ELEN30011 EDM Task

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1.1

For rectangular coordinates and cylindrical coodinates,

$$x = rcos\phi$$

$$y = rsin\phi$$

$$z = z$$

(a)

$x = 1 \times cos(0) = 1, y = 1 \times sin(0) = 0, z = 0$

$$(1, 0, 0)$$

(b)

$x = 1 \times cos(\pi) = -1, y = 1 \times sin(\pi) = 0, z = 0$

$$(-1, 0, 0)$$

(c)

$x = 1 \times cos(-\pi/2) = 0, y = 1 \times sin(-\pi/2) = -1, z = 3$

$$(0, -1, 3)$$

(d)

$x = 0 \times cos(-\pi) = 0, y = 0 \times sin(-\pi) = 0, z = -2$

$$(0, 0, -2)$$

(e)

$x = -1 \times cos(0) = -1, y = -1 \times sin(0) = 0, z = 0$

$$(-1,0,0)$$

1.2

For spherical coordinates and rectangular coordinates,

$$x = rcos\phi sin\theta$$

$$y = rsin\phi sin\theta$$

$$z = rcos\theta$$

It can be noted that, r is the modulus of a vector $r = \sqrt{x^2+y^2+z^2}$;
ϕ is the angle with the x-axis in x-y plane $\phi = arctan(y/x)$;
θ is the angle with the z-axis $\theta = arctan(\sqrt{x^2+y^2}/z)$.
In order to make the answer unique, we assume $r \geq 0, \phi \in [0, 2\pi), \theta \in [0, \pi]$

(a)

$r = \sqrt{1^2+0^2+0^2} = 1$,
$\phi = arctan(0/1) = 0$,
$\theta = arctan(\sqrt{1^2+0^2}/0) = \pi/2$

$$(1, 0, \pi/2)$$

(b)

$r = \sqrt{0^2+1^2+0^2} = 1$,
$\phi = arctan(1/0) = \pi/2$,
$\theta = arctan(\sqrt{0^2+1^2}/0) = \pi/2$

$$(1, \pi/2, \pi/2)$$

(c)

$r = \sqrt{0^2+0^2+1^2} = 1$,
$\phi = arctan(0/0) = undefined$,
$\theta = arctan(\sqrt{0^2+0^2}/1) = 0$

$$(1, 0, 0)$$

- ϕ can be any real number here.

(d)

$r = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2},$
$\phi = arctan(1/0) = \pi/2,$
$\theta = arctan(\sqrt{0^2 + 1^2}/1) = \pi/4$

$$(\sqrt{2}, \pi/2, \pi/4)$$

(e)

$r = \sqrt{0^2 + 0^2 + 0^2} = 0,$
$\phi = arctan(0/0) = undefined,$
$\theta = arctan(\sqrt{0^2 + 0^2}/0) = undefined$

$$(0, 0, 0)$$

- $r = 0$, ϕ and θ can be any real number here.

2.1

(a) Let

$$x = rcos\phi, y = rsin\phi, z = z$$

Jacobin Matrix:

$$J(r, \phi, z) = \begin{pmatrix} cos\phi & -rsin\phi & 0 \\ sin\phi & rcos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = J \begin{pmatrix} dr \\ d\phi \\ dz \end{pmatrix}$$

Hence find expressions for $\hat{r}$ and $\hat{\phi}$ in terms of $\hat{x}$ and $\hat{y}$.
Since $\hat{r}$ and $\hat{\phi}$ are unit vectors, let $r = 1$

$$T(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Or, considering

$$\hat{r} = cos\phi\hat{x} + sin\phi\hat{y}$$
$$\hat{\phi} = -sin\phi\hat{x} + cos\phi\hat{y}$$
$$v = v_r\hat{r} + v_\phi\hat{\phi} + v_z\hat{z}$$

$$v_x = v\cdot\hat{x} = v_r cos\phi - v_\phi sin\phi$$
$$v_y = v\cdot\hat{y} = v_r sin\phi + v_\phi cos\phi$$
$$v_z = v\cdot\hat{z} = v_z$$

In order to make

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = T(\phi)\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}$$

$$T(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)

We've got that

$$T(\phi) = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The (i, j)-cofactor is $C_{i,j} = (-1)^{i+j}M_{i,j}$, where $M_{i,j}$ is the (i, j)-minor.

$$C_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} cos\phi & 0 \\ 0 & 1 \end{vmatrix} = cos\phi$$

