

# $\mathbb{A}^1$ -INVARIANCE FOR UNSTABLE $K_2$

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## 1. AIM OF THE PAPER

The aim of this text is to prove that the non-stable  $K_2$ -functors  $K_2^G$ , where  $G$  is a simply connected Chevalley group of suitable type, satisfy  $\mathbb{A}^1$ -invariance on regular rings  $R$  containing a field  $k$ , that is,

$$(1.1) \quad K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where  $KV_2^G(R)$  is the Karoubi–Villamayor  $K$ -functor associated to  $G$ . This functor originates from [5]. The above equality by e.g. [2, Corollary 4.3.3] implies that  $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$ , i.e. we obtain an explicit presentation for the  $\mathbb{A}^1$ -fundamental group of  $G$  in the sense of Morel–Voevodsky.

”Suitable type” here means that we consider only the cases where we know the centrality of  $K_2$ , or at least the Quillen–Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that  $K_2^G(k[x_1, \dots, x_n]) = K_2^G(k)$ . (Then (1.1) follows by standard geometric methods.) There are two models: Tutenbaev’s proof for the  $SL_n$  case [14] and Stavrova’s proof for  $K_1^G$  [10]. Tutenbaev [14] uses stabilization of the  $K_2$ -functor, and the good properties of the limit=algebraic  $K$ -theory. In [10] stabilization is not used. However, the key steps of both proofs are the same: the case of  $R = k$  (hidden somewhere around [14, p. 140], or, respectively, [10, Theorem 3.1]); Quillen–Suslin lgp;  $\mathbb{P}^1$ -gluing (see [14, Theorem 5.1] or [10, Theorem 1.1]).

### 1.1. Plan of the proof.

(1) (**done**, see ??). Let  $R$  be a local ring. Show that

$$\mathrm{St}^G(R[t^{\pm 1}]) = i_+(\mathrm{St}^G(R[t]))i_-(\mathrm{St}^G(R[t^{-1}]))i_+(\mathrm{St}^G(R[t]))$$

Here  $i_{\pm}$  denote the natural homomorphisms into  $\mathrm{St}^G(R[t^{\pm 1}])$ .

(2) (**done**, see Theorem 3). Show that  $K_2^G(k[t]) = K_2^G(k)$ .

(3) (**done**). Consequently,  $\mathrm{St}^G(k[t]) \rightarrow \mathrm{St}^G(k[t^{\pm 1}])$  is injective and  $\mathrm{St}^G(k[t]) \cap \mathrm{St}^G(k[t^{-1}]) = \mathrm{St}^G(k)$  inside  $\mathrm{St}^G(k[t^{\pm 1}])$ . Also,  $K_2^G(k) = K_2^G(k[t^{\pm 1}])$ .

The first claim uses (2). The second claim follows from (2) and (1).

(4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen–Suslin lgp for  $K_2^G$ .

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plan:Zglu

- (5) (Zariski gluing) for any commutative ring  $A$  and any non-nilpotent  $f, g \in A$  such that  $A = fA + gA$ , the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for  $A = R[t]$  and  $f, g$  non-constant polynomials.)

plan:S-lemma

- (6) (S-lemma) Let  $A$  be a commutative ring,  $S$  a multiplicative subset of  $A$ . If

$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$

for some  $n \geq 1$ , then  $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$  as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu

- (7) (Nisnevich gluing) Assume that  $B$  is a subring of a commutative ring  $A$ , and let  $h \in B$  be a non-nilpotent element. Denote by  $F_h : A \rightarrow A_h$  the localization homomorphism.
- (i) If  $Ah + B = A$ , i.e. the natural map  $B \rightarrow A/Ah$  is surjective, then for any  $x \in \text{St}^G(A_h)$  there exist  $y \in \text{St}^G(A)$  and  $z \in \text{St}^G(B_h)$  such that  $x = F_h(y)z$ .
  - (ii) If moreover  $Ah \cap B = Bh$ , i.e.  $B/Bh \rightarrow A/Ah$  is an isomorphism, and  $h$  is not a zero divisor in  $A$ , then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1, g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [10, Lemma 3.4].

plan:P1

- (8) ( $\mathbb{P}^1$ -gluing) Let  $A$  be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

- (a) Prove Proposition 3.13. This is hard. Have no idea how this can be proved at the moment.
- (b) Prove  $\mathbb{P}^1$ -gluing using all the above facts.

plan:[]f

- (9) (Main corollary of  $\mathbb{P}^1$ -gluing) Let  $A$  be any commutative ring, and let  $f \in A[t]$  be a monic polynomial. Show that  $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$  is injective.

The proof uses (5) and (8).

plan:k(t)

- (10) Prove that  $K_2^G(k(t)) = K_2^G(k)$ . This may be a bit tricky; I will think if we can get rid of it. I don't think Tutenbaev uses it. Unsure whether this is true (?). This is true for  $SK_1$  but in view of Milnor's theorem can not hold for  $K_1$  and  $K_2$ .

plan:k[tn]

- (11) Prove that

$$K_2^G(k[t_1, \dots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of  $K_1^G$ , this uses (9), (2), and (10). There may be other ways.

plan:final

- (12) Final result: let  $R$  be a regular ring containing a field  $k$ . Then  $K_2^G(R[t]) = K_2^G(R)$ . This uses (4), (11) and (7).

## 2. STEINBERG GROUPS: PRELIMINARIES

**2.1. Definition and basic properties.** In what follows  $\Phi$  denotes a reduced irreducible root system and  $\Pi \subseteq \Phi$  denotes its basis (i.e. the set of simple roots). Denote by  $\tilde{\alpha}$ ,  $\Phi^+$  and  $\Phi^-$ , respectively, the maximal root of  $\Phi$  and the subsets of positive and negative roots of  $\Phi$ . The Dynkin diagram and the extended Dynkin diagram of  $\Phi$  corresponding to  $\Pi$  will be denoted by  $D(\Phi)$ ,  $\tilde{D}(\Phi)$ , respectively.

A proper closed root subset  $S \subseteq \Phi$  is called *parabolic* (resp. *reductive*, resp. *special*) if  $\Phi = S \cup -S$  (resp.  $S = -S$ , resp.  $S \cap -S = \emptyset$ ). Any parabolic subset  $S \subseteq \Phi$  can be decomposed into the disjoint union of its reductive and special part, i.e.  $S = \Sigma_S \sqcup \Delta_S$ , where  $\Sigma_S \cap (-\Sigma_S) = \emptyset$ ,  $\Delta_S = -\Delta_S$ .

Denote by  $m_\beta(\alpha)$  the coefficient of  $\beta$  in the expansion of  $\alpha$  in  $\Pi$ , i.e.  $\alpha = \sum_{\beta \in \Pi} m_\beta(\alpha)\beta$ . For  $\beta \in \Pi$  denote by  $\Delta_\beta$  the subsystem of  $\Phi$  spanned by all simple roots except  $\beta$  and by  $\Sigma_\beta$  the set consisting of roots  $\alpha \in \Phi$  such that  $m_\beta(\alpha) > 0$ .

We denote by  $(\alpha, \beta)$  the scalar product of roots and by  $\langle \beta, \alpha \rangle$  the integer  $2(\beta, \alpha)/(\alpha, \alpha)$ . The Weyl group  $W(\Phi)$  is a subgroup of isometries of  $\Phi$  generated by all reflections  $\sigma_\alpha$ , where  $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \cdot \alpha$ . For a subset of roots  $S \subseteq \Phi$  we denote by  $\langle S \rangle$  the root subsystem spanned by  $S$ , i.e. the minimal subset of  $\Phi$  containing  $S$  and invariant under the action of reflections  $\sigma_\alpha$ ,  $\alpha \in S$ .

