$\mathbb{A}^1$-INVARIANCE FOR UNSTABLE K_2

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1. Aim of the paper

The aim of this text is to prove that the non-stable K_2-functors K_2^G, where G is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings R containing a field k, that is,

{eq:A1-main} (1.1) $$K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K-functor associated to G. This functor originates from [5]. The above equality by e.g. [2, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of G in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of K_2, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \ldots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the SL_n case [12] and Stavrova's proof for K_1^G [9]. Tulenbaev [12] uses stabilization of the K_2-functor, and the good properties of the limit=algebraic K-theory. In [9] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [12, p. 140], or, respectively, [9, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [12, Theorem 5.1] or [9, Theorem 1.1]).

1.1. **The case $R = k$.** We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the $\mathbb{A}^1$-invariance of K_1^G, this case should be used to deduce that $K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k)$.)

In Tulenbaev's framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [16, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k. Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt's proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [3], but only for the

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good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [14, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. **Plan of the proof.**

plan:3t (1) (**done**, see Theorem 2). Let R be a local ring. Show that

$$\mathrm{St^G}(R[t^{\pm1}]) = i_+(\mathrm{St^G}(R[t]))i_-(\mathrm{St^G}(R[t^{-1}]))i_+(\mathrm{St^G}(R[t]))$$

Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St^G}(R[t^{\pm1}])$.

plan:k[t] (2) (**done**, see Theorem 4). Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect (3) (**done**). Consequently, $\mathrm{St^G}(k[t]) \to \mathrm{St^G}(k[t^{\pm1}])$ is injective and $\mathrm{St^G}(k[t]) \cap \mathrm{St^G}(k[t^{-1}]) = \mathrm{St^G}(k)$ inside $\mathrm{St^G}(k[t^{\pm1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp (4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen-Suslin lgp for K_2^G.

plan:Zglu (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g\mapsto(g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2)\mapsto g_1 g_2{}^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma (6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A. If

$$K_2^G(A[X_1, \ldots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \ldots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St^G}(A_h)$ there exist $y \in \mathrm{St^G}(A)$ and $z \in \mathrm{St^G}(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and h is not a zero divisor in A, then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g\mapsto(F_h(g),g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2)\mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [9, Lemma 3.4].

plan:P1 (8) ($\mathbb{P}^1$-gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g\mapsto(g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2)\mapsto g_1 g_2{}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

(a) Prove Proposition 5.9. This is hard. Have no idea how this can be proved at the moment.

(b) Prove $\mathbb{P}^1$-glueing using all the above facts.

plan:[]f (9) (Main corollary of $\mathbb{P}^1$-gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.
The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2.

plan:k[tn] (11) Prove that
$$K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k).$$
If we strictly follow the pattern of K_1^G, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let R be a regular ring containing a field k. Then $K_2^G(R[t]) = K_2^G(R)$.
This uses (4), (11) and (7).

2. Steinberg groups: preliminaries

2.1. **Definition and basic properties.** Let G be a split simple Chevalley groups with a root system Φ of rank ≥ 2. Recall that the *Steinberg group* $\mathrm{St}^{\mathrm{G}}(R)$ (also denoted $\mathrm{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi,R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi,R}$ defined as follows:

{rel:add}
$$x_\alpha(s)x_\alpha(t) = x_\alpha(s+t), \tag{2.1}$$

{rel:CCF}
$$[x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta}\left(N_{\alpha\beta ij}\, s^i t^j\right), \quad \alpha \neq -\beta, \quad N_{\alpha\beta ij} \in \mathbb{Z}. \tag{2.2}$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$. Denote by $\mathrm{W}(\Phi, R)$ the subgroup of $\mathrm{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$.

2.2. **Relative Steinberg groups.** Denote by $D(R, I)$ the double of the ring R relative to an ideal I, i.e. the fibered product of rings $R \times_{R/I} R$ with the natural projections $p_1, p_2 \colon D(R, I) \to R$ defined by $p_i(\xi_1, \xi_2) = \xi_i$, $i = 1, 2$. Denote by G_i the kernel of the map $p_i^* \colon \mathrm{St}^{\mathrm{G}}(D(R, I)) \to \mathrm{St}^{\mathrm{G}}(R)$. We define the relative Steinberg group $\mathrm{St}^{\mathrm{G}}(R, I)$ as G_1/C, where $C = [G_1, G_2]$. Clearly, there is an exact sequence.

{eq:suite}
$$1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \mathrm{St}^{\mathrm{G}}(R, I) \xrightarrow{\overline{p_2^*}} \mathrm{St}^{\mathrm{G}}(R) \xrightarrow{\pi^*} \mathrm{St}^{\mathrm{G}}(R/I) \longrightarrow 1 \tag{2.3}$$

Lemma 2.1. *Assume that R and I are such that the canonical projection $R \to R/I$ splits. Then the following facts are true.*

item:st-inj *(i) The map $\mathrm{St}^{\mathrm{G}}(R, I) \to \mathrm{St}^{\mathrm{G}}(R)$ is an injection.*

item:st-semi *(ii) The group $\mathrm{St}^{\mathrm{G}}(R)$ is isomorphic to $\mathrm{St}^{\mathrm{G}}(R/I) \ltimes \mathrm{St}^{\mathrm{G}}(R, I)$.*

