$\mathbb{A}^1$-INVARIANCE FOR UNSTABLE K_2

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1. Aim of the paper

The aim of this text is to prove that the non-stable K_2-functors K_2^G, where G is a simply connected Chevalley group of suitable type, satisfy $\mathbb{A}^1$-invariance on regular rings R containing a field k, that is,

{eq:A1-main} (1.1)
$$K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K-functor associated to G. This functor originates from [7]. The above equality by e.g. [4, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the $\mathbb{A}^1$-fundamental group of G in the sense of Morel–Voevodsky.

"Suitable type" here means that we consider only the cases where we know the centrality of K_2, or at least the Quillen-Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \ldots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev's proof for the SL_n case [15] and Stavrova's proof for K_1^G [11]. Tulenbaev [15] uses stabilization of the K_2-functor, and the good properties of the limit=algebraic K-theory. In [11] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [15, p. 140], or, respectively, [11, Theorem 3.1]); Quillen-Suslin lgp; $\mathbb{P}^1$-gluing (see [15, Theorem 5.1] or [11, Theorem 1.1]).

1.1. **The case $R = k$.** We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the $\mathbb{A}^1$-invariance of K_1^G, this case should be used to deduce that $K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k)$.)

In Tulenbaev's framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [19, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k. Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k. Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt's proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [5], but only for the

2010 *Mathematics Subject Classification.* 19C09, 19C20, 14L15, 20G35.

Key words and phrases. Chevalley group, non-stable K_2-functor, Steinberg group.

good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [17, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

Test, test, test [2, 3, 8, 12].

1.2. Plan of the proof.

plan:3t
(1) (**done**, see Theorem 2). Let R be a local ring. Show that

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t]))$$

Here $i_\pm$ denote the natural homomorphisms into $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

plan:k[t]
(2) (**done**, see Theorem 4). Show that $K_2^G(k[t]) = K_2^G(k)$.

n:k-intersect
(3) (**done**). Consequently, $\mathrm{St}^{\mathrm{G}}(k[t]) \to \mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$ is injective and $\mathrm{St}^{\mathrm{G}}(k[t]) \cap \mathrm{St}^{\mathrm{G}}(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ inside $\mathrm{St}^{\mathrm{G}}(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp
(4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen-Suslin lgp for K_2^G.

plan:Zglu
(5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1,g_2) \mapsto g_1 g_2{}^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma
(6) (S-lemma) Let A be a commutative ring, S a multiplicative subset of A. If

$$K_2^G(A[X_1, \ldots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \ldots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu
(7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A, and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \to A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \to A/Ah$ is surjective, then for any $x \in \mathrm{St}^{\mathrm{G}}(A_h)$ there exist $y \in \mathrm{St}^{\mathrm{G}}(A)$ and $z \in \mathrm{St}^{\mathrm{G}}(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \to A/Ah$ is an isomorphism, and h is not a zero divisor in A, then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1,g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [11, Lemma 3.4].

plan:P1
(8) ($\mathbb{P}^1$-gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g,g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1,g_2) \mapsto g_1 g_2{}^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

(a) Prove Proposition 4.9. This is hard. Have no idea how this can be proved at the moment.
(b) Prove $\mathbb{P}^1$-glueing using all the above facts.

plan:[]f (9) (Main corollary of $\mathbb{P}^1$-gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.
The proof uses (5) and (8).

plan:k(t) (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tulenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2.

plan:k[tn] (11) Prove that

$$K_2^G(k[t_1, \ldots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G, this uses (9), (2), and (10). There may be other ways.

plan:final (12) Final result: let R be a regular ring containing a field k. Then $K_2^G(R[t]) = K_2^G(R)$.
This uses (4), (11) and (7).

2. Steinberg groups: preliminaries

2.1. **Definition and basic properties.** Let G be a split simple Chevalley groups with a root system Φ of rank ≥ 2. Recall that the *Steinberg group* $\mathrm{St}^{\mathrm{G}}(R)$ (also denoted $\mathrm{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi,R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi,R}$ defined as follows:

{rel:add}
$$x_\alpha(s)x_\alpha(t) = x_\alpha(s+t), \tag{2.1}$$

{rel:CCF}
$$[x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta}\left(N_{\alpha\beta ij}\, s^i t^j\right), \quad \alpha \neq -\beta, \quad N_{\alpha\beta ij} \in \mathbb{Z}. \tag{2.2}$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$. Denote by $\mathrm{W}(\Phi, R)$ the subgroup of $\mathrm{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$.

