

\mathbb{A}^1 -INVARIANCE FOR UNSTABLE K_2

S. SINCHUK, A. STAVROVA, AND A. LAVRENOV

1. AIM OF THE PAPER

The aim of this text is to prove that the non-stable K_2 -functors K_2^G , where G is a simply connected Chevalley group of suitable type, satisfy \mathbb{A}^1 -invariance on regular rings R containing a field k , that is,

$$(1.1) \quad K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K -functor associated to G . This functor originates from [5]. The above equality by e.g. [2, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the \mathbb{A}^1 -fundamental group of G in the sense of Morel–Voevodsky.

”Suitable type” here means that we consider only the cases where we know the centrality of K_2 , or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \dots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tutenbaev’s proof for the SL_n case [14] and Stavrova’s proof for K_1^G [10]. Tutenbaev [14] uses stabilization of the K_2 -functor, and the good properties of the limit=algebraic K -theory. In [10] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [14, p. 140], or, respectively, [10, Theorem 3.1]); Quillen-Suslin lgp; \mathbb{P}^1 -gluing (see [14, Theorem 5.1] or [10, Theorem 1.1]).

1.1. Plan of the proof.

- (1) (**done**, see ??). Let R be a local ring. Show that

$$\mathrm{St}^G(R[t^{\pm 1}]) = i_+(\mathrm{St}^G(R[t]))i_-(\mathrm{St}^G(R[t^{-1}]))i_+(\mathrm{St}^G(R[t]))$$

Here i_{\pm} denote the natural homomorphisms into $\mathrm{St}^G(R[t^{\pm 1}])$.

- (2) (**done**, see Theorem 3). Show that $K_2^G(k[t]) = K_2^G(k)$.
 (3) (**done**). Consequently, $\mathrm{St}^G(k[t]) \rightarrow \mathrm{St}^G(k[t^{\pm 1}])$ is injective and $\mathrm{St}^G(k[t]) \cap \mathrm{St}^G(k[t^{-1}]) = \mathrm{St}^G(k)$ inside $\mathrm{St}^G(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

- (4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen-Suslin lgp for K_2^G .

2010 *Mathematics Subject Classification*. 19C09, 19C20, 14L15, 20G35.

Key words and phrases. Chevalley group, non-stable K_2 -functor, Steinberg group.

plan:Zglu

- (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma

- (6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A . If

$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu

- (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A , and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \rightarrow A_h$ the localization homomorphism.
- (i) If $Ah + B = A$, i.e. the natural map $B \rightarrow A/Ah$ is surjective, then for any $x \in \text{St}^G(A_h)$ there exist $y \in \text{St}^G(A)$ and $z \in \text{St}^G(B_h)$ such that $x = F_h(y)z$.
 - (ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \rightarrow A/Ah$ is an isomorphism, and h is not a zero divisor in A , then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1, g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [10, Lemma 3.4].

plan:P1

- (8) (\mathbb{P}^1 -gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

- (a) Prove Proposition 3.13. This is hard. Have no idea how this can be proved at the moment.
- (b) Prove \mathbb{P}^1 -gluing using all the above facts.

plan:[]f

- (9) (Main corollary of \mathbb{P}^1 -gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$ is injective.

The proof uses (5) and (8).

plan:k(t)

- (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tutenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2 .

plan:k[tn]

- (11) Prove that

$$K_2^G(k[t_1, \dots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G , this uses (9), (2), and (10). There may be other ways.

plan:final

- (12) Final result: let R be a regular ring containing a field k . Then $K_2^G(R[t]) = K_2^G(R)$. This uses (4), (11) and (7).

2. STEINBERG GROUPS: PRELIMINARIES

2.1. Definition and basic properties. In what follows Φ denotes a reduced irreducible root system and $\Pi \subseteq \Phi$ denotes its basis (i.e. the set of simple roots). Denote by $\tilde{\alpha}$, Φ^+ and Φ^- , respectively, the maximal root of Φ and the subsets of positive and negative roots of Φ . The Dynkin diagram and the extended Dynkin diagram of Φ corresponding to Π will be denoted by $D(\Phi)$, $\tilde{D}(\Phi)$, respectively.

A proper closed root subset $S \subseteq \Phi$ is called *parabolic* (resp. *reductive*, resp. *special*) if $\Phi = S \cup -S$ (resp. $S = -S$, resp. $S \cap -S = \emptyset$). Any parabolic subset $S \subseteq \Phi$ can be decomposed into the disjoint union of its reductive and special part, i.e. $S = \Sigma_S \sqcup \Delta_S$, where $\Sigma_S \cap (-\Sigma_S) = \emptyset$, $\Delta_S = -\Delta_S$.

Denote by $m_\beta(\alpha)$ the coefficient of β in the expansion of α in Π , i.e. $\alpha = \sum_{\beta \in \Pi} m_\beta(\alpha)\beta$. For $\beta \in \Pi$ denote by Δ_β the subsystem of Φ spanned by all simple roots except β and by Σ_β the set consisting of roots $\alpha \in \Phi$ such that $m_\beta(\alpha) > 0$.

We denote by (α, β) the scalar product of roots and by $\langle \beta, \alpha \rangle$ the integer $2(\beta, \alpha)/(\alpha, \alpha)$. The Weyl group $W(\Phi)$ is a subgroup of isometries of Φ generated by all reflections σ_α , where $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \cdot \alpha$. For a subset of roots $S \subseteq \Phi$ we denote by $\langle S \rangle$ the root subsystem spanned by S , i.e. the minimal subset of Φ containing S and invariant under the action of reflections σ_α , $\alpha \in S$.

Let $G = G(\Phi, -)$ be a (split) simply connected simple Chevalley–Demazure group scheme over R with a root system Φ of rank ≥ 2 . Recall that the *Steinberg group* $\text{St}^G(R)$ (also denoted $\text{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi, R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi, R}$ defined as follows:

$$\boxed{\text{rel:add}} \quad (2.1) \quad x_\alpha(s)x_\alpha(t) = x_\alpha(s+t), \quad \alpha \in \Phi, \quad s, t \in R;$$

$$\boxed{\text{rel:CCF}} \quad (2.2) \quad [x_\alpha(s), x_\beta(t)] = \prod_{i,j \in \mathbb{N}} x_{i\alpha+j\beta} (N_{\alpha\beta ij} s^i t^j), \quad \alpha, \beta \in \Phi, \quad \alpha \neq \pm\beta, \quad s, t \in R.$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$. Denote by $W(\Phi, R)$ the subgroup of $\text{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$, $\alpha \in \Phi$, and by $H(\Phi, R)$ the subgroup generated by all elements $h_\alpha(\varepsilon)$, $\varepsilon \in R^*$, $\alpha \in \Phi$.

