$\mathbb{A}^1$-INVARIANCE FOR UNSTABLE K_2

S. SINCHUK, A. STAVROVA, AND A. LAVRENOV

1. Aim of the paper

The aim of this text is to prove that the non-stable K_2-functors K_2^G, where G is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings R containing a field k, that is,

{eq:A1-main} (1.1) $$K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K-functor associated to G. This functor originates from [5]. The above equality by e.g. [2, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of G in the sense of Morel–Voevodsky.

”Suitable type” here means that we consider only the cases where we know the centrality of K_2, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \ldots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the SL_n case [11] and Stavrova's proof for K_1^G [9]. Tulenbaev [11] uses stabilization of the K_2-functor, and the good properties of the limit=algebraic K-theory. In [9] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [11, p. 140], or, respectively, [9, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [11, Theorem 5.1] or [9, Theorem 1.1]).

1.1. **The case** $R = k$**.** We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the $\mathbb{A}^1$-invariance of K_1^G, this case should be used to deduce that $K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k)$.)

In Tulenbaev's framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [14, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k. Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt's proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [3], but only for the

2010 *Mathematics Subject Classification.* 19C09, 19C20, 14L15, 20G35.

Key words and phrases. Chevalley group, non-stable K_2-functor, Steinberg group.

good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [12, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. **Plan of the proof.**

plan:3t (1) (**done**, see Theorem 2). Let R be a local ring. Show that

$$\mathrm{St^G}(R[t^{\pm 1}]) = i_+(\mathrm{St^G}(R[t]))i_-(\mathrm{St^G}(R[t^{-1}]))i_+(\mathrm{St^G}(R[t]))$$

Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St^G}(R[t^{\pm 1}])$.

plan:k[t] (2) (**done**, see Theorem 4). Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect (3) (**done**). Consequently, $\mathrm{St^G}(k[t]) \to \mathrm{St^G}(k[t^{\pm 1}])$ is injective and $\mathrm{St^G}(k[t]) \cap \mathrm{St^G}(k[t^{-1}]) = \mathrm{St^G}(k)$ inside $\mathrm{St^G}(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp (4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen-Suslin lgp for K_2^G.

plan:Zglu (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2) \mapsto g_1 {g_2}^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma (6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A. If

$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St^G}(A_h)$ there exist $y \in \mathrm{St^G}(A)$ and $z \in \mathrm{St^G}(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and h is not a zero divisor in A, then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [9, Lemma 3.4].

plan:P1 (8) ($\mathbb{P}^1$-gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2) \mapsto g_1 {g_2}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

(a) Prove Proposition 4.6. This is hard. Have no idea how this can be proved at the moment.

(b) Prove $\mathbb{P}^1$-glueing using all the above facts.

plan:[]f (9) (Main corollary of $\mathbb{P}^1$-gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.
The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2.

plan:k[tn] (11) Prove that
$$K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k).$$
If we strictly follow the pattern of K_1^G, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let R be a regular ring containing a field k. Then $K_2^G(R[t]) = K_2^G(R)$.
This uses (4), (11) and (7).

2. Steinberg groups of Chevalley groups: preliminaries

2.1. **Definition and basic properties.** Let G be a split simple Chevalley groups with a root system Φ of rank ≥ 2. Recall that the *Steinberg group* $\mathrm{St}^{\mathrm{G}}(R)$ (also denoted $\mathrm{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi,R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi,R}$ defined as follows:

{rel:add} (2.1)
$$x_\alpha(s)x_\alpha(t) = x_\alpha(s+t),$$

{rel:CCF} (2.2)
$$[x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta}\left(N_{\alpha\beta ij}\, s^i t^j\right), \quad \alpha \neq -\beta, \quad N_{\alpha\beta ij} \in \mathbb{Z}.$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$. Denote by $\mathrm{W}(\Phi, R)$ the subgroup of $\mathrm{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$.

