ELEN30011 Electrical Device Modelling

Workshop 1 - Vector fields

1 Coordinate systems

In representing three dimensional Euclidean space $\mathbb{R}^3$, three coordinate systems are commonly used, i.e. the rectangular, cylindrical, and spherical coordinate systems. As illustrated in Figures 1-3, the coordinates of a point P in these systems can be expressed as

$$
\begin{array}{lll}
\text{(rectangular)} & \mathsf{P} = (x,\, y,\, z), & \text{or} \\
\text{(cylindrical)} & \mathsf{P} = (r,\, \phi,\, z), & \text{or} \\
\text{(spherical)} & \mathsf{P} = (r,\, \phi,\, \theta), &
\end{array}
$$

in which x, y, z are (signed) distances, r is a radius, and ϕ, θ are angles. Transformations between these coordinates for P follow via trigonometry. For example,

$$
\begin{array}{rl}
\text{(rectangular)} \longleftarrow \text{(cylindrical)} & (x,\, y,\, z) = (r\cos\phi,\, r\sin\phi,\, z), \\
\text{(cylindrical)} \longleftarrow \text{(rectangular)} & (r,\, \phi,\, z) = (\sqrt{x^2+y^2},\, \tan^{-1}(\tfrac{y}{x}),\, z), \\
\text{(rectangular)} \longleftarrow \text{(spherical)} & (x,\, y,\, z) = (r\cos\phi\sin\theta,\, r\sin\phi\sin\theta,\, r\cos\theta).
\end{array}
$$

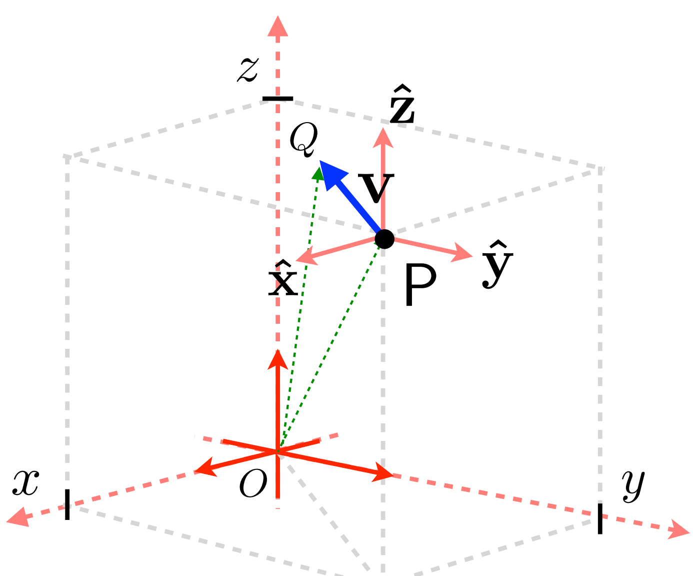

Figure 1: Rectangular coordinate system.

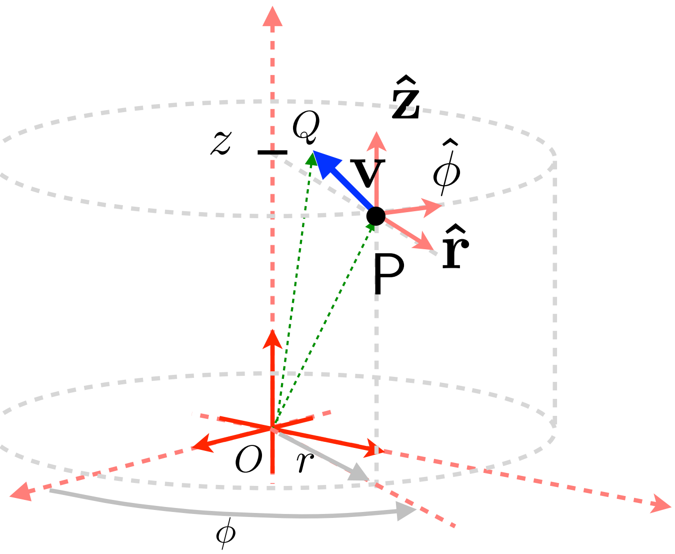

Figure 2: Cylindrical coordinate system.