Following [12], we set

$$\{u, v\}_\alpha = h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}, \quad u, v \in R^*, \quad \alpha \in \Phi,$$

and call these elements *symbols in* $\text{St}(\Phi, R)$. For any ideal I of R , we set

$$\text{Sym}(\Phi, R, I) = \langle \{u, v\}_\alpha, \quad u \in R^*, \quad v \in (1+I)^*, \quad \alpha \in \Phi \rangle \leq \text{St}(\Phi, R).$$

The group $\text{Sym}(\Phi, R, R)$ is denoted by $\text{Sym}(\Phi, R)$. Clearly, one has $\{u, v\}_\alpha \in K_2(\Phi, R)$ for any $u, v \in R^*$, $\alpha \in \Phi$. By [12, Prop. 1.3 (c)] the group $\text{Sym}(\Phi, R, I)$ is generated by all symbols $\{u, v\}_\alpha$, $u \in R^*$, $v \in (1+I)^*$, for any fixed long root $\alpha \in \Phi$.

2.2. Non-standard generation of Steinberg groups.

lem:parab-gen

Lemma 2.1. *Let R be any commutative ring. Let Φ be an irreducible root system of rank ≥ 2 , let Π be a system of simple roots in Φ , and let $J \subseteq \Pi$ be such that $|J| \geq 2$. Set*

$$\alpha_J = \sum_{\beta \in J} m_\beta(\alpha) \beta \text{ for any } \alpha \in \Phi,$$

and

$$\Sigma_J = \{\alpha \in \Phi \mid m_\beta(\alpha) > 0 \text{ for at least one } \beta \in J\}.$$

Let H be the group defined by the generators $x_\alpha(u)$, $\alpha \in \Sigma_J \cup (-\Sigma_J)$, $u \in R$, and the relations (2.1) and (2.2) ranging only over $\alpha \in \Sigma_J \cup (-\Sigma_J)$, and $\beta \in \Sigma_J \cup (-\Sigma_J)$ such that $m\alpha_J \neq -k\beta_J$ for all $m, k \in \mathbb{N}$. Then the natural homomorphism $H \rightarrow \text{St}(\Phi, R)$ is surjective and has central kernel. In particular, if $\text{St}(\Phi, R)$ is centrally closed, then $H \cong \text{St}(\Phi, R)$.

Proof. The group scheme $G = G(\Phi, -)$ over R contains two opposite parabolic R -subgroups P^\pm such that $\pm\Sigma_J$ are the sets of roots corresponding to the unipotent radicals of P^\pm . One can show that $H = \text{St}_{P^+}(R)$ and $\text{St}(\Phi, R) = \text{St}_B(R)$ in the sense of [11]. By [11, Lemma 8] the natural homomorphism $H \rightarrow \text{St}(\Phi, R)$ is surjective. By [11, Lemma 14] its kernel is central. □

2.3. Relative Steinberg groups. Denote by $D(R, I)$ the double of the ring R relative to an ideal I , i.e. the fibered product of rings $R \times_{R/I} R$ with the natural projections $p_1, p_2: D(R, I) \rightarrow R$ defined by $p_i(\xi_1, \xi_2) = \xi_i$, $i = 1, 2$. Denote by G_i the kernel of the map $p_i^*: \text{St}^G(D(R, I)) \rightarrow \text{St}^G(R)$. We define the relative Steinberg group $\text{St}^G(R, I)$ as G_1/C , where $C = [G_1, G_2]$. Clearly, there is an exact sequence.

{eq:suite} (2.3) $1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \text{St}^G(R, I) \xrightarrow{\overline{p_2^*}} \text{St}^G(R) \xrightarrow{\pi^*} \text{St}^G(R/I) \longrightarrow 1$

Lemma 2.2. *Assume that R and I are such that the canonical projection $R \rightarrow R/I$ splits. Then the following facts are true.*

item:st-inj
item:st-semi

- (i) *The map $\text{St}^G(R, I) \rightarrow \text{St}^G(R)$ is an injection.*
- (ii) *The group $\text{St}^G(R, I)$ is isomorphic to $\text{St}^G(R/I) \ltimes \text{St}^G(R, I)$.*

Proof. For the proof of the first assertion see [9, Lemma 8]. Since the group $(G_1 \cap G_2)/C$ vanishes, the sequence (2.3) turns into a split short exact sequence which implies the second assertion. □

lem:Zgen

Lemma 2.3. *Let Σ be the special subset of some parabolic subset of roots $S \subseteq \Phi$. Then the relative Steinberg group $\text{St}^G(R, I)$ admits the following generating set:*

$$\mathcal{Z}(\Sigma, R, I) = \{x_\alpha(0, s) \cdot C \mid s \in I, \alpha \in \Phi\} \cup \{z_\alpha(s, \xi) \mid s \in I, \xi \in R, \alpha \in \Sigma\},$$

where $z_\alpha(s, \xi)$ denotes the element $x_\alpha(0, s)^{x_{-\alpha}(\xi, \xi)} \cdot C$.

Proof. See [9, Lemma 5]. □

2.4. Tulenbaev's lifting property and its corollaries. Throughout this section $I \trianglelefteq A$ is an ideal of arbitrary commutative ring A . For a nonnilpotent element $a \in A$ denote by $\lambda_a: A \rightarrow A_a$ the morphism of principal localization at a . Consider the following commutative square.

$$(2.4) \quad \begin{array}{ccc} A & \xrightarrow{\lambda_a} & A_a \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\bar{\lambda}_a} & A_a/I_a \end{array}$$

Notice that (2.4) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a . Such squares are usually called *Milnor squares* in the literature, see [16, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [14, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp

Definition 2.4. We say that the Steinberg group functor St^G satisfies *Tulenbaev's lifting property* if for every pull-back square (2.4) the following lifting problem has a solution.

$$\begin{array}{ccc} \text{St}^G(A, I) & \xrightarrow{\mu} & \text{St}^G(A) \\ \downarrow & \nearrow & \downarrow \lambda_a^* \\ \text{St}^G(A_a, I) & \xrightarrow{\mu} & \text{St}^G(A_a) \end{array}$$

Theorem 1. Assume that G satisfies Tulenbaev property (2.4) then the following facts are true for arbitrary commutative ring A :

thm:dp

(i) A dilation principle holds for $\text{St}^G(-)$, i. e. if $g \in \text{St}^G(A[t], tA[t])$ is such that equality $\lambda_a^*(h) = 1$ holds in $\text{St}^G(\Phi, R_a[t])$ then for sufficiently large n one has

$$ev_{\left[\frac{R[t] \rightarrow R[t]}{t \mapsto a^n \cdot t} \right]}^*(h) = 1.$$

thm:lg-k2

(ii) A local-global principle holds for $\text{St}^G(-)$, i. e. an element $g \in \text{St}^G(A[t], tA[t])$ is trivial if and only if its image in $\text{St}^G(A_m[t], tA_m[t])$ is trivial for all maximal ideals $m \trianglelefteq A$.

thm:centr

(iii) $K_2^G(A)$ is contained in the centre of $\text{St}^G(A)$.