Proof. For the proof of the first assertion see [8, Lemma 8]. Since the group $(G_1 \cap G_2)/C$ vanishes, the sequence (2.3) turns into a split short exact sequence which implies the second assertion. $\square$

lem:Zgen **Lemma 2.2.** *Let Σ be the special subset of some parabolic subset of roots $S \subseteq \Phi$. Then the relative Steinberg group $\mathrm{St}^{\mathrm{G}}(R, I)$ admits the following generating set:*

$$\mathcal{Z}(\Sigma, R, I) = \{x_\alpha(0, s) \cdot C \mid s \in I, \alpha \in \Phi\} \cup \{z_\alpha(s, \xi) \mid s \in I,\ \xi \in R,\ \alpha \in \Sigma\},$$

where $z_\alpha(s, \xi)$ denotes the element $x_\alpha(0, s)^{x_{-\alpha}(\xi, \xi)} \cdot C$.

Proof. See [8, Lemma 5]. □

2.3. **Tulenbaev's lifting property and its corollaries.** Throughout this section $I \trianglelefteq A$ is an ideal of arbitrary commutative ring A. For a nonnilpotent element $a \in A$ denote by $\lambda_a \colon A \to A_a$ the morphism of principal localization at a. Consider the following commutative square.

{msq} (2.4)
$$\begin{tikzcd} A \arrow[r, "\lambda_a"] \arrow[d, two heads] & A_a \arrow[d, two heads] \\ A/I \arrow[r, "\overline{\lambda_a}"] & A_a/I_a \end{tikzcd}$$

Notice that (2.4) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a. Such squares are usually called *Milnor squares* in the literature, see [15, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [12, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp **Definition 2.3.** We say that the Steinberg group functor St^{G} satisfies *Tulenbaev's lifting property* if for every pull-back square (2.4) the following lifting problem has a solution.

$$\begin{tikzcd} \mathrm{St}^{\mathrm{G}}(A, I) \arrow[r, "\mu"] \arrow[d] & \mathrm{St}^{\mathrm{G}}(A) \arrow[d, "\lambda_a^*"] \\ \mathrm{St}^{\mathrm{G}}(A_a, I) \arrow[r, "\mu"] \arrow[ur, dotted] & \mathrm{St}^{\mathrm{G}}(A_a) \end{tikzcd}$$

Theorem 1. *Assume that G satisfies Tulenbaev property (2.3) then the following facts are true for arbitrary commutative ring A:*

thm:dp *(i) A dilation principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. if $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is such that equality $\lambda_a^*(h) = 1$ holds in $\mathrm{St}^{\mathrm{G}}(\Phi, R_a[t])$ then for sufficiently large n one has*

$$ev^*_{\left[\substack{R[t] \to R[t] \\ t \mapsto a^n \cdot t}\right]}(h) = 1.$$

thm:lg-k2 *(ii) A local-global principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. an element $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is trivial if and only if its image in $\mathrm{St}^{\mathrm{G}}(A_m[t], tA_m[t])$ is trivial for all maximal ideals $m \trianglelefteq A$.*

thm:centr *(iii) $K_2^G(A)$ is contained in the centre of $\mathrm{St}^{\mathrm{G}}(A)$.*

Proof. Follows by the same argument as [12, Theorem 2.1] or [8, Theorem 2] □

3. Decomposition theorems for $\mathrm{St}^{\mathrm{G}}(A[t^{\pm 1}])$ and $\mathrm{St}^{\mathrm{G}}\big(A((t))\big)$.

Lemma 3.1. *Let (R,m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big) i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) = i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big)$$

inside $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

Proof. This is proved exactly as [9, Lemma 5.12]. □

thm:3t

Theorem 2. *Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t])).$$

Proof. This is proved exactly as [9, Theorem 5.1]. □

sec:stbA3

4. Steinberg groups: another presentation

4.1. **Presentation of $\mathrm{St}(4,R)$.** The main goal of this subsection is to show that Tulenbaev's [12, Lemma 2.3] remains valid for the linear Steinberg group of rank ≥ 3. In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [8, Definitions 3.3 and 3.7]).

dfn:yap

Definition 4.1. The relative Steinberg group $\mathrm{St}^*(n,R,I)$ is the group defined by the following two families generators and four families of relations.

- Generators:
  (1) $X^1(u,v)$, where $u \in \mathrm{E}(n,R) \cdot e_1$, $v \in I^n$ such that $v^t \cdot u = 0$;
  (2) $X^2(u,v)$, where $u \in I^n$, $v \in \mathrm{E}(n,R) \cdot e_1$ such that $v^t \cdot u = 0$.
  Notice that ϕ maps both $X^1(u,v)$ and $X^2(u,v)$ to $T(u,v) = e + u \cdot v^t \in \mathrm{E}(n,R,I)$.
- Relations:
  (1) $X^1(u,v) \cdot X^1(u,w) = X^1(u,v+w)$, $u \in \mathrm{E}(n,R) \cdot e_1$, $v,w \in I^n$;
  (2) $X^2(u,v) \cdot X^2(w,v) = X^2(u+w,v)$, $u,w \in I^n$, $v \in \mathrm{E}(n,R) \cdot e_1$;
  (3) ${}^{X^\sigma(u^2,v^2)}X^\tau(u^1,v^1) = X^\tau(T(u^2,v^2) \cdot u^1, T(v^2,u^2)^{-1} \cdot v^1)$, $\sigma,\tau = 1,2$;
  (4) $X^1(g \cdot e_1, g^* \cdot be_2) = X^2(g \cdot be_1, g^* \cdot e_2)$ where $b \in I$ and $g^* = g^{t^{-1}}$ denotes the contragradient matrix.

