$\mathbb{A}^1$-INVARIANCE FOR UNSTABLE K_2

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1. Aim of the paper

The aim of this text is to prove that the non-stable K_2-functors K_2^G, where G is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings R containing a field k, that is,

{eq:A1-main}
$$K_2^G(R[t]) = K_2^G(R). \tag{1.1}$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K-functor associated to G. This functor originates from [5]. The above equality by e.g. [2, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of G in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of K_2, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \ldots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the SL_n case [12] and Stavrova's proof for K_1^G [9]. Tulenbaev [12] uses stabilization of the K_2-functor, and the good properties of the limit=algebraic K-theory. In [9] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [12, p. 140], or, respectively, [9, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [12, Theorem 5.1] or [9, Theorem 1.1]).

1.1. **The case** $R = k$**.** We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the $\mathbb{A}^1$-invariance of K_1^G, this case should be used to deduce that $K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k)$.)

In Tulenbaev's framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [16, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k. Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt's proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [3], but only for the

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good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [14, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. **Plan of the proof.**

plan:3t (1) (**done**, see Theorem 2). Let R be a local ring. Show that

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t]))$$

Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

plan:k[t] (2) (**done**, see Theorem 4). Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect (3) (**done**). Consequently, $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp (4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen-Suslin lgp for K_2^G.

plan:Zglu (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2) \mapsto g_1 g_2{}^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma (6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A. If

$$K_2^G(A[X_1, \ldots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \ldots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St}^{\mathrm{G}}(A_h)$ there exist $y \in \mathrm{St}^{\mathrm{G}}(A)$ and $z \in \mathrm{St}^{\mathrm{G}}(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and h is not a zero divisor in A, then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [9, Lemma 3.4].

plan:P1 (8) ($\mathbb{P}^1$-gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2) \mapsto g_1 g_2{}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

(a) Prove Proposition 5.10. This is hard. Have no idea how this can be proved at the moment.

(b) Prove $\mathbb{P}^1$-glueing using all the above facts.

plan:[]f (9) (Main corollary of $\mathbb{P}^1$-gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.
The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2.

plan:k[tn] (11) Prove that

$$K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let R be a regular ring containing a field k. Then $K_2^G(R[t]) = K_2^G(R)$.
This uses (4), (11) and (7).

2. Steinberg groups of Chevalley groups: preliminaries

2.1. **Definition and basic properties.** Let G be a split simple Chevalley groups with a root system Φ of rank ≥ 2. Recall that the *Steinberg group* $\mathrm{St}^{\mathrm{G}}(R)$ (also denoted $\mathrm{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi,R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi,R}$ defined as follows:

{rel:add}
$$x_\alpha(s)x_\alpha(t) = x_\alpha(s+t), \tag{2.1}$$

{rel:CCF}
$$[x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta}\left(N_{\alpha\beta ij}\, s^i t^j\right), \quad \alpha \neq -\beta, \quad N_{\alpha\beta ij} \in \mathbb{Z}. \tag{2.2}$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$. Denote by $\mathrm{W}(\Phi, R)$ the subgroup of $\mathrm{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$.

2.2. Relative Steinberg groups.

{eq:suite}
$$1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \mathrm{St}^{\mathrm{G}}(R, I) \xrightarrow{\overline{p_2^*}} \mathrm{St}^{\mathrm{G}}(R) \xrightarrow{\pi^*} \mathrm{St}^{\mathrm{G}}(R/I) \longrightarrow 1 \tag{2.3}$$

Lemma 2.1. *Assume that R and I are such that the canonical projection $R \to R/I$ splits. Then the following facts are true.*

item:st-inj (i) *The map $\mathrm{St}^{\mathrm{G}}(R, I) \to \mathrm{St}^{\mathrm{G}}(R)$ is an injection.*
item:st-semi (ii) *The group $\mathrm{St}^{\mathrm{G}}(R)$ is isomorphic to $\mathrm{St}^{\mathrm{G}}(R/I) \ltimes \mathrm{St}^{\mathrm{G}}(R, I)$.*

Proof. For the proof of the first assertion see [8, Lemma 8]. Since the group $(G_1 \cap G_2)/C$ vanishes, the sequence (2.3) turns into a split short exact sequence which implies the second assertion. ∎

2.3. **Tulenbaev's lifting property and its corollaries.** Throughout this section $I \trianglelefteq A$ is an ideal of arbitrary commutative ring A. For a nonnilpotent element $a \in A$ denote by

$\lambda_a\colon A \to A_a$ the morphism of principal localization at a. Consider the following commutative square.

