$\mathbb{A}^1$-INVARIANCE FOR UNSTABLE K_2

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1. Aim of the paper

The aim of this text is to prove that the non-stable K_2-functors K_2^G, where G is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings R containing a field k, that is,

{eq:A1-main}
$$K_2^G(R[t]) = K_2^G(R). \tag{1.1}$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K-functor associated to G. This functor originates from [Jar83]. The above equality by e.g. [AHW15, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of G in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of K_2, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \dots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the SL_n case [Tul83] and Stavrova's proof for K_1^G [Sta14]. Tulenbaev [Tul83] uses stabilization of the K_2-functor, and the good properties of the limit=algebraic K-theory. In [Sta14] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [Tul83, p. 140], or, respectively, [Sta14, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [Tul83, Theorem 5.1] or [Sta14, Theorem 1.1]).

1.1. **The case $R = k$.** We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the $\mathbb{A}^1$-invariance of K_1^G, this case should be used to deduce that $K_2^G(k[t_1, \dots, t_n]) = K_2^G(k)$.)

In Tulenbaev's framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [Wen14, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k. Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt's proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [Deo78], but only for the

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good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [VW12, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. **Plan of the proof.**

plan:3t (1) (**done**, see Theorem 1). Let R be a local ring. Show that

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t]))$$

Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

plan:k[t] (2) (**done**, see Theorem 3). Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect (3) (**done**). Consequently, $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp (4) (**done** for split ACDE, see Theorem 2). Prove Quillen-Suslin lgp for K_2^G.

plan:Zglu (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2)\mapsto g_1 g_2{}^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma (6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A. If

$$K_2^G(A[X_1, \ldots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \ldots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [Abe83, Lemma 3.6].

plan:Nglu (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St}^{\mathrm{G}}(A_h)$ there exist $y \in \mathrm{St}^{\mathrm{G}}(A)$ and $z \in \mathrm{St}^{\mathrm{G}}(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and h is not a zero divisor in A, then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g),g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2)\mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [Sta14, Lemma 3.4].

plan:P1 (8) ($\mathbb{P}^1$-gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2)\mapsto g_1 g_2{}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

(a) **(done)** Prove Corollary 6.2.
(b) Prove Proposition 6.1. This is hard. Have no idea how this can be proved at the moment.
(c) Prove $\mathbb{P}^1$-glueing using all the above facts.

plan:[]f (9) (Main corollary of $\mathbb{P}^1$-gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.
The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2.

plan:k[tn] (11) Prove that

$$K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let R be a regular ring containing a field k. Then $K_2^G(R[t]) = K_2^G(R)$.
This uses (4), (11) and (7).

2. Steinberg groups of Chevalley groups: preliminaries

2.1. **Definition and basic properties.** Let G be a split simple Chevalley groups with a root system Φ of rank ≥ 2. Recall that the *Steinberg group* $\mathrm{St}^{\mathrm{G}}(R)$ (also denoted $\mathrm{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi,R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi,R}$ defined as follows:

{rel:add}
$$x_\alpha(s)x_\alpha(t) = x_\alpha(s+t), \tag{2.1}$$

{rel:CCF}
$$[x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta}\left(N_{\alpha\beta ij}\, s^i t^j\right), \quad \alpha \neq -\beta, \quad N_{\alpha\beta ij} \in \mathbb{Z}. \tag{2.2}$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

2.2. **Tulenbaev's lifting property.** Let $I \trianglelefteq R$ be an ideal of a commutative ring. For a nonnilpotent element $a \in R$ denote by $\lambda_a\colon R \to R_a$ the morphism of principal localization at a. Consider the following commutative square.

{msq}
$$\begin{tikzcd} R \arrow[r, "\lambda_a"] \arrow[d, two heads] & R_a \arrow[d, two heads] \\ R/I \arrow[r, "\overline{\lambda_a}"] & R_a/I_a \end{tikzcd} \tag{2.3}$$

Notice that (2.3) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a. Such squares are usually called *Milnor squares* in the literature, see [Wei13, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [Tul83, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp **Definition 2.1.** We say that the Steinberg group functor St^{G} satisfies *Tulenbaev's lifting property* if for every pull-back square (2.3) the following lifting problem has a solution.

$$\begin{tikzcd} \mathrm{St}^{\mathrm{G}}(R, I) \arrow[r, "\mu"] \arrow[d] & \mathrm{St}^{\mathrm{G}}(R) \arrow[d, "\lambda_a^*"] \\ \mathrm{St}^{\mathrm{G}}(R_a, I) \arrow[r, "\mu"] \arrow[ur, dotted] & \mathrm{St}^{\mathrm{G}}(R_a) \end{tikzcd}$$

2.3. **The action of torus.** Throughout this subsection $G = G_{\mathrm{ad}}$ denotes a split simple Chevalley group of adjoint type with the root system Φ of rank ≥ 2. Denote by $T = T_{\mathrm{ad}}$ the torus of G and by $T(R)$ its group of R-points.

