

\mathbb{A}^1 -INVARIANCE FOR UNSTABLE K_2

S. SINCHUK, A. STAVROVA, AND A. LAVRENOV

1. AIM OF THE PAPER

The aim of this text is to prove that the non-stable K_2 -functors K_2^G , where G is a simply connected Chevalley group of suitable type, satisfy \mathbb{A}^1 -invariance on regular rings R containing a field k , that is,

$$(1.1) \quad K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K -functor associated to G . This functor originates from [Jar83]. The above equality by e.g. [AHW15, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the \mathbb{A}^1 -fundamental group of G in the sense of Morel–Voevodsky.

”Suitable type” here means that we consider only the cases where we know the centrality of K_2 , or at least the Quillen–Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \dots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev’s proof for the SL_n case [Tul83] and Stavrova’s proof for K_1^G [Sta14]. Tulenbaev [Tul83] uses stabilization of the K_2 -functor, and the good properties of the limit=algebraic K -theory. In [Sta14] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [Tul83, p. 140], or, respectively, [Sta14, Theorem 3.1]); Quillen–Suslin lgp; \mathbb{P}^1 -gluing (see [Tul83, Theorem 5.1] or [Sta14, Theorem 1.1]).

1.1. The case $R = k$. We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the \mathbb{A}^1 -invariance of K_1^G , this case should be used to deduce that $K_2^G(k[t_1, \dots, t_n]) = K_2^G(k)$.)

In Tulenbaev’s framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [Wen14, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k . Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k . Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt’s proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2 ? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [Deo78], but only for the

2010 *Mathematics Subject Classification*. 19C09, 19C20, 14L15, 20G35.

Key words and phrases. Chevalley group, non-stable K_2 -functor, Steinberg group.

good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [VW12, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. Plan of the proof.

plan:3t

- (1) (**done**, see Theorem 1). Let R be a local ring. Show that

$$\mathrm{St}^G(R[t^{\pm 1}]) = i_+(\mathrm{St}^G(R[t]))i_-(\mathrm{St}^G(R[t^{-1}]))i_+(\mathrm{St}^G(R[t]))$$

Here i_{\pm} denote the natural homomorphisms into $\mathrm{St}^G(R[t^{\pm 1}])$.

plan:k[t]

- (2) (**done**, see Theorem 3). Show that $K_2^G(k[t]) = K_2^G(k)$.

plan:k-intersect

- (3) (**done**). Consequently, $\mathrm{St}^G(k[t]) \rightarrow \mathrm{St}^G(k[t^{\pm 1}])$ is injective and $\mathrm{St}^G(k[t]) \cap \mathrm{St}^G(k[t^{-1}]) = \mathrm{St}^G(k)$ inside $\mathrm{St}^G(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp

- (4) (**done** for split ACDE, see Theorem 2). Prove Quillen-Suslin lgp for K_2^G .

plan:Zglu

- (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma

- (6) (S -lemma) Let A be a commutative ring, S a multiplicative subset of A . If

$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [Abe83, Lemma 3.6].

plan:Nglu

- (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A , and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \rightarrow A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \rightarrow A/Ah$ is surjective, then for any $x \in \mathrm{St}^G(A_h)$ there exist $y \in \mathrm{St}^G(A)$ and $z \in \mathrm{St}^G(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \rightarrow A/Ah$ is an isomorphism, and h is not a zero divisor in A , then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1, g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [Sta14, Lemma 3.4].

plan:P1

- (8) (\mathbb{P}^1 -gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

- (a) (**done**) Prove Corollary 6.2.

(b) Prove Proposition 6.1. This is hard. Have no idea how this can be proved at the moment.

(c) Prove \mathbb{P}^1 -gluing using all the above facts.

plan:[]f

- (9) (Main corollary of \mathbb{P}^1 -gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$ is injective.

The proof uses (5) and (8).

plan:k(t)

- (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tullenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2 .

plan:k[tn]

- (11) Prove that

$$K_2^G(k[t_1, \dots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G , this uses (9), (2), and (10). There may be other ways.

plan:final

- (12) Final result: let R be a regular ring containing a field k . Then $K_2^G(R[t]) = K_2^G(R)$. This uses (4), (11) and (7).

2. STEINBERG GROUPS OF CHEVALLEY GROUPS: PRELIMINARIES

Definition, functoriality, "congruence subgroups" $\text{St}^G(\Phi, R, I)$ versus $\ker(\text{St}^G(\Phi, R) \rightarrow \text{St}^G(\Phi, R/I))$.

