$\mathbb{A}^1$-INVARIANCE FOR UNSTABLE K_2

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1. Aim of the paper

The aim of this text is to prove that the non-stable K_2-functors K_2^G, where G is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings R containing a field k, that is,

{eq:A1-main}
$$K_2^G(R[t]) = K_2^G(R). \tag{1.1}$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K-functor associated to G. This functor originates from [Jar83]. The above equality by e.g. [AHW15, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of G in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of K_2, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \ldots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the SL_n case [Tul83] and Stavrova's proof for K_1^G [Sta14]. Tulenbaev [Tul83] uses stabilization of the K_2-functor, and the good properties of the limit=algebraic K-theory. In [Sta14] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [Tul83, p. 140], or, respectively, [Sta14, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [Tul83, Theorem 5.1] or [Sta14, Theorem 1.1]).

1.1. The case $R = k$.

We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the $\mathbb{A}^1$-invariance of K_1^G, this case should be used to deduce that $K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k)$.)

In Tulenbaev's framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [Wen14, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k. Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt's proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [Deo78], but only for the

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good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [VW12, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. Plan of the proof.

plan:3t (1) (**done**, see Theorem 1). Let R be a local ring. Show that

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t]))$$

Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

plan:k[t] (2) (**done**, see Theorem 3). Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect (3) (**done**). Consequently, $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp (4) (**done** for split ACDE, see Theorem 2). Prove Quillen-Suslin lgp for K_2^G.

plan:Zglu (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2)\mapsto g_1 g_2{}^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma (6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A. If

$$K_2^G(A[X_1, \ldots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \ldots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [Abe83, Lemma 3.6].

plan:Nglu (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St}^{\mathrm{G}}(A_h)$ there exist $y \in \mathrm{St}^{\mathrm{G}}(A)$ and $z \in \mathrm{St}^{\mathrm{G}}(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and h is not a zero divisor in A, then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g),g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2)\mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [Sta14, Lemma 3.4].

plan:P1 (8) ($\mathbb{P}^1$-gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2)\mapsto g_1 g_2{}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

(a) Prove Lemma 6.1 for CDE and Corollary 6.3. This should be relatively easy.
(b) Prove Proposition 6.2. This is hard. Have no idea how this can be proved at the moment.
(c) Prove $\mathbb{P}^1$-glueing using all the above facts.

plan:[]f (9) (Main corollary of $\mathbb{P}^1$-gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.
The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$.
This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it.

plan:k[tn] (11) Prove that

$$K_2^G(k[t_1,\dots,t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let R be a regular ring containing a field k. Then $K_2^G(R[t]) = K_2^G(R)$.
This uses (4), (11) and (7).

2. Steinberg groups of Chevalley groups: preliminaries

Definition, functoriality, "congruence subgroups" $\mathrm{St^G}(\Phi, R, I)$ versus $\ker(\mathrm{St^G}(\Phi, R) \to \mathrm{St^G}(\Phi, R/I))$.

3. Decomposition theorems for $\mathrm{St^G}(A[t^{\pm 1}])$ and $\mathrm{St^G}\big(A((t))\big)$.

Lemma 3.1. *Let (R, m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St^G}(R[t]) \to \mathrm{St^G}(R[t^{\pm 1}])$ and $i_- : \mathrm{St^G}(R[t^{-1}] \to \mathrm{St^G}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$i_+\big(\mathrm{St^G}(m \cdot R[t])^{\mathrm{St^G}(R[t])}\big) i_-\big(\mathrm{St^G}(R[t^{-1}]\big) = i_-\big(\mathrm{St^G}(R[t^{-1}]\big) i_+\big(\mathrm{St^G}(m \cdot R[t])^{\mathrm{St^G}(R[t])}\big)$$

inside $\mathrm{St^G}(R[t^{\pm 1}])$.

