

ELEN30011 EDM Task

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1.1

$\nabla \cdot \mathbf{F}$ (div \mathbf{F}) is a scalar, $\nabla \times \mathbf{F}$ is a vector field.

Explanation: \Downarrow

1.2

Let $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a vector field with

$$\mathbf{F}(x, y, z) = F_x(x, y, z)\hat{\mathbf{x}} + F_y(x, y, z)\hat{\mathbf{y}} + F_z(x, y, z)\hat{\mathbf{z}}$$

The divergence of \mathbf{F} (div \mathbf{F}) is

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_x, F_y, F_z) \\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\end{aligned}$$

$\nabla \cdot \mathbf{F}$ is a scalar.

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \right) \times (F_x(x, y, z)\hat{\mathbf{x}} + F_y(x, y, z)\hat{\mathbf{y}} + F_z(x, y, z)\hat{\mathbf{z}}) \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{\mathbf{z}}\end{aligned}$$

$\nabla \times \mathbf{F}$ is a vector field.

1.3

(a)

$$\text{grad}f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

(b)

$$\text{grad}f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = 1\hat{\mathbf{x}} + z\hat{\mathbf{y}} + y\hat{\mathbf{z}}$$

(c)

$$\text{grad}f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = x\hat{\mathbf{x}} + (y + \frac{1}{2}z^2\sin y)\hat{\mathbf{y}} - z\cos y\hat{\mathbf{z}}$$

(d)

$$\text{grad}f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} = \frac{-x}{x^2 + y^2 + z^2}\hat{\mathbf{x}} + \frac{-y}{x^2 + y^2 + z^2}\hat{\mathbf{y}} + \frac{-z}{x^2 + y^2 + z^2}\hat{\mathbf{z}}$$

1.4

(a)

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$$

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 0 \end{vmatrix} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

(b)

$$\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = -1 + 1 = 0$$

$$\text{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & 0 & z \end{vmatrix} = 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

(c)

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$$

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 0 & -x \end{vmatrix} = [1 - (-1)]\hat{\mathbf{y}} = 2\hat{\mathbf{y}}$$

(d)

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\begin{aligned} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \hat{\mathbf{x}} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \hat{\mathbf{y}} + \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y} \right) \hat{\mathbf{z}} \\ &= 0\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + 0\hat{\mathbf{z}} \end{aligned}$$