2.2. **Relative Steinberg groups.** Denote by $D(R, I)$ the double of the ring R relative to an ideal I, i. e. the fibered product of rings $R \times_{R/I} R$ with the natural projections $p_1, p_2\colon D(R, I) \to R$ defined by $p_i(\xi_1, \xi_2) = \xi_i$, $i = 1, 2$. Denote by G_i the kernel of the map $p_i^*\colon \mathrm{St}^{\mathrm{G}}(D(R, I)) \to \mathrm{St}^{\mathrm{G}}(R)$. We define the relative Steinberg group $\mathrm{St}^{\mathrm{G}}(R, I)$ as G_1/C, where $C = [G_1, G_2]$. Clearly, there is an exact sequence.

{eq:suite}
$$1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \mathrm{St}^{\mathrm{G}}(R, I) \xrightarrow{\overline{p_2^*}} \mathrm{St}^{\mathrm{G}}(R) \xrightarrow{\pi^*} \mathrm{St}^{\mathrm{G}}(R/I) \longrightarrow 1 \tag{2.3}$$

Lemma 2.1. *Assume that R and I are such that the canonical projection $R \to R/I$ splits. Then the following facts are true.*

item:st-inj (i) *The map $\mathrm{St}^{\mathrm{G}}(R, I) \to \mathrm{St}^{\mathrm{G}}(R)$ is an injection.*

item:st-semi (ii) *The group $\mathrm{St}^{\mathrm{G}}(R)$ is isomorphic to $\mathrm{St}^{\mathrm{G}}(R/I) \ltimes \mathrm{St}^{\mathrm{G}}(R, I)$.*

Proof. For the proof of the first assertion see [10, Lemma 8]. Since the group $(G_1 \cap G_2)/C$ vanishes, the sequence (2.3) turns into a split short exact sequence which implies the second assertion. $\square$

lem:Zgen **Lemma 2.2.** *Let Σ be the special subset of some parabolic subset of roots $S \subseteq \Phi$. Then the relative Steinberg group $\mathrm{St}^{\mathrm{G}}(R, I)$ admits the following generating set:*

$$\mathcal{Z}(\Sigma, R, I) = \{x_\alpha(0, s) \cdot C \mid s \in I, \alpha \in \Phi\} \cup \{z_\alpha(s, \xi) \mid s \in I,\ \xi \in R,\ \alpha \in \Sigma\},$$

where $z_\alpha(s, \xi)$ denotes the element $x_\alpha(0, s)^{x_{-\alpha}(\xi,\xi)} \cdot C$.

Proof. See [10, Lemma 5]. □

2.3. **Tulenbaev's lifting property and its corollaries.** Throughout this section $I \trianglelefteq A$ is an ideal of arbitrary commutative ring A. For a nonnilpotent element $a \in A$ denote by $\lambda_a \colon A \to A_a$ the morphism of principal localization at a. Consider the following commutative square.

{msq} (2.4)
$$\begin{tikzcd} A \arrow[r, "\lambda_a"] \arrow[d, two heads] & A_a \arrow[d, two heads] \\ A/I \arrow[r, "\overline{\lambda_a}"] & A_a/I_a \end{tikzcd}$$

Notice that (2.4) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a. Such squares are usually called *Milnor squares* in the literature, see [18, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [15, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp **Definition 2.3.** We say that the Steinberg group functor St^{G} satisfies *Tulenbaev's lifting property* if for every pull-back square (2.4) the following lifting problem has a solution.

$$\begin{tikzcd} \mathrm{St}^{\mathrm{G}}(A, I) \arrow[r, "\mu"] \arrow[d] & \mathrm{St}^{\mathrm{G}}(A) \arrow[d, "\lambda_a^*"] \\ \mathrm{St}^{\mathrm{G}}(A_a, I) \arrow[r, "\mu"] \arrow[ur, dotted] & \mathrm{St}^{\mathrm{G}}(A_a) \end{tikzcd}$$

