ON CENTRALITY OF ORTHOGONAL K_2

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ABSTRACT. We give a short uniform proof of centrality of $\mathrm{K}_2(\Phi, R)$ for all simply-laced root systems Φ of rank ≥ 3.

INRODUCTION

The main goal of this paper is to establish the local-global principle and centrality of orthogonal K_2 for arbitrary commutative ring R.

Recall that for each reduced root system Φ and a commutative ring R one can associate a split simple group $\mathrm{G}(\Phi, R)$ called the *Chevalley group*. For Φ of rank ≥ 2 the abstract group $\mathrm{G}(\Phi, R)$ contains a large normal subgroup $\mathrm{E}(\Phi, R)$ generated by the elementary root unipotents $t_\alpha(\xi)$, $\xi \in R$, $\alpha \in \Phi$. The quotient group functor $\mathrm{K}_1(\Phi, R) = \mathrm{G}(\Phi, R)/\mathrm{E}(\Phi, R)$ is called the *unstable* K_1-*functor*. This functor shares many properties with its stable counterpart $\mathrm{SK}_1(R) = \varinjlim_{l\to\infty} \mathrm{K}_1(\mathsf{A}_\ell, R)$ which, in turn, is an essential direct summand of $\mathrm{K}_1(R)$.

Similarly to the definition of the algebraic K_2-functor, one can define $\mathrm{K}_2(\Phi, R)$ as the kernel of the canonical projection $\phi\colon \mathrm{St}(\Phi, R) \twoheadrightarrow \mathrm{E}(\Phi, R)$. Here $\mathrm{St}(\Phi, R)$ stands for the *Steinberg group* of type Φ, i.e. the group defined by formal generators $x_\alpha(\xi)$ modeling root unipotents $t_\alpha(\xi)$ and the explicit list of relations called Steinberg relations (see Section 1).

It is classically known that $\mathrm{K}_2(\Phi, R)$ is a central subgroup of $\mathrm{St}(\Phi, R)$ if R is commutative local ring and Φ has rank ≥ 2 (see [7]). However, to prove that centrality of K_2 holds for *arbitrary* R is much harder. The first result in this direction was obtained in 1977 by W. van der Kallen for $\Phi = \mathsf{A}_\ell$, $\ell \geq 3$ using the technique called "another presentation". Using a similar approach, the first-named author has established centrality of $\mathrm{K}_2(\mathsf{C}_\ell, R)$ for $\ell \geq 3$, see [4]. In turn, the key result of this paper is the following theorem.

Theorem 1. *Let R be an arbitrary commutative ring and $\ell \geq 4$ then the group $\mathrm{K}_2(\mathsf{D}_\ell, R)$ is contained in the centre of $\mathrm{St}(\mathsf{D}_\ell, R)$.*

Notice that the analogous statement in the case $\Phi = \mathsf{E}_\ell$, $\ell = 6, 7, 8$ has been demonstrated by the second named author in [6]. Theorem 1 is essentially a corollary of the following local-global principle.

Theorem 2. *Let Φ be an irreducible simply-laced root system of rank ≥ 3 and R be arbitrary commutative ring. Then an element $g \in \mathrm{St}(\Phi, R[X])$ satisfying $g(0) = 1 \in \mathrm{St}(\Phi, R)$ is trivial in $\mathrm{St}(\Phi, R[X])$ if and only if the elements $g_M \in \mathrm{St}(\Phi, R_M[X])$ are trivial for all maximal ideals $M \triangleleft R$.*

The special case $\Phi = \mathsf{A}_\ell$, $\ell \geq 4$ of Theorem 2 was demonstrated for the first time by Tulenbaev, see [9, Theorem 2.1], while similar assertions for $\Phi = \mathsf{C}_\ell$, $\ell \geq 3$ and $\Phi = \mathsf{E}_\ell$, $\ell = 6, 7, 8$ have been

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established in [3] and [6], respectively. The special cases $\Phi = \mathsf{A}_3$ and $\Phi = \mathsf{D}_\ell$, $\ell \geq 4$ of Theorem 2 are new and have not been known before.

There are known counterexamples to centrality of K_2 in the rank 2 case (see [10]). This shows that the assumption on the rank of Φ in Theorem 2 is strict.

Compare Theorem 2 with the local-global principle for K_1 which states that an element of the congruence subgroup $g \in \mathrm{G}(\Phi,\, R[X], XR[X])$ is elementary (i. e. lies in $\mathrm{E}(\Phi, R[X])$) if and only if $g_M = 1 \in \mathrm{E}(\Phi,\, R_M)$ for all maximal ideals $M \triangleleft R$.

Our proof of Theorem 2 follows the same lines as in [6, Theorem 1]. More concretely, using the same patching technique as in [6] we reduce the problem to the case $\Phi = \mathsf{A}_3$. Then, we formulate and prove a new presentation of the linear group $\mathrm{St}(4, R)$ and then utilize it to reprove Tulenbaev's key lemma (cf. [9, Lemma 2.3]) in the rank 3 case.