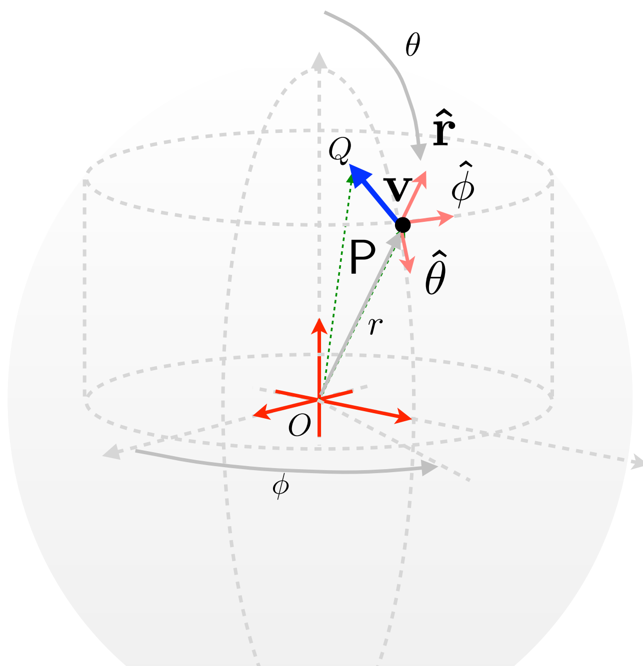

Figure 3: Spherical coordinate system.

Tasks

1.1. Find the rectangular coordinates (x, y, z) for P, given its cylindrical coordinates (r, ϕ, z):

(a) $(r, \phi, z) = (1, 0, 0)$.
(b) $(r, \phi, z) = (1, \pi, 0)$.
(c) $(r, \phi, z) = (1, -\pi/2, 3)$.
(d) $(r, \phi, z) = (0, -\pi, -2)$.
(e) $(r, \phi, z) = (-1, 0, 0)$.

1.2. Find the spherical coordinates (r, ϕ, θ) for P, given its rectangular coordinates (x, y, z):

(a) $(x, y, z) = (1, 0, 0)$.
(b) $(x, y, z) = (0, 1, 0)$.
(c) $(x, y, z) = (0, 0, 1)$.
(d) $(x, y, z) = (0, 1, 1)$.
(e) $(x, y, z) = (0, 0, 0)$.

2 Unit vectors and vector transformations

At every point P in $\mathbb{R}^3$, a triple of unit vectors is defined for each coordinate system, as illustrated in Figures 1-3. These unit vectors are denoted by

$$\begin{aligned} &(\text{ rectangular }) \quad && \hat{\boldsymbol{x}},\ \hat{\boldsymbol{y}},\ \hat{\boldsymbol{z}}, \\ &(\text{ cylindrical }) \quad && \hat{\boldsymbol{r}},\ \hat{\boldsymbol{\phi}},\ \hat{\boldsymbol{z}}, \\ &(\text{ spherical }) \quad && \hat{\boldsymbol{r}},\ \hat{\boldsymbol{\phi}},\ \hat{\boldsymbol{\theta}}. \end{aligned}$$

Any vector $\mathbf{v}$ in $\mathbb{R}^3$ can be expressed with respect to these unit vectors, via P, with

$$(\text{ rectangular }) \quad \mathsf{P} = (x, y, z), \qquad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x\,\hat{\boldsymbol{x}} + v_y\,\hat{\boldsymbol{y}} + v_z\,\hat{\boldsymbol{z}}, \tag{1}$$

$$(\text{ cylindrical }) \quad \mathsf{P} = (r, \phi, z), \qquad \mathbf{v} = \begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = v_r\,\hat{\boldsymbol{r}} + v_\phi\,\hat{\boldsymbol{\phi}} + v_z\,\hat{\boldsymbol{z}}, \tag{2}$$

$$(\text{ spherical }) \quad \mathsf{P} = (r, \phi, \theta), \qquad \mathbf{v} = \begin{pmatrix} v_r \\ v_\phi \\ v_\theta \end{pmatrix} = v_r\,\hat{\boldsymbol{r}} + v_\phi\,\hat{\boldsymbol{\phi}} + v_\theta\,\hat{\boldsymbol{\theta}}. \tag{3}$$

Transformations between these representations of $\mathbf{v}$ can be obtained by trigonometry and algebra. This can be formalized for rectangular and cylindrical coordinates as follows. Note that (x, y, z) and (r, ϕ, z) are used interchangeably to specify the point P.