{msq} (2.4)
$$\begin{tikzcd} A \arrow[r, "\lambda_a"] \arrow[d, two heads] & A_a \arrow[d, two heads] \\ A/I \arrow[r, "\overline{\lambda_a}"] & A_a/I_a \end{tikzcd}$$

Notice that (2.4) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a. Such squares are usually called *Milnor squares* in the literature, see [15, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [12, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp **Definition 2.2.** We say that the Steinberg group functor St^{G} satisfies *Tulenbaev's lifting property* if for every pull-back square (2.4) the following lifting problem has a solution.

$$\begin{tikzcd} \mathrm{St}^{\mathrm{G}}(A, I) \arrow[r, "\mu"] \arrow[d] & \mathrm{St}^{\mathrm{G}}(A) \arrow[d, "\lambda_a^*"] \\ \mathrm{St}^{\mathrm{G}}(A_a, I) \arrow[r, "\mu"] \arrow[ur, dotted] & \mathrm{St}^{\mathrm{G}}(A_a) \end{tikzcd}$$

Theorem 1. *Assume that G satisfies Tulenbaev property (2.2) then the following facts are true for arbitrary commutative ring A:*

thm:dp *(i) A dilation principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. if $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is such that equality $\lambda_a^*(h) = 1$ holds in $\mathrm{St}^{\mathrm{G}}(\Phi, R_a[t])$ then for sufficiently large n one has*
$$ev^*_{\left[\frac{R[t]\to R[t]}{t\mapsto a^n\cdot t}\right]}(h) = 1.$$

thm:lg-k2 *(ii) A local-global principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. an element $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is trivial if and only if its image in $\mathrm{St}^{\mathrm{G}}(A_m[t], tA_m[t])$ is trivial for all maximal ideals $m \trianglelefteq A$.*

thm:centr *(iii) $K_2^G(A)$ is contained in the centre of $\mathrm{St}^{\mathrm{G}}(A)$.*

Proof. Follows by the same argument as [12, Theorem 2.1] or [8, Theorem 2] □

3. Decomposition theorems for $\mathrm{St}^{\mathrm{G}}(A[t^{\pm 1}])$ and $\mathrm{St}^{\mathrm{G}}\big(A((t))\big)$.

Lemma 3.1. *Let (R, m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*
$$i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big) i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) = i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big)$$
inside $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

Proof. This is proved exactly as [9, Lemma 5.12]. □

thm:3t **Theorem 2.** *Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*
$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t])) i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}])) i_+(\mathrm{St}^{\mathrm{G}}(R[t])).$$

Proof. This is proved exactly as [9, Theorem 5.1]. □

4. Steinberg group of type A_3

The main goal of this subsection is to show that Tulenbaev's [12, Lemma 2.3] remains valid for the linear Steinberg group of rank ≥ 3. In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [8, Definitions 3.3 and 3.7]).

Definition 4.1. The relative Steinberg group $\mathrm{St}^*(n, R, I)$ is the group defined by the following two families generators and four families of relations.