Similarly,

$$\begin{matrix} & C_{12} = -sin\phi & C_{13} = 0 \\ C_{21} = sin\phi & C_{22} = cos\phi & C_{23} = 0 \\ C_{31} = 0 & C_{32} = 0 & C_{33} = 1 \end{matrix}$$

Hence,

$$(T^*(\phi))^T = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The adjugate matrix of T is:

$$T^*(\phi) = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$det[T(\phi)] = 1$$

Based on Cramer's rule,

$$T^{-1}(\phi) = \frac{1}{det} T^*(\phi)$$

Hence,

$$T^{-1}(\phi) = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$r = \sqrt{x^2 + y^2}$$
$$cos(\phi) = \frac{x}{\sqrt{x^2 + y^2}}$$
$$sin(\phi) = \frac{y}{\sqrt{x^2 + y^2}}$$

(d)

According to the result of part c, substitute $cos\phi$ and $sin\phi$:

$$T^{-1}(\phi) = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$S(P) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.2

In Question 2.1(d), we've got

$$S(P) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this task, we assume $v_x = -1, v_y = 0, v_z = 0$.

(a) $(0, -1, 0)^T$

When P = (0, -1, 0)

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = S(P) \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

(b) $(-1, 0, 0)^T$

When P = (1, 0, 0)

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = S(P) \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

(c) $(1, 0, 0)^T$

When P = (-1, 0, 0)

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = S(P) \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(d) $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0)^T$

When P = (1, -1, 0)

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = S(P) \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

(e) $(-\lim_{x\to 0, y\to 0} \frac{x}{\sqrt{x^2+y^2}}, \lim_{x\to 0, y\to 0} \frac{y}{\sqrt{x^2+y^2}}, 0)^T$

When P = (0, 0, 0)

We have to compute:

$$\lim_{x\to 0, y\to 0} \frac{x}{\sqrt{x^2+y^2}}$$
$$\lim_{x\to 0, y\to 0} \frac{y}{\sqrt{x^2+y^2}}$$

However, we do not know the path of (x, y) approaching to 0.

We get $S(P)$ and the result:

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = S(P) \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \lim_{x\to 0, y\to 0} \frac{x}{\sqrt{x^2+y^2}} & \lim_{x\to 0, y\to 0} \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\lim_{x\to 0, y\to 0} \frac{y}{\sqrt{x^2+y^2}} & \lim_{x\to 0, y\to 0} \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -\lim_{x\to 0, y\to 0} \frac{x}{\sqrt{x^2+y^2}} \\ \lim_{x\to 0, y\to 0} \frac{y}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix}$$

r and ϕ are undefined, $z = 0$.

3.1

(a)

```
x = -2:.1:2;
y = -2:.1:2;
[xx, yy] = meshgrid(x, y);

size(xx)
size(yy)
```

