

ELEN30011 EDM Task

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1.1

For rectangular coordinates and cylindrical coordinates,

$$x = r \cos \phi$$

$$y = r \sin \phi$$

$$z = z$$

(a)

$$x = 1 \times \cos(0) = 1, y = 1 \times \sin(0) = 0, z = 0$$

$$(1, 0, 0)$$

(b)

$$x = 1 \times \cos(\pi) = -1, y = 1 \times \sin(\pi) = 0, z = 0$$

$$(-1, 0, 0)$$

(c)

$$x = 1 \times \cos(-\pi/2) = 0, y = 1 \times \sin(-\pi/2) = -1, z = 3$$

$$(0, -1, 3)$$

(d)

$$x = 0 \times \cos(-\pi) = 0, y = 0 \times \sin(-\pi) = 0, z = -2$$

$$(0, 0, -2)$$

(e)

$$x = -1 \times \cos(0) = -1, y = -1 \times \sin(0) = 0, z = 0$$

$$(-1, 0, 0)$$

1.2

For spherical coordinates and rectangular coordinates,

$$x = r \cos \phi \sin \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \theta$$

It can be noted that, r is the modulus of a vector $r = \sqrt{x^2 + y^2 + z^2}$;

ϕ is the angle with the x-axis in x-y plane $\phi = \arctan(y/x)$;

θ is the angle with the z-axis $\theta = \arctan(\sqrt{x^2 + y^2}/z)$.

In order to make the answer unique, we assume $r \geq 0, \phi \in [0, 2\pi), \theta \in [0, \pi]$

(a)

$$r = \sqrt{1^2 + 0^2 + 0^2} = 1,$$

$$\phi = \arctan(0/1) = 0,$$

$$\theta = \arctan(\sqrt{1^2 + 0^2}/0) = \pi/2$$

$$(1, 0, \pi/2)$$

(b)

$$r = \sqrt{0^2 + 1^2 + 0^2} = 1,$$

$$\phi = \arctan(1/0) = \pi/2,$$

$$\theta = \arctan(\sqrt{0^2 + 1^2}/0) = \pi/2$$

$$(1, \pi/2, \pi/2)$$

(c)

$$r = \sqrt{0^2 + 0^2 + 1^2} = 1,$$

$$\phi = \arctan(0/0) = \text{undefined},$$

$$\theta = \arctan(\sqrt{0^2 + 0^2}/1) = 0$$

$$(1, 0, 0)$$

- ϕ can be any real number here.

(d)

$$r = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2},$$

$$\phi = \arctan(1/0) = \pi/2,$$

$$\theta = \arctan(\sqrt{0^2 + 1^2}/1) = \pi/4$$

$$(\sqrt{2}, \pi/2, \pi/4)$$

(e)

$$r = \sqrt{0^2 + 0^2 + 0^2} = 0,$$

$$\phi = \arctan(0/0) = \text{undefined},$$

$$\theta = \arctan(\sqrt{0^2 + 0^2}/0) = \text{undefined}$$

$$(0, 0, 0)$$

- $r = 0$, ϕ and θ can be any real number here.

2.1

(a) Let

$$x = r\cos\phi, y = r\sin\phi, z = z$$

Jacobian Matrix:

$$J(r, \phi, z) = \begin{pmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = J \begin{pmatrix} dr \\ d\phi \\ dz \end{pmatrix}$$

Hence find expressions for \hat{r} and $\hat{\phi}$ in terms of \hat{x} and \hat{y} .

Since \hat{r} and $\hat{\phi}$ are unit vectors, let $r = 1$

$$T(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Or, considering

$$\begin{aligned}\hat{r} &= \cos\phi\hat{x} + \sin\phi\hat{y} \\ \hat{\phi} &= -\sin\phi\hat{x} + \cos\phi\hat{y} \\ v &= v_r\hat{r} + v_\phi\hat{\phi} + v_z\hat{z}\end{aligned}$$

$$\begin{aligned}v_x &= v \cdot \hat{x} = v_r \cos\phi - v_\phi \sin\phi \\ v_y &= v \cdot \hat{y} = v_r \sin\phi + v_\phi \cos\phi \\ v_z &= v \cdot \hat{z} = v_z\end{aligned}$$

In order to make

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = T(\phi) \begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}$$

$$T(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)

We've got that

$$T(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The (i, j)-cofactor is $C_{i,j} = (-1)^{i+j} M_{i,j}$, where $M_{i,j}$ is the (i, j)-minor.