Tasks

2.1. (a) Using the geometry illustrated in Figures 1 and 2, find expressions for $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ in terms of $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. Given representations (1)-(2), subsequently show that

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \mathbf{v}\cdot\hat{\boldsymbol{x}} \\ \mathbf{v}\cdot\hat{\boldsymbol{y}} \\ \mathbf{v}\cdot\hat{\boldsymbol{z}} \end{pmatrix} = T(\phi) \begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}, \quad \text{where} \quad T(\phi) \doteq \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Show that the determinant of $T(\phi)$ is $\mathsf{det}\, T(\phi) = 1$. Using Cramer's rule, show that

$$T(\phi)^{-1} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

(c) Using trigonometry involving the location of the point P, write down expressions for r, $\cos\phi$, and $\sin\phi$ in terms of x and y.

(d) Combining your answers above, show that the coordinate representation for vector $\mathbf{v}$ can be transformed from rectangular coordinates to cylindrical coordinates by

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\doteq S(\mathsf{P})} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}. \tag{5}$$

2.2. A common misconception in dealing with the coordinate systems of Figures 1-3 is to confuse the point P with the arbitrary vector $\mathbf{v}$ that is to be represented. For example, in the cylindrical coordinate system of Figure 2, P specifies a point on the surface of the cylinder at which the vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ unit vectors are to be defined, whereas $\mathbf{v}$ is the vector that is to be represented. i.e. P effectively sets the unit vectors. Note in (5) that

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = v_r\,\hat{\boldsymbol{r}} + v_\phi\,\hat{\boldsymbol{\phi}} + v_z\,\hat{\boldsymbol{z}}, \quad \mathsf{P} = (x,\, y,\, z), \quad \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x\,\hat{\boldsymbol{x}} + v_y\,\hat{\boldsymbol{y}} + v_z\,\hat{\boldsymbol{z}},$$

with x, y appearing in the coefficient matrix $S(\mathsf{P})$ in (5) determined by P. To emphasize the relationship between P and the representations of $\mathbf{v}$, for this task, fix a vector $\mathbf{v}$ via the rectangular components

$$v_x = -1, \quad v_y = 0, \quad v_z = 0. \tag{6}$$

With $\mathsf{P} = (x, y, z) = (0, 1, 0)$, for example, substitution of (6) in (5) yields

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = S(\mathsf{P}) \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

That is, for this choice of P, the vector $\mathbf{v}$ has a $\hat{\boldsymbol{\phi}}$ component only, of magnitude 1. It contains no components in the $\hat{\boldsymbol{r}}$ or $\hat{\boldsymbol{z}}$ directions, see Figure 2.

Using (5) and given the same rectangular components for $\mathbf{v}$ specified in (6), compute the cylindrical components v_r, v_ϕ, and v_z for each of the following choices of P. As per the example above, rationalize your answer in each case. Include a sketch if useful.

(a) $\mathsf{P} = (0,\ -1,\ 0)$.
(b) $\mathsf{P} = (1,\ 0,\ 0)$,
(c) $\mathsf{P} = (-1,\ 0,\ 0)$,
(d) $\mathsf{P} = (1,\ -1,\ 0)$,
(e) $\mathsf{P} = (0,\ 0,\ 0)$.

3 Visualization of a scalar function and a vector field

A pair of particles, each carrying charge Q, are located on the z-axis in rectangular coordinates, at positions $(0, 0, d)$ and $(0, 0, -d)$. On the $x-y$ plane, i.e. for $z = 0$, the associated electrostatic potential $V : \mathbb{R}^2 \to \mathbb{R}$ and corresponding electric field $\mathbf{E} : \mathbb{R}^2 \to \mathbb{R}^3$ are given respectively by

$$V(x, y) = \frac{Q}{2\,\pi\,\epsilon}\,\frac{1}{\sqrt{d^2 + x^2 + y^2}} \quad \mathsf{V}, \tag{7}$$

$$\mathbf{E}(x, y) = \frac{Q}{2\,\pi\,\epsilon}\,\frac{x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}}{(d^2 + x^2 + y^2)^{\frac{3}{2}}} \quad \mathsf{V\,m^{-1}}, \tag{8}$$

