$\mathbb{A}^1$-INVARIANCE FOR UNSTABLE K_2

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1. Aim of the paper

The aim of this text is to prove that the non-stable K_2-functors K_2^G, where G is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings R containing a field k, that is,

{eq:A1-main}
$$K_2^G(R[t]) = K_2^G(R). \tag{1.1}$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K-functor associated to G. This functor originates from [5]. The above equality by e.g. [2, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of G in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of K_2, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \ldots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the SL_n case [14] and Stavrova's proof for K_1^G [10]. Tulenbaev [14] uses stabilization of the K_2-functor, and the good properties of the limit=algebraic K-theory. In [10] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [14, p. 140], or, respectively, [10, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [14, Theorem 5.1] or [10, Theorem 1.1]).

1.1. **Plan of the proof.**

plan:3t
(1) (**done**, see **??**). Let R be a local ring. Show that

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t]))$$

Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

plan:k[t]
(2) (**done**, see Theorem 3). Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect
(3) (**done**). Consequently, $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp
(4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen-Suslin lgp for K_2^G.

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plan:Zglu (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2) \mapsto g_1 g_2{}^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma (6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A. If

$$K_2^G(A[X_1, \ldots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \ldots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St}^{\mathrm{G}}(A_h)$ there exist $y \in \mathrm{St}^{\mathrm{G}}(A)$ and $z \in \mathrm{St}^{\mathrm{G}}(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and h is not a zero divisor in A, then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [10, Lemma 3.4].

plan:P1 (8) ($\mathbb{P}^1$-gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2) \mapsto g_1 g_2{}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

(a) Prove Proposition 3.13. This is hard. Have no idea how this can be proved at the moment.

(b) Prove $\mathbb{P}^1$-glueing using all the above facts.

plan:[]f (9) (Main corollary of $\mathbb{P}^1$-gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.

The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2.

plan:k[tn] (11) Prove that

$$K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let R be a regular ring containing a field k. Then $K_2^G(R[t]) = K_2^G(R)$.

This uses (4), (11) and (7).

2. Steinberg groups: preliminaries

2.1. **Definition and basic properties.** In what follows Φ denotes a reduced irreducible root system and $\Pi \subseteq \Phi$ denotes its basis (i.e. the set of simple roots). Denote by $\widetilde{\alpha}$, Φ^+ and Φ^-, respectively, the maximal root of Φ and the subsets of positive and negative roots of Φ. The Dynkin diagram and the extended Dynkin diagram of Φ corresponding to Π will be denoted by $D(\Phi)$, $\widetilde{D}(\Phi)$, respectively.

A proper closed root subset $S \subseteq \Phi$ is called *parabolic* (resp. *reductive*, resp. *special*) if $\Phi = S \cup -S$ (resp. $S = -S$, resp. $S \cap -S = \varnothing$). Any parabolic subset $S \subseteq \Phi$ can be decomposed into the disjoint union of its reductive and special part, i.e. $S = \Sigma_S \sqcup \Delta_S$, where $\Sigma_S \cap (-\Sigma_S) = \varnothing$, $\Delta_S = -\Delta_S$.

Denote by $m_\beta(\alpha)$ the coefficient of β in the expansion of α in Π, i. e. $\alpha = \sum_{\beta\in\Pi} m_\beta(\alpha)\beta$. For $\beta \in \Pi$ denote by Δ_β the subsystem of Φ spanned by all simple roots except β and by Σ_β the set consisting of roots $\alpha \in \Phi$ such that $m_\beta(\alpha) > 0$.

We denote by (α, β) the scalar product of roots and by $\langle \beta, \alpha\rangle$ the integer $2(\beta, \alpha)/(\alpha, \alpha)$. The Weyl group $W(\Phi)$ is a subgroup of isometries of Φ generated by all reflections σ_α, where $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha\rangle \cdot \alpha$. For a subset of roots $S \subseteq \Phi$ we denote by $\langle S\rangle$ the root subsystem spanned by S, i.e. the minimal subset of Φ containing S and invariant under the action of reflections σ_α, $\alpha \in S$.

Let $G = G(\Phi, -)$ be a (split) simply connected simple Chevalley–Demazure group scheme over R with a root system Φ of rank ≥ 2. Recall that the *Steinberg group* $\mathrm{St}^{\mathrm{G}}(R)$ (also denoted $\mathrm{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi,R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi,R}$ defined as follows:

{rel:add} $$x_\alpha(s)x_\alpha(t) = x_\alpha(s + t),\ \alpha \in \Phi,\ s, t \in R; \tag{2.1}$$

{rel:CCF} $$[x_\alpha(s), x_\beta(t)] = \prod_{i,j\in\mathbb{N}} x_{i\alpha+j\beta}\left(N_{\alpha\beta ij}\, s^i t^j\right), \quad \alpha, \beta \in \Phi,\ \alpha \neq \pm\beta,\ s, t \in R. \tag{2.2}$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$. Denote by $\mathrm{W}(\Phi, R)$ the subgroup of $\mathrm{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$, $\alpha \in \Phi$, and by $\mathrm{H}(\Phi, R)$ the subgroup generated by all elements $h_\alpha(\varepsilon)$, $\varepsilon \in R^*$, $\alpha \in \Phi$.

Following [12], we set

$$\{u, v\}_\alpha = h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}, \qquad u, v \in R^*,\ \alpha \in \Phi,$$

and call these elements *symbols in* $\mathrm{St}(\Phi, R)$. For any ideal I of R, we set

$$\mathrm{Sym}(\Phi, R, I) = \langle \{u, v\}_\alpha,\ u \in R^*,\ v \in (1 + I)^*,\ \alpha \in \Phi\rangle \leq \mathrm{St}(\Phi, R).$$

The group $\mathrm{Sym}(\Phi, R, R)$ is denoted by $\mathrm{Sym}(\Phi, R)$. Clearly, one has $\{u, v\}_\alpha \in K_2(\Phi, R)$ for any $u, v \in R^*$, $\alpha \in \Phi$. By [12, Prop. 1.3 (c)] the group $\mathrm{Sym}(\Phi, R, I)$ is generated by all symbols $\{u, v\}_\alpha$, $u \in R^*$, $v \in (1 + I)^*$, for any fixed long root $\alpha \in \Phi$.

