ELEN30011 Electrical Device Modelling

Workshop 2 - Vector calculus

Vector calculus is the language of Maxwell's equations, the fundamental set of four equations that model the interaction of electric and magnetic fields, charge, and current, that together underpin electrical engineering. Maxwell's equations provide the basis for physical models of electrical devices including resistors, capacitors, and inductors. They also underpin modelling of semiconductor devices such as diodes and transistors.

Maxwell's equations can be equivalently expressed in either differential and integral form. This workshop covers the operations of differentiation and integration needed for their application.

1 Differentiation

Essential to understanding and applying Maxwell's equations in differential form is a working knowledge of the basic differential operators used in vector calculus, namely, gradient (grad), divergence (div), and curl (curl). Given a scalar-valued function f and a vector field $\mathbf{F}$, i.e.

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad \mathbf{F} = F_x\,\hat{\boldsymbol{x}} + F_y\,\hat{\boldsymbol{y}} + F_z\,\hat{\boldsymbol{z}} : \mathbb{R}^3 \to \mathbb{R}^3, \tag{1}$$

these operators are defined in rectangular coordinates by

$$\mathsf{grad}\, f \doteq \frac{\partial f}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{z}}, \tag{2}$$

$$\mathsf{div}\,\mathbf{F} \doteq \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\,, \tag{3}$$

$$\mathsf{curl}\,\mathbf{F} \doteq \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}}\,. \tag{4}$$

Tasks

1.1. The gradient operator grad takes a scalar-valued function f as per (1) and maps it to a vector field in $\mathbb{R}^3$. That is, $\mathsf{grad}\, f : \mathbb{R}^3 \to \mathbb{R}^3$. The divergence operator div operates on a vector field $\mathbf{F}$ as per (1). Determine whether div yields a scalar-valued function or a vector field. Repeat for the curl operator.

1.2. The gradient operator grad is often written in coordinate form as

$$\mathsf{grad} = \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\,,$$

so that (for example),

$$\begin{aligned}\nabla f &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right) f = \hat{\boldsymbol{x}}\frac{\partial f}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial f}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial f}{\partial z}\,.\end{aligned}$$

Similarly, by inspection of (3), the divergence of a vector field $\mathbf{F}$ may be written formally as the scalar product $\nabla \cdot \mathbf{F}$. By inspection of (4), write down a corresponding form using this notation for $\mathsf{curl}\,\mathbf{F}$.

1.3. Compute $\mathsf{grad}\, f$ of each of the following scalar-valued functions f:

   (a) $f(x, y, z) = 5$.
   (b) $f(x, y, z) = x + y\,z$.
   (c) $f(x, y, z) = \frac{1}{2}\,x^2 + \frac{1}{2}\,y^2 - \frac{1}{2}\,z^2 \cos y$.
   (d) $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

1.4. Compute $\mathsf{div}\,\mathbf{F}$ and $\mathsf{curl}\,\mathbf{F}$ of each of the following vector fields $\mathbf{F}$:

   (a) $\mathbf{F}(x, y, z) = 3\,\hat{\boldsymbol{x}}$.
   (b) $\mathbf{F}(x, y, z) = -x\,\hat{\boldsymbol{x}} + z\,\hat{\boldsymbol{z}}$.
   (c) $\mathbf{F}(x, y, z) = z\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{z}}$.
   (d) $\mathbf{F}(x, y, z) = \mathsf{grad}\, f(x, y, z)$, for any twice continuously differentiable scalar-valued function f.

1.5. Using the MATLAB command `quiver3`, the vector fields $\mathbf{F}$ of 1.4(b)-(c) may be visualized as per Figures 1 and 2. Do either of the fields appear to exhibit non-zero circulation? Are there any sources or sinks evident? Discuss your observations in view of your specific computations of div and curl above.

   [Note: The *circulation* of $\mathbf{F}$ can be quantified via (5) below. Associating $\mathbf{F}$ with a force field, the circulation of $\mathbf{F}$ around a closed path quantifies whether work must be done in traversing that closed path in the presence of $\mathbf{F}$.]