2.2. **Tulenbaev's lifting property and its corollaries.** Throughout this section $I \trianglelefteq A$ is an ideal of arbitrary commutative ring A. For a nonnilpotent element $a \in A$ denote by $\lambda_a\colon A \to A_a$ the morphism of principal localization at a. Consider the following commutative square.

{msq} (2.3)
$$\begin{tikzcd} A \arrow[r, "\lambda_a"] \arrow[d, two heads] & A_a \arrow[d, two heads] \\ A/I \arrow[r, "\overline{\lambda_a}"] & A_a/I_a \end{tikzcd}$$

Notice that (2.3) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a. Such squares are usually called *Milnor squares* in the literature, see [13, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [11, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

`def:tlp` **Definition 2.1.** We say that the Steinberg group functor St^{G} satisfies *Tulenbaev's lifting property* if for every pull-back square (2.3) the following lifting problem has a solution.

$$\begin{tikzcd} \mathrm{St}^{\mathrm{G}}(A, I) \arrow[r, "\mu"] \arrow[d] & \mathrm{St}^{\mathrm{G}}(A) \arrow[d, "\lambda_a^*"] \\ \mathrm{St}^{\mathrm{G}}(A_a, I) \arrow[r, "\mu"] \arrow[ur, dotted] & \mathrm{St}^{\mathrm{G}}(A_a) \end{tikzcd}$$

Theorem 1. *Assume that G satisfies Tulenbaev property (2.1) then the following facts are true for arbitrary commutative ring A:*

`thm:dp` *(i) A dilation principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. if $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is such that equality $\lambda_a^*(h) = 1$ holds in $\mathrm{St}^{\mathrm{G}}(\Phi, R_a[t])$ then for sufficiently large n one has*

$$ev^*_{\left[\frac{R[t] \to R[t]}{t \mapsto a^n \cdot t}\right]}(h) = 1.$$

`thm:lg-k2` *(ii) A local-global principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. an element $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is trivial if and only if its image in $\mathrm{St}^{\mathrm{G}}(A_m[t], tA_m[t])$ is trivial for all maximal ideals $m \trianglelefteq A$.*

`thm:centr` *(iii) $K_2^G(A)$ is contained in the centre of $\mathrm{St}^{\mathrm{G}}(A)$.*

Proof. Follows by the same argument as [11, Theorem 2.1] or [8, Theorem 2] □

2.3. **The action of torus.** Throughout this subsection $G = G_{\mathrm{ad}}$ denotes a split simple Chevalley group of adjoint type with the root system Φ of rank ≥ 2. Denote by $T = T_{\mathrm{ad}}$ the torus of G and by $T(R)$ its group of R-points.

We identify the root lattice $X^*(T) = \mathrm{Hom}(T, \mathbf{G}_{\mathrm{m}})$ with the lattice $\mathbb{Z}\,\Phi$ in the obvious way. In particular, for $\alpha \in \Phi$ we denote by α_R the corresponding map $T(R) \to R^*$ on R-points. An element $h \in T(R)$ defines a permutation of the set $\mathcal{X}_{\Phi,R}$ of generators of $\mathrm{St}^{\mathrm{G}}(R)$ as follows:

$$h \cdot x_\alpha(\xi) = x_\alpha(\alpha_R(h) \cdot \xi). \tag{2.4}$$

Notice that h preserves the defining relations $\mathcal{R}_{\Phi,R}$ of the Steinberg group (and thus, determines a permutation of $\mathcal{R}_{\Phi,R}$). Indeed, the assertion is immediate for relation (2.1). Verification of the fact that h preserves (2.2) is a routine computation which should use the fact that for $\alpha, \beta \in \mathbb{Z}\,\Phi$ one has $(\alpha+\beta)_R(h) = \alpha_R(h) \cdot \beta_R(h)$.