Proof. This is proved exactly as [Sta14, Lemma 5.12]. □

thm:3t **Theorem 1.** *Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St^G}(R[t]) \to \mathrm{St^G}(R[t^{\pm 1}])$ and $i_- : \mathrm{St^G}(R[t^{-1}] \to \mathrm{St^G}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$\mathrm{St^G}(R[t^{\pm 1}]) = i_+(\mathrm{St^G}(R[t])) i_-(\mathrm{St^G}(R[t^{-1}])) i_+(\mathrm{St^G}(R[t])).$$

Proof. This is proved exactly as [Sta14, Theorem 5.1]. □

4. Quillen-Suslin lgp, Zariski gluing, Nisnevich gluing, S-lemma

thm:lg-k2 **Theorem 2.** *Let R be arbitrary commutative ring and let G be a simple Chevalley group of type $\mathsf{A}_\ell, \mathsf{C}_\ell, \mathsf{D}_\ell$ or E_ℓ and rank $\ell \geq 3$. An element $g \in \mathrm{St^G}(R[t], tR[t])$ is trivial if and only if its image in $\mathrm{St^G}(R_M[t], tR_M[t])$ is trivial for all maximal ideals $M \trianglelefteq R$.*

Proof. In the case $\Phi = \mathsf{C}_\ell$, $\ell \geq 3$ the assertion of the theorem is the main result of [Lav15]. For a simply laced Φ of rank ≥ 3 this can be proved by the same token as [Sin16, Theorem 2] if one uses a stronger variant of Tulenbaev's lemma proved in the appendices below (see Corollary A.5). □

5. The case of $K_2^G(k[t])$ and some corollaries

thm:k[t] **Theorem 3.** *Let k be a field. Let $G = G(\Phi, -)$ be a simply connected simple Chevalley group of rank ≥ 2 then*

$$K_2^G(k[t]) = K_2^G(k).$$

Proof. See Korollar after Satz 1 in [Reh75]. ☐

cor:k[t]inj **Corollary 5.1.** *Let G, k be as in Theorem 3. Then $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$.*

Proof. Clearly, $g \in \ker(\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}]))$ implies $g \in K_2^G(k[t])$. Since $K_2^G(k[t]) = K_2^G(k)$, and there is a section $K_2^G(k[t^{\pm 1}]) \to K_2^G(k)$, the map is injective. Second claim: take $g \in \mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}])$. Then the image $\phi(g)$ belongs to $E(k) = E(k[t]) \cap E(k[t^{-1}])$, and after adjusting g by an element of $\mathrm{St}^{\mathrm{G}}(k)$, we can assume that $g \in K_2^G(k[t]) \cap K_2^G(k[t^{-1}])$. Hence $g \in K_2^G(k) \subseteq \mathrm{St}^{\mathrm{G}}(k)$. ☐

Corollary 5.2. *Let G, k be as in Theorem 3. Then $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.*

Proof. We use Theorem 1. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}^{\mathrm{G}}(k[t]))$, $y \in \mathrm{St}^{\mathrm{G}}(k[t^{-1}]))$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}^{\mathrm{G}}(k) K_2^G(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ and $x_1 x_2 \in \mathrm{St}^{\mathrm{G}}(k)$. That is, $g \in K_2^G(k)$. ☐

6. $\mathbb{P}^1$-gluing

6.1. **Preliminaries.** Let R denote arbitrary commutative local ring with the maximal ideal m and the residue field k. Consider the following commutative diagram of groups.