$$C_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} \cos\phi & 0 \\ 0 & 1 \end{vmatrix} = \cos\phi$$

Similarly,

$$\begin{array}{lll} C_{12} = -\sin\phi & C_{13} = 0 \\ C_{21} = \sin\phi & C_{22} = \cos\phi & C_{23} = 0 \\ C_{31} = 0 & C_{32} = 0 & C_{33} = 1 \end{array}$$

Hence,

$$(T^*(\phi))^T = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The adjugate matrix of T is:

$$T^*(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det[T(\phi)] = 1$$

Based on Cramer's rule,

$$T^{-1}(\phi) = \frac{1}{\det} T^*(\phi)$$

Hence,

$$T^{-1}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \cos(\phi) &= \frac{x}{\sqrt{x^2 + y^2}} \\ \sin(\phi) &= \frac{y}{\sqrt{x^2 + y^2}} \end{aligned}$$

(d)

According to the result of part c, substitute $\cos\phi$ and $\sin\phi$:

$$\begin{aligned} T^{-1}(\phi) &= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ S(P) &= \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

2.2

In Question 2.1(d), we've got

$$S(P) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) $(-1, 0, 0)$

When $P = (0, -1, 0)$

$$S(P) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1}(P) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1}(P)P^T = (-1, 0, 0)^T$$

(b) $(1, 0, 0)$

When $P = (1, 0, 0)$

$$S^{-1}(P) = S(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1}(P)P^T = (1, 0, 0)^T$$

(c) $(1, 0, 0)$

When $P = (-1, 0, 0)$

$$S^{-1}(P) = S(P) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1}(P)P^T = (1, 0, 0)^T$$

(d) $(0, -\sqrt{2}, 0)$

When $P = (1, -1, 0)$

$$S(P) = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1}(P) = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1}(P)P^T = (0, -\sqrt{2}, 0)^T$$

(e) $(0, 0, 0)$

When $P = (0, 0, 0)$

Since,

$$\lim_{x \rightarrow 0, y=0} \frac{x}{\sqrt{x^2 + y^2}} = 1$$

$$\lim_{x=0, y \rightarrow 0} \frac{y}{\sqrt{x^2 + y^2}} = 1$$

We get $S(P)$ and $S^{-1}(P)$:

$$S(P) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} S^{-1}(P) = \begin{pmatrix} 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S^{-1}(P)P^T = (0, 0, 0)^T$$

3.1

(a)

```
x = -2:.1:2;
y = -2:.1:2;
[xx, yy] = meshgrid(x, y);

size(xx)
size(yy)
```

Which output is:

```
ans =

    41    41

ans =

    41    41
```

xx and yy are both 41×41 matrix.

(b)