Proof. Follows by the same argument as [14, Theorem 2.1] or [9, Theorem 2] □

sec:sigma

2.5. The automorphisms σ_i . Our notation and conventions follows [15, § 4]. Let Φ be an irreducible root system with some fixed basis of simple roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. We denote by Φ^\vee the dual root system of Φ consisting of vectors $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \Phi$. As usual, $P(\Phi^\vee)$ denotes the lattice spanned by the fundamental weights ϖ_i . Recall that ϖ_i are uniquely determined by relations $\langle \varpi_i, \alpha_j^\vee \rangle = (\varpi_i, \alpha_j) = \delta_{ij}$.

Notice that for $\varpi \in P(\Phi^\vee)$ and $\beta \in \mathbb{Z}\Phi$ one has $(\varpi, \beta) \in \mathbb{Z}$. Consequently, for $\varepsilon \in R^*$ and $\varpi \in P(\Phi^\vee)$ the identity $\chi_{\varpi, \varepsilon}(\beta) = \varepsilon^{(\varpi, \beta)}$ gives a well-defined character $\chi_{\varpi, \varepsilon} \in \text{Hom}(\mathbb{Z}\Phi, R^*)$.

Consider the action of $H = \text{Hom}(\mathbb{Z}\Phi, R^*)$ on the set of generators $\mathcal{K}_{\Phi, R}$ of the Steinberg group $\text{St}^G(R)$ defined by

$$(2.5) \quad \chi \cdot x_\alpha(\xi) = x_\alpha(\chi(\alpha) \cdot \xi), \quad \chi \in H, \quad \alpha \in \Phi, \quad \xi \in R.$$

Since χ is a character, the above action preserves the set of Steinberg relations $\mathcal{R}_{\Phi, R}$ and, thus, gives a well-defined action of H on $\mathrm{St}^G(R)$.

Example 2.5. The principal example which motivates the above construction is as follows. Let A be a ring, take $R = A[t, t^{-1}]$ to be the ring of Laurent polynomials over A and let $\alpha_i \in \Pi$ be some simple root. Since $t \in R^*$ we can consider the automorphisms σ_i^+ and σ_i^- of $\mathrm{St}(\Phi, R)$ given by $\sigma_i^+ = \chi_{\varpi_i, t}$, $\sigma_i^- = \chi_{\varpi_i, t^{-1}}$. It is easy to see that

`eq:sigma_act}`

$$(2.6) \quad \sigma_i^{\pm}(x_{\alpha}(\xi)) = x_{\alpha}(t^{\pm m_i(\alpha)} \cdot \xi),$$

where $m_k(\alpha)$ denotes the coefficient in the expansion of α in Π , i. e. $\alpha = \sum m_k(\alpha)\alpha_k$.

One of the key steps of our proof of Suslin lemma for K_2 is to define an analogue of σ_i for the group $\mathrm{St}^G(A[t])$. Of course, we cannot expect such map to be automorphism or even be defined on the whole group $\mathrm{St}^G(A[t])$. However, it turns out that for certain i is still possible to define certain subgroups of $\mathrm{St}^G(A[t])$ and the maps modeling σ_i between them.

3. \mathbb{P}^1 -GLUING

Throughout this section $G = G(\Phi, -)$ denotes a simply connected Chevalley–Demazure group scheme of type Φ .

`def:p1g`

Definition 3.1. Let F be a group-valued functor from **CRings** to **Groups** and let A be a commutative ring. Consider the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{i_+} & A[t] \\ i_- \downarrow & & \downarrow j_+ \\ A[t^{-1}] & \xrightarrow{j_-} & A[t, t^{-1}] \end{array}$$

We say that F satisfies the \mathbb{P}^1 -glueing property for A if the following sequence of *pointed sets* is exact in the middle term:

$$F(A) \xrightarrow{\Delta_A^F} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\pm_A^F} F(A[t, t^{-1}]).$$

Here Δ_A^F denotes the (split injective) diagonal map and, by definition, \pm_A^F maps (g^+, g^-) to $F(j_+)(g^+) \cdot F(j_-)(g^-)^{-1}$. Notice that a priori \pm_A^F is only a morphism of pointed sets but if F takes values in abelian groups then \pm_A^F is also a morphism of groups.

An equivalent way to formulate \mathbb{P}^1 -glueing property is as follows: $F(j_+)$ and $F(j_-)$ are injective and the intersection of their images coincides with the image of $F(j_+i_+) = F(j_-i_-)$.

The main result of this section is the following theorem which generalizes [14, Theorem 5.1] to Chevalley groups. Notice that a K_1 -analogue of the result below has been established in a much greater generality by the second-named author (see [10, Theorem 1.1]).

`thm:p1`

Theorem 2. Assume that G satisfies Tulenbaev lifting property 2.4. Then the Steinberg group functor $\mathrm{St}^G(-)$ satisfies \mathbb{P}^1 -glueing property for an arbitrary commutative ring A .

Proof. Let (g^+, g^-) be an element of $\mathrm{St}^G(A[t]) \times \mathrm{St}^G(A[t^{-1}])$ such that the equality $g^+ = g^-$ holds in $\mathrm{St}^G(A[t, t^{-1}])$.