- Generators:
  (1) $X^1(u, v)$, where $u \in \mathrm{E}(n, R) \cdot e_1$, $v \in I^n$ such that $v^t \cdot u = 0$;
  (2) $X^2(u, v)$, where $u \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$ such that $v^t \cdot u = 0$.
  Notice that ϕ maps both $X^1(u, v)$ and $X^2(u, v)$ to $T(u, v) = e + u \cdot v^t \in \mathrm{E}(n, R, I)$.
- Relations:
  (1) $X^1(u, v) \cdot X^1(u, w) = X^1(u, v + w)$, $u \in \mathrm{E}(n, R) \cdot e_1$, $v, w \in I^n$;
  (2) $X^2(u, v) \cdot X^2(w, v) = X^2(u + w, v)$, $u, w \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$;
  (3) ${}^{X^\sigma(u^2,v^2)}X^\tau(u^1, v^1) = X^\tau(T(u^2, v^2) \cdot u^1, T(v^2, u^2)^{-1} \cdot v^1)$, $\sigma, \tau = 1, 2$;
  (4) $X^1(g \cdot e_1, g^* \cdot be_2) = X^2(g \cdot be_1, g^* \cdot e_2)$ where $b \in I$ and $g^* = g^{t^{-1}}$ denotes the contragradient matrix.

Lemma 4.2. *The groups* $\mathrm{St}^*(n, R, I)$ *and* $\mathrm{St}(n, R, I)$ *are isomorphic.*

Proof. **TODO:** □

The next step of the proof is to is construct certain elements in $\mathrm{St}(n, R)$ similar to Tulenbaev's elements $X_{u,v}(a)$ see [12, § 1].

Let $v \in R^n$ be a column. Denote by $O(v)$ the submodule of R^n consisting of all columns w such that $w^t \cdot v = 0$. A column $w \in R^n$ is called v-*decomposable* if it can be presented as a finite sum $w = \sum_{i=1}^{p} w^i$ such that each w^i has at least two zero entries and $v^t \cdot w^i = 0$. Denote by $D(v)$ the submodule of $O(v)$ consisting of all v-decomposable columns. For a column $v \in R^n$ denote by $I(v)$ the ideal of R spanned by its entries $v_1, \ldots, v_n$.

Let $u, v, w \in R^n$ be columns such that $w^t v = 0$. It is easy to check (cf. [6, Lemma 3.2]) that

$$(uv) \cdot w = \sum_{i<j} w_{ij}, \text{ where } w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n.$$

The above decomposition is called the *canonical* decomposition of $(uv) \cdot w$. In particular, this shows that the column $a \cdot w$ is always v-decomposable for $a \in I(v)$, $w \in O(v)$, i. e. $I(v) \cdot O(v) \subseteq D(v)$. It is also straightforward to check that $D(v) \subseteq D(bv)$, $b \cdot D(v) \subseteq D(v)$ for $b \in R$.

Denote by B^1 the subset of $R^n \times R^n \times R$ consisting of triples (u, v, a) such that $v^t \cdot u = 0$, $v \in D(u)$, $a \in I(u)$. Denote by B^2 the set consisting of triples (v, u, a) such that $(u, v, a) \in B^1$.

lem:Zfacts **Lemma 4.3.** *Assume that* $n \geq 4$. *One can define two families of elements* $Z^\tau(u, v, a)$, $\tau = 1, 2$ *of the group* $\mathrm{St}(n, R)$ *parametrized by* $(u, v, a) \in B^\tau$ *satisfying the following properties:*

(1) $\phi(Z^\tau(u, v, a)) = e + uav^t \in \mathrm{E}(n, R)$, $(u, v, a) \in B^\tau$;
(2) $Z^1(u, v + w, a) = Z^1(u, v, a) \cdot Z^1(u, w, a)$;

(3) $Z^2(v+w,u,a) = Z^2(v,u,a)\cdot Z^2(w,u,a)$;

(4) for $\tau = 1,2$ *and* $b \in R$ *if* $(u, vb, a), (ub, v, a) \in B^\tau$ *then one has*

$$Z^\tau(u, vb, a) = Z^\tau(u, v, ab) = Z^\tau(ub, v, a);$$

(5) ${}^gZ^\tau(u,v,a) = Z^\tau(\phi(g)\cdot u, \phi(g)^*\cdot v, a)$, $\tau = 1,2$, $g \in \mathrm{St}^{\mathrm{G}}(n,R)$.