Lemma 4.2. *The groups $\mathrm{St}^*(n,R,I)$ and $\mathrm{St}(n,R,I)$ are isomorphic.*

Proof. **TODO:** □

The next step of the proof is to is construct certain elements in $\mathrm{St}(n,R)$ similar to Tulenbaev's elements $X_{u,v}(a)$ see [12, § 1].

Let $v \in R^n$ be a column. Denote by $O(v)$ the submodule of R^n consisting of all columns w such that $w^t \cdot v = 0$. A column $w \in R^n$ is called *v-decomposable* if it can be presented as a finite sum $w = \sum_{i=1}^{p} w^i$ such that each w^i has at least two zero entries and $v^t \cdot w^i = 0$. Denote by $D(v)$ the submodule of $O(v)$ consisting of all v-decomposable columns. For a column $v \in R^n$ denote by $I(v)$ the ideal of R spanned by its entries $v_1, \ldots, v_n$.

Let $u, v, w \in R^n$ be columns such that $w^t v = 0$. It is easy to check (cf. [6, Lemma 3.2]) that

$$(uv) \cdot w = \sum_{i<j} w_{ij}, \text{ where } w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n.$$

The above decomposition is called the *canonical* decomposition of $(uv) \cdot w$. In particular, this shows that the column $a \cdot w$ is always v-decomposable for $a \in I(v)$, $w \in O(v)$, i. e. $I(v) \cdot O(v) \subseteq D(v)$. It is also straightforward to check that $D(v) \subseteq D(bv)$, $b \cdot D(v) \subseteq D(v)$ for $b \in R$.

Denote by B^1 the subset of $R^n \times R^n \times R$ consisting of triples (u, v, a) such that $v^t \cdot u = 0$, $v \in D(u)$, $a \in I(u)$. Denote by B^2 the set consisting of triples (v, u, a) such that $(u, v, a) \in B^1$.

lem:Zfacts **Lemma 4.3.** *Assume that $n \geq 4$. One can define two families of elements $Z^\tau(u, v, a)$, $\tau = 1, 2$ of the group $\mathrm{St}(n, R)$ parametrized by $(u, v, a) \in B^\tau$ satisfying the following properties:*

(1) $\phi(Z^\tau(u, v, a)) = e + uav^t \in \mathrm{E}(n, R)$, $(u, v, a) \in B^\tau$;
(2) $Z^1(u, v + w, a) = Z^1(u, v, a) \cdot Z^1(u, w, a)$;
(3) $Z^2(v + w, u, a) = Z^2(v, u, a) \cdot Z^2(w, u, a)$;
(4) for $\tau = 1, 2$ and $b \in R$ if $(u, vb, a), (ub, v, a) \in B^\tau$ then one has

$$Z^\tau(u, vb, a) = Z^\tau(u, v, ab) = Z^\tau(ub, v, a);$$

(5) ${}^g Z^\tau(u, v, a) = Z^\tau(\phi(g) \cdot u, \phi(g)^* \cdot v, a)$, $\tau = 1, 2$, $g \in \mathrm{St}^{\mathrm{G}}(n, R)$.

Proof. One constructs the elements $Z^1(u, v, a)$ in exactly the same way as Tulenbaev constructs his elements $X_{u,v}(a)$ (see definitions preceding [12, Lemma 1.2]). Indeed, set

$$Z^1(v, w, a) = \prod_{k=1}^{p} X(v, a \cdot w^k), \quad Z^2(w, v, a) = \prod_{k=1}^{p} X(a \cdot w^k, v). \tag{4.1}$$

where $X(u, v)$ denotes the elements defined by Tulenbaev before [12, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case $n = 4$) can be proved by the same token as in [12, Lemma 1.3]. □

For the rest of this section a denotes a nonnilpotent element of R and $\lambda_a \colon R \to R_a$ is the morphism of principal localization at a.

lem:rk3rels **Lemma 4.4.** *For any $g \in \mathrm{E}(n, R_a)$ there exist $u, v \in R^n$ and sufficiently large natural numbers k, m such that the following facts hold:*

