ELEN30011 Electrical Device Modelling

Workshop 3 - Maxwell's Equations

Maxwell's equations describe a fundamental field-based model that describes interactions between charge particles. This model assumes the existence of force fields (electric and magnetic) that facilitate these interactions. The four equations are empirical in nature (i.e. based on compiled experimental evidence), and essentially describe the forces experienced by charged particles due to other charged particles, both stationary and moving. Action at long distances is possible through the admissibility (in the model) of propagating electromagnetic fields, which form the basis of radio communications for example. (There, a moving electron in the transmitter antenna induces motion in an electron in the receiving antenna, mediated by a propagating electromagnetic field.) Maxwell's equations accurately predict the behaviour of "classical" charge particle interactions (i.e. above the quantum scale, and below relativistic speeds). These equations have been in use for over one hundred years, and form the physical basis for models of electrical devices such as resistors, capacitors, inductors, transformers, etc.

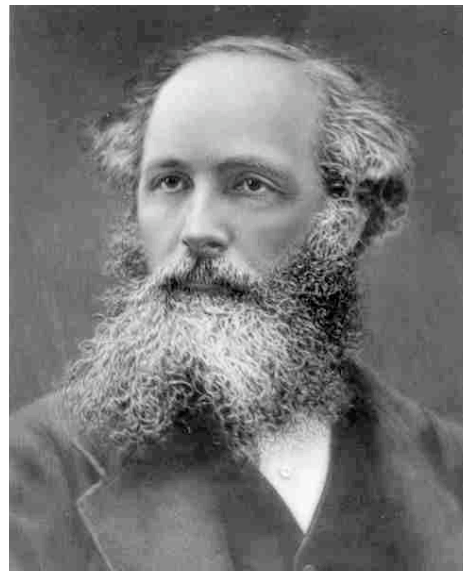

Figure 1: James Clerk Maxwell, 1831 - 1879. Courtesy Cavendish Laboratory, University of Cambridge.

In differential form, the four Maxwell's equations may be written as follows:

$$\text{Faraday's Law} \qquad \mathsf{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\text{Gauss' Law of Electricity} \qquad \mathsf{div}\,\mathbf{D} = \rho \tag{2}$$

$$\text{Ampere's Law} \qquad \mathsf{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{3}$$

$$\text{Gauss' Law of Magnetism} \qquad \mathsf{div}\,\mathbf{B} = 0 \tag{4}$$

There, $\mathbf{E}$ and $\mathbf{H}$ denote the electric and magnetic (vector) fields respectively, whilst $\mathbf{D}$ and $\mathbf{B}$ denote the electric and magnetic flux densities respectively. The fields and flux densities are related explicitly by $\mathbf{D} = \epsilon\,\mathbf{E}$ and $\mathbf{B} = \mu\,\mathbf{H}$, where ϵ and μ denote the permittivity and permeability of the material in which the fields and fluxes are present. $\mathbf{J}$ is the current density due to the motion of any charged particles locally.

The four equations above may be expressed in integral form via application of Gauss' Divergence Theorem and Stokes' Theorem, two famous results in vector calculus. Explicitly the equations may be written as follows:

$$\oint_{\ell(A)} \mathbf{E} \cdot d\mathbf{l} = -\iint_{A(\ell)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} \tag{5}$$

$$\iint_{A(v)} \mathbf{D} \cdot d\mathbf{A} = \iiint_{v(A)} \rho \, dv \tag{6}$$

$$\oint_{\ell(A)} \mathbf{H} \cdot d\mathbf{l} = \iint_{A(\ell)} \mathbf{J} \cdot d\mathbf{A} + \iint_{A(\ell)} \frac{d\mathbf{D}}{dt} \cdot d\mathbf{A} \tag{7}$$

$$\iint \mathbf{B} \cdot d\mathbf{A} = 0 \tag{8}$$

There, $A(\ell)$ denotes a cross-sectional area enclosed by some path $\ell(A)$ in three dimensional space, $\mathbb{R}^3$. Similarly, $v(A)$ denotes the volume of some closed surface with surface area $A(v)$.

Respectively, the four equations are known historically by their discoverers.

Maxwell's equations may be applied directly to the computation of the electric and magnetic fields associated with charge distributions and currents in materials. The tasks that follow concern examples of such computations.