Let m be a maximal ideal of A . By Proposition 3.14 below the functor $\mathrm{St}^G(-)$ satisfies \mathbb{P}^1 -glueing property for the local ring A_m hence $(\lambda_m^*(g_+), \lambda_m^*(g_-)) = \Delta_{A_m}(\lambda_m^*(g^+)(0))$ and in the groups $\mathrm{St}^G(A_m[t])$ and $\mathrm{St}^G(A_m[t^{-1}])$ we have the equalities:

$$\lambda_m^*(g^+ \cdot g^+(0)^{-1}) = \lambda_m^*(g^+) \cdot \lambda_m^*(g^+)(0)^{-1} = 1; \quad \lambda_m^*(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for $\mathrm{St}^G(-)$ (see Theorem 1.(ii)) these equalities hold globally and $(g^+, g^-) = \Delta_A(g^+(0))$, as claimed. \square

rem:stk2

Remark 3.2. It is clear that if the functor St^G satisfies \mathbb{P}^1 -glueing property for A then so does the functor K_2^G . The converse statement also holds, indeed, if $(g^+, g^-) \in \mathrm{Ker}(\pm_A^{\mathrm{St}})$ then inside $E^G(A[t, t^{-1}])$ we have the equality:

$$\varphi(\mathrm{St}^G(j_+)(g^+)) = \varphi(\mathrm{St}^G(j_-)(g^-)) \in E^G(A[t]) \cap E^G(A[t^{-1}]) = E^G(A).$$

Consequently, we can find $g_0 \in \mathrm{St}^G(A)$ so that $(g^+ g_0^{-1}, g^- g_0^{-1}) \in \mathrm{Ker}(\pm_A^{K_2})$ and it remains to apply the \mathbb{P}^1 -glueing property for K_2^G .

Corollary 3.3. *Let A be any commutative ring and $f \in A[t]$ be a monic polynomial. Then the map $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$ is injective.*

Proof. \square

3.1. The field case. Throughout this section k denotes an arbitrary field.

thm:k[t]

Theorem 3. *Assume that $G = G(\Phi, -)$ and Φ is any irreducible root system of rank ≥ 2 .*

satz1

- (i) *The subgroup $K_2(\Phi, k[t]) \leq \mathrm{St}^G(\Phi, k[t])$ is generated by symbols $\{u, v\}_\alpha$, $u, v \in k^*$, $\alpha \in \Phi$.*
- (ii) *As a consequence, the canonical injection $K_2(\Phi, k) \hookrightarrow K_2(\Phi, k[t])$ is an isomorphism, and $K_2(k[t]) = K_2(k)$ is central in $\mathrm{St}(\Phi, k[t])$.*

Proof. See [8, Satz 1] and the corollary after it. \square

cor:k[t]inj

Corollary 3.4. *Let G be as in Theorem 3. Then the functors St^G , K_2^G satisfy \mathbb{P}^1 -glueing property for k .*

Proof. By Remark 3.2 it suffices to prove the assertion only for the functor K_2^G . By the previous theorem $K_2^G(i_+)$ and $K_2^G(i_-)$ are isomorphisms hence the morphisms $K_2^G(j_+)$ and $K_2^G(j_-)$ are split injective and $\mathrm{Im}(j_+ i_+) = \mathrm{Im}(j_- i_-) = \mathrm{Im}(j_+) = \mathrm{Im}(j_-)$. \square

thm:k[t+-1]

Theorem 4. *Let Φ be an irreducible root system of rank ≥ 2 and let α be a long root of Φ . Denote by H the subgroup of $K_2(\Phi, k[t^{\pm 1}])$ generated by symbols $\{t, u\}_\alpha$, $u \in k^*$.*

- (i) *For $\Phi \neq G_2$ one has*

$$K_2(\Phi, k[t^{\pm 1}]) = K_2(\Phi, k) \oplus H.$$

In particular, $K_2(\Phi, k[t^{\pm 1}])$ is central in $\mathrm{St}(\Phi, k[t^{\pm 1}])$. Here $K_2(\Phi, k)$ is considered as a subgroup of $K_2(\Phi, k[t^{\pm 1}])$ via the natural injection.

- (ii) *The map $\{t, -\}_\alpha: k^* \rightarrow H$ is injective.*

Proof. By [4, Korollar 4] the group $K_2(\Phi, k[t^{\pm 1}])$ is generated by $K_2(\Phi, k)$ and H . By [12, Prop. 1.1 (S1)] one has $\{1, u\}_\alpha = 1$ for any $\alpha \in \Phi$. Hence H is in the kernel of the natural projection $K_2(\Phi, k[t^{\pm 1}]) \rightarrow K_2(\Phi, k)$ sending t to 1, and $H \cap K_2(\Phi, k) = 1$. By [12, Prop. 1.3 (a)] symbols are central in $\mathrm{St}(\Phi, k[t^{\pm 1}])$, hence $K_2(\Phi, k[t^{\pm 1}]) = K_2(\Phi, k) \oplus H$.

Now consider the following commutative diagram.

$$\begin{array}{ccc} k^* \cong K_1(k) & \xrightarrow{\{t, -\}_\alpha} & K_2(\Phi, k[t^{\pm 1}]) \longrightarrow K_2(\Phi, k(t)) \\ & \searrow \{t, -\} & \downarrow \\ & & K_2^M(k(t)) \end{array}$$

By Matsumoto's theorem $K_2(\Phi, k(t))$ is isomorphic to $K_2^M(k(t))$ if Φ is nonsymplectic, otherwise $K_2(\Phi, k(t)) \cong K_2^{MW}(k(t))$, see [6, Corollaire 5.11]. Thus, the vertical arrow is either an isomorphism or the canonical projection $K_2^{MW}(k(t)) \rightarrow K_2^M(k(t))$, see [13, § 6]. On the other hand, the diagonal arrow is injective by Milnor's theorem, see [7, § 2]. Thus the left top horizontal arrow is injective as well. \square

Remark 3.5. Notice that in the symplectic case the composite map $\{t, -\}_\alpha: k^* \rightarrow K_2^{MW}(k(t))$ is *not* a morphism of groups because the Steinberg symbol $\{-, -\}_\alpha$ corresponding to the long root α is not bilinear. If we defined this map using a short root α , this map would be a morphism of groups, however, the diagonal map would equal $\{t^2, -\}$ and would not be injective.

3.2. Tulenbaev's section 3. For the rest of this section A denotes an arbitrary commutative local ring with the maximal ideal m and the residue field k . We denote by π the canonical projection $A \rightarrow k$. Throughout this section we will employ the following notation:

- R denotes the Laurent polynomial ring $A[t, t^{-1}]$;
- B denotes the subring $A[t] + m[t^{-1}]$ of R consisting of Laurent polynomials $f(t, t^{-1})$ whose coefficients of terms of negative degree belong to m ;
- I denotes the ideal $m[t, t^{-1}]$ of R (which can be also considered as an ideal of B).

Note that since A is local, then $K_2(\Phi, A) = \text{Sym}(\Phi, A)$ by [12, Theorem 2.13]. In particular, $K_2(\Phi, A)$ is central in $\text{St}(\Phi, A)$, and $W(\Phi, A)/H(\Phi, A)$ is naturally isomorphic to the Weyl group $W(\Phi)$.