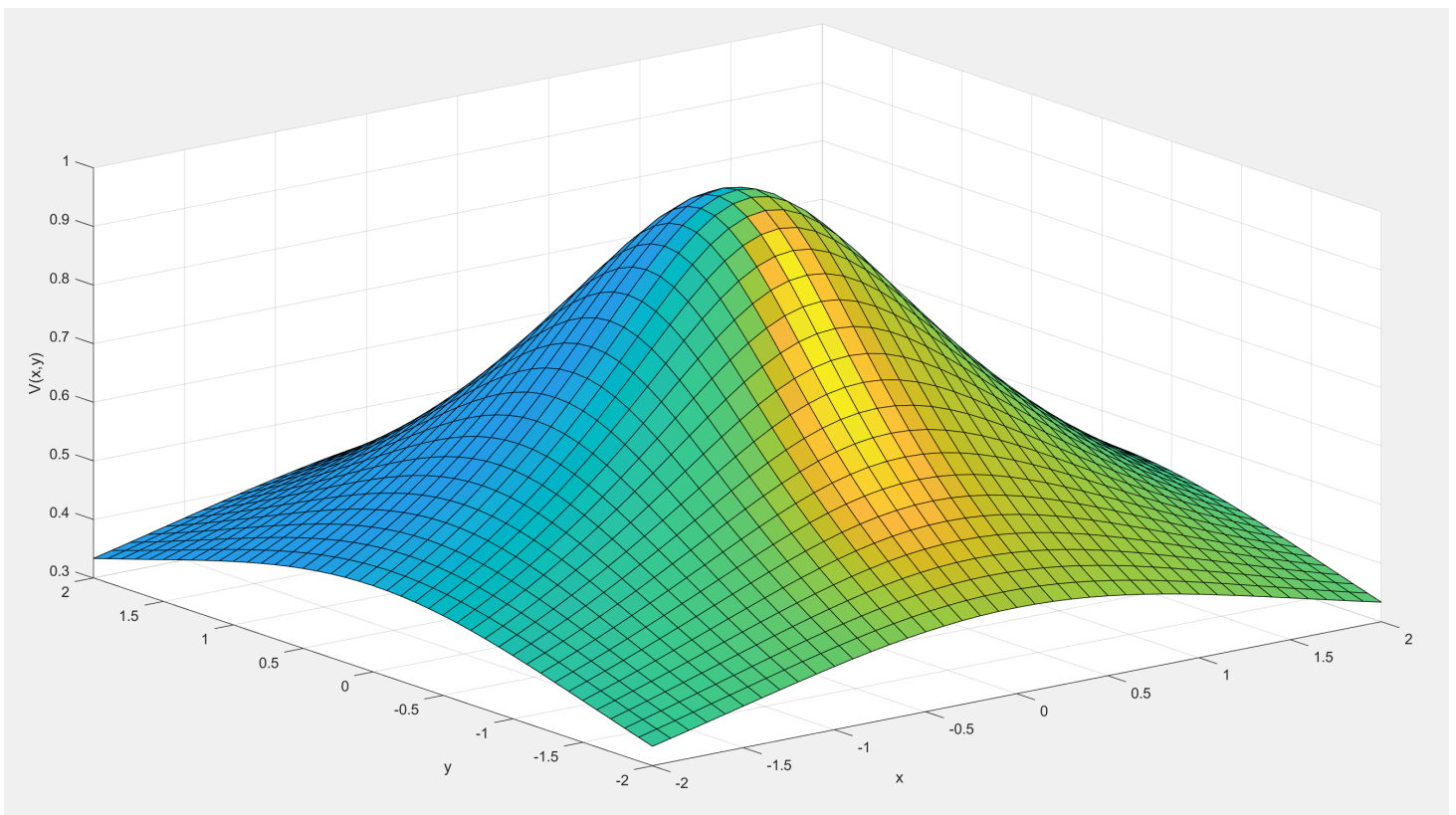
```

x = -2:.1:2;
y = -2:.1:2;
[xx, yy] = meshgrid(x, y);

size(xx)
size(yy)

zz = 1./sqrt(1 + xx.^2 + yy.^2);
figure(1);
surf1(xx, yy, zz);
xlabel('x');
ylabel('y');
zlabel('V(x,y)');
grid on;

```



Based on the picture above, it has been shown that the surface exhibit a maximum.

After checked the value "zz" in workspace, we get the maximum point is $(0, 0, 1)$.

For certain plane, origin can always be the point with the highest electrostatic potential. If a charge moves in any direction on its x-y plane, the electric field does positive work on it.

(c)

A circle.

