

# $\mathbb{A}^1$ -INVARIANCE FOR UNSTABLE $K_2$

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## 1. AIM OF THE PAPER

The aim of this text is to prove that the non-stable  $K_2$ -functors  $K_2^G$ , where  $G$  is a simply connected Chevalley group of suitable type, satisfy  $\mathbb{A}^1$ -invariance on regular rings  $R$  containing a field  $k$ , that is,

$$(1.1) \quad K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where  $KV_2^G(R)$  is the Karoubi–Villamayor  $K$ -functor associated to  $G$ . This functor originates from [Jar83]. The above equality by e.g. [AHW15, Corollary 4.3.3] implies that  $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$ , i.e. we obtain an explicit presentation for the  $\mathbb{A}^1$ -fundamental group of  $G$  in the sense of Morel–Voevodsky.

”Suitable type” here means that we consider only the cases where we know the centrality of  $K_2$ , or at least the Quillen–Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that  $K_2^G(k[x_1, \dots, x_n]) = K_2^G(k)$ . (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev’s proof for the  $SL_n$  case [Tul83] and Stavrova’s proof for  $K_1^G$  [Sta14]. Tulenbaev [Tul83] uses stabilization of the  $K_2$ -functor, and the good properties of the limit=algebraic  $K$ -theory. In [Sta14] stabilization is not used. However, the key steps of both proofs are the same: the case of  $R = k$  (hidden somewhere around [Tul83, p. 140], or, respectively, [Sta14, Theorem 3.1]); Quillen–Suslin lgp;  $\mathbb{P}^1$ -gluing (see [Tul83, Theorem 5.1] or [Sta14, Theorem 1.1]).

**1.1. The case  $R = k$ .** We consider the case of  $K_2^G(k[t])$  vs.  $K_2^G(k)$ . (As in the  $\mathbb{A}^1$ -invariance of  $K_1^G$ , this case should be used to deduce that  $K_2^G(k[t_1, \dots, t_n]) = K_2^G(k)$ .)

In Tulenbaev’s framework, it follows from stabilization. However, the equality  $K_2^G(k[t]) = K_2^G(k)$  is sort of known for all groups. Namely, in [Wen14, Theorem 5.1]: let  $k$  be an infinite field and let  $G$  be a connected reductive group over  $k$ . Then the inclusion  $k \hookrightarrow k[t]$  induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of  $G$  is invertible in  $k$ . Once we know that the homology  $H_2$  coincides with  $K_2^G$  (on both sides), this gives the result. It would be nice to check Wendt’s proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that  $K_2^G$  coincides with  $H_2$ ? If yes, then we probably know it for  $K_2^G(k)$  even for isotropic groups [Deo78], but only for the

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good Chevalley groups for  $K_2^G(k[t])$ . We should try to understand what is proved in [VW12, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

## 1.2. Plan of the proof.

plan:3t

- (1) (**done**, see Theorem 1). Let  $R$  be a local ring. Show that

$$\mathrm{St}^G(R[t^{\pm 1}]) = i_+(\mathrm{St}^G(R[t]))i_-(\mathrm{St}^G(R[t^{-1}]))i_+(\mathrm{St}^G(R[t]))$$

Here  $i_{\pm}$  denote the natural homomorphisms into  $\mathrm{St}^G(R[t^{\pm 1}])$ .

plan:k[t]

plan:k-intersect

- (2) (**done**, see Theorem 3). Show that  $K_2^G(k[t]) = K_2^G(k)$ .  
 (3) (**done**). Consequently,  $\mathrm{St}^G(k[t]) \rightarrow \mathrm{St}^G(k[t^{\pm 1}])$  is injective and  $\mathrm{St}^G(k[t]) \cap \mathrm{St}^G(k[t^{-1}]) = \mathrm{St}^G(k)$  inside  $\mathrm{St}^G(k[t^{\pm 1}])$ . Also,  $K_2^G(k) = K_2^G(k[t^{\pm 1}])$ .

