

# HOMOTOPY INVARIANCE FOR NONSTABLE $K_2$ -FUNCTORS

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## 1. AIM OF THE PAPER

The aim of this text is to prove that the non-stable  $K_2$ -functors  $K_2^G$ , where  $G$  is a simply connected Chevalley group of suitable type, satisfy  $\mathbb{A}^1$ -invariance on regular rings  $R$  containing a field  $k$ , that is,

$$(1.1) \quad K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where  $KV_2^G(R)$  is the Karoubi–Villamayor  $K$ -functor associated to  $G$ . This functor originates from [?]. The above equality by e.g. [?, Corollary 4.3.3] implies that  $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$ , i.e. we obtain an explicit presentation for the  $\mathbb{A}^1$ -fundamental group of  $G$  in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of  $K_2$ , or at least the Quillen–Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that  $K_2^G(k[x_1, \dots, x_n]) = K_2^G(k)$ . (Then (??) follows by standard geometric methods.) There are two models: Tulenbaev’s proof for the  $SL_n$  case [?] and Stavrova’s proof for  $K_1^G$  [?]. Tulenbaev [?] uses stabilization of the  $K_2$ -functor, and the good properties of the limit=algebraic  $K$ -theory. In [?] stabilization is not used. However, the key steps of both proofs are the same: the case of  $R = k$  (hidden somewhere around [?, p. 140], or, respectively, [?, Theorem 3.1]); Quillen–Suslin lgp;  $\mathbb{P}^1$ -gluing (see [?, Theorem 5.1] or [?, Theorem 1.1]).

**1.1. The case  $R = k$ .** We consider the case of  $K_2^G(k[t])$  vs.  $K_2^G(k)$ . (As in the  $\mathbb{A}^1$ -invariance of  $K_1^G$ , this case should be used to deduce that  $K_2^G(k[t_1, \dots, t_n]) = K_2^G(k)$ .)

In Tulenbaev’s framework, it follows from stabilization. However, the equality  $K_2^G(k[t]) = K_2^G(k)$  is sort of known for all groups. Namely, in [?, Theorem 5.1]: let  $k$  be an infinite field and let  $G$  be a connected reductive group over  $k$ . Then the inclusion  $k \hookrightarrow k[t]$  induces an isomorphism

$$H_{\bullet}(G(k), \mathbb{Z}) \xrightarrow{\cong} H_{\bullet}(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of  $G$  is invertible in  $k$ . Once we know that the homology  $H_2$  coincides with  $K_2^G$  (on both sides), this gives the result. It would be nice to check Wendt’s proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that  $K_2^G$  coincides with  $H_2$ ? If yes, then we probably know it for  $K_2^G(k)$  even for isotropic groups [?], but only for the

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good Chevalley groups for  $K_2^G(k[t])$ . We should try to understand what is proved in [?, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

## 1.2. Plan of the proof.

plan:3t

- (1) **(done)** Let  $R$  be a local ring. Show that

$$\mathrm{St}(R[t^{\pm 1}]) = i_+(\mathrm{St}(R[t]))i_-(\mathrm{St}(R[t^{-1}]))i_+(\mathrm{St}(R[t]))$$

Here  $i_{\pm}$  denote the natural homomorphisms into  $\mathrm{St}(R[t^{\pm 1}])$ .

plan:k[t]

n:k-intersect

- (2) Show that  $K_2^G(k[t]) = K_2^G(k)$ .  
 (3) Consequently,  $\mathrm{St}(k[t]) \rightarrow \mathrm{St}(k[t^{\pm 1}])$  is injective and  $\mathrm{St}(k[t]) \cap \mathrm{St}(k[t^{-1}]) = \mathrm{St}(k)$  inside  $\mathrm{St}(k[t^{\pm 1}])$ . Also,  $K_2^G(k) = K_2^G(k[t^{\pm 1}])$ .

The first claim uses (??). The second claim follows from (??) and (??).

plan:QSlgp

plan:Zglu

- (4) **(done)** Prove the Quillen-Suslin lgp for  $K_2^G$ . See ??.  
 (5) (Zariski gluing) for any commutative ring  $A$  and any non-nilpotent  $f, g \in A$  such that  $A = fA + gA$ , the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (??).

(Remark. It seems that we need this property only for  $A = R[t]$  and  $f, g$  non-constant polynomials.)

plan:S-lemma

- (6) ( $S$ -lemma) Let  $A$  be a commutative ring,  $S$  a multiplicative subset of  $A$ . If

$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$

for some  $n \geq 1$ , then  $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$  as well.

