c) = \frac{1}{2}$$

2.2

2.2.1

$$curl\mathbf{H} = \mathbf{\nabla} \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & 0 \end{vmatrix} = -\hat{\mathbf{y}} + \hat{\mathbf{z}}$$

2.2.2

In part 2.2.1, we've got $curl\mathbf{H} = -\hat{\mathbf{y}} + \hat{\mathbf{z}}$. Since the x component of $\mathbf{H}$ is 0, the flux of A_{left} and A_{left} equals 0.

$$\int curl\mathbf{H} \cdot d\mathbf{A_{left}} = \int (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot dA_{left}\hat{\mathbf{x}} = 0$$
$$\int curl\mathbf{H} \cdot d\mathbf{A_{right}} = \int (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot dA_{right}\hat{\mathbf{x}} = 0$$

2.2.3

The surface of A_{angle} is $z - y = 0$

Choose $x = 2s$, $y = 2t$, $s, t \in [0, 1]$

$\Rightarrow z = y = 2t$

$$\mathbf{l_{angle}}(s, t) = 2s\hat{\mathbf{x}} + 2t\hat{\mathbf{y}} + 2t\hat{\mathbf{z}} \quad s, t \in [0, 1]$$

$$
\begin{aligned}
d\mathbf{A_{angle}} &= (\frac{d\mathbf{l}}{ds} \times \frac{d\mathbf{l}}{dt})dsdt \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{vmatrix} dsdt \\
&= (4\hat{\mathbf{z}} - 4\hat{\mathbf{y}})dsdt
\end{aligned}
$$

$$
\begin{aligned}
\int_A curl\mathbf{H} \cdot d\mathbf{A_{angle}} &= \iint (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot (4\hat{\mathbf{z}} - 4\hat{\mathbf{y}})dsdt \\
&= 8 \int_0^1 \int_0^1 dsdt \\
&= 8
\end{aligned}
$$

2.2.4

$$
d\mathbf{A_{front}} = dA_{front}\hat{\mathbf{y}}
$$

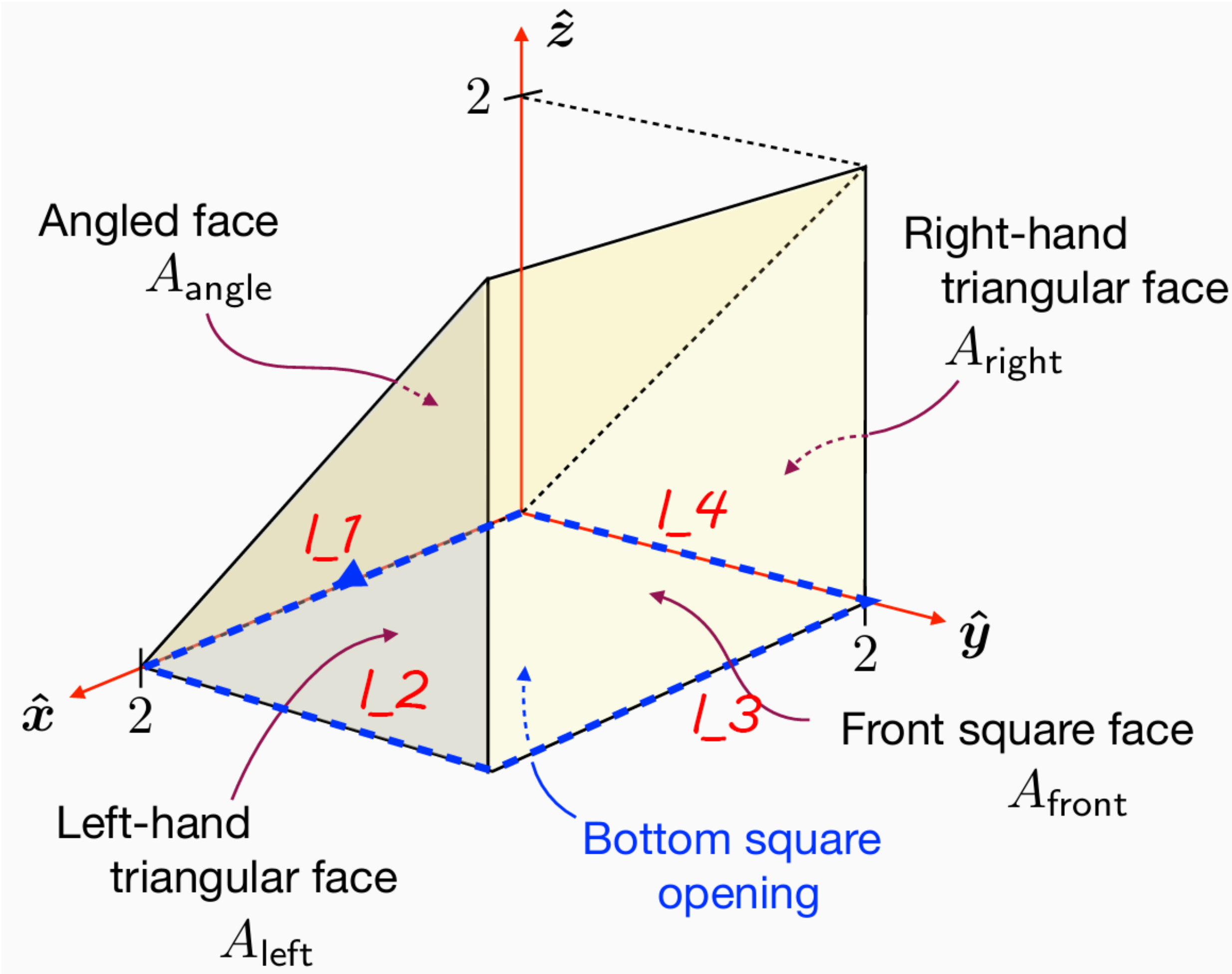
$$
\begin{aligned}
\int_A curl\mathbf{H} \cdot d\mathbf{A_{front}} &= \int_A (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot (\hat{\mathbf{y}})dA_{front} \\
&= -\int_A dA \\
&= -4
\end{aligned}
$$