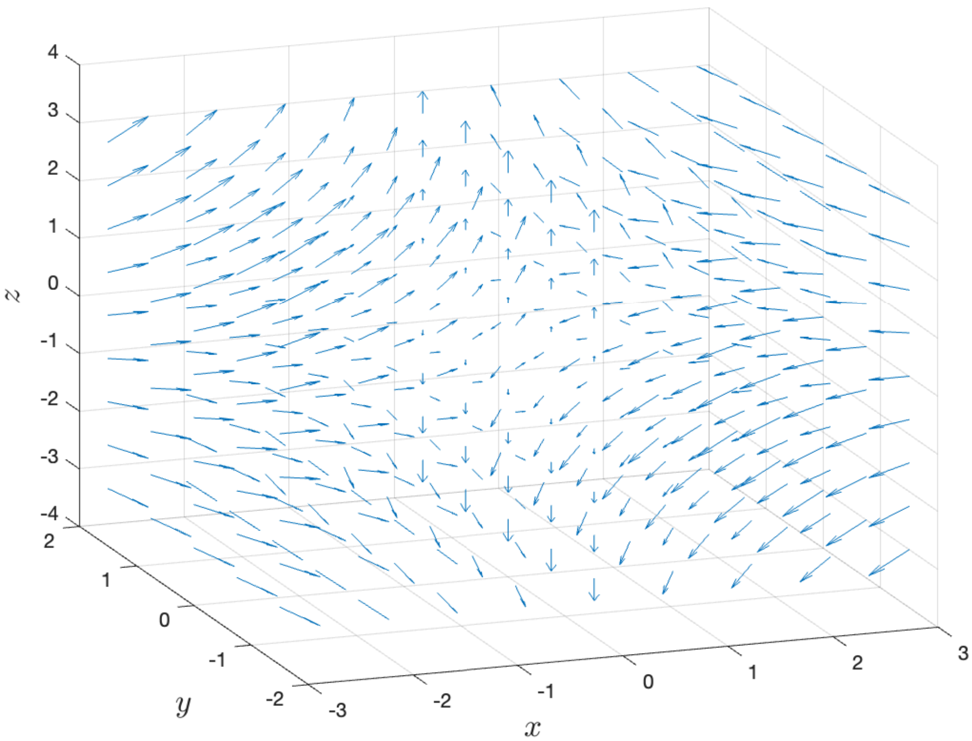

Figure 1: Visualization of the field **F** in 1.4(b) [Section 1].