Our first result is analogous to [14, Lemma 3.1(e)] (cf. also with [3, § 2.3A]).

lem:bruhat

Lemma 3.6. *Let Φ be any irreducible root system. Let $\Phi^+, \Phi^{+'}$ be two systems of positive roots in Φ .*

- (i) *The Steinberg group $\text{St}(\Phi, A)$ admits the following analogue of the Bruhat decomposition:*

{eq:bruhat}

$$(3.1) \quad \text{St}(\Phi, A) = \bigsqcup_{w \in W(\Phi)} \left(U(\Phi^{+'}, A) \cdot wH(\Phi, A) \cdot U(\Phi^+, A) \cdot \ker(\text{St}(\Phi, A) \xrightarrow{\pi^*} \text{St}(\Phi, k)) \right).$$

- (ii) *Assume that $uw h v l = u' w' h' v' l'$ for some $u, u' \in U(\Phi^{+'}, A)$, $w, w' \in W(\Phi)$, $h, h' \in H(\Phi, A)$, and $l, l' \in \ker(\text{St}(\Phi, A) \xrightarrow{\pi^*} \text{St}(\Phi, k))$. Then $w = w'$,*

$$h^{-1} h' \in \text{Sym}(\Phi, A) \cdot \ker(H(\Phi, A) \xrightarrow{\pi^*} H(\Phi, k)),$$

and there exist $a \in U(\Phi^{+'}, m)$ such that $w^{-1}(u^{-1} u' a) w \in U(\Phi^+, R)$, and $b \in U(\Phi^+, m)$ such that

$$b = v^{-1}(u^{-1} u')^{wh} v' = l(l')^{-1}.$$

Proof. For any field k , the group $G(\Phi, k) = E(\Phi, k)$ admits Bruhat decomposition, hence $K_2(\Phi, k) \leq H(\Phi, k)$ implies

$$\mathrm{St}(\Phi, k) = \bigsqcup_{w \in W(\Phi)} \mathrm{U}(\Phi^{+'}, k) w H(\Phi, k) \mathrm{U}(\Phi^+, k).$$

Let $w_0 \in W(\Phi)$ be such that $v_0(\Phi^{+'}) = \Phi^+$. Then also

$$\mathrm{St}(\Phi, k) = w_0^{-1} H(\Phi, k) \mathrm{St}(\Phi, k) = \bigsqcup_{w \in W(\Phi)} \mathrm{U}(\Phi^{+'}, k) w H(\Phi, k) \mathrm{U}(\Phi^+, k).$$

As a consequence, the first three factors in the right hand side of the decomposition (3.1) are mapped epimorphically onto $\mathrm{St}(\Phi, k)$ and the last factor coincides with $\mathrm{Ker}(\pi^*)$ from which the first assertion of the lemma follows. The second assertion follows from the unicity of the Bruhat decomposition in $G(\Phi, k)$. \square

lem:tul3.1zh

Lemma 3.7. *Assume that A is regular and the rank of Φ is at least 2. Then there is an injective map $\phi: (1 + m)^* \rightarrow \mathrm{St}(\Phi, R, I)$, natural in (A, m) , such that $j\phi = \{t, -\}_\alpha$, where $j: \mathrm{St}(\Phi, R, I) \rightarrow \mathrm{St}(\Phi, R)$ is the canonical map and α is a long root of Φ . Moreover, if Φ is nonsymplectic then ϕ is a homomorphism of groups.*

Proof. The map ϕ can be defined as follows:

$$\begin{aligned} \phi(u) &= y_\alpha(t(u-1)) y_{-\alpha}(-t^{-1}(u^{-1}-1))^{x_\alpha(-t)} \\ &\quad \cdot y_\alpha(t(u-1))^{w_\alpha(-t)} y_\alpha(u-1) y_{-\alpha}(-(u^{-1}-1))^{x_\alpha(-1)} y_\alpha(u-1)^{w_\alpha(-1)}. \end{aligned}$$

It is easy to check that $j\phi = \{t, -\}_\alpha$. If A is regular then it is integral and, therefore, can be embedded into its field of fractions K . Now the injectivity of ϕ follows from Theorem 4 and the commutativity of the following diagram.

$$\begin{array}{ccc} 1 + m & \hookrightarrow & K^* \\ \downarrow \phi & & \downarrow \\ \mathrm{St}(\Phi, R, I) & \xrightarrow{j} & \mathrm{St}(\Phi, K[t, t^{-1}]) \end{array}$$

\square

lem:sigma-X

Lemma 3.8. *Let Φ be an irreducible root system of rank $l \geq 3$ and of type A_l ($l \geq 3$), C_l ($l \geq 3$), D_l ($l \geq 4$), E_6 or E_7 . Let $\alpha_i \in \Pi$, $1 \leq i \leq l$, be a simple root of Φ such that the parabolic subgroup P_i of $G(\Phi, -)$ has abelian unipotent radical. Denote by $\mathrm{St}P_i^-(\Phi, A)$ the subgroup of $\mathrm{St}(\Phi, A)$ generated by $x_\alpha(u)$, $\alpha \in \Delta_i \cup (-\Sigma_i)$, $u \in A$, and by $H(\Phi, A)$ (?). Then there exists a group homomorphism*

$$\delta_i: \mathrm{St}(\Phi, A[t], tA[t]) \rightarrow \mathrm{St}P_i^-(\Phi, A) \cdot \mathrm{St}(\Phi, A[t], tA[t])$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{St}(\Phi, A[t], tA[t]) & \xrightarrow{j_+} & \mathrm{St}(\Phi, A[t, t^{-1}]) \\ \downarrow \delta_i & & \downarrow \chi_{\varpi_i, t} \\ \mathrm{St}P_i^-(\Phi, A) \cdot \mathrm{St}(\Phi, A[t], tA[t]) & \xrightarrow{j_+} & \mathrm{St}(\Phi, A[t, t^{-1}]) \end{array}$$

Proof. The cases A_l and C_l are done using another presentation. Other cases are done by amalgamation of A_3 -pieces.... \square

n:parab-pairs

Lemma 3.9. *Let Φ be an irreducible root system of rank $l \geq 3$ and of type A_l ($l \geq 3$), C_l ($l \geq 3$), D_l ($l \geq 4$), E_6 , E_7 or E_8 . Let $\alpha_i \in \Pi$ be the simple root adjacent to α_l in the Dynkin diagram of Φ (note that P_l has abelian unipotent radical if $\Phi \neq E_8$, and extraspecial if $\Phi = E_8$). Set $\Phi^{+'} = w_{\alpha_l}(\Phi^+)$. Then $\Phi^{+'} \setminus \Phi^+ = -\alpha_l$, $\Phi^+ \setminus \Phi^{+'} = \alpha_l$, and $\Sigma_i \setminus \Sigma_l \subseteq w_{\alpha_l}(\Sigma_l)$.*