$$
\int_A curl\mathbf{H} \cdot d\mathbf{A} = 8 - 4 = 4
$$

2.3

2.3.1

l_1, l_2, l_3, l_4 forms a square, anticlockwise.



2.3.2

Given:

$$\mathbf{H}(x, y, z) = -z\hat{\mathbf{x}} + x\hat{\mathbf{y}}$$

- $\mathbf{l}_1(s) = 2s\hat{\mathbf{x}}$

$$\mathbf{H}_1(x, y, z) = 2s\hat{\mathbf{y}}$$

- $\mathbf{l}_2(s) = 2\hat{\mathbf{x}} + 2s\hat{\mathbf{y}}$

$$\mathbf{H}_2(x, y, z) = 2\hat{\mathbf{y}}$$

- $\mathbf{l}_3(s) = 2(1 - s)\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$

$$\mathbf{H}_3(x, y, z) = 2(1 - s)\hat{\mathbf{y}}$$

- $\mathbf{l}_4(s) = 2(1 - s)\hat{\mathbf{y}}$

$$\mathbf{H}_4(x,y,z) = \mathbf{0}$$

2.3.3

- l_1

$$\begin{aligned}\int_{l_1} \mathbf{H}\cdot d\mathbf{l_1} &= \int_0^1 \mathbf{H}_1(s)\cdot\frac{d\mathbf{l_1}(s)}{ds}ds \\ &= \int_0^1 (0,2s,0)\cdot(2,0,0)ds \\ &= 0\end{aligned}$$

- l_2

$$\begin{aligned}\int_{l_2} \mathbf{H}\cdot d\mathbf{l_2} &= \int_0^1 \mathbf{H}_2(s)\cdot\frac{d\mathbf{l_2}(s)}{ds}ds \\ &= \int_0^1 4ds \\ &= 4\end{aligned}$$

- l_3

$$\begin{aligned}\int_{l_3} \mathbf{H}\cdot d\mathbf{l_3} &= \int_0^1 \mathbf{H}_3(s)\cdot\frac{d\mathbf{l_3}(s)}{ds}ds \\ &= \int_0^1 (0,2(1-s),0)\cdot(-2,0,0)ds \\ &= 0\end{aligned}$$

- l_4

$$\begin{aligned}\int_{l_4} \mathbf{H}\cdot d\mathbf{l_4} &= \int_0^1 \mathbf{H}_4(s)\cdot\frac{d\mathbf{l_4}(s)}{ds}ds \\ &= 0\end{aligned}$$

Hence,

$$\begin{aligned}\oint_l \mathbf{H}\cdot d\mathbf{l} &= \int_{l_1}\mathbf{H}\cdot d\mathbf{l_1} + \int_{l_2}\mathbf{H}\cdot d\mathbf{l_2} + \int_{l_3}\mathbf{H}\cdot d\mathbf{l_3} + \int_{l_4}\mathbf{H}\cdot d\mathbf{l_4} \\ &= 0+4+0+0 \\ &= 4\end{aligned}$$

2.3.4

In 2.2.4, we get

$$\int_A curl\mathbf{H} \cdot d\mathbf{A} = 4$$

In 2.3.3,

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = 4$$

It shows the Stokes' Theorem:

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = \int_A curl\mathbf{H} \cdot d\mathbf{A}$$

$\mathbf{A}$ is an oriented smooth open surface bounded by Jordan curve $\mathbf{l}$. Hence, Stokes' theoremholds for the field permeating through the configuration.

2.4

2.4.1

$$\begin{aligned} Q = \iiint_V \rho dV &= \int_0^1 \int_0^1 \int_0^1 \rho(x, y, z) dx dy dz \\ &= \int_0^1 \int_0^1 \int_0^1 (\pi z)^2 sin(\pi x) cos(\frac{\pi y}{2}) dx dy dz \\ &= 2\pi \int_0^1 \int_0^1 z^2 cos(\frac{\pi y}{2}) dy dz \\ &= 4 \int_0^1 z^2 sin(\frac{\pi y}{2})|_{y=0}^1 dz \\ &= \frac{4}{3} z^3|_{z=0}^1 \\ &= \frac{4}{3}(C) \end{aligned}$$

2.4.2

$$\begin{aligned}\oiint_A \mathbf{E}(x,y,z)d\mathbf{A} &= \oiint_A div\mathbf{E}dxdydz \\ &= \iiint_V \frac{\rho}{\epsilon}dxdydz \\ &= \frac{4}{3\epsilon}(Vm)\end{aligned}$$

Gauss's law shows:

$$\nabla \cdot \mathbf{E} = \frac{Q}{\epsilon}$$

Hence, the total flux of the electric field $\mathbf{E}$ out through the surface of the volume V is $\frac{4}{3\epsilon}$ Vm.