$$V = \frac{1}{\sqrt{1+x^2+y^2}}$$

$$\sqrt{1+x^2+y^2} = \frac{1}{V}$$

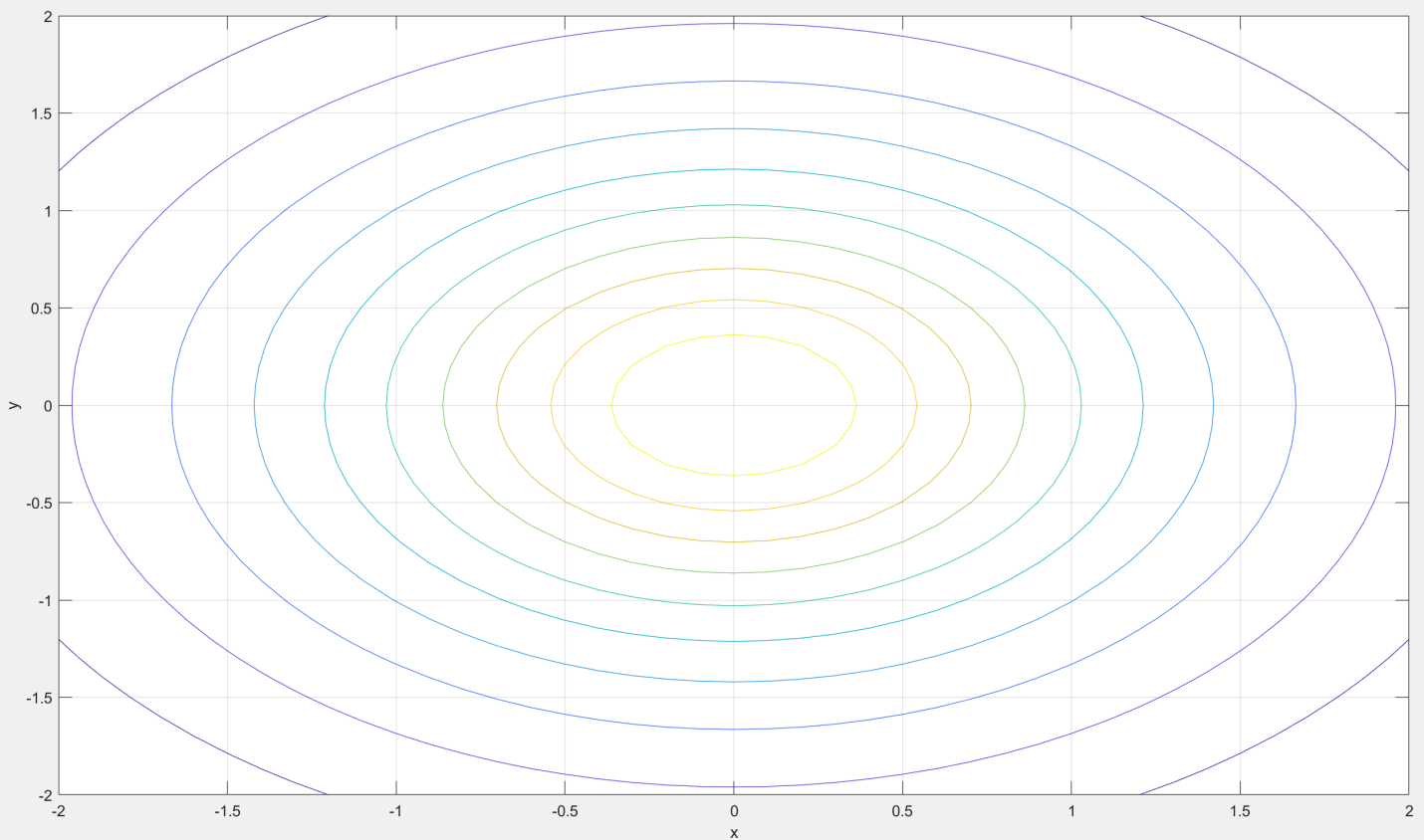
$$r = \sqrt{x^2+y^2} = \sqrt{\frac{1}{V^2} - 1}$$

Its radius is $\sqrt{\frac{1}{V^2} - 1}$

If $V = c > 1$, radius will be an imaginary number, which is impossible here. Hence, c will never be greater than 1.

(d)

```
figure(2);
contour(xx, yy, zz, 10);
xlabel('x');
ylabel('y');
grid on;
```