Theorem 1. *Assume that G satisfies Tulenbaev property (2.3) then the following facts are true for arbitrary commutative ring A:*

thm:dp *(i) A dilation principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. if $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is such that equality $\lambda_a^*(h) = 1$ holds in $\mathrm{St}^{\mathrm{G}}(\Phi, R_a[t])$ then for sufficiently large n one has*

$$ev^*_{\left[\frac{R[t] \to R[t]}{t \mapsto a^n \cdot t}\right]}(h) = 1.$$

thm:lg-k2 *(ii) A local-global principle holds for $\mathrm{St}^{\mathrm{G}}(-)$, i. e. an element $g \in \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ is trivial if and only if its image in $\mathrm{St}^{\mathrm{G}}(A_m[t], tA_m[t])$ is trivial for all maximal ideals $m \trianglelefteq A$.*

thm:centr *(iii) $K_2^G(A)$ is contained in the centre of $\mathrm{St}^{\mathrm{G}}(A)$.*

Proof. Follows by the same argument as [15, Theorem 2.1] or [10, Theorem 2] □

3. Decomposition theorems for $\mathrm{St}^{\mathrm{G}}(A[t^{\pm 1}])$ and $\mathrm{St}^{\mathrm{G}}\big(A((t))\big)$.

Lemma 3.1. *Let (R, m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big) i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) = i_-\big(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]\big) i_+\big(\mathrm{St}^{\mathrm{G}}(m \cdot R[t])^{\mathrm{St}^{\mathrm{G}}(R[t])}\big)$$

inside $\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$.

Proof. This is proved exactly as [11, Lemma 5.12]. □

thm:3t **Theorem 2.** *Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2. Let $i_+ : \mathrm{St}^{\mathrm{G}}(R[t]) \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ and $i_- : \mathrm{St}^{\mathrm{G}}(R[t^{-1}] \to \mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$\mathrm{St}^{\mathrm{G}}(R[t^{\pm 1}]) = i_+(\mathrm{St}^{\mathrm{G}}(R[t]))i_-(\mathrm{St}^{\mathrm{G}}(R[t^{-1}]))i_+(\mathrm{St}^{\mathrm{G}}(R[t])).$$

Proof. This is proved exactly as [11, Theorem 5.1]. □

sec:sigma

3.1. **The automorphisms σ_i.** Our notation and conventions follows [16, § 4]. Let Φ be an irreducible root system with some fixed basis of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$. We denote by $\Phi^\vee$ the *dual root system of* Φ consisting of vectors $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \Phi$. As usual, $P(\Phi^\vee)$ denotes the lattice spanned by the *fundamental weights* ϖ_i. Recall that ϖ_i are uniquely determined by relations $\langle \varpi_i, \alpha_j^\vee \rangle = (\varpi_i, \alpha_j) = \delta_{ij}$.

Notice that for $\varpi \in P(\Phi^\vee)$ and $\beta \in \mathbb{Z}\,\Phi$ one has $(\varpi, \beta) \in \mathbb{Z}$. Consequently, for $\varepsilon \in R^*$ and $\varpi \in P(\Phi^\vee)$ the identity $\chi_{\varpi,\varepsilon}(\beta) = \varepsilon^{(\varpi,\beta)}$ gives a well-defined character $\chi_{\varpi,\varepsilon} \in \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$.

Consider the action of $H = \mathrm{Hom}(\mathbb{Z}\,\Phi, R^*)$ on the set of generators $\mathcal{X}_{\Phi,R}$ of the Steinberg group $\mathrm{St}^{\mathrm{G}}(R)$ defined by

$$\chi \cdot x_\alpha(\xi) = x_\alpha(\chi(\alpha) \cdot \xi),\ \chi \in H,\ \alpha \in \Phi,\ \xi \in R. \tag{3.1}$$

Since χ is a character, the above action preserves the set of Steinberg relations $\mathcal{R}_{\Phi,R}$ and, thus, gives a well-defined action of H on $\mathrm{St}^{\mathrm{G}}(R)$.