2.2. **Non-standard generation of Steinberg groups.**

lem:parab-gen **Lemma 2.1.** *Let R be any commutative ring. Let Φ be an irreducible root system of rank ≥ 2, let Π be a system of simple roots in Φ, and let $J \subseteq \Pi$ be such that $|J| \geq 2$. Set*

$$\alpha_J = \sum_{\beta \in J} m_\beta(\alpha)\beta \text{ for any } \alpha \in \Phi,$$

and

$$\Sigma_J = \{\alpha \in \Phi \mid m_\beta(\alpha) > 0 \text{ for at least one } \beta \in J\}.$$

Let H be the group defined by the generators $x_\alpha(u)$, $\alpha \in \Sigma_J \cup (-\Sigma_J)$, $u \in R$, and the relations (2.1) and (2.2) ranging only over $\alpha \in \Sigma_J \cup (-\Sigma_J)$, and $\beta \in \Sigma_J \cup (-\Sigma_J)$ such that $m\alpha_J \neq -k\beta_J$ for all $m, k \in \mathbb{N}$. Then the natural homomorphism $H \to \mathrm{St}(\Phi, R)$ is surjective and has central kernel. In particular, if $\mathrm{St}(\Phi, R)$ is centrally closed, then $H \cong \mathrm{St}(\Phi, R)$.

Proof. The group scheme $G = G(\Phi, -)$ over R contais two opposite parabolic R-subgroups $P^\pm$ such that $\pm\Sigma_J$ are the sets of roots corresponding to the unipotent radicals of $P^\pm$. One can show that $H = \mathrm{St}_{P^+}(R)$ and $\mathrm{St}(\Phi, R) = \mathrm{St}_B(R)$ in the sense of [11]. By [11, Lemma 8] the natural homomorphism $H \to \mathrm{St}(\Phi, R)$ is surjective. By [11, Lemma 14] its kernel is central. □

2.3. **Relative Steinberg groups.** Denote by $D(R, I)$ the double of the ring R relative to an ideal I, i.e. the fibered product of rings $R \times_{R/I} R$ with the natural projections $p_1, p_2 \colon D(R, I) \to R$ defined by $p_i(\xi_1, \xi_2) = \xi_i$, $i = 1, 2$. Denote by G_i the kernel of the map $p_i^* \colon \mathrm{St}^{\mathrm{G}}(D(R, I)) \to \mathrm{St}^{\mathrm{G}}(R)$. We define the relative Steinberg group $\mathrm{St}^{\mathrm{G}}(R, I)$ as G_1/C, where $C = [G_1, G_2]$. Clearly, there is an exact sequence.

{eq:suite}
$$1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \mathrm{St}^{\mathrm{G}}(R, I) \xrightarrow{\overline{p_2^*}} \mathrm{St}^{\mathrm{G}}(R) \xrightarrow{\pi^*} \mathrm{St}^{\mathrm{G}}(R/I) \longrightarrow 1 \qquad (2.3)$$

Lemma 2.2. *Assume that R and I are such that the canonical projection $R \to R/I$ splits. Then the following facts are true.*

item:st-inj (i) *The map $\mathrm{St}^{\mathrm{G}}(R, I) \to \mathrm{St}^{\mathrm{G}}(R)$ is an injection.*
item:st-semi (ii) *The group $\mathrm{St}^{\mathrm{G}}(R)$ is isomorphic to $\mathrm{St}^{\mathrm{G}}(R/I) \ltimes \mathrm{St}^{\mathrm{G}}(R, I)$.*

Proof. For the proof of the first assertion see [9, Lemma 8]. Since the group $(G_1 \cap G_2)/C$ vanishes, the sequence (2.3) turns into a split short exact sequence which implies the second assertion. □

lem:Zgen **Lemma 2.3.** *Let Σ be the special subset of some parabolic subset of roots $S \subseteq \Phi$. Then the relative Steinberg group $\mathrm{St}^{\mathrm{G}}(R, I)$ admits the following generating set:*

$$\mathcal{Z}(\Sigma, R, I) = \{x_\alpha(0, s) \cdot C \mid s \in I, \alpha \in \Phi\} \cup \{z_\alpha(s, \xi) \mid s \in I,\ \xi \in R,\ \alpha \in \Sigma\},$$

where $z_\alpha(s, \xi)$ denotes the element $x_\alpha(0, s)^{x_{-\alpha}(\xi, \xi)} \cdot C$.

Proof. See [9, Lemma 5]. □

2.4. **Tulenbaev's lifting property and its corollaries.** Throughout this section $I \trianglelefteq A$ is an ideal of arbitrary commutative ring A. For a nonnilpotent element $a \in A$ denote by $\lambda_a \colon A \to A_a$ the morphism of principal localization at a. Consider the following commutative square.

{msq} (2.4)
$$\begin{tikzcd} A \arrow[r, "\lambda_a"] \arrow[d, two heads] & A_a \arrow[d, two heads] \\ A/I \arrow[r, "\overline{\lambda_a}"] & A_a/I_a \end{tikzcd}$$

Notice that (2.4) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a. Such squares are usually called *Milnor squares* in the literature, see [16, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [14, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp **Definition 2.4.** We say that the Steinberg group functor St^{G} satisfies *Tulenbaev's lifting property* if for every pull-back square (2.4) the following lifting problem has a solution.

$$\begin{tikzcd} \mathrm{St}^{\mathrm{G}}(A, I) \arrow[r, "\mu"] \arrow[d] & \mathrm{St}^{\mathrm{G}}(A) \arrow[d, "\lambda_a^*"] \\ \mathrm{St}^{\mathrm{G}}(A_a, I) \arrow[r, "\mu"] \arrow[ur, dotted] & \mathrm{St}^{\mathrm{G}}(A_a) \end{tikzcd}$$

Theorem 1. *Assume that G satisfies Tulenbaev property (2.4) then the following facts are true for arbitrary commutative ring A:*

thm:dp *(i) A dilation principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. if $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is such that equality $\lambda_a^*(h) = 1$ holds in $\mathrm{St}^{\mathrm{G}}(\Phi, R_a[t])$ then for sufficiently large n one has*

$$ev^*_{\left[\frac{R[t]\to R[t]}{t\mapsto a^n\cdot t}\right]}(h) = 1.$$

thm:lg-k2 *(ii) A local-global principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. an element $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is trivial if and only if its image in $\mathrm{St}^{\mathrm{G}}(A_m[t], tA_m[t])$ is trivial for all maximal ideals $m \trianglelefteq A$.*

thm:centr *(iii) $K_2^G(A)$ is contained in the centre of $\mathrm{St}^{\mathrm{G}}(A)$.*

Proof. Follows by the same argument as [14, Theorem 2.1] or [9, Theorem 2] □

sec:sigma

2.5. **The automorphisms σ_i.** Our notation and conventions follows [15, § 4]. Let Φ be an irreducible root system with some fixed basis of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. We denote by $\Phi^\vee$ the *dual root system of* Φ consisting of vectors $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \Phi$. As usual, $P(\Phi^\vee)$ denotes the lattice spanned by the *fundamental weights* ϖ_i. Recall that ϖ_i are uniquely determined by relations $\langle \varpi_i, \alpha_j^\vee \rangle = (\varpi_i, \alpha_j) = \delta_{ij}$.