1.5

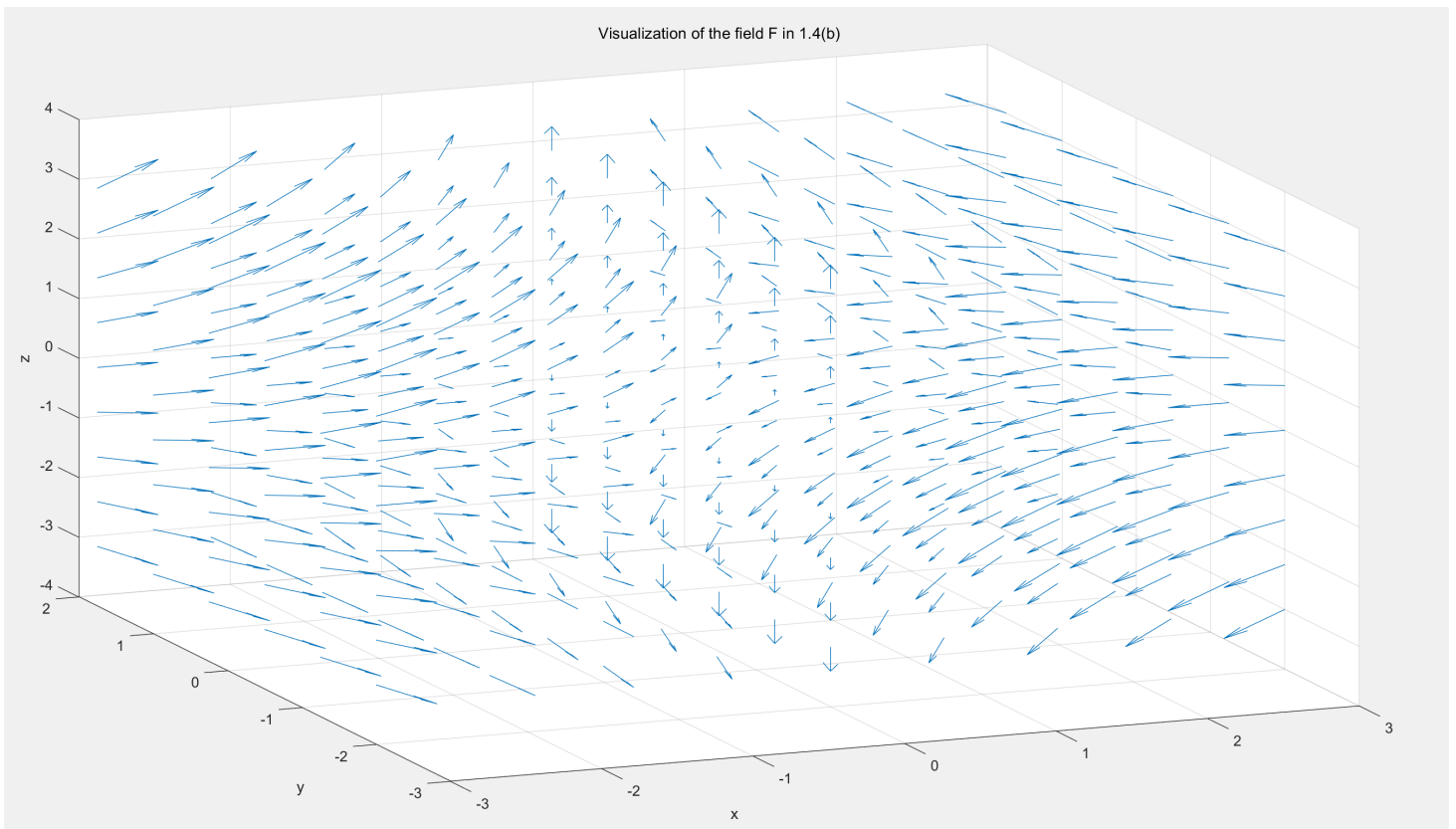
```
close all
clear
clc

x = -3:.75:3;
y = -2:.75:2;
z = -3:.75:3;

[X, Y, Z] = meshgrid(x, y, z);

FX = -X;
FY = 0.*Y;
FZ = Z;

figure(1);
quiver3(X,Y,Z,FX,FY,FZ)
xlabel("x");
ylabel("y");
zlabel("z");
title("Visualization of the field F in 1.4(b)")
```



