ON CENTRALITY OF ORTHOGONAL K_2

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ABSTRACT. We give a short uniform proof of centrality of $\mathrm{K}_2(\Phi, R)$ for all simply-laced root systems Φ of rank ≥ 3.

INTRODUCTION

The aim of this paper is to establish Quillen's local-global principle and the centrality of even orthogonal K_2 for arbitrary commutative ring R.

Recall that to each reduced irreducible root system Φ and a commutative ring R one can associate a split simple group $\mathrm{G}(\Phi, R)$ called a *Chevalley group*. The *elementary group* $\mathrm{E}(\Phi, R)$ is defined as the subgroup of $\mathrm{G}(\Phi, R)$ generated by elementary root unipotents $t_\alpha(\xi)$, $\xi \in R$, $\alpha \in \Phi$. By definition, the Steinberg group $\mathrm{St}(\Phi, R)$ is presented by generators $x_\alpha(\xi)$, which model the unipotents $t_\alpha(\xi)$, subjected to Steinberg relations (see Section 1). There is a natural map $\phi\colon \mathrm{St}(\Phi, R) \twoheadrightarrow \mathrm{G}(\Phi, R)$ sending $x_\alpha(\xi)$ to $t_\alpha(\xi)$ whose image equals $\mathrm{E}(\Phi, R)$.

By a theorem of Taddei (see [1]) the elementary group $\mathrm{E}(\Phi, R)$ is a normal subgroup of $\mathrm{G}(\Phi, R)$ if Φ has rank at least 2. Thus, similarly to the definition of algebraic SK_1 and K_2-functors, one can define the groups $\mathrm{K}_1(\Phi, R)$, $\mathrm{K}_2(\Phi, R)$ as the cokernel and kernel of ϕ:

$$1 \longrightarrow \mathrm{K}_2(\Phi, R) \longrightarrow \mathrm{St}(\Phi, R) \longrightarrow \mathrm{G}(\Phi, R) \longrightarrow \mathrm{K}_1(\Phi, R) \longrightarrow 1.$$

It is a classical theorem of M. Kervaire and R. Steinberg that the stable Steinberg group $\mathrm{St}(R)$ is the universal central extension of $\mathrm{E}(R)$ for arbitrary R, see [11, Theorem III.5.5]. The unstable analogue of the "universality" part of this theorem was obtained by M. Stein and W. van der Kallen in 1970's, see [3, 7]. However, until recently, the analogue of the "centrality" part of the theorem was only known for $\Phi = \mathsf{A}_\ell$, $\ell \geq 4$, see [2, Corollary 2] and for other classical Φ "in the stable range", i. e. when the dimension of R is suffiently small with respect to the rank of Φ, see [8, Corollary 3.4].

Recently the authors of this paper have established the centrality of $\mathrm{K}_2(\mathsf{C}_\ell, R)$ ($\ell \geq 3$) and $\mathrm{K}_2(\mathsf{E}_\ell, R)$ ($\ell = 6, 7, 8$) for arbitrary commutative ring R, see [5, 6]. In turn, the key result of this paper is the following theorem.

Theorem 1. *Let R be an arbitrary commutative ring and $\ell \geq 4$ then the group $\mathrm{K}_2(\mathsf{D}_\ell, R)$ is a central subgroup of $\mathrm{St}(\mathsf{D}_\ell, R)$.*

Theorem 1 follows from the following local-global principle. We refer the reader to the proof [6, Theorem]

Theorem 2. *Let Φ be an irreducible simply-laced root system of rank ≥ 3 and R be arbitrary commutative ring. Then an element $g \in \mathrm{St}(\Phi,\, R[X])$ satisfying $g(0) = 1 \in \mathrm{St}(\Phi,\, R)$ is trivial*

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in $\mathrm{St}(\Phi,\, R[X])$ *if and only if the images* $g_M \in \mathrm{St}(\Phi,\, R_M[X])$ *of* g *under localisation morphisms* $R \to R_M$ *are trivial for all maximal ideals* $M \trianglelefteq R$*.*

The special case $\Phi = \mathsf{A}_\ell$, $\ell \geq 4$ of Theorem 2 was demonstrated for the first time by Tulenbaev, see [10, Theorem 2.1], while cases $\Phi = \mathsf{C}_\ell$, $\ell \geq 3$ and $\Phi = \mathsf{E}_\ell$, $\ell = 6, 7, 8$ have been analyzed in [4] and [6], respectively. The special cases $\Phi = \mathsf{A}_3$ and $\Phi = \mathsf{D}_\ell$, $\ell \geq 4$ of Theorem 2 are new and have not been known before.

There are known counterexamples to centrality of K_2 in rank 2 (see [12]). This shows that our assumption on the rank of Φ in Theorem 2 is strict.

Compare Theorem 2 with the local-global principle for K_1 which states that an element of the congruence subgroup $g \in \mathrm{G}(\Phi,\, R[X], XR[X])$ is elementary (i. e. lies in $\mathrm{E}(\Phi, R[X])$) if and only if $g_M = 1 \in \mathrm{E}(\Phi,\, R_M)$ for all maximal ideals $M \trianglelefteq R$.