3. Decomposition theorems for $\mathrm{St}^{\mathrm{G}}(A[t^{\pm 1}])$ and $\mathrm{St}^{\mathrm{G}}\big(A((t))\big)$.

Lemma 3.1. *Let (R, m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big) i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) = i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big)$$

inside $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

Proof. This is proved exactly as [9, Lemma 5.12]. □

`thm:3t` **Theorem 2.** *Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t])).$$

Proof. This is proved exactly as [9, Theorem 5.1]. □

4. $\mathbb{P}^1$-GLUING

Throughout this section $G = G(\Phi, -)$ denotes an arbitrary Chevalley group scheme of type Φ. We also assume everywhere below that Φ is irreducible and has rank ≥ 2.

def:p1g **Definition 4.1.** We say that the Steinberg group functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-*glueing property* for a ring A if the following sequence of groups is exact in the middle term

$$\mathrm{St}^{\mathrm{G}}(A) \xhookrightarrow{\Delta_A} \mathrm{St}^{\mathrm{G}}(A[t]) \times \mathrm{St}^{\mathrm{G}}(A[t^{-1}]) \xrightarrow{\pm_A} \mathrm{St}^{\mathrm{G}}(A[t, t^{-1}]).$$

Here we denote by Δ_A the split injection $g \mapsto (g, g)$ and $\pm_A$ denotes the map $(g, h) \to g \cdot h^{-1}$.

The main result of this section is the following theorem which plays a crucial role in the sequel.

thm:p1 **Theorem 3.** *The Steinberg group functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for an arbitrary commutative ring A.*

Proof. Let (g^+, g^-) be an element of $\mathrm{St}^{\mathrm{G}}(A[t]) \times \mathrm{St}^{\mathrm{G}}(A[t^{-1}])$ such that the equality $g^+ = g^-$ holds in $\mathrm{St}^{\mathrm{G}}(A[t, t^{-1}])$.

Let M be a maximal ideal of A. By Proposition 4.8 below the functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for the local ring A_M hence $(\lambda^*_M(g_+), \lambda^*_M(g_-)) = \Delta_{A_M}(\lambda^*_M(g^+)(0))$ and in the groups $\mathrm{St}^{\mathrm{G}}(A_M[t])$ and $\mathrm{St}^{\mathrm{G}}(A_M[t^{-1}])$ we have the equalities:

$$\lambda^*_M(g^+ \cdot g^+(0)^{-1}) = \lambda^*_M(g^+) \cdot \lambda^*_M(g^+)(0)^{-1} = 1; \qquad \lambda^*_M(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for $\mathrm{St}^{\mathrm{G}}(-)$ (see Theorem 1.(ii)) these equalities hold globally and $(g^+, g^-) = \Delta_A(g^+(0))$, as claimed. □

Corollary 4.2. *Let A be any commutative ring and $f \in A[t]$ be a monic polynomial. Then the map $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.*

Proof. □

4.1. **The case of a field.** Throughout this section k denotes arbitrary field k.

thm:k[t] **Theorem 4.** *The following statements are true.*

satz1 (i) *For $A = k, k[t]$ the subgroup $K_2^G(A) \trianglelefteq \mathrm{St}^{\mathrm{G}}(A)$ is generated by elements of the form*

$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1},\ u, v \in k^*.$$

(ii) *As a consequence, the canonical injection $K_2^G(k) \hookrightarrow K_2^G(k[t])$ is an isomorphism.*

Proof. See [7, Satz 1] and the corollary after it. □

cor:k[t]inj **Corollary 4.3.** *Let G, k be as in Theorem 4. Then $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$.*

Proof. Clearly, $g \in \ker(\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}]))$ implies $g \in K_2^G(k[t])$. Since $K_2^G(k[t]) = K_2^G(k)$, and there is a section $K_2^G(k[t^{\pm 1}]) \to K_2^G(k)$, the map is injective. Second claim: take $g \in \mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}])$. Then the image $\phi(g)$ belongs to $E(k) = E(k[t]) \cap E(k[t^{-1}])$, and after adjusting g by an element of $\mathrm{St}^{\mathrm{G}}(k)$, we can assume that $g \in K_2^G(k[t]) \cap K_2^G(k[t^{-1}])$. Hence $g \in K_2^G(k) \subseteq \mathrm{St}^{\mathrm{G}}(k)$. □