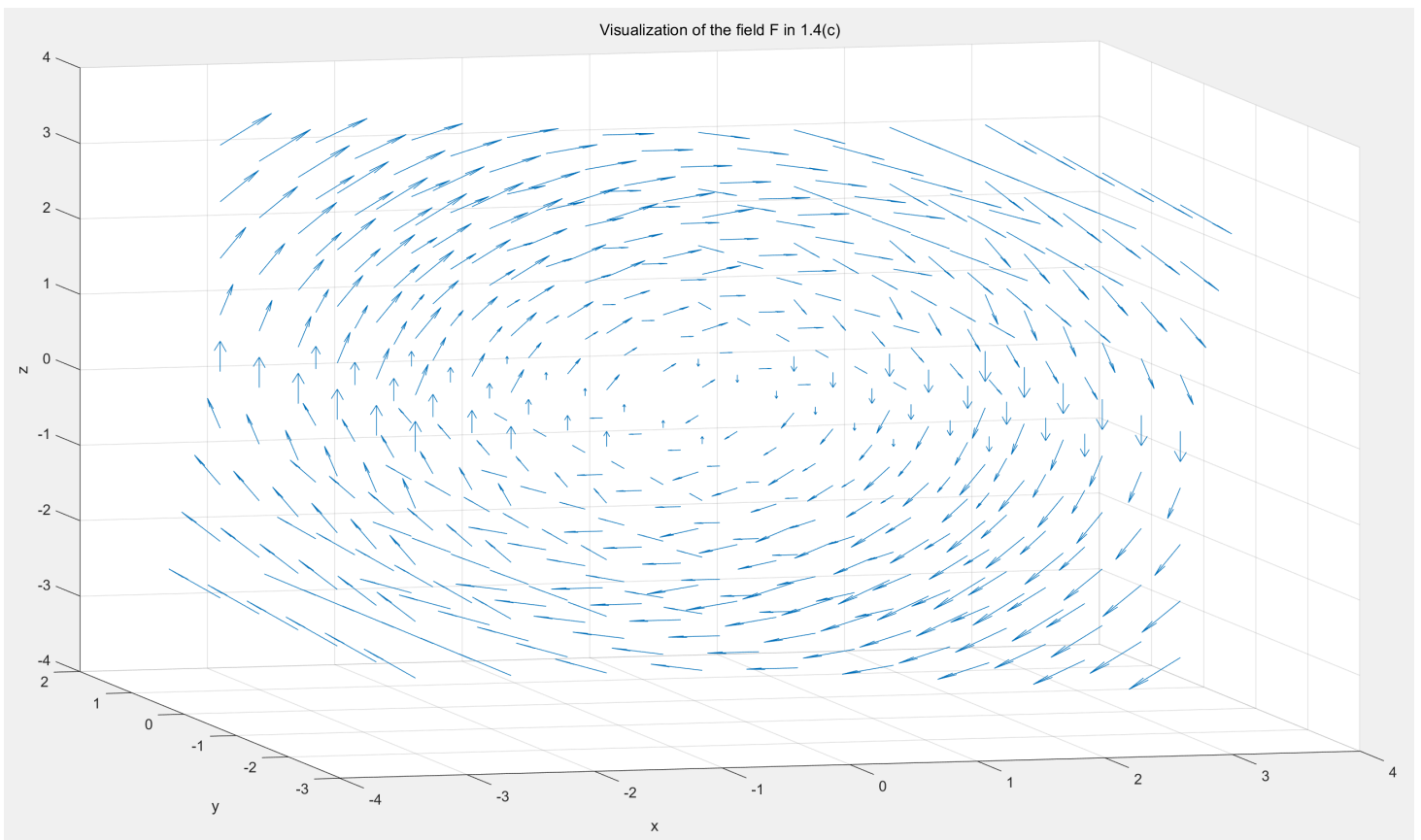
```
close all
clear
clc

x = -3:.75:3;
y = -2:.75:2;
z = -3:.75:3;

[X, Y, Z] = meshgrid(x, y, z);

FX = Z;
FY = 0.*Y;
FZ = -X;

figure(1);
quiver3(X,Y,Z,FX,FY,FZ)
xlabel("x");
ylabel("y");
zlabel("z");
title("Visualization of the field F in 1.4(c)")
```



2.1

2.1.1

\mathbf{l}_1 is a straight line.

$$\mathbf{l}_1(s) = (1 - s)\mathbf{P}_1 + s\mathbf{P}_2$$

\mathbf{l}_2 is circular arc of radius 1.

Let $x = \cos(\frac{\pi s}{4})$, $y = \sin(\frac{\pi s}{4})$.

$$x^2 + y^2 = \cos^2(\frac{\pi s}{4}) + \sin^2(\frac{\pi s}{4}) = 1$$

In cartesian coordinates, $x^2 + y^2 = 1$ is describing a circle with a radius of 1.

Plus, since $s \in [0, 1]$, $x \in [\frac{\sqrt{2}}{2}, 1]$, $y \in [0, \frac{\sqrt{2}}{2}]$.

When s increases, \mathbf{l}_2 moves from $(1, 0)$ to $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ along this circle, anticlockwise.