The rest of the paper is organised as follows. In Section 2 we formulate "yet another presentation" for the rank 3 Steinberg groups. Finally, in Section 3 we prove our main results.

1. Preliminaries

Throughout this paper R denotes an associative commutative ring with identity. All commutators are left-normed, i. e. $[x,\, y] = xyx^{-1}y^{-1}$.

We denote by R^n the free R-module with basis $e_1, \ldots, e_n$ and by $\mathrm{Um}(n,\, R)$ the subset of unimodular columns $v \in R^n$ whose entries generate R as an ideal.

As usual, $\mathrm{E}(n, R) = \mathrm{E}(\mathsf{A}_{n-1}, R)$ denotes the elementary group, i. e. the subgroup of the general linear group $\mathrm{GL}(n, R)$ generated by transvections $t_{ij}(r) = 1 + r \cdot e_{ij}$, $1 \leq i \neq j \leq n$, $r \in R$, where 1 stands for the identity matrix and e_{ij} for the matrix unit.

The Steinberg group $\mathrm{St}(n, R) = \mathrm{St}(\mathsf{A}_{n-1}, R)$ is the group presented by generators $x_{ij}(r)$, $1 \leq i \neq j \leq n$, $r \in R\}$ and relations

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s), \tag{S1}$$

$$[x_{ij}(r),\, x_{hk}(s)] = 1, \text{ for } h \neq j,\ k \neq i, \tag{S2}$$

$$[x_{ij}(r),\, x_{jk}(s)] = x_{ik}(rs). \tag{S3}$$

The natural projection $\phi\colon\ \mathrm{St}(n,\, R) \to \mathrm{E}(n,\, R)$ sends $x_{ij}(r)$ to $t_{ij}(r)$.

In [8] Suslin showed for $n \geq 3$ that the elementary group $\mathrm{E}(n, R)$ coincides with the subgroup of $\mathrm{GL}(n,\, R)$ generated by matrices of the form $t(u,\, v) = 1 + uv^t$ where $u \in \mathrm{Um}(n,\, R)$, $v \in R^n$ and $u^tv = 0$. Here u^t stands for the transpose of u. This result clearly implies that $\mathrm{E}(n,\, R)$ is normal inside $\mathrm{GL}(n,\, R)$. Afterwards, W. van der Kallen developed Suslin's ideas and showed for $n \geq 4$ that the Steinberg group $\mathrm{St}(n,\, R)$ is isomorphic to the group presented by generators

$$\{X(u,\, v) \mid u \in \mathrm{Um}(n,\, R),\ v \in R^n,\ u^tv = 0\}$$

and the following list of relations (see [1]):

$$X(u,\, v)X(u,\, w) = X(u,\, v+w), \tag{K1}$$

$$X(u,\, v)X(u',\, v')X(u,\, v)^{-1} = X(t(u,\, v)u',\, t(v,\, u)^{-1}v'). \tag{K2}$$

This presentation clearly implies that $\phi\colon\ \mathrm{St}(n,\, R) \to \mathrm{E}(n,\, R)$ is a central extension. Notice that our notation for the generators slightly differs from that of [1], e. g. our elements $X(u, v)$ correspond to van der Kallen's (u, v^t).

Recall that an ideal $I \trianglelefteq R$ is called a *splitting ideal* if the canonical projection $R \twoheadrightarrow R/I$ splits as a unital ring morphism. For a splitting ideal $I \trianglelefteq R$ the *relative Steinberg group* $\mathrm{St}(n,\, R,\, I) = \mathrm{St}(\mathsf{A}_{n-1}, R, I)$ can be defined as the kernel $\mathrm{Ker}\big(\mathrm{St}(n,\, R) \twoheadrightarrow \mathrm{St}(n,\, R/I)\big)$.

For a general I the relative Steinberg group $\mathrm{St}(\Phi, R, I)$ is no more a subgroup of $\mathrm{St}(\Phi, R)$ but rather is its central extension. For the purposes of the present text it suffices to consider only relative Steinberg groups corresponding to splitting ideals I. For more information regarding general relative Steinberg groups we refer the reader to [6, Section 3].

2. Yet another presentation for $\mathrm{St}(4, R)$.

The key ingredient in Tulenbaev's proof of the local-global principle is the construction of the homomorphism T which fits into the following commutative diagram.

$$\begin{tikzcd}
\mathrm{St}(n,\, R \ltimes XR_a[X],\, XR_a[X]) \arrow[r, hook] \arrow[d] & \mathrm{St}(n, R \ltimes XR_a[X]) \arrow[d, "\lambda_a^*"'] \\
\mathrm{St}(n,\, R_a[X],\, XR_a[X]) \arrow[r, hook] \arrow[ur, dashed, "T"'] & \mathrm{St}(n, R_a[X])
\end{tikzcd}$$

Here $n \geq 5$, a is arbitrary nonnilpotent element of R and λ_a^* is the morphism induced by the morphism $\lambda_a\colon R \to R_a$ of principal localisation at a. Notice that $XR_a[X]$ is a splitting ideal for both $R_a[X]$ and $R \ltimes XR_a[X]$. It is clear that the image of T is contained in the image of the top arrow, therefore this implies that the relative Steinberg groups in the left-hand side of the diagram are isomoprhic.