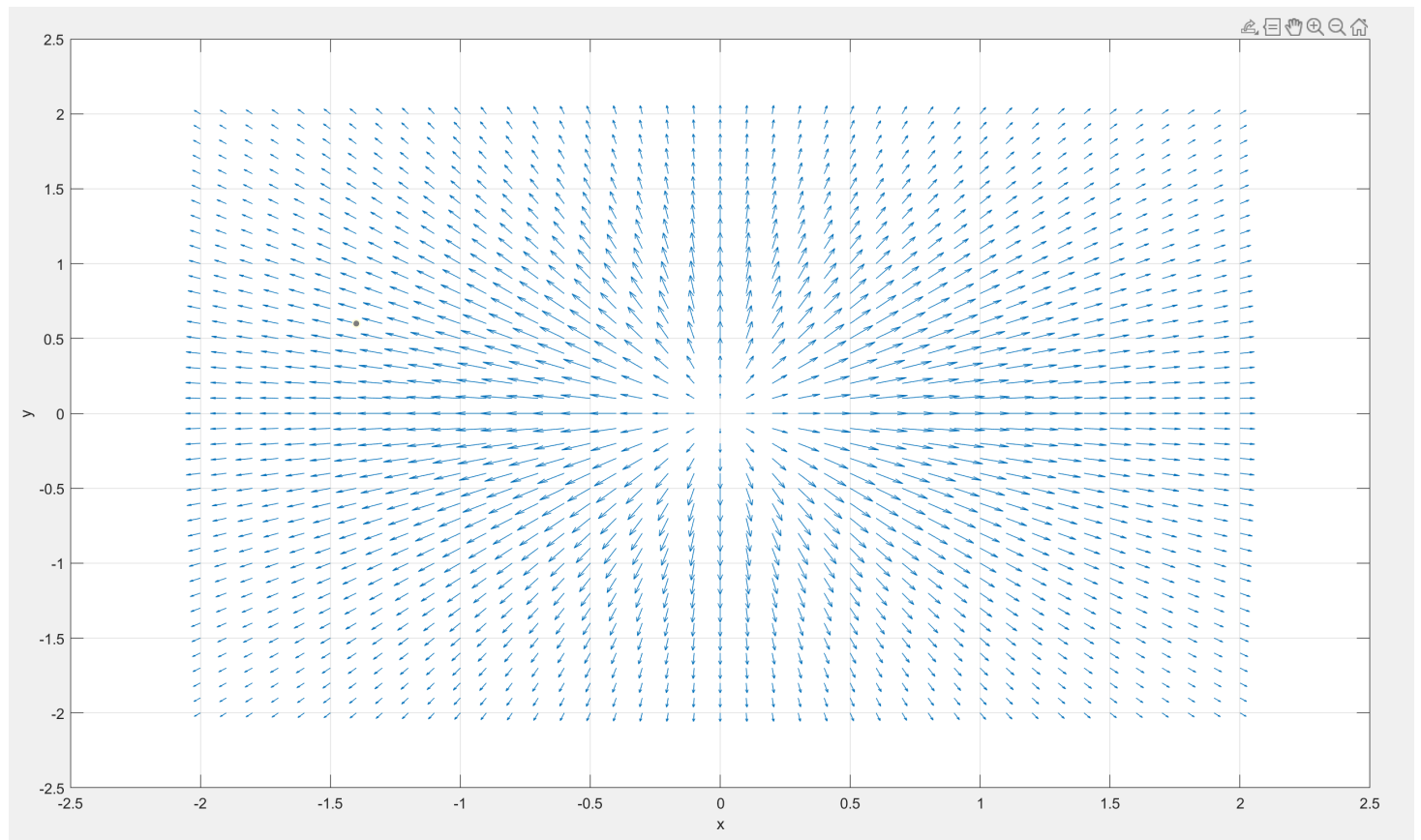


"10" means: Display 10 contour lines at automatically chosen levels (heights).

3.2

(a)

```
exx = xx./(1 + xx.^2 + yy.^2).^(3/2);  
eyy = yy./(1 + xx.^2 + yy.^2).^(3/2);  
  
figure(3);  
quiver(xx,yy,exx,eyy);  
xlabel('x');  
ylabel('y');  
grid on;
```