Let  $G = G(\Phi, -)$  be a (split) simply connected simple Chevalley–Demazure group scheme over  $R$  with a root system  $\Phi$  of rank  $\geq 2$ . Recall that the *Steinberg group*  $\text{St}^G(R)$  (also denoted  $\text{St}(\Phi, R)$ ) is defined by means of generators  $\mathcal{X}_{\Phi, R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$  and the set of relations  $\mathcal{R}_{\Phi, R}$  defined as follows:

$$\boxed{\text{rel:add}} \quad (2.1) \quad x_\alpha(s)x_\alpha(t) = x_\alpha(s+t), \quad \alpha \in \Phi, \quad s, t \in R;$$

$$\boxed{\text{rel:CCF}} \quad (2.2) \quad [x_\alpha(s), x_\beta(t)] = \prod_{i,j \in \mathbb{N}} x_{i\alpha+j\beta} (N_{\alpha\beta ij} s^i t^j), \quad \alpha, \beta \in \Phi, \quad \alpha \neq \pm\beta, \quad s, t \in R.$$

The indices  $i, j$  appearing in the right-hand side of the above relation range over all positive natural numbers such that  $i\alpha + j\beta \in \Phi$ . The structure constants  $N_{\alpha\beta ij} = \pm 1, 2, 3$  appearing in (2.2) depend only on  $\Phi$  and can be computed precisely.

Recall that for  $\alpha \in \Phi$ ,  $\varepsilon \in R^*$  the semisimple root elements  $h_\alpha(\varepsilon)$  are defined as  $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$ . Denote by  $W(\Phi, R)$  the subgroup of  $\text{St}(\Phi, R)$  generated by all elements  $w_\alpha(\varepsilon)$ ,  $\varepsilon \in R^*$ ,  $\alpha \in \Phi$ , and by  $H(\Phi, R)$  the subgroup generated by all elements  $h_\alpha(\varepsilon)$ ,  $\varepsilon \in R^*$ ,  $\alpha \in \Phi$ .

Following [12], we set

$$\{u, v\}_\alpha = h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}, \quad u, v \in R^*, \quad \alpha \in \Phi,$$

and call these elements *symbols in*  $\text{St}(\Phi, R)$ . For any ideal  $I$  of  $R$ , we set

$$\text{Sym}(\Phi, R, I) = \langle \{u, v\}_\alpha, \quad u \in R^*, \quad v \in (1+I)^*, \quad \alpha \in \Phi \rangle \leq \text{St}(\Phi, R).$$

The group  $\text{Sym}(\Phi, R, R)$  is denoted by  $\text{Sym}(\Phi, R)$ . Clearly, one has  $\{u, v\}_\alpha \in K_2(\Phi, R)$  for any  $u, v \in R^*$ ,  $\alpha \in \Phi$ . By [12, Prop. 1.3 (c)] the group  $\text{Sym}(\Phi, R, I)$  is generated by all symbols  $\{u, v\}_\alpha$ ,  $u \in R^*$ ,  $v \in (1+I)^*$ , for any fixed long root  $\alpha \in \Phi$ .

## 2.2. Non-standard generation of Steinberg groups.

lem:parab-gen

**Lemma 2.1.** *Let  $R$  be any commutative ring. Let  $\Phi$  be an irreducible root system of rank  $\geq 2$ , let  $\Pi$  be a system of simple roots in  $\Phi$ , and let  $J \subseteq \Pi$  be such that  $|J| \geq 2$ . Set*

$$\alpha_J = \sum_{\beta \in J} m_\beta(\alpha) \beta \text{ for any } \alpha \in \Phi,$$

and

$$\Sigma_J = \{\alpha \in \Phi \mid m_\beta(\alpha) > 0 \text{ for at least one } \beta \in J\}.$$

Let  $H$  be the group defined by the generators  $x_\alpha(u)$ ,  $\alpha \in \Sigma_J \cup (-\Sigma_J)$ ,  $u \in R$ , and the relations (2.1) and (2.2) ranging only over  $\alpha \in \Sigma_J \cup (-\Sigma_J)$ , and  $\beta \in \Sigma_J \cup (-\Sigma_J)$  such that  $m\alpha_J \neq -k\beta_J$  for all  $m, k \in \mathbb{N}$ . Then the natural homomorphism  $H \rightarrow \text{St}(\Phi, R)$  is surjective and has central kernel. In particular, if  $\text{St}(\Phi, R)$  is centrally closed, then  $H \cong \text{St}(\Phi, R)$ .

*Proof.* The group scheme  $G = G(\Phi, -)$  over  $R$  contains two opposite parabolic  $R$ -subgroups  $P^\pm$  such that  $\pm\Sigma_J$  are the sets of roots corresponding to the unipotent radicals of  $P^\pm$ . One can show that  $H = \text{St}_{P^+}(R)$  and  $\text{St}(\Phi, R) = \text{St}_B(R)$  in the sense of [11]. By [11, Lemma 8] the natural homomorphism  $H \rightarrow \text{St}(\Phi, R)$  is surjective. By [11, Lemma 14] its kernel is central. □

**2.3. Relative Steinberg groups.** Denote by  $D(R, I)$  the double of the ring  $R$  relative to an ideal  $I$ , i.e. the fibered product of rings  $R \times_{R/I} R$  with the natural projections  $p_1, p_2: D(R, I) \rightarrow R$  defined by  $p_i(\xi_1, \xi_2) = \xi_i$ ,  $i = 1, 2$ . Denote by  $G_i$  the kernel of the map  $p_i^*: \text{St}^G(D(R, I)) \rightarrow \text{St}^G(R)$ . We define the relative Steinberg group  $\text{St}^G(R, I)$  as  $G_1/C$ , where  $C = [G_1, G_2]$ . Clearly, there is an exact sequence.

$$(2.3) \quad 1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \text{St}^G(R, I) \xrightarrow{\overline{p_2^*}} \text{St}^G(R) \xrightarrow{\pi^*} \text{St}^G(R/I) \longrightarrow 1$$

**Lemma 2.2.** *Assume that  $R$  and  $I$  are such that the canonical projection  $R \rightarrow R/I$  splits. Then the following facts are true.*

- (i) *The map  $\text{St}^G(R, I) \rightarrow \text{St}^G(R)$  is an injection.*
- (ii) *The group  $\text{St}^G(R, I)$  is isomorphic to  $\text{St}^G(R/I) \ltimes \text{St}^G(R, I)$ .*

*Proof.* For the proof of the first assertion see [9, Lemma 8]. Since the group  $(G_1 \cap G_2)/C$  vanishes, the sequence (2.3) turns into a split short exact sequence which implies the second assertion. □

lem:Zgen

**Lemma 2.3.** *Let  $\Sigma$  be the special subset of some parabolic subset of roots  $S \subseteq \Phi$ . Then the relative Steinberg group  $\text{St}^G(R, I)$  admits the following generating set:*

$$\mathcal{Z}(\Sigma, R, I) = \{x_\alpha(0, s) \cdot C \mid s \in I, \alpha \in \Phi\} \cup \{z_\alpha(s, \xi) \mid s \in I, \xi \in R, \alpha \in \Sigma\},$$

where  $z_\alpha(s, \xi)$  denotes the element  $x_\alpha(0, s)^{x_{-\alpha}(\xi, \xi)} \cdot C$ .