Notice that for $\varpi \in P(\Phi^\vee)$ and $\beta \in \mathbb{Z}\,\Phi$ one has $(\varpi, \beta) \in \mathbb{Z}$. Consequently, for $\varepsilon \in R^*$ and $\varpi \in P(\Phi^\vee)$ the identity $\chi_{\varpi,\varepsilon}(\beta) = \varepsilon^{(\varpi,\beta)}$ gives a well-defined character $\chi_{\varpi,\varepsilon} \in \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$.

Consider the action of $H = \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$ on the set of generators $\mathcal{X}_{\Phi,R}$ of the Steinberg group $\mathrm{St}^{\mathrm{G}}(R)$ defined by

$$\chi \cdot x_\alpha(\xi) = x_\alpha(\chi(\alpha) \cdot \xi),\ \chi \in H,\ \alpha \in \Phi,\ \xi \in R. \tag{2.5}$$

Since χ is a character, the above action preserves the set of Steinberg relations $\mathcal{R}_{\Phi,R}$ and, thus, gives a well-defined action of H on $\mathrm{St}^{\mathrm{G}}(R)$.

Example 2.5. The principal example which motivates the above construction is as follows. Let A be a ring, take $R = A[t, t^{-1}]$ to be the ring of Laurent polynomials over A and let $\alpha_i \in \Pi$ be some simple root. Since $t \in R^*$ we can consider the automorphisms σ_i^+ and σ_i^- of $\mathrm{St}(\Phi, R)$ given by $\sigma_i^+ = \chi_{\varpi_i,t}$, $\sigma_i^- = \chi_{\varpi_i,t^{-1}}$. It is easy to see that

eq:sigma_act}
$$\sigma_i^{\pm}(x_\alpha(\xi)) = x_\alpha(t^{\pm m_i(\alpha)} \cdot \xi), \tag{2.6}$$

where $m_k(\alpha)$ denotes the coefficient in the expansion of α in Π, i.e. $\alpha = \sum m_k(\alpha)\alpha_k$.

One of the key steps of our proof of Suslin lemma for K_2 is to define an analogue of σ_i for the group $\mathrm{St}^{\mathrm{G}}(A[t])$. Of course, we cannot expect such map to be automorphism or even be defined on the whole group $\mathrm{St}^{\mathrm{G}}(A[t])$. However, it turns out that for certain i is still possible to define certain subgroups of $\mathrm{St}^{\mathrm{G}}(A[t])$ and the maps modeling σ_i between them.

3. $\mathbb{P}^1$-GLUING

Throughout this section $G = G(\Phi, -)$ denotes a simply connected Chevalley–Demazure group scheme of type Φ.

def:p1g **Definition 3.1.** Let F be a group-valued functor from **CRings** to **Groups** and let A be a commutative ring. Consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "i_+"] \arrow[d, "i_-"'] & A[t] \arrow[d, "j_+"] \\
A[t^{-1}] \arrow[r, "j_-"] & A[t, t^{-1}]
\end{tikzcd}$$

We say that F satisfies the $\mathbb{P}^1$*-glueing property for A* if the following sequence of *pointed sets* is exact in the middle term:

$$F(A) \xhookrightarrow{\Delta_A^F} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\pm_A^F} F(A[t, t^{-1}]).$$

Here Δ_A^F denotes the (split injective) diagonal map and, by definition, $\pm_A^F$ maps (g^+, g^-) to $F(j_+)(g^+) \cdot F(j_-)(g^-)^{-1}$. Notice that a priori $\pm_A^F$ is only a morphism of pointed sets but if F takes values in abelian groups then $\pm_A^F$ is also a morphism of groups.

An equivalent way to formulate $\mathbb{P}^1$-glueing property is as follows: $F(j_+)$ and $F(j_-)$ are injective and the intersection of their images coincides with the image of $F(j_+ i_+) = F(j_- i_-)$.

The main result of this section is the following theorem which generalizes [14, Theorem 5.1] to Chevalley groups. Notice that a K_1-analogue of the result below has been established in a much greater generality by the second-named author (see [10, Theorem 1.1]).

thm:p1 **Theorem 2.** *Assume that G satisfies Tulenbaev lifting property 2.4. Then the Steinberg group functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for an arbitrary commutative ring A.*

Proof. Let (g^+, g^-) be an element of $\mathrm{St}^{\mathrm{G}}(A[t]) \times \mathrm{St}^{\mathrm{G}}(A[t^{-1}])$ such that the equality $g^+ = g^-$ holds in $\mathrm{St}^{\mathrm{G}}(A[t, t^{-1}])$.

Let m be a maximal ideal of A. By Proposition 3.14 below the functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for the local ring A_m hence $(\lambda^*_m(g_+), \lambda^*_m(g_-)) = \Delta_{A_m}(\lambda^*_m(g^+)(0))$ and in the groups $\mathrm{St}^{\mathrm{G}}(A_m[t])$ and $\mathrm{St}^{\mathrm{G}}(A_m[t^{-1}])$ we have the equalities:

$$\lambda^*_m(g^+ \cdot g^+(0)^{-1}) = \lambda^*_m(g^+) \cdot \lambda^*_m(g^+)(0)^{-1} = 1; \qquad \lambda^*_m(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for $\mathrm{St}^{\mathrm{G}}(-)$ (see Theorem 1.(ii)) these equalities hold globally and $(g^+, g^-) = \Delta_A(g^+(0))$, as claimed. □

rem:stk2 *Remark* 3.2. It is clear that if the functor St^{G} satisfies $\mathbb{P}^1$-glueing property for A then so does the functor K_2^G. The converse statement also holds, indeed, if $(g^+, g^-) \in \mathrm{Ker}\,(\pm^{\mathrm{St}}_A)$ then inside $E^G(A[t, t^{-1}])$ we have the equality:

$$\varphi(\mathrm{St}^{\mathrm{G}}(j_+)(g^+)) = \varphi(\mathrm{St}^{\mathrm{G}}(j_-)(g^-)) \in E^G(A[t]) \cap E^G(A[t^{-1}]) = E^G(A).$$

Consequently, we can find $g_0 \in \mathrm{St}^{\mathrm{G}}(A)$ so that $(g^+ g_0^{-1}, g^- g_0^{-1}) \in \mathrm{Ker}\,(\pm^{K_2}_A)$ and it remains to apply the $\mathbb{P}^1$-glueing property for K_2^G.

