Let A be arbitrary commutative unital ring and $\mathfrak{m}$ be its ideal. Denote by B the ring $\mathfrak{m}[t^{-1}] + A[t]$ with the obvious $\mathbb{Z}$-grading. Clearly, the k-th homogeneous component B_k of B equals $A \cdot t^k$ for $k \geq 0$ or $B_k = \mathfrak{m} \cdot t^k$ for $k < 0$.

For $n \geq 1$ consider the following collections of generators:

$$\mathcal{X}_n = \{x_\alpha(\xi) \mid \xi \in B_k,\ \text{for } k \leq n\}.$$

It is obvious that $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$, denote by $\mathcal{X}_\infty$ the union of all $\mathcal{X}_n$'s.

Let Φ be a simply-laced irreducible root system of rank ≥ 3. Denote by G_m the group presented by generators $\mathcal{X}_m$ and the set of relations $\mathcal{R}_m$ consisting of the following three families of relations:

$$(a_m)\qquad x_\alpha(\xi)x_\alpha(\eta) = x_\alpha(\xi+\eta), \qquad \xi, \eta \in B_k,\ k \leq m;$$

$$(b_m)\qquad [x_\alpha(\xi), x_{\alpha'}(\eta))] = 1, \qquad \text{in the case } \alpha + \alpha' \notin \Phi \cup \{0\},\ \xi \in B_k,\ \eta \in B_l,\ l, k \leq m;$$

$$(c_m)\qquad [x_\alpha(\xi), x_{\alpha'}(\eta)] = x_{\alpha+\alpha'}(N_{\alpha,\alpha'}\xi\eta), \qquad \text{in the case } \alpha + \alpha' \in \Phi,\ \xi \in B_k,\ \eta \in B_l,\ l, k, l+k \leq m.$$

The aim of this subsection is to the following analogue of [3, Lemma 3.3].

Lemma 1. *For all simply-laced root systems of rank ≥ 3 the natural inclusion of generators induces an isomorphism $G_1 \to \mathrm{St}(\Phi, B)$.*

Unfortunately, the lemma cannot be proved by a slight modification of the original short argument of Tulenbayev. Instead, we have to employ a variant of much longer argument of U. Rehmann and C. Soulé, see [1, 2].

To prove Lemma 1 it suffices to show that the natural map $\theta_m\colon G_m \to G_{m+1}$ is an isomorphism for $m \geq 1$. Indeed, this would give the required isomorphism $G_1 \cong \mathrm{St}(\Phi, B)$ since $\langle \mathcal{X}_\infty \mid \mathcal{R}_\infty \rangle$ is evidently isomorphic to $\mathrm{St}(\Phi, B)$.

Remark 2. There are two reasons why we can't simply refer to [1] or [2] for the proof of Lemma 1. The first is that Rehmann and Soulé prove that $G_m \to G_{m+1}$ is an isomorphism for $m \geq 2$, while we want to establish this fact starting from $m = 1$. The second is that they prove their results only in the special case $B = A[t]$, moreover, they make assumptions about the ground ring A (it is assumed that $A = k$ is a field in [1] and that $A = \mathbb{Z}$ in [2]).

We will use the following commutator identities (cf. [1, H1]):

$$[ab, c] = {}^a[b, c] \cdot [a, c]; \tag{1}$$

$$[a, c] = 1 \text{ implies } [a, [b, c]] = [[a, b], {}^b c]. \tag{2}$$

The following lemma is reminiscent of [1, Proposition 1.1] and [2, Proposition 3.2.2].

Lemma 3. *Suppose $m \geq 1$. Let $\alpha, \beta, \alpha', \beta'$ be such that $\alpha + \beta = \alpha' + \beta'$. Assume, moreover, that $\xi \in B_k$, $\xi' \in B_{k'}$, $\eta \in B_l$, $\eta' \in B_{l'}$ are such that $N_{\alpha,\beta}\xi\eta = N_{\alpha',\beta'}\xi'\eta'$ for some $k, k', l, l' \leq m$ satisfying $k + l = k' + l' = m + 1$. Then in the group G_m the following relations hold:*

$$[x_\alpha(\xi), x_\beta(\eta)] = [x_{\alpha'}(\xi'), x_{\beta'}(\eta')] \tag{3}$$

$$[[x_\gamma(\zeta), [x_\alpha(\xi), x_\beta(\eta)]] = 1 \qquad \textit{in the case } \gamma \in \{\alpha, \beta, \alpha+\beta\},\ \zeta \in B_{k''},\ k'' \leq m. \tag{4}$$

Proof. Notice that $k+l=m+1,\ k,l\leq m$ imply $k,l>0$, hence $B_i=t^i\cdot A$ for $i=k,k',l,'l'$. Therefore, we can repeat the argument of [1, Proposition 1.1] verbatim. $\square$

To prove that θ_m is an isomorphism we construct the missing generators of G_{m+1} in G_m and then show that they satisfy relations $\mathcal{R}_{m+1}$.