*Proof.* See [9, Lemma 5]. □

**2.4. Tulenbaev's lifting property and its corollaries.** Throughout this section  $I \trianglelefteq A$  is an ideal of arbitrary commutative ring  $A$ . For a nonnilpotent element  $a \in A$  denote by  $\lambda_a: A \rightarrow A_a$  the morphism of principal localization at  $a$ . Consider the following commutative square.

$$(2.4) \quad \begin{array}{ccc} A & \xrightarrow{\lambda_a} & A_a \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\overline{\lambda_a}} & A_a/I_a \end{array}$$

Notice that (2.4) is a pull-back square if and only if  $\lambda_a$  induces an isomorphism of  $I$  and  $I_a$ . Such squares are usually called *Milnor squares* in the literature, see [16, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [14, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

**def:tlp**

**Definition 2.4.** We say that the Steinberg group functor  $\text{St}^G$  satisfies *Tulenbaev's lifting property* if for every pull-back square (2.4) the following lifting problem has a solution.

$$\begin{array}{ccc} \text{St}^G(A, I) & \xrightarrow{\mu} & \text{St}^G(A) \\ \downarrow & \nearrow & \downarrow \lambda_a^* \\ \text{St}^G(A_a, I) & \xrightarrow{\mu} & \text{St}^G(A_a) \end{array}$$

**Theorem 1.** Assume that  $G$  satisfies Tulenbaev property (2.4) then the following facts are true for arbitrary commutative ring  $A$ :

**thm:dp**

(i) A dilation principle holds for  $\text{St}^G(-)$ , i. e. if  $g \in \text{St}^G(A[t], tA[t])$  is such that equality  $\lambda_a^*(h) = 1$  holds in  $\text{St}^G(\Phi, R_a[t])$  then for sufficiently large  $n$  one has

$$ev_{\left[ \begin{smallmatrix} R[t] \rightarrow R[t] \\ t \mapsto a^n \cdot t \end{smallmatrix} \right]}^*(h) = 1.$$

**thm:lg-k2**

(ii) A local-global principle holds for  $\text{St}^G(-)$ , i. e. an element  $g \in \text{St}^G(A[t], tA[t])$  is trivial if and only if its image in  $\text{St}^G(A_m[t], tA_m[t])$  is trivial for all maximal ideals  $m \trianglelefteq A$ .

**thm:centr**

(iii)  $K_2^G(A)$  is contained in the centre of  $\text{St}^G(A)$ .

*Proof.* Follows by the same argument as [14, Theorem 2.1] or [9, Theorem 2] □

**sec:sigma**

**2.5. The automorphisms  $\sigma_i$ .** Our notation and conventions follows [15, § 4]. Let  $\Phi$  be an irreducible root system with some fixed basis of simple roots  $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ . We denote by  $\Phi^\vee$  the dual root system of  $\Phi$  consisting of vectors  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ ,  $\alpha \in \Phi$ . As usual,  $P(\Phi^\vee)$  denotes the lattice spanned by the fundamental weights  $\varpi_i$ . Recall that  $\varpi_i$  are uniquely determined by relations  $\langle \varpi_i, \alpha_j^\vee \rangle = (\varpi_i, \alpha_j) = \delta_{ij}$ .

Notice that for  $\varpi \in P(\Phi^\vee)$  and  $\beta \in \mathbb{Z}\Phi$  one has  $(\varpi, \beta) \in \mathbb{Z}$ . Consequently, for  $\varepsilon \in R^*$  and  $\varpi \in P(\Phi^\vee)$  the identity  $\chi_{\varpi, \varepsilon}(\beta) = \varepsilon^{(\varpi, \beta)}$  gives a well-defined character  $\chi_{\varpi, \varepsilon} \in \text{Hom}(\mathbb{Z}\Phi, R^*)$ .

Consider the action of  $H = \text{Hom}(\mathbb{Z}\Phi, R^*)$  on the set of generators  $\mathcal{X}_{\Phi, R}$  of the Steinberg group  $\text{St}^G(R)$  defined by

$$(2.5) \quad \chi \cdot x_\alpha(\xi) = x_\alpha(\chi(\alpha) \cdot \xi), \quad \chi \in H, \quad \alpha \in \Phi, \quad \xi \in R.$$

Since  $\chi$  is a character, the above action preserves the set of Steinberg relations  $\mathcal{R}_{\Phi, R}$  and, thus, gives a well-defined action of  $H$  on  $\mathrm{St}^G(R)$ .

**Example 2.5.** The principal example which motivates the above construction is as follows. Let  $A$  be a ring, take  $R = A[t, t^{-1}]$  to be the ring of Laurent polynomials over  $A$  and let  $\alpha_i \in \Pi$  be some simple root. Since  $t \in R^*$  we can consider the automorphisms  $\sigma_i^+$  and  $\sigma_i^-$  of  $\mathrm{St}(\Phi, R)$  given by  $\sigma_i^+ = \chi_{\varpi_i, t}$ ,  $\sigma_i^- = \chi_{\varpi_i, t^{-1}}$ . It is easy to see that

`eq:sigma_act}`

$$(2.6) \quad \sigma_i^{\pm}(x_{\alpha}(\xi)) = x_{\alpha}(t^{\pm m_i(\alpha)} \cdot \xi),$$

where  $m_k(\alpha)$  denotes the coefficient in the expansion of  $\alpha$  in  $\Pi$ , i. e.  $\alpha = \sum m_k(\alpha)\alpha_k$ .

One of the key steps of our proof of Suslin lemma for  $K_2$  is to define an analogue of  $\sigma_i$  for the group  $\mathrm{St}^G(A[t])$ . Of course, we cannot expect such map to be automorphism or even be defined on the whole group  $\mathrm{St}^G(A[t])$ . However, it turns out that for certain  $i$  is still possible to define certain subgroups of  $\mathrm{St}^G(A[t])$  and the maps modeling  $\sigma_i$  between them.

### 3. $\mathbb{P}^1$ -GLUING

Throughout this section  $G = G(\Phi, -)$  denotes a simply connected Chevalley–Demazure group scheme of type  $\Phi$ .

`def:p1g`

**Definition 3.1.** Let  $F$  be a group-valued functor from **CRings** to **Groups** and let  $A$  be a commutative ring. Consider the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{i_+} & A[t] \\ i_- \downarrow & & \downarrow j_+ \\ A[t^{-1}] & \xrightarrow{j_-} & A[t, t^{-1}] \end{array}$$

We say that  $F$  satisfies the  $\mathbb{P}^1$ -glueing property for  $A$  if the following sequence of *pointed sets* is exact in the middle term:

$$F(A) \xrightarrow{\Delta_A^F} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\pm_A^F} F(A[t, t^{-1}]).$$

Here  $\Delta_A^F$  denotes the (split injective) diagonal map and, by definition,  $\pm_A^F$  maps  $(g^+, g^-)$  to  $F(j_+)(g^+) \cdot F(j_-)(g^-)^{-1}$ . Notice that a priori  $\pm_A^F$  is only a morphism of pointed sets but if  $F$  takes values in abelian groups then  $\pm_A^F$  is also a morphism of groups.

An equivalent way to formulate  $\mathbb{P}^1$ -glueing property is as follows:  $F(j_+)$  and  $F(j_-)$  are injective and the intersection of their images coincides with the image of  $F(j_+i_+) = F(j_-i_-)$ .

The main result of this section is the following theorem which generalizes [14, Theorem 5.1] to Chevalley groups. Notice that a  $K_1$ -analogue of the result below has been established in a much greater generality by the second-named author (see [10, Theorem 1.1]).