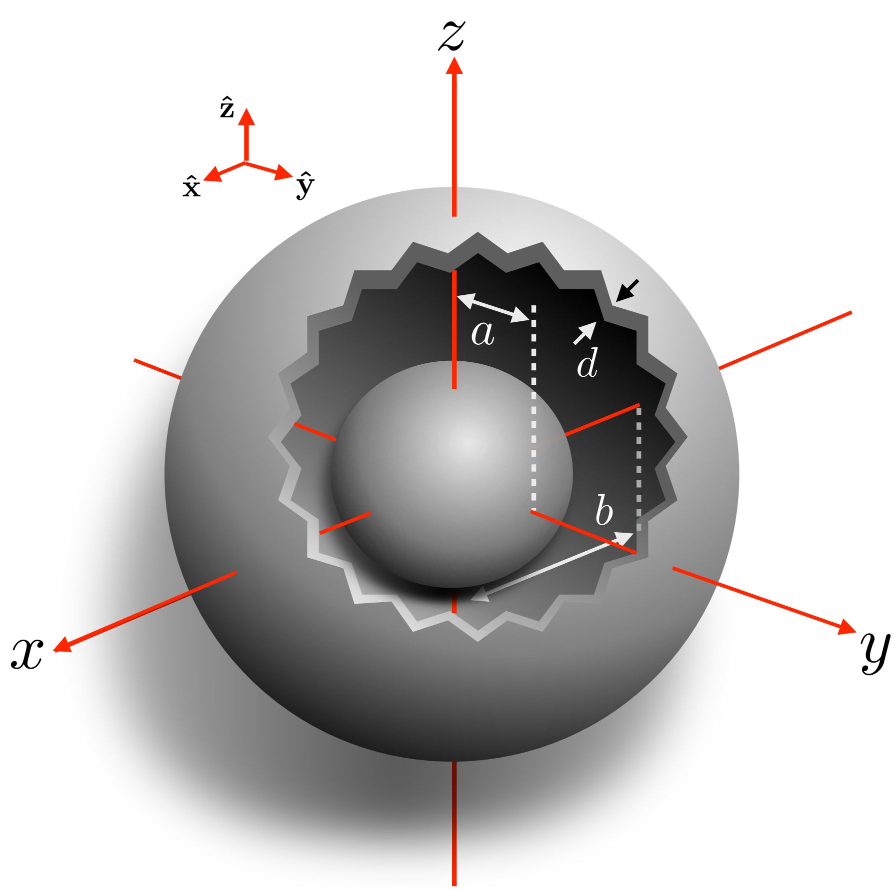

Figure 2: Solid sphere and concentric spherical shell [Section 1].

1 Electric field due to a charged spherical configuration

A solid sphere of radius a is located concentrically in the otherwise interior void of a spherical shell of thickness $d > 0$ and inner radius b, with $a < b$, as shown in Figure 2. The inner solid sphere carriers a radially-varying charge distribution given by

$$\rho_{\mathsf{in}}(r) \doteq \left(\frac{Q}{\pi\, a^4}\right) r \qquad \mathsf{C\, m^{-3}}, \tag{9}$$

for all $r \in [0, a]$, in which $Q > 0$ is a constant (with units C). No current can flow to or from the solid sphere or the concentric spherical shell, and no current can flow between them.

The outer spherical shell is constructed from an excellent conducting material (e.g. copper), and carries zero net charge. The permittivity of the configuration is assumed to be known and constant throughout, and is denoted by ϵ.

1.1 Inner solid sphere

By examining the charge carried by the inner solid sphere, due to the charge distribution (9), the electric field in its interior and immediate surroundings can be determined. The integral form (6) of Gauss' law of electricity can be applied to this end.

Tasks

1.1.1. With a view to applying Gauss' law of electricity, explain why the electric field $\mathbf{E}$ associated with the configuration of Figure 2 must be radially directed and uniform with respect to angles in the spherical coordinate system, i.e. taking the form

$$\mathbf{E} = \mathbf{E}(r, \phi, \theta) = E(r)\, \hat{\boldsymbol{r}}, \tag{10}$$

for all $r \geq 0$, given some (as yet unknown) scalar-valued function $E : \mathbb{R} \to \mathbb{R}$, and the radial unit vector $\hat{\boldsymbol{r}}$. [HINT: Symmetry.]