In order to construct the arrow T Tulenbaev uses the following presentation for the relative Steinberg group, see [9, Proposition 1.6].

Proposition 2.1. *For a splitting ideal I the group $\mathrm{St}(n,\, R,\, I)$ is isomorphic to the group given by generators*

$$\{X(u,\, v) \mid u \in \mathrm{E}(n,\, R)e_1,\ v \in I^n,\ u^t v = 0\}$$

subject to relations

(T1) $$X(u,\, v)X(u,\, w) = X(u,\, v + w),$$

(T2) $$X(u,\, v)X(u',\, v')X(u,\, v)^{-1} = X(t(u,\, v)u',\, t(v,\, u)^{-1}v'),$$

(T3) $$X(ur + w,\, v) = X(u,\, vr)X(w,\, v)\ \textit{ for } r \in R,\ (u,\, w) \in \mathrm{Um}_{n\times 2}(R).$$

Here $\mathrm{Um}_{n\times 2}(R)$ denotes the set of $n \times 2$ *unimodular matrices*, i. e. matrices M such that there exists a $2 \times n$ matrix N satisfying $NM = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Both "another presentations" of van der Kallen and Tulenbaev are given in terms of generators parametrised by pairs of vectors, where the first one is in some sense "good" (i. e. unimodular or a column of an elementary matrix) and the second one is arbitrary. It is easy to formulate the additivity property in the second argument for these generators (cf. (K1), (T1)), while it is not so easy when it comes for the additivity in the first argument (cf. (T3)).

The construction of the homomorphism $T\colon \mathrm{St}(n,\, R_a[X],\, XR_a[X]) \to \mathrm{St}(n,\, R \ltimes XR_a[X]])$ amounts to choosing certain elements in the target group $\mathrm{St}(n,\, R \ltimes XR_a[X]])$ and proving that these elements satisfy the relations (T1)–(T3), see [9, Lemmas 1.2 and 1.3 c)]. The main problem with this recipe is that the assumption $n \geq 5$ is essential to check that the relation (T3) holds.

It is not possible to choose arbitrary orthogonal vectors u, v as parameters for the generators because in this situation it is not even known that $t(u,\, v) \in \mathrm{E}(n,\, R)$, see [**Rao**].

To generalise Tulenbaev's results for $n = 4$ we will use a more "symmetric" presentation with two types of generators: $F(u,\, v)$ with u "good" and v arbitrary and $S(u,\, v)$ with u arbitrary and v "good". The generators $F(u,\, v)$ will be additive in the second component, while $S(u,\, v)$ will be additive in the first one. When u and v are both "good" we require these two generators to coincide. More formally, we have the following presentation.

Definition 2.2. For $I \trianglelefteq R$ and $n \geq 4$ define $\mathrm{St}^*(n, R, I)$ to be the group with the set of generators

$$\{F(u, v) \mid u \in \mathrm{E}(n, R)e_1,\ v \in I^n,\ u^t v = 0\} \cup \{S(u, v) \mid u \in I^n,\ v \in \mathrm{E}(n, R)e_1,\ u^t v = 0\}$$

subject to relations

(R1) $$F(u, v)F(u, w) = F(u, v + w),$$

(R2) $$S(u, v)S(w, v) = S(u + w, v),$$

(R3) $$F(u, v)F(u', v')F(u, v)^{-1} = F(t(u, v)u', t(v, u)^{-1}v'),$$

(R4) $$F(u, va) = S(ua, v)\ \forall\, a \in I,\ (u, v^t) = (Me_1, e_2^t M^{-1}),\ M \in \mathrm{E}(n, R).$$

Notice that we only described how a generator $F(u,v)$ acts via conjugation on another generator $F(u',v')$ and omitted three similar relations involving generators $S(u,v)$ of the second type. The reason for this is the following lemma which asserts that the "missing" relations will follow automatically from (R1)–(R4).