`thm:p1`

**Theorem 2.** Assume that  $G$  satisfies Tulenbaev lifting property 2.4. Then the Steinberg group functor  $\mathrm{St}^G(-)$  satisfies  $\mathbb{P}^1$ -glueing property for an arbitrary commutative ring  $A$ .

*Proof.* Let  $(g^+, g^-)$  be an element of  $\mathrm{St}^G(A[t]) \times \mathrm{St}^G(A[t^{-1}])$  such that the equality  $g^+ = g^-$  holds in  $\mathrm{St}^G(A[t, t^{-1}])$ .

Let  $m$  be a maximal ideal of  $A$ . By Proposition 3.14 below the functor  $\mathrm{St}^G(-)$  satisfies  $\mathbb{P}^1$ -glueing property for the local ring  $A_m$  hence  $(\lambda_m^*(g_+), \lambda_m^*(g_-)) = \Delta_{A_m}(\lambda_m^*(g^+)(0))$  and in the groups  $\mathrm{St}^G(A_m[t])$  and  $\mathrm{St}^G(A_m[t^{-1}])$  we have the equalities:

$$\lambda_m^*(g^+ \cdot g^+(0)^{-1}) = \lambda_m^*(g^+) \cdot \lambda_m^*(g^+)(0)^{-1} = 1; \quad \lambda_m^*(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for  $\mathrm{St}^G(-)$  (see Theorem 1.(ii)) these equalities hold globally and  $(g^+, g^-) = \Delta_A(g^+(0))$ , as claimed.  $\square$

rem:stk2

*Remark 3.2.* It is clear that if the functor  $\mathrm{St}^G$  satisfies  $\mathbb{P}^1$ -glueing property for  $A$  then so does the functor  $K_2^G$ . The converse statement also holds, indeed, if  $(g^+, g^-) \in \mathrm{Ker}(\pm_A^{\mathrm{St}})$  then inside  $E^G(A[t, t^{-1}])$  we have the equality:

$$\varphi(\mathrm{St}^G(j_+)(g^+)) = \varphi(\mathrm{St}^G(j_-)(g^-)) \in E^G(A[t]) \cap E^G(A[t^{-1}]) = E^G(A).$$

Consequently, we can find  $g_0 \in \mathrm{St}^G(A)$  so that  $(g^+g_0^{-1}, g^-g_0^{-1}) \in \mathrm{Ker}(\pm_A^{K_2})$  and it remains to apply the  $\mathbb{P}^1$ -glueing property for  $K_2^G$ .

**Corollary 3.3.** *Let  $A$  be any commutative ring and  $f \in A[t]$  be a monic polynomial. Then the map  $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$  is injective.*

*Proof.*  $\square$

**3.1. The field case.** Throughout this section  $k$  denotes an arbitrary field.

thm:k[t]

**Theorem 3.** *Assume that  $G = G(\Phi, -)$  and  $\Phi$  is any irreducible root system of rank  $\geq 2$ .*

satz1

- (i) *The subgroup  $K_2(\Phi, k[t]) \leq \mathrm{St}^G(\Phi, k[t])$  is generated by symbols  $\{u, v\}_\alpha$ ,  $u, v \in k^*$ ,  $\alpha \in \Phi$ .*
- (ii) *As a consequence, the canonical injection  $K_2(\Phi, k) \hookrightarrow K_2(\Phi, k[t])$  is an isomorphism, and  $K_2(k[t]) = K_2(k)$  is central in  $\mathrm{St}(\Phi, k[t])$ .*

*Proof.* See [8, Satz 1] and the corollary after it.  $\square$

cor:k[t]inj

**Corollary 3.4.** *Let  $G$  be as in Theorem 3. Then the functors  $\mathrm{St}^G$ ,  $K_2^G$  satisfy  $\mathbb{P}^1$ -glueing property for  $k$ .*

*Proof.* By Remark 3.2 it suffices to prove the assertion only for the functor  $K_2^G$ . By the previous theorem  $K_2^G(i_+)$  and  $K_2^G(i_-)$  are isomorphisms hence the morphisms  $K_2^G(j_+)$  and  $K_2^G(j_-)$  are split injective and  $\mathrm{Im}(j_+i_+) = \mathrm{Im}(j_-i_-) = \mathrm{Im}(j_+) = \mathrm{Im}(j_-)$ .  $\square$

thm:k[t+-1]

**Theorem 4.** *Let  $\Phi$  be an irreducible root system of rank  $\geq 2$  and let  $\alpha$  be a long root of  $\Phi$ . Denote by  $H$  the subgroup of  $K_2(\Phi, k[t^{\pm 1}])$  generated by symbols  $\{t, u\}_\alpha$ ,  $u \in k^*$ .*

- (i) *For  $\Phi \neq G_2$  one has*

$$K_2(\Phi, k[t^{\pm 1}]) = K_2(\Phi, k) \oplus H.$$

*In particular,  $K_2(\Phi, k[t^{\pm 1}])$  is central in  $\mathrm{St}(\Phi, k[t^{\pm 1}])$ . Here  $K_2(\Phi, k)$  is considered as a subgroup of  $K_2(\Phi, k[t^{\pm 1}])$  via the natural injection.*

- (ii) *The map  $\{t, -\}_\alpha: k^* \rightarrow H$  is injective.*

*Proof.* By [4, Korollar 4] the group  $K_2(\Phi, k[t^{\pm 1}])$  is generated by  $K_2(\Phi, k)$  and  $H$ . By [12, Prop. 1.1 (S1)] one has  $\{1, u\}_\alpha = 1$  for any  $\alpha \in \Phi$ . Hence  $H$  is in the kernel of the natural projection  $K_2(\Phi, k[t^{\pm 1}]) \rightarrow K_2(\Phi, k)$  sending  $t$  to 1, and  $H \cap K_2(\Phi, k) = 1$ . By [12, Prop. 1.3 (a)] symbols are central in  $\mathrm{St}(\Phi, k[t^{\pm 1}])$ , hence  $K_2(\Phi, k[t^{\pm 1}]) = K_2(\Phi, k) \oplus H$ .



Now consider the following commutative diagram.

$$\begin{array}{ccc} k^* \cong K_1(k) & \xrightarrow{\{t, -\}_\alpha} & K_2(\Phi, k[t^{\pm 1}]) \longrightarrow K_2(\Phi, k(t)) \\ & \searrow_{\{t, -\}} & \downarrow \\ & & K_2^M(k(t)) \end{array}$$

By Matsumoto's theorem  $K_2(\Phi, k(t))$  is isomorphic to  $K_2^M(k(t))$  if  $\Phi$  is nonsymplectic, otherwise  $K_2(\Phi, k(t)) \cong K_2^{MW}(k(t))$ , see [6, Corollaire 5.11]. Thus, the vertical arrow is either an isomorphism or the canonical projection  $K_2^{MW}(k(t)) \rightarrow K_2^M(k(t))$ , see [13, § 6]. On the other hand, the diagonal arrow is injective by Milnor's theorem, see [7, § 2]. Thus the left top horizontal arrow is injective as well.  $\square$

*Remark 3.5.* Notice that in the symplectic case the composite map  $\{t, -\}_\alpha: k^* \rightarrow K_2^{MW}(k(t))$  is *not* a morphism of groups because the Steinberg symbol  $\{-, -\}_\alpha$  corresponding to the long root  $\alpha$  is not bilinear. If we defined this map using a short root  $\alpha$ , this map would be a morphism of groups, however, the diagonal map would equal  $\{t^2, -\}$  and would not be injective.