Our proof of Theorem 2 is similar to that of [6, Theorem 1]. Using the same patching technique we reduce the problem to the case $\Phi = \mathsf{A}_3$. Then, using a new presentation of the relative Steinberg group $\mathrm{St}(4, R, I)$ we reprove Tulenbaev's key lemma (cf. [10, Lemma 2.3]) in the rank 3 case.

The rest of the paper is organised as follows. In Section 2 we describe a new symmetric presentation for the relative Steinberg group of rank 3. This presentation is then used in Section 3 to establish the particular case $\Phi = \mathsf{A}_3$ of Theorem 2. Finally, in **??** we derive our principal results from this special case and the amalgamation theorem for relative Steinberg groups from [6].

1. Preliminaries

Throughout this paper R denotes an associative commutative ring with identity. All commutators are left-normed, i. e. $[x,\, y] = xyx^{-1}y^{-1}$.

We denote by R^n the free R-module with basis $e_1, \ldots, e_n$ and by $\mathrm{Um}(n,\, R)$ the subset of unimodular columns $v \in R^n$ whose entries generate R as an ideal.

As usual, $\mathrm{E}(n, R) = \mathrm{E}(\mathsf{A}_{n-1}, R)$ denotes the elementary group, i. e. the subgroup of the general linear group $\mathrm{GL}(n, R)$ generated by transvections $t_{ij}(r) = 1 + r \cdot e_{ij}$, $1 \leq i \neq j \leq n$, $r \in R$, where 1 stands for the identity matrix and e_{ij} for the matrix unit.

The Steinberg group $\mathrm{St}(n, R) = \mathrm{St}(\mathsf{A}_{n-1}, R)$ is the group defined by generators $x_{ij}(r)$, $1 \leq i \neq j \leq n$, $r \in R\}$ and relations

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s), \tag{S1}$$

$$[x_{ij}(r),\, x_{hk}(s)] = 1, \text{ for } h \neq j,\ k \neq i, \tag{S2}$$

$$[x_{ij}(r),\, x_{jk}(s)] = x_{ik}(rs). \tag{S3}$$

The natural projection $\phi\colon\ \mathrm{St}(n,\, R) \to \mathrm{E}(n,\, R)$ sends $x_{ij}(r)$ to $t_{ij}(r)$.

In [9] Suslin showed for $n \geq 3$ that the elementary group $\mathrm{E}(n, R)$ coincides with the subgroup of $\mathrm{GL}(n,\, R)$ generated by matrices of the form $t(u,\, v) = 1 + uv^t$ where $u \in \mathrm{Um}(n,\, R)$, $v \in R^n$ and $u^tv = 0$. Here u^t stands for the transpose of u. This result clearly implies that $\mathrm{E}(n,\, R)$ is normal inside $\mathrm{GL}(n,\, R)$. Afterwards, W. van der Kallen developed Suslin's ideas and showed for $n \geq 4$ that the Steinberg group $\mathrm{St}(n,\, R)$ is isomorphic to the group presented by generators

$$\{X(u,\, v) \mid u \in \mathrm{Um}(n,\, R),\ v \in R^n,\ u^tv = 0\}$$

and the following list of relations (see [2, Theorem 1]):

$$X(u,\, v)X(u,\, w) = X(u,\, v+w), \tag{K1}$$

$$X(u,\, v)X(u',\, v')X(u,\, v)^{-1} = X(t(u,\, v)u',\, t(v,\, u)^{-1}v'). \tag{K2}$$

This presentation clearly implies that $\phi\colon \mathrm{St}(n, R) \to \mathrm{E}(n, R)$ is a central extension. Notice that we parametrize elements $X(u, v)$ with two columns rather than with one column and one row as it is done in [2]. Our generator $X(u, v)$ corresponds to van der Kallen's generator $X(u, v^t)$.

Recall that an ideal $I \trianglelefteq R$ is called a *splitting ideal* if the canonical projection $R \twoheadrightarrow R/I$ splits as a unital ring morphism, i. e. one has $\pi\sigma = 1$.

$$0 \longrightarrow I \longrightarrow R \underset{\sigma}{\overset{\pi}{\rightleftarrows}} R/I \longrightarrow 0$$

For a splitting ideal $I \trianglelefteq R$ the *relative Steinberg group* $\mathrm{St}(n, R, I) = \mathrm{St}(\mathsf{A}_{n-1}, R, I)$ can be defined as the kernel $\mathrm{Ker}\,\big(\mathrm{St}(n, R) \twoheadrightarrow \mathrm{St}(n, R/I)\big)$.

For a general I the relative Steinberg group $\mathrm{St}(\Phi, R, I)$ is no more a subgroup of $\mathrm{St}(\Phi, R)$ and should be defined as a central extension of the above kernel. For the purposes of the present text it suffices to consider only relative Steinberg groups corresponding to splitting ideals. For more information regarding the general case we refer the reader to [6, Section 3].