Proof. The first two claims are obvious. The last claim follows from the fact that for any root $\alpha \in \Sigma_i \setminus \Sigma_l$ one has $w_{\alpha_l}(\alpha) = \alpha + \alpha_l$. \square

lem:tul3.3

Lemma 3.10. *Let Φ be an irreducible root system of rank ≥ 2 , let Π be a system of simple roots in Φ , and let $J \subseteq \Pi$ be such that $|J| \geq 2$. Let S be any group, and $n \in \mathbb{N}$. For any $b \in B$ let $|b| \in \mathbb{Z}$ be such that $t^{|b|}$ is the smallest power of t occurring in b . Assume that there are elements $s_\alpha(u) \in S$ for all $\alpha \in \Sigma_J \cup (-\Sigma_J)$ and $u \in B$ such that $|u| \geq -n$, satisfying the relations*

- (1) $s_\alpha(u)s_\alpha(v) = s_\alpha(u+v)$ for all $\alpha \in \Sigma_J \cup (-\Sigma_J)$, $u, v \in B$, $|u|, |v| \geq -n$;
- (2) $[s_\alpha(u), s_\beta(v)] = \prod_{i,j \in \mathbb{N}} s_{i\alpha+j\beta}(N_{\alpha\beta ij} u^i v^j)$ for all $\alpha, \beta \in \Sigma_J \cup (-\Sigma_J)$ such that $i\alpha_J \neq -j\beta_J$
for all $i, j \in \mathbb{N}$, and all $u, v \in B$ such that $i|u| + j|v| \geq -n$ whenever $N_{\alpha\beta ij} \neq 0$.

If $n = 1$, then the map $x_\alpha(u) \mapsto s_\alpha(u)$ extends to a group homomorphism $\text{St}(\Phi, B) \rightarrow S$.

Proof. \square

3.3. Tulenbaev's section 4.

thm:tul4.1

Theorem 5. *Let Φ be an irreducible root system of rank $l \geq 3$ of type A_l , C_l , D_l ($l \geq 4$), E_6 or E_7 . Then $\ker(\text{St}(\Phi, A[t], m[t]) \rightarrow \text{St}(\Phi, A[t]))$ surjects onto $\ker(\text{St}(\Phi, A[t^{\pm 1}], m[t^{\pm 1}]) \rightarrow \text{St}(\Phi, A[t^{\pm 1}]))$.*

Proof. Set

$$\tilde{B} = H(\Phi, A[t^{\pm 1}]) \cdot U(\Phi^+, A[t^{\pm 1}]) \leq \text{St}(\Phi, A[t^{\pm 1}]),$$

and

$$\tilde{D} = \phi(\text{Sym}^t(\Phi, A, m)) \leq \text{St}(\Phi, A[t^{\pm 1}], m[t^{\pm 1}])$$

in the notation of Lemma 3.7. Consider the set of equivalence classes

$$V = \text{St}(\Phi, A[t]) \times \tilde{B} \times \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}]) / \sim,$$

where $(a, b, \beta) \sim (a', b', \beta')$ if and only if there is $\gamma \in \text{St}(\Phi, A[t], mA[t])$, $p \in U(\Phi^+, A[t])$ and $\mu \in \tilde{D} \cdot U(\Phi^+, mA[t^{\pm 1}])$ such that

$$\tilde{a} = a\gamma_1^{-1}t^{-1}, \quad \tilde{b} = pt\mu, \quad \tilde{\beta} = \mu^{-1}(\gamma_2)b\beta,$$

where γ_1 and γ_2 are the images of γ in the respective groups. The elements of V will be denoted $[a, b, \beta]$.

We will define a map $\sigma : V \rightarrow V$. By Lemma 3.6 and the definition of V every element of V can be written in the form $[aw, b, \beta]$ for some $a \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+, A)$, $w \in W(\Phi, A) \leq \text{St}(\Phi, A)$, $b \in \tilde{B}$ and $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$. We define

$$\sigma|_{\text{St}(\Phi, A[t], tA[t])} = \delta_l : \text{St}(\Phi, A[t], tA[t]) \rightarrow \text{St}(\Phi, A[t], tA[t])\text{St}P_l^-(\Phi, A) \leq [\text{St}(\Phi, A[t]), 1, 1],$$

where δ_l the homomorphism constructed in Lemma 3.8. For any $\alpha \in \Phi$, $u \in A[t^{\pm 1}]$ we have a homomorphism

$$\chi_{\varpi_l, t} : X_\alpha(A[t^{\pm 1}]) \rightarrow X_\alpha(A[t^{\pm 1}]), \quad x_\alpha(u) \mapsto x_\alpha(t^{m_{\alpha_l}(\alpha)}u).$$

In particular, this induces a homomorphism $\sigma_U = \chi_{\varpi_l, t}|_{U(\Phi^+, A)} : U(\Phi^+, A) \rightarrow U(\Phi^+, A[t])$. Combining δ_l and σ_U , we obtain a homomorphism (check!!!)

$$\sigma : \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+, A) \rightarrow \text{St}(\Phi, A[t]).$$

Now we define

$$\sigma \cdot [aw, b, \beta] = [\sigma(a)w, w^{-1}\chi_{\varpi_l, t}(wb), \chi_{\varpi_l, t}(\beta)].$$

Note that, clearly, $w^{-1}\chi_{\varpi_l, t}(w) \in H(\Phi, A[t^{\pm 1}])$.

Next we define a map $\sigma' : V \rightarrow V$. Set $\Phi^{+'} = w_{\alpha_l}(\Phi^+)$. By Lemma 3.6 and the definition of V every element of V can be written in the form $[aw, b, \beta]$ for some $a \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^{+'}, A)$, $w \in W(\Phi, A) \leq \text{St}(\Phi, A)$, $b \in \tilde{B}$ and $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$. Note that $w_{\alpha_l}(\Pi)$ is a set of simple roots of Φ contained in $\Phi^{+'}$, with $w_{\alpha_l}(\alpha_l) = -\alpha_l$ playing the role of α_l . Then by Lemma 3.8 there is a homomorphism

$$\delta'_l : \text{St}(\Phi, A[t], tA[t]) \rightarrow \text{St}(\Phi, A[t], tA[t])\text{St}P_l'^-(\Phi, A),$$

compatible with $\chi_{w_{\alpha_l}(\varpi_l), t}$, where $\text{St}P_l'^-(\Phi, A)$ denotes the subgroup of $\text{St}(\Phi, A)$ corresponding to the parabolic set of roots $w_{\alpha_l}(\Delta_l \cup \Sigma_l)$. We define

$$\sigma'|_{\text{St}(\Phi, A[t], tA[t])} = \delta'_l : \text{St}(\Phi, A[t], tA[t]) \rightarrow \text{St}(\Phi, A[t], tA[t])\text{St}P_l'^-(\Phi, A) \leq [\text{St}(\Phi, A[t]), 1, 1].$$