**3.2. Tulenbaev's section 3.** For the rest of this section  $A$  denotes an arbitrary commutative local ring with the maximal ideal  $m$  and the residue field  $k$ . We denote by  $\pi$  the canonical projection  $A \rightarrow k$ . Throughout this section we will employ the following notation:

- $R$  denotes the Laurent polynomial ring  $A[t, t^{-1}]$ ;
- $B$  denotes the subring  $A[t] + m[t^{-1}]$  of  $R$  consisting of Laurent polynomials  $f(t, t^{-1})$  whose coefficients of terms of negative degree belong to  $m$ ;
- $I$  denotes the ideal  $m[t, t^{-1}]$  of  $R$  (which can be also considered as an ideal of  $B$ ).

Note that since  $A$  is local, then  $K_2(\Phi, A) = \text{Sym}(\Phi, A)$  by [12, Theorem 2.13]. In particular,  $K_2(\Phi, A)$  is central in  $\text{St}(\Phi, A)$ , and  $W(\Phi, A)/H(\Phi, A)$  is naturally isomorphic to the Weyl group  $W(\Phi)$ .

Our first result is analogous to [14, Lemma 3.1(e)] (cf. also with [3, § 2.3A]).

**lem:bruhat**

**Lemma 3.6.** *Let  $\Phi$  be any irreducible root system. Let  $\Phi^+, \Phi^{+'}$  be two systems of positive roots in  $\Phi$ .*

- (i) *The Steinberg group  $\text{St}(\Phi, A)$  admits the following analogue of the Bruhat decomposition:*

$$(3.1) \quad \text{St}(\Phi, A) = \bigsqcup_{w \in W(\Phi)} \left( U(\Phi^{+'}, A) \cdot wH(\Phi, A) \cdot U(\Phi^+, A) \cdot \ker(\text{St}(\Phi, A) \xrightarrow{\pi^*} \text{St}(\Phi, k)) \right).$$

- (ii) *Assume that  $uw h v l = u' w' h' v' l'$  for some  $u, u' \in U(\Phi^{+'}, A)$ ,  $w, w' \in W(\Phi)$ ,  $h, h' \in H(\Phi, A)$ , and  $l, l' \in \ker(\text{St}(\Phi, A) \xrightarrow{\pi^*} \text{St}(\Phi, k))$ . Then  $w = w'$ ,*

$$h^{-1} h' \in \text{Sym}(\Phi, A) \cdot \ker(H(\Phi, A) \xrightarrow{\pi^*} H(\Phi, k)),$$

*and there exist  $a \in U(\Phi^{+'}, m)$  such that  $w^{-1}(u^{-1} u' a) w \in U(\Phi^+, R)$ , and  $b \in U(\Phi^+, m)$  such that*

$$b = v^{-1}(u^{-1} u')^{wh} v' = l(l')^{-1}.$$

**eq:bruhat**



*Proof.* For any field  $k$ , the group  $G(\Phi, k) = E(\Phi, k)$  admits Bruhat decomposition, hence  $K_2(\Phi, k) \leq H(\Phi, k)$  implies

$$\mathrm{St}(\Phi, k) = \bigsqcup_{w \in W(\Phi)} \mathrm{U}(\Phi^{+'}, k) w H(\Phi, k) \mathrm{U}(\Phi^+, k).$$

Let  $w_0 \in W(\Phi)$  be such that  $v_0(\Phi^{+'}) = \Phi^+$ . Then also

$$\mathrm{St}(\Phi, k) = w_0^{-1} H(\Phi, k) \mathrm{St}(\Phi, k) = \bigsqcup_{w \in W(\Phi)} \mathrm{U}(\Phi^{+'}, k) w H(\Phi, k) \mathrm{U}(\Phi^+, k).$$

As a consequence, the first three factors in the right hand side of the decomposition (3.1) are mapped epimorphically onto  $\mathrm{St}(\Phi, k)$  and the last factor coincides with  $\mathrm{Ker}(\pi^*)$  from which the first assertion of the lemma follows. The second assertion follows from the unicity of the Bruhat decomposition in  $G(\Phi, k)$ .  $\square$

lem:tul3.1zh

**Lemma 3.7.** *Assume that  $A$  is regular and  $\Phi$  has rank at least 2. Then there is an injective map  $\phi: (1+m)^* \rightarrow \mathrm{St}(\Phi, R, I)$ , natural in  $(A, m)$ , such that  $j\phi = \{t, -\}_\alpha$ , where  $j: \mathrm{St}(\Phi, R, I) \rightarrow \mathrm{St}(\Phi, R)$  is the canonical map and  $\alpha$  is a long root of  $\Phi$ . Moreover, if  $\Phi$  is nonsymplectic then  $\phi$  is a homomorphism of groups.*

*Proof.* The map  $\phi$  can be defined as follows:

$$\begin{aligned} \phi(u) &= y_\alpha(t(u-1)) y_{-\alpha}(-t^{-1}(u^{-1}-1))^{x_\alpha(-t)} \\ &\quad \cdot y_\alpha(t(u-1))^{w_\alpha(-t)} y_\alpha(u-1) y_{-\alpha}(-(u^{-1}-1))^{x_\alpha(-1)} y_\alpha(u-1)^{w_\alpha(-1)}. \end{aligned}$$

It is easy to check that  $j\phi = \{t, -\}_\alpha$ . If  $A$  is regular then it is integral and, therefore, can be embedded into its field of fractions  $K$ . Now the injectivity of  $\phi$  follows from Theorem 4 and the commutativity of the following diagram.

$$\begin{array}{ccc} (1+m)^* & \hookrightarrow & K^* \\ \downarrow \phi & & \downarrow \\ \mathrm{St}(\Phi, R, I) & \longrightarrow & \mathrm{St}(\Phi, K[t^{\pm 1}]) \end{array}$$

Here the bottom arrow is the composition of  $j$  and the map of Steinberg groups induced by the embedding  $R \hookrightarrow K[t^{\pm 1}]$ .  $\square$

lem:sigma-X

**Lemma 3.8.** *Let  $\Phi$  be an irreducible root system of rank  $l \geq 3$  and of type  $A_l$  ( $l \geq 3$ ),  $C_l$  ( $l \geq 3$ ),  $D_l$  ( $l \geq 4$ ),  $E_6$  or  $E_7$ . Let  $\alpha_i \in \Pi$ ,  $1 \leq i \leq l$ , be a simple root of  $\Phi$  such that the parabolic subgroup  $P_i$  of  $G(\Phi, -)$  has abelian unipotent radical. Denote by  $\mathrm{St}P_i^-(\Phi, A)$  the subgroup of  $\mathrm{St}(\Phi, A)$  generated by  $x_\alpha(u)$ ,  $\alpha \in \Delta_i \cup (-\Sigma_i)$ ,  $u \in A$ , and by  $H(\Phi, A)$  (?). Then there exists a group homomorphism*

$$\delta_i: \mathrm{St}(\Phi, A[t], tA[t]) \rightarrow \mathrm{St}P_i^-(\Phi, A) \cdot \mathrm{St}(\Phi, A[t], tA[t])$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{St}(\Phi, A[t], tA[t]) & \xrightarrow{j_+} & \mathrm{St}(\Phi, A[t, t^{-1}]) \\ \downarrow \delta_i & & \downarrow \chi_{\varpi_i, t} \\ \mathrm{St}P_i^-(\Phi, A) \cdot \mathrm{St}(\Phi, A[t], tA[t]) & \xrightarrow{j_+} & \mathrm{St}(\Phi, A[t, t^{-1}]) \end{array}$$

