ON CENTRALITY OF ORTHOGONAL K_2

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Abstract. We give a short uniform proof of centrality of $\mathrm{K}_2(\Phi, R)$ for all simply-laced root systems Φ of rank ≥ 3.

Inroduction

The main goal of this paper is to establish the local-global principle and centrality of orthogonal K_2 for arbitrary commutative ring R.

Recall that to each reduced root system Φ and a commutative ring R one can associate a split simple group $\mathrm{G}(\Phi, R)$ called the *Chevalley group*. For Φ of rank ≥ 2 the abstract group $\mathrm{G}(\Phi, R)$ contains a large normal subgroup $\mathrm{E}(\Phi, R)$ generated by the elementary root unipotents $t_\alpha(\xi)$, $\xi \in R$, $\alpha \in \Phi$. The quotient group $\mathrm{K}_1(\Phi, R) = \mathrm{G}(\Phi, R)/\mathrm{E}(\Phi, R)$ is called the *unstable* K_1*-group*. This group shares many properties with its stable counterpart $\mathrm{SK}_1(R) = \varinjlim_{l\to\infty} \mathrm{K}_1(\mathsf{A}_\ell, R)$ which, in turn, is an essential direct summand of $\mathrm{K}_1(R)$.

Similarly to the definition of the algebraic K_2-functor, one can define $\mathrm{K}_2(\Phi, R)$ as the kernel of the canonical projection $\phi\colon \mathrm{St}(\Phi, R) \twoheadrightarrow \mathrm{E}(\Phi, R)$. Here $\mathrm{St}(\Phi, R)$ stands for the *Steinberg group* of type Φ, i. e. the group defined by formal generators $x_\alpha(\xi)$ modeling root unipotents $t_\alpha(\xi)$ and the explicit list of relations called Steinberg relations (see Section 1).

It is classically known that $\mathrm{K}_2(\Phi, R)$ is a central subgroup of $\mathrm{St}(\Phi, R)$ if R is commutative local ring and Φ has rank ≥ 2 (see [5]). However, to prove that centrality of K_2 holds for *arbitrary* R is much harder. The first result in this direction was obtained in 1977 by W. van der Kallen for $\Phi = \mathsf{A}_\ell$, $\ell \geq 3$ using the technique called "another presentation". Using a similar approach, the first-named author has established centrality of $\mathrm{K}_2(\mathsf{C}_\ell, R)$ for $\ell \geq 3$, see [3]. In turn, the key result of this paper is the following theorem.

Theorem 1. *Let R be an arbitrary commutative ring and $\ell \geq 4$ then the group $\mathrm{K}_2(\mathsf{D}_\ell, R)$ is contained in the centre of $\mathrm{St}(\mathsf{D}_\ell, R)$.*

Notice that the analogous statement in the case $\Phi = \mathsf{E}_\ell$, $\ell = 6, 7, 8$ has been demonstrated by the second named author in [4]. Theorem 1 is essentially a corollary of the following local-global principle.

Theorem 2. *Let Φ be an irreducible simply-laced root system of rank ≥ 3 and R be arbitrary commutative ring. Then an element $g \in \mathrm{St}(\Phi, R[X])$ satisfying $g(0) = 1 \in \mathrm{St}(\Phi, R)$ is trivial in $\mathrm{St}(\Phi, R[X])$ if and only if the elements $g_M \in \mathrm{St}(\Phi, R_M[X])$ are trivial for all maximal ideals $M \triangleleft R$.*

The special case $\Phi = \mathsf{A}_\ell$, $\ell \geq 4$ of Theorem 2 was demonstrated for the first time by Tulenbaev, see [7, Theorem 2.1], while cases $\Phi = \mathsf{C}_\ell$, $\ell \geq 3$ and $\Phi = \mathsf{E}_\ell$, $\ell = 6, 7, 8$ have been analyzed in [2]

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and [4], respectively. The special cases $\Phi = \mathsf{A}_3$ and $\Phi = \mathsf{D}_\ell$, $\ell \geq 4$ of Theorem 2 are new and have not been known before.

There are known counterexamples to centrality of K_2 in rank 2 (see [8]). This shows that our assumption on the rank of Φ in Theorem 2 is strict.

Compare Theorem 2 with the local-global principle for K_1 which states that an element of the congruence subgroup $g \in \mathrm{G}(\Phi,\, R[X], XR[X])$ is elementary (i. e. lies in $\mathrm{E}(\Phi, R[X])$) if and only if $g_M = 1 \in \mathrm{E}(\Phi,\, R_M)$ for all maximal ideals $M \lhd R$.

Our proof of Theorem 2 is similar to that of [4, Theorem 1]. Using the same patching technique we reduce the problem to the case $\Phi = \mathsf{A}_3$. Then, using a new presentation of the relative Steinberg group $\mathrm{St}(4, R, I)$ we reprove Tulenbaev's key lemma (cf. [7, Lemma 2.3]) in the rank 3 case.