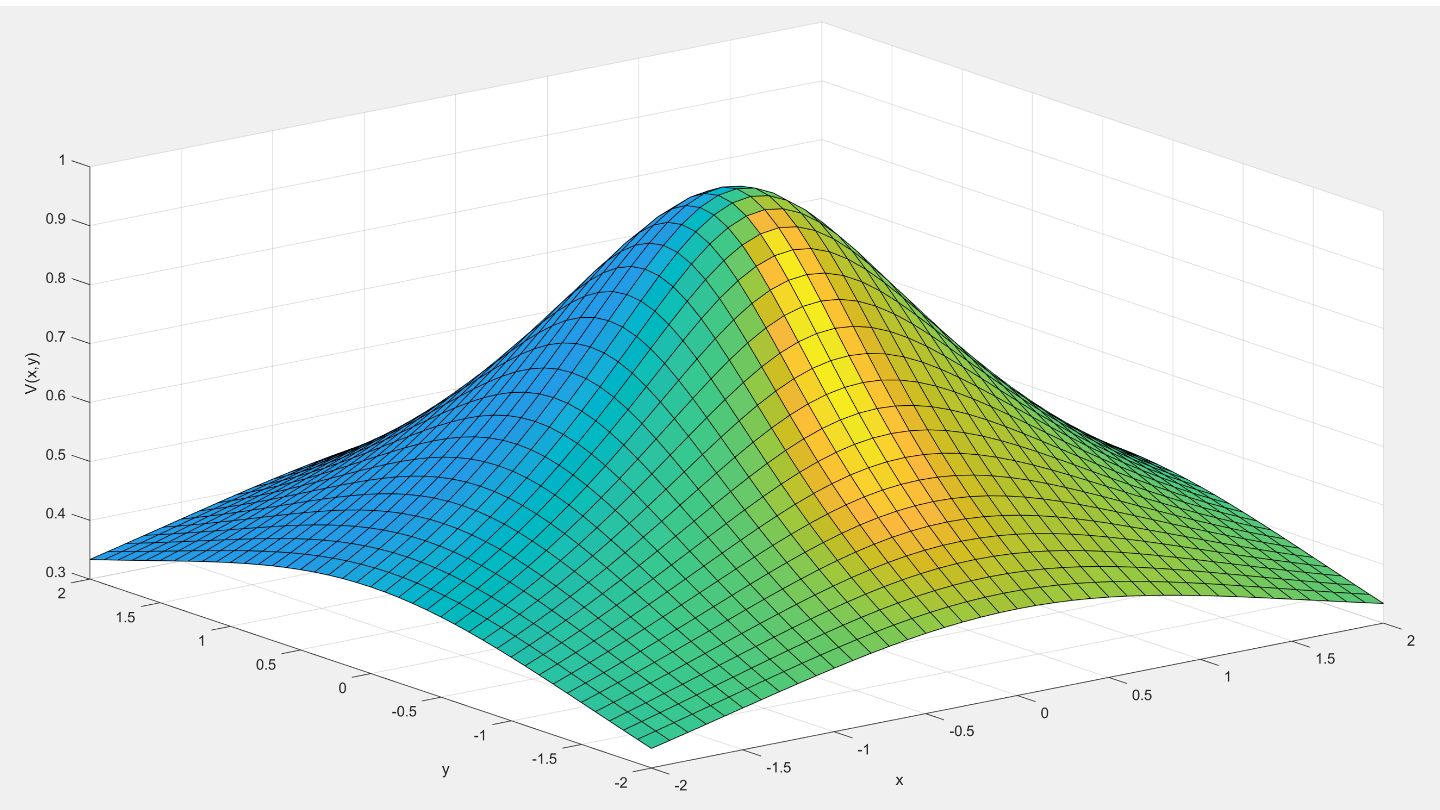
Which output is:

```
ans =

    41    41

ans =

    41    41
```

xx and yy are both 41×41 matrix.

(b)

```
x = -2:.1:2;
y = -2:.1:2;
[xx, yy] = meshgrid(x, y);

size(xx)
size(yy)

zz = 1./sqrt(1 + xx.^2 + yy.^2);
figure(1);
surfl(xx, yy, zz);
xlabel('x');
ylabel('y');
zlabel('V(x,y)');
grid on;
```



Based on the picture above, it has been shown that the surface exhibit a maximum.

After checked the value "zz" in workspace, we get the maximum point is $(0, 0, 1)$.

For certain plane, origin can always be the point with the highest electrostatic potential. If a charge moves in any direction on its x-y plane, the electric field does positive work on it.

(c)
A circle.