for all $(x, y) \in \mathbb{R}^2$. Note that V is a scalar(-valued) function, while $\mathbf{E}$ is a vector field. For the purpose of computation and visualisation, select

$$\frac{Q}{2\,\pi\,\epsilon} \doteq 1, \qquad d \doteq 1. \tag{9}$$

Tasks

3.1. The electrostatic potential (7) defines a two dimensional surface, given by $z = V(x, y)$.

(a) Start MATLAB. At the $\gg$ prompt, enter x and y axis grids by typing the following:

```
≫ x = -2:.1:2;
≫ y = -2:.1:2;
≫ [xx,yy] = meshgrid(x,y);
```

Determine the size of the `xx` matrix by typing

```
≫ size(xx)
```

Do the same for `yy`.

(b) Using the following commands, compute and plot V in (7) subject to selection (9) over the aforementioned grid, annotate with axis labels, and add a grid.

```
≫ zz = 1./sqrt(1 + xx.^2 + yy.^2);
≫ figure(1);
≫ surfl(xx,yy,zz);
≫ xlabel('x');
≫ ylabel('y');
≫ zlabel('V(x,y)');
≫ grid on;
```

Does the surface exhibit a minimum or a maximum? If so, where is it located, what value does V take there, and what does its existence mean physically?

(c) A contour of the electrostatic potential V is the set of all points $(x, y) \in \mathbb{R}^2$ such that $V(x, y) = c$, for some $c \in \mathbb{R}$. This contour describes the set of all points with the same electrostatic potential c. With $c \leq 1$, and using (7), (9), determine the shape of the corresponding contours of V. What happens for $c > 1$?
[HINT: Solve $V(x, y) = c$ for $x^2 + y^2$.]

(d) Plot the contours of V in a separate figure, via the following commands:

```
≫ figure(2);
≫ contour(xx,yy,zz,10);
≫ xlabel('x');
≫ ylabel('y');
≫ grid on;
```

Compare your answer to 3.1(c). What is the purpose of the last argument "`10`" in the `contour` command above?

3.2. The electric field (8) defines a vector field, i.e. $(e_x, e_y, 0) = \mathbf{E}(x, y)$ in rectangular coordinates. [Note that (8) follows by finding the gradient of (7), with $\mathbf{E}(x, y) = -\nabla V(x, y)$.]

(a) In order to plot $\mathbf{E}$, its components in (8) must first computed across the entire grid. To do this, use the commands

```
≫ exx = xx./(1 + xx.^2 + yy.^2).^(3/2);
≫ exx = yy./(1 + xx.^2 + yy.^2).^(3/2);
```

To see what this vector field looks like, plot it via

```
≫ figure(3); quiver(xx,yy,exx,eyy);
```

(b) Try the following commands:

```
≫ xnew = -2:.25:2;
≫ ynew = xnew;
≫ [xxnew, yynew] = meshgrid(xnew,ynew);
≫ exxnew = xxnew./(1 + xxnew.^2 + yynew.^2).^(3/2);
≫ eyynew = yynew./(1 + xxnew.^2 + yynew.^2).^(3/2);
≫ figure(4);
≫ quiver(xxnew,yynew,exxnew,eyynew);
```

What is the difference between this vector field plot and the one in 3.2(a) above?

(c) Now overlay this vector field plot of $\mathbf{E}$ on top of the earlier contour plot of V.

```
≫ figure(2);
≫ hold on;
≫ quiver(xxnew,yynew,exxnew,eyynew);
```

By inspection of the overlaid plot obtained, what is the relationship between the gradient vector field and the contours?

3.3. (a) A tangent to a contour describes a line (or plane) that is tangent to that contour. As a contour of V is a curve, all points on it can be parametrised by a real variable $s \in \mathbb{R}$. Given $c \in \mathbb{R}$, such a contour consists of all points $(x(s), y(s)) \in \mathbb{R}^2$ that satisfy

$$c = V(x(s), y(s)), \qquad s \in \mathbb{R}. \tag{10}$$

Tangents to this contour are described by the vector field $\mathbf{T}(s) = \dot{x}(s)\,\hat{\boldsymbol{x}} + \dot{y}(s)\,\hat{\boldsymbol{y}}$. In order to find $\mathbf{T}(s)$, differentiate (10) with respect to s. Confirm via the chain rule that