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp

plan:Zglu

- (4) (**done** for split ACDE, see Theorem 2). Prove Quillen-Suslin lgp for  $K_2^G$ .  
 (5) (Zariski gluing) for any commutative ring  $A$  and any non-nilpotent  $f, g \in A$  such that  $A = fA + gA$ , the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for  $A = R[t]$  and  $f, g$  non-constant polynomials.)

plan:S-lemma

- (6) ( $S$ -lemma) Let  $A$  be a commutative ring,  $S$  a multiplicative subset of  $A$ . If

$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$

for some  $n \geq 1$ , then  $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$  as well.

This should be easy; see [Abe83, Lemma 3.6].

plan:Nglu

- (7) (Nisnevich gluing) Assume that  $B$  is a subring of a commutative ring  $A$ , and let  $h \in B$  be a non-nilpotent element. Denote by  $F_h : A \rightarrow A_h$  the localization homomorphism.

(i) If  $Ah + B = A$ , i.e. the natural map  $B \rightarrow A/Ah$  is surjective, then for any  $x \in \mathrm{St}^G(A_h)$  there exist  $y \in \mathrm{St}^G(A)$  and  $z \in \mathrm{St}^G(B_h)$  such that  $x = F_h(y)z$ .

(ii) If moreover  $Ah \cap B = Bh$ , i.e.  $B/Bh \rightarrow A/Ah$  is an isomorphism, and  $h$  is not a zero divisor in  $A$ , then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1, g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [Sta14, Lemma 3.4].

plan:P1

- (8) ( $\mathbb{P}^1$ -gluing) Let  $A$  be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

- (a) Prove Proposition 6.1. This is hard. Have no idea how this can be proved at the moment.  
 (b) Prove  $\mathbb{P}^1$ -gluing using all the above facts.

`plan:[]f`

- (9) (Main corollary of  $\mathbb{P}^1$ -gluing) Let  $A$  be any commutative ring, and let  $f \in A[t]$  be a monic polynomial. Show that  $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$  is injective.

The proof uses (5) and (8).

`plan:k(t)`

- (10) Prove that  $K_2^G(k(t)) = K_2^G(k)$ . This may be a bit tricky; I will think if we can get rid of it. I don't think Tullenbaev uses it. Unsure whether this is true (?). This is true for  $SK_1$  but in view of Milnor's theorem can not hold for  $K_1$  and  $K_2$ .

`plan:k[tn]`

- (11) Prove that

$$K_2^G(k[t_1, \dots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of  $K_1^G$ , this uses (9), (2), and (10). There may be other ways.

`plan:final`

- (12) Final result: let  $R$  be a regular ring containing a field  $k$ . Then  $K_2^G(R[t]) = K_2^G(R)$ . This uses (4), (11) and (7).

## 2. STEINBERG GROUPS OF CHEVALLEY GROUPS: PRELIMINARIES

**2.1. Definition and basic properties.** Let  $G$  be a split simple Chevalley groups with a root system  $\Phi$  of rank  $\geq 2$ . Recall that the *Steinberg group*  $\text{St}^G(R)$  (also denoted  $\text{St}(\Phi, R)$ ) is defined by means of generators  $\mathcal{X}_{\Phi, R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$  and the set of relations  $\mathcal{R}_{\Phi, R}$  defined as follows:

`{rel:add}`

$$(2.1) \quad x_\alpha(s)x_\alpha(t) = x_\alpha(s+t),$$

`{rel:CCF}`

$$(2.2) \quad [x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta} (N_{\alpha\beta ij} s^i t^j), \quad \alpha \neq -\beta, \quad N_{\alpha\beta ij} \in \mathbb{Z}.$$

The indices  $i, j$  appearing in the right-hand side of the above relation range over all positive natural numbers such that  $i\alpha + j\beta \in \Phi$ . The structure constants  $N_{\alpha\beta ij} = \pm 1, 2, 3$  appearing in (2.2) depend only on  $\Phi$  and can be computed precisely.