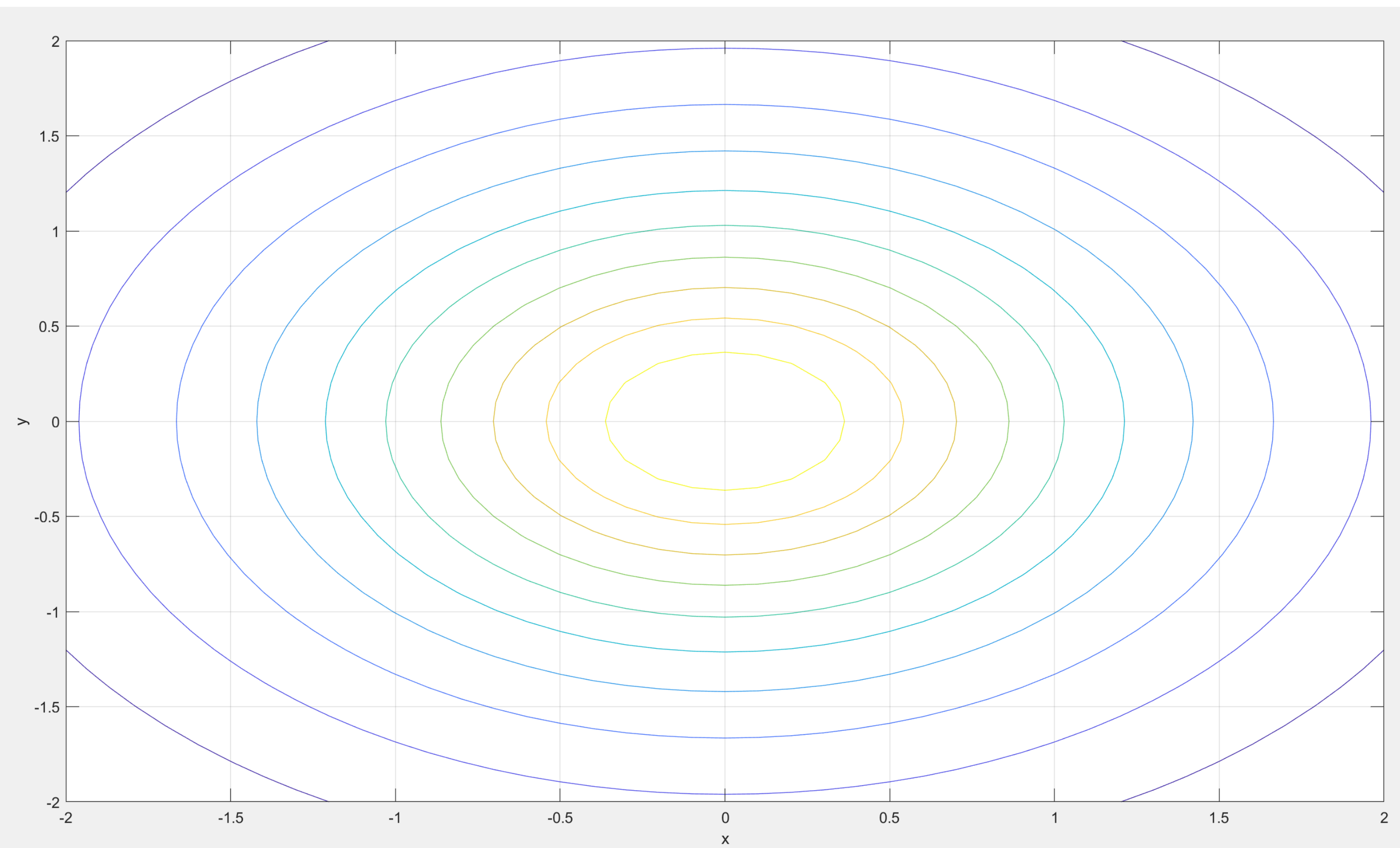
$$V = \frac{1}{\sqrt{1 + x^2 + y^2}}$$
$$\sqrt{1 + x^2 + y^2} = \frac{1}{V}$$
$$r = \sqrt{x^2 + y^2} = \sqrt{\frac{1}{V^2} - 1}$$

Its radius is $\sqrt{\frac{1}{V^2} - 1}$

If $V = c > 1$, radius will be an imaginary number, which is impossible here. Hence, c will never be greater than 1.

(d)

```
figure(2);
contour(xx, yy,zz, 10);
xlabel('x');
ylabel('y');
grid on;
```