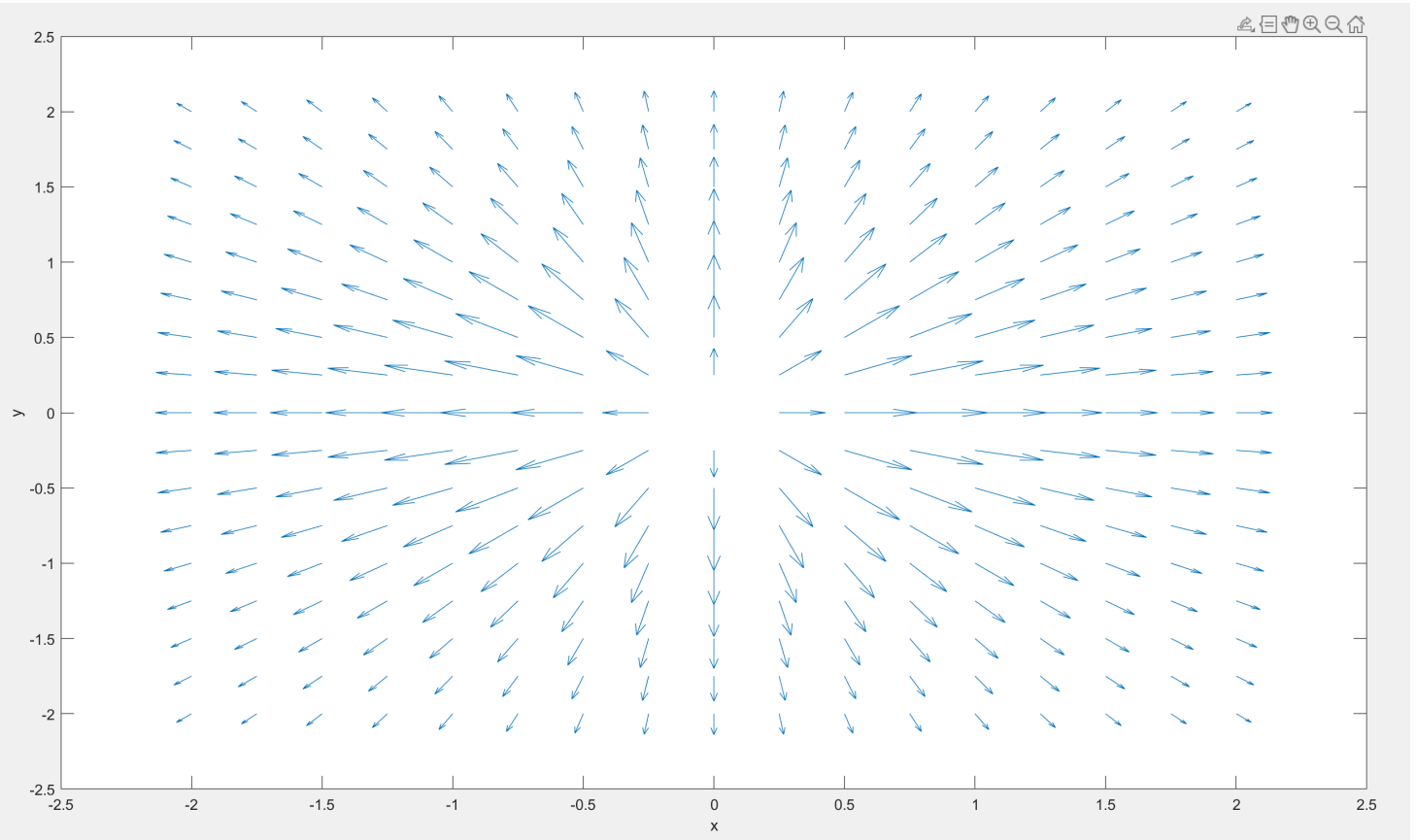
(b)

```

xnew = -2:.25:2;
ynew = xnew;
[xxnew, yynew] = meshgrid(xnew, ynew);
exxnew = xxnew./(1 + xxnew.^2 + yynew.^2).^(3/2);
eeynew = yynew./(1 + xxnew.^2 + yynew.^2).^(3/2);

figure(4);
quiver(xxnew, yynew, exxnew, eeynew);
hold on;

```



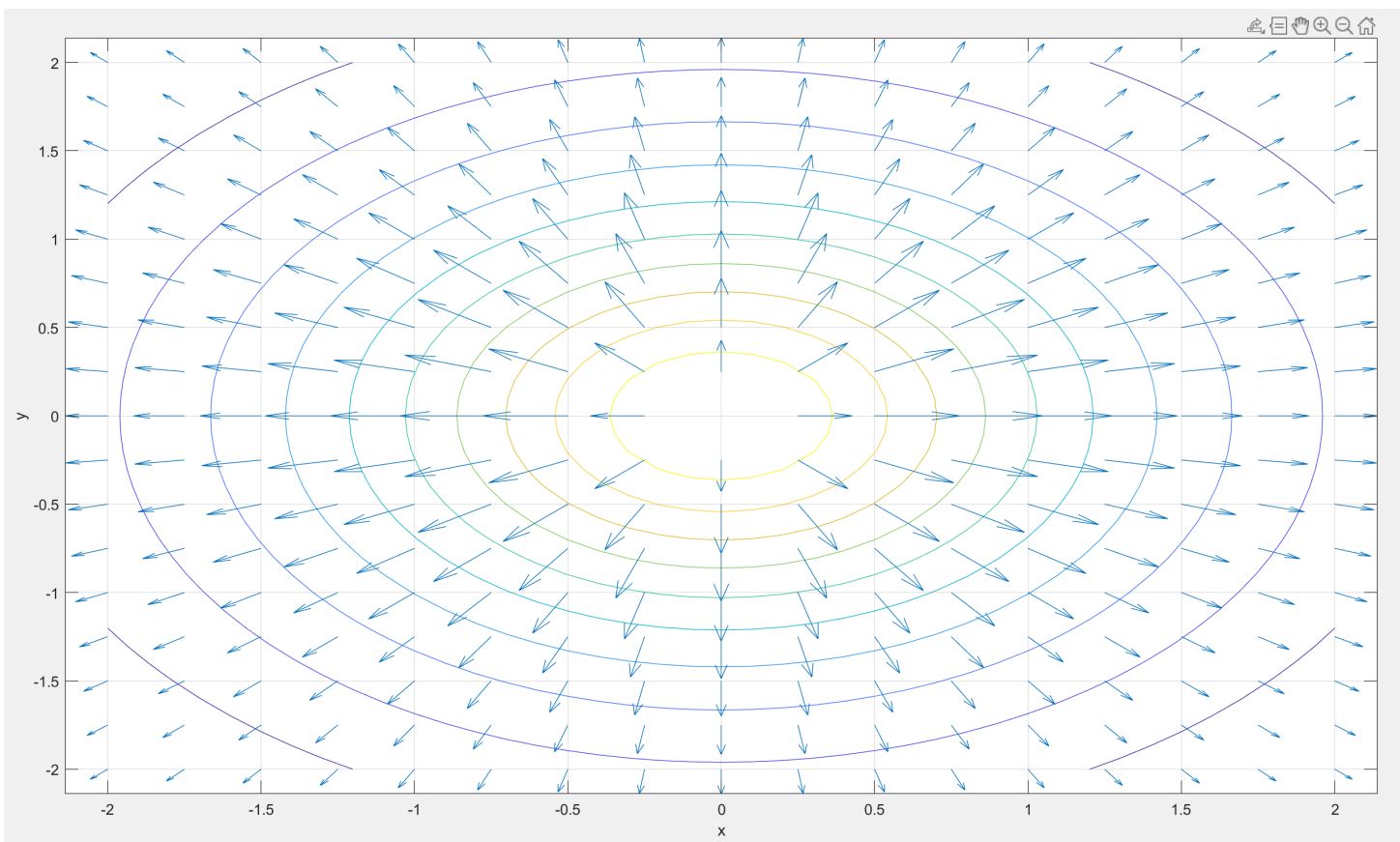
- More "sparse": xx and yy are 41×41 matrix (interval 0.1), while $xxnew$ and $yynew$ are 17×17 matrix (interval 0.25).