2.1. Tullenbaev's lifting property. The following property of linear Steinberg groups was discovered for the first time by Tullenbaev (see [Tul83, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp

Definition 2.1. We say that the Steinberg group functor St^G satisfies *Tullenbaev's lifting property* if for every pair (R, I) and every nonnilpotent $a \in R$ such that the localization map of modules $\lambda_a: I \rightarrow I_a$ is an isomorphism the following lifting problem has a solution.

$$\begin{array}{ccc} \text{St}^G(R, I) & \xrightarrow{\mu} & \text{St}^G(R) \\ \downarrow \lambda_a^* & \nearrow & \downarrow \lambda_a^* \\ \text{St}^G(R_a, I) & \xrightarrow{\mu} & \text{St}^G(R_a) \end{array}$$

3. DECOMPOSITION THEOREMS FOR $\text{St}^G(A[t^{\pm 1}])$ AND $\text{St}^G(A((t)))$.

Lemma 3.1. Let (R, m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2 . Let $i_+ : \text{St}^G(R[t]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ and $i_- : \text{St}^G(R[t^{-1}]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ be the natural homomorphisms. Then

$$i_+(\text{St}^G(m \cdot R[t])^{\text{St}^G(R[t])}) i_-(\text{St}^G(R[t^{-1}])) = i_-(\text{St}^G(R[t^{-1}])) i_+(\text{St}^G(m \cdot R[t])^{\text{St}^G(R[t])})$$

inside $\text{St}^G(R[t^{\pm 1}])$.

Proof. This is proved exactly as [Sta14, Lemma 5.12]. □

thm:3t

Theorem 1. Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2 . Let $i_+ : \text{St}^G(R[t]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ and $i_- : \text{St}^G(R[t^{-1}]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ be the natural homomorphisms. Then

$$\text{St}^G(R[t^{\pm 1}]) = i_+(\text{St}^G(R[t])) i_-(\text{St}^G(R[t^{-1}])) i_+(\text{St}^G(R[t])).$$

Proof. This is proved exactly as [Sta14, Theorem 5.1]. □

4. QUILLEN-SUSLIN LGP, ZARISKI GLUING, NISNEVICH GLUING, S -LEMMA

thm:lg-k2

Theorem 2. *Assume that the Steinberg group functor St^G satisfies Tulenbaev's lifting property (cf. Definition 2.1). Then an element $g \in \mathrm{St}^G(R[t], tR[t])$ is trivial if and only if its image in $\mathrm{St}^G(R_M[t], tR_M[t])$ is trivial for all maximal ideals $M \trianglelefteq R$.*

Proof. Follows by the same argument as [Tul83, Theorem 2.1] or [Sin16, Theorem 2] \square

5. THE CASE OF $K_2^G(k[t])$ AND SOME COROLLARIES

Let $G = G(\Phi, -)$ be the simply-connected Chevalley—Demazure group scheme of type Φ .

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$.

thm:k[t]

Theorem 3. *Let k be a field. Assuming that Φ is irreducible of rank at least 2 the following facts are true.*

satz1

(i) *For $A = k, k[t]$ the subgroup $K_2^G(A) \leq \mathrm{St}^G(A)$ is generated by elements of the form*

$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}, \quad u, v \in k^*.$$

(ii) *As a consequence, the canonical map $K_2^G(k) \hookrightarrow K_2^G(k[t])$ is an isomorphism.*

Proof. See [Reh75, Satz 1] and the corollary after it. \square

cor:k[t]inj

Corollary 5.1. *Let G, k be as in Theorem 3. Then $\mathrm{St}^G(k[t]) \rightarrow \mathrm{St}^G(k[t^{\pm 1}])$ is injective and $\mathrm{St}^G(k[t]) \cap \mathrm{St}^G(k[t^{-1}]) = \mathrm{St}^G(k)$ inside $\mathrm{St}^G(k[t^{\pm 1}])$.*