*Proof.* The cases  $A_l$  and  $C_l$  are done using another presentation. Other cases are done by amalgamation of  $A_3$ -pieces....  $\square$

n:parab-pairs

**Lemma 3.9.** *Let  $\Phi$  be an irreducible root system of rank  $l \geq 3$  and of type  $A_l$  ( $l \geq 3$ ),  $C_l$  ( $l \geq 3$ ),  $D_l$  ( $l \geq 4$ ),  $E_6$ ,  $E_7$  or  $E_8$ . Let  $\alpha_i \in \Pi$  be the simple root adjacent to  $\alpha_l$  in the Dynkin diagram of  $\Phi$  (note that  $P_l$  has abelian unipotent radical if  $\Phi \neq E_8$ , and extraspecial if  $\Phi = E_8$ ). Set  $\Phi^{+'} = w_{\alpha_l}(\Phi^+)$ . Then  $\Phi^{+'} \setminus \Phi^+ = -\alpha_l$ ,  $\Phi^+ \setminus \Phi^{+'} = \alpha_l$ , and  $\Sigma_i \setminus \Sigma_l \subseteq w_{\alpha_l}(\Sigma_l)$ .*

*Proof.* The first two claims are obvious. The last claim follows from the fact that for any root  $\alpha \in \Sigma_i \setminus \Sigma_l$  one has  $w_{\alpha_l}(\alpha) = \alpha + \alpha_l$ .  $\square$

lem:tul3.3

**Lemma 3.10.** *Let  $\Phi$  be an irreducible root system of rank  $\geq 2$ , let  $\Pi$  be a system of simple roots in  $\Phi$ , and let  $J \subseteq \Pi$  be such that  $|J| \geq 2$ . Let  $S$  be any group, and  $n \in \mathbb{N}$ . For any  $b \in B$  let  $|b| \in \mathbb{Z}$  be such that  $t^{|b|}$  is the smallest power of  $t$  occuring in  $b$ . Assume that there are elements  $s_\alpha(u) \in S$  for all  $\alpha \in \Sigma_J \cup (-\Sigma_J)$  and  $u \in B$  such that  $|u| \geq -n$ , satisfying the relations*

- (1)  $s_\alpha(u)s_\alpha(v) = s_\alpha(u+v)$  for all  $\alpha \in \Sigma_J \cup (-\Sigma_J)$ ,  $u, v \in B$ ,  $|u|, |v| \geq -n$ ;
- (2)  $[s_\alpha(u), s_\beta(v)] = \prod_{i,j \in \mathbb{N}} s_{i\alpha+j\beta}(N_{\alpha\beta ij} u^i v^j)$  for all  $\alpha, \beta \in \Sigma_J \cup (-\Sigma_J)$  such that  $i\alpha_J \neq -j\beta_J$   
for all  $i, j \in \mathbb{N}$ , and all  $u, v \in B$  such that  $i|u| + j|v| \geq -n$  whenever  $N_{\alpha\beta ij} \neq 0$ .

If  $n = 1$ , then the map  $x_\alpha(u) \mapsto s_\alpha(u)$  extends to a group homomorphism  $\text{St}(\Phi, B) \rightarrow S$ .

*Proof.*  $\square$

### 3.3. Tulenbaev's section 4.

thm:tul4.1

**Theorem 5.** *Let  $\Phi$  be an irreducible root system of rank  $l \geq 3$  of type  $A_l$ ,  $C_l$ ,  $D_l$  ( $l \geq 4$ ),  $E_6$  or  $E_7$ . Then  $\ker(\text{St}(\Phi, A[t], m[t]) \rightarrow \text{St}(\Phi, A[t]))$  surjects onto  $\ker(\text{St}(\Phi, A[t^{\pm 1}], m[t^{\pm 1}]) \rightarrow \text{St}(\Phi, A[t^{\pm 1}]))$ .*

*Proof.* Set

$$\tilde{B} = H(\Phi, A[t^{\pm 1}]) \cdot U(\Phi^+, A[t^{\pm 1}]) \leq \text{St}(\Phi, A[t^{\pm 1}]),$$

and

$$\tilde{D} = \phi(\text{Sym}^t(\Phi, A, m)) \leq \text{St}(\Phi, A[t^{\pm 1}], m[t^{\pm 1}])$$

in the notation of Lemma 3.7. Consider the set of equivalence classes

$$V = \text{St}(\Phi, A[t]) \times \tilde{B} \times \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}]) / \sim,$$

where  $(a, b, \beta) \sim (a', b', \beta')$  if and only if there is  $\gamma \in \text{St}(\Phi, A[t], mA[t])$ ,  $p \in U(\Phi^+, A[t])$  and  $\mu \in \tilde{D} \cdot U(\Phi^+, mA[t^{\pm 1}])$  such that

$$\tilde{a} = a\gamma_1^{-1}t^{-1}, \quad \tilde{b} = pt\mu, \quad \tilde{\beta} = \mu^{-1}(\gamma_2)b\beta,$$

where  $\gamma_1$  and  $\gamma_2$  are the images of  $\gamma$  in the respective groups. The elements of  $V$  will be denoted  $[a, b, \beta]$ .

**We will define a map  $\sigma : V \rightarrow V$ .** By Lemma 3.6 and the definition of  $V$  every element of  $V$  can be written in the form  $[aw, b, \beta]$  for some  $a \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+, A)$ ,  $w \in W(\Phi, A) \leq \text{St}(\Phi, A)$ ,  $b \in \tilde{B}$  and  $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$ . We define

$$\sigma|_{\text{St}(\Phi, A[t], tA[t])} = \delta_l : \text{St}(\Phi, A[t], tA[t]) \rightarrow \text{St}(\Phi, A[t], tA[t])\text{St}P_l^-(\Phi, A) \leq [\text{St}(\Phi, A[t]), 1, 1],$$

where  $\delta_l$  the homomorphism constructed in Lemma 3.8. For any  $\alpha \in \Phi$ ,  $u \in A[t^{\pm 1}]$  we have a homomorphism

$$\chi_{\varpi_l, t} : X_\alpha(A[t^{\pm 1}]) \rightarrow X_\alpha(A[t^{\pm 1}]), \quad x_\alpha(u) \mapsto x_\alpha(t^{m_{\alpha_l}(\alpha)}u).$$

In particular, this induces a homomorphism  $\sigma_U = \chi_{\varpi_l, t}|_{U(\Phi^+, A)} : U(\Phi^+, A) \rightarrow U(\Phi^+, A[t])$ . Combining  $\delta_l$  and  $\sigma_U$ , we obtain a homomorphism (check!!!)

$$\sigma : \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+, A) \rightarrow \text{St}(\Phi, A[t]).$$

Now we define

$$\sigma \cdot [aw, b, \beta] = [\sigma(a)w, w^{-1}\chi_{\varpi_l, t}(wb), \chi_{\varpi_l, t}(\beta)].$$

Note that, clearly,  $w^{-1}\chi_{\varpi_l, t}(w) \in H(\Phi, A[t^{\pm 1}])$ .