Lemma 2.3. *Denote by* $\phi\colon \mathrm{St}^*(n, R, I) \to \mathrm{E}(n, R)$ *the natural map sending* $F(u, v) \mapsto t(u, v)$ *and* $S(u, v) \mapsto t(u, v)$*. Then the following facts are true.*

(i) $\mathrm{St}^*(n, R, I)$ *is generated as an abstract group by the set of elements*

$$\{F(u, va) \mid a \in I,\ (u, v^t) = (Me_1, e_2^t M^{-1}),\ M \in \mathrm{E}(n, R)\};$$

(ii) *for any* $g \in \mathrm{St}^*(n, R, I)$ *one has*

(R3') $$gF(u, v)g^{-1} = F(\phi(g)u, \phi(g^{-1})^t v);$$

(iii) *for any* $g \in \mathrm{St}^*(n, R, I)$ *one has*

(R3") $$gS(u, v)g^{-1} = S(\phi(g)u, \phi(g^{-1})^t v);$$

(iv) *there is a "transpose automorphism" defined on* $\mathrm{St}^*(n, R, I)$ *satisfying*

$$F(u, v)^t = S(v, u), \quad S(u, v)^t = F(v, u).$$

Proof. Let $F(u, v)$ be an arbitrary generator of the first type and let $M \in \mathrm{E}(n, R)$ be such that $F(u, v) = F(Me_1, M^*\tilde{v})$. Here we denote by M^* the contragradient matrix, $M^* = (M^{-1})^t$. By (R1) we have

$$F(u, v) = \prod_{k\neq 1} F(Me_1, M^* e_k \tilde{v}_k)$$

where $\tilde{v}_k$ stands for k-th coordinate of $\tilde{v} = \sum e_i \tilde{v}_i$. Applying relations (R2), (R4) we get

$$S(u, v) = \prod_{k\neq 1} S(Ne_k\tilde{u}_k, N^*e_1) = \prod_{k\neq 1} F(Ne_k, N^*e_1\tilde{u}_k)$$

and, thus, obtain (i). Obviously, (ii) follows from (i). To prove (iii) it suffices to show that

$$F(u, v)S(u', v')F(u, v)^{-1} = S(t(u, v)u', t(u, v)^* v').$$

For $S(u', v') = \prod_{k\neq 1} F(Ne_k, N^*e_1\tilde{u}_k)$ we have

$$\begin{aligned} F(u, v)S(u', v')F(u, v)^{-1} &= F(u, v)\prod F(Ne_k, N^*e_1\tilde{u}_k)F(u, v)^{-1} = \\ &= \prod F(t(u, v)Ne_k, t(u, v)^* N^* e_1 \tilde{u}_k) = \\ &= \prod S(t(u, v)Ne_k\tilde{u}_k, t(u, v)^* N^* e_1) = S(t(u, v)u', t(u, v)^* v'). \end{aligned}$$

Finally, (iv) follows from (iii). □

Next, we show that for a splitting ideal $I \trianglelefteq R$ the group $\mathrm{St}^*(n, R, I)$ is isomorphic to $\mathrm{St}(n, R, I) = \mathrm{Ker}\big(\mathrm{St}(n, R) \twoheadrightarrow \mathrm{St}(n, R/I)\big)$. With this end we construct two inverse homomorphisms

$$\mathrm{St}^*(n, R, I) \underset{\kappa}{\overset{\iota}{\rightleftarrows}} \mathrm{St}(n, R, I).$$

Firstly, we define ι as a homomorhism to an absolute group

$$\mathrm{St}^*(n, R, I) \xrightarrow{\iota} \mathrm{St}(n, R).$$

We do not need an assumption that I is a splitting ideal for that.

Clearly, we map elements $F(u, v)$ to van der Kallen's another generators $X(u, v)$. To find images for $S(u, v)$ as well we recall that van der Kallen in [1, pp. 3.8–3.10] defines also elements

$$x(u, v) \in \mathrm{St}(n, R),\ u^t v = 0,\ u_i = 0 \text{ or } v_i = 0 \text{ for some } 1 \le i \le n.$$

Now, "transpose" the definition [1, p. 3.13].

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Definition 2.4. For $u \in R^n$, $v \in \mathrm{E}(n, R)e_1$, $u^t v = 0$, consider the set $\overline{Y}(u, v) \subseteq \mathrm{St}(n, R)$ consisiting of all elements $y \in \mathrm{St}(n, R)$ which could be decomposed as a product $\prod x(w^k, v)$, where $\sum w^k = u$ and $w^k = (e_p v_q - e_q v_p)c^k$ for some $c^k \in R$, $1 \le p \ne q \le n$.

Since coloumns of elementary matrices are unimodular, $\overline{Y}(u, v)$ is not empty (see [1, pp. 3.1–3.2]). Obviously, for $x \in \overline{Y}(u, v)$ and $y \in \overline{Y}(w, v)$ holds $xy \in \overline{Y}(u + w, v)$.

Repeating [1, pp. 3.14–3.15] verbatim one shows that

Lemma 2.5. *for $g \in \mathrm{St}(n, R)$ holds*

a) $g\overline{Y}(u, v)g^{-1} \subseteq \overline{Y}(\phi(g)u, \phi(g)^* v)$;

b) $\overline{Y}(u, v)$ consists of exactly one element.

The only element of $\overline{Y}(u, v)$ we denote by $Y(u, v)$. We map elements $S(u, v) \in \mathrm{St}^*(n, R, I)$ to $Y(u, v)$. To show that such a homomorphism is well-defined we check that $X(u, v)$ and $Y(u, v)$ satisfy relations R1–R4, but R1–R3 are already checked.