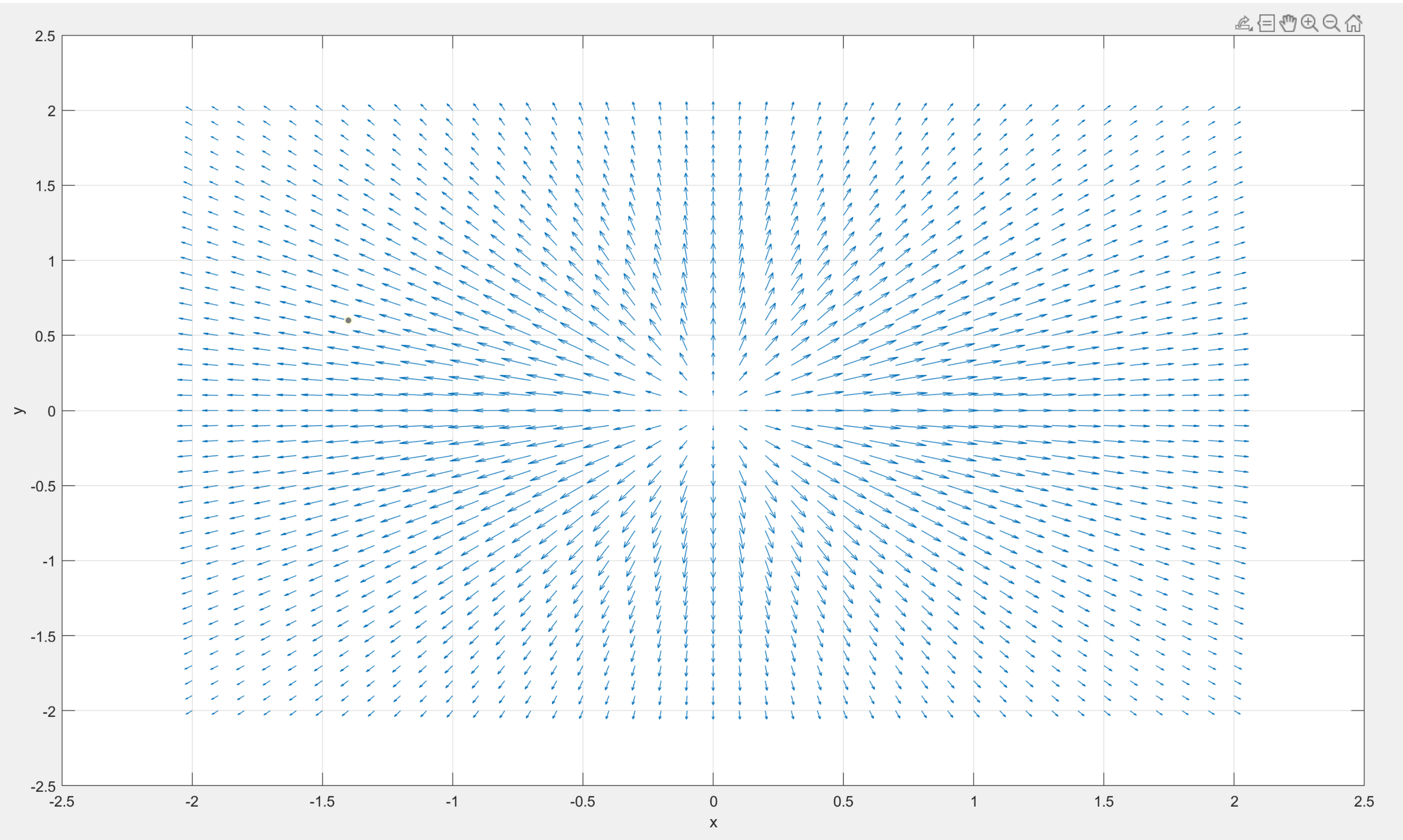


"10" means: Display 10 contour lines at automatically chosen levels (heights).

3.2

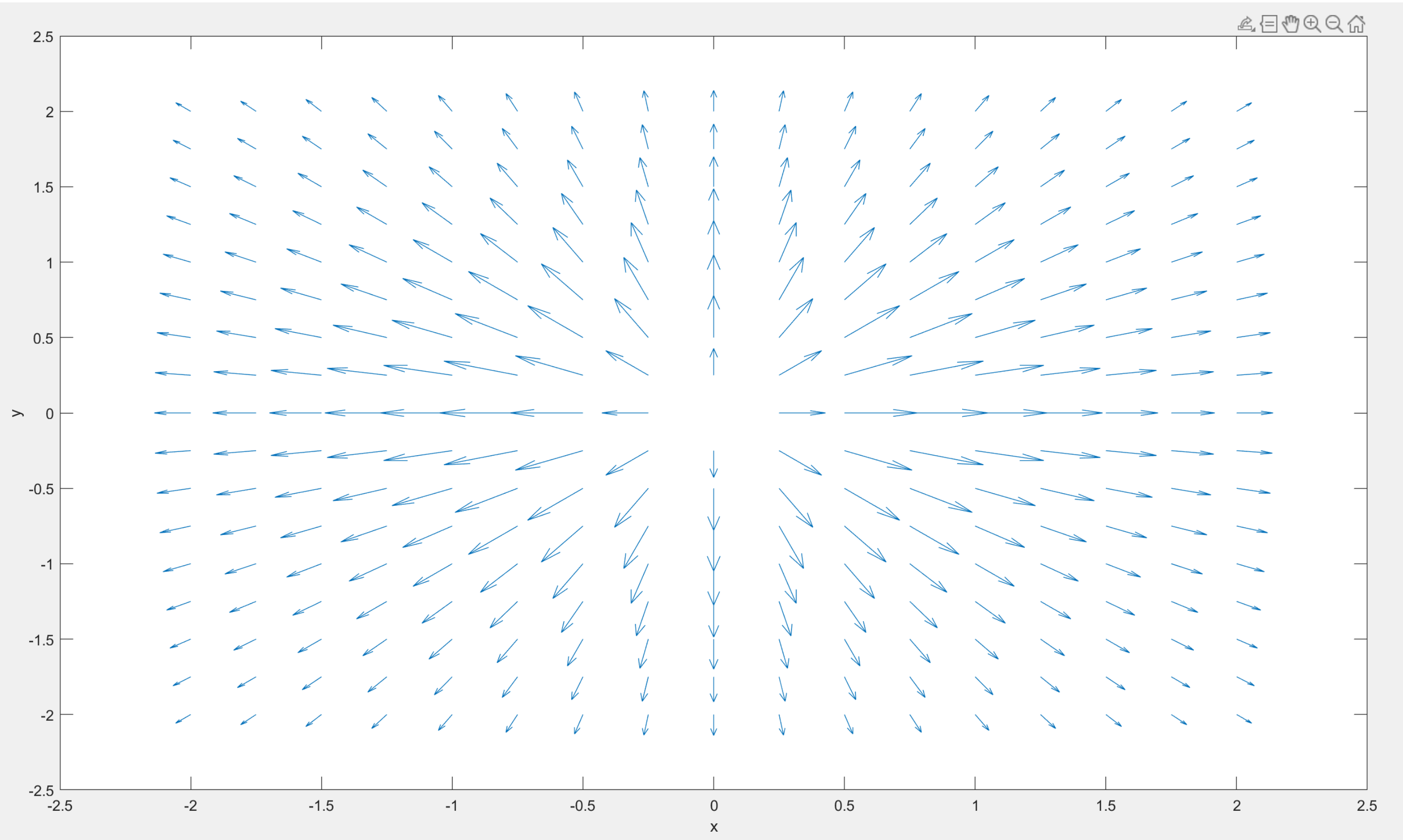
(a)

```
exx = xx./(1 + xx.^2 + yy.^2).^(3/2);
eyy = yy./(1 + xx.^2 + yy.^2).^(3/2);

figure(3);
quiver(xx,yy,exx,eyy);
xlabel('x');
ylabel('y');
grid on;
```