We identify the root lattice $X^*(T) = \mathrm{Hom}(T, \mathbf{G}_{\mathrm{m}})$ with the lattice $\mathbb{Z}\,\Phi$ in the obvious way. In particular, for $\alpha \in \Phi$ we denote by α_R the corresponding map $T(R) \to R^*$ on R-points. An element $h \in T(R)$ defines a permutation of the set $\mathcal{X}_{\Phi,R}$ of generators of $\mathrm{St}^{\mathrm{G}}(R)$ as follows:

$$h \cdot x_\alpha(\xi) = x_\alpha(\alpha_R(h) \cdot \xi). \tag{2.4}$$

Notice that h preserves the defining relations $\mathcal{R}_{\Phi,R}$ of the Steinberg group (and thus, determines a permutation of $\mathcal{R}_{\Phi,R}$). Indeed, the assertion is immediate for relation (2.1). Verification of the fact that h preserves (2.2) is a routine computation which should use the fact that for $\alpha, \beta \in \mathbb{Z}\,\Phi$ one has $(\alpha+\beta)_R(h) = \alpha_R(h) \cdot \beta_R(h)$.

3. Decomposition theorems for $\mathrm{St}^{\mathrm{G}}(A[t^{\pm 1}])$ and $\mathrm{St}^{\mathrm{G}}\big(A((t))\big)$.

Lemma 3.1. *Let (R, m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big) i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) = i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big)$$

inside $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

Proof. This is proved exactly as [Sta14, Lemma 5.12]. □

thm:3t **Theorem 1.** *Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t])) i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}])) i_+(\mathrm{St}^{\mathrm{G}}(R[t])).$$

Proof. This is proved exactly as [Sta14, Theorem 5.1]. □

4. Quillen-Suslin lgp, Zariski gluing, Nisnevich gluing, S-lemma

thm:lg-k2 **Theorem 2.** *Assume that the Steinberg group functor St^{G} satisfies Tulenbaev's lifting property (cf. Definition 2.1). Then an element $g \in \mathrm{St}^{\mathrm{G}}(R[t], tR[t])$ is trivial if and only if its image in $\mathrm{St}^{\mathrm{G}}(R_M[t], tR_M[t])$ is trivial for all maximal ideals $M \trianglelefteq R$.*

Proof. Follows by the same argument as [Tul83, Theorem 2.1] or [Sin16, Theorem 2] □

5. The case of $K_2^G(k[t])$ and some corollaries

Let $G = G(\Phi, -)$ be the simply-connected Chevalley—Demazure group scheme of type Φ. Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$.

thm:k[t] **Theorem 3.** *Let k be a field. Assuming that Φ is irreducible of rank at least 2 the following facts are true.*

satz1 *(i) For $A = k, k[t]$ the subgroup $K_2^G(A) \leq \mathrm{St}^{\mathrm{G}}(A)$ is generated by elements of the form*

$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1},\ u, v \in k^*.$$

(ii) As a consequence, the canonical map $K_2^G(k) \hookrightarrow K_2^G(k[t])$ is an isomorphism.

Proof. See [Reh75, Satz 1] and the corollary after it. □

cor:k[t]inj **Corollary 5.1.** *Let G, k be as in Theorem 3. Then $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm1}])$.*

Proof. Clearly, $g \in \ker(\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm1}]))$ implies $g \in K_2^G(k[t])$. Since $K_2^G(k[t]) = K_2^G(k)$, and there is a section $K_2^G(k[t^{\pm1}]) \to K_2^G(k)$, the map is injective. Second claim: take $g \in \mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}])$. Then the image $\phi(g)$ belongs to $E(k) = E(k[t]) \cap E(k[t^{-1}])$, and after adjusting g by an element of $\mathrm{St}^{\mathrm{G}}(k)$, we can assume that $g \in K_2^G(k[t]) \cap K_2^G(k[t^{-1}])$. Hence $g \in K_2^G(k) \subseteq \mathrm{St}^{\mathrm{G}}(k)$. □

Corollary 5.2. *Let G, k be as in Theorem 3. Then $K_2^G(k[t^{\pm1}]) = K_2^G(k)$.*