Corollary 4.4. *Let G, k be as in Theorem 4. Then $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.*

Proof. We use Theorem 2. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}^{\mathrm{G}}(k[t]))$, $y \in \mathrm{St}^{\mathrm{G}}(k[t^{-1}]))$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}^{\mathrm{G}}(k) K_2^G(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ and $x_1 x_2 \in \mathrm{St}^{\mathrm{G}}(k)$. That is, $g \in K_2^G(k)$. $\square$

4.2. **The case of a local ring.** For the rest of this section A denotes an arbitrary commutative local ring with the maximal ideal m and the residue field k. We denote by π the canonical projection $A \to k$. Throughout this section we will employ the following notation:

- R denotes the Laurent polynomial ring $A[t, t^{-1}]$;
- B denotes the subring $A[t] + m[t^{-1}]$ of R consisting of Laurent polynomials $f(t, t^{-1})$ whose coefficients of terms of negative degree belong to m;
- I denotes the ideal $m[t, t^{-1}]$ of R (which can be also considered as an ideal of B).

Consider the following commutative diagram of groups.

{diag:cs}
$$\begin{tikzcd} C_B \arrow[r, hook] \arrow[d, two heads, "k^+"] & \mathrm{St}^{\mathrm{G}}(B, I) \arrow[r, "\mu^{+0}"] \arrow[d, "j^+"] & \mathrm{St}^{\mathrm{G}}(B) \arrow[r, two heads, "p^{+0}"] \arrow[d, "i^+"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[d, hook, "\overline{i^+}"] \\ C_R \arrow[r, hook] & \mathrm{St}^{\mathrm{G}}(R, I) \arrow[r, "\mu^{\pm}"'] \arrow[ur, dashed, "\varphi"] & \mathrm{St}^{\mathrm{G}}(R) \arrow[r, two heads, "p^{\pm}"'] & \mathrm{St}^{\mathrm{G}}(k[t, t^{-1}]) \end{tikzcd} \tag{4.1}$$

Notice that by Corollary 4.3 the map $\overline{i^+}$ in the right-hand side of the above diagram is injective. Invoking Tulenbaev's property 2.1 we also find a lifting map in the central square of the diagram.

Our next result is analogous to [11, Lemma 3.1(e)] (cf. also with [4, § 2.3A]).

cor:bruhat **Lemma 4.5.** *The Steinberg group $\mathrm{St}^{\mathrm{G}}(A)$ admits the following analogue of the Bruhat decomposition:*

$$\mathrm{St}^{\mathrm{G}}(A) = \mathrm{U}(\Phi^+, A) \cdot \mathrm{W}(\Phi, A) \cdot \mathrm{U}(\Phi^+, A) \cdot \mathrm{Im}\,(\mathrm{St}^{\mathrm{G}}(A, m) \to \mathrm{St}^{\mathrm{G}}(A)).$$

Proof. First of all, notice that $\mathrm{St}^{\mathrm{G}}(k)$ admits Bruhat decomposition for arbitrary field k:

$$\mathrm{St}^{\mathrm{G}}(k) = \mathrm{U}(\Phi^+, k) \cdot \mathrm{W}(\Phi, k) \cdot \mathrm{U}(\Phi^+, k).$$

Indeed, this is an immediate consequence of the classical Bruhat decomposition for the linear groups (see e. g. [10, Theorem 4]) and the fact that $K_2^G(k)$ is central in $\mathrm{St}^{\mathrm{G}}(k)$ and is generated by symbols $h_\alpha(u)$ lying in $\mathrm{W}(\Phi, k)$ (cf. Theorem 4.(i)).