Lemma 2.6. *Consider $(u, v^t) = (Me_1, e_2^t M^{-1})$, $M \in \mathrm{E}(n, R)$, $a \in I$. Then $X(u, va) = Y(ua, v)$.*

Proof. It suffices to show that $X(e_1, e_2 a) = Y(e_1 a, e_2)$. Compute the commutator $[Y(-e_3, e_2), X(e_1, e_3 a)]$ in two ways:

$$Y(-e_3, e_2)X(e_1, e_3 a)Y(-e_3, e_2)^{-1} \cdot X(e_1, -e_3 a) = X(e_1, e_2 a),$$

and

$$Y(-e_3, e_2) \cdot X(e_1, e_3 a)Y(e_3, e_2)X(e_1, e_3 a)^{-1} = Y(e_1 a, e_2).$$

□

Therefore, the homomorphism $\iota\colon \mathrm{St}^*(n, R, I) \to \mathrm{St}(n, R)$ sending $F(u, v)$ to $X(u, v)$ and $S(u, v)$ to $Y(u, v)$ is well-defined. Denote $\rho_I\colon R \twoheadrightarrow R/I$, then one can easily see that induced morphism on Steinberg groups

$$\rho_I^*\colon \mathrm{St}(n, R) \to \mathrm{St}(n, R/I)$$

sends $X(u, v)$ to $X(\rho_I(u), \rho_I(v))$ and $Y(u, v)$ to $Y(\rho_I(u), \rho_I(v))$. Observe that $\mathrm{E}(n, R) \twoheadrightarrow \mathrm{E}(n, R/I)$ is surjective so that for $x \in \mathrm{E}(n, R)e_1$ holds that $\rho_I(x) \in \mathrm{E}(n, R/I)e_1$. It follows from additivity relations that $X(u, 0) = 1 = Y(0, v)$, thus $\mathrm{Im}(\iota) \subseteq \mathrm{Ker}(\rho_I^*) = \mathrm{St}(n, R, I)$.

We constructed ι for an arbitrary $I \trianglelefteq R$. For I a splitting ideal we construct an inverse homomorphism
$$\kappa\colon \mathrm{St}(n,\, R,\, I) \to \mathrm{St}^*(n,\, R,\, I).$$
With this end we recall a presentation of $\mathrm{St}(n,\, R,\, I)$ as a group with action of $\mathrm{St}(n,\, R)$. For a splitting ideal such a presentation was firstly obtained by Swan [**Swa1**, **Swa2**, **Swa3**], and for an arbitrary I by Keune [2] and Loday [5].

For a group G acting on a group H from the left denote by ${}^g h$ the image of $h \in H$ under homomorphism corresponding by $g \in G$.

Let $I \trianglelefteq R$ be a splitting ideal then $\mathrm{St}(n,\, R,\, I)$ can be presented as a group with action of $\mathrm{St}(n,\, R)$ with the set of (relative) generators
$$\{y_{ij}(a) \mid a \in I,\ 1 \leq i \neq j \leq n\}$$
subject to relations

$$\text{(KL1)} \qquad y_{ij}(a)y_{ij}(b) = y_{ij}(a+b),$$
$$\text{(KL2)} \qquad {}^{x_{ij}(r)}y_{hk}(b) = y_{hk}(b),\ \text{ for } h \neq j,\ k \neq i,$$
$$\text{(KL3)} \qquad {}^{x_{ij}(r)}y_{jk}(b) \cdot y_{jk}(-b) = y_{ik}(rb),$$
$$\text{(KL4)} \qquad y_{ij}(a) \cdot {}^{x_{jk}(s)}y_{ij}(-a) = y_{ik}(as),$$
$$\text{(KL5)} \qquad {}^{x_{hk}(a)}\big({}^g y_{ij}(b)\big) = y_{hk}(a) \cdot {}^g y_{ij}(b) \cdot y_{hk}(-a)\ \text{ for } a \in I,\ g \in \mathrm{St}(n,\, R).$$

In other words, $\mathrm{St}(n,\, R,\, I)$ is isomorphic to a quotient group of a free group F on the set of generators
$$\mathrm{St}(n,\, R) \times \{y_{ij}(a) \mid a \in I,\ 1 \leq i \neq j \leq n\}$$
(the natural action of $\mathrm{St}(n,\, R)$ on F is given by ${}^f(g,\, y) = (fg,\, y)$) modulo the normal equivariant subgroup generated by KL1–KL5 (where $y_{ij}(a)$ stands for $(1,\, y_{ij}(a))$. Keune in Loday consider only stable Steinberg group $\mathrm{St}(R) = \varinjlim \mathrm{St}(n,\, R)$. This presentation for unstable group firstly appeared in [6] in the context of Chevalley groups. Observe that the definition of the relative Steinberg group in [6, Def. 3.3] differs from ours but remark after this definition and [6, Lem. 8] immediantly imply that for a splitting ideal $I \trianglelefteq R$ it coincides with the one given in the present paper. The existance of the presentation is proven in [6, Prop. 6]. Relation 5 in [6, Prop. 6] joins relations KL3 and KL4, and relations 2 and 3 follow from KL4 and KL5 with the use of KL6.