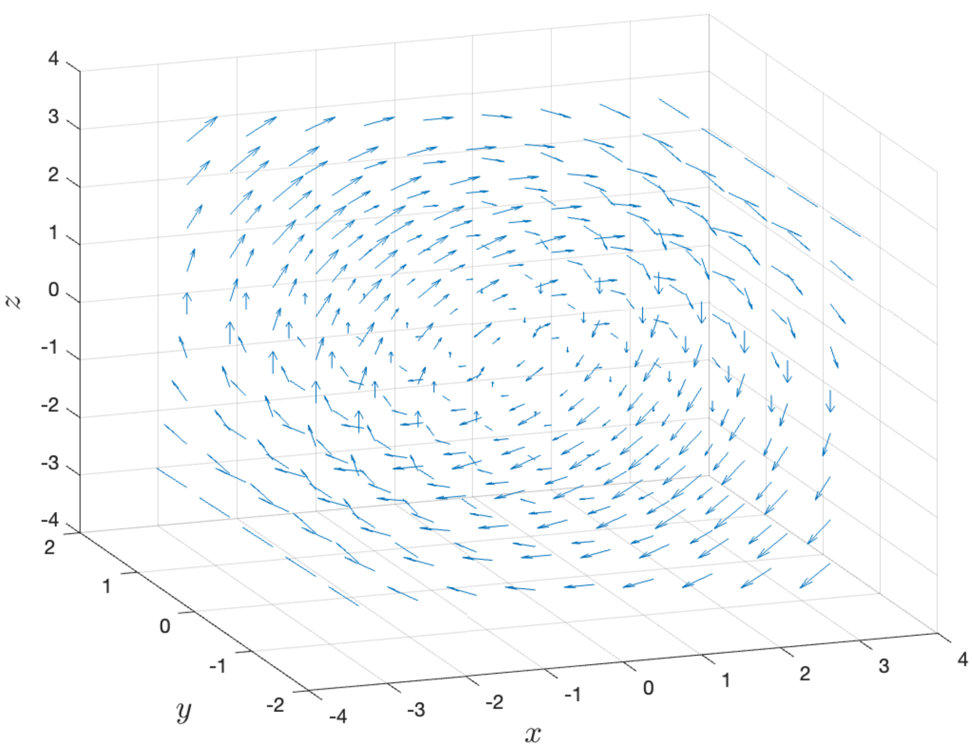

Figure 2: Visualization of the field **F** in 1.4(c) [Section 1].

2 Integration

Essential to understanding and applying Maxwell's equations in integral form is a working knowledge of line, surface, and volume integration. For example, line and surface integrals arise naturally in the derivation of physical models for inductance. Similarly, surface and volume integrals are required in modelling capacitance. In particular, line integrals are required in evaluation circulation and potential difference, whilst surface integrals are required for computing flux. Volume integrals are required when a quantity (such as charge, or resistance) is spatially distributed in a non-uniform way or over a region of space with non-trivial geometry. Integrals of these types are as follows:

$$V = -\int_\ell \mathbf{E}\cdot d\mathbf{l}\,, \quad C = \oint_\ell \mathbf{F}\cdot d\mathbf{l}, \tag{5}$$

$$\Phi = \iint_A \mathbf{F}\cdot d\mathbf{A}, \tag{6}$$

$$Q = \iiint_v \rho\, dv. \tag{7}$$

In the first line integral in (5), $\mathbf{E}$ is associated with an electric field, ℓ with some path in space, and V with the work per unit charge required to move a charge along the path ℓ in the presence of $\mathbf{E}$. By inspection, V is obtained by integrating along the path ℓ the component of the electric field $\mathbf{E}$ that is tangential to that path.

The second line integral in (5) defines the *circulation* C of field $\mathbf{F}$ around a *closed* path ℓ. The circulation C is a scalar by definition. It may be positive, negative, or zero for any given ℓ. Field $\mathbf{F}$ is conservative if its circulation is zero for all closed paths.

2.1 Line integration

An electric field permeating $\mathbb{R}^3$ is given by

$$\mathbf{E}(x,y,z) = -y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}} \qquad \mathsf{V\,m^{-1}} \tag{8}$$

for all $(x,y,z)\in\mathbb{R}^3$. Two points are located in rectangular coordinates by the vectors

$$\mathbf{P}_1 = \hat{\boldsymbol{x}}\,, \quad \mathbf{P}_2 = \frac{1}{\sqrt{2}}\,(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})\,.$$

A pair of paths, both starting at $\mathbf{P}_1$ and both ending at $\mathbf{P}_2$, are considered. Path ℓ_1 is a straight line segment, while path ℓ_2 is a circular arc of radius 1. Vector fields defining all points along these paths are given by

$$\ell_1 \quad : \quad \mathbf{l}_1(s) \doteq (1-s)\,\hat{\boldsymbol{x}} + \frac{s}{\sqrt{2}}\,(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}})\,, \tag{9}$$

$$\ell_2 \quad : \quad \mathbf{l}_2(s) \doteq \cos\left(\frac{\pi\, s}{4}\right)\,\hat{\boldsymbol{x}} + \sin\left(\frac{\pi\, s}{4}\right)\,\hat{\boldsymbol{y}}\,, \tag{10}$$

for all $s\in[0,1]$.