The rest of the paper is organised as follows. In Section 2 we describe a new symmetric presentation for the relative Steinberg group of rank 3. This presentation is then used in Section 3 to establish the particular case $\Phi = \mathsf{A}_3$ of Theorem 2. Finally, in Section 4 we derive our principal results from this special case and the amalgamation theorem for relative Steinberg groups from [4].

1. Preliminaries

Throughout this paper R denotes an associative commutative ring with identity. All commutators are left-normed, i. e. $[x,\, y] = xyx^{-1}y^{-1}$.

We denote by R^n the free R-module with basis $e_1, \ldots, e_n$ and by $\mathrm{Um}(n,\, R)$ the subset of unimodular columns $v \in R^n$ whose entries generate R as an ideal.

As usual, $\mathrm{E}(n, R) = \mathrm{E}(\mathsf{A}_{n-1}, R)$ denotes the elementary group, i. e. the subgroup of the general linear group $\mathrm{GL}(n, R)$ generated by transvections $t_{ij}(r) = 1 + r \cdot e_{ij}$, $1 \leq i \neq j \leq n$, $r \in R$, where 1 stands for the identity matrix and e_{ij} for the matrix unit.

The Steinberg group $\mathrm{St}(n, R) = \mathrm{St}(\mathsf{A}_{n-1}, R)$ is the group presented by generators $x_{ij}(r)$, $1 \leq i \neq j \leq n$, $r \in R\}$ and relations

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s), \tag{S1}$$

$$[x_{ij}(r),\, x_{hk}(s)] = 1, \text{ for } h \neq j,\ k \neq i, \tag{S2}$$

$$[x_{ij}(r),\, x_{jk}(s)] = x_{ik}(rs). \tag{S3}$$

The natural projection $\phi\colon\ \mathrm{St}(n,\, R) \to \mathrm{E}(n,\, R)$ sends $x_{ij}(r)$ to $t_{ij}(r)$.

In [6] Suslin showed for $n \geq 3$ that the elementary group $\mathrm{E}(n, R)$ coincides with the subgroup of $\mathrm{GL}(n,\, R)$ generated by matrices of the form $t(u,\, v) = 1 + uv^t$ where $u \in \mathrm{Um}(n,\, R)$, $v \in R^n$ and $u^tv = 0$. Here u^t stands for the transpose of u. This result clearly implies that $\mathrm{E}(n,\, R)$ is normal inside $\mathrm{GL}(n,\, R)$. Afterwards, W. van der Kallen developed Suslin's ideas and showed for $n \geq 4$ that the Steinberg group $\mathrm{St}(n,\, R)$ is isomorphic to the group presented by generators

$$\{X(u,\, v) \mid u \in \mathrm{Um}(n,\, R),\ v \in R^n,\ u^tv = 0\}$$

and the following list of relations (see [1, Theorem 1]):

$$X(u,\, v)X(u,\, w) = X(u,\, v+w), \tag{K1}$$

$$X(u,\, v)X(u',\, v')X(u,\, v)^{-1} = X(t(u,\, v)u',\, t(v,\, u)^{-1}v'). \tag{K2}$$

This presentation clearly implies that $\phi\colon\ \mathrm{St}(n,\, R) \to \mathrm{E}(n,\, R)$ is a central extension. Notice that our notation for the generators of $\mathrm{St}(n, R)$ slightly differs from that of [1], e. g. we parametrize elements $X(u, v)$ with two columns rather than with one column and one row. Our generator $X(u, v)$ corresponds to van der Kallen's generator (u, v^t).

Recall that an ideal $I \trianglelefteq R$ is called a *splitting ideal* if the canonical projection $R \twoheadrightarrow R/I$ splits as a unital ring morphism, i. e. one has $\pi\sigma = 1$.

$$0 \longrightarrow I \longrightarrow R \underset{\sigma}{\overset{\pi}{\rightleftarrows}} R/I \longrightarrow 0$$

For a splitting ideal $I \trianglelefteq R$ the *relative Steinberg group* $\mathrm{St}(n,\, R,\, I) = \mathrm{St}(\mathsf{A}_{n-1}, R, I)$ can be defined as the kernel $\mathrm{Ker}\,\big(\mathrm{St}(n,\, R) \twoheadrightarrow \mathrm{St}(n,\, R/I)\big)$.

For a general I the relative Steinberg group $\mathrm{St}(\Phi, R, I)$ is no more a subgroup of $\mathrm{St}(\Phi, R)$ but rather is its central extension. For the purposes of the present text it suffices to consider only relative Steinberg groups corresponding to splitting ideals I. For more information regarding the general case we refer the reader to [4, Section 3].