One could probably wonder why Tulenbaev needs his own presentation for $\mathrm{St}(n,\, R,\, I)$ and does not use the one of Keune–Loday. The reason is the following. To construct a map from Keune–Loday group one should firstly define an action of the absolute Steinberg group on the target group, in our context, the action of $\mathrm{St}(n,\, B_a)$ on $\mathrm{St}(n,\, B,\, I)$, what is probably possible, but seems to be much harder, then giving another presentation (in particular, one should define ${}^{x_{ij}(r/a^m)}y_{ji}(c)$).

Now, to construct $\kappa\colon \mathrm{St}(n,\, R,\, I) \to \mathrm{St}^*(n,\, R,\, I)$ we need to define an action of $\mathrm{St}(n,\, R)$ on $\mathrm{St}^*(n,\, R,\, I)$ and find elements $y^*_{ij}(a) \in \mathrm{St}^*(n,\, R,\, I)$ subject to KL1–KL5. To define a action of the absolute group we use van der Kallen's another presentaion for it. For $u \in \mathrm{Um}(n,\, R)$, $v \in R^n$, $u^t v = 0$ define $\alpha(u,\, v)\colon \mathrm{St}^*(n,\, R,\, I) \to \mathrm{St}^*(n,\, R,\, I)$ by $\alpha(u,\, v)\big(F(u',\, v')\big) = F(t(u,\, v)u',\, t(u,\, v)^* v')$, and $\alpha(u,\, v)\big(S(u',\, v')\big) = S(t(u,\, v)u',\, t(u,\, v)^* v')$. Obviously, the images of the generators satisfy R1–R4, so that $\alpha(u,\, v)$ is a well-defined automorphism. Also, α's themselves clearly satisfy K1–K2, thus $X(u,\, v) \mapsto \alpha(u,\, v)$ is a well-defined action of $\mathrm{St}(n,\, R)$ on $\mathrm{St}^*(n,\, R,\, I)$.

Next, define $y^*_{ij}(a) = F(e_i,\, e_j a)$, $a \in I$. These elements obviously satisfy KL1, KL2 and KL5. Check KL3:

$$\begin{aligned}{}^{x_{ij}(r)}y^*_{jk}(b) &= F(t_{ij}(r)e_j,\, t_{ji}(-r)e_k b) = F(e_i r + e_j,\, e_k b) = \\ &= S(e_i r b + e_j b,\, e_k) = S(e_i r b,\, e_k) S(e_j b,\, e_k) = F(e_i,\, e_k r b) F(e_j,\, e_k b).\end{aligned}$$

KL4 is similar. Finally, we have a well-defined map

$$\kappa\colon\ \mathrm{St}(n,\, R,\, I) \to \mathrm{St}^*(n,\, R,\, I)$$

sending $y_{ij}(a)$ to $y^*_{ij}(a)$. Obviously, $\iota \circ \kappa = \mathrm{id}$ (cf. [1, 3.6 d)]), thus κ is injective. Surjectivity of κ follows from Lemma 2.3 a), thus it is an isomorphism and ι is inverse to it.

At this point we obtained the presentation $\mathrm{St}^*(n,\, R,\, I)$ for the $\mathrm{St}(n,\, R,\, I)$ where $I \trianglelefteq R$ is a splitting ideal, $n \geq 4$.

Below we obtain the main result of this section, namely, construct a map

$$\mathrm{T}\colon\ \mathrm{St}(n,\, R_a[X],\, XR_a[X]) \to \mathrm{St}(n,\, R \ltimes XR_a[X],\, XR_a[X]),$$

for $n \geq 4$. A local-global principle for Steinberg group and centrality of K_2 formally follow from existance of this map [6, Lem. 15, Lem. 16, proof of Th. 2].

We work in a more general situation.

Theorem 3. *Let B be a ring, $I \trianglelefteq B$, $a \in B$ such that $\forall\, x \in I$ there exists a unique $y \in I$ such that $ya = x$ (we denote this y by $\frac{x}{a}$, elements $\frac{x}{a^m}$ are also well defined). This requirement is equivalent to say that the principle localisation $\lambda_a\colon I \to I_a$ of ideal I is an isomorphism. Then there exists a map*

$$\mathrm{T}\colon\ \mathrm{St}(n,\, B_a,\, I) \to \mathrm{St}(n,\, B)$$

making the diagram

$$\begin{tikzcd} \mathrm{St}^*(2n,\, B,\, I) \arrow[r, "\iota"] \arrow[d, "\lambda^*_a"'] & \mathrm{St}(2n,\, B) \arrow[d, "\lambda^*_a"] \\ \mathrm{St}^*(2n,\, B_a,\, I) \arrow[r, "\iota"'] \arrow[ur, dashed, "\mathrm{T}"'] & \mathrm{St}(2n,\, B_a) \end{tikzcd}$$

commutative.