2. Yet another presentation for the relative Steinberg group.

The key ingredient in Tulenbaev's proof of the local-global principle is the construction of the homomorphism T which fits into the following commutative diagram.

$$\begin{tikzcd} \mathrm{St}(n,\, R \ltimes XR_a[X],\, XR_a[X]) \arrow[r, hook] \arrow[d, "\lambda_a^*"'] & \mathrm{St}(n, R \ltimes XR_a[X]) \arrow[d, "\lambda_a^*"'] \\ \mathrm{St}(n,\, R_a[X],\, XR_a[X]) \arrow[r, hook] \arrow[ur, dashed, "T"'] & \mathrm{St}(n, R_a[X]) \end{tikzcd} \tag{2.1}$$

Here $n \geq 5$, a is arbitrary non-nilpotent element of R and λ_a^* is the morphism induced by the morphism $\lambda_a\colon R \to R_a$ of principal localisation (i. e. localisation by the powers of a). Notice that $XR_a[X]$ is a splitting ideal for both $R_a[X]$ and $R \ltimes XR_a[X]$. It is clear that the image of T is contained in the image of the top arrow, therefore this implies that the relative Steinberg groups in the left-hand side of the diagram are isomoprhic.

In order to construct the arrow T Tulenbaev uses the following presentation for the relative Steinberg group, see [10, Proposition 1.6].

Proposition 2.2. *For a splitting ideal I and $n \geq 4$ the group $\mathrm{St}(n, R, I)$ is isomorphic to the group defined by generators*

$$\{X(u, v) \mid u \in \mathrm{E}(n, R)e_1,\ v \in I^n,\ u^t v = 0\}$$

and the following set of relations

$$X(u, v)X(u, w) = X(u, v + w), \tag{T1}$$

$$X(u, v)X(u', v')X(u, v)^{-1} = X(t(u, v)u', t(v, u)^{-1}v'), \tag{T2}$$

$$X(ur + w, v) = X(u, vr)X(w, v) \text{ for } r \in R,\ (u, w) \in \mathrm{Um}_{n\times 2}(R). \tag{T3}$$

Moreover, the group $\mathrm{St}(n, R)$ decomposes into the semidirect product $\mathrm{St}(n, R, I) \rtimes \mathrm{St}(n, R/I)$.

Here $\mathrm{Um}_{n\times 2}(R)$ denotes the set of $n \times 2$ *unimodular matrices*, i. e. matrices M such that there exists a $2 \times n$ matrix N satisfying $NM = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the definition of the semidirect product the group $\mathrm{St}(n, R/I)$ acts on $\mathrm{St}(n, R, I)$ by conjugation via the splitting map, i. e. ${}^gX(u, v) = X(\phi(\sigma(g))u, \phi(\sigma(g))^*v)$ for $g \in \mathrm{St}(n, R/I)$.

Proof. For the sake of completeness we present below a more detailed version of Tulenbaev's proof. Tulenbaev constructs a pair of mutually inverse maps

$$\mathrm{St}(n,R,I) \rtimes \mathrm{St}(n,R/I) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{St}(n,R).$$

Define the map ψ by $\psi(x_{ij}(\xi)) = (X(e_i,(\xi-\pi(\xi)e_j),\ x_{ij}(\pi(\xi)))$. We have to check that ψ preserves relations (S1)–(S3). For example, let us check (S3). First of all, recall that in any semidirect product of groups one can compute the commutator of two elements in the following way:

$$[(a,b),(c,d)] = (a\cdot {}^{b}c\cdot {}^{bdb^{-1}}a^{-1}\cdot {}^{[b,d]}c^{-1}, [b,d]).$$

For $\xi \in R$ set $\xi' = \xi - \pi(\xi)$. Specializing $a = X(e_i, \xi' e_j)$, $b = x_{ij}(\pi(\xi))$, $c = X(e_j, \eta' e_k)$, $d = x_{jk}(\pi(\eta))$ and using the definition of the conjugation action one computes the commutator $[\psi(x_{ij}(\xi), \psi(x_{jk}(\eta))]$ as follows:

$$\left(X(e_i,\xi' e_j)\cdot X(e_j+\pi(\xi)e_i,\eta' e_k)\cdot X(e_i,\pi(\eta)\xi' e_k-\xi' e_j)\cdot X(e_j,-\eta' e_k),\ x_{ik}(\pi(\xi\eta))\right). \tag{2.3}$$

After applying additivity relations (T1), (T3) and moving the factor $X(e_i, -\xi' e_j)$ to the left hand side of the formula by means of relation (T2) one gets the following expression

$$(X(e_j,\eta' e_k)\cdot X(e_i,(\xi'\eta'+\pi(\xi)\eta'+\pi(\eta)\xi')e_k\cdot X(e_j,-\eta' e_k),\ x_{ik}(\pi(\xi\eta)).$$

Since $\xi'\eta' + \pi(\xi)\eta' + \pi(\eta)\xi' = \xi\eta - \pi(\xi\eta)$ the above expression clearly simplifies to $\psi(x_{ik}(\xi\eta))$, as claimed. Verification of the fact that ψ preserves relations (S1), (S2) is similar to the above computation but is easier. In particular, one does not need to use relation (T3).

