# HOMOTOPY INVARIANCE FOR NONSTABLE $K_2$-FUNCTORS

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## 1. Aim of the paper

The aim of this text is to prove that the non-stable $K_2$-functors $K_2^G$, where $G$ is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings $R$ containing a field $k$, that is,

{eq:A1-main}

$$K_2^G(R[t]) = K_2^G(R). \tag{1.1}$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor $K$-functor associated to $G$. This functor originates from [J]. The above equality by e.g. [AHW, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of $G$ in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of $K_2$, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1,\dots,x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the $\mathrm{SL}_n$ case [Tul82] and Stavrova's proof for $K_1^G$ [St14]. Tulenbaev [Tul82] uses stabilization of the $K_2$-functor, and the good properties of the limit=algebraic $K$-theory. In [St14] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [Tul82, p. 140], or, respectively, [St14, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [Tul82, Theorem 5.1] or [St14, Theorem 1.1]).

1.1. **The case $R = k$.** We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the $\mathbb{A}^1$-invariance of $K_1^G$, this case should be used to deduce that $K_2^G(k[t_1,\dots,t_n]) = K_2^G(k)$.)

In Tulenbaev's framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [Wen14, Theorem 5.1]: let $k$ be an infinite field and let $G$ be a connected reductive group over $k$. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k),\mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]),\mathbb{Z}),$$

if the order of the fundamental group of $G$ is invertible in $k$. Once we know that the homology $H_2$ coincides with $K_2^G$ (on both sides), this gives the result. It would be nice to check Wendt's proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that $K_2^G$ coincides with $H_2$? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [Deo], but only for the

2010 *Mathematics Subject Classification.* 19C09, 19C20, 14L15, 20G35.

*Key words and phrases.* Chevalley group, non-stable $K_2$-functor, Steinberg group.

good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [VW, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. **Plan of the proof.**

plan:3t
(1) **(done)** Let $R$ be a local ring. Show that
$$\mathrm{St}(R[t^{\pm 1}]) = i_+(\mathrm{St}(R[t]))i_-(\mathrm{St}(R[t^{-1}]))i_+(\mathrm{St}(R[t]))$$
Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St}(R[t^{\pm 1}])$.

plan:k[t]
(2) Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect
(3) Consequently, $\mathrm{St}(k[t]) \to \mathrm{St}(k[t^{\pm 1}])$ is injective and $\mathrm{St}(k[t]) \cap \mathrm{St}(k[t^{-1}]) = \mathrm{St}(k)$ inside $\mathrm{St}(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.
The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp
(4) **(done)** Prove the Quillen-Suslin lgp for $K_2^G$. See Theorem 2.

plan:Zglu
(5) (Zariski gluing) for any commutative ring $A$ and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets
$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2)\mapsto g_1 {g_2}^{-1}} K_2^G(A_{fg})$$
is exact. The proof is usually almost the same as for (4).
(Remark. It seems that we need this property only for $A = R[t]$ and $f, g$ non-constant polynomials.)

plan:S-lemma
(6) ($S$-lemma) Let $A$ be a commutative ring, $S$ a multiplicative subset of $A$. If
$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$
for some $n \geq 1$, then $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$ as well.
This should be easy; see [A, Lemma 3.6].

plan:Nglu
(7) (Nisnevich gluing) Assume that $B$ is a subring of a commutative ring $A$, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.
(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St}(A_h)$ there exist $y \in \mathrm{St}(A)$ and $z \in \mathrm{St}(B_h)$ such that $x = F_h(y)z$.
(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and $h$ is not a zero divisor in $A$, then the sequence of pointed sets
$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2)\mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$
is exact.
This should use something from the proof of (4) or (5); see [St14, Lemma 3.4].

plan:P1
(8) ($\mathbb{P}^1$-gluing) Let $A$ be any commutative ring. Show that the sequence of pointed sets
$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2)\mapsto g_1 {g_2}^{-1}} K_2^G(A[t, t^{-1}])$$
is exact.
This should use (2), (4), (1), (3).

plan:[]f
(9) (Main corollary of $\mathbb{P}^1$-gluing) Let $A$ be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.
The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$.
This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it.

plan:k[tn] (11) Prove that
$$K_2^G(k[t_1, \dots, t_n]) = K_2^G(k).$$
If we strictly follow the pattern of $K_1^G$, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let $R$ be a regular ring containing a field $k$. Then $K_2^G(R[t]) = K_2^G(R)$.
This uses (4), (11) and (7).

## 2. Steinberg groups of Chevalley groups: preliminaries

Definition, functoriality, "congruence subgroups" $\mathrm{St}(\Phi, R, I)$ versus $\ker(\mathrm{St}(\Phi, R) \to \mathrm{St}(\Phi, R/I))$.

## 3. Decomposition theorems for $\mathrm{St}(A[t^{\pm 1}])$ and $\mathrm{St}\big(A((t))\big)$.

**Lemma 3.1.** *Let $(R, m)$ be a local ring, and let $G$ be a simply connected simple group over $R$ of isotropic rank $\geq 2$. Let $i_+ : \mathrm{St}(R[t]) \to \mathrm{St}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}(R[t^{-1}] \to \mathrm{St}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*
$$i_+\big(\mathrm{St}(m \cdot R[t])^{\mathrm{St}(R[t])}\big) i_-\big(\mathrm{St}(R[t^{-1}]\big) = i_-\big(\mathrm{St}(R[t^{-1}]\big) i_+\big(\mathrm{St}(m \cdot R[t])^{\mathrm{St}(R[t])}\big)$$
*inside* $\mathrm{St}(R[t^{\pm 1}])$.