1.1.2. Show that the total charge $Q_{\mathsf{in}}(r)$ contained within any concentric spherical test surface of radius $r \in [0, a]$, i.e. contained within the inner solid sphere, is given by

$$Q_{\mathsf{in}}(r) = \int_0^r \int_0^{2\pi} \int_0^{\pi} \rho_{\mathsf{in}}(\alpha)\, \alpha^2\, \sin\theta\, d\theta\, d\phi\, d\alpha = Q \left(\frac{r}{a}\right)^4 \qquad \mathsf{C}. \tag{11}$$

Include your working in both equalities.

1.1.3. Noting that (11) is applicable for $r \in [0, a]$ only, argue why the total charge contained within any concentric spherical test surface of radius $r \in (a, b)$ that completely contains the inner solid sphere (but remains inside the concentric spherical shell) must be $+Q$.

1.1.4. By applying Gauss' law of electricity (twice) in its integral form (6), and in view of (10), show that the electric field $\mathbf{E}$ in the interior of and immediately surrounding the inner solid sphere in Figure 2 is given by

$$\mathbf{E} = \mathbf{E}(r) = \frac{Q}{4\,\pi\,\epsilon\,a^2} \begin{cases} \left(\dfrac{r}{a}\right)^2 \hat{\boldsymbol{r}}, & r \in [0, a], \\ \left(\dfrac{a}{r}\right)^2 \hat{\boldsymbol{r}}, & r \in (a, b), \end{cases} \tag{12}$$

with units of $\mathsf{V\,m^{-1}}$.

1.2 Concentric spherical shell

The concentric spherical shell of inner radius b and outer radius $b + d$ is designated as carrying zero net charge. However, the fact that it is constructed from an excellent conductor means that it has an effect on how the electric field due to the inner solid sphere permeates the region.

Tasks

1.2.1. Given that the spherical shell is constructed from an excellent conducting material, explain why this implies that the electric field must be zero inside that material, i.e. for $r \in (b, b + d)$.

1.2.2. Consider a concentric spherical test surface of radius $r \in (b, b + d)$, i.e. with its surface lying within the material of the concentric spherical shell. In view of 1.2.1, what is the total flux of the electric field out through this test surface? Applying Gauss' law of electricity in integral form (6), argue why the total charge contained with this test surface must be zero.

1.2.3. In view of 1.2.2, and the fact that the inner solid sphere carries charge $+Q$, argue why a charge separation must occur in the excellent conducting material of the concentric spherical shell. Further argue why a total charge of $-Q$ must be uniformly distributed on the inner surface of the spherical shell, i.e. at radius $r = b$. Explain where the residual charge $+Q$ must be stored in the concentric spherical shell.

[Note: the residual charge $+Q$ must be stored somewhere in or on the concentric spherical shell, as no current can flow to or from it.]

1.3 Complete configuration

With the electric field and associated charge separation in the concentric spherical understood, the electric field for the complete configuration of Figure 2 can now be understood.

Tasks

1.3.1. Consider a concentric spherical test surface of radius $r > b + d$, i.e. with its surface containing the entirety of the complete configuration of Figure 2. Argue why the total charge contained within this test surface, i.e. containing the entirety of both the solid sphere and concentric spherical shell, must be $+Q$.

1.3.2. A further application of Gauss' law of electricity in its integral form (6) yields the electric field $\mathbf{E}$ due to the complete configuration of Figure 2. In particular,

$$\mathbf{E} = \mathbf{E}(r) = \frac{Q}{4\,\pi\,\epsilon\,a^2}\begin{cases} \left(\dfrac{r}{a}\right)^2 \hat{\boldsymbol{r}}, & r \in [0, a], \\ \left(\dfrac{a}{r}\right)^2 \hat{\boldsymbol{r}}, & r \in [a, b], \\ 0\,\hat{\boldsymbol{r}}, & r \in (b, b+d), \\ \left(\dfrac{a}{r}\right)^2 \hat{\boldsymbol{r}}, & r \geq b+d. \end{cases} \tag{13}$$

Given $Q \doteq 4\,\pi\,\epsilon$, $a \doteq 1\,\mathsf{m}$, $b \doteq 2\,\mathsf{m}$, and $d \doteq 1\,\mathsf{m}$, Figure 3 illustrates a three dimensional sector of radius $3.5\,\mathsf{m}$ centred at the origin. Sample MATLAB code is shown in Figure 4. Is the visualised field of Figure 3 consistent with (13)?