It remains to check that the map φ given by $\varphi\left((X(u,v),1)\right) = X(u,v)$, $\varphi\left((1,x_{ij}(\xi))\right) = X(e_i,\sigma(\xi)e_j)$ is well-defined and inverse to ψ. ☐

Remark 2.4. Notice that from the proof of the above proposition it follows that one could replace the family of relations (T3) in the definition of $\mathrm{St}(n,R,I)$ with the following smaller family:

(T3') $\quad X(e_i r + e_j,\, e_k) = X(e_i,\, rae_k)X(e_j,\, ae_k)$ for $r \in R,\ a \in I$ and i,j,k distinct.

This follows from the fact that only this smaller family of relations was actually used in the construction of the map ψ (more precisely, it was used to simplify the second factor of (2.3)).

Both "another presentations" of van der Kallen and Tulenbaev are given in terms of generators parametrised by pairs of vectors, where the first one is "good" in some sense (i. e. unimodular or a column of an elementary matrix) while the second one is arbitrary. It is easy to formulate the additivity property in the second argument for these generators (cf. (K1), (T1)), while it is not so easy when it comes to the additivity in the first argument (cf. (T3)).

The construction of the homomorphism $T\colon \mathrm{St}(n,\, R_a[X],\, XR_a[X]) \to \mathrm{St}(n,\, R \ltimes XR_a[X])$ amounts to choosing certain elements in the group $\mathrm{St}(n,\, R \ltimes XR_a[X])$ and proving that these elements satisfy relations (T1)–(T3), see [10, Lemmas 1.2 and 1.3 c)]. The main problem with this recipe is that the assumption $n \geq 5$ is essential to check that the relation (T3) holds.

It is not possible to choose arbitrary orthogonal vectors u, v as parameters for the generators because in this situation it is not even known that $t(u,\, v) \in \mathrm{E}(n,\, R)$, see [**Rao**].

To generalise Tulenbaev's results for $n = 4$ we use a more symmetric presentation with two types of generators: $F(u,\, v)$ with u good and v arbitrary and $S(u,\, v)$ with u arbitrary and v good. The generators $F(u,\, v)$ are additive in the second component, while $S(u,\, v)$ are additive in the first one. When u and v are both good we require that these two generators coincide. More formally, we make the following definition.

Definition 2.5. For $I \trianglelefteq R$ and $n \geq 4$ define $\mathrm{St}^*(n, R, I)$ to be the group with the set of generators

$$\{F(u, v) \mid u \in \mathrm{E}(n, R)e_1,\ v \in I^n,\ u^t v = 0\} \cup \{S(u, v) \mid u \in I^n,\ v \in \mathrm{E}(n, R)e_1,\ u^t v = 0\}$$

subject to relations

(R1) $\quad F(u, v)F(u, w) = F(u, v + w),$

(R2) $\quad S(u, v)S(w, v) = S(u + w, v),$

(R3) $\quad F(u, v)F(u', v')F(u, v)^{-1} = F(t(u, v)u', t(v, u)^{-1}v'),$

(R4) $\quad F(u, va) = S(ua, v),$ for all $a \in I$, $(u, v^t) = (Me_1, e_2^t M^{-1})$, $M \in \mathrm{E}(n, R)$.

Notice that we only described how a generator $F(u, v)$ acts via conjugation on another generator $F(u', v')$ and have omitted three similar relations involving generators $S(u, v)$ of the second type. The reason for this is the following lemma which asserts that the "missing" relations will follow automatically from (R1)–(R4).

Lemma 2.6. *Denote by* $\phi\colon \mathrm{St}^*(n, R, I) \to \mathrm{E}(n, R)$ *the natural map sending* $F(u, v) \mapsto t(u, v)$ *and* $S(u, v) \mapsto t(u, v)$*. Then the following facts are true.*

a) $\mathrm{St}^*(n, R, I)$ *is generated as an abstract group by the set of elements*

$$\{F(u, va) \mid a \in I,\ (u, v^t) = (Me_1, e_2^t M^{-1}),\ M \in \mathrm{E}(n, R)\};$$

b) for any $g \in \mathrm{St}^*(n, R, I)$ *one has*

(R3') $$gF(u, v)g^{-1} = F(\phi(g)u, \phi(g^{-1})^t v);$$

c) for any $g \in \mathrm{St}^*(n, R, I)$ *one has*

(R3") $$gS(u, v)g^{-1} = S(\phi(g)u, \phi(g^{-1})^t v);$$

d) there is a "transpose automorphism" defined on $\mathrm{St}^*(n, R, I)$ *satisfying*