*Proof.* This is proved exactly as [St14, Lemma 5.12]. ☐

thm:3t **Theorem 1.** *Let $R$ be a local ring, and let $G$ be a simply connected simple group over $R$ of isotropic rank $\geq 2$. Let $i_+ : \mathrm{St}(R[t]) \to \mathrm{St}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}(R[t^{-1}] \to \mathrm{St}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*
$$\mathrm{St}(R[t^{\pm 1}]) = i_+(\mathrm{St}(R[t])) i_-(\mathrm{St}(R[t^{-1}])) i_+(\mathrm{St}(R[t])).$$

*Proof.* This is proved exactly as [St14, Theorem 5.1]. ☐

## 4. Quillen-Suslin lgp, Zariski gluing, Nisnevich gluing, $S$-lemma

thm:lg-k2 **Theorem 2.** *Let $R$ be arbitrary commutative ring and let $\Phi$ be a root system of type $\mathsf{A}_\ell, \mathsf{C}_\ell, \mathsf{D}_\ell$ or $\mathsf{E}_\ell$ and rank $\ell \geq 3$. An element $g \in \mathrm{St}(\Phi, R[t], tR[t])$ is trivial if and only if its image in $\mathrm{St}(\Phi, R_M[t], tR_M[t])$ is trivial for all maximal ideals $M \trianglelefteq R$.*

*Proof.* The case $\Phi = \mathsf{C}_\ell$, $\ell \geq 3$ is contained in some future Lavrenov's solo paper. It think that the case of a simply laced $\Phi$ of rank $\geq 3$ can be settled in a similar way as in [S15]. We already have a written proof that [Tul82, Lemma 2.3] extends to rank 3 groups, see here. It should be inserted into this article at a some place. ☐

## 5. The case of $K_2^G(k[t])$ and some corollaries

thm:k[t] **Theorem 3.** *Let $k$ be a field. Let $G = G(\Phi, -)$ be a simply connected simple Chevalley group of rank $\geq 2$ such that ... Then*
$$K_2^G(k[t]) = K_2^G(k).$$

*Proof.* ??? ☐

cor:k[t]inj **Corollary 5.1.** *Let $G$, $k$ be as in Theorem 3. Then* $\mathrm{St}(k[t]) \to \mathrm{St}(k[t^{\pm 1}])$ *is injective and* $\mathrm{St}(k[t]) \cap \mathrm{St}(k[t^{-1}]) = \mathrm{St}(k)$ *inside* $\mathrm{St}(k[t^{\pm 1}])$.

*Proof.* Clearly, $g \in \ker(\mathrm{St}(k[t]) \to \mathrm{St}(k[t^{\pm 1}]))$ implies $g \in K_2^G(k[t])$. Since $K_2^G(k[t]) = K_2^G(k)$, and there is a section $K_2^G(k[t^{\pm 1}]) \to K_2^G(k)$, the map is injective. Second claim: take $g \in \mathrm{St}(k[t]) \cap \mathrm{St}(k[t^{-1}])$. Then the image $\phi(g)$ belongs to $E(k) = E(k[t]) \cap E(k[t^{-1}])$, and after adjusting $g$ by an element of $\mathrm{St}(k)$, we can assume that $g \in K_2^G(k[t]) \cap K_2^G(k[t^{-1}])$. Hence $g \in K_2^G(k) \subseteq \mathrm{St}(k)$. □

**Corollary 5.2.** *Let $G$, $k$ be as in Theorem 3. Then* $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.

*Proof.* We use Theorem 1. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}(k[t]))$, $y \in \mathrm{St}(k[t^{-1}]))$ . Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}(k) K_2^G(k[t^{-1}]) = \mathrm{St}(k)$ and $x_1 x_2 \in \mathrm{St}(k)$. That is, $g \in K_2^G(k)$. □

## 6. $\mathbb{P}^1$-gluing

The following lemma is an analog of [Tul82, Proposition 4.3 (a)].

**Lemma 6.1.** *Let $R$ be a local ring, $k = R/m$, and let $k$, $G$ be as in Theorem 3. The natural homomorphism* $\mathrm{St}(R[t]) \to \mathrm{St}(R[t^{\pm 1}])$ *is injective.*

*Proof.* Let $I$ be the maximal ideal of $R$, $l = R/I$, and consider the natural maps $\rho : \mathrm{St}(R[t, t^{-1}]) \to \mathrm{St}(l[t, t^{-1}])$, $\rho_+ : \mathrm{St}(R[t]) \to \mathrm{St}(l[t])$, $\rho_- : \mathrm{St}(R[t^{-1}]) \to \mathrm{St}(l[t^{-1}])$. Take $x \in \ker(\mathrm{St}(R[t]) \to \mathrm{St}(R[t^{\pm 1}])$. By the field case Corollary 5.1 one has $\rho_+(x) = 1$, hence $x \in \mathrm{St}(I \cdot R[t])^{\mathrm{St}(R[t])}$.

??????? □

**Lemma 6.2.** *Let $G$, $k$ be as in Theorem 3. Let $(R, m)$ be a local ring such that $R/m = k$. Then*

$$\mathrm{St}(R[t]) \cap \mathrm{St}(R[t^{-1}]) = \mathrm{St}(R)$$

*inside* $\mathrm{St}(R[t^{\pm 1}])$.

*Proof.* ?????? □

**Theorem 4.** *Let $A$ be any commutative ring. Then the sequence of pointed sets*

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2) \mapsto g_1 {g_2}^{-1}} K_2^G(A[t, t^{-1}])$$

*is exact.*

*Proof.* ???? □

**Corollary 6.3.** *Let $A$ be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that* $K_2^G(A[t]) \to K_2^G(A[t]_f)$ *is injective.*

*Proof.* ???? □

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