Proof. We use Theorem 1. Take $g \in K_2^G(k[t^{\pm1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}^{\mathrm{G}}(k[t]))$, $y \in \mathrm{St}^{\mathrm{G}}(k[t^{-1}]))$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}^{\mathrm{G}}(k)K_2^G(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ and $x_1x_2 \in \mathrm{St}^{\mathrm{G}}(k)$. That is, $g \in K_2^G(k)$. □

Denote by $\mathrm{W}(\Phi, R)$ the subgroup of $\mathrm{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$. The following result is a Steinberg level analogue of the well-known Bruhat decomposition.

cor:bruhat **Corollary 5.3.** *For Φ of rank ≥ 2 and a field k one has*

$$\mathrm{St}(\Phi, k) = \mathrm{U}(\Phi^+, k) \cdot \mathrm{W}(\Phi, k) \cdot \mathrm{U}(\Phi^+, k).$$

Proof. From Theorem 3.(i) it follows that $K_2(\Phi, k)$ is contained in $\mathrm{W}(\Phi, k)$, therefore the reqired assertion follows from the classical Bruhat decomposition (see e. g. [SFW67, Theorem 4]). For a direct proof in the linear case see also [HO89, § 2.3A]. □

6. $\mathbb{P}^1$-GLUING

6.1. **Preliminaries.** Let R denote arbitrary commutative local ring with the maximal ideal m and the residue field k. Consider the following commutative diagram of groups.

$$\begin{tikzcd}
C_+ \arrow[r, hook] \arrow[dd, two heads, "k^+"] & \mathrm{St}^{\mathrm{G}}(R[t], m[t]) \arrow[r, "\mu^+"] \arrow[d, "j_1^+"] & \mathrm{St}^{\mathrm{G}}(R[t]) \arrow[r, two heads, "p^+"] \arrow[d, "i_1^+"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[d, equal] \\
 & \mathrm{St}^{\mathrm{G}}(R[t] + m[t^{-1}], m[t^{\pm 1}]) \arrow[r, "\mu^{+\varepsilon}"] \arrow[d, "j_2^+"] & \mathrm{St}^{\mathrm{G}}(R[t] + m[t^{-1}]) \arrow[r, two heads, "p^{+\varepsilon}"] \arrow[d, "i_2^+"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[d, hook, "\overline{i_2^+}"] \\
C_\pm \arrow[r, hook] & \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}], m[t^{\pm 1}]) \arrow[r, "\mu^\pm"'] \arrow[ur, dashed, "\varphi"] & \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) \arrow[r, two heads, "p^\pm"] & \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])
\end{tikzcd}$$

ulKernSurject **Proposition 6.1.** *The map k^+ is surjective.*

Proof. Compare with [Tul83, Proposition 4.1]. □

cor:TulInj **Corollary 6.2.** *The map i_2^+ is injective.*

Proof. First of all, notice that by invoking Tulenbaev's lifting property 2.1 we find a map φ such that $i_2^+ \varphi = \mu^\pm$.

Now let $g \in \mathrm{St}^{\mathrm{G}}(R[t] + m[t^{-1}])$ be an element of $\ker(i_2^+)$. By Corollary 5.1 g also lies in $\ker(p^{+\varepsilon})$ and hence comes from some $\widetilde{g} \in \mathrm{St}^{\mathrm{G}}(A[t] + m[t^{-1}], m[t^{\pm 1}])$ via $\mu^{+\varepsilon}$. Since $j_2^+(\widetilde{g})$ lies in $C_\pm$ by Proposition 6.1 it comes from some $\widehat{g} \in C_+$ via k^+. The statement of corollary now follows from a simple diagram chasing, indeed:

$$g = \varphi(j_2^+(\widetilde{g})) = \varphi(k^+(\widehat{g})) = \varphi(j_2^+(j_1^+(\widehat{g}))) = \mu^{+\varepsilon}(j_1^+(\widehat{g})) = i_1^+(\mu^+(\widehat{g})) = i_1^+(1) = 1. \quad \square$$

6.2. **Main result.** The following lemma is an analog of [Tul83, Proposition 4.3 (a)].

Lemma 6.3. *Let R be a local ring, $k = R/m$, and let k, G be as in Theorem 3. The natural homomorphism $\mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ is injective.*

Proof. Let I be the maximal ideal of R, $l = R/I$, and consider the natural maps $\rho : \mathrm{St}^{\mathrm{G}}(R[t, t^{-1}]) \to \mathrm{St}^{\mathrm{G}}(l[t, t^{-1}])$, $\rho_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(l[t])$, $\rho_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}]) \to \mathrm{St}^{\mathrm{G}}(l[t^{-1}])$. Take $x \in \ker(\mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$. By the field case Corollary 5.1 one has $\rho_+(x) = 1$, hence $x \in \mathrm{St}^{\mathrm{G}}(I \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}$.