To prove this theorem we need to find some elements "$X(u,\, v)$" inside $\mathrm{St}(n,\, B)$ for $u \in \mathrm{E}(n,\, B_a)e_1$, $v \in I$. The idea is the following. Say a is not a zero divisor, then $ua^m \in B$ for some m and we will construct some element "$X(ua^m,\, va^{-m})$". One can not garantee that $ua^m \in \mathrm{E}(n,\, B)e_1 a^k$, however, the ideal generated by entries of ua^m contains some power of a (this is equivalent to say that ua^m becomes unimodular after the principle localisation in a). Denote $\mathrm{I}(u)$ the ideal generated by entries of $u \in R^n$, $\mathrm{I}(u) = \sum\limits_{k=1}^{n} u_k R$. Then, we need generators parametrised by pairs $(u,\, v)$ with $u^t v = 0$, $v \in I$, $a^m \in \mathrm{I}(u)$ for some $m \in \mathbb{N}$. Such generators are defined in [9].

Definition 2.7 (Tulenbaev)**.** For $u,\, v \in B^n$, $a \in \mathrm{I}(u)$ and $v_1, \ldots, v_N \in B^n$ such that $u^t v_k = 0$ $\forall\, k$, each v_k has at least two zero coordinates and $\sum_{k=1}^{N} v_k = v$ define $X_{u,v}(a) = \prod\limits_{k=1}^{N} x(u,\, v_k a)$. Tulenbaev shows that factors $x(u,\, v_k a)$ commute [9, Lem. 1.1 e)] and that for any other decomposition of v as a sum $v = \sum_{j=1}^{M} v'_j$ of vectors orthogonal to u and having two zero coordinates holds $\prod_{k=1}^{N} x(u,\, v_k a) = \prod_{j=1}^{M} x(u,\, v'_j a)$ [9, p. 3], i.e., elements $X_{u,v}(a)$ are well-defined. Obviously, $\phi(X_{u,v}(a)) = t(u,\, va)$.

Observe that Tulenbaev uses different notation for van der Kallen elements. He writes $X_{u,v}$ instead of $X(u,\,v)$ and $X(u,\,v)$ instead of $x(u,\,v)$. We keep van der Kallen's notation.

Lemma 2.8. *For $u,\,v,\,v'$ and $w \in B^n$, such that v and v' have decomposition as in above definition, $u^t w = 0$, $a,\,b \in \mathrm{I}(u)$, $c \in B$, $g \in \mathrm{St}(n,\,B)$ holds*

a) $X_{u,vc}(a) = X_{u,v}(ca)$,

b) $X_{uc,v}(ca) = X_{u,vc^2}(a)$,

c) $X_{u,v}(a)X_{u,v'}(a) = X_{u,v+v'}(a)$,

d) $g\,X_{u,wb}(a)g^{-1} = X_{\phi(g)u,\phi(g)^*wb}(a)$.

Proof. The statement of a) is obvious from the definition, b) follows from [9, Lem. 1.1 d)], c) is [9, Lem. 1.3 a)], d) is proven for $n \geq 5$ in [9, Lem. 1.3 b)] and afterwards Tulenbaev makes a remark that for $n = 4$ the statement is also true. Indeed, take $z \in B^n$ such that $z^t u = b$ and decompose

$$(z^t u)w = \sum_{i<j} w_{ij},$$

where $w_{ij} = (e_i u_j - e_j u_i)(w_i z_j - w_j z_i) = w_{ji}$. Each w_{ij} is orthogonal to u and has two zero coordinates ($n \geq 4$). Thus, $X_{u,wb}(a) = \prod_{i<j} x(u,\,w_{ij}a)$. It is enough to prove the statement of d) for $g = x_{hk}(r)$. If $h \neq i, j$ or $\{h, k\} = \{i, j\}$, $\phi(g)^* w_{ij} a$ still has two zero coordinates. Consider the case $j = h$, $i \neq k$. With [1, p. 3.12] we get $g\,x(u,\,w_{ij}a)g^{-1} = x(\phi(g)u,\,\phi(g)^* w_{ij}a)$. Denote $u_{ij} = e_i u_j - e_j u_i$ and $c_{ij} = w_i z_j - w_j z_i$. Using

$$\phi(g)^* u_{ij} = (\phi(g)u)_{ij} + (\phi(g)u)_{ki}\,r$$

and [1, p. 3.11] one obtains

$$x(\phi(g)u,\,\phi(g)^* w_{ij}a) = x(\phi(g)u,\,(\phi(g)u)_{ij}c_{ij}a) \cdot x(\phi(g)u,\,(\phi(g)u)_{ki}\,rc_{ij}a).$$

Decomposing in such a way each $\phi(g)^* w_{ij} a$ which does not have two zero coordinates we finally get a product from the definition of $X_{\phi(g)u,\phi(g)^*wb}(a)$. □

Siilarly, one can define "transposed" version of Tulenbaev's elements $X_{u,v}(a)$.