For every $\xi\in B_{m+1}$ and $\alpha\in\Phi$ there exist $\xi'\in B_m$ and $\alpha'\in\Phi$ such that $\xi=t\xi'$ and $\alpha-\alpha'\in\Phi$. Thus, we can make the following definition:

$$x_\alpha(\xi):=[x_{\alpha-\alpha'}(N_{\alpha-\alpha',\alpha'}\xi'),x_{\alpha'}(t)],\tag{5}$$

and from (3) its correctness (i. e. the independence of the choice of α') follows.

We now turn to the verification of relations (a_{m+1}), (b_{m+1}), (c_{m+1}). Notice first that (1) and (4) immediately imply (a_{m+1}) and hence (b_{m+1}) in the special case $\alpha=\alpha'$.

To verify relations (c_{m+1}) it suffices to show that

$$[x_\alpha(\xi),x_{\alpha'}(at^{m+1})]=[x_\alpha(t\xi),x_{\alpha'}(at^m)],\ a\in A,\ \xi\in B_k,\ k\leq 0.\tag{6}$$

We can find root subsystem $\Psi\subseteq\Phi$ of type A_3 containing the roots α,α'. Choose basis $\{\alpha,\beta,\gamma\}$ of Ψ so that $\alpha'=\beta$ and the Dynkin diagram of Ψ looks as follows:

$$\circ_\alpha \;\text{———}\; \circ_\beta \;\text{———}\; \circ_\gamma\ .\tag{7}$$

Let us verify (6):

$$\begin{aligned}
[x_\alpha(\xi),x_\beta(at^{m+1})]&=[x_\alpha(\xi),[x_{\beta+\gamma}(t),x_{-\gamma}(a't^m)]] \text{ by (5) for a suitable } a'\in A\\
&=[x_{\alpha+\beta+\gamma}(\epsilon t\xi),{}^{x_{\beta+\gamma}(t)}x_{-\gamma}(a't^m)] \text{ by (2), for } \epsilon=N_{\alpha,\beta+\gamma}\\
&={}^{x_{\beta+\gamma}(t)}[x_{\alpha+\beta+\gamma}(\epsilon t\xi),x_{-\gamma}(a't^m)] \text{ by } (b_1)\\
&={}^{x_{\beta+\gamma}(t)}[x_{\alpha+\beta+\gamma}(\epsilon t^2\xi),x_{-\gamma}(a't^{m-1})] \text{ by (3) if } k=0 \text{ or } (c_m) \text{ if } k<0\\
&=[[x_\alpha(t\xi),x_{\beta+\gamma}(t)],{}^{x_{\beta+\gamma}(t)}x_{-\gamma}(a't^{m-1})] \text{ by } (b_2),\ (c_2) \text{ or by (4),(5) if } m=1\\
&=[x_\alpha(t\xi),x_\beta(at^m)] \text{ by (2).}
\end{aligned}$$

Now, let us show that relations (b_{m+1}) hold. Without loss of generality we may also assume $k\leq l=m+1$, thus $\eta=bt^{m+1}$ for some $b\in A$.

1. First consider the case $k\leq 0$. There are two further subcases.
   (a) Case $\alpha\not\perp\alpha'$. As before, without loss of generality we may assume that α,α' are contained in a root system Ψ of type A_3, whose simple roots are labeled as in (7) and, moreover, $\alpha'=\alpha+\beta$. Using (2) and (b_m) we get (recall that $\xi\in B_k$):

$$\begin{aligned}
&[x_\alpha(\xi),x_{\alpha+\beta}(bt^{m+1})]=[x_\alpha(\xi),[x_{\alpha+\beta+\gamma}(bt^m),x_{-\gamma}(t)]]=\\
&\qquad=[[x_\alpha(\xi),x_{\alpha+\beta+\gamma}(bt^m)],{}^{x_{\alpha+\beta+\gamma}(bt^m)}x_{-\gamma}(t)]=1.
\end{aligned}$$

   (b) Case $\alpha\perp\alpha'$. As before, we set $\alpha'=\gamma$ and use (2),(b_m) and (c_m):

$$\begin{aligned}
&[x_\alpha(\xi),x_\gamma(bt^{m+1})]=[x_\alpha(\xi),[x_{\beta+\gamma}(bt^m),x_{-\beta}(t)]]=\\
&=[[x_\alpha(\xi),x_{\beta+\gamma}(bt^m)],{}^{x_{\beta+\gamma}(bt^m)}x_{-\beta}(t)]={}^{x_{\beta+\gamma}(bt^m)}[x_{\alpha+\beta+\gamma}(bt^m\xi),x_{-\beta}(t)]=1.
\end{aligned}$$

2. Now assume $1\leq k\leq m+1$. At first we want to prove (b_{m+1}) only in the special case $\xi=t^k$. We proceed by induction on k starting with $k=1$. Again, there are two cases.