Example 3.2. The principal example which motivates the above construction is as follows. Let A be a ring, take $R = A[t, t^{-1}]$ to be the ring of Laurent polynomials over A and let $\alpha_i \in \Pi$ be some simple root. Since $t \in R^*$ we can consider the automorphisms σ_i^+ and σ_i^- of $\mathrm{St}(\Phi, R)$ given by $\sigma_i^+ = \chi_{\varpi_i,t}$, $\sigma_i^- = \chi_{\varpi_i,t^{-1}}$. It is easy to see that

eq:sigma_act}

$$\sigma_i^\pm(x_\alpha(\xi)) = x_\alpha(t^{\pm m_i(\alpha)} \cdot \xi), \tag{3.2}$$

where $m_k(\alpha)$ denotes the coefficient in the expansion of α in Π, i. e. $\alpha = \sum m_k(\alpha)\alpha_k$.

One of the key steps of our proof of Suslin lemma for K_2 is to define an analogue of σ_i for the group $\mathrm{St}^{\mathrm{G}}(A[t])$. Of course, we cannot expect such map to be automorphism or even be defined on the whole group $\mathrm{St}^{\mathrm{G}}(A[t])$. However, it turns out that for certain i is still possible to define certain subgroups of $\mathrm{St}^{\mathrm{G}}(A[t])$ and the maps modeling σ_i between them. First, we settle the case $\Phi = \mathsf{A}_3$ invoking the presentation obtained in **??**. Then we study the general case using the Curtis-Tits presentation.

lem:sigma **Lemma 3.3.** *Let A be a local commutative ring, $G = G(\Phi, -)$ where Φ is an irreducible root system. Assume that on the Dynkin diagram of Φ one can find an endnode numbered i such that i is contained in a subdiagram of type A_3.*

Then there exists subgroups N_i^+, N_i^- *of* $\mathrm{St}^{\mathrm{G}}(A[t])$ *and homomorphisms* $\widetilde{\sigma}_i^+ \colon N_i^+ \to N_i^-$, $\widetilde{\sigma}_i^- \colon N_i^- \to N_i^+$ *compatible with the action of* $\sigma_i^{\pm}$ *i. e. such that the following diagram commutes.*

$$\begin{array}{ccc} N_i^{\pm} & \xrightarrow{\lambda_t^*} & \mathrm{St}^{\mathrm{G}}(A[t,t^{-1}]) \\ \big\downarrow{\scriptstyle \widetilde{\sigma}_i^{\pm}} & & \big\downarrow{\scriptstyle \sigma_i^{\pm}} \\ N_i^{\mp} & \xrightarrow{\lambda_t^*} & \mathrm{St}^{\mathrm{G}}(A[t,t^{-1}]) \end{array}$$

Proof for $\Phi = \mathsf{A}_\ell$, $\ell \geq 3$ *and* $i = 1$. For $\alpha_i \in \Pi$ denote by P_i^+ (resp. P_i^-) the subgroup of $\mathrm{St}^{\mathrm{G}}(A)$ generated by $x_\alpha(\xi)$ for $\xi \in A$, $\alpha \in \Sigma_i^+ \cup \Delta_i$ (resp. $\alpha \in \Sigma_i^- \cup \Delta_i$).

First, we define the subgroups $N_i^{\pm}$ (this definition also works for Φ such that $m_i(\widetilde{\alpha}) = 1$). Define N_i^+ (resp. N_i^-) to be the subgroup consisting of $g \in \mathrm{St}^{\mathrm{G}}(A[t])$ such that $g(0) \in P_i^+$ (resp. $g(0) \in P_i^-$).

Denote by j_ℓ the natural map $\mathrm{St}(\Delta_1, A) \to \mathrm{St}(\Phi, A)$. By the Levi decomposition $P_1^{\pm}$ is isomorphic to $\mathrm{U}(\Sigma_1^{\pm}, A) \rtimes \mathrm{Im}\,(j_\ell)$.

