

\mathbb{A}^1 -INVARIANCE FOR UNSTABLE K_2

S. SINCHUK, A. STAVROVA, AND A. LAVRENOV

1. AIM OF THE PAPER

The aim of this text is to prove that the non-stable K_2 -functors K_2^G , where G is a simply connected Chevalley group of suitable type, satisfy \mathbb{A}^1 -invariance on regular rings R containing a field k , that is,

$$(1.1) \quad K_2^G(R[t]) = K_2^G(R).$$

As a corollary, we should easily deduce that

$$K_2^G(R) = KV_2^G(R),$$

where $KV_2^G(R)$ is the Karoubi–Villamayor K -functor associated to G . This functor originates from [5]. The above equality by e.g. [2, Corollary 4.3.3] implies that $\pi_1^{\mathbb{A}^1}(G)(R) = K_2^G(R)$, i.e. we obtain an explicit presentation for the \mathbb{A}^1 -fundamental group of G in the sense of Morel–Voevodsky.

”Suitable type” here means that we consider only the cases where we know the centrality of K_2 , or at least the Quillen–Suslin lgp. Some intermediate steps can be proved in larger generality.

Essentially, we need to prove that $K_2^G(k[x_1, \dots, x_n]) = K_2^G(k)$. (Then (1.1) follows by standard geometric methods.) There are two models: Tulenbaev’s proof for the SL_n case [11] and Stavrova’s proof for K_1^G [9]. Tulenbaev [11] uses stabilization of the K_2 -functor, and the good properties of the limit=algebraic K -theory. In [9] stabilization is not used. However, the key steps of both proofs are the same: the case of $R = k$ (hidden somewhere around [11, p. 140], or, respectively, [9, Theorem 3.1]); Quillen–Suslin lgp; \mathbb{P}^1 -gluing (see [11, Theorem 5.1] or [9, Theorem 1.1]).

1.1. The case $R = k$. We consider the case of $K_2^G(k[t])$ vs. $K_2^G(k)$. (As in the \mathbb{A}^1 -invariance of K_1^G , this case should be used to deduce that $K_2^G(k[t_1, \dots, t_n]) = K_2^G(k)$.)

In Tulenbaev’s framework, it follows from stabilization. However, the equality $K_2^G(k[t]) = K_2^G(k)$ is sort of known for all groups. Namely, in [15, Theorem 5.1]: let k be an infinite field and let G be a connected reductive group over k . Then the inclusion $k \hookrightarrow k[t]$ induces an isomorphism

$$H_\bullet(G(k), \mathbb{Z}) \xrightarrow{\cong} H_\bullet(G(k[t]), \mathbb{Z}),$$

if the order of the fundamental group of G is invertible in k . Once we know that the homology H_2 coincides with K_2^G (on both sides), this gives the result. It would be nice to check Wendt’s proof; maybe, discuss it in a seminar?

Is it necessary to know the centrality in order to show that K_2^G coincides with H_2 ? If yes, then we probably know it for $K_2^G(k)$ even for isotropic groups [3], but only for the

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good Chevalley groups for $K_2^G(k[t])$. We should try to understand what is proved in [13, Proposition 5.3] using only the universality of the Steinberg group. This may be useful. The paper is unpublished, so again everything should be double-checked if you want to refer to it.

1.2. Plan of the proof.

plan:3t

- (1) (**done**, see Theorem 2). Let R be a local ring. Show that

$$\mathrm{St}^G(R[t^{\pm 1}]) = i_+(\mathrm{St}^G(R[t]))i_-(\mathrm{St}^G(R[t^{-1}]))i_+(\mathrm{St}^G(R[t]))$$

Here i_{\pm} denote the natural homomorphisms into $\mathrm{St}^G(R[t^{\pm 1}])$.

plan:k[t]

plan:k-intersect

- (2) (**done**, see Theorem 4). Show that $K_2^G(k[t]) = K_2^G(k)$.
 (3) (**done**). Consequently, $\mathrm{St}^G(k[t]) \rightarrow \mathrm{St}^G(k[t^{\pm 1}])$ is injective and $\mathrm{St}^G(k[t]) \cap \mathrm{St}^G(k[t^{-1}]) = \mathrm{St}^G(k)$ inside $\mathrm{St}^G(k[t^{\pm 1}])$. Also, $K_2^G(k) = K_2^G(k[t^{\pm 1}])$.