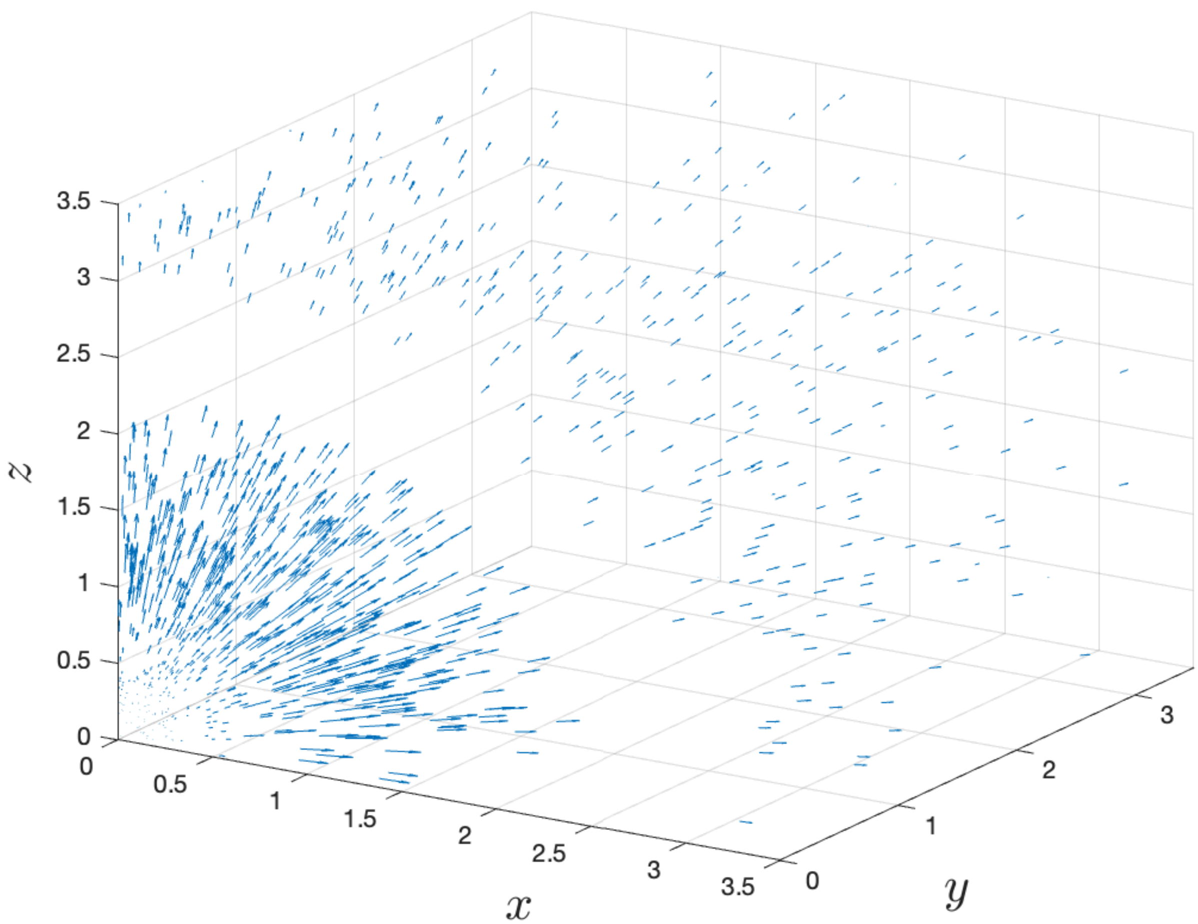

Figure 3: Electric field $\mathbf{E}$ [Section 1].

```
Q_on_four_pi_eps = 1;
a = 1;
b = 2;
d = 1;

n_points = 2000;
coords = rand(n_points,3);
r = coords(:,1)*6;
phi = coords(:,2)*pi/2;
theta = coords(:,3)*pi/2;

E_x = zeros(n_points,1);
E_y = zeros(n_points,1);
E_z = zeros(n_points,1);
x = zeros(n_points,1);
y = zeros(n_points,1);
z = zeros(n_points,1);

for i=1:n_points
        x_i = r(i)*sin(theta(i))*cos(phi(i));
        y_i = r(i)*sin(theta(i))*sin(phi(i));
        z_i = r(i)*cos(theta(i));
        r_i = r(i);

        x(i) = x_i;
        y(i) = y_i;
        z(i) = z_i;

        E_x_LO = Q_on_four_pi_eps/a*(r_i/a)^2*x_i*(r_i<=a);
        E_y_LO = Q_on_four_pi_eps/a*(r_i/a)^2*y_i*(r_i<=a);
        E_z_LO = Q_on_four_pi_eps/a*(r_i/a)^2*z_i*(r_i<=a);

        E_x_HI = Q_on_four_pi_eps/a*(a./r_i)^2*x_i;
        E_y_HI = Q_on_four_pi_eps/a*(a./r_i)^2*y_i;
        E_z_HI = Q_on_four_pi_eps/a*(a./r_i)^2*z_i;

        E_x_HI = E_x_HI*(r_i>=a & r_i<=b | r_i>=(b+d));
        E_y_HI = E_y_HI*(r_i>=a & r_i<=b | r_i>=(b+d));
        E_z_HI = E_z_HI*(r_i>=a & r_i<=b | r_i>=(b+d));

        E_x(i) = E_x_LO + E_x_HI;
        E_y(i) = E_y_LO + E_y_HI;
        E_z(i) = E_z_LO + E_z_HI;