2. Yet another presentation for the relative Steinberg group.

The key ingredient in Tulenbaev's proof of the local-global principle is the construction of the homomorphism T which fits into the following commutative diagram.

$$\begin{tikzcd}
\mathrm{St}(n,\, R \ltimes XR_a[X],\, XR_a[X]) \arrow[r, hook] \arrow[d] & \mathrm{St}(n, R \ltimes XR_a[X]) \arrow[d, "\lambda_a^*"'] \\
\mathrm{St}(n,\, R_a[X],\, XR_a[X]) \arrow[r, hook] \arrow[ur, dashed, "T"'] & \mathrm{St}(n, R_a[X])
\end{tikzcd} \tag{2.1}$$

Here $n \geq 5$, a is arbitrary nonnilpotent element of R and λ_a^* is the morphism induced by the morphism $\lambda_a \colon R \to R_a$ of principal localisation at a. Notice that $XR_a[X]$ is a splitting ideal for both $R_a[X]$ and $R \ltimes XR_a[X]$. It is clear that the image of T is contained in the image of the top arrow, therefore this implies that the relative Steinberg groups in the left-hand side of the diagram are isomoprhic.

In order to construct the arrow T Tulenbaev uses the following presentation for the relative Steinberg group, see [7, Proposition 1.6].

Proposition 2.2. *For a splitting ideal I and $n \geq 4$ the group $\mathrm{St}(n,\, R,\, I)$ is isomorphic to the group defined by generators*

$$\{X(u,\, v) \mid u \in \mathrm{E}(n,\, R)e_1,\ v \in I^n,\ u^t v = 0\}$$

and the following set of relations

$$X(u,\, v)X(u,\, w) = X(u,\, v + w), \tag{T1}$$

$$X(u,\, v)X(u',\, v')X(u,\, v)^{-1} = X(t(u,\, v)u',\, t(v,\, u)^{-1}v'), \tag{T2}$$

$$X(ur + w,\, v) = X(u,\, vr)X(w,\, v) \ \text{ for } r \in R,\ (u,\, w) \in \mathrm{Um}_{n\times 2}(R). \tag{T3}$$

Moreover, the group $\mathrm{St}(n, R)$ decomposes into the semidirect product $\mathrm{St}(n,\, R, I) \rtimes \mathrm{St}(n,\, R/I)$.

Here $\mathrm{Um}_{n\times 2}(R)$ denotes the set of $n \times 2$ *unimodular matrices*, i. e. matrices M such that there exists a $2 \times n$ matrix N satisfying $NM = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the definition of the semidirect product the group $\mathrm{St}(n, R/I)$ acts on $\mathrm{St}(n, R, I)$ by conjugation via the splitting map, i. e. ${}^g X(u,\, v) = X(\phi(\sigma(g))u, \phi(\sigma(g))^* v)$ for $g \in \mathrm{St}(n, R/I)$.

Proof. For the sake of completeness we present below a more detailed version of Tulenbaev's proof. Tulenbaev constructs a pair of mutually inverse maps

$$\mathrm{St}(n, R, I) \rtimes \mathrm{St}(n, R/I) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{St}(n, R).$$

Define the map ψ by $\psi(x_{ij}(\xi)) = (X(e_i, (\xi - \pi(\xi)e_j),\ x_{ij}(\pi(\xi)))$. We have to check that ψ preserves the relations (S1)–(S3). For example, let us check (S2). First of all, recall that in any semidirect product of groups one can compute the commutator of two elements as follows:

$$[(a, b), (c, d)] = (a \cdot {}^{b}c \cdot {}^{bdb^{-1}}a^{-1} \cdot {}^{[b,d]}c^{-1}, [b, d]).$$

For $\xi \in R$ set $\xi' = \xi - \pi(\xi)$. Specializing $a = X(e_i, \xi' e_j)$, $b = x_{ij}(\pi(\xi))$, $c = X(e_j, \eta' e_k)$, $d = x_{jk}(\pi(\eta))$ and using the definition of the conjugation action one computes the commutator $[\psi(x_{ij}(\xi), \psi(x_{jk}(\eta))]$ as follows:

$$\left(X(e_i, \xi' e_j) \cdot X(e_j + \pi(\xi)e_i, \eta' e_k) \cdot X(e_i, \pi(\eta)\xi' e_k - \xi' e_j) \cdot X(e_j, -\eta' e_k),\ x_{ik}(\pi(\xi\eta))\right).$$

After applying additivity relations (T1), (T3) and moving the factor $X(e_i, -\xi' e_j)$ to the left hand side of the formula by means of relation (T2) one gets the following expression

$$(X(e_j, \eta' e_k) \cdot X(e_i, (\xi'\eta' + \pi(\xi)\eta' + \pi(\eta)\xi')e_k \cdot X(e_j, -\eta' e_k),\ x_{ik}(\pi(\xi\eta)).$$

Since $\xi'\eta' + \pi(\xi)\eta' + \pi(\eta)\xi' = \xi\eta - \pi(\xi\eta)$ the above expression clearly simplifies to $\psi(x_{ik}(\xi\eta))$, as claimed.