(1) $\lambda_a(u) = g \cdot a^k e_1$, $\lambda_a(v) = g^* \cdot a^k e_2$ and $u^t \cdot v = 0$;
(2) $(u, v, a^m) \in B^1 \cap B^2$;
(3) for $b \in R$ divisible by some sufficiently large power of a one has

$$Z^1(u, b \cdot v, a^m) = Z^2(b \cdot u, v, a^m).$$

Proof. It is straightforward to choose u and v satisfying the first requirement of the lemma. We can even choose u, v in such a way that $u \in D(v)$ and $v \in D(u)$. Indeed, notice that $I(u) = a^{k_1}$, $I(v) = a^{k_2}$ for some natural k_1, k_2 hence for $u' = a^{k_2} \cdot u$ and $v' = a^{k_1} \cdot v$ one has

$$u' \in I(v) \cdot O(v) \subseteq D(v) \subseteq D(v'), \quad v' \in I(u) \cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns $x, y \in R^n$ and a large natural p in such a way that vectors u, v, x, y additionally satisfy the following properties

$$\lambda_a(x) = g^* \cdot a^k e_3,\ \ \lambda_a(y) = g \cdot a^k e_3,\ \ y^t \cdot x = a^p \in R,$$
$$u^t \cdot x = 0,\ \ u^t \cdot v = 0,\ \ y^t \cdot v = 0,$$
$$(u, x, a^m) \in B^1,\ \ (y, v, a^m) \in B^2.$$

Now direct computation using Lemma 4.3 shows that

$$Z^2(a^{m+p}b \cdot u, v, a^m) = Z^2(b \cdot (e + a^m \cdot ux^t)y, (e - a^m \cdot xu^t)v, a^m) \cdot Z^2(-by, v, a^m) =$$
$$= [Z^1(u, x, a^m), Z^2(b \cdot y, v, a^m)] =$$
$$= Z^1(u, x, a^m) \cdot Z^1((e + a^m b \cdot yv^t)u, -(e - a^m b \cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b \cdot v, a^m), \quad \square$$

hence the third assertion of the lemma follows.

cor:tlpA3 **Corollary 4.5.** *For $G = G(\mathsf{A}_3, -)$ the Steinberg group functor $\mathrm{St}^{\mathrm{G}}(R)$ satisfies Tulenbaev lifting property 2.3.*

Proof. Follows from Lemma 4.4 by the same token as in [12, Lemma 2.3]. $\square$

4.2. **Application: centrality of K_2 in the simply-laced case.**

sec:sigma

4.3. **Application: the automorphisms σ_i.** Our notation and conventions follows [13, § 4]. Let Φ be an irreducible root system with some fixed basis of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. We denote by $\Phi^\vee$ the *dual root system of* Φ consisting of vectors $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \Phi$. As usual, $P(\Phi^\vee)$ denotes the lattice spanned by the *fundamental weights* ϖ_i. Recall that ϖ_i are uniquely determined by relations $\langle \varpi_i, \alpha_j^\vee \rangle = (\varpi_i, \alpha_j) = \delta_{ij}$.

Notice that for $\varpi \in P(\Phi^\vee)$ and $\beta \in \mathbb{Z}\,\Phi$ one has $(\varpi, \beta) \in \mathbb{Z}$. Consequently, for $\varepsilon \in R^*$ and $\varpi \in P(\Phi^\vee)$ the identity $\chi_{\varpi,\varepsilon}(\beta) = \varepsilon^{(\varpi,\beta)}$ gives a well-defined character $\chi_{\varpi,\varepsilon} \in \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$.

Consider the action of $H = \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$ on the set of generators $\mathcal{X}_{\Phi,R}$ of the Steinberg group $\mathrm{St}^{\mathrm{G}}(R)$ defined by

$$\chi \cdot x_\alpha(\xi) = x_\alpha(\chi(\alpha) \cdot \xi),\ \ \chi \in H,\ \ \alpha \in \Phi,\ \ \xi \in R. \tag{4.2}$$

Since χ is a character, the above action preserves the set of Steinberg relations $\mathcal{R}_{\Phi,R}$ and, thus, gives a well-defined action of H on $\mathrm{St}^{\mathrm{G}}(R)$.

Example 4.6. The principal example which motivates the above construction is as follows. Let A be a ring, take $R = A[t, t^{-1}]$ to be the ring of Laurent polynomials over A and let $\alpha_i \in \Pi$ be some simple root. Since $t \in R^*$ we can consider the automorphisms σ_i^+ and σ_i^- of $\mathrm{St}(\Phi, R)$ given by $\sigma_i^+ = \chi_{\varpi_i,t}$, $\sigma_i^- = \chi_{\varpi_i,t^{-1}}$. It is easy to see that

eq:sigma_act}

$$\sigma_i^\pm(x_\alpha(\xi)) = x_\alpha(t^{\pm m_i(\alpha)} \cdot \xi), \tag{4.3}$$

where $m_k(\alpha)$ denotes the coefficient in the expansion of α in Π, i. e. $\alpha = \sum m_k(\alpha)\alpha_k$.