The first claim uses (2). The second claim follows from (2) and (1).

plan:QSlgp

plan:Zglu

- (4) (**done** for split ACDE, see Theorem 1.(ii)). Prove Quillen-Suslin lgp for K_2^G .
 (5) (Zariski gluing) for any commutative ring A and any non-nilpotent $f, g \in A$ such that $A = fA + gA$, the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A_f) \times K_2^G(A_g) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A_{fg})$$

is exact. The proof is usually almost the same as for (4).

(Remark. It seems that we need this property only for $A = R[t]$ and f, g non-constant polynomials.)

plan:S-lemma

- (6) (S -lemma) Let A be a commutative ring, S a multiplicative subset of A . If

$$K_2^G(A[X_1, \dots, X_n]) = K_2^G(A)$$

for some $n \geq 1$, then $K_2^G(A_S[X_1, \dots, X_n]) = K_2^G(A_S)$ as well.

This should be easy; see [1, Lemma 3.6].

plan:Nglu

- (7) (Nisnevich gluing) Assume that B is a subring of a commutative ring A , and let $h \in B$ be a non-nilpotent element. Denote by $F_h : A \rightarrow A_h$ the localization homomorphism.

(i) If $Ah + B = A$, i.e. the natural map $B \rightarrow A/Ah$ is surjective, then for any $x \in \mathrm{St}^G(A_h)$ there exist $y \in \mathrm{St}^G(A)$ and $z \in \mathrm{St}^G(B_h)$ such that $x = F_h(y)z$.

(ii) If moreover $Ah \cap B = Bh$, i.e. $B/Bh \rightarrow A/Ah$ is an isomorphism, and h is not a zero divisor in A , then the sequence of pointed sets

$$K_2^G(B) \xrightarrow{g \mapsto (F_h(g), g)} K_2^G(B_h) \times K_2^G(A) \xrightarrow{(g_1, g_2) \mapsto g_1 F_h(g_2)^{-1}} K_2^G(A_h)$$

is exact.

This should use something from the proof of (4) or (5); see [9, Lemma 3.4].

plan:P1

- (8) (\mathbb{P}^1 -gluing) Let A be any commutative ring. Show that the sequence of pointed sets

$$1 \longrightarrow K_2^G(A) \xrightarrow{g \mapsto (g, g)} K_2^G(A[t]) \times K_2^G(A[t^{-1}]) \xrightarrow{(g_1, g_2) \mapsto g_1 g_2^{-1}} K_2^G(A[t, t^{-1}])$$

is exact.

This should use (1)–(4).

- (a) Prove Proposition 4.8. This is hard. Have no idea how this can be proved at the moment.
 (b) Prove \mathbb{P}^1 -gluing using all the above facts.

plan:[]f

- (9) (Main corollary of \mathbb{P}^1 -gluing) Let A be any commutative ring, and let $f \in A[t]$ be a monic polynomial. Show that $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$ is injective.

The proof uses (5) and (8).

plan:k(t)

- (10) Prove that $K_2^G(k(t)) = K_2^G(k)$. This may be a bit tricky; I will think if we can get rid of it. I don't think Tullenbaev uses it. Unsure whether this is true (?). This is true for SK_1 but in view of Milnor's theorem can not hold for K_1 and K_2 .

plan:k[tn]

- (11) Prove that

$$K_2^G(k[t_1, \dots, t_n]) = K_2^G(k).$$

If we strictly follow the pattern of K_1^G , this uses (9), (2), and (10). There may be other ways.

plan:final

- (12) Final result: let R be a regular ring containing a field k . Then $K_2^G(R[t]) = K_2^G(R)$. This uses (4), (11) and (7).

2. STEINBERG GROUPS OF CHEVALLEY GROUPS: PRELIMINARIES

2.1. Definition and basic properties. Let G be a split simple Chevalley groups with a root system Φ of rank ≥ 2 . Recall that the *Steinberg group* $\text{St}^G(R)$ (also denoted $\text{St}(\Phi, R)$) is defined by means of generators $\mathcal{X}_{\Phi, R} = \{x_\alpha(\xi) \mid \xi \in R, \alpha \in \Phi\}$ and the set of relations $\mathcal{R}_{\Phi, R}$ defined as follows:

{rel:add}

$$(2.1) \quad x_\alpha(s)x_\alpha(t) = x_\alpha(s+t),$$

{rel:CCF}

$$(2.2) \quad [x_\alpha(s), x_\beta(t)] = \prod x_{i\alpha+j\beta} (N_{\alpha\beta ij} s^i t^j), \quad \alpha \neq -\beta, \quad N_{\alpha\beta ij} \in \mathbb{Z}.$$

The indices i, j appearing in the right-hand side of the above relation range over all positive natural numbers such that $i\alpha + j\beta \in \Phi$. The structure constants $N_{\alpha\beta ij} = \pm 1, 2, 3$ appearing in (2.2) depend only on Φ and can be computed precisely.