Recall that for  $\alpha \in \Phi$ ,  $\varepsilon \in R^*$  the semisimple root elements  $h_\alpha(\varepsilon)$  are defined as  $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$ . Denote by  $W(\Phi, R)$  the subgroup of  $\text{St}(\Phi, R)$  generated by all elements  $w_\alpha(\varepsilon)$ ,  $\varepsilon \in R^*$ .

**2.2. Tullenbaev's lifting property.** Let  $I \trianglelefteq R$  be an ideal of a commutative ring. For a nonnilpotent element  $a \in R$  denote by  $\lambda_a: R \rightarrow R_a$  the morphism of principal localization at  $a$ . Consider the following commutative square.

`{msq}`

(2.3)

$$\begin{array}{ccc} R & \xrightarrow{\lambda_a} & R_a \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{\overline{\lambda_a}} & R_a/I_a \end{array}$$

Notice that (2.3) is a pull-back square if and only if  $\lambda_a$  induces an isomorphism of  $I$  and  $I_a$ . Such squares are usually called *Milnor squares* in the literature, see [Wei13, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tullenbaev (see [Tul83, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

**def:tlp**

**Definition 2.1.** We say that the Steinberg group functor  $\mathrm{St}^G$  satisfies *Tulenbaev's lifting property* if for every pull-back square (2.3) the following lifting problem has a solution.

$$\begin{array}{ccc} \mathrm{St}^G(R, I) & \xrightarrow{\mu} & \mathrm{St}^G(R) \\ \downarrow & \nearrow & \downarrow \lambda_a^* \\ \mathrm{St}^G(R_a, I) & \xrightarrow{\mu} & \mathrm{St}^G(R_a) \end{array}$$

**2.3. The action of torus.** Throughout this subsection  $G = G_{\mathrm{ad}}$  denotes a split simple Chevalley group of adjoint type with the root system  $\Phi$  of rank  $\geq 2$ . Denote by  $T = T_{\mathrm{ad}}$  the torus of  $G$  and by  $T(R)$  its group of  $R$ -points.

We identify the root lattice  $X^*(T) = \mathrm{Hom}(T, \mathbf{G}_m)$  with the lattice  $\mathbb{Z}\Phi$  in the obvious way. In particular, for  $\alpha \in \Phi$  we denote by  $\alpha_R$  the corresponding map  $T(R) \rightarrow R^*$  on  $R$ -points. An element  $h \in T(R)$  defines a permutation of the set  $\mathcal{X}_{\Phi, R}$  of generators of  $\mathrm{St}^G(R)$  as follows:

$$(2.4) \quad h \cdot x_\alpha(\xi) = x_\alpha(\alpha_R(h) \cdot \xi).$$

Notice that  $h$  preserves the defining relations  $\mathcal{R}_{\Phi, R}$  of the Steinberg group (and thus, determines a permutation of  $\mathcal{R}_{\Phi, R}$ ). Indeed, the assertion is immediate for relation (2.1). Verification of the fact that  $h$  preserves (2.2) is a routine computation which should use the fact that for  $\alpha, \beta \in \mathbb{Z}\Phi$  one has  $(\alpha + \beta)_R(h) = \alpha_R(h) \cdot \beta_R(h)$ .

### 3. DECOMPOSITION THEOREMS FOR $\mathrm{St}^G(A[t^{\pm 1}])$ AND $\mathrm{St}^G(A((t)))$ .

**Lemma 3.1.** *Let  $(R, m)$  be a local ring, and let  $G$  be a simply connected simple group over  $R$  of isotropic rank  $\geq 2$ . Let  $i_+ : \mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}])$  and  $i_- : \mathrm{St}^G(R[t^{-1}]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}])$  be the natural homomorphisms. Then*

$$i_+(\mathrm{St}^G(m \cdot R[t])^{\mathrm{St}^G(R[t])})i_-(\mathrm{St}^G(R[t^{-1}])) = i_-(\mathrm{St}^G(R[t^{-1}]))i_+(\mathrm{St}^G(m \cdot R[t])^{\mathrm{St}^G(R[t])})$$

inside  $\mathrm{St}^G(R[t^{\pm 1}])$ .