One of the key steps of our proof of Suslin lemma for K_2 is to define an analogue of σ_i for the group $\mathrm{St}^{\mathrm{G}}(A[t])$. Of course, we cannot expect such map to be automorphism or even be defined on the whole group $\mathrm{St}^{\mathrm{G}}(A[t])$. However, it turns out that for certain i is still possible to define certain subgroups of $\mathrm{St}^{\mathrm{G}}(A[t])$ and the maps modeling σ_i between them. First, we settle the case $\Phi = \mathsf{A}_3$ invoking the presentation obtained in 4.1. Then we study the general case using the Curtis-Tits presentation.

lem:sigma **Lemma 4.7.** *Let A be a local commutative ring, $G = G(\Phi, -)$ where Φ is an irreducible root system. Assume that on the Dynkin diagram of Φ one can find an endnode numbered i such that i is contained in a subdiagram of type A_3.*

Then there exists subgroups N_i^+, N_i^- of $\mathrm{St}^{\mathrm{G}}(A[t])$ and homomorphisms $\widetilde{\sigma}_i^+ \colon N_i^+ \to N_i^-$, $\widetilde{\sigma}_i^- \colon N_i^- \to N_i^+$ compatible with the action of $\sigma_i^\pm$ i. e. such that the following diagram commutes.

$$\begin{tikzcd} N_i^\pm \arrow[r, "\lambda_t^*"] \arrow[d, "\widetilde{\sigma}_i^\pm"] & \mathrm{St}^{\mathrm{G}}(A[t,t^{-1}]) \arrow[d, "\sigma_i^\pm"] \\ N_i^\mp \arrow[r, "\lambda_t^*"] & \mathrm{St}^{\mathrm{G}}(A[t,t^{-1}]) \end{tikzcd}$$

Proof for $\Phi = \mathsf{A}_\ell$, $\ell \geq 3$ and $i = 1$. For $\alpha_i \in \Pi$ denote by P_i^+ (resp. P_i^-) the subgroup of $\mathrm{St}^{\mathrm{G}}(A)$ generated by $x_\alpha(\xi)$ for $\xi \in A$, $\alpha \in \Sigma_i^+ \cup \Delta_i$ (resp. $\alpha \in \Sigma_i^- \cup \Delta_i$).

First, we define the subgroups $N_i^\pm$ (this definition also works for Φ such that $m_i(\widetilde{\alpha}) = 1$). Define N_i^+ (resp. N_i^-) to be the subgroup consisting of $g \in \mathrm{St}^{\mathrm{G}}(A[t])$ such that $g(0) \in P_i^+$ (resp. $g(0) \in P_i^-$).

Denote by j_ℓ the natural map $\mathrm{St}(\Delta_1, A) \to \mathrm{St}(\Phi, A)$. By the Levi decomposition $P_1^\pm$ is isomorphic to $\mathrm{U}(\Sigma_1^\pm, A) \rtimes \mathrm{Im}\,(j_\ell)$.

The stable rank of A equals 1 hence from the injective stability theorem for K_2 (see [11, Theorem 4.1]) it follows that the map j_ℓ is injective for $\ell \geq 3$ and that $P_1^\pm = \mathrm{U}(\Sigma_1^\pm, A) \rtimes \mathrm{St}(\Delta_1, A)$. Clearly, there exists a unique group homomorphism $\sigma_1^\pm \colon P_1^\pm \to \mathrm{St}^{\mathrm{G}}(\Phi, A[t])$ acting identically on $\mathrm{St}(\Delta_1, A)$ and sending any generator $x_\alpha(\xi) \in \mathrm{U}(\Sigma_1^\pm, A)$ (i. e. $\alpha \in \Sigma_1^\pm$) to $x_\alpha(t \cdot \xi)$.

Set $H = \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$. By Lemma 2.1.(ii) we have $\mathrm{St}^{\mathrm{G}}(A[t]) = \mathrm{St}^{\mathrm{G}}(A) \ltimes H$ hence $N_1^\pm \cong P_1^\pm \ltimes H$. Define the map $\widetilde{\sigma}_1^\pm \colon H \to \mathrm{St}^{\mathrm{G}}(A[t])$ on the generators of $\mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ as follows ?????. A routine check shows that the defining relations ?????? of H are satisfied and that $\sigma_1^\pm$ preserves the action of $P_1^\pm$ on H. Thus, we obtain a well-defined map $\sigma_1^\pm$: $N_i^\pm \to \mathrm{St}^{\mathrm{G}}(A[t])$ fitting into the above commutative diagram. The fact that the image of $\sigma_1^\pm$ is contained in $N_i^\mp$ is obvious.

Proof in the general case. □

5. $\mathbb{P}^1$-GLUING

Throughout this section $G = G(\Phi, -)$ denotes a Chevalley group scheme of type Φ.

def:p1g **Definition 5.1.** Let F be a group-valued functor from **CRings** to **Groups** and let A be a commutative ring. Consider the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "i_+"] \arrow[d, "i_-"'] & A[t] \arrow[d, "j_+"] \\ A[t^{-1}] \arrow[r, "j_-"] & A[t,t^{-1}] \end{tikzcd}$$

We say that F satisfies the $\mathbb{P}^1$*-glueing property for A* if the following sequence of *pointed sets* is exact in the middle term:

$$F(A) \xhookrightarrow{\Delta_A^F} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\pm_A^F} F(A[t,t^{-1}]).$$

Here Δ_A^F denotes the (split injective) diagonal map and, by definition, $\pm_A^F$ maps (g^+, g^-) to $F(j_+)(g^+) \cdot F(j_-)(g^-)^{-1}$. Notice that a priori $\pm_A^F$ is only a morphism of pointed sets but if F takes values in abelian groups then $\pm_A^F$ is also a morphism of groups.

An equivalent way to formulate $\mathbb{P}^1$-glueing property is as follows: $F(j_+)$ and $F(j_-)$ are injective and the intersection of their images coincides with the image of $F(j_+i_+) = F(j_-i_-)$.