$$\begin{tikzcd}
C_+ \arrow[r, hook] \arrow[dddd, two heads, "k^+"] & \mathrm{St}^{\mathrm{G}}(R[t], m[t]) \arrow[rr, "\mu^+"] \arrow[dr, two heads] \arrow[dd, "j_1^+"] & & \mathrm{St}^{\mathrm{G}}(R[t]) \arrow[r, two heads, "p^+"] \arrow[dd, "i_1^+"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[dd, equal] \\
& & \mathrm{Ker}(p^+) \arrow[ur, hook] \arrow[dd, "\overline{j_1^+}"] & & \\
& \mathrm{St}^{\mathrm{G}}(R[t] + m[t^{-1}], m[t^{\pm 1}]) \arrow[rr, "\mu^{+\varepsilon}"] \arrow[dr, two heads] \arrow[dd, "j_2^+"] & & \mathrm{St}^{\mathrm{G}}(R[t] + m[t^{-1}]) \arrow[r, two heads, "p^{+\varepsilon}"] \arrow[dd, "i_2^+"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[dd, hook] \\
& & \mathrm{Ker}(p^{+\varepsilon}) \arrow[ur, hook] \arrow[dd, two heads, "\overline{j_2^+}"] & & \\
C_\pm \arrow[r, hook] & \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}], m[t^{\pm 1}]) \arrow[rr, "\mu^\pm"'] \arrow[dr, two heads] \arrow[uurr, dashed, "\varphi"] & & \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) \arrow[r, two heads, "p^\pm"] & \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}]) \\
& & \mathrm{Ker}(p^\pm) \arrow[ur, hook] & &
\end{tikzcd}$$

Notice that the map $\overline{j_2^+}$ is surjective (this is what A. Stavrova argument from 08/11/15 letter actually proves).

em:TulDiagram **Lemma 6.1.** *There exists a map φ such that $i_2^+ \varphi = \mu^\pm$.*

Proof. Compare with [Tul83, Lemma 3.2]. In Tulenbaev's paper this lemma invokes "another presentation". I am pretty confident that this result can be demonstrated for $\Phi = \mathsf{D}_\ell, \mathsf{E}_\ell$ using the same "amalgamation" technique as in the proof of lgp. $\square$

ulKernSurject **Proposition 6.2.** *The map k^+ is surjective.*

Proof. Compare with [Tul83, Proposition 4.1]. $\square$

:TulMapInject **Corollary 6.3.** *The map i_2^+ is injective.*

Proof. Compare with [Tul83, Corollary 4.2]. This should simultaneously invoke Theorem 3, Lemma 6.1 and Proposition 6.2. $\square$

6.2. **Main result.** The following lemma is an analog of [Tul83, Proposition 4.3 (a)].

Lemma 6.4. *Let R be a local ring, $k = R/m$, and let k, G be as in Theorem 3. The natural homomorphism $\mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ is injective.*

Proof. Let I be the maximal ideal of R, $l = R/I$, and consider the natural maps $\rho : \mathrm{St}^{\mathrm{G}}(R[t, t^{-1}]) \to \mathrm{St}^{\mathrm{G}}(l[t, t^{-1}])$, $\rho_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(l[t])$, $\rho_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}]) \to \mathrm{St}^{\mathrm{G}}(l[t^{-1}])$. Take $x \in \ker(\mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$. By the field case Corollary 5.1 one has $\rho_+(x) = 1$, hence $x \in \mathrm{St}^{\mathrm{G}}(I \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}$.

??????? $\square$

Lemma 6.5. *Let G, k be as in Theorem 3. Let (R, m) be a local ring such that $R/m = k$. Then*

$$\mathrm{St}^{\mathrm{G}}(R[t]) \cap \mathrm{St}^{\mathrm{G}}(R[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(R)$$

inside $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

Proof. ?????? $\square$

Theorem 4. *Let A be any commutative ring. Then the sequence of pointed sets*

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2) \mapsto g_1 {g_2}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

Proof. Follows from the above lemmas. $\square$

Corollary 6.6. *Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.*

Proof. ???? $\square$

Appendices

A. Linear Steinberg group in rank 3

The main goal of this subsection is to show that Tulenbaev's [Tul83, Lemma 2.3] remains valid for the linear Steinberg group of rank ≥ 3. In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [Sin16, Definitions 3.3 and 3.7]).

Definition A.1. The relative Steinberg group $\mathrm{St}^*(n,R,I)$ is the group defined by the following two families generators and four families of relations.