**Next we define a map  $\sigma' : V \rightarrow V$ .** Set  $\Phi^{+'} = w_{\alpha_l}(\Phi^+)$ . By Lemma 3.6 and the definition of  $V$  every element of  $V$  can be written in the form  $[aw, b, \beta]$  for some  $a \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^{+'}, A)$ ,  $w \in W(\Phi, A) \leq \text{St}(\Phi, A)$ ,  $b \in \tilde{B}$  and  $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$ . Note that  $w_{\alpha_l}(\Pi)$  is a set of simple roots of  $\Phi$  contained in  $\Phi^{+'}$ , with  $w_{\alpha_l}(\alpha_l) = -\alpha_l$  playing the role of  $\alpha_l$ . Then by Lemma 3.8 there is a homomorphism

$$\delta'_l : \text{St}(\Phi, A[t], tA[t]) \rightarrow \text{St}(\Phi, A[t], tA[t])\text{St}P_l'^-(\Phi, A),$$

compatible with  $\chi_{w_{\alpha_l}(\varpi_l), t}$ , where  $\text{St}P_l'^-(\Phi, A)$  denotes the subgroup of  $\text{St}(\Phi, A)$  corresponding to the parabolic set of roots  $w_{\alpha_l}(\Delta_l \cup \Sigma_l)$ . We define

$$\sigma'|_{\text{St}(\Phi, A[t], tA[t])} = \delta'_l : \text{St}(\Phi, A[t], tA[t]) \rightarrow \text{St}(\Phi, A[t], tA[t])\text{St}P_l'^-(\Phi, A) \leq [\text{St}(\Phi, A[t]), 1, 1].$$

Similarly, for any  $\alpha \in \Phi$ ,  $u \in A[t^{\pm 1}]$  we consider a homomorphism

$$\chi_{w_{\alpha_l}(\varpi_l), t} : X_\alpha(A[t^{\pm 1}]) \rightarrow X_\alpha(A[t^{\pm 1}]), \quad x_\alpha(u) \mapsto x_\alpha(t^{m'_{-\alpha_l}(\alpha)}u),$$

where  $m'_{w_{\alpha_l}(\alpha_i)}(\alpha)$ ,  $1 \leq i \leq l$ , is the coefficient of  $w_{\alpha_l}(\alpha_i)$  in the decomposition of  $\alpha \in \Phi$  with respect to  $w_{\alpha_l}(\Pi)$ . We also consider a homomorphism  $\sigma'_U = \chi_{w_{\alpha_l}(\varpi_l), t}|_{U(\Phi^{+'}, A)} : U(\Phi^{+'}, A) \rightarrow U(\Phi^{+'}, A[t])$ . Combining  $\delta'_l$  and  $\sigma'_U$ , we obtain a homomorphism

$$\sigma' : \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^{+'}, A) \rightarrow \text{St}(\Phi, A[t]).$$

Now we define

$$\sigma' \cdot [aw, b, \beta] = [\sigma'(a)w, w^{-1}\chi_{w_{\alpha_l}(\varpi_l), t}(wb), \chi_{w_{\alpha_l}(\varpi_l), t}(\beta)].$$

As in the case of  $\sigma$ , we have  $w^{-1}\chi_{w_{\alpha_l}(\varpi_l), t}(w) \in H(\Phi, A[t^{\pm 1}])$ .

Now we show that  $\sigma, \sigma' : V \rightarrow V$  are correctly defined and bijective. We prove that for  $\sigma'$ , the case of  $\sigma$  being analogous....

**Next we show that  $\sigma$  and  $\sigma'$  commute.** First we note that if  $\sigma \cdot [a, 1, 1] = [x, y, 1]$ , then for any  $b \in \tilde{B}$  and  $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$  one has

$$\sigma \cdot [a, b, \beta] = [x, y\chi_{\varpi_l, t}(b), \chi_{\varpi_l, t}(\beta)],$$

and a similar equality holds for  $\sigma'$ . Since  $\chi_{\varpi_l, t}$  and  $\chi_{w_{\alpha_l}(\varpi_l), t}$  commute on  $\text{St}(\Phi, A[t^{\pm 1}])$  and on  $\text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$ , it is enough to check that  $\sigma$  and  $\sigma'$  commute on any element of the form  $[aw, 1, 1] \in V$ , where  $a \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+, A)$ ,  $w \in W(\Phi, A)$ . We can write  $a = a_0 \cdot x_{\alpha_l}(u)$ , where  $a_0 \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+ \cap \Phi^{+'}, A)$ .....

Now we define the action of  $\text{St}(\Phi, A[t^{\pm 1}])$  on  $V$ . For any  $\alpha \in \Phi$  and  $u \in A[t] \subseteq A[t^{\pm 1}]$  we set

$$x_\alpha(u) \cdot [a, b, \beta] = [x_\alpha(u)a, b, \beta].$$

Next we want to use Lemma 3.10 applied to the set  $J = \{\alpha_i, \alpha_l\}$ , where  $\alpha_i$  is the simple root adjacent to  $\alpha_l$ . For any  $\alpha \in \pm \Sigma_l$  and any  $u \in A$  we set

$$x_\alpha(t^{-1}u) \cdot [a, b, \beta] = \sigma^{\mp 1} \cdot x_\alpha(u) \cdot \sigma^{\pm 1}[a, b, \beta].$$

For any  $\alpha \in \pm \Sigma_i$  and any  $u \in A$  we set

$$x_\alpha(t^{-1}u) \cdot [a, b, \beta] = \sigma'^{\mp 1} \cdot x_\alpha(u) \cdot \sigma'^{\pm 1}[a, b, \beta]$$

(cf. Lemma 3.9). Then we check that this action satisfies the properties required in Lemma 3.10. This defines an action of  $\text{St}(\Phi, A[t^{\pm 1}])$  on  $V$ .

The last step is to check that for any  $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$  the action of its image in  $\text{St}(\Phi, A[t^{\pm 1}])$  on  $[1, 1, 1]$  gives  $[1, 1, \beta]$ .  $\square$

### 3.4. Suslin's lemma.

lem:tulinj

**Lemma 3.11.** *Assume that  $G$  satisfies Tulenbaev lifting property 2.4. Then the map  $i$  in the following commutative diagram of groups is injective.*

{diag:cs}

(3.2)

$$\begin{array}{ccccccc} C_B & \hookrightarrow & \text{St}^G(B, I) & \xrightarrow{\mu_B} & \text{St}^G(B) & \xrightarrow{\pi_B} \twoheadrightarrow & \text{St}^G(k[t]) \\ \downarrow k & & \downarrow j & \nearrow \varphi & \downarrow i & & \downarrow \\ C_R & \hookrightarrow & \text{St}^G(R, I) & \xrightarrow{\mu_R} & \text{St}^G(R) & \xrightarrow{\pi_R} \twoheadrightarrow & \text{St}^G(k[t, t^{-1}]) \end{array}$$

*Proof.* First of all, notice that by Corollary 3.4 the vertical map in the right-hand side of the diagram is injective. Invoking Tulenbaev's property 2.4 we also find a lifting map  $\varphi$  in the central square of the diagram.

Let  $g \in \text{St}^G(B)$  be an element of  $\text{Ker}(i)$ . Since  $g$  also lies in  $\text{Ker}(\pi_B)$  it comes from some  $\tilde{g} \in \text{St}^G(B, I)$  via  $\mu_B$ . But  $j(\tilde{g})$  lies in  $C_R$ , hence, by Proposition 3.13 below it comes from some  $\hat{g} \in C_B$  via  $k$ . Finally,  $g = \varphi(j(\tilde{g})) = \varphi(k(\hat{g})) = \mu_B(\hat{g}) = 1$ , as claimed.  $\square$

The following result is analogous to [14, Proposition 4.1].

**Lemma 3.12.** *The map  $j$  in the diagram (3.2) is surjective for any commutative ring  $A$ .*

*Proof.* Let  $\alpha_i$  be arbitrary simple root from  $\Pi$  and let  $\Sigma_i$  (resp.  $\Sigma_i^-$ ) denote the special subsets of roots consisting of  $\alpha \in \Phi$  such that  $m_i(\alpha) > 0$  (resp.  $m_i(\alpha) < 0$ ).