The stable rank of A equals 1 hence from the injective stability theorem for K_2 (see [14, Theorem 4.1]) it follows that the map j_ℓ is injective for $\ell \geq 3$ and that $P_1^{\pm} = \mathrm{U}(\Sigma_1^{\pm}, A) \rtimes \mathrm{St}(\Delta_1, A)$. Clearly, there exists a unique group homomorphism $\sigma_1^{\pm} \colon P_1^{\pm} \to \mathrm{St}^{\mathrm{G}}(\Phi, A[t])$ acting identically on $\mathrm{St}(\Delta_1, A)$ and sending any generator $x_\alpha(\xi) \in \mathrm{U}(\Sigma_1^{\pm}, A)$ (i. e. $\alpha \in \Sigma_1^{\pm}$) to $x_\alpha(t \cdot \xi)$.

Set $H = \mathrm{St}^{\mathrm{G}}(A[t], tA[t])$. By Lemma 2.1.(ii) we have $\mathrm{St}^{\mathrm{G}}(A[t]) = \mathrm{St}^{\mathrm{G}}(A) \ltimes H$ hence $N_1^{\pm} \cong P_1^{\pm} \ltimes H$. Define the map $\widetilde{\sigma}_1^{\pm} \colon H \to \mathrm{St}^{\mathrm{G}}(A[t])$ on the generators of $\mathrm{St}^{\mathrm{G}}(A[t], tA[t])$ as follows ?????. A routine check shows that the defining relations ?????? of H are satisfied and that $\sigma_1^{\pm}$ preserves the action of $P_1^{\pm}$ on H. Thus, we obtain a well-defined map $\sigma_1^{\pm}$: $N_i^{\pm} \to \mathrm{St}^{\mathrm{G}}(A[t])$ fitting into the above commutative diagram. The fact that the image of $\sigma_1^{\pm}$ is contained in $N_i^{\mp}$ is obvious.

4. $\mathbb{P}^1$-GLUING

Throughout this section $G = G(\Phi, -)$ denotes a Chevalley group scheme of type Φ.

def:p1g **Definition 4.1.** Let F be a group-valued functor from **CRings** to **Groups** and let A be a commutative ring. Consider the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{i_+} & A[t] \\ {\scriptstyle i_-}\big\downarrow & & \big\downarrow{\scriptstyle j_+} \\ A[t^{-1}] & \xrightarrow{j_-} & A[t,t^{-1}] \end{array}$$

We say that F satisfies the $\mathbb{P}^1$-*glueing property for* A if the following sequence of *pointed sets* is exact in the middle term:

$$F(A) \xhookrightarrow{\Delta_A^F} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\pm_A^F} F(A[t,t^{-1}]).$$

Here Δ_A^F denotes the (split injective) diagonal map and, by definition, $\pm_A^F$ maps (g^+, g^-) to $F(j_+)(g^+) \cdot F(j_-)(g^-)^{-1}$. Notice that a priori $\pm_A^F$ is only a morphism of pointed sets but if F takes values in abelian groups then $\pm_A^F$ is also a morphism of groups.

An equivalent way to formulate $\mathbb{P}^1$-glueing property is as follows: $F(j_+)$ and $F(j_-)$ are injective and the intersection of their images coincides with the image of $F(j_+i_+) = F(j_-i_-)$.

The main result of this section is the following theorem which generalizes [15, Theorem 5.1] to Chevalley groups. Notice that a K_1-analogue of the result below has been established in a much greater generality by the second-named author (see [11, Theorem 1.1]).

thm:p1 **Theorem 3.** *Assume that G satisfies Tulenbaev lifting property 2.3. Then the Steinberg group functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for an arbitrary commutative ring A.*

Proof. Let (g^+, g^-) be an element of $\mathrm{St}^{\mathrm{G}}(A[t]) \times \mathrm{St}^{\mathrm{G}}(A[t^{-1}])$ such that the equality $g^+ = g^-$ holds in $\mathrm{St}^{\mathrm{G}}(A[t, t^{-1}])$.