Proof. Clearly, $g \in \ker(\mathrm{St}^G(k[t]) \rightarrow \mathrm{St}^G(k[t^{\pm 1}]))$ implies $g \in K_2^G(k[t])$. Since $K_2^G(k[t]) = K_2^G(k)$, and there is a section $K_2^G(k[t^{\pm 1}]) \rightarrow K_2^G(k)$, the map is injective. Second claim: take $g \in \mathrm{St}^G(k[t]) \cap \mathrm{St}^G(k[t^{-1}])$. Then the image $\phi(g)$ belongs to $E(k) = E(k[t]) \cap E(k[t^{-1}])$, and after adjusting g by an element of $\mathrm{St}^G(k)$, we can assume that $g \in K_2^G(k[t]) \cap K_2^G(k[t^{-1}])$. Hence $g \in K_2^G(k) \subseteq \mathrm{St}^G(k)$. \square

Corollary 5.2. *Let G, k be as in Theorem 3. Then $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.*

Proof. We use Theorem 1. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}^G(k[t])$, $y \in \mathrm{St}^G(k[t^{-1}])$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}^G(k) K_2^G(k[t^{-1}]) = \mathrm{St}^G(k)$ and $x_1 x_2 \in \mathrm{St}^G(k)$. That is, $g \in K_2^G(k)$. \square

Denote by $W(\Phi, R)$ the subgroup of $\mathrm{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$. The following result is a Steinberg level analogue of the well-known Bruhat decomposition.

cor:bruhat

Corollary 5.3. *For Φ of rank ≥ 2 and a field k one has*

$$\mathrm{St}(\Phi, k) = U(\Phi^+, k) \cdot W(\Phi, k) \cdot U(\Phi^+, k).$$

Proof. From Theorem 3.(i) it follows that $K_2(\Phi, k)$ is contained in $W(\Phi, k)$, therefore the required assertion follows from the classical Bruhat decomposition (see e.g. [SFW67, Theorem 4]). For a direct proof in the linear case see also [HO89, § 2.3A]. \square

6. \mathbb{P}^1 -GLUING

6.1. Preliminaries. Let R denote arbitrary commutative local ring with the maximal ideal m and the residue field k . Consider the following commutative diagram of groups.

$$\begin{array}{ccccccc}
 C_+ & \hookrightarrow & \mathrm{St}^G(R[t], m[t]) & \xrightarrow{\mu^+} & \mathrm{St}^G(R[t]) & \xrightarrow{p^+} & \mathrm{St}^G(k[t]) \\
 \downarrow k^+ & & \downarrow j_1^+ & \searrow & \downarrow i_1^+ & & \parallel \\
 & & \mathrm{St}^G(R[t] + m[t^{-1}], m[t^{\pm 1}]) & \xrightarrow{\mu^{+\varepsilon}} & \mathrm{St}^G(R[t] + m[t^{-1}]) & \xrightarrow{p^{+\varepsilon}} & \mathrm{St}^G(k[t]) \\
 & & \downarrow j_2^+ & \searrow & \downarrow i_2^+ & & \downarrow \overline{i}_2^+ \\
 C_{\pm} & \hookrightarrow & \mathrm{St}^G(R[t^{\pm 1}], m[t^{\pm 1}]) & \xrightarrow{\mu^{\pm}} & \mathrm{St}^G(R[t^{\pm 1}]) & \xrightarrow{p^{\pm}} & \mathrm{St}^G(k[t^{\pm 1}]) \\
 & & \downarrow j_2^+ & \searrow & \downarrow i_2^+ & & \downarrow \overline{i}_2^+ \\
 & & \mathrm{Ker}(p^{\pm}) & & & &
 \end{array}$$

Additional arrows in the diagram include: $\mathrm{Ker}(p^+)$ to $\mathrm{Ker}(p^{+\varepsilon})$ via \overline{j}_1^+ ; $\mathrm{Ker}(p^{+\varepsilon})$ to $\mathrm{Ker}(p^{\pm})$ via \overline{j}_2^+ ; a dashed arrow φ from $\mathrm{St}^G(R[t] + m[t^{-1}])$ to $\mathrm{St}^G(R[t^{\pm 1}])$; and a curved arrow from $\mathrm{St}^G(R[t], m[t])$ to $\mathrm{Ker}(p^+)$.

$\mathrm{Ker}(p^+)$

Proposition 6.1. *The map k^+ is surjective.*

Proof. First of all, notice that using Tulenbaev's lifting property 2.1 we find a map φ such that $i_2^+ \varphi = \mu^{\pm}$.

Compare with [Tul83, Proposition 4.1]. □

$\mathrm{cor}: \mathrm{TulInj}$

Corollary 6.2. *The map i_2^+ is injective.*

Proof. Follows from the above proposition by a simple diagram chasing (cf. [Tul83, Cor. 4.2]).