This should be easy; see [?, Lemma 3.6].

plan:Nglu

- (7) (Nisnevich gluing) Assume that  $B$  is a subring of a commutative ring  $A$ , and let  $h \in B$  be a non-nilpotent element. Denote by  $F_h : A \rightarrow A_h$  the localization homomorphism.

(i) If  $Ah + B = A$ , i.e. the natural map  $B \rightarrow A/Ah$  is surjective, then for any  $x \in \mathrm{St}(A_h)$  there exist  $y \in \mathrm{St}(A)$  and  $z \in \mathrm{St}(B_h)$  such that  $x = F_h(y)z$ .

(ii) If moreover  $Ah \cap B = Bh$ , i.e.  $B/Bh \rightarrow A/Ah$  is an isomorphism, and  $h$  is not a zero divisor in  $A$ , then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1, g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (??) or (??); see [?, Lemma 3.4].

plan:P1

- (8) ( $\mathbb{P}^1$ -gluing) Let  $A$  be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (??), (??), (??), (??).

plan:[ ]f

- (9) (Main corollary of  $\mathbb{P}^1$ -gluing) Let  $A$  be any commutative ring, and let  $f \in A[t]$  be a monic polynomial. Show that  $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$  is injective.

The proof uses (??) and (??).

**plan:k(t)**(10) Prove that  $K_2^G(k(t)) = K_2^G(k)$ .

This may be a bit tricky; I will think if we can get rid of it. I don't think Tullenbaev uses it.

**plan:k(tn)**

(11) Prove that

$$K_2^G(k[t_1, \dots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of  $K_1^G$ , this uses (??), (??), and (??). There may be other ways.

**plan:final**(12) Final result: let  $R$  be a regular ring containing a field  $k$ . Then  $K_2^G(R[t]) = K_2^G(R)$ .

This uses (??), (??) and (??).

## 2. STEINBERG GROUPS OF CHEVALLEY GROUPS: PRELIMINARIES

Definition, functoriality, "congruence subgroups"  $\text{St}(\Phi, R, I)$  versus  $\ker(\text{St}(\Phi, R) \rightarrow \text{St}(\Phi, R/I))$ .

## 3. DECOMPOSITION THEOREMS FOR $\text{St}(A[t^{\pm 1}])$ AND $\text{St}(A((t)))$ .

**Lemma 3.1.** *Let  $(R, m)$  be a local ring, and let  $G$  be a simply connected simple group over  $R$  of isotropic rank  $\geq 2$ . Let  $i_+ : \text{St}(R[t]) \rightarrow \text{St}(R[t^{\pm 1}])$  and  $i_- : \text{St}(R[t^{-1}]) \rightarrow \text{St}(R[t^{\pm 1}])$  be the natural homomorphisms. Then*

$$i_+(\text{St}(m \cdot R[t])^{\text{St}(R[t])})i_-(\text{St}(R[t^{-1}])) = i_-(\text{St}(R[t^{-1}]))i_+(\text{St}(m \cdot R[t])^{\text{St}(R[t])})$$

inside  $\text{St}(R[t^{\pm 1}])$ .

*Proof.* This is proved exactly as [?, Lemma 5.12]. □

**thm:3t**

**Theorem 1.** *Let  $R$  be a local ring, and let  $G$  be a simply connected simple group over  $R$  of isotropic rank  $\geq 2$ . Let  $i_+ : \text{St}(R[t]) \rightarrow \text{St}(R[t^{\pm 1}])$  and  $i_- : \text{St}(R[t^{-1}]) \rightarrow \text{St}(R[t^{\pm 1}])$  be the natural homomorphisms. Then*

$$\text{St}(R[t^{\pm 1}]) = i_+(\text{St}(R[t]))i_-(\text{St}(R[t^{-1}]))i_+(\text{St}(R[t])).$$

*Proof.* This is proved exactly as [?, Theorem 5.1]. □

## 4. QUILLEN-SUSLIN LGP, ZARISKI GLUING, NISNEVICH GLUING, $S$ -LEMMA

**thm:lg-k2**

**Theorem 2.** *Let  $R$  be arbitrary commutative ring and let  $\Phi$  be a root system of type  $A_\ell, C_\ell, D_\ell$  or  $E_\ell$  and rank  $\ell \geq 3$ . An element  $g \in \text{St}(\Phi, R[t], tR[t])$  is trivial if and only if its image in  $\text{St}(\Phi, R_M[t], tR_M[t])$  is trivial for all maximal ideals  $M \leq R$ .*

*Proof.* The case  $\Phi = C_\ell, \ell \geq 3$  is contained in some future Lavrenov's solo paper. It think that the case of a simply laced  $\Phi$  of rank  $\geq 3$  can be settled in a similar way as in [?]. We already have a written proof that [?, Lemma 2.3] extends to rank 3 groups, see here. It should be inserted into this article at a some place. □