Let m be a maximal ideal of A. By Proposition 4.10 below the functor $\mathrm{St}^{\mathrm{G}}(-)$ satisfies $\mathbb{P}^1$-glueing property for the local ring A_m hence $(\lambda_m^*(g_+), \lambda_m^*(g_-)) = \Delta_{A_m}(\lambda_m^*(g^+)(0))$ and in the groups $\mathrm{St}^{\mathrm{G}}(A_m[t])$ and $\mathrm{St}^{\mathrm{G}}(A_m[t^{-1}])$ we have the equalities:

$$\lambda_m^*(g^+ \cdot g^+(0)^{-1}) = \lambda_m^*(g^+) \cdot \lambda_m^*(g^+)(0)^{-1} = 1; \qquad \lambda_m^*(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for $\mathrm{St}^{\mathrm{G}}(-)$ (see Theorem 1.(ii)) these equalities hold globally and $(g^+, g^-) = \Delta_A(g^+(0))$, as claimed. ☐

rem:stk2 *Remark* 4.2. It is clear that if the functor St^{G} satisfies $\mathbb{P}^1$-glueing property for A then so does the functor K_2^G. The converse statement also holds, indeed, if $(g^+, g^-) \in \mathrm{Ker}\,(\pm_A^{\mathrm{St}})$ then inside $E^G(A[t, t^{-1}])$ we have the equality:

$$\varphi(\mathrm{St}^{\mathrm{G}}(j_+)(g^+)) = \varphi(\mathrm{St}^{\mathrm{G}}(j_-)(g^-)) \in E^G(A[t]) \cap E^G(A[t^{-1}]) = E^G(A).$$

Consequently, we can find $g_0 \in \mathrm{St}^{\mathrm{G}}(A)$ so that $(g^+g_0^{-1}, g^-g_0^{-1}) \in \mathrm{Ker}\,(\pm_A^{K_2})$ and it remains to apply the $\mathbb{P}^1$-glueing property for K_2^G.

Corollary 4.3. *Let A be any commutative ring and $f \in A[t]$ be a monic polynomial. Then the map $K_2^G(A[t]) \to K_2^G(A[t]_f)$ is injective.*

Proof. ☐

4.1. **The case of a field.** Throughout this section k denotes arbitrary field k.

thm:k[t] **Theorem 4.** *Assume that $G = G(\Phi, -)$ and Φ is irreducible of rank ≥ 2.*

satz1 *(i) The subgroup $K_2^G(k[t]) \trianglelefteq \mathrm{St}^{\mathrm{G}}(k[t])$ is generated by elements of the form*

$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1},\ u, v \in k^*.$$

(ii) As a consequence, the canonical injection $K_2^G(k) \hookrightarrow K_2^G(k[t])$ is an isomorphism.

Proof. See [9, Satz 1] and the corollary after it. ☐

cor:k[t]inj **Corollary 4.4.** *Let G be as in the above theorem. Then the functors St^{G}, K_2^G satisfy $\mathbb{P}^1$-glueing property for k.*

Proof. By Remark 4.2 it suffices to prove the assertion only for the functor K_2^G. By the previous theorem $K_2^G(i_+)$ and $K_2^G(i_-)$ are isomorphisms hence the morphisms $K_2^G(j_+)$ and $K_2^G(j_-)$ are split injective and $\mathrm{Im}\,(j_+i_+) = \mathrm{Im}\,(j_-i_-) = \mathrm{Im}\,(j_+) = \mathrm{Im}\,(j_-)$. ☐

Corollary 4.5. *Let G, k be as in Theorem 4. Then $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.*

Proof. We use Theorem 2. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \mathrm{St}^{\mathrm{G}}(k[t]))$, $y \in \mathrm{St}^{\mathrm{G}}(k[t^{-1}]))$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \mathrm{St}^{\mathrm{G}}(k)K_2^G(k[t^{-1}]) = \mathrm{St}^{\mathrm{G}}(k)$ and $x_1x_2 \in \mathrm{St}^{\mathrm{G}}(k)$. That is, $g \in K_2^G(k)$. ☐

4.2. **Suslin lemma for K_2.** For the rest of this section A denotes an arbitrary commutative local ring with the maximal ideal m and the residue field k. We denote by π the canonical projection $A \to k$. Throughout this section we will employ the following notation:

- R denotes the Laurent polynomial ring $A[t, t^{-1}]$;
- B denotes the subring $A[t] + m[t^{-1}]$ of R consisting of Laurent polynomials $f(t, t^{-1})$ whose coefficients of terms of negative degree belong to m;
- I denotes the ideal $m[t, t^{-1}]$ of R (which can be also considered as an ideal of B).