Indeed, let $g \in \mathrm{St}^G(R[t] + m[t^{-1}])$ be an element of $\ker(i_2^+)$. By Corollary 5.1 g also lies in $\ker(p^{+\varepsilon})$ and hence comes from some $\tilde{g} \in \mathrm{St}^G(A[t] + m[t^{-1}], m[t^{\pm 1}])$ via $\mu^{+\varepsilon}$. Since $j_2^+(\tilde{g})$ lies in C_{\pm} by Proposition 6.1 it comes from some $\hat{g} \in C_+$ via k^+ . It remains to notice that $g = \mu^{+\varepsilon}(\tilde{g}) = i_1^+ \mu^+(\hat{g}) = i_1^+(1) = 1$, as claimed. □

6.2. Main result. The following lemma is an analog of [Tul83, Proposition 4.3 (a)].

Lemma 6.3. *Let R be a local ring, $k = R/m$, and let k, G be as in Theorem 3. The natural homomorphism $\mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}])$ is injective.*

Proof. Let I be the maximal ideal of R , $l = R/I$, and consider the natural maps $\rho : \mathrm{St}^G(R[t, t^{-1}]) \rightarrow \mathrm{St}^G(l[t, t^{-1}])$, $\rho_+ : \mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(l[t])$, $\rho_- : \mathrm{St}^G(R[t^{-1}]) \rightarrow \mathrm{St}^G(l[t^{-1}])$. Take $x \in \ker(\mathrm{St}^G(R[t]) \rightarrow \mathrm{St}^G(R[t^{\pm 1}]))$. By the field case Corollary 5.1 one has $\rho_+(x) = 1$, hence $x \in \mathrm{St}^G(I \cdot R[t])^{\mathrm{St}^G(R[t])}$.

???????

□

Lemma 6.4. *Let G, k be as in Theorem 3. Let (R, m) be a local ring such that $R/m = k$. Then*

$$\mathrm{St}^G(R[t]) \cap \mathrm{St}^G(R[t^{-1}]) = \mathrm{St}^G(R)$$

inside $\mathrm{St}^G(R[t^{\pm 1}])$.

Proof. ?????? □

Theorem 4. *Let A be any commutative ring. Then the sequence of pointed sets*

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

Proof. Follows from the above lemmas. □

Corollary 6.5. *Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$ is injective.*

Proof. ??? □

Appendices

A. LINEAR STEINBERG GROUP IN RANK 3

The main goal of this subsection is to show that Tulenbaev's [Tul83, Lemma 2.3] remains valid for the linear Steinberg group of rank ≥ 3 . In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [Sin16, Definitions 3.3 and 3.7]).

Definition A.1. The relative Steinberg group $\text{St}^*(n, R, I)$ is the group defined by the following two families generators and four families of relations.

- Generators:

- (1) $X^1(u, v)$, where $u \in E(n, R) \cdot e_1$, $v \in I^n$ such that $v^t \cdot u = 0$;
- (2) $X^2(u, v)$, where $u \in I^n$, $v \in E(n, R) \cdot e_1$ such that $v^t \cdot u = 0$.

Notice that ϕ maps both $X^1(u, v)$ and $X^2(u, v)$ to $T(u, v) = e + u \cdot v^t \in E(n, R, I)$.

- Relations:

- (1) $X^1(u, v) \cdot X^1(u, w) = X^1(u, v + w)$, $u \in E(n, R) \cdot e_1$, $v, w \in I^n$;
- (2) $X^2(u, v) \cdot X^2(w, v) = X^2(u + w, v)$, $u, w \in I^n$, $v \in E(n, R) \cdot e_1$;
- (3) $X^\sigma(u^2, v^2) X^\tau(u^1, v^1) = X^\tau(T(u^2, v^2) \cdot u^1, T(v^2, u^2)^{-1} \cdot v^1)$, $\sigma, \tau = 1, 2$;
- (4) $X^1(g \cdot e_1, g^* \cdot b e_2) = X^2(g \cdot b e_1, g^* \cdot e_2)$ where $b \in I$ and $g^* = g^{t^{-1}}$ denotes the contragradient matrix.

Lemma A.2. *The groups $\text{St}^*(n, R, I)$ and $\text{St}(n, R, I)$ are isomorphic.*

Proof. **TODO:** □

The next step of the proof is to construct certain elements in $\text{St}(n, R)$ similar to Tulenbaev's elements $X_{u,v}(a)$ see [Tul83, § 1].