(b)

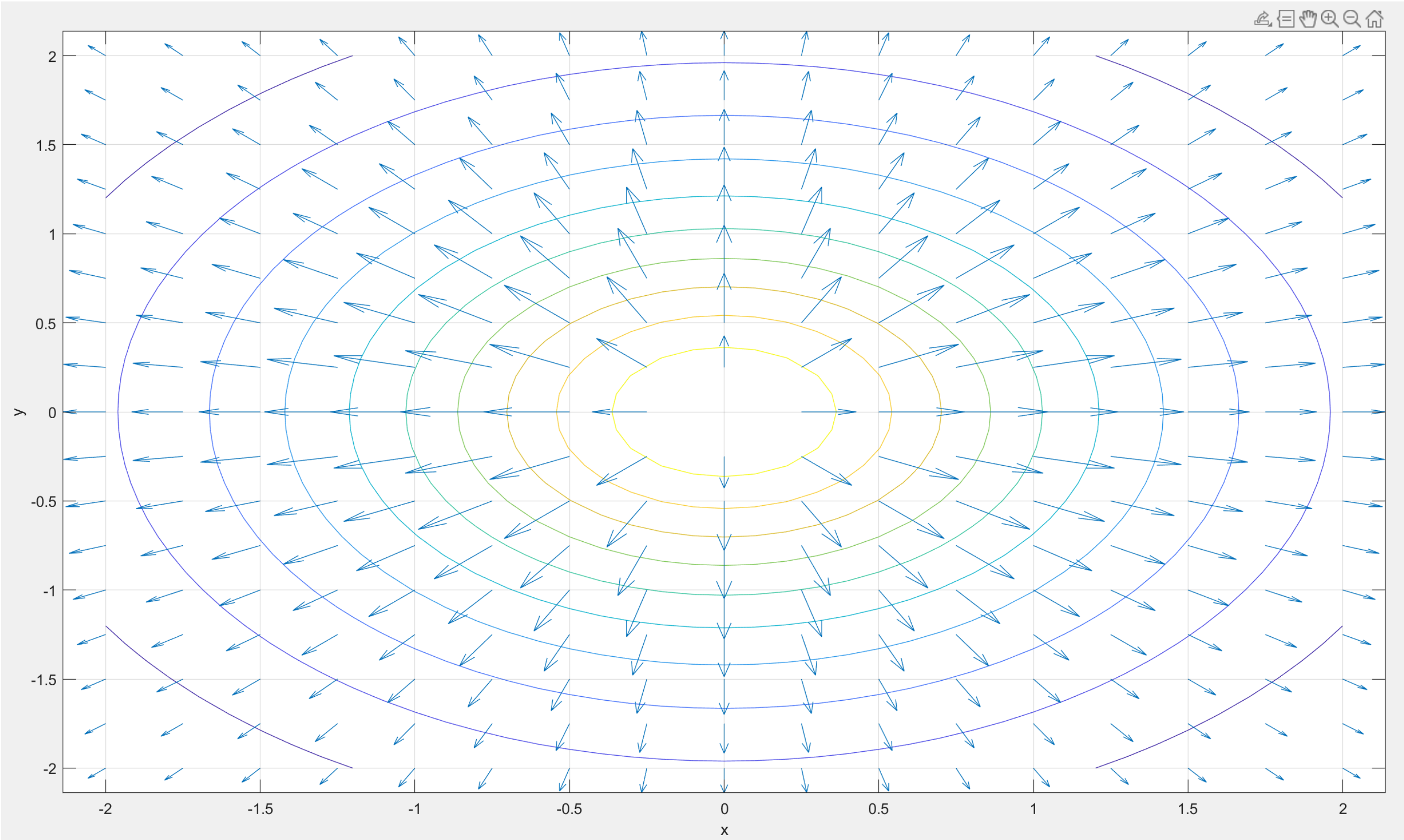
```
xnew = -2:.25:2;
ynew = xnew;
[xxnew, yynew] = meshgrid(xnew, ynew);
exxnew = xxnew./(1 + xxnew.^2 + yynew.^2).^(3/2);
eyynew = yynew./(1 + xxnew.^2 + yynew.^2).^(3/2);

figure(4);
quiver(xxnew, yynew, exxnew, eyynew);
hold on;
```