*Proof.* This is proved exactly as [Sta14, Lemma 5.12]. □

**thm:3t**

**Theorem 1.** *Let  $R$  be a local ring, and let  $G$  be a simply connected simple group over  $R$  of isotropic rank  $\geq 2$ . Let  $i_+ : \mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}])$  and  $i_- : \mathrm{St}^G(R[t^{-1}]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}])$  be the natural homomorphisms. Then*

$$\mathrm{St}^G(R[t^{\pm 1}]) = i_+(\mathrm{St}^G(R[t]))i_-(\mathrm{St}^G(R[t^{-1}]))i_+(\mathrm{St}^G(R[t])).$$

*Proof.* This is proved exactly as [Sta14, Theorem 5.1]. □

### 4. QUILLEN-SUSLIN LGP, ZARISKI GLUING, NISNEVICH GLUING, $S$ -LEMMA

**thm:lg-k2**

**Theorem 2.** *Assume that the Steinberg group functor  $\mathrm{St}^G$  satisfies Tulenbaev's lifting property (cf. Definition 2.1). Then an element  $g \in \mathrm{St}^G(R[t], tR[t])$  is trivial if and only if its image in  $\mathrm{St}^G(R_M[t], tR_M[t])$  is trivial for all maximal ideals  $M \trianglelefteq R$ .*

*Proof.* Follows by the same argument as [Tul83, Theorem 2.1] or [Sin16, Theorem 2] □

## 5. THE CASE OF $K_2^G(k[t])$ AND SOME COROLLARIES

Throughout this section  $G = G(\Phi, -)$  denotes the (split) simply-connected Chevalley—Demazure group scheme of type  $\Phi$ .

**thm:k[t]**

**Theorem 3.** *Let  $k$  be a field. Assuming that  $\Phi$  is irreducible of rank at least 2 the following facts are true.*

**satz1**

(i) *For  $A = k, k[t]$  the subgroup  $K_2^G(A) \leq \text{St}^G(A)$  is generated by elements of the form*

$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}, \quad u, v \in k^*.$$

(ii) *As a consequence, the canonical map  $K_2^G(k) \hookrightarrow K_2^G(k[t])$  is an isomorphism.*

*Proof.* See [Reh75, Satz 1] and the corollary after it.  $\square$

**cor:k[t]inj**

**Corollary 5.1.** *Let  $G, k$  be as in Theorem 3. Then  $\text{St}^G(k[t]) \rightarrow \text{St}^G(k[t^{\pm 1}])$  is injective and  $\text{St}^G(k[t]) \cap \text{St}^G(k[t^{-1}]) = \text{St}^G(k)$  inside  $\text{St}^G(k[t^{\pm 1}])$ .*

*Proof.* Clearly,  $g \in \ker(\text{St}^G(k[t]) \rightarrow \text{St}^G(k[t^{\pm 1}]))$  implies  $g \in K_2^G(k[t])$ . Since  $K_2^G(k[t]) = K_2^G(k)$ , and there is a section  $K_2^G(k[t^{\pm 1}]) \rightarrow K_2^G(k)$ , the map is injective. Second claim: take  $g \in \text{St}^G(k[t]) \cap \text{St}^G(k[t^{-1}])$ . Then the image  $\phi(g)$  belongs to  $E(k) = E(k[t]) \cap E(k[t^{-1}])$ , and after adjusting  $g$  by an element of  $\text{St}^G(k)$ , we can assume that  $g \in K_2^G(k[t]) \cap K_2^G(k[t^{-1}])$ . Hence  $g \in K_2^G(k) \subseteq \text{St}^G(k)$ .  $\square$