It remains to check that the map φ given by $\varphi\left((X(u, v), 1)\right) = X(u, v)$, $\varphi\left((1, x_{ij}(\xi))\right) = X(e_i, \sigma(\xi)e_j)$ is well-defined and inverse to ψ. $\square$

Remark 2.3. Notice that from the proof of the above proposition it follows that one could replace the family of relations (T3) in the definition of $\mathrm{St}(n, R, I)$ with the following smaller family:

(T3') $\quad X(e_i r + e_j,\, e_k) = X(e_i,\, rae_k)X(e_j,\, ae_k)$ for $r \in R,\ a \in I$ and i, j, k distinct.

Both "another presentations" of van der Kallen and Tulenbaev are given in terms of generators parametrised by pairs of vectors, where the first one is "good" in some sense (i. e. unimodular or a column of an elementary matrix) while the second one is arbitrary. It is easy to formulate the additivity property in the second argument for these generators (cf. (K1), (T1)), while it is not so easy when it comes to the additivity in the first argument (cf. (T3)).

The construction of the homomorphism $T\colon \mathrm{St}(n,\, R_a[X],\, XR_a[X]) \to \mathrm{St}(n,\, R \ltimes XR_a[X])$ amounts to choosing certain elements in the group $\mathrm{St}(n,\, R \ltimes XR_a[X])$ and proving that these elements satisfy relations (T1)–(T3), see [7, Lemmas 1.2 and 1.3 c)]. The main problem with this recipe is that the assumption $n \geq 5$ is essential to check that the relation (T3) holds.

It is not possible to choose arbitrary orthogonal vectors u, v as parameters for the generators because in this situation it is not even known that $t(u,\, v) \in \mathrm{E}(n,\, R)$, see [**Rao**].

To generalise Tulenbaev's results for $n = 4$ we will use a more "symmetric" presentation with two types of generators: $F(u,\, v)$ with u "good" and v arbitrary and $S(u,\, v)$ with u arbitrary and v "good". The generators $F(u,\, v)$ will be additive in the second component, while $S(u,\, v)$ will be additive in the first one. When u and v are both "good" we require these two generators to coincide. More formally, we have the following presentation.

Definition 2.4. For $I \trianglelefteq R$ and $n \geq 4$ define $\mathrm{St}^*(n,\, R,\, I)$ to be the group with the set of generators

$$\{F(u,\, v) \mid u \in \mathrm{E}(n,\, R)e_1,\ v \in I^n,\ u^t v = 0\} \cup \{S(u,\, v) \mid u \in I^n,\ v \in \mathrm{E}(n,\, R)e_1,\ u^t v = 0\}$$

subject to relations

(R1) $\quad F(u,\, v)F(u,\, w) = F(u,\, v + w)$,

(R2) $\quad S(u,\, v)S(w,\, v) = S(u + w,\, v)$,

(R3) $\quad F(u,\, v)F(u',\, v')F(u,\, v)^{-1} = F(t(u,\, v)u',\, t(v,\, u)^{-1}v')$,

(R4) $\quad F(u,\, va) = S(ua,\, v)$, for all $a \in I$, $(u,\, v^t) = (Me_1,\, e_2^t M^{-1})$, $M \in \mathrm{E}(n,\, R)$.

Notice that we only described how a generator $F(u,v)$ acts via conjugation on another generator $F(u',v')$ and have omitted three similar relations involving generators $S(u,v)$ of the second type. The reason for this is the following lemma which asserts that the "missing" relations will follow automatically from (R1)–(R4).

Lemma 2.5. *Denote by* $\phi\colon \mathrm{St}^*(n,\,R,\,I) \to \mathrm{E}(n,\,R)$ *the natural map sending* $F(u,\,v) \mapsto t(u,\,v)$ *and* $S(u,\,v) \mapsto t(u,\,v)$*. Then the following facts are true.*

a) $\mathrm{St}^*(n,\,R,\,I)$ *is generated as an abstract group by the set of elements*

$$\{F(u,\,va) \mid a \in I,\ (u,\,v^t) = (Me_1,\,e_2^tM^{-1}),\ M \in \mathrm{E}(n,\,R)\};$$

b) for any $g \in \mathrm{St}^*(n,\,R,\,I)$ *one has*

$$gF(u,\,v)g^{-1} = F(\phi(g)u,\,\phi(g^{-1})^tv); \tag{R3'}$$

c) for any $g \in \mathrm{St}^*(n,\,R,\,I)$ *one has*

$$gS(u,\,v)g^{-1} = S(\phi(g)u,\,\phi(g^{-1})^tv); \tag{R3''}$$

d) there is a "transpose automorphism" defined on $\mathrm{St}^*(n,R,I)$ *satisfying*

$$F(u,\,v)^t = S(v,\,u), \quad S(u,\,v)^t = F(v,\,u).$$

Proof. Let $F(u,\,v)$ be an arbitrary generator of the first type and let $M \in \mathrm{E}(n,\,R)$ be such that $F(u,\,v) = F(Me_1,\,M^*\tilde{v})$. Here we denote by M^* the contragradient matrix, $M^* = (M^{-1})^t$. By (R1) we have