Tasks

2.1.1. Confirm that the vector fields (9)-(10) describe the specified path geometries. i.e. ℓ_1 is a straight line segment, ℓ_2 is circular arc of radius 1, and both traverse from $\mathbf{P}_1$ to $\mathbf{P}_2$ by increasing s from $s = 0$ to $s = 1$.

2.1.2. Show that the electric field (8) evaluated along these two paths (9)-(10) is

$$\mathbf{E}(\mathbf{l}_1(s)) \doteq -\frac{s}{\sqrt{2}}\,\hat{\boldsymbol{x}} - \left(1 - s + \frac{s}{\sqrt{2}}\right)\,\hat{\boldsymbol{y}}\,, \tag{11}$$

$$\mathbf{E}(\mathbf{l}_2(s)) = -\sin\left(\frac{\pi\,s}{4}\right)\,\hat{\boldsymbol{x}} - \cos\left(\frac{\pi\,s}{4}\right)\,\hat{\boldsymbol{y}}\,, \tag{12}$$

for all $s \in [0, 1]$.

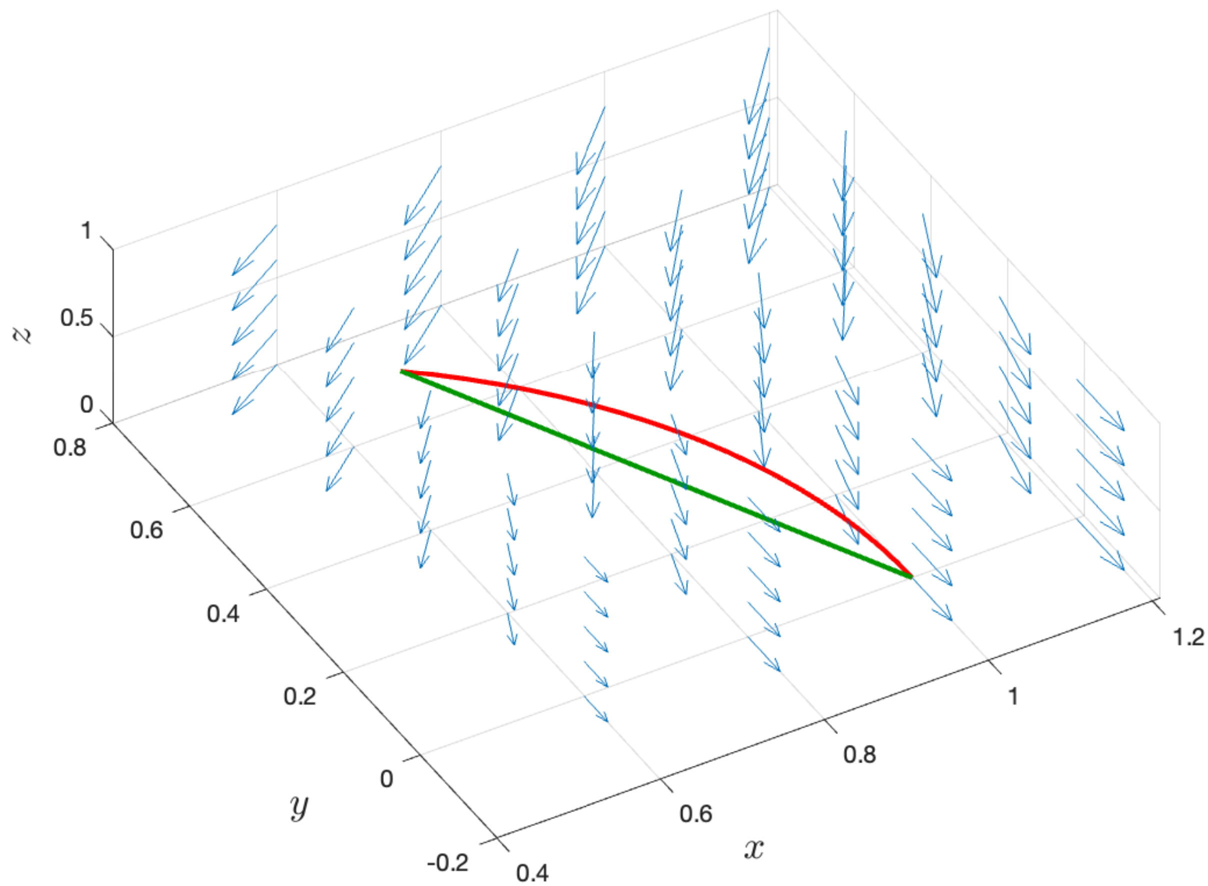

Figure 3: Visualization of electric field $\mathbf{E}$ and paths ℓ_1 and ℓ_2 [Section 2.1].

2.1.3. Using MATLAB, visualize the electric field and both paths via the following commands:

```
≫ x = 0.6:.2:1.2; y = 0:.2:.8; z = 0:.2:.8;
≫ [xx,yy,zz] = meshgrid(x,y,z);
≫ Exx = -yy; Eyy = -xx; Ezz = 0*xx;
≫ figure(2);
≫ quiver3(xx,yy,zz,Exx,Eyy,Ezz);
≫ grid on; hold on;
≫ xhat = [1;0;0]; yhat = [0;1;0];
≫ s = 0:.05:1;
≫ L1 = (1 - s).*xhat + s.*(xhat + yhat)/sqrt(2);
≫ L2 = ??
≫ plot3(L1(1,:),L1(2,:),L1(3,:),'Color',[0 .6 0],'LineWidth',2);
≫ plot3(L2(1,:),L2(2,:),L2(3,:),'Color',[1 0 0],'LineWidth',2);
```

The plot obtained should be similar to that of Figure 3. Label the path ends appropriately as either $\mathbf{P}_1$ or $\mathbf{P}_2$.

2.1.4. Using (9)-(12), by direction computation, show that the line integrals evaluate to