Our first result is analogous to [15, Lemma 3.1(e)] (cf. also with [6, § 2.3A]).

cor:bruhat

Lemma 4.6. *The Steinberg group $\mathrm{St}^{\mathrm{G}}(A)$ admits the following analogue of the Bruhat decomposition:*

$$\mathrm{St}^{\mathrm{G}}(A) = \mathrm{U}(\Phi^+, A) \cdot \mathrm{W}(\Phi, A) \cdot \mathrm{U}(\Phi^+, A) \cdot \mathrm{Im}\,(\mathrm{St}^{\mathrm{G}}(A, m) \to \mathrm{St}^{\mathrm{G}}(A)).$$

Proof. First of all, notice that $\mathrm{St}^{\mathrm{G}}(k)$ admits Bruhat decomposition for arbitrary field k:

$$\mathrm{St}^{\mathrm{G}}(k) = \mathrm{U}(\Phi^+, k) \cdot \mathrm{W}(\Phi, k) \cdot \mathrm{U}(\Phi^+, k).$$

Indeed, this is an immediate consequence of the classical Bruhat decomposition for the linear groups (see e. g. [13, Theorem 4]) and the fact that $K_2^G(k)$ is central in $\mathrm{St}^{\mathrm{G}}(k)$ and is generated by symbols $h_\alpha(u)$ lying in $\mathrm{W}(\Phi, k)$ (cf. Theorem 4.(i)).

As a consequence, the first three factors in the right hand side of the decomposition are mapped epimorphically onto $\mathrm{St}^{\mathrm{G}}(k)$ and the last factor coincides with $\mathrm{Ker}\,(\pi_*)$ from which the assertion of the lemma follows. □

lem:tulinj

Lemma 4.7. *Assume that G satisfies Tulenbaev lifting property 2.3. Then the map i in the following commutative diagram of groups is injective.*

{diag:cs} (4.1)

$$\begin{tikzcd}
C_B \arrow[r, hook] \arrow[d, two heads, "k"] & \mathrm{St}^{\mathrm{G}}(B, I) \arrow[r, "\mu_B"] \arrow[d, "j"] & \mathrm{St}^{\mathrm{G}}(B) \arrow[r, two heads, "\pi_B"] \arrow[d, "i"] & \mathrm{St}^{\mathrm{G}}(k[t]) \arrow[d, hook] \\
C_R \arrow[r, hook] & \mathrm{St}^{\mathrm{G}}(R, I) \arrow[r, "\mu_R"'] \arrow[ur, dashed, "\varphi"] & \mathrm{St}^{\mathrm{G}}(R) \arrow[r, two heads, "\pi_R"'] & \mathrm{St}^{\mathrm{G}}(k[t, t^{-1}])
\end{tikzcd}$$

Proof. First of all, notice that by Corollary 4.4 the vertical map in the right-hand side of the diagram is injective. Invoking Tulenbaev's property 2.3 we also find a lifting map φ in the central square of the diagram.

Let $g \in \mathrm{St}^{\mathrm{G}}(B)$ be an element of $\mathrm{Ker}\,(i)$. Since g also lies in $\mathrm{Ker}\,(\pi_B)$ it comes from some $\widetilde{g} \in \mathrm{St}^{\mathrm{G}}(B, I)$ via μ_B. But $j(\widetilde{g})$ lies in C_R, hence, by Proposition 4.9 below it comes from some $\widehat{g} \in C_B$ via k. Finally, $g = \varphi(j(\widetilde{g})) = \varphi(k(\widehat{g})) = \mu_B(\widehat{g}) = 1$, as claimed. □

The following result is analogous to [15, Proposition 4.1].

Lemma 4.8. *The map j in the diagram (4.1) is surjective for any commutative ring A.*

Proof. Let α_i be arbitrary simple root from Π and let Σ_i (resp. Σ_i^-) denote the special subsets of roots consisting of $\alpha \in \Phi$ such that $m_i(\alpha) > 0$ (resp. $m_i(\alpha) < 0$).