Denote the map $A \to k$ by π. By ??? the first factor of the decomposition coincides with $\mathrm{Ker}\,(\pi_*)$ whereas by Lemma 4.5 the product of the last three factors is mapped epimorphically onto $\mathrm{St}^{\mathrm{G}}(k)$. $\square$

The following result is analogous to [11, Proposition 4.1].

prop:kersurj **Proposition 4.6.** *The map k^+ is surjective.*

Proof. $\square$

Using a simple diagram chasing argument we are able to obtain the following result.

cor:tulinj **Corollary 4.7.** *The canonical map $i^+ \colon \mathrm{St}^{\mathrm{G}}(B) \to \mathrm{St}^{\mathrm{G}}(R)$ is injective.*

Proof. Let $g \in \mathrm{St}^{\mathrm{G}}(B^+)$ be an element of $\mathrm{Ker}\,(i^+)$. Since g also lies in $\mathrm{Ker}\,(p^{+0})$ it comes from some $\widetilde{g} \in \mathrm{St}^{\mathrm{G}}(B^+, I)$ via μ^{+0}. Now $j^+(\widetilde{g})$ lies in $C_\pm$, hence, by Proposition 4.6 it comes from some $\widehat{g} \in C_{+0}$ via k^+. Finally, $g = \varphi(j^+(\widetilde{g})) = \varphi(k^+(\widehat{g})) = \mu^{+0}(\widehat{g}) = 1$, as claimed. ☐

prop:p1g **Proposition 4.8** ($\mathbb{P}^1$-glueing in the local case)**.** *Blah-blah-blah*

Proof. ☐

Appendices

A. Linear Steinberg group in rank 3

The main goal of this subsection is to show that Tulenbaev's [11, Lemma 2.3] remains valid for the linear Steinberg group of rank ≥ 3. In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [8, Definitions 3.3 and 3.7]).

Definition A.1. The relative Steinberg group $\mathrm{St}^*(n, R, I)$ is the group defined by the following two families generators and four families of relations.

- Generators:
  (1) $X^1(u, v)$, where $u \in \mathrm{E}(n, R) \cdot e_1$, $v \in I^n$ such that $v^t \cdot u = 0$;
  (2) $X^2(u, v)$, where $u \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$ such that $v^t \cdot u = 0$.
  Notice that ϕ maps both $X^1(u, v)$ and $X^2(u, v)$ to $T(u, v) = e + u \cdot v^t \in \mathrm{E}(n, R, I)$.
- Relations:
  (1) $X^1(u, v) \cdot X^1(u, w) = X^1(u, v + w)$, $u \in \mathrm{E}(n, R) \cdot e_1$, $v, w \in I^n$;
  (2) $X^2(u, v) \cdot X^2(w, v) = X^2(u + w, v)$, $u, w \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$;
  (3) ${}^{X^\sigma(u^2, v^2)}X^\tau(u^1, v^1) = X^\tau(T(u^2, v^2) \cdot u^1, T(v^2, u^2)^{-1} \cdot v^1)$, $\sigma, \tau = 1, 2$;
  (4) $X^1(g \cdot e_1, g^* \cdot be_2) = X^2(g \cdot be_1, g^* \cdot e_2)$ where $b \in I$ and $g^* = g^{t^{-1}}$ denotes the contragradient matrix.

Lemma A.2. *The groups* $\mathrm{St}^*(n, R, I)$ *and* $\mathrm{St}(n, R, I)$ *are isomorphic.*

Proof. **TODO:** ☐

The next step of the proof is to is construct certain elements in $\mathrm{St}(n, R)$ similar to Tulenbaev's elements $X_{u,v}(a)$ see [11, § 1].

Let $v \in R^n$ be a column. Denote by $O(v)$ the submodule of R^n consisting of all columns w such that $w^t \cdot v = 0$. A column $w \in R^n$ is called v-*decomposable* if it can be presented as a finite sum $w = \sum\limits_{i=1}^{p} w^i$ such that each w^i has at least two zero entries and $v^t \cdot w^i = 0$. Denote by $D(v)$ the submodule of $O(v)$ consisting of all v-decomposable columns. For a column $v \in R^n$ denote by $I(v)$ the ideal of R spanned by its entries $v_1, \ldots, v_n$.