$$V_{\ell_1} = -\int_{\ell_1} \mathbf{E} \cdot d\mathbf{l}_1 = -\int_0^1 \mathbf{E}(\mathbf{l}_1(s)) \cdot \frac{d\mathbf{l}_1}{ds}(s)\, ds = \frac{1}{2}, \tag{13}$$

$$V_{\ell_2} = -\int_{\ell_2} \mathbf{E} \cdot d\mathbf{l}_1 = -\int_0^1 \mathbf{E}(\mathbf{l}_2(s)) \cdot \frac{d\mathbf{l}_2}{ds}(s)\, ds = \frac{1}{2}. \tag{14}$$

In terms of work, force, and charge, what do these computed quantities V_{ℓ_1} and V_{ℓ_2} correspond to physically?

2.1.5. Using (8) only, show that the scalar valued function $U : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by

$$U(x, y, z) \doteq c + x\, y, \qquad c \doteq \text{constant}, \tag{15}$$

satisfies $\mathbf{E}(x, y, z) \equiv -\nabla U(x, y, z)$ for all $(x, y, z) \in \mathbb{R}^3$. Using (15), called a *potential*, evaluate the *potential difference* $U(\mathbf{P}_2) - U(\mathbf{P}_1)$, and compare this to your evaluation of (13)-(14) above. Why are all three equal?

2.2 Surface integration

A magnetic field is given by

$$\mathbf{H}(x, y, z) = -z\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}}, \tag{16}$$

An open prism A is rests with its square open base located in the x-y plane, as shown in Figure 4, with its angled face A_{angle} contained within the $z = y$ plane, left-hand and right-hand triangular faces A_{left} and A_{right} parallel to the $y - z$ plane, and front square face A_{front} located in the $y = 2$ plane. The perimeter of the square opening, illustrated in blue, has side length 2 and describes a square path ℓ with orientation indicated by the blue arrow shown.