(c)

```

figure(2);
hold on;
quiver(xxnew, yynew, exxnew, eeynew);

```



Perpendicular to each other.

3.3

(a)

The contour V is a curve, its tangents is $T(s) = \dot{x}(s)\hat{x} + \dot{y}(s)\hat{y}$

Based on the chain rule, which is:

$$\frac{d}{ds} = \frac{d}{dx} \frac{dx}{ds}$$

Since $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j$, the gradient of V is

$$\nabla V = \frac{\partial V}{\partial x}i + \frac{\partial V}{\partial y}j$$

$$\nabla V \cdot T(s) = \nabla V \cdot \frac{d}{ds}(x(s), y(s)) = \frac{d}{ds}V(x(s), y(s))$$

Since V is constant along the contour, its derivative is zero.

$$\nabla V \cdot T(s) = 0$$

In Task 3.2, we got $E(x, y) = -\nabla V(x, y)$

Hence, we get the fomular 11.

According to the electric field (8) and (9),

$$E(x(s), y(s)) \approx \frac{x(s)\hat{x} + y(s)\hat{y}}{(1 + x^2(s) + y^2(s))^{\frac{3}{2}}}$$

In formula (12),

$$T(s) = \hat{T}(x(s), y(s)) \approx \frac{y(s)\hat{x} - x(s)\hat{y}}{\sqrt{1 + x^2(s) + y^2(s)}}$$

$$\vec{E} \cdot \vec{T} = \frac{x(s)y(s) - y(s)x(s)}{(1 + x^2(s) + y^2(s))^2} = 0$$

The dot product of vectors is 0 means these vectors are perpendicular.

For $\forall (x(s), y(s)) \in \mathbb{R}^2, s \in \mathbb{R}, \vec{E} \cdot \vec{T} = 0$, which means $T(s)$ must be everywhere perpendicular to $E(x(s), y(s))$.