Similarly, for any $\alpha \in \Phi$, $u \in A[t^{\pm 1}]$ we consider a homomorphism

$$\chi_{w_{\alpha_l}(\varpi_l), t} : X_{\alpha}(A[t^{\pm 1}]) \rightarrow X_{\alpha}(A[t^{\pm 1}]), \quad x_{\alpha}(u) \mapsto x_{\alpha}(t^{m'_{- \alpha_l}(\alpha)}u),$$

where $m'_{w_{\alpha_l}(\alpha_i)}(\alpha)$, $1 \leq i \leq l$, is the coefficient of $w_{\alpha_l}(\alpha_i)$ in the decomposition of $\alpha \in \Phi$ with respect to $w_{\alpha_l}(\Pi)$. We also consider a homomorphism $\sigma'_U = \chi_{w_{\alpha_l}(\varpi_l), t}|_{U(\Phi^{+'}, A)} : U(\Phi^{+'}, A) \rightarrow U(\Phi^{+'}, A[t])$. Combining δ'_l and σ'_U , we obtain a homomorphism

$$\sigma' : \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^{+'}, A) \rightarrow \text{St}(\Phi, A[t]).$$

Now we define

$$\sigma' \cdot [aw, b, \beta] = [\sigma'(a)w, w^{-1}\chi_{w_{\alpha_l}(\varpi_l), t}(wb), \chi_{w_{\alpha_l}(\varpi_l), t}(\beta)].$$

As in the case of σ , we have $w^{-1}\chi_{w_{\alpha_l}(\varpi_l), t}(w) \in H(\Phi, A[t^{\pm 1}])$.

Now we show that $\sigma, \sigma' : V \rightarrow V$ are correctly defined and bijective. We prove that for σ' , the case of σ being analogous.....

Next we show that σ and σ' commute. First we note that if $\sigma \cdot [a, 1, 1] = [x, y, 1]$, then for any $b \in \tilde{B}$ and $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$ one has

$$\sigma \cdot [a, b, \beta] = [x, y\chi_{\varpi_l, t}(b), \chi_{\varpi_l, t}(\beta)],$$

and a similar equality holds for σ' . Since $\chi_{\varpi_l, t}$ and $\chi_{w_{\alpha_l}(\varpi_l), t}$ commute on $\text{St}(\Phi, A[t^{\pm 1}])$ and on $\text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$, it is enough to check that σ and σ' commute on any element of the form $[aw, 1, 1] \in V$, where $a \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+, A)$, $w \in W(\Phi, A)$. We can write $a = a_0 \cdot x_{\alpha_l}(u)$, where $a_0 \in \text{St}(\Phi, A[t], tA[t]) \cdot U(\Phi^+ \cap \Phi^{+'}, A)$

Now we define the action of $\text{St}(\Phi, A[t^{\pm 1}])$ on V . For any $\alpha \in \Phi$ and $u \in A[t] \subseteq A[t^{\pm 1}]$ we set

$$x_{\alpha}(u) \cdot [a, b, \beta] = [x_{\alpha}(u)a, b, \beta].$$

Next we want to use Lemma 3.10 applied to the set $J = \{\alpha_i, \alpha_l\}$, where α_i is the simple root adjacent to α_l . For any $\alpha \in \pm\Sigma_l$ and any $u \in A$ we set

$$x_\alpha(t^{-1}u) \cdot [a, b, \beta] = \sigma^{\mp 1} \cdot x_\alpha(u) \cdot \sigma^{\pm 1}[a, b, \beta].$$

For any $\alpha \in \pm\Sigma_i$ and any $u \in A$ we set

$$x_\alpha(t^{-1}u) \cdot [a, b, \beta] = \sigma'^{\mp 1} \cdot x_\alpha(u) \cdot \sigma'^{\pm 1}[a, b, \beta]$$

(cf. Lemma 3.9). Then we check that this action satisfies the properties required in Lemma 3.10. This defines an action of $\text{St}(\Phi, A[t^{\pm 1}])$ on V .

The last step is to check that for any $\beta \in \text{St}(\Phi, A[t^{\pm 1}], mA[t^{\pm 1}])$ the action of its image in $\text{St}(\Phi, A[t^{\pm 1}])$ on $[1, 1, 1]$ gives $[1, 1, \beta]$. \square

3.4. Suslin's lemma.

lem:tulinj

Lemma 3.11. *Assume that G satisfies Tulenbaev lifting property 2.4. Then the map i in the following commutative diagram of groups is injective.*

{diag:cs}

(3.2)

$$\begin{array}{ccccccc} C_B & \hookrightarrow & \text{St}^G(B, I) & \xrightarrow{\mu_B} & \text{St}^G(B) & \xrightarrow{\pi_B} & \text{St}^G(k[t]) \\ \downarrow k & & \downarrow j & \nearrow \varphi & \downarrow i & & \downarrow \\ C_R & \hookrightarrow & \text{St}^G(R, I) & \xrightarrow{\mu_R} & \text{St}^G(R) & \xrightarrow{\pi_R} & \text{St}^G(k[t, t^{-1}]) \end{array}$$

Proof. First of all, notice that by Corollary 3.4 the vertical map in the right-hand side of the diagram is injective. Invoking Tulenbaev's property 2.4 we also find a lifting map φ in the central square of the diagram.

Let $g \in \text{St}^G(B)$ be an element of $\text{Ker}(i)$. Since g also lies in $\text{Ker}(\pi_B)$ it comes from some $\tilde{g} \in \text{St}^G(B, I)$ via μ_B . But $j(\tilde{g})$ lies in C_R , hence, by Proposition 3.13 below it comes from some $\hat{g} \in C_B$ via k . Finally, $g = \varphi(j(\tilde{g})) = \varphi(k(\hat{g})) = \mu_B(\hat{g}) = 1$, as claimed. \square

The following result is analogous to [14, Proposition 4.1].

Lemma 3.12. *The map j in the diagram (3.2) is surjective for any commutative ring A .*

Proof. Let α_i be arbitrary simple root from Π and let Σ_i (resp. Σ_i^-) denote the special subsets of roots consisting of $\alpha \in \Phi$ such that $m_i(\alpha) > 0$ (resp. $m_i(\alpha) < 0$).