Corollary 3.3. *Let A be any commutative ring and $f \in A[t]$ be a monic polynomial. Then the map $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.*

Proof. □

3.1. **The field case.** Throughout this section k denotes an arbitrary field.

thm:k[t] **Theorem 3.** *Assume that $G = G(\Phi, -)$ and Φ is any irreducible root system of rank ≥ 2.*

satz1 *(i) The subgroup $K_2(\Phi, k[t]) \trianglelefteq \mathrm{St}^{\mathrm{G}}(\Phi, k[t])$ is generated by symbols $\{u, v\}_\alpha$, $u, v \in k^*, \alpha \in \Phi$.*

(ii) As a consequence, the canonical injection $K_2(\Phi, k) \hookrightarrow K_2(\Phi, k[t])$ is an isomorphism, and $K_2(k[t]) = K_2(k)$ is central in $\mathrm{St}(\Phi, k[t])$.

Proof. See [8, Satz 1] and the corollary after it. □

cor:k[t]inj **Corollary 3.4.** *Let G be as in Theorem 3. Then the functors St^{G}, K_2^G satisfy $\mathbb{P}^1$-glueing property for k.*

Proof. By Remark 3.2 it suffices to prove the assertion only for the functor K_2^G. By the previous theorem $K_2^G(i_+)$ and $K_2^G(i_-)$ are isomorphisms hence the morphisms $K_2^G(j_+)$ and $K_2^G(j_-)$ are split injective and $\mathrm{Im}\,(j_+ i_+) = \mathrm{Im}\,(j_- i_-) = \mathrm{Im}\,(j_+) = \mathrm{Im}\,(j_-)$. □

thm:k[t+-1] **Theorem 4.** *Let Φ be an irreducible root system of rank ≥ 2 and let α be a long root of Φ. Denote by H the subgroup of $K_2(\Phi, k[t^{\pm 1}])$ generated by symbols $\{t, u\}_\alpha$, $u \in k^*$.*

(i) For $\Phi \neq G_2$ one has

$$K_2(\Phi, k[t^{\pm 1}]) = K_2(\Phi, k) \oplus H.$$

In particular, $K_2(\Phi, k[t^{\pm 1}])$ is central in $\mathrm{St}(\Phi, k[t^{\pm 1}])$. Here $K_2(\Phi, k)$ is considered as a subgroup of $K_2(\Phi, k[t^{\pm 1}])$ via the natural injection.

(ii) The map $\{t, -\}_\alpha \colon k^* \to H$ is injective.

Proof. By [4, Korollar 4] the group $K_2(\Phi, k[t^{\pm 1}])$ is generated by $K_2(\Phi, k)$ and H. By [12, Prop. 1.1 (S1)] one has $\{1, u\}_\alpha = 1$ for any $\alpha \in \Phi$. Hence H is in the kernel of the natural projection $K_2(\Phi, k[t^{\pm 1}]) \to K_2(\Phi, k)$ sending t to 1, and $H \cap K_2(\Phi, k) = 1$. By [12, Prop. 1.3 (a)] symbols are central in $\mathrm{St}(\Phi, k[t^{\pm 1}])$, hence $K_2(\Phi, k[t^{\pm 1}]) = K_2(\Phi, k) \oplus H$.

Now consider the following commutative diagram.

$$\begin{tikzcd}
k^* \cong K_1(k) \arrow[r, "\{t,-\}_\alpha"] \arrow[rrd, hook, "\{t,-\}_\alpha"'] & K_2(\Phi, k[t^{\pm 1}]) \arrow[r] & K_2(\Phi, k(t)) \arrow[d] \\
& & K_2^M(k(t))
\end{tikzcd}$$

By Matsumoto's theorem $K_2(\Phi, k(t))$ is isomorphic to $K_2^M(k(t))$ if Φ is nonsymplectic, otherwise $K_2(\Phi, k(t)) \cong K_2^{MW}(k(t))$, see [6, Corollaire 5.11]. Thus, the vertical arrow is either an isomorphism or the canonical projection $K_2^{MW}(k(t)) \to K_2^M(k(t))$, see [13, § 6]. On the other hand, the diagonal arrow is injective by Milnor's theorem, see [7, § 2]. Thus the left top horizontal arrow is injective as well. □

Remark 3.5. Notice that in the symplectic case the composite map $\{t,-\}_\alpha \colon k^* \to K_2^{MW}(k(t))$ is *not* a morphism of groups because the Steinberg symbol $\{-,-\}_\alpha$ corresponding to the long root α is not bilinear. If we defined this map using a short root α, this map would be a morphism of groups, however, the diagonal map would equal $\{t^2,-\}$ and would not be injective.

3.2. **Tulenbaev's section 3.** For the rest of this section A denotes an arbitrary commutative local ring with the maximal ideal m and the residue field k. We denote by π the canonical projection $A \to k$. Throughout this section we will employ the following notation:

- R denotes the Laurent polynomial ring $A[t, t^{-1}]$;
- B denotes the subring $A[t] + m[t^{-1}]$ of R consisting of Laurent polynomials $f(t, t^{-1})$ whose coefficients of terms of negative degree belong to m;
- I denotes the ideal $m[t, t^{-1}]$ of R (which can be also considered as an ideal of B).

Note that since A is local, then $K_2(\Phi, A) = \mathrm{Sym}(\Phi, A)$ by [12, Theorem 2.13]. In particular, $K_2(\Phi, A)$ is central in $\mathrm{St}(\Phi, A)$, and $W(\Phi, A)/H(\Phi, A)$ is naturally isomorphic to the Weyl group $W(\Phi)$.