Let $u, v, w \in R^n$ be columns such that $w^t v = 0$. It is easy to check (cf. [6, Lemma 3.2]) that

$$(uv) \cdot w = \sum_{i<j} w_{ij}, \text{ where } w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n.$$

The above decomposition is called the *canonical* decomposition of $(uv) \cdot w$. In particular, this shows that the column $a \cdot w$ is always v-decomposable for $a \in I(v)$, $w \in O(v)$, i. e.

$I(v) \cdot O(v) \subseteq D(v)$. It is also straightforward to check that $D(v) \subseteq D(bv)$, $b \cdot D(v) \subseteq D(v)$ for $b \in R$.

Denote by B^1 the subset of $R^n \times R^n \times R$ consisting of triples (u, v, a) such that $v^t \cdot u = 0$, $v \in D(u)$, $a \in I(u)$. Denote by B^2 the set consisting of triples (v, u, a) such that $(u, v, a) \in B^1$.

lem:Zfacts **Lemma A.3.** *Assume that $n \geq 4$. One can define two families of elements $Z^\tau(u, v, a)$, $\tau = 1, 2$ of the group $\mathrm{St}(n, R)$ parametrized by $(u, v, a) \in B^\tau$ satisfying the following properties:*

(1) $\phi(Z^\tau(u, v, a)) = e + uav^t \in \mathrm{E}(n, R)$, $(u, v, a) \in B^\tau$;
(2) $Z^1(u, v + w, a) = Z^1(u, v, a) \cdot Z^1(u, w, a)$;
(3) $Z^2(v + w, u, a) = Z^2(v, u, a) \cdot Z^2(w, u, a)$;
(4) for $\tau = 1, 2$ and $b \in R$ if $(u, vb, a), (ub, v, a) \in B^\tau$ then one has

$$Z^\tau(u, vb, a) = Z^\tau(u, v, ab) = Z^\tau(ub, v, a);$$

(5) ${}^g Z^\tau(u, v, a) = Z^\tau(\phi(g) \cdot u, \phi(g)^* \cdot v, a)$, $\tau = 1, 2$, $g \in \mathrm{St}^{\mathrm{G}}(n, R)$.

Proof. One constructs the elements $Z^1(u, v, a)$ in exactly the same way as Tulenbaev constructs his elements $X_{u,v}(a)$ (see definitions preceding [11, Lemma 1.2]). Indeed, set

$$Z^1(v, w, a) = \prod_{k=1}^{p} X(v, a \cdot w^k), \quad Z^2(w, v, a) = \prod_{k=1}^{p} X(a \cdot w^k, v). \tag{A.1}$$

where $X(u, v)$ denotes the elements defined by Tulenbaev before [11, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case $n = 4$) can be proved by the same token as in [11, Lemma 1.3]. □

For the rest of this section a denotes a nonnilpotent element of R and $\lambda_a \colon R \to R_a$ is the morphism of principal localization at a.

lem:rk3rels **Lemma A.4.** *For any $g \in \mathrm{E}(n, R_a)$ there exist $u, v \in R^n$ and sufficiently large natural numbers k, m such that the following facts hold:*

(1) $\lambda_a(u) = g \cdot a^k e_1$, $\lambda_a(v) = g^* \cdot a^k e_2$ and $u^t \cdot v = 0$;
(2) $(u, v, a^m) \in B^1 \cap B^2$;
(3) for $b \in R$ divisible by some sufficiently large power of a one has

$$Z^1(u, b \cdot v, a^m) = Z^2(b \cdot u, v, a^m).$$

Proof. It is straightforward to choose u and v satisfying the first requirement of the lemma. We can even choose u, v in such a way that $u \in D(v)$ and $v \in D(u)$. Indeed, notice that $I(u) = a^{k_1}$, $I(v) = a^{k_2}$ for some natural k_1, k_2 hence for $u' = a^{k_2} \cdot u$ and $v' = a^{k_1} \cdot v$ one has

$$u' \in I(v) \cdot O(v) \subseteq D(v) \subseteq D(v'), \quad v' \in I(u) \cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns $x, y \in R^n$ and a large natural p in such a way that vectors u, v, x, y additionally satisfy the following properties