**Corollary 5.2.** *Let  $G, k$  be as in Theorem 3. Then  $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$ .*

*Proof.* We use Theorem 1. Take  $g \in K_2^G(k[t^{\pm 1}])$ , then  $g = x_1 y x_2$ ,  $x_i \in \text{St}^G(k[t])$ ,  $y \in \text{St}^G(k[t^{-1}])$ . Since  $E(k[t]) \cap E(k[t^{-1}]) = E(k)$ , we have  $y \in \text{St}^G(k) K_2^G(k[t^{-1}]) = \text{St}^G(k)$  and  $x_1 x_2 \in \text{St}^G(k)$ . That is,  $g \in K_2^G(k)$ .  $\square$

Our next result is a Steinberg level analogue of the well-known Bruhat decomposition. For a direct proof of this statement in the linear case see [HO89, § 2.3A].

**cor:bruhat**

**Corollary 5.3.** *For  $\Phi$  of rank  $\geq 2$  and a field  $k$  one has*

$$\text{St}^G(k) = U(\Phi^+, k) \cdot W(\Phi, k) \cdot U(\Phi^+, k).$$

*Proof.* It is classically known that  $K_2(\Phi, k)$  is generated by symbols  $h_\alpha(u)$  lying in  $W(\Phi, k)$  (see e.g. Theorem 3.(i)). Thus, the required assertion follows from the classical Bruhat decomposition (see e.g. [SFW67, Theorem 4]).  $\square$

**cor:bllocal**

**Corollary 5.4.** *For  $\Phi$  of rank  $\geq 2$  and a local ring  $(A, m, k)$  one has*

$$\text{St}^G(A) = U(\Phi^+, A) \cdot W(\Phi, A) \cdot U(\Phi^+, A) \cdot \text{Im}(\text{St}^G(A, m) \rightarrow \text{St}^G(A)).$$

*Proof.* Denote the map  $A \rightarrow k$  by  $\pi$ . By ??? the first factor of the decomposition coincides with  $\text{Ker}(\pi_*)$  whereas by Corollary 5.3 the product of the last three factors is mapped epimorphically onto  $\text{St}^G(k)$ .  $\square$

## 6. $\mathbb{P}^1$ -GLUING

**6.1. Preliminaries.** Let  $A$  denote arbitrary commutative local ring with the maximal ideal  $m$  and the residue field  $k$ . Throughout this section we will employ the following notation:

- $R$  denotes the Laurent polynomial ring  $A[t, t^{-1}]$ ;

- $B^+$  (resp.  $B^-$ ) denotes the subring  $A[t] + m[t^{-1}]$  (resp.  $m[t] + A[t^{-1}]$ ) of  $R$  consisting of Laurent polynomials  $f(t, t^{-1})$  whose coefficients of terms of negative (resp. positive) degree belong to  $m$ ;
- $I$  denotes the ideal  $m[t, t^{-1}]$  of  $R$  (which can be also considered as an ideal of both  $B^+$  and  $B^-$ ).

Consider the following commutative diagram of groups.

{diag:cs}

(6.1)

$$\begin{array}{ccccccc}
 C_{+0} & \hookrightarrow & \mathrm{St}^G(B^+, I) & \xrightarrow{\mu^{+0}} & \mathrm{St}^G(B^+) & \xrightarrow{p^{+0}} \twoheadrightarrow & \mathrm{St}^G(k[t]) \\
 \downarrow k^+ & & \downarrow j^+ & \nearrow \varphi & \downarrow i^+ & & \downarrow \bar{i}^+ \\
 C_{\pm} & \hookrightarrow & \mathrm{St}^G(R, I) & \xrightarrow{\mu^{\pm}} & \mathrm{St}^G(R) & \xrightarrow{p^{\pm}} \twoheadrightarrow & \mathrm{St}^G(k[t, t^{-1}])
 \end{array}$$

Notice that by Corollary 5.1 the map  $\bar{i}^+$  in the right-hand side of the above diagram is injective. Invoking Tullenbaev's property 2.1 we also find a lifting map in the central square of the diagram.