$$F(u, v)^t = S(v, u), \quad S(u, v)^t = F(v, u).$$

Proof. Let $F(u, v)$ be an arbitrary generator of the first type and let $M \in \mathrm{E}(n, R)$ be such that $F(u, v) = F(Me_1, M^*\tilde{v})$. Here we denote by M^* the contragradient matrix, $M^* = (M^{-1})^t$. By (R1) we have

$$F(u, v) = \prod_{k \neq 1} F(Me_1, M^* e_k \tilde{v}_k)$$

where $\tilde{v}_k$ stands for k-th coordinate of $\tilde{v} = \sum e_i \tilde{v}_i$. Applying relations (R2), (R4) we get

$$S(u, v) = \prod_{k \neq 1} S(Ne_k \tilde{u}_k, N^* e_1) = \prod_{k \neq 1} F(Ne_k, N^* e_1 \tilde{u}_k)$$

and, thus, obtain (a). Obviously, (b) follows from (a). To prove (c) it suffices to show that

$$F(u, v)S(u', v')F(u, v)^{-1} = S(t(u, v)u', t(u, v)^* v').$$

For $S(u', v') = \prod_{k \neq 1} F(Ne_k, N^* e_1 \tilde{u}_k)$ we have

$$\begin{aligned} F(u, v)S(u', v')F(u, v)^{-1} &= F(u, v) \prod F(Ne_k, N^* e_1 \tilde{u}_k) F(u, v)^{-1} = \\ &= \prod F(t(u, v)Ne_k, t(u, v)^* N^* e_1 \tilde{u}_k) = \\ &= \prod S(t(u, v)Ne_k \tilde{u}_k, t(u, v)^* N^* e_1) = S(t(u, v)u', t(u, v)^* v'). \end{aligned}$$

Finally, (d) follows from (c). $\square$

Our next goal is to show for a splitting ideal $I \trianglelefteq R$ that the group $\mathrm{St}^*(n,\, R,\, I)$ is isomorphic to $\mathrm{St}(n,\, R,\, I) = \mathrm{Ker}\,\big(\mathrm{St}(n,\, R) \twoheadrightarrow \mathrm{St}(n,\, R/I)\big)$. With this end we construct two mutually inverse homomorphisms

$$\mathrm{St}^*(n,\, R,\, I) \underset{\kappa}{\overset{\iota}{\rightleftarrows}} \mathrm{St}(n,\, R,\, I).$$

In order to construct the arrow κ we use the presentation from Proposition 2.2. We set

$$\kappa(X(u,v)) = F(u,v),\ u \in \mathrm{E}(n,R)e_1,\ v \in I^n.$$

In view of Remark 2.4 it suffices to check that $F(u,v)$ satisfy the relations (T1), (T2), (T3'). The validity of relations (T1), (T2) is obvious while (T3') follows from (R2) and (R4), indeed:

$$F(e_i, rae_k)F(e_j, ae_k) = S(rae_i, e_k)S(ae_j, e_k) = S(a(re_i + e_j), e_k) = F(e_ir + e_j, ae_k).$$

As for the map ι, we firsty define it as a homomorphism to the absolute group. Then we prove that the image of ι is contained in $\mathrm{St}(n,R,I)$. Clearly, ι should map the elements $F(u,\, v)$ to van der Kallen's generators $X(u,\, v)$. We should find the images for the elements $S(u,\, v)$ as well. First of all, recall that van der Kallen in [2, pp. 3.8–3.10] constructs the following elements:

$$x(u,\, v) \in \mathrm{St}(n,\, R),\ u^tv = 0,\ u_i = 0\ \text{ or }\ v_i = 0\ \text{ for some }\ 1 \le i \le n.$$

The following definition is the "transpose" of [2, p. 3.13].

Definition 2.7. For $u \in R^n$, $v \in \mathrm{E}(n,\, R)e_1$, $u^tv = 0$, consider the set $\overline{Y}(u,\, v) \subseteq \mathrm{St}(n,\, R)$ consisiting of all elements $y \in \mathrm{St}(n,\, R)$ that admit a decomposition $y = \prod x(w^k,\, v)$, where $\sum u^k = u$ and $u^k = (e_pv_q - e_qv_p)c^k$ for some $c^k \in R$, $1 \le p \ne q \le n$.

Since columns of elementary matrices are unimodular, one can find $w \in R^t$ such that $w^tv = 1$. It is not hard to check that

$$(w^t \cdot v) \cdot u = \sum_{p<q} u_{pq},\ \text{ where } u_{pq} = (e_pv_q - e_qv_p) \cdot (u_pw_q - u_qw_p) \in R^n, \tag{2.7}$$

therefore $\overline{Y}(u,\, v)$ is not empty (cf. [2, pp. 3.1–3.2]). Obviously, for $x \in \overline{Y}(u,\, v)$ and $y \in \overline{Y}(w,\, v)$ one has $xy \in \overline{Y}(u + w,\, v)$.

Repeating [2, pp. 3.14–3.15] verbatim one shows the following.