Proof. One constructs the elements $Z^1(u,v,a)$ in exactly the same way as Tulenbaev constructs his elements $X_{u,v}(a)$ (see definitions preceding [12, Lemma 1.2]). Indeed, set

$$Z^1(v,w,a) = \prod_{k=1}^{p} X(v, a\cdot w^k),\quad Z^2(w,v,a) = \prod_{k=1}^{p} X(a\cdot w^k, v). \tag{4.1}$$

where $X(u,v)$ denotes the elements defined by Tulenbaev before [12, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case $n=4$) can be proved by the same token as in [12, Lemma 1.3]. $\square$

For the rest of this section a denotes a nonnilpotent element of R and $\lambda_a\colon R \to R_a$ is the morphism of principal localization at a.

lem:rk3rels **Lemma 4.4.** *For any* $g \in \mathrm{E}(n, R_a)$ *there exist* $u, v \in R^n$ *and sufficiently large natural numbers* k, m *such that the following facts hold:*

(1) $\lambda_a(u) = g\cdot a^k e_1$, $\lambda_a(v) = g^*\cdot a^k e_2$ *and* $u^t\cdot v = 0$;

(2) $(u, v, a^m) \in B^1 \cap B^2$;

(3) for $b \in R$ *divisible by some sufficiently large power of* a *one has*

$$Z^1(u, b\cdot v, a^m) = Z^2(b\cdot u, v, a^m).$$

Proof. It is straightforward to choose u and v satisfying the first requirement of the lemma. We can even choose u, v in such a way that $u \in D(v)$ and $v \in D(u)$. Indeed, notice that $I(u) = a^{k_1}$, $I(v) = a^{k_2}$ for some natural k_1, k_2 hence for $u' = a^{k_2}\cdot u$ and $v' = a^{k_1}\cdot v$ one has

$$u' \in I(v)\cdot O(v) \subseteq D(v) \subseteq D(v'),\quad v' \in I(u)\cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns $x, y \in R^n$ and a large natural p in such a way that vectors u, v, x, y additionally satisfy the following properties

$$\lambda_a(x) = g^*\cdot a^k e_3,\ \lambda_a(y) = g\cdot a^k e_3,\ y^t\cdot x = a^p \in R,$$
$$u^t\cdot x = 0,\ u^t\cdot v = 0,\ y^t\cdot v = 0,$$
$$(u, x, a^m) \in B^1,\ (y, v, a^m) \in B^2.$$

Now direct computation using Lemma 4.3 shows that

$$Z^2(a^{m+p}b\cdot u, v, a^m) = Z^2(b\cdot(e + a^m\cdot ux^t)y, (e - a^m\cdot xu^t)v, a^m)\cdot Z^2(-by, v, a^m) =$$
$$= [Z^1(u, x, a^m), Z^2(b\cdot y, v, a^m)] =$$
$$= Z^1(u, x, a^m)\cdot Z^1((e + a^m b\cdot yv^t)u, -(e - a^m b\cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b\cdot v, a^m),\quad \square$$

hence the third assertion of the lemma follows.

cor:tlpA3 **Corollary 4.5.** *For* $G = G(\mathsf{A}_3, -)$ *the Steinberg group functor* $\mathrm{St}^{\mathrm{G}}(R)$ *satisfies Tulenbaev lifting property 2.2.*

Proof. Follows from Lemma 4.4 by the same token as in [12, Lemma 2.3]. $\square$

5. $\mathbb{P}^1$-GLUING

Throughout this section $G = G(\Phi, -)$ denotes a Chevalley group scheme of type Φ.

def:p1g **Definition 5.1.** Let F be a group-valued functor from **CRings** to **Groups** and let A be a commutative ring. Consider the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "i_+"] \arrow[d, "i_-"'] & A[t] \arrow[d, "j_+"] \\ A[t^{-1}] \arrow[r, "j_-"] & A[t, t^{-1}] \end{tikzcd}$$

We say that F satisfies the $\mathbb{P}^1$*-glueing property for* A if the following sequence of *pointed sets* is exact in the middle term:

$$F(A) \xhookrightarrow{\Delta_A^F} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\pm_A^F} F(A[t, t^{-1}]).$$

Here Δ_A^F denotes the (split injective) diagonal map and, by definition, $\pm_A^F$ maps (g^+, g^-) to $F(j_+)(g^+) \cdot F(j_-)(g^-)^{-1}$. Notice that a priori $\pm_A^F$ is only a morphism of pointed sets but if F takes values in abelian groups then $\pm_A^F$ is also a morphism of groups.