The main result of this subsection is the following proposition which is analogous to [Tul83, Proposition 4.1].

prop:kersurj

**Proposition 6.1.** *The map  $k^+$  is surjective.*

*Proof.* Compare with [Tul83, Proposition 4.1]. □

Now using a simple diagram chasing argument we derive the main result of this subsection.

cor:tulinj

**Corollary 6.2.** *The map  $i^+$  is injective.*

*Proof.* Let  $g \in \mathrm{St}^G(B^+)$  be an element of  $\mathrm{Ker}(i^+)$ . Since  $g$  also lies in  $\mathrm{Ker}(p^{+0})$  it comes from some  $\tilde{g} \in \mathrm{St}^G(B^+, I)$  via  $\mu^{+0}$ . Now  $j^+(\tilde{g})$  lies in  $C_{\pm}$ , hence, by Proposition 6.1 it comes from some  $\hat{g} \in C_{+0}$  via  $k^+$ . Finally,  $g = \varphi(j^+(\tilde{g})) = \varphi(k^+(\hat{g})) = \mu^{+0}(\hat{g}) = 1$ , as claimed. □

**6.2. Main result.** The following lemma is an analog of [Tul83, Proposition 4.3 (a)].

**Lemma 6.3.** *Let  $R$  be a local ring,  $k = R/m$ , and let  $k, G$  be as in Theorem 3. The natural homomorphism  $\mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}])$  is injective.*

*Proof.* Let  $I$  be the maximal ideal of  $R$ ,  $l = R/I$ , and consider the natural maps  $\rho : \mathrm{St}^G(R[t, t^{-1}]) \rightarrow \mathrm{St}^G(l[t, t^{-1}])$ ,  $\rho_+ : \mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(l[t])$ ,  $\rho_- : \mathrm{St}^G(R[t^{-1}]) \rightarrow \mathrm{St}^G(l[t^{-1}])$ . Take  $x \in \ker(\mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}]))$ . By the field case Corollary 5.1 one has  $\rho_+(x) = 1$ , hence  $x \in \mathrm{St}^G(I \cdot R[t])^{\mathrm{St}^G(R[t])}$ .

???????

□

**Lemma 6.4.** *Let  $G, k$  be as in Theorem 3. Let  $(R, m)$  be a local ring such that  $R/m = k$ . Then*

$$\mathrm{St}^G(R[t]) \cap \mathrm{St}^G(R[t^{-1}]) = \mathrm{St}^G(R)$$

*inside  $\mathrm{St}^G(R[t^{\pm 1}])$ .*

*Proof.* ?????? □

**Theorem 4.** *Let  $A$  be any commutative ring. Then the sequence of pointed sets*

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

*is exact.*

*Proof.* Follows from the above lemmas. □

**Corollary 6.5.** *Let  $A$  be any commutative ring, and let  $f \in A[t]$  be a monic polynomial. Show that  $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$  is injective.*

*Proof.* ??? □

# Appendices

## A. LINEAR STEINBERG GROUP IN RANK 3

The main goal of this subsection is to show that Tulenbaev's [Tul83, Lemma 2.3] remains valid for the linear Steinberg group of rank  $\geq 3$ . In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [Sin16, Definitions 3.3 and 3.7]).

**Definition A.1.** The relative Steinberg group  $\text{St}^*(n, R, I)$  is the group defined by the following two families generators and four families of relations.

- Generators:

- (1)  $X^1(u, v)$ , where  $u \in E(n, R) \cdot e_1$ ,  $v \in I^n$  such that  $v^t \cdot u = 0$ ;
- (2)  $X^2(u, v)$ , where  $u \in I^n$ ,  $v \in E(n, R) \cdot e_1$  such that  $v^t \cdot u = 0$ .

Notice that  $\phi$  maps both  $X^1(u, v)$  and  $X^2(u, v)$  to  $T(u, v) = e + u \cdot v^t \in E(n, R, I)$ .