(b)

```

close all
clear
clc

x = -2:.1:2;
y = -2:.1:2;
[xx, yy] = meshgrid(x, y);

zz = 1./sqrt(1 + xx.^2 + yy.^2);

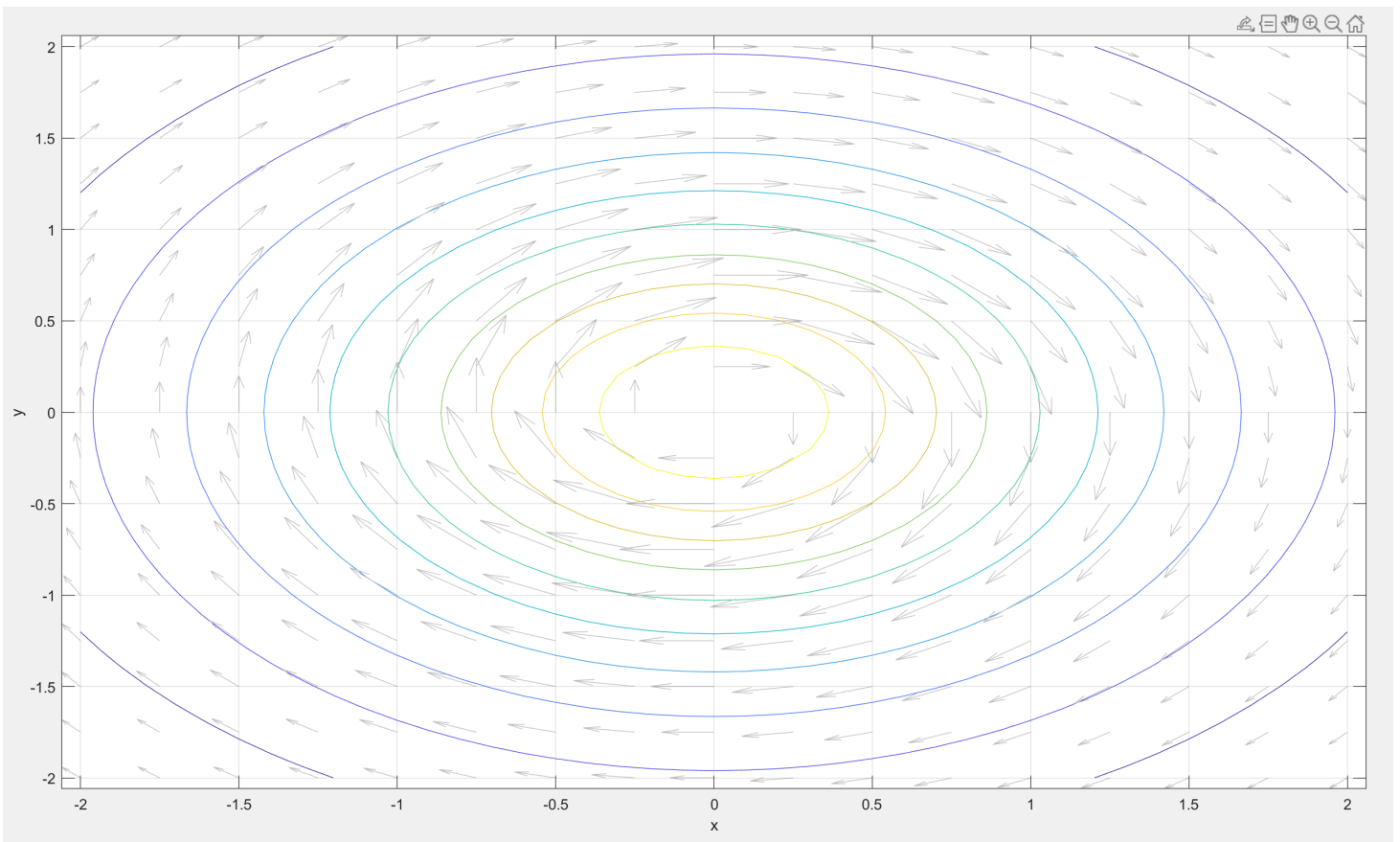
figure(4);
contour(xx, yy, zz, 10);
xlabel('x');
ylabel('y');
grid on;

exx = xx./(1 + xx.^2 + yy.^2).^(3/2);
eyy = yy./(1 + xx.^2 + yy.^2).^(3/2);

xnew = -2:.25:2;
ynew = xnew;
[xxnew, yynew] = meshgrid(xnew, ynew);
exxnew = yynew./(1 + xxnew.^2 + yynew.^2).^(3/2);
eyynew = -xxnew./(1 + xxnew.^2 + yynew.^2).^(3/2);

figure(4);
hold on;
quiver(xxnew, yynew, exxnew, eyynew, "Color", "#C0C0C0");
hold off;

```



(c)

We've already found that the electric field $E(x, y) = -\nabla V(x, y)$, this figure shows:

- \hat{T} is the tangent vector of V , and it is perpendicular to the electric field E .
- Also, the electric field E is perpendicular to the contours V .

This means the electric field is not divergence, its curl will be 0.

$$\nabla \times E = 0$$

It is a special case of Maxwell's Equations, in this case $E = -\text{grad}(V)$.