end;

figure(1);
quiver3(x,y,z,E_x,E_y,E_z);
grid on;
xlabel('$x$','Interpreter','latex','FontSize',20);
ylabel('$y$','Interpreter','latex','FontSize',20);
zlabel('$z$','Interpreter','latex','FontSize',20);
axis(3.5*[0 1 0 1 0 1]);
view(32,23);
```

Figure 4: Example MATLAB code for visualising $\mathbf{E}$ [Section 1].

2 Magnetic field due to coaxial current densities

As illustrated in Figure 5, an infinite length solid cylindrical core of radius a is located with its axis positioned along the z-axis in $\mathbb{R}^3$. It is surrounded by a coaxial cylindrical shield of inner radius $b > a$ and outer radius $b + d$, where $d > 0$ is the shield thickness. The core supports a current density $\mathbf{J}_{\mathsf{core}}$ while the shield supports a corresponding current density $\mathbf{J}_{\mathsf{shield}}$, where

$$\mathbf{J}_{\mathsf{core}}(r) = \frac{3\,I\,r}{2\,\pi\,a^3}\,\hat{\boldsymbol{z}}, \qquad r \in [0, a], \tag{14}$$

$$\mathbf{J}_{\mathsf{shield}}(r) = \frac{-I}{\pi\,[(b+d)^2 - b^2]}\,\hat{\boldsymbol{z}}, \qquad r \in [b, b+d], \tag{15}$$

with units of $\mathsf{A\,m^{-2}}$, in which $I > 0$. The core and shield are metallic, and are separated and surrounded by air.