The main result of this section is the following theorem which generalizes [12, Theorem 5.1] to Chevalley groups. Notice that a K_1-analogue of the result below has been established in a much greater generality by the second-named author (see [9, Theorem 1.1]).

thm:p1 **Theorem 3.** *Assume that G satisfies Tulenbaev lifting property 2.3. Then the Steinberg group functor* $\mathrm{St}^{\mathrm{G}}(-)$ *satisfies $\mathbb{P}^1$-glueing property for an arbitrary commutative ring A.*

Proof. Let (g^+, g^-) be an element of $\mathrm{St}^{\mathrm{G}}(A[t]) \times \mathrm{St}^{\mathrm{G}}(A[t^{-1}])$ such that the equality $g^+ = g^-$ holds in $\mathrm{St}^{\mathrm{G}}(A[t, t^{-1}])$.

Let m be a maximal ideal of A. By Proposition 5.10 below the functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for the local ring A_m hence $(\lambda_m^*(g_+), \lambda_m^*(g_-)) = \Delta_{A_m}(\lambda_m^*(g^+)(0))$ and in the groups $\mathrm{St}^{\mathrm{G}}(A_m[t])$ and $\mathrm{St}^{\mathrm{G}}(A_m[t^{-1}])$ we have the equalities:

$$\lambda_m^*(g^+ \cdot g^+(0)^{-1}) = \lambda_m^*(g^+) \cdot \lambda_m^*(g^+)(0)^{-1} = 1; \qquad \lambda_m^*(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for $\mathrm{St}^{\mathrm{G}}(-)$ (see Theorem 1.(ii)) these equalities hold globally and $(g^+, g^-) = \Delta_A(g^+(0))$, as claimed. $\square$

rem:stk2 *Remark* 5.2. It is clear that if the functor St^{G} satisfies $\mathbb{P}^1$-glueing property for A then so does the functor K_2^G. The converse statement also holds, indeed, if $(g^+, g^-) \in \mathrm{Ker}\,(\pm_A^{\mathrm{St}})$ then inside $E^G(A[t, t^{-1}])$ we have the equality:

$$\varphi(\mathrm{St}^{\mathrm{G}}(j_+)(g^+)) = \varphi(\mathrm{St}^{\mathrm{G}}(j_-)(g^-)) \in E^G(A[t]) \cap E^G(A[t^{-1}]) = E^G(A).$$

Consequently, we can find $g_0 \in \mathrm{St}^{\mathrm{G}}(A)$ so that $(g^+g_0^{-1}, g^-g_0^{-1}) \in \mathrm{Ker}\,(\pm_A^{K_2})$ and it remains to apply the $\mathbb{P}^1$-glueing property for K_2^G.

Corollary 5.3. *Let A be any commutative ring and $f \in A[t]$ be a monic polynomial. Then the map $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.*

Proof. $\square$

5.1. **The case of a field.** Throughout this section k denotes arbitrary field k.

thm:k[t] **Theorem 4.** *Assume that $G = G(\Phi, -)$ and Φ is irreducible of rank ≥ 2.*

satz1 *(i) The subgroup $K_2^G(k[t]) \trianglelefteq \mathrm{St}^{\mathrm{G}}(k[t])$ is generated by elements of the form*

$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1},\ u, v \in k^*.$$

(ii) As a consequence, the canonical injection $K_2^G(k) \hookrightarrow K_2^G(k[t])$ is an isomorphism.

Proof. See [7, Satz 1] and the corollary after it. $\square$

cor:k[t]inj **Corollary 5.4.** *Let G be as in the above theorem. Then the functors* St^{G}, K_2^G *satisfy $\mathbb{P}^1$-glueing property for k.*

Proof. By Remark 5.2 it suffices to prove the assertion only for the functor K_2^G. By the previous theorem $K_2^G(i_+)$ and $K_2^G(i_-)$ are isomorphisms hence the morphisms $K_2^G(j_+)$ and $K_2^G(j_-)$ are split injective and $\mathrm{Im}\,(j_+i_+) = \mathrm{Im}\,(j_-i_-) = \mathrm{Im}\,(j_+) = \mathrm{Im}\,(j_-)$. $\square$

Corollary 5.5. *Let G, k be as in Theorem 4. Then $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.*

Proof. We use Theorem 2. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}^{\mathrm{G}}(k[t]))$, $y \in \mathrm{St}^{\mathrm{G}}(k[t^{-1}]))$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}^{\mathrm{G}}(k) K_2^G(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ and $x_1 x_2 \in \mathrm{St}^{\mathrm{G}}(k)$. That is, $g \in K_2^G(k)$. ☐

5.2. **Suslin lemma for K_2.** For the rest of this section A denotes an arbitrary commutative local ring with the maximal ideal m and the residue field k. We denote by π the canonical projection $A \to k$. Throughout this section we will employ the following notation:

- R denotes the Laurent polynomial ring $A[t, t^{-1}]$;
- B denotes the subring $A[t] + m[t^{-1}]$ of R consisting of Laurent polynomials $f(t, t^{-1})$ whose coefficients of terms of negative degree belong to m;
- I denotes the ideal $m[t, t^{-1}]$ of R (which can be also considered as an ideal of B).