By Lemma 2.2 the sets $\mathcal{Z}(\Sigma_i, R, I)$ and $\mathcal{Z}(\Sigma_i^-, R, I)$ both generate $\mathrm{St}^{\mathrm{G}}(R, I)$. Similarly, the subsets $\mathcal{Z}(\Sigma_i, B, I) \subseteq \mathcal{Z}(\Sigma_i, R, I)$, $\mathcal{Z}(\Sigma_i^-, B, I) \subseteq \mathcal{Z}(\Sigma_I, R, I)$ both generate the image of $j\colon \mathrm{St}^{\mathrm{G}}(B, I) \to \mathrm{St}^{\mathrm{G}}(R, I)$.

Using formula (3.2) we can calculate how the powers of the automorphism σ_i act on these generating sets. Indeed, for $z_\alpha(s,\xi) \in \mathcal{Z}(\Sigma_i^-, R, I)$ we have

$$\sigma_i^N(z_\alpha(s,\xi)) = \sigma_i^N(x_\alpha(0,s)^{x_{-\alpha}(\Delta(\xi))}) = z_\alpha(t^{Nm_i(\alpha)} \cdot s, t^{-Nm_i(\alpha)} \cdot \xi).$$

It is clear that for sufficiently large N the element $t^{-Nm_i(\alpha)} \cdot \xi$ belongs to $A[t] \subseteq B$. Consequently, for arbitrary $g \in \mathrm{St}^{\mathrm{G}}(R, I)$ there exists $N > 0$ such that $\sigma^N(g)$ lies in the image of j. On the other hand, by a similar calculation we get for $z = z_\alpha(s,\xi) \in \mathcal{Z}(\Sigma_i, B, I)$ that the element $\sigma^{-N}(z)$ $(N > 0)$ still lies in $\mathcal{Z}(\Sigma_i, B, I)$. Clearly, this implies that the image of j is preserved by σ_i^{-1}. The assertion of the lemma now follows from these two statements:

$$g = \sigma^{-N}\sigma^N(g) \in \sigma^{-N}(\mathrm{Im}\,(j)) \subseteq \mathrm{Im}\,(j). \qquad \square$$

prop:kersurj **Proposition 4.9.** *Under the assumptions of Lemma 4.7 the map k in the diagram (4.1) is surjective.*

Sketch/draft of the proof. The argument presented below only may work under additional assumption that $m_i(\widetilde{\alpha}) = 1$ (i is as in the statement of Lemma 3.3).

Consider the following set

$$X = \mathrm{St}^{\mathrm{G}}(B) \times \widetilde{B}(R) \times \mathrm{St}^{\mathrm{G}}(R, I)/\simeq.$$

The congruence relation we impose should be similar to that used in Tulenbaev's paper. In particular, for $h \in \mathrm{St}^{\mathrm{G}}(B, I)$ we should have the following relation in X

$$[\mu_B(h), 1, g] \simeq [1, 1, j(h)g].$$

Notice that instead of $\mathrm{St}^{\mathrm{G}}(A[t])$ (used by Tulenbayev in the first factor) we use $\mathrm{St}^{\mathrm{G}}(B)$.

TODO: We should prove that there is a well-defined action of $\mathrm{St}^{\mathrm{G}}(R)$ on X.

Now let g be an element of C_R. By the previous lemma we have for some $g' \in \mathrm{St}^{\mathrm{G}}(B, I)$

$$[1,1,g] = [1,1,j(g')] = [\mu_B(g'), 1, 1] = i\mu_B(g')[1,1,1] = [1,1,1]. \qquad \square$$

4.3. **Proof of the main result.** The following result is analogous to [15, Proposition 4.3]. It plays the same role in our proof of $\mathbb{P}^1$-glueing for K_2 as generalized Suslin lemma (cf. [1, Theorem 2.16]) does in the corresponding proof for K_1.

prop:p1g **Proposition 4.10.** *The functors St^{G}, K_2^G satisfy $\mathbb{P}^1$-glueing property for arbitrary local ring A.*

Proof. $\square$

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