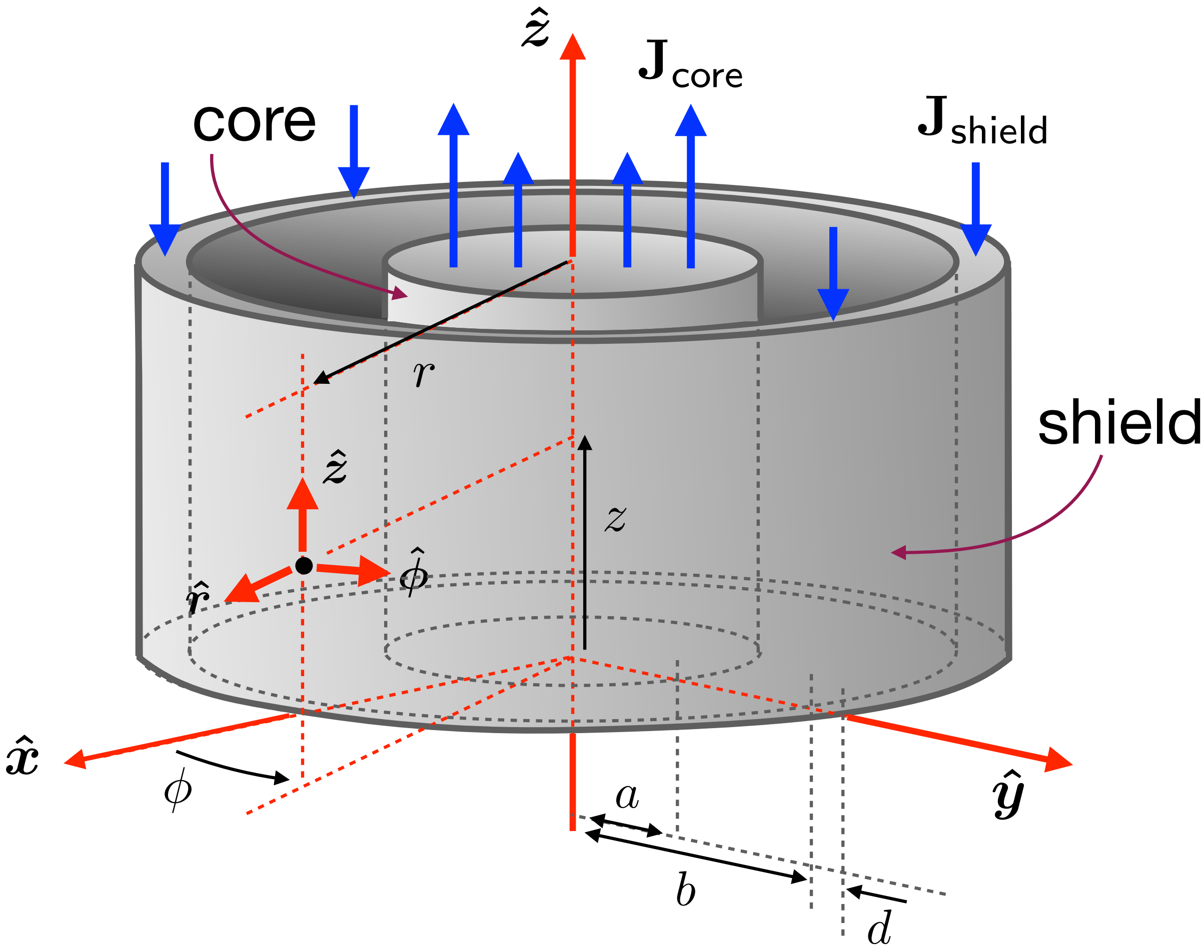

Figure 5: A section of an infinite length coaxial core and shield [Section 2].

2.1 Solid cylindrical core

By examining the current carried by the core, due to the current density (14), the magnetic field in its interior and immediate surroundings can be determined. The integral form (7) of Ampere's law can be applied to this end.

Tasks

2.1.1. With a view to applying Ampere's law, argue why the magnetic field $\mathbf{H}$ associated with the configuration of Figure 5 must be of the form

$$\mathbf{H} = \mathbf{H}(r, \phi, z) = H(r)\,\hat{\boldsymbol{\phi}} \qquad \mathsf{A\,m^{-1}}, \tag{16}$$

for all $r \geq 0$, given some (as yet unknown) scalar-valued function $H : \mathbb{R} \to \mathbb{R}$, and the angular coordinate vector $\hat{\boldsymbol{\phi}}$ in the cylindrical coordinate system. [HINT: Symmetry.]

2.1.2. Show that the current flowing through a coaxial test cylinder of radius $r \in [0, a]$ is

$$I_{\mathsf{core}}(r) = \int_0^r \mathbf{J}_{\mathsf{core}}(\alpha) \cdot (2\,\pi\,\alpha\,d\alpha\,\hat{\boldsymbol{z}}) = I\left(\frac{r}{a}\right)^3 \qquad \mathsf{A\,m^{-2}}. \tag{17}$$

By inspection of (17), what is the total current flowing in the core?

2.1.3. By applying the integral form of Ampere's law (twice), show that the magnetic field $\mathbf{H}$ in the interior of the core, and in the region between the core and shield, is given by

$$\mathbf{H}(r) = \begin{cases} \dfrac{I}{2\,\pi\,a}\left(\dfrac{r}{a}\right)^2 \hat{\boldsymbol{\phi}}, & r \in [0, a]\,, \\ \dfrac{I}{2\,\pi\,a}\left(\dfrac{a}{r}\right) \hat{\boldsymbol{\phi}}, & r \in (a, b], \end{cases} \tag{18}$$

in which $\hat{\boldsymbol{\phi}}$ is a angular unit vector in the cylindrical coordinate system.