2.1.2

Formular (8) shows that,

$$\mathbf{E}(x, y, z) = -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$$

For \mathbf{l}_1 , $x = 1 - s + \frac{s}{\sqrt{2}}$, $y = \frac{s}{\sqrt{2}}$

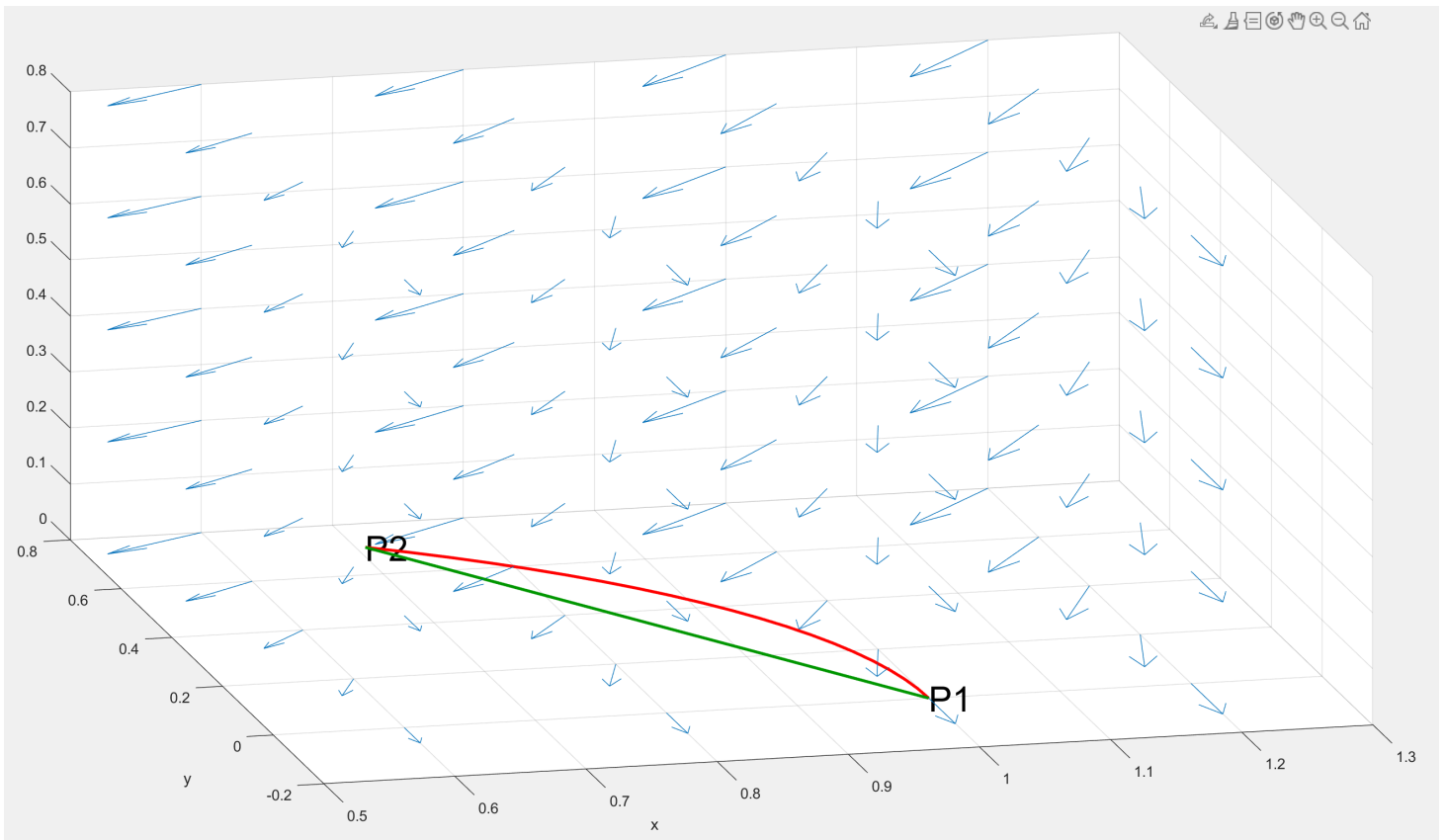
$$\mathbf{E}(\mathbf{l}_1(s)) = -\frac{s}{\sqrt{2}}\hat{\mathbf{x}} - (1 - s + \frac{s}{\sqrt{2}})\hat{\mathbf{y}}$$