Lemma 2.8. *For $g \in \mathrm{St}(n,\, R)$ one has*

a) $g\overline{Y}(u,\, v)g^{-1} \subseteq \overline{Y}(\phi(g)u,\, \phi(g)^*v)$;

b) $\overline{Y}(u,\, v)$ *consists of exactly one element.*

We denote the only element of $\overline{Y}(u,\, v)$ by $Y(u,\, v)$. Now we can finish the definition of ι by requiring that $\iota(S(u,\, v)) = Y(u,\, v)$ for $S(u,\, v) \in \mathrm{St}^*(n,\, R,\, I)$. To show that the map ι is well-defined we should check that elements $X(u,\, v)$ and $Y(u,\, v)$ satisfy relations (R1)–(R4) (with F's and S's replaced with X's and Y's). Only the relation (R4) is not immediately obvious.

Lemma 2.9. *For $(u,\, v^t) = (Me_1,\, e_2^tM^{-1})$, $M \in \mathrm{E}(n,\, R)$, $a \in I$ one has $X(u,\, va) = Y(ua,\, v)$.*

Proof. In view of the above lemma we only need to show that $X(e_1,\, e_2a) = Y(e_1a,\, e_2)$. This equality can be obtained by computing the commutator $[Y(-e_3,\, e_2),\, X(e_1,\, e_3a)]$ in two ways:

$$Y(-e_3,\, e_2)X(e_1,\, e_3a)Y(-e_3,\, e_2)^{-1} \cdot X(e_1,\, -e_3a) = X(e_1,\, e_2a),\ \text{ and}$$

$$Y(-e_3,\, e_2) \cdot X(e_1,\, e_3a)Y(e_3,\, e_2)X(e_1,\, e_3a)^{-1} = Y(e_1a,\, e_2). \quad \square$$

Since $\pi^*(\iota(F(u,v))) = X(\pi(u),\, 0) = 1$ and $\pi^*(\iota(S(u,v)) = Y(0,\, \pi(v)) = 1$ we get that $\mathrm{Im}(\iota) \subseteq \mathrm{Ker}(\pi^*) = \mathrm{St}(n,\, R,\, I)$. It is clear that $\iota\kappa = \mathrm{id}$ hence κ is injective. On the other hand, κ is surjective by Lemma 2.6 (a). Thus, we have the following result.

Proposition 2.10. *For a splitting ideal $I \trianglelefteq R$ and $n \geq 4$ the groups $\mathrm{St}^*(n,\, R,\, I)$ and $\mathrm{St}(n,\, R, I)$ are isomorphic.*

3. The local-global principle for K_2.

Now we turn to the main result of this paper, namely, we construct for $n \geq 4$ the map

$$T\colon\ \mathrm{St}(n,\, R_a[X],\, XR_a[X]) \to \mathrm{St}(n,\, R \ltimes XR_a[X]),$$

that fits into the diagram (2.1). The local-global principle for the Steinberg group and centrality of K_2 will formally follow from the existence of this map, see [6, Lemmas 15–16, and the proof of Theorem 2].

We will prove a somewhat more general result. Let B be a commutative unital ring. We call an ideal I of B *uniquely r-divisible* if for every $m \in I$ there exists only one $m' \in I$ such that $rm' = m$ (we denote such m' by $\frac{m}{r}$). Clearly, I is uniquely r-divisible if and only if the restriction of the morphism of principal localisation $\lambda_r\colon R \to R_r$ to I is an isomorphism.

Theorem 3. *Let B be a ring, $a \in B$ and let I be an ideal of B that is uniquely a-divisible. Then for $n \geq 4$ there exists a map $T\colon\ \mathrm{St}^*(n,\, B_a,\, I) \to \mathrm{St}(n,\, B)$ which makes the following diagram commute.*

(3.1)
$$\begin{tikzcd}
\mathrm{St}^*(n,\, B,\, I) \arrow[r, "\iota"] \arrow[d, "\lambda_a^*"'] & \mathrm{St}(n,\, B) \arrow[d, "\lambda_a^*"] \\
\mathrm{St}^*(n,\, B_a,\, I) \arrow[r, "\iota"'] \arrow[ur, dashed, "T"'] & \mathrm{St}(n,\, B_a)
\end{tikzcd}$$

For a vector $u \in R^n$ we denote by $I(u)$ the ideal generated by the entries of $u \in R^n$, i. e. $I(u) = \sum\limits_{k=1}^{n} u_k R$. To prove Theorem 3 we follow the approach of Tulenbaev and construct yet another family of elements $X_{u,v}(a) \in \mathrm{St}(n,\, B)$, $a \in B$ which contains van der Kallen's generators $X(u, v)$ as a subfamily (i. e. $X_{u,v}(1) = X(u, v)$). The key feature of this definition is that the assumption $u \in \mathrm{Um}(n, B)$ is replaced with a weaker condition $a \in I(u)$ (this is equivalent to saying that u becomes unimodular after localisation in a).