An equivalent way to formulate $\mathbb{P}^1$-glueing property is as follows: $F(j_+)$ and $F(j_-)$ are injective and the intersection of their images coincides with the image of $F(j_+ i_+) = F(j_- i_-)$.

The main result of this section is the following theorem which generalizes [12, Theorem 5.1] to Chevalley groups. Notice that a K_1-analogue of the result below has been established in a much greater generality by the second-named author (see [9, Theorem 1.1]).

thm:p1 **Theorem 3.** *Assume that G satisfies Tulenbaev lifting property 2.2. Then the Steinberg group functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for an arbitrary commutative ring A.*

Proof. Let (g^+, g^-) be an element of $\mathrm{St}^{\mathrm{G}}(A[t]) \times \mathrm{St}^{\mathrm{G}}(A[t^{-1}])$ such that the equality $g^+ = g^-$ holds in $\mathrm{St}^{\mathrm{G}}(A[t, t^{-1}])$.

Let M be a maximal ideal of A. By Proposition 5.11 below the functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for the local ring A_M hence $(\lambda_M^*(g_+), \lambda_M^*(g_-)) = \Delta_{A_M}(\lambda_M^*(g^+)(0))$ and in the groups $\mathrm{St}^{\mathrm{G}}(A_M[t])$ and $\mathrm{St}^{\mathrm{G}}(A_M[t^{-1}])$ we have the equalities:

$$\lambda_M^*(g^+ \cdot g^+(0)^{-1}) = \lambda_M^*(g^+) \cdot \lambda_M^*(g^+)(0)^{-1} = 1; \qquad \lambda_M^*(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for $\mathrm{St}^{\mathrm{G}}(-)$ (see Theorem 1.(ii)) these equalities hold globally and $(g^+, g^-) = \Delta_A(g^+(0))$, as claimed. □

rem:stk2 *Remark* 5.2. It is clear that if the functor St^{G} satisfies $\mathbb{P}^1$-glueing property for A then so does the functor K_2^G. The converse statement also holds, indeed, if $(g^+, g^-) \in \mathrm{Ker}\,(\pm_A^{\mathrm{St}})$ then inside $\mathrm{E}^G(A[t, t^{-1}])$ we have the equality:

$$\varphi(\mathrm{St}^{\mathrm{G}}(j_+)(g^+)) = \varphi(\mathrm{St}^{\mathrm{G}}(j_-)(g^-)) \in \mathrm{E}^G(A[t]) \cap \mathrm{E}^G(A[t^{-1}]) = \mathrm{E}^G(A).$$

Consequently, we can find $g_0 \in \mathrm{St}^{\mathrm{G}}(A)$ so that $(g^+ g_0^{-1}, g^- g_0^{-1}) \in \mathrm{Ker}\,(\pm_A^{K_2})$ and it remains to apply the $\mathbb{P}^1$-glueing property for K_2^G.

Corollary 5.3. *Let A be any commutative ring and $f \in A[t]$ be a monic polynomial. Then the map $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.*

Proof. □

5.1. **The case of a field.** Throughout this section k denotes arbitrary field k.

thm:k[t] **Theorem 4.** *Assume that $G = G(\Phi, -)$ and Φ is irreducible of rank ≥ 2.*

satz1 *(i) The subgroup $K_2^G(k[t]) \trianglelefteq \mathrm{St}^{\mathrm{G}}(k[t])$ is generated by elements of the form*
$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1},\ u, v \in k^*.$$
(ii) As a consequence, the canonical injection $K_2^G(k) \hookrightarrow K_2^G(k[t])$ is an isomorphism.