- More "sparse": xx and yy are 41×41 matrix (interval 0.1), while xxnew and yynew are 17×17 matrix (interval 0.25).

(c)

```
figure(2);
hold on;
quiver(xxnew, yynew, exxnew, eyynew);
```



Perpendicular to each other.

3.3

(a)

The contour V is a curve, its tangents is $T(s) = \dot{x}(s)\hat{x} + \dot{y}(s)\hat{y}$

Based on the chain rule, which is:

$$\frac{d}{ds} = \frac{d}{dx}\frac{dx}{ds}$$

Since $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j$, the gradient of V is

$$\nabla V = \frac{\partial V}{\partial x}i + \frac{\partial V}{\partial y}j$$

$$\nabla V \cdot T(s) = \nabla V \cdot \frac{d}{ds}(x(s), y(s)) = \frac{d}{ds}V(x(s), y(s))$$

Since V is constant along the contour, its derivative is zero.

$$\nabla V \cdot T(s) = 0$$

In Task 3.2, we got $E(x,y) = -\nabla V(x,y)$

Hence, we get the fomular 11.

According to the electric field (8) and (9),

$$E(x(s),y(s)) \approx \frac{x(s)\hat{x} + y(s)\hat{y}}{(1+x^2(s)+y^2(s))^{\frac{3}{2}}}$$

In formula (12),

$$T(s) = \hat{T}(x(s),y(s)) \approx \frac{y(s)\hat{x} - x(s)\hat{y}}{\sqrt{1+x^2(s)+y^2(s)}}$$