$$F(u,\,v) = \prod_{k\neq 1} F(Me_1,\,M^*e_k\tilde{v}_k)$$

where $\tilde{v}_k$ stands for k-th coordinate of $\tilde{v} = \sum e_i\tilde{v}_i$. Applying relations (R2), (R4) we get

$$S(u,\,v) = \prod_{k\neq 1} S(Ne_k\tilde{u}_k,\,N^*e_1) = \prod_{k\neq 1} F(Ne_k,\,N^*e_1\tilde{u}_k)$$

and, thus, obtain (a). Obviously, (b) follows from (a). To prove (c) it suffices to show that

$$F(u,\,v)S(u',\,v')F(u,\,v)^{-1} = S(t(u,\,v)u',\,t(u,\,v)^*v').$$

For $S(u',\,v') = \prod_{k\neq 1} F(Ne_k,\,N^*e_1\tilde{u}_k)$ we have

$$\begin{aligned} F(u,\,v)S(u',\,v')F(u,\,v)^{-1} &= F(u,\,v)\prod F(Ne_k,\,N^*e_1\tilde{u}_k)F(u,\,v)^{-1} = \\ &= \prod F(t(u,\,v)Ne_k,\,t(u,\,v)^*N^*e_1\tilde{u}_k) = \\ &= \prod S(t(u,\,v)Ne_k\tilde{u}_k,\,t(u,\,v)^*N^*e_1) = S(t(u,\,v)u',\,t(u,\,v)^*v'). \end{aligned}$$

Finally, (d) follows from (c). $\square$

Our next goal is to show for a splitting ideal $I \trianglelefteq R$ that the group $\mathrm{St}^*(n,\,R,\,I)$ is isomorphic to $\mathrm{St}(n,\,R,\,I) = \mathrm{Ker}\big(\mathrm{St}(n,\,R) \twoheadrightarrow \mathrm{St}(n,\,R/I)\big)$. With this end we construct two mutually inverse homomorphisms

$$\mathrm{St}^*(n,\,R,\,I) \underset{\kappa}{\overset{\iota}{\rightleftarrows}} \mathrm{St}(n,\,R,\,I).$$

In order to construct the arrow κ we use the presentation from Proposition 2.2. We set

$$\kappa(X(u,v)) = F(u,v),\ u \in \mathrm{E}(n,R)e_1,\ v \in I^n.$$

In view of Remark 2.3 it suffices to check that $F(u,v)$ satisfy the relations (T1), (T2), (T3'). The validity of relations (T1), (T2) is obvious while (T3') follows from (R2) and (R4), indeed:

$$F(e_i, rae_k)F(e_j, ae_k) = S(rae_i, e_k)S(ae_j, e_k) = S(a(re_i + e_j), e_k) = F(e_i r + e_j, ae_k).$$

As for the map ι we firstly define it as a homomorphism to the absolute group (for which we de not need the assumption that I is splitting).

$$\mathrm{St}^*(n,\, R,\, I) \xrightarrow{\iota} \mathrm{St}(n,\, R).$$

Clearly, we must map the elements $F(u,\, v)$ to van der Kallen's generators $X(u,\, v)$. We should find the images for the elements $S(u,\, v)$ as well. First of all, recall that van der Kallen in [1, pp. 3.8–3.10] constructs the following elements

$$x(u,\, v) \in \mathrm{St}(n,\, R),\ u^t v = 0,\ u_i = 0 \text{ or } v_i = 0 \text{ for some } 1 \leq i \leq n.$$

The following definition is the "transpose" of [1, p. 3.13].

Definition 2.6. For $u \in R^n$, $v \in \mathrm{E}(n,\, R)e_1$, $u^t v = 0$, consider the set $\overline{Y}(u,\, v) \subseteq \mathrm{St}(n,\, R)$ consisiting of all elements $y \in \mathrm{St}(n,\, R)$ that admit the decomposition $y = \prod x(w^k,\, v)$, where $\sum u^k = u$ and $u^k = c^k \cdot (e_p v_q - e_q v_p)$ for some $c^k \in R$, $1 \leq p \neq q \leq n$.

Since columns of elementary matrices are unimodular, one can find $w \in R^t$ such that $w^t v = 1$. It is not hard to check that

$$(w^t \cdot v) \cdot u = \sum_{p<q} u_{pq},\ \text{where } u_{pq} = (u_p w_q - u_q w_p) \cdot (v_q e_p - v_p e_q) \in R^n, \tag{2.7}$$

therefore $\overline{Y}(u,\, v)$ is not empty (cf. [1, pp. 3.1–3.2]). Obviously, for $x \in \overline{Y}(u,\, v)$ and $y \in \overline{Y}(w,\, v)$ one has $xy \in \overline{Y}(u + w,\, v)$.

Repeating [1, pp. 3.14–3.15] verbatim one shows the following.