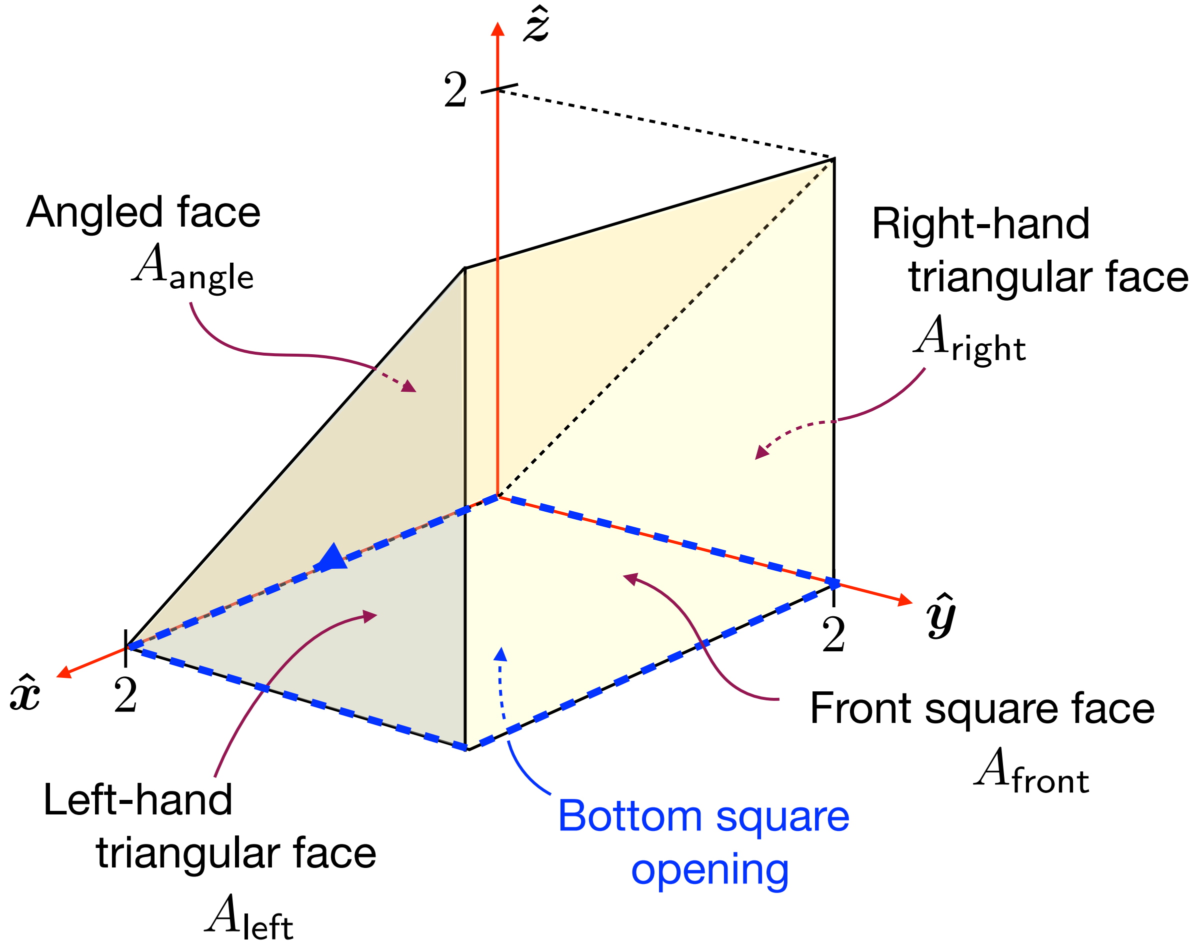

Figure 4: Open prism A [Section 2.2].

The objective is to compute the flux of $\mathsf{curl}\,\mathbf{H}$ up and out through the surface of the open prism, and to compare that computation to circulation of $\mathbf{H}$ around the perimeter ℓ of A.