(a) Case $\alpha \not\perp \alpha'$ is handled similarly to the Case (1a) with the only difference that we have to refer to the inductive assumption rather than (b_m) in the case $k = m+1$.

$$[x_\alpha(t^k), x_{\alpha+\beta}(bt^{m+1})] = [x_\alpha(t^k), [x_{\alpha+\beta+\gamma}(bt^m), x_{-\gamma}(t)]] = \\ = [[x_\alpha(t^k), x_{\alpha+\beta+\gamma}(bt^m)], {}^{x_{\alpha+\beta+\gamma}(bt^m)}x_{-\gamma}(t)] = 1.$$

(b) Case $\alpha \perp \alpha'$. As before, we may assume $\alpha' = \gamma$.

$$\begin{aligned} {}^{x_\gamma(t^k)}x_\alpha(bt^{m+1}) &= {}^{x_\gamma(t^k)}[x_{-\beta}(b't^{m+1}), x_{\alpha+\beta}(1)] \text{ by } (c_{m+1}) \\ &= [x_{-\beta}(b't^{m+1}), {}^{x_\gamma(t^k)}x_{\alpha+\beta}(1)] \text{ by Case (2a) since } \gamma \not\perp -\beta \\ &= [[x_{-\beta-\gamma}(b''t^{m+1-k}), x_\gamma(t^k)], {}^{x_\gamma(t^k)}x_{\alpha+\beta}(1)] \text{ by } (c_{m+1}) \\ &= [x_{-\beta-\gamma}(b''t^{m+1-k})), [x_\gamma(t^k), x_{\alpha+\beta}(1)]] \text{ by (2) and } (b_m) \text{ since } m+1-k \le m \\ &= x_\alpha(b'''t^{m+1}) \text{ by } (c_{m+1}). \end{aligned}$$

Usual identities for structure constants imply (cf. [1, p. 12]):

$$b''' = N_{-\beta-\gamma,\alpha+\beta+\gamma} \cdot N_{\gamma,\alpha+\beta} \cdot N_{-\beta-\gamma,\gamma} \cdot N_{-\beta,\alpha+\beta} \cdot b = b,$$

Thus we have demonstrated (b_{m+1}) in the special case $a = 1$.

(3) Now let us prove (b_{m+1}) for arbitrary $\xi = at^k$, $a \in A$ and $1 \le k \le m+1$. Again, there are two subcases.

(a) Case $\alpha \not\perp \alpha'$.

$$\begin{aligned} [x_\alpha(at^k), x_{\alpha+\beta}(bt^{m+1})] &= [x_\alpha(at^k), [x_{\alpha+\beta+\gamma}(t^{m+1}), x_{-\gamma}(b)]] \text{ by } (c_{m+1}) \\ &= [[x_\alpha(at^k), x_{\alpha+\beta+\gamma}(t^{m+1})], {}^{x_{\alpha+\beta+\gamma}(t^{m+1})}x_{-\gamma}(b)] \text{ by } (b_k) \text{ or Case (1) if } k = m+1 \\ &= 1 \text{ by Case (2).} \end{aligned}$$

(b) Case $\alpha \perp \alpha'$. The required identity follows from the following chain of equalities:

$$\begin{aligned} x_{\alpha+\beta+\gamma}(abt^{m+1}) &= {}^{x_{-\beta}(t^k)}x_{\alpha+\beta+\gamma}(abt^{m+1}) \text{ by Case (2)} \\ &= {}^{x_{-\beta}(t^k)}[x_{\alpha+\beta}(bt^{m+1-k}), x_\gamma(at^k)] \text{ by } (c_{m+1}) \\ &= [x_\alpha(bt^{m+1})x_{\alpha+\beta}(bt^{m+1-k}), x_\gamma(at^k)] \text{ by } (b_k) \text{ or Case (2) if } k = m+1 \\ &= {}^{x_\alpha(bt^{m+1})}x_{\alpha+\beta+\gamma}(abt^{m+1})[x_\alpha(bt^{m+1}), x_\gamma(at^k)] \text{ by (1) and } (c_{m+1}) \\ &= x_{\alpha+\beta+\gamma}(abt^{m+1})[x_\alpha(bt^{m+1}), x_\gamma(at^k)] \text{ by Case (3a).} \end{aligned}$$

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