- Relations:

- (1)  $X^1(u, v) \cdot X^1(u, w) = X^1(u, v + w)$ ,  $u \in E(n, R) \cdot e_1$ ,  $v, w \in I^n$ ;
- (2)  $X^2(u, v) \cdot X^2(w, v) = X^2(u + w, v)$ ,  $u, w \in I^n$ ,  $v \in E(n, R) \cdot e_1$ ;
- (3)  $X^{\sigma(u^2, v^2)} X^{\tau}(u^1, v^1) = X^{\tau}(T(u^2, v^2) \cdot u^1, T(v^2, u^2)^{-1} \cdot v^1)$ ,  $\sigma, \tau = 1, 2$ ;
- (4)  $X^1(g \cdot e_1, g^* \cdot be_2) = X^2(g \cdot be_1, g^* \cdot e_2)$  where  $b \in I$  and  $g^* = g^{t^{-1}}$  denotes the contragradient matrix.

**Lemma A.2.** *The groups  $\text{St}^*(n, R, I)$  and  $\text{St}(n, R, I)$  are isomorphic.*

*Proof.* **TODO:** □

The next step of the proof is to construct certain elements in  $\text{St}(n, R)$  similar to Tulenbaev's elements  $X_{u,v}(a)$  see [Tul83, § 1].

Let  $v \in R^n$  be a column. Denote by  $O(v)$  the submodule of  $R^n$  consisting of all columns  $w$  such that  $w^t \cdot v = 0$ . A column  $w \in R^n$  is called *v-decomposable* if it can be presented as a finite sum  $w = \sum_{i=1}^p w^i$  such that each  $w^i$  has at least two zero entries and  $v^t \cdot w^i = 0$ . Denote by  $D(v)$  the submodule of  $O(v)$  consisting of all *v-decomposable* columns. For a column  $v \in R^n$  denote by  $I(v)$  the ideal of  $R$  spanned by its entries  $v_1, \dots, v_n$ .

Let  $u, v, w \in R^n$  be columns such that  $w^t v = 0$ . It is easy to check (cf. [Kal77, Lemma 3.2]) that

$$(uv) \cdot w = \sum_{i < j} w_{ij}, \text{ where } w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n.$$



The above decomposition is called the *canonical* decomposition of  $(uv) \cdot w$ . In particular, this shows that the column  $a \cdot w$  is always  $v$ -decomposable for  $a \in I(v)$ ,  $w \in O(v)$ , i.e.  $I(v) \cdot O(v) \subseteq D(v)$ . It is also straightforward to check that  $D(v) \subseteq D(bv)$ ,  $b \cdot D(v) \subseteq D(v)$  for  $b \in R$ .

Denote by  $B^1$  the subset of  $R^n \times R^n \times R$  consisting of triples  $(u, v, a)$  such that  $v^t \cdot u = 0$ ,  $v \in D(u)$ ,  $a \in I(u)$ . Denote by  $B^2$  the set consisting of triples  $(v, u, a)$  such that  $(u, v, a) \in B^1$ .

**lem:Zfacts**

**Lemma A.3.** *Assume that  $n \geq 4$ . One can define two families of elements  $Z^\tau(u, v, a)$ ,  $\tau = 1, 2$  of the group  $\text{St}(n, R)$  parametrized by  $(u, v, a) \in B^\tau$  satisfying the following properties:*

- (1)  $\phi(Z^\tau(u, v, a)) = e + uav^t \in E(n, R)$ ,  $(u, v, a) \in B^\tau$ ;
- (2)  $Z^1(u, v + w, a) = Z^1(u, v, a) \cdot Z^1(u, w, a)$ ;
- (3)  $Z^2(v + w, u, a) = Z^2(v, u, a) \cdot Z^2(w, u, a)$ ;
- (4) for  $\tau = 1, 2$  and  $b \in R$  if  $(u, vb, a), (ub, v, a) \in B^\tau$  then one has

$$Z^\tau(u, vb, a) = Z^\tau(u, v, ab) = Z^\tau(ub, v, a);$$

- (5)  ${}^gZ^\tau(u, v, a) = Z^\tau(\phi(g) \cdot u, \phi(g)^* \cdot v, a)$ ,  $\tau = 1, 2$ ,  $g \in \text{St}^G(n, R)$ .