The flux of $\mathsf{curl}\,\mathbf{H}$ through the open prism A is the surface integral

$$\iint_A \mathsf{curl}\,\mathbf{H} \cdot d\mathbf{A}\,, \tag{17}$$

where $d\mathbf{A}$ is the outward pointing unit normal scaled by the incremental cross-sectional area. As the open prism A consists of 4 separate faces, the surface integral of (17) may be decomposed into a sum of 4 component surface integrals, i.e.

$$\begin{aligned}\iint_{A} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A} &= \iint_{A_{\mathsf{left}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{left}} + \iint_{A_{\mathsf{right}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{right}} \\ &\quad + \iint_{A_{\mathsf{angle}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{angle}} + \iint_{A_{\mathsf{front}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{front}}. \end{aligned} \tag{18}$$

Tasks

2.2.1. Show that $\mathsf{curl}\,\mathbf{H} = -\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$.

2.2.2. Argue why the outward unit pointing normals for the left-hand face A_{left} and the right-hand face A_{right} can be written respectively as $\hat{\boldsymbol{x}}$ and $-\hat{\boldsymbol{x}}$, so that the corresponding incremental area vectors are

$$d\mathbf{A}_{\mathsf{left}} = dA_{\mathsf{left}}\,\hat{\boldsymbol{x}}, \qquad d\mathbf{A}_{\mathsf{right}} = -dA_{\mathsf{right}}\,\hat{\boldsymbol{x}}, \qquad dA_{\mathsf{left}},\, dA_{\mathsf{right}} > 0.$$

Subsequently show that

$$\iint_{A_{\mathsf{left}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{left}} = 0 = \iint_{A_{\mathsf{right}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{right}}.$$

2.2.3. In order to compute the flux through the angled face, first show that all points on the angled face A_{angle} can be described by the vector field

$$\mathbf{l}_{\mathsf{angle}}(s,t) \doteq 2\,s\,\hat{\boldsymbol{x}} + 2\,t\,\hat{\boldsymbol{y}} + 2\,t\,\hat{\boldsymbol{z}}, \qquad s,t \in [0,1].$$

By computing suitable partial derivatives of this vector field, and evaluating a cross-product, further show that

$$d\mathbf{A}_{\mathsf{angle}} = (-4\,\hat{\boldsymbol{y}} + 4\,\hat{\boldsymbol{z}})\,ds\,dt.$$

Finally, by evaluating an ensuing definite double integral, show that

$$\iint_{A_{\mathsf{angle}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{angle}} = +8.$$

2.2.4. An analogous calculation for the front face A_{front} yields that

$$\iint_{A_{\mathsf{front}}} \mathsf{curl}\,\mathbf{H}\cdot d\mathbf{A}_{\mathsf{angle}} = -4.$$

Using this, your answers above, and (18), find the total flux of $\mathsf{curl}\,\mathbf{H}$ out through the open prism A.

2.3 An application of Stokes' theorem

The circulation of $\mathbf{H}$ around the perimeter ℓ of the open prism in Figure 4 is given by the second line integral in (5). As ℓ is a square, this perimeter (path) may be divided into 4 sub-paths, with one sub-path corresponding to each side. Hence, the circulation can be expressed as the sum of 4 component line integrals, with

$$\oint_{\ell} \mathbf{H} \cdot d\mathbf{l} = \int_{\ell_1} \mathbf{H} \cdot d\mathbf{l}_1 + \int_{\ell_2} \mathbf{H} \cdot d\mathbf{l}_2 + \int_{\ell_3} \mathbf{H} \cdot d\mathbf{l}_3 + \int_{\ell_4} \mathbf{H} \cdot d\mathbf{l}_4, \tag{19}$$

in which, for example,

$$\int_{\ell_1} \mathbf{H} \cdot d\mathbf{l}_1 = \int_0^1 \mathbf{H}_1(s) \cdot \frac{d\mathbf{l}_1(s)}{ds}\, ds\,,$$

and $\mathbf{H}_1(s) \doteq \mathbf{H}(\mathbf{l}_1(s))$ is the magnetic field (16) along the path ℓ_1. Similarly define $\mathbf{H}_{2,3,4}$.