Definition 3.2. For $u \in R^n$ we denote by $D(u)$ the set consisting of all $v \in B^n$ decomposing into a sum $v = \sum_{k=1}^{N} v_k$ where $v_1, \ldots, v_N \in B^n$ are such that $u^t v_k = 0$ and each v_k has at least two zero coordinates. Now for $a \in I(u)$ and $v \in D(u)$ set $X_{u,v}(a) = \prod\limits_{k=1}^{N} x(u,\, v_k a)$.

Tulenbaev shows that the factors $x(u,\, v_k a)$, $1 \leq k \leq N$ commute, see [10, Lemma 1.1 e)]. He also shows that if v admits another decomposition $v = \sum_{j=1}^{M} v'_j$ satisfying the assumptions of Definition 3.2 then one has $\prod_{k=1}^{N} x(u,\, v_k a) = \prod_{j=1}^{M} x(u,\, v'_j a)$, see the discussion following [10, Lemma 1.1]. Thus, the elements $X_{u,v}(a)$ are well-defined. Obviously, $\phi(X_{u,v}(a)) = t(u,\, va)$.

Remark 3.3. From the canonical decomposition (2.7) it follows that $vb \in D(u)$ for any $b \in I(u)$ and v such that $v^t u = 0$. In particular, if I is uniquely a-divisible and $a^k \in I(u)$, then every $v \in I^n$ satisfying $v^t u = 0$ is contained in $D(u)$.

Notice that Tulenbaev uses different notation for van der Kallen elements, e. g. he writes $X_{u,v}$ instead of $X(u,\, v)$ and $X(u,\, v)$ instead of $x(u,\, v)$. We stick to van der Kallen's notation.

Lemma 3.4. *For $u,\, w \in B^n$, $v, v' \in D(u)$ such that $u^t w = 0$, for $a,\, b \in I(u)$, $c \in B$, $g \in \mathrm{St}(n,\, B)$ the following are true:*

a) $X_{u,vc}(a) = X_{u,v}(ca)$,
b) $X_{uc,v}(ca) = X_{u,vc^2}(a)$,
c) $X_{u,v}(a)X_{u,v'}(a) = X_{u,v+v'}(a)$,
d) $g\,X_{u,wb}(a)g^{-1} = X_{\phi(g)u,\phi(g)^*wb}(a)$.

Proof. The statement of (a) is obvious from the definition, (b) follows from [10, Lemma 1.1 d)], (c) is exactly the statement of [10, Lemma 1.3 a)].

The assertion of (d) is proven in [10, Lemma 1.3 b)] under the assumption $n \geq 5$. Afterwards Tulenbaev remarks that the assertion remains true for $n = 4$. Indeed, take $z \in B^n$ such that $z^t u = b$ and write the canonical decomposition (2.7):

$$(z^t u)w = \sum_{i<j} u_{ij}c_{ij}, \text{ where } u_{ij} = e_i u_j - e_j u_i, \text{ and } c_{ij} = w_i z_j - w_j z_i.$$

Each $u_{ij}c_{ij}$ is orthogonal to u and, because of the assumption $n \geq 4$, has at least two zero coordinates. Thus, $X_{u,wb}(a) = \prod_{i<j} x(u,\, u_{ij}ac_{ij})$. It is enough to prove the assertion of (d) in the special case $g = x_{hk}(r)$. If $h \neq i, j$ or $\{h, k\} = \{i, j\}$ the vector $\phi(g)^* u_{ij}ac_{ij}$ still has two zero coordinates.

Now consider the case $j = h$, $i \neq k$ By [2, p. 3.12] one has

$$g\,x(u,\, u_{ij}ac_{ij})g^{-1} = x(\phi(g)u,\, \phi(g)^* u_{ij}ac_{ij}).$$

Set $u' = \phi(g)u$, since $\phi(g)^* u_{ij} = t_{kj}(-r) \cdot u_{ij} = e_i u_j - e_j u_i + e_k r u_i = u'_{ij} + u'_{ki} r$, we obtain from [2, p. 3.11] that

$$x(\phi(g)u,\, \phi(g)^* u_{ij}ac_{ij}) = x\left(u',\, u'_{ij}c_{ij}a\right) \cdot x\left(u',\, u'_{ki}rc_{ij}a\right).$$

Decomposing in this fashion each factor $g\,x(u, u_{ij}ac_{ij})g^{-1}$ for which $\phi(g)^* u_{ij}ac_{ij}$ does not have two zero coordinates, we arrive at a product satisfying the requirements of the definition of $X_{\phi(g)u,\phi(g)^*wb}(a)$.

Since $u_{ij} = -u_{ji}$, $c_{ij} = -c_{ji}$ the case $i = h$, $j \neq k$ formally follows from the one just considered. ☐

We will also need the "transposed" analogue of Tulenbaev's elements $X_{u,v}(a)$.

Definition 3.5. Let $v \in B^n$, $a \in I(v)$, and $u \in D(v)$, i. e. $u = \sum\limits_{i=1}^{M} u_i$ for some u_k having two zero components such that $v^t u_k = 0$. Set $Y_{u,v}(a) = \prod_{k=1}^{M} x(u_k a,\, v)$. Similarly to Definition 3.2 this definition does not depend neither on the order of factors nor on the choice of decomposition for u.