??????? □

Lemma 6.4. *Let G, k be as in Theorem 3. Let (R, m) be a local ring such that $R/m = k$. Then*

$$\mathrm{St}^{\mathrm{G}}(R[t]) \cap \mathrm{St}^{\mathrm{G}}(R[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(R)$$

inside $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

Proof. ?????? □

Theorem 4. *Let A be any commutative ring. Then the sequence of pointed sets*

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2) \mapsto g_1 g_2{}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

Proof. Follows from the above lemmas. □

Corollary 6.5. *Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.*

Proof. ???? □

Appendices

A. Linear Steinberg group in rank 3

The main goal of this subsection is to show that Tulenbaev's [Tul83, Lemma 2.3] remains valid for the linear Steinberg group of rank ≥ 3. In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [Sin16, Definitions 3.3 and 3.7]).

Definition A.1. The relative Steinberg group $\mathrm{St}^*(n, R, I)$ is the group defined by the following two families generators and four families of relations.

- Generators:
  (1) $X^1(u, v)$, where $u \in \mathrm{E}(n, R) \cdot e_1$, $v \in I^n$ such that $v^t \cdot u = 0$;
  (2) $X^2(u, v)$, where $u \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$ such that $v^t \cdot u = 0$.
  Notice that ϕ maps both $X^1(u, v)$ and $X^2(u, v)$ to $T(u, v) = e + u \cdot v^t \in \mathrm{E}(n, R, I)$.
- Relations:
  (1) $X^1(u, v) \cdot X^1(u, w) = X^1(u, v + w)$, $u \in \mathrm{E}(n, R) \cdot e_1$, $v, w \in I^n$;
  (2) $X^2(u, v) \cdot X^2(w, v) = X^2(u + w, v)$, $u, w \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$;
  (3) ${}^{X^\sigma(u^2,v^2)}X^\tau(u^1, v^1) = X^\tau(T(u^2, v^2) \cdot u^1, T(v^2, u^2)^{-1} \cdot v^1)$, $\sigma, \tau = 1, 2$;
  (4) $X^1(g \cdot e_1, g^* \cdot be_2) = X^2(g \cdot be_1, g^* \cdot e_2)$ where $b \in I$ and $g^* = g^{t^{-1}}$ denotes the contragradient matrix.

Lemma A.2. *The groups* $\mathrm{St}^*(n, R, I)$ *and* $\mathrm{St}(n, R, I)$ *are isomorphic.*

Proof. **TODO:** □

The next step of the proof is to is construct certain elements in $\mathrm{St}(n, R)$ similar to Tulenbaev's elements $X_{u,v}(a)$ see [Tul83, § 1].

Let $v \in R^n$ be a column. Denote by $O(v)$ the submodule of R^n consisting of all columns w such that $w^t \cdot v = 0$. A column $w \in R^n$ is called v-*decomposable* if it can be presented as a finite sum $w = \sum_{i=1}^{p} w^i$ such that each w^i has at least two zero entries and $v^t \cdot w^i = 0$. Denote by $D(v)$ the submodule of $O(v)$ consisting of all v-decomposable columns. For a column $v \in R^n$ denote by $I(v)$ the ideal of R spanned by its entries $v_1, \ldots, v_n$.

Let $u, v, w \in R^n$ be columns such that $w^t v = 0$. It is easy to check (cf. [Kal77, Lemma 3.2]) that

$$(uv) \cdot w = \sum_{i<j} w_{ij}, \text{ where } w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n.$$

The above decomposition is called the *canonical* decomposition of $(uv) \cdot w$. In particular, this shows that the column $a \cdot w$ is always v-decomposable for $a \in I(v)$, $w \in O(v)$, i. e. $I(v) \cdot O(v) \subseteq D(v)$. It is also straightforward to check that $D(v) \subseteq D(bv)$, $b \cdot D(v) \subseteq D(v)$ for $b \in R$.