2.3.1. Show that points on these 4 sub-paths can be described via the respective vector fields

$$\begin{aligned} \ell_1 &: \quad \mathbf{l}_1(s) \doteq 2\, s\, \hat{\boldsymbol{x}}, \\ \ell_2 &: \quad \mathbf{l}_2(s) \doteq 2\, \hat{\boldsymbol{x}} + 2\, s\, \hat{\boldsymbol{y}}, \\ \ell_3 &: \quad \mathbf{l}_3(s) \doteq 2\,(1-s)\, \hat{\boldsymbol{x}} + 2\, \hat{\boldsymbol{y}}, \\ \ell_4 &: \quad \mathbf{l}_4(s) \doteq 2\,(1-s)\, \hat{\boldsymbol{y}}, \end{aligned}$$

in which $s \in [0, 1]$ on each sub-path. Be sure to confirm that the orientation of the sub-paths match the direction indicated by the blue arrow in Figure 4.

2.3.2. Show that

$$\begin{aligned} \mathbf{H}_1(s) &= 2\, s\, \hat{\boldsymbol{y}}, \\ \mathbf{H}_2(s) &= 2\, \hat{\boldsymbol{y}}, \\ \mathbf{H}_3(s) &= 2\,(1-s)\, \hat{\boldsymbol{y}}, \end{aligned}$$

and find the corresponding expression for the magnetic field $\mathbf{H}_4(s)$ along sub-path ℓ_4.

2.3.3. Using your answers above, evaluate the four line integrals in (19), i.e. find

$$\int_{\ell_1} \mathbf{H} \cdot d\mathbf{l}_1, \quad \int_{\ell_2} \mathbf{H} \cdot d\mathbf{l}_2, \quad \int_{\ell_3} \mathbf{H} \cdot d\mathbf{l}_3, \quad \int_{\ell_4} \mathbf{H} \cdot d\mathbf{l}_4,$$

and subsequently compute the circulation (19) of $\mathbf{H}$ around the open prism perimeter ℓ.

2.3.4. Compare your computed curl flux (18) in 2.2.4 with your computed circulation (19) in 2.2.3. Verify that Stokes' theorem holds for the field (16) permeating through the configuration of Figure 4.

2.4 Volume integration

A cubic volume $V \doteq [0,1] \times [0,1] \times [0,1] \subset \mathbb{R}^3$ of space has sides of length $1\,\mathsf{m}$, with one corner located at the origin. It contains a charge distribution with density $\rho : \mathbb{R}^3 \to \mathbb{R}$ given by

$$\rho(x,y,z) \doteq (\pi\, z)^2 \, \sin(\pi x) \, \cos\left(\frac{\pi\, y}{2}\right) \qquad \mathsf{C\,m^{-3}} \tag{20}$$

for all $(x,y,z) \in V$. The total charge Q contained within the volume V is determined by the definite volume integral

$$Q \doteq \iiint_V \rho \, dv = \int_0^1 \int_0^1 \int_0^1 \rho(x,y,z)\, dx\, dy\, dz \qquad \mathsf{C}. \tag{21}$$

Tasks

2.4.1. Determine the total charge Q in the cubic volume V by explicitly evaluating the definite volume integral (21).

2.4.2. Where the volume V is constructed from a material with constant (electrical) *permittivity* ϵ, Gauss' law of electricity (one of Maxwell's equations) states that the electric field $\mathbf{E}$ due to the charge density ρ defined in (20) must satisfy

$$\mathsf{div}\, \mathbf{E}(x,y,z) = \frac{\rho(x,y,z)}{\epsilon}\,, \tag{22}$$

for all $(x,y,z) \in V$. Given (21) and (22), what does Gauss' divergence theorem tell us about the total flux of the electric field $\mathbf{E}$ out through the surface of the volume V?