Our first result is analogous to [14, Lemma 3.1(e)] (cf. also with [3, § 2.3A]).

lem:bruhat **Lemma 3.6.** *Let Φ be any irreducible root system. Let Φ^+, $\Phi^{+'}$ be two systems of positive roots in Φ.*

(i) The Steinberg group $\mathrm{St}(\Phi, A)$ admits the following analogue of the Bruhat decomposition:

{eq:bruhat}
$$\mathrm{St}(\Phi, A) = \bigsqcup_{w \in W(\Phi)} \Big(\mathrm{U}(\Phi^{+'}, A) \cdot w H(\Phi, A) \cdot \mathrm{U}(\Phi^+, A) \cdot \ker\big(\mathrm{St}(\Phi, A) \xrightarrow{\pi^*} \mathrm{St}(\Phi, k)\big) \Big). \tag{3.1}$$

(ii) Assume that $uwhvl = u'w'h'v'l'$ for some $u, u' \in \mathrm{U}(\Phi^{+'}, A)$, $w, w' \in W(\Phi)$, $h, h' \in H(\Phi, A)$, and $l, l' \in \ker\big(\mathrm{St}(\Phi, A) \xrightarrow{\pi^*} \mathrm{St}(\Phi, k)\big)$. Then $w = w'$,

$$h^{-1}h' \in \mathrm{Sym}(\Phi, A) \cdot \ker\big(H(\Phi, A) \xrightarrow{\pi^*} H(\Phi, k)\big),$$

and there exist $a \in \mathrm{U}(\Phi^{+'}, m)$ such that $w^{-1}(u^{-1}u'a)w \in \mathrm{U}(\Phi^+, R)$, and $b \in \mathrm{U}(\Phi^+, m)$ such that

$$b = v^{-1}(u^{-1}u')^{wh}v' = l(l')^{-1}.$$

Proof. For any field k, the group $G(\Phi, k) = E(\Phi, k)$ admits Bruhat decomposition, hence $K_2(\Phi, k) \leq H(\Phi, k)$ implies

$$\mathrm{St}(\Phi, k) = \bigsqcup_{w \in \mathrm{W}(\Phi)} \mathrm{U}(\Phi^{+'}, k) w H(\Phi, k)\, \mathrm{U}(\Phi^{+}, k).$$

Let $w_0 \in W(\Phi)$ be such that $v_0(\Phi^{+'}) = \Phi^{+}$. Then also

$$\mathrm{St}(\Phi, k) = w_0^{-1} H(\Phi, k) \mathrm{St}(\Phi, k) = \bigsqcup_{w \in \mathrm{W}(\Phi)} \mathrm{U}(\Phi^{+'}, k) w H(\Phi, k)\, \mathrm{U}(\Phi^{+}, k).$$

As a consequence, the first three factors in the right hand side of the decomposition (3.1) are mapped epimorphically onto $\mathrm{St}(\Phi, k)$ and the last factor coincides with $\mathrm{Ker}\,(\pi^*)$ from which the first assertion of the lemma follows. The second assertion follows from the unicity of the Bruhat decomposition in $G(\Phi, k)$. ☐

lem:tul3.1zh **Lemma 3.7.** *Denote by* $\mathrm{Sym}^{\mathrm{t}}(\Phi, A, m)$ *the subgroup of* $\mathrm{Sym}(\Phi, A[t^{\pm 1}])$ *generated by all symbols of the form* $\{t, u\}_\alpha$, $\alpha \in \Phi$, $u \in 1 + m$.

Then there is an injective homomorphism $\phi : \mathrm{Sym}^{\mathrm{t}}(\Phi, A, m) \to \mathrm{St}(\Phi, A[t^{\pm 1}], I)$, *natural in* (A, m), *such that the composition of the canonical map* $\iota\colon \mathrm{St}(\Phi, A[t^{\pm 1}], I) \to \mathrm{St}(\Phi, A[t^{\pm 1}])$ *with* ϕ *equals* $\mathrm{id}_{\mathrm{Sym}^{\mathrm{t}}(\Phi, A, m)}$.

Proof. The map ϕ can be defined as follows:

$$\phi(\{t, u\}_\alpha) = y_\alpha(t(u-1)) y_{-\alpha}(-t^{-1}(u^{-1}-1))^{x_\alpha(-t)} \cdot$$
$$\cdot\, y_\alpha(t(u-1))^{w_\alpha(-t)} y_\alpha(u-1) y_{-\alpha}(-(u^{-1}-1))^{x_\alpha(-1)} y_\alpha(u-1)^{w_\alpha(-1)}.$$

If A is an integral domain (e.g. A regular), this follows from the injectivity of $A^* \to K^*$, where K the fraction field of A, together with the injectivity of $K^* \to \mathrm{St}(A_2, K(t))$ (see e.g. [**DeSte-dvr**]) and the stability [**DeSte-dvr**] applied to $A_2 \to \Phi$ and the field $K(t)$. ☐

lem:sigma-X **Lemma 3.8.** *Let* Φ *be an irreducible root system of rank* $l \geq 3$ *and of type* A_l $(l \geq 3)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_6 *or* E_7. *Let* $\alpha_i \in \Pi$, $1 \leq i \leq l$, *be a simple root of* Φ *such that the parabolic subgroup* P_i *of* $G(\Phi, -)$ *has abelian unipotent radical. Denote by* $\mathrm{St}P_i^-(\Phi, A)$ *the subgroup of* $\mathrm{St}(\Phi, A)$ *generated by* $x_\alpha(u)$, $\alpha \in \Delta_i \cup (-\Sigma_i)$, $u \in A$, *and by* $H(\Phi, A)$ *(?). Then there exists a group homomorphism*

$$\delta_i : \mathrm{St}(\Phi, A[t], tA[t]) \to \mathrm{St}P_i^-(\Phi, A) \cdot \mathrm{St}(\Phi, A[t], tA[t])$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{St}(\Phi, A[t], tA[t]) \arrow[r, "j_+"] \arrow[d, "\delta_i"] & \mathrm{St}(\Phi, A[t, t^{-1}]) \arrow[d, "\chi_{\varpi_i, t}"] \\
\mathrm{St}P_i^-(\Phi, A) \cdot \mathrm{St}(\Phi, A[t], tA[t]) \arrow[r, "j_+"] & \mathrm{St}(\Phi, A[t, t^{-1}])
\end{tikzcd}$$

Proof. The cases A_l and C_l are done using another presentation. Other cases are done by amalgamation of A_3-pieces.... ☐

n:parab-pairs **Lemma 3.9.** *Let* Φ *be an irreducible root system of rank* $l \geq 3$ *and of type* A_l $(l \geq 3)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_6, E_7 *or* E_8. *Let* $\alpha_i \in \Pi$ *be the simple root adjacent to* α_l *in the Dynkin diagram of* Φ *(note that* P_l *has abelian unipotent radical if* $\Phi \neq E_8$, *and extraspecial if* $\Phi = E_8$). *Set* $\Phi^{+'} = w_{\alpha_l}(\Phi^+)$. *Then* $\Phi^{+'} \setminus \Phi^+ = -\alpha_l$, $\Phi^+ \setminus \Phi^{+'} = \alpha_l$, *and* $\Sigma_i \setminus \Sigma_l \subseteq w_{\alpha_l}(\Sigma_l)$.