Let $v \in R^n$ be a column. Denote by $O(v)$ the submodule of R^n consisting of all columns w such that $w^t \cdot v = 0$. A column $w \in R^n$ is called *v-decomposable* if it can be presented as a finite sum $w = \sum_{i=1}^p w^i$ such that each w^i has at least two zero entries and $v^t \cdot w^i = 0$. Denote by $D(v)$ the submodule of $O(v)$ consisting of all *v-decomposable* columns. For a column $v \in R^n$ denote by $I(v)$ the ideal of R spanned by its entries v_1, \dots, v_n .

Let $u, v, w \in R^n$ be columns such that $w^t v = 0$. It is easy to check (cf. [Kal77, Lemma 3.2]) that

$$(uv) \cdot w = \sum_{i < j} w_{ij}, \text{ where } w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n.$$

The above decomposition is called the *canonical* decomposition of $(uv) \cdot w$. In particular, this shows that the column $a \cdot w$ is always v -decomposable for $a \in I(v)$, $w \in O(v)$, i.e. $I(v) \cdot O(v) \subseteq D(v)$. It is also straightforward to check that $D(v) \subseteq D(bv)$, $b \cdot D(v) \subseteq D(v)$ for $b \in R$.

Denote by B^1 the subset of $R^n \times R^n \times R$ consisting of triples (u, v, a) such that $v^t \cdot u = 0$, $v \in D(u)$, $a \in I(u)$. Denote by B^2 the set consisting of triples (v, u, a) such that $(u, v, a) \in B^1$.

lem:Zfacts

Lemma A.3. *Assume that $n \geq 4$. One can define two families of elements $Z^\tau(u, v, a)$, $\tau = 1, 2$ of the group $\text{St}(n, R)$ parametrized by $(u, v, a) \in B^\tau$ satisfying the following properties:*

- (1) $\phi(Z^\tau(u, v, a)) = e + uav^t \in E(n, R)$, $(u, v, a) \in B^\tau$;
- (2) $Z^1(u, v + w, a) = Z^1(u, v, a) \cdot Z^1(u, w, a)$;
- (3) $Z^2(v + w, u, a) = Z^2(v, u, a) \cdot Z^2(w, u, a)$;
- (4) for $\tau = 1, 2$ and $b \in R$ if $(u, vb, a), (ub, v, a) \in B^\tau$ then one has

$$Z^\tau(u, vb, a) = Z^\tau(u, v, ab) = Z^\tau(ub, v, a);$$

- (5) ${}^g Z^\tau(u, v, a) = Z^\tau(\phi(g) \cdot u, \phi(g)^* \cdot v, a)$, $\tau = 1, 2$, $g \in \text{St}^G(n, R)$.

Proof. One constructs the elements $Z^1(u, v, a)$ in exactly the same way as Tulenbaev constructs his elements $X_{u,v}(a)$ (see definitions preceding [Tul83, Lemma 1.2]). Indeed, set

$$(A.1) \quad Z^1(v, w, a) = \prod_{k=1}^p X(v, a \cdot w^k), \quad Z^2(w, v, a) = \prod_{k=1}^p X(a \cdot w^k, v).$$

where $X(u, v)$ denotes the elements defined by Tulenbaev before [Tul83, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case $n = 4$) can be proved by the same token as in [Tul83, Lemma 1.3]. \square

For the rest of this section a denotes a nonnilpotent element of R and $\lambda_a: R \rightarrow R_a$ is the morphism of principal localization at a .

lem:rk3rels

Lemma A.4. *For any $g \in E(n, R_a)$ there exist $u, v \in R^n$ and sufficiently large natural numbers k, m such that the following facts hold:*

- (1) $\lambda_a(u) = g \cdot a^k e_1$, $\lambda_a(v) = g^* \cdot a^k e_2$ and $u^t \cdot v = 0$;
- (2) $(u, v, a^m) \in B^1 \cap B^2$;
- (3) for $b \in R$ divisible by some sufficiently large power of a one has

$$Z^1(u, b \cdot v, a^m) = Z^2(b \cdot u, v, a^m).$$

Proof. It is straightforward to choose u and v satisfying the first requirement of the lemma. We can even choose u, v in such a way that $u \in D(v)$ and $v \in D(u)$. Indeed, notice that $I(u) = a^{k_1}$, $I(v) = a^{k_2}$ for some natural k_1, k_2 hence for $u' = a^{k_2} \cdot u$ and $v' = a^{k_1} \cdot v$ one has