2.2 Coaxial cylindrical shield

The coaxial shield in Figure 5 also a carries a non-zero current density (15), in the opposite direction to that in core, which subtracts from the magnetic field further away from the axis of the configuration.

Tasks

2.2.1. Determine the total current flowing in the shield.

2.2.2. With $a \doteq 0.5$, $b \doteq 0.75$, and $d \doteq 0.25$, the magnetic field throughout the configuration is shown in Figure 6. This visualization suggests that the magnetic field is zero outside the shield, i.e. for $r > 1$. Explain (in words) why this is justified by Ampere's law.

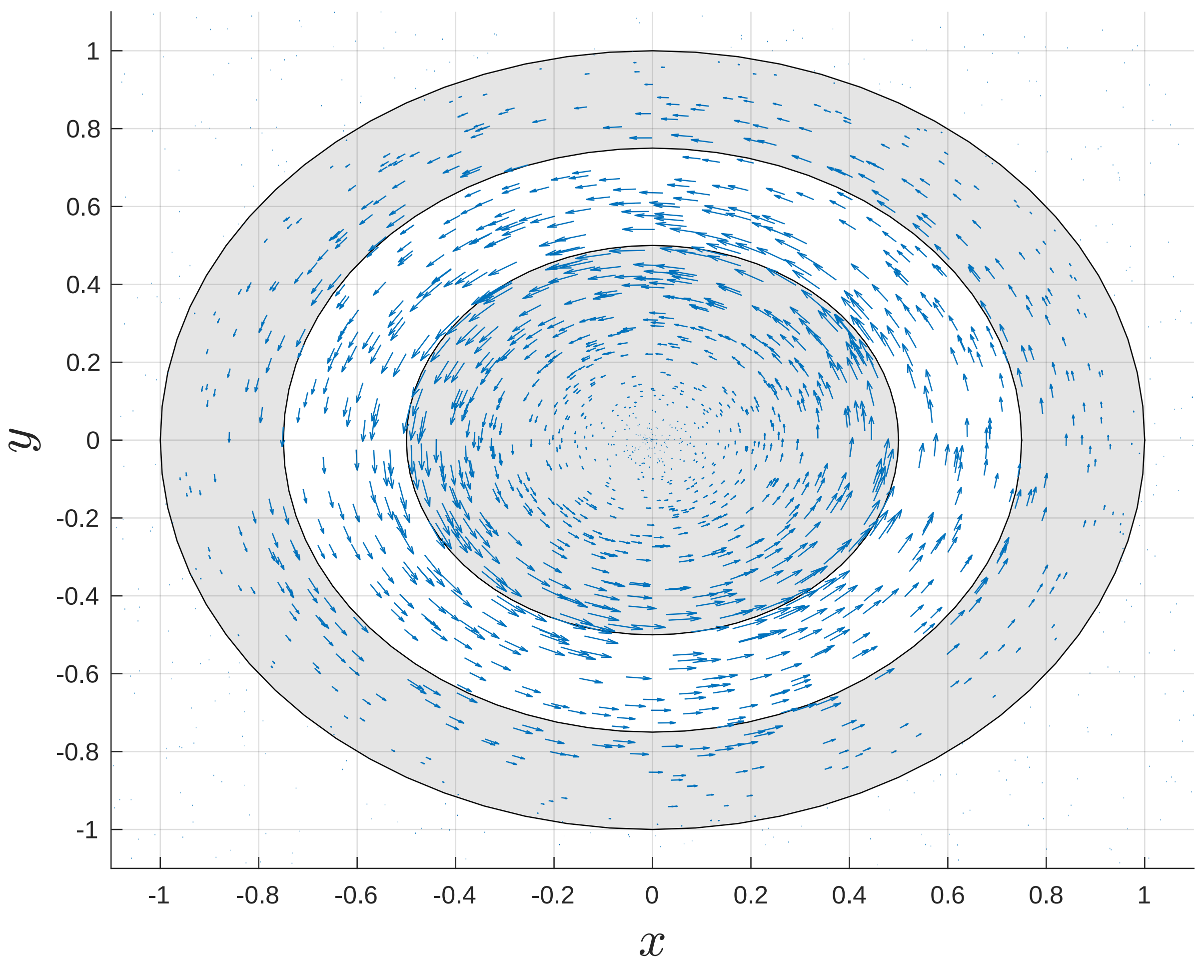

Figure 6: Magnetic field $\mathbf{H}$ [Section 2].