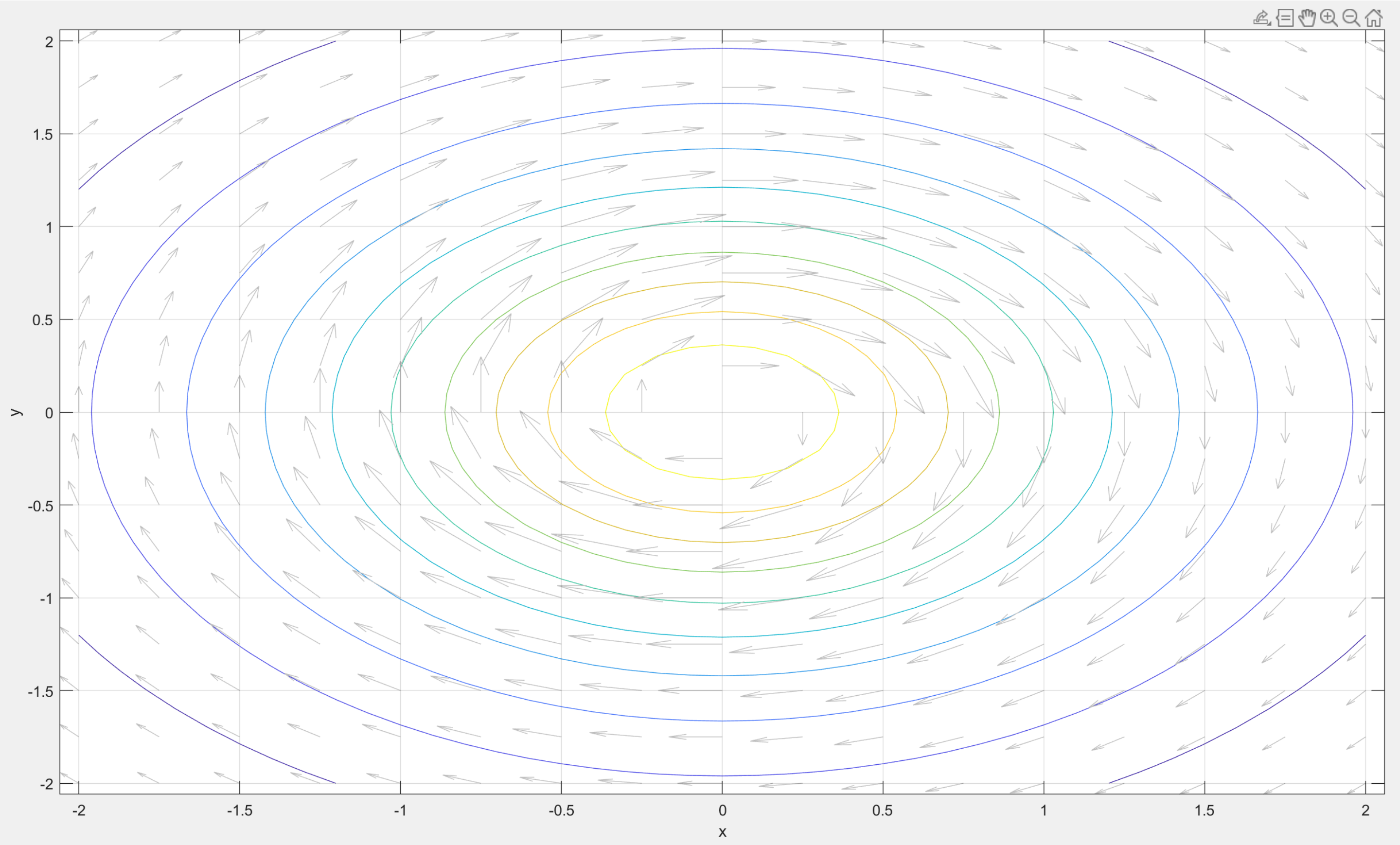
$$\overrightarrow{E} \cdot \overrightarrow{T} = \frac{x(s)y(s) - y(s)x(s)}{(1+x^2(s)+y^2(s))^2} = 0$$

The dot product of vectors is 0 means these vectors are perpendicular.

For $\forall (x(s),y(s)) \in \mathbb{R}^2, s \in \mathbb{R}$, $\overrightarrow{E} \cdot \overrightarrow{T} = 0$, which means $T(s)$ must be everywhere perpendicular to $E(x(s),y(s))$.

(b)

```
close all
clear
clc

x = -2:.1:2;
y = -2:.1:2;
[xx, yy] = meshgrid(x, y);

zz = 1./sqrt(1 + xx.^2 + yy.^2);

figure(4);
contour(xx, yy,zz, 10);
xlabel('x');
ylabel('y');
grid on;

exx = xx./(1 + xx.^2 + yy.^2).^(3/2);
eyy = yy./(1 + xx.^2 + yy.^2).^(3/2);

xnew = -2:.25:2;
ynew = xnew;
[xxnew, yynew] = meshgrid(xnew, ynew);
exxnew = yynew./(1 + xxnew.^2 + yynew.^2).^(3/2);
eyynew = -xxnew./(1 + xxnew.^2 + yynew.^2).^(3/2);

figure(4);
hold on;
quiver(xxnew, yynew, exxnew, eyynew, "Color","#C0C0C0");
hold off;
```



(c)

We've already found that the electric field $E(x,y) = -\nabla V(x,y)$, this figure shows:

- $\hat{T}$ is the tangent vector of V, and it is perpendicular to the electric field E.
- Also, the electric field E is perpendicular to the contours V.

This means the electric field is not divergence, its curl will be 0.

$$\nabla \times E = 0$$

It is a special case of Maxwell's Equations, in this case $E = -grad(V)$.