For \mathbf{l}_2 , $x = \cos(\frac{\pi s}{4})$, $y = \sin(\frac{\pi s}{4})$

$$\mathbf{E}(\mathbf{l}_2(s)) = -\sin(\frac{\pi s}{4})\hat{\mathbf{x}} - \cos(\frac{\pi s}{4})\hat{\mathbf{y}}$$

2.1.3

```
x = 0.6:.2:1.2; y = 0:.2:.8; z = 0:.2:.8;
[xx,yy,zz] = meshgrid(x,y,z);
Exx =-yy; Eyy =-xx; Ezz = 0*xx;
figure(2);
quiver3(xx,yy,zz,Exx,Eyy,Ezz);
grid on; hold on;
xhat = [1;0;0]; yhat = [0;1;0];
s = 0:.05:1;
L1 = (1- s).*xhat + s.*(xhat + yhat)/sqrt(2);
L2 = cos(pi .*s ./4).*xhat + sin(pi .*s ./4).*yhat;
plot3(L1(1,:),L1(2,:),L1(3:),'Color',[0 .6 0],'LineWidth',2);
plot3(L2(1,:),L2(2,:),L2(3:),'Color',[1 0 0],'LineWidth',2);
xlabel("x")
ylabel("y")
text(1,0,0,"P1","FontSize",24)
text(1/sqrt(2), 1/sqrt(2), 0, "P2", "FontSize",24)
```



2.1.4

There is no singularity here, the result is not related to the path.

$$\begin{aligned}
 V_{l_1} &= - \int_{l_1} \mathbf{E} \cdot d\mathbf{l}_1 \\
 &= - \int_{l_1} \mathbf{E}(\mathbf{l}_1(s)) \cdot \frac{d\mathbf{l}_1}{ds}(s) ds \\
 &= - \int_s (-\frac{s}{\sqrt{2}}, -(1-s+\frac{s}{\sqrt{2}})) (-1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) ds \\
 &= - \int_0^1 \sqrt{2}s - s - \frac{\sqrt{2}}{2} ds \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
V_{l_2} &= - \int_{l_2} \mathbf{E} \cdot d\mathbf{l}_2 \\
&= - \int_{l_2} \mathbf{E}(\mathbf{l}_2(s)) \cdot \frac{d\mathbf{l}_2}{ds}(s) ds \\
&= - \int_s (-\sin(\frac{\pi s}{4}), -\cos(\frac{\pi s}{4})) (-\frac{\pi}{4} \sin(\frac{\pi s}{4}), \frac{\pi}{4} \cos(\frac{\pi s}{4})) ds \\
&= \int_0^1 \cos^2(\frac{\pi s}{4}) - \sin^2(\frac{\pi s}{4}) ds \\
&= \frac{\pi}{4} \int_0^1 \cos(\frac{\pi s}{2}) ds \\
&= \frac{\pi}{4} \times \frac{2}{\pi} \sin(\frac{\pi s}{2}) \Big|_{s=0}^{s=1} \\
&= \frac{1}{2}
\end{aligned}$$

- Force:

$$\mathbf{F} = q\mathbf{E}$$

- Work:

$$W = - \int_l \mathbf{F} d\mathbf{l} = -q \int_l \mathbf{E} d\mathbf{l}$$

- Voltage

$$V = \frac{W}{q} = - \int_l \mathbf{E} d\mathbf{l}$$

2.1.5

Since $\mathbf{E}(x, y, z) = -y\hat{\mathbf{x}} - x\hat{\mathbf{y}}$ (shown in Fomular 8)

This field is static,

$$-\nabla U = \mathbf{E}$$

$$\frac{\partial U}{\partial x} = y$$

$$\frac{\partial U}{\partial y} = x$$

Hence,

$$U = xy + c$$

where c is constant.

$$\mathbf{P}_1 = (1, 0), \mathbf{P}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$U(\mathbf{P}_2) - U(\mathbf{P}_1) = (1/2 + c) - (0 + c) = \frac{1}{2}$$

2.2

2.2.1

$$\text{curl} \mathbf{H} = \nabla \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & 0 \end{vmatrix} = -\hat{\mathbf{y}} + \hat{\mathbf{z}}$$