By Lemma 2.3 the sets $\mathcal{Z}(\Sigma_i, R, I)$ and $\mathcal{Z}(\Sigma_i^-, R, I)$ both generate $\text{St}^G(R, I)$. Similarly, the subsets $\mathcal{Z}(\Sigma_i, B, I) \subseteq \mathcal{Z}(\Sigma_i, R, I)$, $\mathcal{Z}(\Sigma_i^-, B, I) \subseteq \mathcal{Z}(\Sigma_i^-, R, I)$ both generate the image of $j: \text{St}^G(B, I) \rightarrow \text{St}^G(R, I)$.

Using formula (2.6) we can calculate how the powers of the automorphism σ_i act on these generating sets. Indeed, for $z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i^-, R, I)$ we have

$$\sigma_i^N(z_\alpha(s, \xi)) = \sigma_i^N(x_\alpha(0, s)^{x_\alpha(\Delta(\xi))}) = z_\alpha(t^{Nm_i(\alpha)} \cdot s, t^{-Nm_i(\alpha)} \cdot \xi).$$

It is clear that for sufficiently large N the element $t^{-Nm_i(\alpha)} \cdot \xi$ belongs to $A[t] \subseteq B$. Consequently, for arbitrary $g \in \text{St}^G(R, I)$ there exists $N > 0$ such that $\sigma^N(g)$ lies in the image of j . On the other hand, by a similar calculation we get for $z = z_\alpha(s, \xi) \in \mathcal{Z}(\Sigma_i, B, I)$ that the element $\sigma^{-N}(z)$ ($N > 0$) still lies in $\mathcal{Z}(\Sigma_i, B, I)$. Clearly, this implies that the image of j is preserved by σ_i^{-1} . The assertion of the lemma now follows from these two statements:

$$g = \sigma^{-N} \sigma^N(g) \in \sigma^{-N}(\text{Im}(j)) \subseteq \text{Im}(j). \quad \square$$

prop:kursurj

Proposition 3.13. *Under the assumptions of Lemma 3.11 the map k in the diagram (3.2) is surjective.*

Sketch/draft of the proof. The argument presented below only may work under additional assumption that $m_i(\tilde{\alpha}) = 1$ (i is as in the statement of ??).

Consider the following set

$$X = \mathrm{St}^G(B) \times \tilde{B}(R) \times \mathrm{St}^G(R, I) / \simeq .$$

The congruence relation we impose should be similar to that used in Tulenbaev’s paper. In particular, for $h \in \mathrm{St}^G(B, I)$ we should have the following relation in X

$$[\mu_B(h), 1, g] \simeq [1, 1, j(h)g].$$

Notice that instead of $\mathrm{St}^G(A[t])$ (used by Tulenbayev in the first factor) we use $\mathrm{St}^G(B)$.

TODO: We should prove that there is a well-defined action of $\mathrm{St}^G(R)$ on X .

Now let g be an element of C_R . By the previous lemma we have for some $g' \in \mathrm{St}^G(B, I)$

$$[1, 1, g] = [1, 1, j(g')] = [\mu_B(g'), 1, 1] = i\mu_B(g')[1, 1, 1] = [1, 1, 1]. \quad \square$$

3.5. Proof of the main result. The following result is analogous to [14, Proposition 4.3]. It plays the same role in our proof of \mathbb{P}^1 -glueing for K_2 as generalized Suslin lemma (cf. [1, Theorem 2.16]) does in the corresponding proof for K_1 .

prop:p1g

Proposition 3.14. *The functors St^G , K_2^G satisfy \mathbb{P}^1 -glueing property for arbitrary local ring A .*

Proof. □

REFERENCES

- [1] E. Abe. “Whitehead groups of Chevalley groups over polynomial rings”. *Comm. Algebra* **11**, no. 12, (1983), pp. 1271–1307.
- [2] A. Asok, M. Hoyois, and M. Wendt. “Affine representability results in \mathbb{A}^1 -homotopy theory II: principal bundles and homogeneous spaces”. 2015. arXiv: [1507.08020](https://arxiv.org/abs/1507.08020) [math.AG].
- [3] A. J. Hahn and T. O’Meara. *The classical groups and K-theory*. Springer, 1989.
- [4] J. Hurrelbrink. “Endlich präsentierte arithmetische Gruppen und K_2 über Laurent-Polynomringen”. *Math. Ann.* **225**, no. 2, (1977), pp. 123–129. ISSN: 0025-5831.
- [5] J. F. Jardine. “On the homotopy groups of algebraic groups”. *J. Algebra* **81**, no. 1, (1983), pp. 180–201.
- [6] H. Matsumoto. “Sur les sous-groupes arithmétiques des groupes semi-simples déployés”. *Ann. Sci. ENS* **2**, no. 1, (1969), pp. 1–62.
- [7] J. Milnor. “Algebraic K-theory and quadratic forms”. *Invent. math.* **9**, no. 4, (1970), pp. 318–344.
- [8] U. Rehmann. “Präsentationen von Chevalleygruppen über $k[t]$ ”. 1975. URL: <http://www.math.uni-bielefeld.de/~rehmann/Publ/presentation.pdf>.
- [9] S. Sinchuk. “On centrality of K_2 for Chevalley groups of type E_ℓ ”. *J. Pure Appl. Algebra* **220**, no. 2, (2016), pp. 857–875.
- [10] A. Stavrova. “Homotopy invariance of non-stable K_1 -functors”. *J. K-theory* **13**, no. 2, (2014), pp. 199–248.
- [11] A. Stavrova. “On the congruence kernel of isotropic groups over rings”. *preprint* (2014).
- [12] M. R. Stein. “Surjective stability in dimension 0 for K_2 and related functors”. *Trans. Amer. Math. Soc.* **178**, (Apr. 1973), pp. 165–191.
- [13] A. A. Suslin. “Torsion in K_2 of fields”. *K-theory* **1**, no. 1, (1987), pp. 5–29.
- [14] M. Tulenbaev. “The Steinberg group of a polynomial ring”. *Math. USSR Sb.* **45**, (1983), pp. 131–144.
- [15] N. Vavilov. “Weight elements of Chevalley groups”. *St. Petersburg Math. J.* **20**, no. 1, (2009), pp. 23–57.

- Kbook** [16] C. A. Weibel. *The K-book: An introduction to algebraic K-theory*. Vol. 145. 2013.

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA

E-mail address: `sinchukss@gmail.com`

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA

E-mail address: `anastasia.stavrova@gmail.com`

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, D-80333 MÜNCHEN

E-mail address: `avlavrenov@gmail.com`