Our first result is analogous to [12, Lemma 3.1(e)] (cf. also with [4, § 2.3A]).

cor:bruhat **Lemma 5.6.** *The Steinberg group $\mathrm{St}^{\mathrm{G}}(A)$ admits the following analogue of the Bruhat decomposition:*

$$\mathrm{St}^{\mathrm{G}}(A) = \mathrm{U}(\Phi^+, A) \cdot \mathrm{W}(\Phi, A) \cdot \mathrm{U}(\Phi^+, A) \cdot \mathrm{Im}\,(\mathrm{St}^{\mathrm{G}}(A, m) \to \mathrm{St}^{\mathrm{G}}(A)).$$

Proof. First of all, notice that $\mathrm{St}^{\mathrm{G}}(k)$ admits Bruhat decomposition for arbitrary field k:

$$\mathrm{St}^{\mathrm{G}}(k) = \mathrm{U}(\Phi^+, k) \cdot \mathrm{W}(\Phi, k) \cdot \mathrm{U}(\Phi^+, k).$$

Indeed, this is an immediate consequence of the classical Bruhat decomposition for the linear groups (see e. g. [10, Theorem 4]) and the fact that $K_2^G(k)$ is central in $\mathrm{St}^{\mathrm{G}}(k)$ and is generated by symbols $h_\alpha(u)$ lying in $\mathrm{W}(\Phi, k)$ (cf. Theorem 4.(i)).

As a consequence, the first three factors in the right hand side of the decomposition are mapped epimorphically onto $\mathrm{St}^{\mathrm{G}}(k)$ and the last factor coincides with $\mathrm{Ker}\,(\pi_*)$ from which the assertion of the lemma follows. ☐

lem:tulinj **Lemma 5.7.** *Assume that G satisfies Tulenbaev lifting property 2.3. Then the map i in the following commutative diagram of groups is injective.*

{diag:cs} (5.1)
$$\begin{tikzcd}
C_B \arrow[r, hook] \arrow[d, two heads, "k"] & \mathrm{St}^{\mathrm{G}}(B, I) \arrow[r, "\mu_B"] \arrow[d, "j"] & \mathrm{St}^{\mathrm{G}}(B) \arrow[r, two heads, "\pi_B"] \arrow[d, "i"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[d, hook] \\
C_R \arrow[r, hook] & \mathrm{St}^{\mathrm{G}}(R, I) \arrow[r, "\mu_R"'] \arrow[ur, dashed, "\varphi"] & \mathrm{St}^{\mathrm{G}}(R) \arrow[r, two heads, "\pi_R"'] & \mathrm{St}^{\mathrm{G}}(k[t, t^{-1}])
\end{tikzcd}$$

Proof. First of all, notice that by Corollary 5.4 the vertical map in the right-hand side of the diagram is injective. Invoking Tulenbaev's property 2.3 we also find a lifting map φ in the central square of the diagram.

Let $g \in \mathrm{St}^{\mathrm{G}}(B)$ be an element of $\mathrm{Ker}\,(i)$. Since g also lies in $\mathrm{Ker}\,(\pi_B)$ it comes from some $\widetilde{g} \in \mathrm{St}^{\mathrm{G}}(B, I)$ via μ_B. But $j(\widetilde{g})$ lies in C_R, hence, by Proposition 5.9 below it comes from some $\widehat{g} \in C_B$ via k. Finally, $g = \varphi(j(\widetilde{g})) = \varphi(k(\widehat{g})) = \mu_B(\widehat{g}) = 1$, as claimed. ☐

The following result is analogous to [12, Proposition 4.1].

Lemma 5.8. *The map j in the diagram (5.1) is surjective for any commutative ring A.*

Proof. Let α_i be arbitrary simple root from Π and let Σ_i (resp. Σ_i^-) denote the special subsets of roots consisting of $\alpha \in \Phi$ such that $m_i(\alpha) > 0$ (resp. $m_i(\alpha) < 0$).

By Lemma 2.2 the sets $\mathcal{Z}(\Sigma_i, R, I)$ and $\mathcal{Z}(\Sigma_i^-, R, I)$ both generate $\mathrm{St}^{\mathrm{G}}(R, I)$. Similarly, the subsets $\mathcal{Z}(\Sigma_i, B, I) \subseteq \mathcal{Z}(\Sigma_i, R, I)$, $\mathcal{Z}(\Sigma_i^-, B, I) \subseteq \mathcal{Z}(\Sigma_I, R, I)$ both generate the image of $j\colon \mathrm{St}^{\mathrm{G}}(B, I) \to \mathrm{St}^{\mathrm{G}}(R, I)$.