Lemma 2.8. *For $g \in \mathrm{St}(n,\, R)$ one has*

a) $g\overline{Y}(u,\, v)g^{-1} \subseteq \overline{Y}(\phi(g)u,\, \phi(g)^* v)$;
b) $\overline{Y}(u,\, v)$ *consists of exactly one element.*

We denote the only element of $\overline{Y}(u,\, v)$ by $Y(u,\, v)$. Now we can finish the definition of ι by requiring that $\iota(S(u,\, v)) = Y(u,\, v)$ for $S(u,\, v) \in \mathrm{St}^*(n,\, R,\, I)$. To show that the map ι is well-defined we should check that elements $X(u,\, v)$ and $Y(u,\, v)$ satisfy the relations (R1)–(R4) (with F's and S's replaced with X's and Y's). Only the validity of relation (R4) is not immediately obvious.

Lemma 2.9. *For $(u,\, v^t) = (Me_1,\, e_2^t M^{-1})$, $M \in \mathrm{E}(n,\, R)$, $a \in I$ one has $X(u,\, va) = Y(ua,\, v)$.*

Proof. In view of the above lemma we only need to show that $X(e_1,\, e_2 a) = Y(e_1 a,\, e_2)$. This equality can be obtained by computing the commutator $[Y(-e_3,\, e_2),\, X(e_1,\, e_3 a)]$ in two ways:

$$Y(-e_3,\, e_2)X(e_1,\, e_3 a)Y(-e_3,\, e_2)^{-1} \cdot X(e_1,\, -e_3 a) = X(e_1,\, e_2 a),\ \text{and}$$

$$Y(-e_3,\, e_2) \cdot X(e_1,\, e_3 a)Y(e_3,\, e_2)X(e_1,\, e_3 a)^{-1} = Y(e_1 a,\, e_2). \quad \square$$

Since $\pi^*(\iota(F(u,v))) = X(\pi(u),\, 0) = 1$ and $\pi^*(\iota(S(u,v)) = Y(0,\, \pi(v)) = 1$ we get that $\mathrm{Im}(\iota) \subseteq \mathrm{Ker}(\pi^*) = \mathrm{St}(n,\, R,\, I)$. Thus, we have demonstrated the following result.

Proposition 2.10. *For a splitting ideal $I \trianglelefteq R$ and $n \geq 4$ the groups $\mathrm{St}^*(n,\, R,\, I)$ and $\mathrm{St}(n,\, R, I)$ are isomorphic.*

3. The local-global principle for K_2.

Now we turn to the main result of this paper, namely, we construct for $n \geq 4$ the map

$$T\colon \mathrm{St}(n,\, R_a[X],\, XR_a[X]) \to \mathrm{St}(n,\, R \ltimes XR_a[X]),$$

that fits into the diagram (2.1). The local-global principle for the Steinberg group and centrality of K_2 will formally follow from the existence of this map, see [4, Lemmas 15–16, and the proof of Theorem 2].

We will prove a somewhat more general result. Let B be a commutative unital ring. A B-module M is called *uniquely r-divisible* if for every $m \in M$ there exists a sole $m' \in M$ such that $rm' = m$ (we denote such m' by $\frac{m}{a}$). Clearly, M is uniquely r-divisible if and only if the morphism of principal localisation $\lambda_r\colon M \to M_r$ is an isomorphism.

Theorem 3. *Let B be a ring, $a \in B$ and let I be an ideal B that is uniquely a-divisible. Then for $n \geq 4$ there exists a map $T\colon \mathrm{St}(n,\, B_a,\, I) \to \mathrm{St}(n,\, B)$ which makes the following diagram commute.*

$$\begin{tikzcd} \mathrm{St}^*(n,\, B,\, I) \arrow[r, "\iota"] \arrow[d, "\lambda_a^*"'] & \mathrm{St}(n,\, B) \arrow[d, "\lambda_a^*"] \\ \mathrm{St}^*(n,\, B_a,\, I) \arrow[r, "\iota"'] \arrow[ur, dashed, "T"'] & \mathrm{St}(n,\, B_a) \end{tikzcd} \tag{3.1}$$

For a vector $u \in R^n$ we denote by $I(u)$ the ideal generated by the entries of $u \in R^n$, i. e. $I(u) = \sum\limits_{k=1}^{n} u_k R$. To prove Theorem 3 we follow the approach of Tulenbaev and construct yet another family of elements $X_{u,v}(a) \in \mathrm{St}(n,\, B)$, $a \in B$ which contains van der Kallen's generators $X(u,v)$ as a subfamily (i. e. $X_{u,v}(1) = X(u,v)$). The key feature of this definition is that the assumption $u \in \mathrm{Um}(n,B)$ is replaced with a weaker condition $a \in I(u)$ (this is equivalent to saying that u becomes unimodular after localisation in a).

Definition 3.2. For $u \in R^n$ we denote by $D(u)$ the set consisting of all $v \in B^n$ decomposing into a sum $v = \sum_{k=1}^{N} v_k$ where $v_1, \ldots, v_N \in B^n$ are such that $u^t v_k = 0$ and each v_k has at least two zero coordinates. Now for $a \in I(u)$ and $v \in D(u)$ set $X_{u,v}(a) = \prod\limits_{k=1}^{N} x(u,\, v_k a)$.