$$u' \in I(v) \cdot O(v) \subseteq D(v) \subseteq D(v'), \quad v' \in I(u) \cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns $x, y \in R^n$ and a large natural p in such a way that vectors u, v, x, y additionally satisfy the following properties

$$\lambda_a(x) = g^* \cdot a^k e_3, \quad \lambda_a(y) = g \cdot a^k e_3, \quad y^t \cdot x = a^p \in R,$$

$$u^t \cdot x = 0, \quad u^t \cdot v = 0, \quad y^t \cdot v = 0,$$

$$(u, x, a^m) \in B^1, \quad (y, v, a^m) \in B^2.$$

Now direct computation using Lemma A.3 shows that

$$\begin{aligned} Z^2(a^{m+p}b \cdot u, v, a^m) &= Z^2(b \cdot (e + a^m \cdot ux^t)y, (e - a^m \cdot xu^t)v, a^m) \cdot Z^2(-by, v, a^m) = \\ &= [Z^1(u, x, a^m), Z^2(b \cdot y, v, a^m)] = \\ &= Z^1(u, x, a^m) \cdot Z^1((e + a^m b \cdot yv^t)u, -(e - a^m b \cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b \cdot v, a^m), \quad \square \end{aligned}$$

hence the third assertion of the lemma follows.

cor:tulmap

Corollary A.5. *[Tulenbaev’s lemma] For $n \geq 4$ there is a map T_n so that the following diagram commutes.*

Proof. Follows from Lemma A.4 by the same token as in [Tul83, Lemma 2.3]. \square

REFERENCES

- Abe [Abe83] E. Abe. “Whitehead groups of Chevalley groups over polynomial rings”. *Comm. Algebra* **11**, no. 12, (1983), pp. 1271–1307.
- AHW15 [AHW15] A. Asok, M. Hoyois, and M. Wendt. “Affine representability results in \mathbb{A}^1 -homotopy theory II: principal bundles and homogeneous spaces”. 2015. arXiv: [1507.08020](https://arxiv.org/abs/1507.08020) [math.AG].
- Deo [Deo78] V. V. Deodhar. “On central extensions of rational points of algebraic groups”. *Amer. J. Math.* (1978), pp. 303–386.
- HOM [HO89] A. J. Hahn and T. O’Meara. *The classical groups and K-theory*. Springer, 1989.
- J [Jar83] J. F. Jardine. “On the homotopy groups of algebraic groups”. *J. Algebra* **81**, no. 1, (1983), pp. 180–201.
- Ka [Kal77] W. van der Kallen. “Another presentation for Steinberg groups”. *Indag. Math.* **80**, no. 4, (1977), pp. 304–312.
- La15 [Lav15] A. Lavrenov. “Local-global principle for symplectic K_2 ”. 2015.
- Re75 [Reh75] U. Rehmann. “Präsentationen von Chevalleygruppen über $k[t]$ ”. 1975. URL: <http://www.math.uni-bielefeld.de/~rehmann/Publ/praesentation.pdf>.
- St-lect [SFW67] R. Steinberg, J. Faulkner, and R. Wilson. *Lectures on Chevalley groups*. Yale University New Haven, 1967.
- S15 [Sin16] S. Sinchuk. “On centrality of K_2 for Chevalley groups of type E_ℓ ”. *J. Pure Appl. Algebra* **220**, no. 2, (2016), pp. 857–875.
- St-poly [Sta14] A. Stavrova. “Homotopy invariance of non-stable K_1 -functors”. *J. K-theory* **13**, no. 2, (2014), pp. 199–248.
- Tu [Tul83] M. Tulenbaev. “The Steinberg group of a polynomial ring”. *Math. USSR Sb.* **45**, (1983), pp. 131–144.
- VW [VW12] K. Völkel and M. Wendt. “On \mathbb{A}^1 -fundamental groups of isotropic reductive groups”. 2012. arXiv: [1207.2364](https://arxiv.org/abs/1207.2364) [math.KT].
- W-k[t] [Wen14] M. Wendt. “On homology of linear groups over $k[t]$ ”. *Math. Res. Lett.* **21**, no. 6, (2014).

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA

E-mail address: `sinchukss@gmail.com`

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA

E-mail address: `anastasia.stavrova@gmail.com`

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, D-80333 MÜNCHEN

E-mail address: `avlavrenov@gmail.com`