Proof. See [7, Satz 1] and the corollary after it. □

cor:k[t]inj **Corollary 5.4.** *Let G be as in the above theorem. Then the functors St^{G}, K_2^G satisfy $\mathbb{P}^1$-glueing property for k.*

Proof. By Remark 5.2 it suffices to prove the assertion only for the functor K_2^G. By the previous theorem $K_2^G(i_+)$ and $K_2^G(i_-)$ are isomorphisms hence the morphisms $K_2^G(j_+)$ and $K_2^G(j_-)$ are split injective and $\mathrm{Im}\,(j_+i_+) = \mathrm{Im}\,(j_-i_-) = \mathrm{Im}\,(j_+) = \mathrm{Im}\,(j_-)$. □

Corollary 5.5. *Let G, k be as in Theorem 4. Then $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.*

Proof. We use Theorem 2. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}^{\mathrm{G}}(k[t]))$, $y \in \mathrm{St}^{\mathrm{G}}(k[t^{-1}]))$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}^{\mathrm{G}}(k)K_2^G(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ and $x_1x_2 \in \mathrm{St}^{\mathrm{G}}(k)$. That is, $g \in K_2^G(k)$. □

5.2. **The action of torus.** Our notation and conventions follows [13, § 4]. Let Φ be an irreducible root system with some fixed basis of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. We denote by $\Phi^\vee$ the *dual root system of* Φ consisting of vectors $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \Phi$. As usual, $P(\Phi^\vee)$ denotes the lattice spanned by the *fundamental weights* ϖ_i. Recall that ϖ_i are uniquely determined by relations $\langle \varpi_i, \alpha_j^\vee \rangle = (\varpi_i, \alpha_j) = \delta_{ij}$.

Notice that for $\varpi \in P(\Phi^\vee)$ and $\beta \in \mathbb{Z}\,\Phi$ one has $(\varpi, \beta) \in \mathbb{Z}$. Consequently, for $\varepsilon \in R^*$ and $\varpi \in P(\Phi^\vee)$ the identity $\chi_{\varpi,\varepsilon}(\beta) = \varepsilon^{(\varpi,\beta)}$ gives a well-defined character $\chi_{\varpi,\varepsilon} \in \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$.

Consider the action of $H = \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$ on the set of generators $\mathcal{X}_{\Phi,R}$ of the Steinberg group $\mathrm{St}^{\mathrm{G}}(R)$ defined by
$$\chi \cdot x_\alpha(\xi) = x_\alpha(\chi(\alpha) \cdot \xi),\ \chi \in H,\ \alpha \in \Phi,\ \xi \in R. \tag{5.1}$$
Since χ is a character, the above action preserves the set of Steinberg relations $\mathcal{R}_{\Phi,R}$ and, thus, gives a well-defined action of H on $\mathrm{St}^{\mathrm{G}}(R)$.

Example 5.6. The principal example which motivates the above construction is as follows. Let A be a ring, take $R = A[t, t^{-1}]$ to be the ring of Laurent polynomials over A and let $\alpha_i \in \Pi$ be some simple root. Since $t \in R^*$ we can consider the automorphism σ_i of $\mathrm{St}(\Phi, R)$ given by $\sigma_i = \chi_{\varpi_i,t}$. It is easy to see that
$$\sigma_i(x_\alpha(\xi)) = x_\alpha(t^{m_i(\alpha)} \cdot \xi), \tag{5.2}$$
where $m_k(\alpha)$ denotes the coefficient in the expansion of α in Π, i. e. $\alpha = \sum m_k(\alpha)\alpha_k$.