## 5. THE CASE OF $K_2^G(k[t])$ AND SOME COROLLARIES

**thm:k[t]**

**Theorem 3.** *Let  $k$  be a field. Let  $G = G(\Phi, -)$  be a simply connected simple Chevalley group of rank  $\geq 2$  such that ... Then*

$$K_2^G(k[t]) = K_2^G(k).$$

*Proof.* ??? □

cor:k[t]inj

**Corollary 5.1.** *Let  $G, k$  be as in ???. Then  $\mathrm{St}(k[t]) \rightarrow \mathrm{St}(k[t^{\pm 1}])$  is injective and  $\mathrm{St}(k[t]) \cap \mathrm{St}(k[t^{-1}]) = \mathrm{St}(k)$  inside  $\mathrm{St}(k[t^{\pm 1}])$ .*

*Proof.* Clearly,  $g \in \ker(\mathrm{St}(k[t]) \rightarrow \mathrm{St}(k[t^{\pm 1}]))$  implies  $g \in K_2^G(k[t])$ . Since  $K_2^G(k[t]) = K_2^G(k)$ , and there is a section  $K_2^G(k[t^{\pm 1}]) \rightarrow K_2^G(k)$ , the map is injective. Second claim: take  $g \in \mathrm{St}(k[t]) \cap \mathrm{St}(k[t^{-1}])$ . Then the image  $\phi(g)$  belongs to  $E(k) = E(k[t]) \cap E(k[t^{-1}])$ , and after adjusting  $g$  by an element of  $\mathrm{St}(k)$ , we can assume that  $g \in K_2^G(k[t]) \cap K_2^G(k[t^{-1}])$ . Hence  $g \in K_2^G(k) \subseteq \mathrm{St}(k)$ .  $\square$

**Corollary 5.2.** *Let  $G, k$  be as in ???. Then  $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$ .*

*Proof.* We use ??. Take  $g \in K_2^G(k[t^{\pm 1}])$ , then  $g = x_1 y x_2$ ,  $x_i \in \mathrm{St}(k[t])$ ,  $y \in \mathrm{St}(k[t^{-1}])$ . Since  $E(k[t]) \cap E(k[t^{-1}]) = E(k)$ , we have  $y \in \mathrm{St}(k) K_2^G(k[t^{-1}]) = \mathrm{St}(k)$  and  $x_1 x_2 \in \mathrm{St}(k)$ . That is,  $g \in K_2^G(k)$ .  $\square$

## 6. $\mathbb{P}^1$ -GLUING

The following lemma is an analog of [?, Proposition 4.3 (a)].

**Lemma 6.1.** *Let  $R$  be a local ring,  $k = R/m$ , and let  $k, G$  be as in ??. The natural homomorphism  $\mathrm{St}(R[t]) \rightarrow \mathrm{St}(R[t^{\pm 1}])$  is injective.*

*Proof.* Let  $I$  be the maximal ideal of  $R$ ,  $l = R/I$ , and consider the natural maps  $\rho : \mathrm{St}(R[t, t^{-1}]) \rightarrow \mathrm{St}(l[t, t^{-1}])$ ,  $\rho_+ : \mathrm{St}(R[t]) \rightarrow \mathrm{St}(l[t])$ ,  $\rho_- : \mathrm{St}(R[t^{-1}]) \rightarrow \mathrm{St}(l[t^{-1}])$ . Take  $x \in \ker(\mathrm{St}(R[t]) \rightarrow \mathrm{St}(R[t^{\pm 1}]))$ . By the field case ?? one has  $\rho_+(x) = 1$ , hence  $x \in \mathrm{St}(I \cdot R[t])^{\mathrm{St}(R[t])}$ .  
???????  $\square$

**Lemma 6.2.** *Let  $G, k$  be as in ??. Let  $(R, m)$  be a local ring such that  $R/m = k$ . Then*

$$\mathrm{St}(R[t]) \cap \mathrm{St}(R[t^{-1}]) = \mathrm{St}(R)$$

*inside  $\mathrm{St}(R[t^{\pm 1}])$ .*

*Proof.* ??????  $\square$

**Theorem 4.** *Let  $A$  be any commutative ring. Then the sequence of pointed sets*

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

*is exact.*

*Proof.* ???  $\square$

**Corollary 6.3.** *Let  $A$  be any commutative ring, and let  $f \in A[t]$  be a monic polynomial. Show that  $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$  is injective.*

*Proof.* ???  $\square$

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