One can repeat van der Kallen's and Tulenbaev's arguments and prove the following transposed version of Lemma 3.4. We leave the proof of this lemma to the reader.

Lemma 3.6. *For u, u', w and $v \in B^n$, such that u and u' have decomposition as in above definition, $w^t v = 0$, a, $b \in I(v)$, $c \in B$, $g \in \mathrm{St}(n, B)$ holds*

a) $Y_{uc,v}(a) = Y_{u,v}(ca)$,
b) $Y_{u,vc}(ca) = Y_{uc^2,v}(a)$,
c) $Y_{u,v}(a)X_{u',v}(a) = Y_{u+u',v}(a)$,
d) $g\,Y_{wb,v}(a)g^{-1} = Y_{\phi(g)wb,\phi(g)^*v}(a)$.

Finally, it remains to show that for a "good" pair $(u,\, v)$ the elements X and Y coincide.

Lemma 3.7. *Let u, v, x, y be elements of B^n and $b, r \in B$ be such that $u^t v = 0$, $x^t y = b$, $x^t v = 0$, $u^t y = 0$ and $b \in I(u) \cap I(v)$. Then one has $X_{u,vb^4r}(b) = Y_{ub^4r,v}(b)$.*

Proof. Compute $g = [Y_{-xbr,v}(b),\, X_{u,yb}(b)]$ in two different ways using Lemmas 3.4 and 3.6:

$$g = X_{t(xb^2r,-v)u,\, t(xb^2r,-v)^*yb}(b)X_{u,-yb}(b) = X_{u,\, yb+vb^4r}(b)\, X_{u,-yb}(b) = X_{u,vb^4r}(b),$$

$$g = Y_{-xbr,v}(b)Y_{t(u,yb^2)xbr,\, t(u,yb^2)^*v}(b) = Y_{-xbr,v}(b)\, Y_{xbr+ub^4r,\, v}(b) = Y_{ub^4r,v}(b). \quad \square$$

Now, we are ready to construct the desired map $T\colon \mathrm{St}(n,\, B_a,\, I) \to \mathrm{St}(n,\, B)$.

Proof of Theorem 3. Consider $u = Me_1$, $M \in \mathrm{E}(n,\, B_a)$ and $v \in I^n$ such that $u^tv = 0$. Set $w = M^*e_1$. Since $w^tu = 1$ there exist vectors $\tilde{w}$, $\tilde{u} \in B^n$ and a natural number m such that

$$\lambda_a(\tilde{w}) = wa^m,\ \lambda_a(\tilde{u}) = ua^m \text{ and moreover } \tilde{u}^tv = 0 \text{ and } \tilde{w}^t\tilde{u} = a^{2m}.$$

It is clear that $a^{2m} \in I(\tilde{u})$, moreover by Remark 3.3 we have $v/a^{3m} \in D(\tilde{u})$, therefore we are allowed to set $T(F(u,\, v)) = X_{\tilde{u},v/a^{3m}}(a^{2m})$.

The first two assertions of Lemma 3.4 guarantee that this definition does not depend on the choice of m and the liftings $\tilde{u}$ and $\tilde{w}$. Similarly, we can define $T(S(u,\, v)) = Y_{u/a^{3m},\tilde{v}}(a^{2m})$. In view of Lemmas 3.4 and 3.6 the map T preserves relations (R1)–(R3).

It remains to check for $u = Me_1$, $v = M^*e_2$ and $c \in I$ that

$$T(F(u, cv)) = X_{\tilde{u},\, \tilde{v}c/a^{4m}}(a^{2m}) = Y_{\tilde{u}c/a^{4m},\, \tilde{v}}(a^{2m}) = T(S(uc, v)). \tag{3.2}$$

Here $\tilde{u}, \tilde{v} \in B^n$ are liftings of u and v such that $\lambda_a(\tilde{u}) = ua^m$, $\lambda_a(\tilde{v}) = va^m$ and $\tilde{u}^t\tilde{v} = 0$. We can also assume that m is so large that $a^{2m} \in I(\tilde{u}) \cap I(\tilde{v})$ and there exist $\tilde{x}$, $\tilde{y}$ such that the following equations hold:

$$\lambda_a(\tilde{x}) = Me_3a^m,\ \lambda_a(\tilde{y}) = M^*e_3a^m \text{ and } \tilde{x}^t\tilde{y} = a^{2m},\ \tilde{x}^t\tilde{v} = 0,\ \tilde{u}^t\tilde{y} = 0.$$

To obtain (3.2) it remains to apply Lemma 3.7 taking $b = a^{2m}$, $r = c/a^{12m}$. Thus, the map T is well-defined.

Commutativity of the diagram (3.1) follows directly from the definitions of elements $X(u,\, v)$, $Y(u,\, v)$ and $X_{u,v}(a)$, $Y_{u,v}(a)$. $\square$

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