Our next goal is to define an analogue of σ_i for the group $\mathrm{St}^{\mathrm{G}}(A[t])$. Obviously, we cannot expect such map to be automorphism or even be defined on the whole group $\mathrm{St}^{\mathrm{G}}(A[t])$. However, it turns out that in some cases it is still possible to define it on a certain subgroup of $\mathrm{St}^{\mathrm{G}}(A[t])$.

lem:sigma **Lemma 5.7.** *Assume that A is a local commutative ring, $G = G(\Phi, -)$ and (Φ, i) is as follows:*

- $\Phi = \mathsf{A}_\ell$, $\ell \geq 3$, $i = 1$;
- Φ *is classical of rank* $\ell \geq 4$, $i = 1$;
- $\Phi = \mathsf{E}_\ell$, $\ell = 6, 7, 8$, $i = \ell$.

Then there exists subgroups N_i^+, N_i^- *of* $\mathrm{St}^{\mathrm{G}}(A[t])$ *and a homomorphism* $\sigma_i' \colon N_i^+ \to N_i^-$ *compatible with the action of* σ_i *i. e. such that the following diagram commutes:*

$$\begin{tikzcd} N_i^+ \arrow[r, "\lambda_t^*"] \arrow[d, "\sigma_i'"] & \mathrm{St}^{\mathrm{G}}(A[t, t^{-1}]) \arrow[d, "\sigma_i"] \\ N_i^- \arrow[r, "\lambda_t^*"] & \mathrm{St}^{\mathrm{G}}(A[t, t^{-1}]) \end{tikzcd}$$

We first prove the assertion of the lemma in the linear case and then deduce all the other cases from it.

In the case when $m_i(\widetilde{\alpha}) = 1$ the subgroups $N_i^{\pm}$ can be constructed explicitly as follows. For $\alpha_i \in \Pi$ consider the subgroup P_i^+ (resp. P_i^-) of $\mathrm{St}^{\mathrm{G}}(A)$ generated by $x_\alpha(\xi)$ for $\xi \in A$, $\alpha \in \Sigma_i^+ \cup \Delta_i$ (resp. $\alpha \in \Sigma_i^- \cup \Delta_i$). Now, let N_i^+ (resp. N_i^-) be the subgroup consisting of $g \in \mathrm{St}^{\mathrm{G}}(A[t])$ such that $g(0) \in P_i^+$ (resp. $g(0) \in P_i^-$).

Proof in the case $\Phi = \mathsf{A}_\ell$, $\ell \geq 3$. Denote by i_ℓ the natural map $\mathrm{St}^{G(\Delta_1)}(A) \to \mathrm{St}^{G(\Phi)}(A)$. Notice that $\Delta_1 \cong \mathsf{A}_{\ell-1}$. By the Levi decomposition P_1^+ is isomorphic to $\mathrm{U}(\Sigma_1, A) \rtimes \mathrm{Im}\,(i_\ell)$.

The stable rank of A equals 1 hence from the injective stability theorem for K_2 (see [11, Theorem 4.1]) it follows that the map i_ℓ is injective for $\ell \geq 3$ and that $P_1^+ \cong \mathrm{U}(\Sigma_1, A) \rtimes \mathrm{St}^{G(\Delta_1)}(A)$.

By Lemma 2.1.(ii) $\mathrm{St}^{\mathrm{G}}(A[t]) = \mathrm{St}^{\mathrm{G}}(A) \ltimes \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ hence $N_1^{\pm} = P_1^{\pm} \ltimes \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$. Define σ_1' on the generators $X^1(u, v)$, $X^2(v, u)$ of $\mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ ($u \in \mathrm{E}(\ell + 1, A[t]) \cdot e_1$, $v \in tA[t]^{\ell+1}$) by

$$\sigma_1'(X^1(u, v)) = Z^1()$$

$\square$

5.3. **The case of a local ring.** For the rest of this section A denotes an arbitrary commutative local ring with the maximal ideal m and the residue field k. We denote by π the canonical projection $A \to k$. Throughout this section we will employ the following notation:

- R denotes the Laurent polynomial ring $A[t, t^{-1}]$;
- B denotes the subring $A[t] + m[t^{-1}]$ of R consisting of Laurent polynomials $f(t, t^{-1})$ whose coefficients of terms of negative degree belong to m;
- I denotes the ideal $m[t, t^{-1}]$ of R (which can be also considered as an ideal of B).