Tulenbaev shows that factors $x(u,\, v_k a)$, $1 \leq k \leq N$ commute [7, Lemma 1.1 e)] and that for any other decomposition $v = \sum_{j=1}^{M} v'_j$ one has $\prod_{k=1}^{N} x(u,\, v_k a) = \prod_{j=1}^{M} x(u,\, v'_j a)$, see the discussion following [7, Lemma 1.1]. Thus, the elements $X_{u,v}(a)$ are well-defined. Obviously, $\phi(X_{u,v}(a)) = t(u,\, va)$.

Remark 3.3. From the canonical decomposition (2.7) it follows that $bv \in D(u)$ for any $b \in I(u)$ and v such that $v^t u = 0$. In particular, if I is uniquely a-divisible and $a^k \in I(u)$, then every $v \in I^n$ satisfying $v^t u = 0$ is contained in $D(u)$.

Notice that Tulenbaev uses different notation for van der Kallen elements, e. g. he writes $X_{u,v}$ instead of $X(u,\, v)$ and $X(u,\, v)$ instead of $x(u,\, v)$. We stick to van der Kallen's notation.

Lemma 3.4. *For $u,\, w \in B^n$, $v, v' \in D(u)$ such that $u^t w = 0$, for $a,\, b \in I(u)$, $c \in B$, $g \in \mathrm{St}(n,\, B)$ the following are true:*

a) $X_{u,vc}(a) = X_{u,v}(ca)$,
b) $X_{uc,v}(ca) = X_{u,vc^2}(a)$,
c) $X_{u,v}(a) X_{u,v'}(a) = X_{u,v+v'}(a)$,

d) $g\,X_{u,wb}(a)g^{-1} = X_{\phi(g)u,\phi(g)^*wb}(a)$.

Proof. The statement of (a) is obvious from the definition, (b) follows from [7, Lemma 1.1 d)], (c) is exactly the statement of [7, Lemma 1.3 a)].

The assertion of (d) is proven in [7, Lemma 1.3 b)] under the assumption $n \geq 5$. Afterwards Tulenbaev remarks that the assertion remains true for $n = 4$. We will give a detailed proof of this fact. Indeed, take $z \in B^n$ such that $z^t u = b$ and write the canonical decomposition (2.7):

$$(z^t u)w = \sum_{i<j} c_{ij}u_{ij}, \text{ where } u_{ij} = u_j e_i - u_i e_j, \text{ and } c_{ij} = w_i z_j - w_j z_i.$$

Each $c_{ij}u_{ij}$ is orthogonal to u and, because of the assumption $n \geq 4$, has at least two zero coordinates. Thus, $X_{u,wb}(a) = \prod_{i<j} x(u,\, ac_{ij}u_{ij})$. It is enough to prove the assertion of (d) in the special case $g = x_{hk}(r)$. If $h \neq i, j$ or $\{h, k\} = \{i, j\}$ the vector $\phi(g)^* ac_{ij}u_{ij}$ still has two zero coordinates.

Now consider the case $j = h$, $i \neq k$. By [1, p. 3.12] one has

$$g\,x(u,\, ac_{ij}u_{ij})g^{-1} = x(\phi(g)u,\, ac_{ij}\,\phi(g)^* u_{ij}).$$

Set $u' = \phi(g)u$, since $\phi(g)^* u_{ij} = t_{kj}(-r)\cdot u_{ij} = u_j e_i - u_i e_j + r u_i e_k = u'_{ij} + r\,u'_{ki}$, we obtain from [1, p. 3.11] that

$$x(\phi(g)u,\, ac_{ij}\,\phi(g)^* u_{ij}) = x\left(u',\, c_{ij}a\,u'_{ij}\right)\cdot x\left(u',\, rc_{ij}a\,u'_{ki}\right).$$

Decomposing in this fashion each factor $g\,x(u,\, ac_{ij}u_{ij})g^{-1}$ for which $\phi(g)^* ac_{ij}\,u_{ij}$ does not have two zero coordinates, we arrive at a product satisfying the requirements of the definition of $X_{\phi(g)u,\phi(g)^*wb}(a)$. □

We will also need the "transposed" analogue of Tulenbaev's elements $X_{u,v}(a)$.

Definition 3.5. Let $v \in B^n$, $a \in I(v)$, and $u \in D(v)$, i.e. $u = \sum\limits_{i=1}^{M} u_i$ for some u_k having two zero components such that $v^t u_k = 0$. Set $Y_{u,v}(a) = \prod_{k=1}^{M} x(u_k a,\, v)$. Similarly to Definition 3.2 this definition does not depend neither on the order of factors nor on the choice of decomposition for u.

One can repeat van der Kallen's and Tulenbaev's arguments and prove the following transposed version of Lemma 3.4. We leave the proof of this lemma to the reader.