Recall that for $\alpha \in \Phi$, $\varepsilon \in R^*$ the semisimple root elements $h_\alpha(\varepsilon)$ are defined as $h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1)$. Denote by $W(\Phi, R)$ the subgroup of $\text{St}(\Phi, R)$ generated by all elements $w_\alpha(\varepsilon)$, $\varepsilon \in R^*$.

2.2. Relative Steinberg groups.

{eq:suite}

$$(2.3) \quad 1 \longrightarrow (G_1 \cap G_2)/C \longrightarrow \text{St}^G(R, I) \xrightarrow{\overline{p_2^*}} \text{St}^G(R) \xrightarrow{\pi^*} \text{St}^G(R/I) \longrightarrow 1$$

Lemma 2.1. *Assume that R and I are such that the canonical projection $R \rightarrow R/I$ splits. Then the following facts are true.*

- (i) *The map $\text{St}^G(R, I) \rightarrow \text{St}^G(R)$ is an injection.*
- (ii) *The group $\text{St}^G(R)$ is isomorphic to $\text{St}^G(R/I) \ltimes \text{St}^G(R, I)$.*

Proof. For the proof of the first assertion see [8, Lemma 8]. Since the group $(G_1 \cap G_2)/C$ vanishes, the sequence (2.3) turns into a split short exact sequence which implies the second assertion. \square

2.3. Tullenbaev's lifting property and its corollaries. Throughout this section $I \trianglelefteq A$ is an ideal of arbitrary commutative ring A . For a nonnilpotent element $a \in A$ denote by

$\lambda_a: A \rightarrow A_a$ the morphism of principal localization at a . Consider the following commutative square.

$$\begin{array}{ccc} A & \xrightarrow{\lambda_a} & A_a \\ \downarrow & & \downarrow \\ A/I & \xrightarrow{\bar{\lambda}_a} & A_a/I_a \end{array} \quad (2.4)$$

Notice that (2.4) is a pull-back square if and only if λ_a induces an isomorphism of I and I_a . Such squares are usually called *Milnor squares* in the literature, see [14, Ch. I, § 2].

The following property of linear Steinberg groups was discovered for the first time by Tulenbaev (see [11, Lemmas 2.3, 3.2]) and plays a key role in the sequel.

def:tlp

Definition 2.2. We say that the Steinberg group functor St^G satisfies *Tulenbaev's lifting property* if for every pull-back square (2.4) the following lifting problem has a solution.

$$\begin{array}{ccc} \text{St}^G(A, I) & \xrightarrow{\mu} & \text{St}^G(A) \\ \downarrow & \nearrow & \downarrow \lambda_a^* \\ \text{St}^G(A_a, I) & \xrightarrow{\mu} & \text{St}^G(A_a) \end{array}$$

Theorem 1. Assume that G satisfies Tulenbaev property (2.2) then the following facts are true for arbitrary commutative ring A :

thm:dp

(i) A dilation principle holds for $\text{St}^G(-)$, i. e. if $g \in \text{St}^G(A[t], tA[t])$ is such that equality $\lambda_a^*(h) = 1$ holds in $\text{St}^G(\Phi, R_a[t])$ then for sufficiently large n one has

$$\text{ev}_{\left[\begin{smallmatrix} R[t] \rightarrow R[t] \\ t \mapsto a^n \cdot t \end{smallmatrix} \right]}^*(h) = 1.$$

thm:lg-k2

(ii) A local-global principle holds for $\text{St}^G(-)$, i. e. an element $g \in \text{St}^G(A[t], tA[t])$ is trivial if and only if its image in $\text{St}^G(A_m[t], tA_m[t])$ is trivial for all maximal ideals $m \trianglelefteq A$.

thm:centr

(iii) $K_2^G(A)$ is contained in the centre of $\text{St}^G(A)$.

Proof. Follows by the same argument as [11, Theorem 2.1] or [8, Theorem 2] □

2.4. The action of torus. Our notation and conventions follows [12, § 4]. Let Φ be an irreducible root system with some fixed basis of simple roots $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. We denote by Φ^\vee the *dual root system* of Φ consisting of vectors $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, $\alpha \in \Phi$. As usual, $P(\Phi^\vee)$ denotes the lattice spanned by the *fundamental weights* ϖ_i . Recall that ϖ_i are uniquely determined by relations $\langle \varpi_i, \alpha_j^\vee \rangle = (\varpi_i, \alpha_j) = \delta_{ij}$.