- Generators:
  (1) $X^1(u,v)$, where $u \in \mathrm{E}(n,R)\cdot e_1$, $v \in I^n$ such that $v^t\cdot u = 0$;
  (2) $X^2(u,v)$, where $u \in I^n$, $v \in \mathrm{E}(n,R)\cdot e_1$ such that $v^t \cdot u = 0$.
  Notice that ϕ maps both $X^1(u,v)$ and $X^2(u,v)$ to $T(u,v) = e + u\cdot v^t \in \mathrm{E}(n,R,I)$.
- Relations:
  (1) $X^1(u,v)\cdot X^1(u,w) = X^1(u,v+w)$, $u \in \mathrm{E}(n,R)\cdot e_1$, $v,w \in I^n$;
  (2) $X^2(u,v)\cdot X^2(w,v) = X^2(u+w,v)$, $u,w \in I^n$, $v \in \mathrm{E}(n,R)\cdot e_1$;
  (3) ${}^{X^\sigma(u^2,v^2)}X^\tau(u^1,v^1) = X^\tau(T(u^2,v^2)\cdot u^1, T(v^2,u^2)^{-1}\cdot v^1)$, $\sigma,\tau = 1,2$;
  (4) $X^1(g\cdot e_1, g^*\cdot be_2) = X^2(g\cdot be_1, g^*\cdot e_2)$ where $b \in I$ and $g^* = g^{t^{-1}}$ denotes the contragradient matrix.

Lemma A.2. *The groups* $\mathrm{St}^*(n,R,I)$ *and* $\mathrm{St}(n,R,I)$ *are isomorphic.*

Proof. **TODO:** □

The next step of the proof is to is construct certain elements in $\mathrm{St}(n,R)$ similar to Tulenbaev's elements $X_{u,v}(a)$ see [Tul83, § 1].

Let $v \in R^n$ be a column. Denote by $O(v)$ the submodule of R^n consisting of all columns w such that $w^t\cdot v = 0$. A column $w \in R^n$ is called *v-decomposable* if it can be presented as a finite sum $w = \sum\limits_{i=1}^{p} w^i$ such that each w^i has at least two zero entries and $v^t\cdot w^i = 0$. Denote by $D(v)$ the submodule of $O(v)$ consisting of all v-decomposable columns. For a column $v \in R^n$ denote by $I(v)$ the ideal of R spanned by its entries $v_1,\ldots,v_n$.

Let $u,v,w \in R^n$ be columns such that $w^tv = 0$. It is easy to check (cf. [Kal77, Lemma 3.2]) that

$$(uv)\cdot w = \sum_{i<j} w_{ij}, \text{ where } w_{ij} = (w_iu_j - w_ju_i)(v_je_i - v_ie_j) \in A^n.$$

The above decomposition is called the *canonical* decomposition of $(uv)\cdot w$. In particular, this shows that the column $a\cdot w$ is always v-decomposable for $a \in I(v)$, $w \in O(v)$, i.e. $I(v)\cdot O(v) \subseteq D(v)$. It is also straightforward to check that $D(v) \subseteq D(bv)$, $b\cdot D(v) \subseteq D(v)$ for $b \in R$.

Denote by B^1 the subset of $R^n\times R^n \times R$ consisting of triples (u,v,a) such that $v^t\cdot u = 0$, $v \in D(u)$, $a \in I(u)$. Denote by B^2 the set consisting of triples (v,u,a) such that $(u,v,a) \in B^1$.

lem:Zfacts **Lemma A.3.** *Assume that $n \geq 4$. One can define two families of elements $Z^\tau(u,v,a)$, $\tau = 1,2$ of the group $\mathrm{St}(n,R)$ parametrized by $(u,v,a) \in B^\tau$ satisfying the following properties:*

(1) $\phi(Z^\tau(u,v,a)) = e + uav^t \in \mathrm{E}(n,R)$, $(u,v,a) \in B^\tau$;
(2) $Z^1(u,v+w,a) = Z^1(u,v,a)\cdot Z^1(u,w,a)$;
(3) $Z^2(v+w,u,a) = Z^2(v,u,a)\cdot Z^2(w,u,a)$;
(4) for $\tau = 1,2$ and $b \in R$ if $(u,vb,a),(ub,v,a) \in B^\tau$ then one has

$$Z^\tau(u,vb,a) = Z^\tau(u,v,ab) = Z^\tau(ub,v,a);$$

(5) ${}^gZ^\tau(u,v,a) = Z^\tau(\phi(g)\cdot u, \phi(g)^*\cdot v, a)$, $\tau = 1,2$, $g \in \mathrm{St}^{\mathrm{G}}(n,R)$.