$$0 = \nabla V(x(s), y(s)) \cdot \mathbf{T}(s) = -\mathbf{E}(x(s), y(s)) \cdot \mathbf{T}(s), \tag{11}$$

in which $\mathbf{E}$ is as per (8). Explain by (11) why $\mathbf{T}(s)$ must be everywhere perpendicular to $\mathbf{E}(x(s), y(s))$. Show that one such choice of tangent vector field is given by

$$\mathbf{T}(s) = \hat{\mathbf{T}}(x(s), y(s)), \qquad \hat{\mathbf{T}}(x, y) \doteq \frac{y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}}{\sqrt{1 + x^2 + y^2}}. \tag{12}$$

(b) Following similar steps to that of Tasks 3.1-3.2, reproduce the Figure 4 below, illustrating the contours of the electrostatic potential V, and the tangent vector field $\hat{\mathbf{T}}$ defined in (12), on the same axes.

(c) Summarize the relationship between the vector fields $\mathbf{E}$ and $\hat{\mathbf{T}}$, and the contours of V.

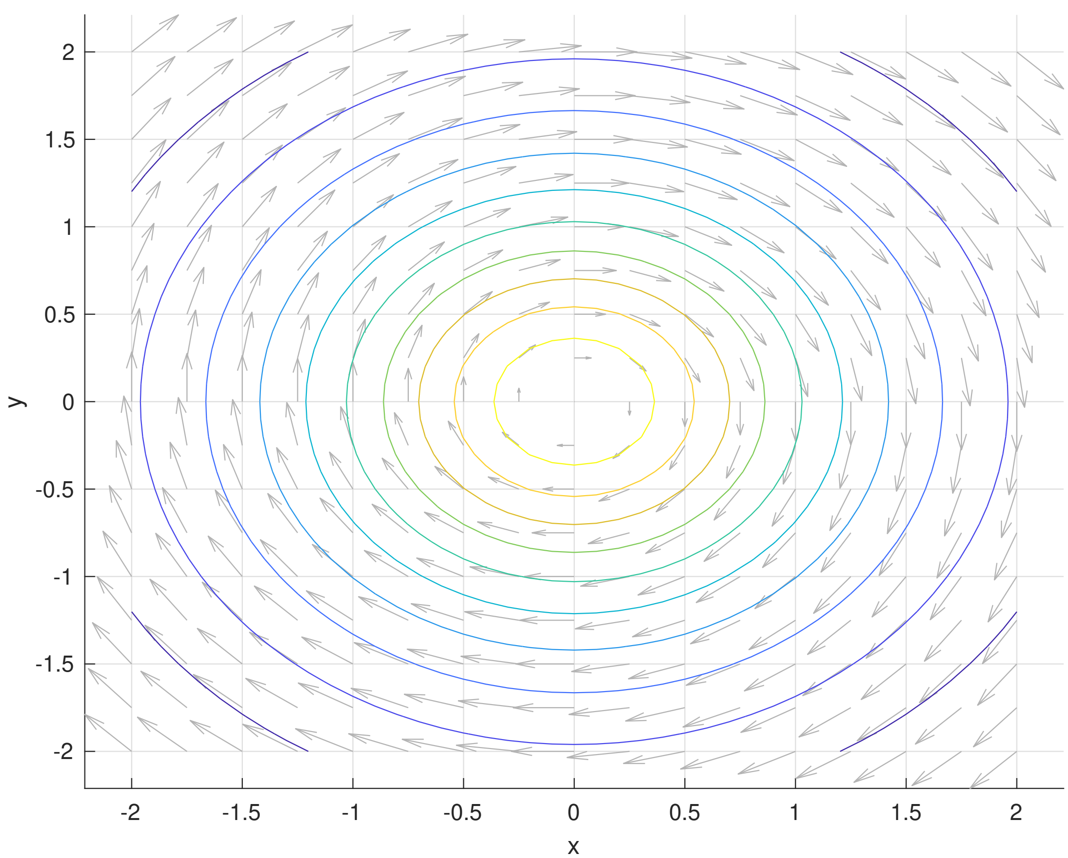

Figure 4: Contours of V and tangent vector field $\hat{\mathbf{T}}$.