Proof. The first two claims are obvious. The last claim follows from the fact that for any root $\alpha \in \Sigma_i \setminus \Sigma_l$ one has $w_{\alpha_l}(\alpha) = \alpha + \alpha_l$. □

lem:tul3.3 **Lemma 3.10.** *Let Φ be an irreducible root system of rank ≥ 2, let Π be a system of simple roots in Φ, and let $J \subseteq \Pi$ be such that $|J| \geq 2$. Let S be any group, and $n \in \mathbb{N}$. For any $b \in B$ let $|b| \in \mathbb{Z}$ be such that $t^{|b|}$ is the smallest power of t occuring in b. Assume that there are elements $s_\alpha(u) \in S$ for all $\alpha \in \Sigma_J \cup (-\Sigma_J)$ and $u \in B$ such that $|u| \geq -n$, satisfying the relations*

(1) $s_\alpha(u)s_\alpha(v) = s_\alpha(u+v)$ for all $\alpha \in \Sigma_J \cup (-\Sigma_J)$, $u, v \in B$, $|u|, |v| \geq -n$;
(2) $[s_\alpha(u), s_\beta(v)] = \prod_{i,j\in\mathbb{N}} s_{i\alpha+j\beta}(N_{\alpha\beta ij}u^iv^j)$ for all $\alpha, \beta \in \Sigma_J \cup (-\Sigma_J)$ such that $i\alpha_J \neq -j\beta_J$ for all $i, j \in \mathbb{N}$, and all $u, v \in B$ such that $i|u| + j|v| \geq -n$ whenever $N_{\alpha\beta ij} \neq 0$.

If $n = 1$, then the map $x_\alpha(u) \mapsto s_\alpha(u)$ extends to a group homomorphism $\mathrm{St}(\Phi, B) \to S$.

Proof. □

3.3. **Tulenbaev's section 4.**

thm:tul4.1 **Theorem 5.** *Let Φ be an irreducible root system of rank $l \geq 3$ of type A_l, C_l, D_l ($l \geq 4$), E_6 or E_7. Then $\ker\big(\mathrm{St}(\Phi, A[t], m[t]) \to \mathrm{St}(\Phi, A[t])\big)$ surjects onto $\ker\big(\mathrm{St}(\Phi, A[t^{\pm1}], m[t^{\pm1}]) \to \mathrm{St}(\Phi, A[t^{\pm1}])\big)$.*

Proof. Set
$$\tilde{B} = H(\Phi, A[t^{\pm1}]) \cdot \mathrm{U}(\Phi^+, A[t^{\pm1}]) \leq \mathrm{St}(\Phi, A[t^{\pm1}]),$$
and
$$\tilde{D} = \phi(\mathrm{Sym}^{\mathrm{t}}(\Phi, A, m)) \leq \mathrm{St}(\Phi, A[t^{\pm1}], m[t^{\pm1}])$$
in the notation of Lemma 3.7. Consider the set ofequivalence classes
$$V = \mathrm{St}(\Phi, A[t]) \times \tilde{B} \times \mathrm{St}(\Phi, A[t^{\pm1}], mA[t^{\pm1}]) / \sim,$$
where $(a, b, \beta) \sim (a', b', \beta')$ if and only if there is $\gamma \in \mathrm{St}(\Phi, A[t], mA[t])$, $p \in \mathrm{U}(\Phi^+, A[t])$ and $\mu \in \tilde{D} \cdot \mathrm{U}(\Phi^+, mA[t^{\pm1}])$ such that
$$\tilde{a} = a\gamma_1^{-1}t^{-1}, \quad \tilde{b} = pt\mu, \quad \tilde{\beta} = \mu^{-1}(\gamma_2)b\beta,$$
where γ_1 and γ_2 are the images of γ in the respective groups. The elements of V will be denoted $[a, b, \beta]$.

We will define a map $\sigma : V \to V$. By Lemma 3.6 and the definition of V every element of V can be written in the form $[aw, b, \beta]$ for some $a \in \mathrm{St}(\Phi, A[t], tA[t]) \cdot \mathrm{U}(\Phi^+, A)$, $w \in W(\Phi, A) \leq \mathrm{St}(\Phi, A)$, $b \in \tilde{B}$ and $\beta \in \mathrm{St}(\Phi, A[t^{\pm1}], mA[t^{\pm1}])$. We define
$$\sigma|_{\mathrm{St}(\Phi, A[t], tA[t])} = \delta_l : \mathrm{St}(\Phi, A[t], tA[t]) \to \mathrm{St}(\Phi, A[t], tA[t])\mathrm{St}P_l^-(\Phi, A) \leq [\mathrm{St}(\Phi, A[t]), 1, 1],$$
where δ_l the homomorphism constructed in Lemma 3.8. For any $\alpha \in \Phi$, $u \in A[t^{\pm1}]$ we have a homomorphism
$$\chi_{\varpi_l, t} : X_\alpha(A[t^{\pm1}]) \to X_\alpha(A[t^{\pm1}]), \quad x_\alpha(u) \mapsto x_\alpha(t^{m_{\alpha_l}(\alpha)}u).$$
In particular, this induces a homomorphism $\sigma_U = \chi_{\varpi_l, t}|_{\mathrm{U}(\Phi^+, A)} : \mathrm{U}(\Phi^+, A) \to \mathrm{U}(\Phi^+, A[t])$. Combining δ_l and σ_U, we obtain a homomorphism (check!!!)
$$\sigma : \mathrm{St}(\Phi, A[t], tA[t]) \cdot \mathrm{U}(\Phi^+, A) \to \mathrm{St}(\Phi, A[t]).$$

Now we define

$$\sigma\cdot[aw,b,\beta]=[\sigma(a)w,w^{-1}\chi_{\varpi_l,t}(wb),\chi_{\varpi_l,t}(\beta)].$$

Note that, clearly, $w^{-1}\chi_{\varpi_l,t}(w)\in H(\Phi,A[t^{\pm1}])$.