Denote by B^1 the subset of $R^n \times R^n \times R$ consisting of triples (u, v, a) such that $v^t \cdot u = 0$, $v \in D(u)$, $a \in I(u)$. Denote by B^2 the set consisting of triples (v, u, a) such that $(u, v, a) \in B^1$.

lem:Zfacts **Lemma A.3.** *Assume that $n \geq 4$. One can define two families of elements $Z^\tau(u, v, a)$, $\tau = 1, 2$ of the group $\mathrm{St}(n, R)$ parametrized by $(u, v, a) \in B^\tau$ satisfying the following properties:*

(1) $\phi(Z^\tau(u, v, a)) = e + uav^t \in \mathrm{E}(n, R)$, $(u, v, a) \in B^\tau$;
(2) $Z^1(u, v + w, a) = Z^1(u, v, a) \cdot Z^1(u, w, a)$;
(3) $Z^2(v + w, u, a) = Z^2(v, u, a) \cdot Z^2(w, u, a)$;
(4) for $\tau = 1, 2$ and $b \in R$ if $(u, vb, a), (ub, v, a) \in B^\tau$ then one has

$$Z^\tau(u, vb, a) = Z^\tau(u, v, ab) = Z^\tau(ub, v, a);$$

(5) ${}^g Z^\tau(u, v, a) = Z^\tau(\phi(g) \cdot u, \phi(g)^* \cdot v, a)$, $\tau = 1, 2$, $g \in \mathrm{St}^{\mathrm{G}}(n, R)$.

Proof. One constructs the elements $Z^1(u, v, a)$ in exactly the same way as Tulenbaev constructs his elements $X_{u,v}(a)$ (see definitions preceding [Tul83, Lemma 1.2]). Indeed, set

$$Z^1(v, w, a) = \prod_{k=1}^{p} X(v, a \cdot w^k), \quad Z^2(w, v, a) = \prod_{k=1}^{p} X(a \cdot w^k, v). \tag{A.1}$$

where $X(u, v)$ denotes the elements defined by Tulenbaev before [Tul83, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case $n = 4$) can be proved by the same token as in [Tul83, Lemma 1.3]. ☐

For the rest of this section a denotes a nonnilpotent element of R and $\lambda_a \colon R \to R_a$ is the morphism of principal localization at a.

lem:rk3rels **Lemma A.4.** *For any $g \in \mathrm{E}(n, R_a)$ there exist $u, v \in R^n$ and sufficiently large natural numbers k, m such that the following facts hold:*

(1) $\lambda_a(u) = g \cdot a^k e_1$, $\lambda_a(v) = g^* \cdot a^k e_2$ and $u^t \cdot v = 0$;
(2) $(u, v, a^m) \in B^1 \cap B^2$;
(3) for $b \in R$ divisible by some sufficiently large power of a one has

$$Z^1(u, b \cdot v, a^m) = Z^2(b \cdot u, v, a^m).$$

Proof. It is straightforward to choose u and v satisfying the first requirement of the lemma. We can even choose u, v in such a way that $u \in D(v)$ and $v \in D(u)$. Indeed, notice that $I(u) = a^{k_1}$, $I(v) = a^{k_2}$ for some natural k_1, k_2 hence for $u' = a^{k_2} \cdot u$ and $v' = a^{k_1} \cdot v$ one has

$$u' \in I(v) \cdot O(v) \subseteq D(v) \subseteq D(v'), \quad v' \in I(u) \cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns $x, y \in R^n$ and a large natural p in such a way that vectors u, v, x, y additionally satisfy the following properties

$$\lambda_a(x) = g^* \cdot a^k e_3,\ \lambda_a(y) = g \cdot a^k e_3,\ y^t \cdot x = a^p \in R,$$

$$u^t \cdot x = 0,\ u^t \cdot v = 0,\ y^t \cdot v = 0,$$

$$(u, x, a^m) \in B^1,\ (y, v, a^m) \in B^2.$$

Now direct computation using Lemma A.3 shows that

$$\begin{aligned} Z^2(a^{m+p}b\cdot u, v, a^m) &= Z^2(b\cdot(e+a^m\cdot ux^t)y, (e-a^m\cdot xu^t)v, a^m)\cdot Z^2(-by, v, a^m) = \\ &= [Z^1(u, x, a^m), Z^2(b\cdot y, v, a^m)] = \\ &= Z^1(u, x, a^m)\cdot Z^1((e+a^m b\cdot yv^t)u, -(e-a^m b\cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b\cdot v, a^m), \quad \square \end{aligned}$$

hence the third assertion of the lemma follows.

cor:tulmap **Corollary A.5.** *[Tulenbaev's lemma] For $n \geq 4$ there is a map T_n so that the following diagram commutes.*

Proof. Follows from Lemma A.4 by the same token as in [Tul83, Lemma 2.3]. □

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