2.2.2

In part 2.2.1, we've got $\text{curl} \mathbf{H} = -\hat{\mathbf{y}} + \hat{\mathbf{z}}$. Since the x component of \mathbf{H} is 0, the flux of A_{left} and A_{left} equals 0.

$$\begin{aligned} \int \text{curl} \mathbf{H} \cdot d\mathbf{A}_{\text{left}} &= \int (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot dA_{\text{left}} \hat{\mathbf{x}} = 0 \\ \int \text{curl} \mathbf{H} \cdot d\mathbf{A}_{\text{right}} &= \int (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot dA_{\text{right}} \hat{\mathbf{x}} = 0 \end{aligned}$$

2.2.3

The surface of A_{angle} is $z - y = 0$

Choose $x = 2s, y = 2t, s, t \in [0, 1]$

$$\Rightarrow z = y = 2t$$

$$\mathbf{l}_{\text{angle}}(s, t) = 2s\hat{\mathbf{x}} + 2t\hat{\mathbf{y}} + 2t\hat{\mathbf{z}} \quad s, t \in [0, 1]$$

$$\begin{aligned}
d\mathbf{A}_{\text{angle}} &= \left(\frac{d\mathbf{l}}{ds} \times \frac{d\mathbf{l}}{dt} \right) ds dt \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{vmatrix} ds dt \\
&= (4\hat{\mathbf{z}} - 4\hat{\mathbf{y}}) ds dt
\end{aligned}$$

$$\begin{aligned}
\int_A \text{curl} \mathbf{H} \cdot d\mathbf{A}_{\text{angle}} &= \iiint (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot (4\hat{\mathbf{z}} - 4\hat{\mathbf{y}}) ds dt \\
&= 8 \int_0^1 \int_0^1 ds dt \\
&= 8
\end{aligned}$$

2.2.4

$$d\mathbf{A}_{\text{front}} = dA_{\text{front}} \hat{\mathbf{y}}$$

$$\begin{aligned}
\int_A \text{curl} \mathbf{H} \cdot d\mathbf{A}_{\text{front}} &= \int_A (-\hat{\mathbf{y}} + \hat{\mathbf{z}}) \cdot (\hat{\mathbf{y}}) dA_{\text{front}} \\
&= - \int_A dA \\
&= -4
\end{aligned}$$

$$\int_A \text{curl} \mathbf{H} \cdot d\mathbf{A} = 8 - 4 = 4$$

2.3

2.4

2.4.1

$$\int_0^1 \int_0^1 \int_0^1 (\pi z^2) \sin(\pi x) \cos\left(\frac{\pi y}{2}\right) dx dy dz$$

integrate x: $\int_0^1 \sin(\pi x) dx = \left[-\frac{1}{\pi} \cos(\pi x) \right]_0^1 = -\frac{1}{\pi} \cos(\pi) - \left(-\frac{1}{\pi} \cos(0) \right) = \frac{2}{\pi}$

integrate y: $\int_0^1 \cos\left(\frac{\pi y}{2}\right) dy = \left[\frac{2}{\pi} \sin\left(\frac{\pi y}{2}\right) \right]_0^1 = \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi} \sin(0) = \frac{2}{\pi}$

integrate z: $\int_0^1 (\pi z^2) dz = \pi \int_0^1 z^2 dz = \pi \left[\frac{z^3}{3} \right]_0^1 = \pi \left(\frac{1}{3} - \frac{0}{3} \right) = \frac{\pi^2}{3}$

$$Q = \frac{2}{\pi} \cdot \frac{2}{\pi} \cdot \frac{\pi^2}{3} = \frac{4}{3}$$

2.4.2.

Gauss theorem: $\oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon}$

from (2.4.1), we knew that $Q = \frac{4}{3}$

According to formula (2.2):

$$\text{div} \mathbf{E}(x, y, z) = \frac{\rho(x, y, z)}{\epsilon}$$

let (2.2) in Gauss theorem, we can have:

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon} \int_V \rho(x, y, z) dV$$