By Lemma 2.3 the sets  $\mathcal{Z}(\Sigma_i, R, I)$  and  $\mathcal{Z}(\Sigma_i^-, R, I)$  both generate  $\text{St}^G(R, I)$ . Similarly, the subsets  $\mathcal{Z}(\Sigma_i, B, I) \subseteq \mathcal{Z}(\Sigma_i, R, I)$ ,  $\mathcal{Z}(\Sigma_i^-, B, I) \subseteq \mathcal{Z}(\Sigma_i^-, R, I)$  both generate the image of  $j: \text{St}^G(B, I) \rightarrow \text{St}^G(R, I)$ .

Using formula (2.6) we can calculate how the powers of the automorphism  $\sigma_i$  act on these generating sets. Indeed, for  $z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i^-, R, I)$  we have

$$\sigma_i^N(z_\alpha(s, \xi)) = \sigma_i^N(x_\alpha(0, s)^{x-\alpha(\Delta(\xi))}) = z_\alpha(t^{Nm_i(\alpha)} \cdot s, t^{-Nm_i(\alpha)} \cdot \xi).$$

It is clear that for sufficiently large  $N$  the element  $t^{-Nm_i(\alpha)} \cdot \xi$  belongs to  $A[t] \subseteq B$ . Consequently, for arbitrary  $g \in \text{St}^G(R, I)$  there exists  $N > 0$  such that  $\sigma^N(g)$  lies in the image of  $j$ . On the other hand, by a similar calculation we get for  $z = z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i, B, I)$  that the

element  $\sigma^{-N}(z)$  ( $N > 0$ ) still lies in  $\mathcal{Z}(\Sigma_i, B, I)$ . Clearly, this implies that the image of  $j$  is preserved by  $\sigma_i^{-1}$ . The assertion of the lemma now follows from these two statements:

$$g = \sigma^{-N} \sigma^N(g) \in \sigma^{-N}(\text{Im}(j)) \subseteq \text{Im}(j). \quad \square$$

prop:kersurj

**Proposition 3.13.** *Under the assumptions of Lemma 3.11 the map  $k$  in the diagram (3.2) is surjective.*

*Sketch/draft of the proof.* The argument presented below only may work under additional assumption that  $m_i(\tilde{\alpha}) = 1$  ( $i$  is as in the statement of ??).

Consider the following set

$$X = \text{St}^G(B) \times \tilde{B}(R) \times \text{St}^G(R, I) / \simeq.$$

The congruence relation we impose should be similar to that used in Tulenbaev's paper. In particular, for  $h \in \text{St}^G(B, I)$  we should have the following relation in  $X$

$$[\mu_B(h), 1, g] \simeq [1, 1, j(h)g].$$

Notice that instead of  $\text{St}^G(A[t])$  (used by Tulenbayev in the first factor) we use  $\text{St}^G(B)$ .

TODO: We should prove that there is a well-defined action of  $\text{St}^G(R)$  on  $X$ .

Now let  $g$  be an element of  $C_R$ . By the previous lemma we have for some  $g' \in \text{St}^G(B, I)$

$$[1, 1, g] = [1, 1, j(g')] = [\mu_B(g'), 1, 1] = i\mu_B(g')[1, 1, 1] = [1, 1, 1]. \quad \square$$

**3.5. Proof of the main result.** The following result is analogous to [14, Proposition 4.3]. It plays the same role in our proof of  $\mathbb{P}^1$ -glueing for  $K_2$  as generalized Suslin lemma (cf. [1, Theorem 2.16]) does in the corresponding proof for  $K_1$ .

prop:p1g

**Proposition 3.14.** *The functors  $\text{St}^G$ ,  $K_2^G$  satisfy  $\mathbb{P}^1$ -glueing property for arbitrary local ring  $A$ .*

*Proof.*  $\square$

## REFERENCES

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| <div style="border: 1px solid black; padding: 2px; display: inline-block;">Abe</div><br><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">AHW15</div><br><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">HOM</div><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">Hur77</div><br><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">J</div><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">Ma69</div><br><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">Mil70</div><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">Re75</div><br><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">S15</div><br><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">St-poly</div><br><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">St-cong</div><br><div style="border: 1px solid black; padding: 2px; display: inline-block;">Ste73</div> | <ul style="list-style-type: none"> <li>[1] E. Abe. “Whitehead groups of Chevalley groups over polynomial rings”. <i>Comm. Algebra</i> <b>11</b>, no. 12, (1983), pp. 1271–1307.</li> <li>[2] A. Asok, M. Hoyois, and M. Wendt. “Affine representability results in <math>\mathbb{A}^1</math>-homotopy theory II: principal bundles and homogeneous spaces”. 2015. arXiv: <a href="https://arxiv.org/abs/1507.08020">1507.08020</a> [math.AG].</li> <li>[3] A. J. Hahn and T. O’Meara. <i>The classical groups and K-theory</i>. Springer, 1989.</li> <li>[4] J. Hurrelbrink. “Endlich präsentierte arithmetische Gruppen und <math>K_2</math> über Laurent-Polynomringen”. <i>Math. Ann.</i> <b>225</b>, no. 2, (1977), pp. 123–129. ISSN: 0025-5831.</li> <li>[5] J. F. Jardine. “On the homotopy groups of algebraic groups”. <i>J. Algebra</i> <b>81</b>, no. 1, (1983), pp. 180–201.</li> <li>[6] H. Matsumoto. “Sur les sous-groupes arithmétiques des groupes semi-simples déployés”. <i>Ann. Sci. ENS</i> <b>2</b>, no. 1, (1969), pp. 1–62.</li> <li>[7] J. Milnor. “Algebraic K-theory and quadratic forms”. <i>Invent. math.</i> <b>9</b>, no. 4, (1970), pp. 318–344.</li> <li>[8] U. Rehmann. “Präsentationen von Chevalleygruppen über <math>k[t]</math>”. 1975. URL: <a href="http://www.math.uni-bielefeld.de/~rehmann/Publ/presentation.pdf">http://www.math.uni-bielefeld.de/~rehmann/Publ/presentation.pdf</a>.</li> <li>[9] S. Sinchuk. “On centrality of <math>K_2</math> for Chevalley groups of type <math>E_\ell</math>”. <i>J. Pure Appl. Algebra</i> <b>220</b>, no. 2, (2016), pp. 857–875.</li> <li>[10] A. Stavrova. “Homotopy invariance of non-stable <math>K_1</math>-functors”. <i>J. K-theory</i> <b>13</b>, no. 2, (2014), pp. 199–248.</li> <li>[11] A. Stavrova. “On the congruence kernel of isotropic groups over rings”. <i>preprint</i> (2014).</li> <li>[12] M. R. Stein. “Surjective stability in dimension 0 for <math>K_2</math> and related functors”. <i>Trans. Amer. Math. Soc.</i> <b>178</b>, (Apr. 1973), pp. 165–191.</li> </ul> |
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| Kbook |
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- [13] A. A. Suslin. “Torsion in  $K_2$  of fields”. *K-theory* **1**, no. 1, (1987), pp. 5–29.
- [14] M. Tulenbaev. “The Steinberg group of a polynomial ring”. *Math. USSR Sb.* **45**, (1983), pp. 131–144.
- [15] N. Vavilov. “Weight elements of Chevalley groups”. *St. Petersburg Math. J.* **20**, no. 1, (2009), pp. 23–57.
- [16] C. A. Weibel. *The K-book: An introduction to algebraic K-theory*. Vol. 145. 2013.

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