Notice that for $\varpi \in P(\Phi^\vee)$ and $\beta \in \mathbb{Z}\Phi$ one has $(\varpi, \beta) \in \mathbb{Z}$. Consequently, for $\varepsilon \in R^*$ and $\varpi \in P(\Phi^\vee)$ the identity $\chi_{\varpi, \varepsilon}(\beta) = \varepsilon^{(\varpi, \beta)}$ gives a well-defined character $\chi_{\varpi, \varepsilon} \in \text{Hom}(\mathbb{Z}\Phi, R^*)$.

Consider the action of $H = \text{Hom}(\mathbb{Z}\Phi, R^*)$ on the set of generators $\mathcal{X}_{\Phi, R}$ of the Steinberg group $\text{St}^G(R)$ defined by

$$(2.5) \quad \chi \cdot x_\alpha(\xi) = x_\alpha(\chi(\alpha) \cdot \xi), \quad \chi \in H, \quad \alpha \in \Phi, \quad \xi \in R.$$

Since χ is a character, the above action preserves the set of Steinberg relations $\mathcal{R}_{\Phi, R}$ and, thus, gives a well-defined action of H on $\text{St}^G(R)$.

Example 2.3. The principal example which motivates the above construction is as follows. Let A be a ring, take $R = A[t, t^{-1}]$ to be the ring of Laurent polynomials over A and let $\alpha_i \in \Pi$ be some simple root. Since $t \in R^*$ we can consider the automorphism σ_i of $\text{St}(\Phi, R)$ given by $\sigma_i = \chi_{\varpi_i, t}$. It is easy to see that

$$(2.6) \quad \sigma_i(x_\alpha(\xi)) = x_\alpha(t^{m_i(\alpha)} \cdot \xi),$$

where $m_k(\alpha)$ denotes the coefficient in the expansion of α in Π , i.e. $\alpha = \sum m_k(\alpha)\alpha_k$.

Our next goal is to define an analogue of σ_i for the group $\text{St}^G(A[t])$. Obviously, we cannot expect such map to be automorphism or even be defined on the whole $\text{St}^G(A[t])$. However, it turns out that in some cases it is still possible to define it on a certain subgroup of $\text{St}^G(A[t])$. Indeed, for $\alpha_i \in \Pi$ consider the subgroup N_i of $\text{St}^G(A[t])$ generated by all $x_\alpha(\xi)$ ($\xi \in A[t]$, $\alpha \in \Phi$) such that ξ is divisible by t if $\alpha \in \Sigma_i^-$.

lem:sigma

Lemma 2.4. *Assume that for a ring A the canonical embedding $i: \Delta_i \hookrightarrow \Phi$ induces an injection $K_2^{G(\Delta_i, -)}(A) \rightarrow K_2^{G(\Phi, -)}(A)$. Then there exists a homomorphism $\sigma'_i: N_i \rightarrow \text{St}^G(A[t])$ compatible with the action of σ_i , i.e. such that the following diagram commutes:*

$$\begin{array}{ccc} N_i & \xrightarrow{\lambda_t^*} & \text{St}^G(A[t, t^{-1}]) \\ \downarrow \sigma'_i & & \downarrow \sigma_i \\ \text{St}^G(A[t]) & \xrightarrow{\lambda_t^*} & \text{St}^G(A[t, t^{-1}]) \end{array}$$

Proof. Consider the subgroup N_i^0 of $\text{St}^G(A)$ generated by $x_\alpha(\xi)$ for $\xi \in A$, $\alpha \in \Sigma_i \cup \Delta_i$. By the Levi decomposition N_i^0 is isomorphic to $U(\Sigma_i, A) \rtimes \text{Im}(\text{St}(\Delta_i, A) \rightarrow \text{St}(\Phi, A))$. By our assumption, the right factor in the above product is isomorphic to $\text{St}(\Delta_i, A)$, hence $N_i^0 \cong U(\Sigma_i, A) \rtimes \text{St}(\Delta_i, A)$.