*Proof.* One constructs the elements  $Z^1(u, v, a)$  in exactly the same way as Tulenbaev constructs his elements  $X_{u,v}(a)$  (see definitions preceding [Tul83, Lemma 1.2]). Indeed, set

$$(A.1) \quad Z^1(v, w, a) = \prod_{k=1}^p X(v, a \cdot w^k), \quad Z^2(w, v, a) = \prod_{k=1}^p X(a \cdot w^k, v).$$

where  $X(u, v)$  denotes the elements defined by Tulenbaev before [Tul83, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case  $n = 4$ ) can be proved by the same token as in [Tul83, Lemma 1.3].  $\square$

For the rest of this section  $a$  denotes a nonnilpotent element of  $R$  and  $\lambda_a: R \rightarrow R_a$  is the morphism of principal localization at  $a$ .

**lem:rk3rels**

**Lemma A.4.** *For any  $g \in E(n, R_a)$  there exist  $u, v \in R^n$  and sufficiently large natural numbers  $k, m$  such that the following facts hold:*

- (1)  $\lambda_a(u) = g \cdot a^k e_1$ ,  $\lambda_a(v) = g^* \cdot a^k e_2$  and  $u^t \cdot v = 0$ ;
- (2)  $(u, v, a^m) \in B^1 \cap B^2$ ;
- (3) for  $b \in R$  divisible by some sufficiently large power of  $a$  one has

$$Z^1(u, b \cdot v, a^m) = Z^2(b \cdot u, v, a^m).$$

*Proof.* It is straightforward to choose  $u$  and  $v$  satisfying the first requirement of the lemma. We can even choose  $u, v$  in such a way that  $u \in D(v)$  and  $v \in D(u)$ . Indeed, notice that  $I(u) = a^{k_1}$ ,  $I(v) = a^{k_2}$  for some natural  $k_1, k_2$  hence for  $u' = a^{k_2} \cdot u$  and  $v' = a^{k_1} \cdot v$  one has

$$u' \in I(v) \cdot O(v) \subseteq D(v) \subseteq D(v'), \quad v' \in I(u) \cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns  $x, y \in R^n$  and a large natural  $p$  in such a way that vectors  $u, v, x, y$  additionally satisfy the following properties

$$\begin{aligned} \lambda_a(x) &= g^* \cdot a^k e_3, \quad \lambda_a(y) = g \cdot a^k e_3, \quad y^t \cdot x = a^p \in R, \\ u^t \cdot x &= 0, \quad u^t \cdot v = 0, \quad y^t \cdot v = 0, \end{aligned}$$



$$(u, x, a^m) \in B^1, (y, v, a^m) \in B^2.$$

Now direct computation using Lemma A.3 shows that

$$\begin{aligned} Z^2(a^{m+p}b \cdot u, v, a^m) &= Z^2(b \cdot (e + a^m \cdot ux^t)y, (e - a^m \cdot xu^t)v, a^m) \cdot Z^2(-by, v, a^m) = \\ &= [Z^1(u, x, a^m), Z^2(b \cdot y, v, a^m)] = \\ &= Z^1(u, x, a^m) \cdot Z^1((e + a^m b \cdot yv^t)u, -(e - a^m b \cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b \cdot v, a^m), \quad \square \end{aligned}$$

hence the third assertion of the lemma follows.

cor:tulmap

**Corollary A.5.** *[Tulenbaev’s lemma] For  $n \geq 4$  there is a map  $T_n$  so that the following diagram commutes.*

*Proof.* Follows from Lemma A.4 by the same token as in [Tul83, Lemma 2.3].  $\square$

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