Lemma 3.6. *For u, u', w and $v \in B^n$, such that u and u' have decomposition as in above definition, $w^t v = 0$, a, $b \in I(v)$, $c \in B$, $g \in \mathrm{St}(n, B)$ holds*

a) $Y_{uc,v}(a) = Y_{u,v}(ca)$,
b) $Y_{u,vc}(ca) = Y_{uc^2,v}(a)$,
c) $Y_{u,v}(a)X_{u',v}(a) = Y_{u+u',v}(a)$,
d) $g\,Y_{wb,v}(a)g^{-1} = Y_{\phi(g)wb,\phi(g)^*v}(a)$.

Finally, it remains to show that for a "good" pair (u, v) the elements X and Y coincide.

Lemma 3.7. *Let u, v, x, y be elements of B^n and $b, r \in B$ be such that $u^t v = 0$, $x^t y = b$, $x^t v = 0$, $u^t y = 0$ and $b \in I(u) \cap I(v)$. Then one has $X_{u,b^4rv}(b) = Y_{b^4ru,v}(b)$.*

Proof. Compute $g = [Y_{-arx,v}(b),\, X_{u,ay}(b)]$ in two different ways using Lemmas 3.4 and 3.6:

$$g = X_{t(b^2rx,-v)u,\,t(b^2rx,-v)^*yb}(b)X_{u,-yb}(b) = X_{u,\,by+b^4rv}(b)\,X_{u,-by}(b) = X_{u,b^4rv}(b),$$

$$g = Y_{-brx,v}(b)Y_{t(u,b^2y)arx,\,t(u,b^2y)^*v}(b) = Y_{-brx,v}(b)\,Y_{brx+b^4ru,\,v}(b) = Y_{b^4ru,v}(b). \quad □$$

Now, we are ready to construct the desired map $T\colon \mathrm{St}(n,\, B_a,\, I) \to \mathrm{St}(n,\, B)$.

Proof of Theorem 3. Consider $u = Me_1$, $M \in \mathrm{E}(n, B_a)$ and $v \in I^n$ such that $u^t v = 0$. Set $w = M^* e_1$. Since $w^t u = 1$ there exist vectors $\tilde{w}$, $\tilde{u} \in B^n$ and a natural number m such that

$$\lambda_a(\tilde{w}) = wa^m,\ \lambda_a(\tilde{u}) = ua^m \text{ and moreover } \tilde{u}^t v = 0 \text{ and } \tilde{w}^t \tilde{u} = a^{2m}.$$

It is clear that $a^{2m} \in I(\tilde{u})$, moreover by Remark 3.3 we have $v/a^{3m} \in D(\tilde{u})$, therefore we are allowed to set $T(F(u,\, v)) = X_{\tilde{u}, v/a^{3m}}(a^{2m})$.

The first two assertions of Lemma 3.4 guarantee that this definition does not depend on the choice of m and the liftings $\tilde{u}$ and $\tilde{w}$. Similarly, we can define $T(S(u,\, v)) = Y_{u/a^{3m}, \tilde{v}}(a^{2m})$. In view of Lemmas 3.4 and 3.6 the map T preserves relations (R1)–(R3).

It remains to check for $u = Me_1$, $v = M^* e_2$ and $c \in I$ that

$$T(F(u, bv)) = X_{\tilde{u}, c/a^{4m}\, \tilde{v}}(a^{2m}) = Y_{c/a^{4m}\, \tilde{u}, \tilde{v}}(a^{2m}) = T(S(ub, v)). \tag{3.2}$$

Here $\tilde{u}, \tilde{v} \in B^n$ are liftings of u and v such that $\lambda_a(\tilde{u}) = a^m u$, $\lambda_a(\tilde{v}) = a^m v$ and $\tilde{u}^t \tilde{v} = 0$. We can also assume that m is so large that $a^{2m} \in I(\tilde{u}) \cap I(\tilde{v})$ and there exist $\tilde{x}$, $\tilde{y}$ such that the following equations hold:

$$\lambda_a(\tilde{x}) = a^m Me_3,\ \lambda_a(\tilde{y}) = a^m M^* e_3 \text{ and } \tilde{x}^t \tilde{y} = a^{2m},\ \tilde{x}^t \tilde{v} = 0,\ \tilde{u}^t \tilde{y} = 0.$$

To obtain (3.2) it remains to apply Lemma 3.7 taking $b = a^{2m}$, $r = c/a^{12m}$. Thus, the map T is well-defined.

Commutativity of the diagram (3.1) follows directly from the definitions of elements $X(u,\, v)$, $Y(u,\, v)$ and $X_{u,v}(a)$, $Y_{u,v}(a)$. $\square$

4. Patching Tulenbaev maps (to be written)

The next section is devoted to the proof of Theorems 1–2 for Steinberg groups corresponding to arbitrary simply-laced root system of rank $l \geq 3$. To do so, we glue maps T constructed in the previous section following the approach of [4].

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