Next we define a map $\sigma':V\to V$. Set $\Phi^{+'}=w_{\alpha_l}(\Phi^+)$. By Lemma 3.6 and the definition of V every element of V can be written in the form $[aw,b,\beta]$ for some $a\in \mathrm{St}(\Phi,A[t],tA[t])\cdot \mathrm{U}(\Phi^{+'},A)$, $w\in W(\Phi,A)\le \mathrm{St}(\Phi,A)$, $b\in\tilde{B}$ and $\beta\in \mathrm{St}(\Phi,A[t^{\pm1}],mA[t^{\pm1}])$. Note that $w_{\alpha_l}(\Pi)$ is a set of simple roots of Φ contained in $\Phi^{+'}$, with $w_{\alpha_l}(\alpha_l)=-\alpha_l$ playing the role of α_l. Then by Lemma 3.8 there is a homomorphism

$$\delta_l':\mathrm{St}(\Phi,A[t],tA[t])\to \mathrm{St}(\Phi,A[t],tA[t])\mathrm{St}P_l'^{-}(\Phi,A),$$

compatible with $\chi_{w_{\alpha_l}(\varpi_l),t}$, where $\mathrm{St}P_l'^{-}(\Phi,A)$ denotes the subgroup of $\mathrm{St}(\Phi,A)$ corresponding to the parabolic set of roots $w_{\alpha_l}(\Delta_l\cup\Sigma_l)$. We define

$$\sigma'|_{\mathrm{St}(\Phi,A[t],tA[t])}=\delta_l':\mathrm{St}(\Phi,A[t],tA[t])\to \mathrm{St}(\Phi,A[t],tA[t])\mathrm{St}P_l'^{-}(\Phi,A)\le[\mathrm{St}(\Phi,A[t]),1,1].$$

Similarly, for any $\alpha\in\Phi$, $u\in A[t^{\pm1}]$ we consider a homomorphism

$$\chi_{w_{\alpha_l}(\varpi_l),t}:X_\alpha(A[t^{\pm1}])\to X_\alpha(A[t^{\pm1}]),\quad x_\alpha(u)\mapsto x_\alpha(t^{m'_{-\alpha_l}(\alpha)}u),$$

where $m'_{w_{\alpha_l}(\alpha_i)}(\alpha)$, $1\le i\le l$, is the coefficient of $w_{\alpha_l}(\alpha_i)$ in the decomposition of $\alpha\in\Phi$ with respect to $w_{\alpha_l}(\Pi)$. We also consider a homomorphism $\sigma_U'=\chi_{w_{\alpha_l}(\varpi_l),t}|_{\mathrm{U}(\Phi^{+'},A)}:\mathrm{U}(\Phi^{+'},A)\to \mathrm{U}(\Phi^{+'},A[t])$. Combining δ_l' and σ_U', we obtain a homomorphism

$$\sigma':\mathrm{St}(\Phi,A[t],tA[t])\cdot\mathrm{U}(\Phi^{+'},A)\to \mathrm{St}(\Phi,A[t]).$$

Now we define

$$\sigma'\cdot[aw,b,\beta]=[\sigma'(a)w,w^{-1}\chi_{w_{\alpha_l}(\varpi_l),t}(wb),\chi_{w_{\alpha_l}(\varpi_l),t}(\beta)].$$

As in the case of σ, we have $w^{-1}\chi_{w_{\alpha_l}(\varpi_l),t}(w)\in H(\Phi,A[t^{\pm1}])$.

Now we show that $\sigma,\sigma':V\to V$ are correctly defined and bijective. We prove that for σ', the case of σ being analogous.....

Next we show that σ and σ' commute. First we note that if $\sigma\cdot[a,1,1]=[x,y,1]$, then for any $b\in\tilde{B}$ and $\beta\in\mathrm{St}(\Phi,A[t^{\pm1}],mA[t^{\pm1}])$ one has

$$\sigma\cdot[a,b,\beta]=[x,y\chi_{\varpi_l,t}(b),\chi_{\varpi_l,t}(\beta)],$$

and a similar equality holds for σ'. Since $\chi_{\varpi_l,t}$ and $\chi_{w_{\alpha_l}(\varpi_l),t}$ commute on $\mathrm{St}(\Phi,A[t^{\pm1}])$ and on $\mathrm{St}(\Phi,A[t^{\pm1}],mA[t^{\pm1}])$, it is enough to check that σ and σ' commute on any element of the form $[aw,1,1]\in V$, where $a\in\mathrm{St}(\Phi,A[t],tA[t])\cdot\mathrm{U}(\Phi^+,A)$, $w\in W(\Phi,A)$. We can write $a=a_0\cdot x_{\alpha_l}(u)$, where $a_0\in\mathrm{St}(\Phi,A[t],tA[t])\cdot\mathrm{U}(\Phi^+\cap\Phi^{+'},A)$.....

Now we define the action of $\mathrm{St}(\Phi,A[t^{\pm1}]))$ on V. For any $\alpha\in\Phi$ and $u\in A[t]\subseteq A[t^{\pm1}]$ we set

$$x_\alpha(u)\cdot[a,b,\beta]=[x_\alpha(u)a,b,\beta].$$

Next we want to use Lemma 3.10 applied to the set $J=\{\alpha_i,\alpha_l\}$, where α_i is the simple root adjacent to α_l. For any $\alpha\in\pm\Sigma_l$ and any $u\in A$ we set

$$x_\alpha(t^{-1}u)\cdot[a,b,\beta]=\sigma^{\mp1}\cdot x_\alpha(u)\cdot\sigma^{\pm1}[a,b,\beta].$$

For any $\alpha\in\pm\Sigma_i$ and any $u\in A$ we set

$$x_\alpha(t^{-1}u)\cdot[a,b,\beta]={\sigma'}^{\mp1}\cdot x_\alpha(u)\cdot{\sigma'}^{\pm1}[a,b,\beta]$$

(cf. Lemma 3.9). Then we check that this action satisfies the properties required in Lemma 3.10. This defines an action of $\mathrm{St}(\Phi, A[t^{\pm 1}])$ on V.