Definition 2.9. For $u,\,v \in B^n$, $a \in \mathrm{I}(v)$, and $u_1, \ldots, u_N \in B^n$ such that $u_k^t v = 0$, each u_k has at least two zero coordinates and $u = \sum_{k=1}^N u_k$ define $Y_{u,v}(a) = \prod\limits_{k=1}^{N} x(u_k a,\,v)$. Repeating Tulenbaev's argumentation [9, p. 3] one can show that the definition does not depend on the order of factors and on choice of decomposition of u.

Now, one can repeat van der Kallen's and Tulenbaev's argumentation to prove transposed version of Lemma 2.8. We leave this to the reader.

Lemma 2.10. *For $u,\,u',\,w$ and $v \in B^n$, such that u and u' have decomposition as in above definition, $w^t v = 0$, $a,\,b \in \mathrm{I}(v)$, $c \in B$, $g \in \mathrm{St}(n,\,B)$ holds*

a) $Y_{uc,v}(a) = Y_{u,v}(ca)$,

b) $Y_{u,vc}(ca) = Y_{uc^2,v}(a)$,

c) $Y_{u,v}(a)X_{u',v}(a) = Y_{u+u',v}(a)$,

d) $g\,Y_{wb,v}(a)g^{-1} = Y_{\phi(g)wb,\phi(g)^*v}(a)$.

Finally, it only remains to show that for a "good" pair $(u,\,v)$ elements X and Y coincide.

Lemma 2.11. *Consider $w,\,u,\,z,\,v,\,x,\,y \in B^n$ $a,\,r \in B$ such that $w^t u = a$, $z^t v = a$, $x^t y = a$ and pairs $(w,\,u)$, $(z,\,v)$ and $(x,\,y)$ are mutually orthogonal. Then*

$$X_{u,v}(ra^3) = Y_{u,v}(ra^3).$$

Proof. On one hand,

$$[Y_{x,-v}(ra),\, X_{u,y}(a)] = X_{t(x,-vra)u,t(x,-vra)^*y}(a)X_{u,-y}(a) = \\ = X_{u,y+vra^2}(a)X_{u,-y}(a) = X_{u,v}(ra^3).$$

On the other,

$$[Y_{x,-v}(ra),\, X_{u,y}(a)] = Y_{x,-v}(ra)Y_{t(u,ya)x,t(u,ya)^*v}(ra) = \\ = Y_{-x,v}(ra)Y_{x+ua^2,v}(ra) = Y_{u,v}(ra^3).$$

□

Now, we are ready to construct a map $\mathrm{T}\colon \mathrm{St}(n,\, B_a,\, I) \to \mathrm{St}(n,\, B)$.

Proof of Theorem 3. For $u = Me_1$, $M \in \mathrm{E}(n,\, B_a)$, $v \in I^n$, $u^tv = 0$ denote $w = M^*e_1$ (then $wu = 1$) and choose $m \in \mathbb{N}$ such that some $\tilde{w}$, $\tilde{u} \in B^n$ localise to wa^m and ua^m, $\tilde{u}^tv = 0$ and $\tilde{w}^t\tilde{u} = a^{2m}$. Now set $\mathrm{T}(F(u,\, v)) = X_{\tilde{u},v/a^{3m}}(a^{2m})$. Lemma 2.8 a) and b) garantees that this definition does not depend on the choice of m and the lifts $\tilde{u}$ and $\tilde{w}$. Similarly, set $\mathrm{T}(S(u,\, v)) = Y_{u/a^{3m},\tilde{v}}(a^{2m})$. These elements satisfy R1–R3 by Lemmas 2.8 and 2.10. For $u = Me_1$, $v = M^*e_2$, $M \in \mathrm{E}(n,\, B_a)$ consider $w = M^*e_1$, $z = Me_2$, $x = Me_3$ and $y = M^*e_3$. Then, multiply these element on appropriate power of a, take their lifts to B and apply Lemma 2.11 to get R4. So that, the map T is well-defined.

Commutativity of the diagram follows directly from the definitions of elements $X(u,\, v)$, $Y(u,\, v)$ and $X_{u,v}(a)$, $Y_{u,v}(a)$ and the possibility to redistribute the powers of a after localisation. □

3. Patching Tulenbaev maps

The next section is devoted to the proof of Theorem 3 for Steinberg groups corresponding to an arbitrary simply-laced root system of rank $l \geq 3$. To do so, we will "glue" maps T constructed in the present section for systems of type A_3. All main results of the present paper follow from the existence of such a map.

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