Our first result is analogous to [12, Lemma 3.1(e)] (cf. also with [4, § 2.3A]).

cor:bruhat **Lemma 5.8.** *The Steinberg group* $\mathrm{St}^{\mathrm{G}}(A)$ *admits the following analogue of the Bruhat decomposition:*

$$\mathrm{St}^{\mathrm{G}}(A) = \mathrm{U}(\Phi^+, A) \cdot \mathrm{W}(\Phi, A) \cdot \mathrm{U}(\Phi^+, A) \cdot \mathrm{Im}\,(\mathrm{St}^{\mathrm{G}}(A, m) \to \mathrm{St}^{\mathrm{G}}(A)).$$

Proof. First of all, notice that $\mathrm{St}^{\mathrm{G}}(k)$ admits Bruhat decomposition for arbitrary field k:

$$\mathrm{St}^{\mathrm{G}}(k) = \mathrm{U}(\Phi^+, k) \cdot \mathrm{W}(\Phi, k) \cdot \mathrm{U}(\Phi^+, k).$$

Indeed, this is an immediate consequence of the classical Bruhat decomposition for the linear groups (see e. g. [10, Theorem 4]) and the fact that $K_2^G(k)$ is central in $\mathrm{St}^{\mathrm{G}}(k)$ and is generated by symbols $h_\alpha(u)$ lying in $\mathrm{W}(\Phi, k)$ (cf. Theorem 4.(i)).

As a consequence, the first three factors in the right hand side of the decomposition are mapped epimorphically onto $\mathrm{St}^{\mathrm{G}}(k)$ and the last factor coincides with $\mathrm{Ker}\,(\pi_*)$ from which the assertion of the lemma follows. ☐

lem:tulinj **Lemma 5.9.** *Assume that G satisfies Tulenbaev lifting property 2.2. Then the map i in the following commutative diagram of groups is injective.*

{diag:cs} (5.3)
$$\begin{tikzcd}
C_B \arrow[r, hook] \arrow[d, "k"] & \mathrm{St}^{\mathrm{G}}(B,I) \arrow[r, "\mu_B"] \arrow[d, "j"] & \mathrm{St}^{\mathrm{G}}(B) \arrow[r, two heads, "\pi_B"] \arrow[d, "i"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[d, hook] \\
C_R \arrow[r, hook] & \mathrm{St}^{\mathrm{G}}(R,I) \arrow[r, "\mu_R"'] \arrow[ur, dashed, "\varphi"] & \mathrm{St}^{\mathrm{G}}(R) \arrow[r, two heads, "\pi_R"'] & \mathrm{St}^{\mathrm{G}}(k[t,t^{-1}])
\end{tikzcd}$$

Proof. First of all, notice that by Corollary 5.4 the vertical map in the right-hand side of the diagram is injective. Invoking Tulenbaev's property 2.2 we also find a lifting map φ in the central square of the diagram.

Let $g \in \mathrm{St}^{\mathrm{G}}(B)$ be an element of $\mathrm{Ker}\,(i)$. Since g also lies in $\mathrm{Ker}\,(\pi_B)$ it comes from some $\widetilde{g} \in \mathrm{St}^{\mathrm{G}}(B,I)$ via μ_B. But $j(\widetilde{g})$ lies in C_R, hence, by Proposition 5.10 below it comes from some $\widehat{g} \in C_B$ via k. Finally, $g = \varphi(j(\widetilde{g})) = \varphi(k(\widehat{g})) = \mu_B(\widehat{g}) = 1$, as claimed. ☐

The following result is analogous to [12, Proposition 4.1].

prop:kersurj **Proposition 5.10.** *Under the assumptions of Lemma 5.9 the map k in the diagram (5.3) is surjective.*

Proof. ☐

The following result is analogous to [12, Proposition 4.3].

prop:p1g **Proposition 5.11.** *The functors St^{G}, K_2^G satisfy $\mathbb{P}^1$-glueing property for arbitrary local ring A.*

Proof. ☐

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