The last step is to check that for any $\beta \in \mathrm{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$ the action of its image in $\mathrm{St}(\Phi, A[t^{\pm 1}])$ on $[1, 1, 1]$ gives $[1, 1, \beta]$. □

3.4. **Suslin's lemma.**

lem:tulinj **Lemma 3.11.** *Assume that G satisfies Tulenbaev lifting property 2.4. Then the map i in the following commutative diagram of groups is injective.*

{diag:cs} (3.2)

$$\begin{tikzcd}
C_B \arrow[r, hook] \arrow[d, two heads, "k"] & \mathrm{St}^{\mathrm{G}}(B, I) \arrow[r, "\mu_B"] \arrow[d, "j"] & \mathrm{St}^{\mathrm{G}}(B) \arrow[r, two heads, "\pi_B"] \arrow[d, "i"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[d, hook] \\
C_R \arrow[r, hook] & \mathrm{St}^{\mathrm{G}}(R, I) \arrow[r, "\mu_R"'] \arrow[ur, dashed, "\varphi"] & \mathrm{St}^{\mathrm{G}}(R) \arrow[r, two heads, "\pi_R"'] & \mathrm{St}^{\mathrm{G}}(k[t, t^{-1}])
\end{tikzcd}$$

Proof. First of all, notice that by Corollary 3.4 the vertical map in the right-hand side of the diagram is injective. Invoking Tulenbaev's property 2.4 we also find a lifting map φ in the central square of the diagram.

Let $g \in \mathrm{St}^{\mathrm{G}}(B)$ be an element of $\mathrm{Ker}\,(i)$. Since g also lies in $\mathrm{Ker}\,(\pi_B)$ it comes from some $\widetilde{g} \in \mathrm{St}^{\mathrm{G}}(B, I)$ via μ_B. But $j(\widetilde{g})$ lies in C_R, hence, by Proposition 3.13 below it comes from some $\widehat{g} \in C_B$ via k. Finally, $g = \varphi(j(\widetilde{g})) = \varphi(k(\widehat{g})) = \mu_B(\widehat{g}) = 1$, as claimed. □

The following result is analogous to [14, Proposition 4.1].

Lemma 3.12. *The map j in the diagram (3.2) is surjective for any commutative ring A.*

Proof. Let α_i be arbitrary simple root from Π and let Σ_i (resp. Σ_i^-) denote the special subsets of roots consisting of $\alpha \in \Phi$ such that $m_i(\alpha) > 0$ (resp. $m_i(\alpha) < 0$).

By Lemma 2.3 the sets $\mathcal{Z}(\Sigma_i, R, I)$ and $\mathcal{Z}(\Sigma_i^-, R, I)$ both generate $\mathrm{St}^{\mathrm{G}}(R, I)$. Similarly, the subsets $\mathcal{Z}(\Sigma_i, B, I) \subseteq \mathcal{Z}(\Sigma_i, R, I)$, $\mathcal{Z}(\Sigma_i^-, B, I) \subseteq \mathcal{Z}(\Sigma_I, R, I)$ both generate the image of $j\colon \mathrm{St}^{\mathrm{G}}(B, I) \to \mathrm{St}^{\mathrm{G}}(R, I)$.

Using formula (2.6) we can calculate how the powers of the automorphism σ_i act on these generating sets. Indeed, for $z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i^-, R, I)$ we have

$$\sigma_i^N(z_\alpha(s, \xi)) = \sigma_i^N(x_\alpha(0, s)^{x_{-\alpha}(\Delta(\xi))}) = z_\alpha(t^{N m_i(\alpha)} \cdot s, t^{-N m_i(\alpha)} \cdot \xi).$$

It is clear that for sufficiently large N the element $t^{-N m_i(\alpha)} \cdot \xi$ belongs to $A[t] \subseteq B$. Consequently, for arbitrary $g \in \mathrm{St}^{\mathrm{G}}(R, I)$ there exists $N > 0$ such that $\sigma^N(g)$ lies in the image of j. On the other hand, by a similar calculation we get for $z = z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i, B, I)$ that the element $\sigma^{-N}(z)$ $(N > 0)$ still lies in $\mathcal{Z}(\Sigma_i, B, I)$. Clearly, this implies that the image of j is preserved by σ_i^{-1}. The assertion of the lemma now follows from these two statements:

$$g = \sigma^{-N}\sigma^N(g) \in \sigma^{-N}(\mathrm{Im}\,(j)) \subseteq \mathrm{Im}\,(j).$$ □

prop:kersurj **Proposition 3.13.** *Under the assumptions of Lemma 3.11 the map k in the diagram (3.2) is surjective.*

Sketch/draft of the proof. The argument presented below only may work under additional assumption that $m_i(\widetilde{\alpha}) = 1$ (i is as in the statement of **??**).

Consider the following set

$$X = \mathrm{St}^{\mathrm{G}}(B) \times \widetilde{B}(R) \times \mathrm{St}^{\mathrm{G}}(R, I)/\simeq .$$

The congruence relation we impose should be similar to that used in Tulenbaev's paper. In particular, for $h \in \mathrm{St}^{\mathrm{G}}(B, I)$ we should have the following relation in X

$$[\mu_B(h), 1, g] \simeq [1, 1, j(h)g].$$

Notice that instead of $\mathrm{St}^{\mathrm{G}}(A[t])$ (used by Tulenbayev in the first factor) we use $\mathrm{St}^{\mathrm{G}}(B)$.

TODO: We should prove that there is a well-defined action of $\mathrm{St}^{\mathrm{G}}(R)$ on X.

Now let g be an element of C_R. By the previous lemma we have for some $g' \in \mathrm{St}^{\mathrm{G}}(B, I)$

$$[1, 1, g] = [1, 1, j(g')] = [\mu_B(g'), 1, 1] = i\mu_B(g')[1, 1, 1] = [1, 1, 1]. \qquad \square$$

3.5. **Proof of the main result.** The following result is analogous to [14, Proposition 4.3]. It plays the same role in our proof of $\mathbb{P}^1$-glueing for K_2 as generalized Suslin lemma (cf. [1, Theorem 2.16]) does in the corresponding proof for K_1.

prop:p1g **Proposition 3.14.** *The functors* St^{G}, K_2^G *satisfy* $\mathbb{P}^1$*-glueing property for arbitrary local ring* A.

Proof. $\square$

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