Using formula (4.3) we can calculate how the powers of the automorphism σ_i act on these generating sets. Indeed, for $z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i^-, R, I)$ we have

$$\sigma_i^N(z_\alpha(s, \xi)) = \sigma_i^N(x_\alpha(0, s)^{x_{-\alpha}(\Delta(\xi))}) = z_\alpha(t^{Nm_i(\alpha)} \cdot s, t^{-Nm_i(\alpha)} \cdot \xi).$$

It is clear that for sufficiently large N the element $t^{-Nm_i(\alpha)} \cdot \xi$ belongs to $A[t] \subseteq B$. Consequently, for arbitrary $g \in \mathrm{St}^{\mathrm{G}}(R, I)$ there exists $N > 0$ such that $\sigma^N(g)$ lies in the image of j. On the other hand, by a similar calculation we get for $z = z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i, B, I)$ that the element $\sigma^{-N}(z)$ $(N > 0)$ still lies in $\mathcal{Z}(\Sigma_i, B, I)$. Clearly, this implies that the image of j is preserved by σ_i^{-1}. The assertion of the lemma now follows from these two statements:

$$g = \sigma^{-N}\sigma^N(g) \in \sigma^{-N}(\mathrm{Im}\,(j)) \subseteq \mathrm{Im}\,(j). \qquad \square$$

prop:kersurj

Proposition 5.9. *Under the assumptions of Lemma 5.7 the map k in the diagram (5.1) is surjective.*

Sketch/draft of the proof. The argument presented below only may work under additional assumption that $m_i(\widetilde{\alpha}) = 1$ (i is as in the statement of Lemma 4.7).

Consider the following set

$$X = \mathrm{St}^{\mathrm{G}}(B) \times \widetilde{B}(R) \times \mathrm{St}^{\mathrm{G}}(R, I)/\simeq.$$

The congruence relation we impose should be similar to that used in Tulenbaev's paper. In particular, there are two maps $\psi_1\colon \mathrm{St}^{\mathrm{G}}(B, I) \to \mathrm{St}^{\mathrm{G}}(R, I)$, $p_{2B}^*\colon \mathrm{St}^{\mathrm{G}}(B, I) \to \mathrm{St}^{\mathrm{G}}(B)$ satisfying: (p_2^* is the canonical projection from $\mathrm{St}^{\mathrm{G}}(B, I)$ to $\mathrm{St}^{\mathrm{G}}(B)$ induced by the projection from $D(B, I)$ on the second coordinate, cf. section 3.1 in [8]),

$$[p_2^*(h), 1, g] \simeq [1, 1, \psi_1(h)g].$$

Notice that instead of $\mathrm{St}^{\mathrm{G}}(A[t])$ (used by Tulenbayev in the first factor) we use $\mathrm{St}^{\mathrm{G}}(B)$.

TODO: We should prove that there is a well-defined action of $\mathrm{St}^{\mathrm{G}}(R)$ on X.

We also assume that there is an automorphism σ_i defined on $\mathrm{St}^{\mathrm{G}}(R, I)$ (this should be simple: first define it on $\mathrm{St}^{\mathrm{G}}(D(R, I))$, then show that it is an automorphism of the subgroup $\ker(p_2^*)$ then show that it is an automorphism of the relative group $\mathrm{St}^{\mathrm{G}}(R, I)$).

Now assume that $g \in \mathrm{St}^{\mathrm{G}}(R, I)$ is such that its image $p_{2R}^*(g)$ in $\mathrm{St}^{\mathrm{G}}(R)$ is trivial

Using [8, Lemma 5] we can rewrite g as a product of $Z_\alpha((0, s), (\xi_1, \xi_2))$, $(\xi_1, \xi_2) \in D(R, I)$, $\alpha \in \Sigma_1^+$, $s \in I$ and $x_\alpha((0, s))$, $s \in I$, $\alpha \in \Phi$. Clearly, for $\alpha \in \Sigma_1^+$ we have

$$\sigma^{-N}(Z_\alpha((0, s), (\xi_1, \xi_2))) = Z_\alpha((0, s \cdot t^{-N}), (\xi_1 \cdot t^N, \xi_2 \cdot t^N)).$$

$\square$

By the above formula, for sufficiently large N we have that $\sigma^{-N}(g)$ is a product of $Z_\alpha((0, s), (\xi_1, \xi_2))$ for $\xi_1, \xi_2 \in A[t] \subseteq B$. In particular, this shows that $g \in \mathrm{Im}\,(\psi_1)$ hence for some $g' \in \mathrm{St}^{\mathrm{G}}(B, I)$ we have

$$[1, 1, g] = \sigma^N \cdot \sigma^{-N}[1, 1, g] = \sigma^N[1, 1, \psi_1(g')] = \sigma^N[p_{2B}^*(g'), 1, 1] = \sigma^N p_{2B}^*(g') \cdot \sigma^{-N} \cdot [1, 1, 1].$$

At this point we should be able to prove that $\sigma^N p_{2B}^*(g') \cdot \sigma^{-N} = p_{2R}^*(g) = 1$ (cf. definition of Tulenbaev's action). Finally, we get $[1,1,g] = [1,1,1]$ from which the assertion of the Lemma should follow.

5.3. **Proof of the main result.** The following result is analogous to [12, Proposition 4.3]. It plays the same role in our proof of $\mathbb{P}^1$-glueing for K_2 as generalized Suslin lemma (cf. [1, Theorem 2.16]) does in the corresponding proof for K_1.

prop:p1g **Proposition 5.10.** *The functors* St^{G}*,* K_2^G *satisfy* $\mathbb{P}^1$*-glueing property for arbitrary local ring* A*.*

Proof. □

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