$$\lambda_a(x) = g^* \cdot a^k e_3,\ \lambda_a(y) = g \cdot a^k e_3,\ y^t \cdot x = a^p \in R,$$

$$u^t \cdot x = 0,\ u^t \cdot v = 0,\ y^t \cdot v = 0,$$

$$(u, x, a^m) \in B^1,\ (y, v, a^m) \in B^2.$$

Now direct computation using Lemma A.3 shows that

$$Z^2(a^{m+p}b\cdot u, v, a^m) = Z^2(b\cdot(e+a^m\cdot ux^t)y, (e-a^m\cdot xu^t)v, a^m)\cdot Z^2(-by, v, a^m) =$$
$$= [Z^1(u, x, a^m), Z^2(b\cdot y, v, a^m)] =$$
$$= Z^1(u, x, a^m)\cdot Z^1((e+a^mb\cdot yv^t)u, -(e-a^mb\cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b\cdot v, a^m), \quad \square$$

hence the third assertion of the lemma follows.

cor:tulmap **Corollary A.5.** *[Tulenbaev's lemma] For $n \geq 4$ there is a map T_n so that the following diagram commutes.*

Proof. Follows from Lemma A.4 by the same token as in [11, Lemma 2.3]. □

References

Abe [1] E. Abe. "Whitehead groups of Chevalley groups over polynomial rings". *Comm. Algebra* **11**, no. 12, (1983), pp. 1271–1307.

AHW15 [2] A. Asok, M. Hoyois, and M. Wendt. "Affine representability results in $\mathbb{A}^1$-homotopy theory II: principal bundles and homogeneous spaces". 2015. arXiv: 1507.08020 [math.AG].

Deo [3] V. V. Deodhar. "On central extensions of rational points of algebraic groups". *Amer. J. Math.* (1978), pp. 303–386.

HOM [4] A. J. Hahn and T. O'Meara. *The classical groups and K-theory.* Springer, 1989.

J [5] J. F. Jardine. "On the homotopy groups of algebraic groups". *J. Algebra* **81**, no. 1, (1983), pp. 180–201.

Ka [6] W. van der Kallen. "Another presentation for Steinberg groups". *Indag. Math.* **80**, no. 4, (1977), pp. 304–312.

Re75 [7] U. Rehmann. "Präsentationen von Chevalleygruppen über $k[t]$". 1975. URL: http://www.math.uni-bielefeld.de/~rehmann/Publ/praesentation.pdf.

S15 [8] S. Sinchuk. "On centrality of K_2 for Chevalley groups of type E_ℓ". *J. Pure Appl. Algebra* **220**, no. 2, (2016), pp. 857–875.

St-poly [9] A. Stavrova. "Homotopy invariance of non-stable K_1-functors". *J. K-theory* **13**, no. 2, (2014), pp. 199–248.

St-lect [10] R. Steinberg, J. Faulkner, and R. Wilson. *Lectures on Chevalley groups.* Yale University New Haven, 1967.

Tu [11] M. Tulenbaev. "The Steinberg group of a polynomial ring". *Math. USSR Sb.* **45**, (1983), pp. 131–144.

VW [12] K. Völkel and M. Wendt. "On $\mathbb{A}^1$-fundamental groups of isotropic reductive groups". 2012. arXiv: 1207.2364 [math.KT].

Kbook [13] C. A. Weibel. *The K-book: An introduction to algebraic K-theory.* Vol. 145. 2013.

W-k[t] [14] M. Wendt. "On homology of linear groups over $k[t]$". *Math. Res. Lett.* **21**, no. 6, (2014).

Department of Mathematics and Mechanics, St. Petersburg State University, St. Petersburg, Russia
E-mail address: sinchukss@gmail.com

Department of Mathematics and Mechanics, St. Petersburg State University, St. Petersburg, Russia
E-mail address: anastasia.stavrova@gmail.com

Mathematisches Institut der Universität München, Theresienstr. 39, D-80333 München
E-mail address: avlavrenov@gmail.com