Proof. One constructs the elements $Z^1(u,v,a)$ in exactly the same way as Tulenbaev constructs his elements $X_{u,v}(a)$ (see definitions preceding [Tul83, Lemma 1.2]). Indeed, set

$$Z^1(v,w,a) = \prod_{k=1}^{p} X(v, a\cdot w^k), \quad Z^2(w,v,a) = \prod_{k=1}^{p} X(a\cdot w^k, v). \tag{A.1}$$

where $X(u,v)$ denotes the elements defined by Tulenbaev before [Tul83, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case $n=4$) can be proved by the same token as in [Tul83, Lemma 1.3]. $\square$

For the rest of this section a denotes a nonnilpotent element of R and $\lambda_a\colon R \to R_a$ is the morphism of principal localization at a.

lem:rk3rels

Lemma A.4. *For any $g \in \mathrm{E}(n, R_a)$ there exist $u, v \in R^n$ and sufficiently large natural numbers k, m such that the following facts hold:*

(1) $\lambda_a(u) = g\cdot a^k e_1$, $\lambda_a(v) = g^*\cdot a^k e_2$ and $u^t\cdot v = 0$;
(2) $(u, v, a^m) \in B^1 \cap B^2$;
(3) for $b \in R$ divisible by some sufficiently large power of a one has

$$Z^1(u, b\cdot v, a^m) = Z^2(b\cdot u, v, a^m).$$

Proof. It is straightforward to choose u and v satisfying the first requirement of the lemma. We can even choose u, v in such a way that $u \in D(v)$ and $v \in D(u)$. Indeed, notice that $I(u) = a^{k_1}$, $I(v) = a^{k_2}$ for some natural k_1, k_2 hence for $u' = a^{k_2}\cdot u$ and $v' = a^{k_1}\cdot v$ one has

$$u' \in I(v)\cdot O(v) \subseteq D(v) \subseteq D(v'), \quad v' \in I(u)\cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns $x, y \in R^n$ and a large natural p in such a way that vectors u, v, x, y additionally satisfy the following properties

$$\lambda_a(x) = g^*\cdot a^k e_3,\ \lambda_a(y) = g\cdot a^k e_3,\ y^t\cdot x = a^p \in R,$$
$$u^t\cdot x = 0,\ u^t\cdot v = 0,\ y^t\cdot v = 0,$$
$$(u, x, a^m) \in B^1,\ (y, v, a^m) \in B^2.$$

Now direct computation using Lemma A.3 shows that

$$\begin{aligned} Z^2(a^{m+p}b\cdot u, v, a^m) &= Z^2(b\cdot(e + a^m\cdot ux^t)y, (e - a^m\cdot xu^t)v, a^m)\cdot Z^2(-by, v, a^m) = \\ &= [Z^1(u, x, a^m), Z^2(b\cdot y, v, a^m)] = \\ &= Z^1(u, x, a^m)\cdot Z^1((e + a^m b\cdot yv^t)u, -(e - a^m b\cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b\cdot v, a^m), \end{aligned}$$ $\square$

hence the third assertion of the lemma follows.

cor:tulmap

Corollary A.5. *[Tulenbaev's lemma] For $n \geq 4$ there is a map T_n so that the following diagram commutes.*

$$\begin{tikzcd} \mathrm{St}(n, R[t], tR[t]) \arrow[rr] \arrow[dr, "\lambda_a"'] & & \mathrm{St}(n, R \ltimes tR_a[t], tR_a[t]) \\ & \mathrm{St}(n, R_a[t], tR_a[t]) \arrow[ur, dotted, "T_n"'] & \end{tikzcd}$$

Proof. Follows from Lemma A.4 by the same token as in [Tul83, Lemma 2.3]. $\square$

We refer the reader to [Kal77]...

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