By Lemma 2.1.(ii) $\text{St}^G(A[t]) \cong \text{St}^G(A) \rtimes \text{St}^G(A, tA[t])$ hence $N_i \cong N_i^0 \rtimes \text{St}^G(A, tA[t])$. \square

3. DECOMPOSITION THEOREMS FOR $\text{St}^G(A[t^{\pm 1}])$ AND $\text{St}^G(A((t)))$.

Lemma 3.1. *Let (R, m) be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2 . Let $i_+ : \text{St}^G(R[t]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ and $i_- : \text{St}^G(R[t^{-1}]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$i_+(\text{St}^G(m \cdot R[t])^{\text{St}^G(R[t])})i_-(\text{St}^G(R[t^{-1}])) = i_-(\text{St}^G(R[t^{-1}]))i_+(\text{St}^G(m \cdot R[t])^{\text{St}^G(R[t])})$$

inside $\text{St}^G(R[t^{\pm 1}])$.

Proof. This is proved exactly as [9, Lemma 5.12]. \square

thm:3t

Theorem 2. *Let R be a local ring, and let G be a simply connected simple group over R of isotropic rank ≥ 2 . Let $i_+ : \text{St}^G(R[t]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ and $i_- : \text{St}^G(R[t^{-1}]) \rightarrow \text{St}^G(R[t^{\pm 1}])$ be the natural homomorphisms. Then*

$$\text{St}^G(R[t^{\pm 1}]) = i_+(\text{St}^G(R[t]))i_-(\text{St}^G(R[t^{-1}]))i_+(\text{St}^G(R[t])).$$

Proof. This is proved exactly as [9, Theorem 5.1]. \square

4. \mathbb{P}^1 -GLUING

Throughout this section $G = G(\Phi, -)$ denotes a Chevalley group scheme of type Φ .

def:p1g

Definition 4.1. Let F be a group-valued functor from **CRings** to **Groups** and let A be a commutative ring. Consider the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{i_+} & A[t] \\ i_- \downarrow & & \downarrow j_+ \\ A[t^{-1}] & \xrightarrow{j_-} & A[t, t^{-1}] \end{array}$$

We say that F satisfies the \mathbb{P}^1 -glueing property for A if the following sequence of pointed sets is exact in the middle term:

$$F(A) \xrightarrow{\Delta_A^F} F(A[t]) \times F(A[t^{-1}]) \xrightarrow{\pm_A^F} F(A[t, t^{-1}]).$$

Here Δ_A^F denotes the (split injective) diagonal map and, by definition, \pm_A^F maps (g^+, g^-) to $F(j_+)(g^+) \cdot F(j_-)(g^-)^{-1}$. Notice that a priori \pm_A^F is only a morphism of pointed sets but if F takes values in abelian groups then \pm_A^F is also a morphism of groups.

An equivalent way to formulate \mathbb{P}^1 -glueing property is as follows: $F(j_+)$ and $F(j_-)$ are injective and the intersection of their images coincides with the image of $F(j_+i_+) = F(j_-i_-)$.

The main result of this section is the following theorem which generalizes [11, Theorem 5.1] to Chevalley groups. Notice that a K_1 -analogue of the result below has been established in a much greater generality by the second-named author (see [9, Theorem 1.1]).

thm:p1

Theorem 3. Assume that G satisfies Tulenbaev lifting property 2.2. Then the Steinberg group functor $\text{St}^G(-)$ satisfies \mathbb{P}^1 -glueing property for an arbitrary commutative ring A .

Proof. Let (g^+, g^-) be an element of $\text{St}^G(A[t]) \times \text{St}^G(A[t^{-1}])$ such that the equality $g^+ = g^-$ holds in $\text{St}^G(A[t, t^{-1}])$.

Let M be a maximal ideal of A . By Proposition 4.9 below the functor $\text{St}^G(-)$ satisfies \mathbb{P}^1 -glueing property for the local ring A_M hence $(\lambda_M^*(g_+), \lambda_M^*(g_-)) = \Delta_{A_M}(\lambda_M^*(g^+)(0))$ and in the groups $\text{St}^G(A_M[t])$ and $\text{St}^G(A_M[t^{-1}])$ we have the equalities:

$$\lambda_M^*(g^+ \cdot g^+(0)^{-1}) = \lambda_M^*(g^+) \cdot \lambda_M^*(g^+(0))^{-1} = 1; \quad \lambda_M^*(g^- \cdot g^+(0)^{-1}) = 1.$$

Now, by the local-global principle for $\text{St}^G(-)$ (see Theorem 1.(ii)) these equalities hold globally and $(g^+, g^-) = \Delta_A(g^+(0))$, as claimed. \square

rem:stk2

Remark 4.2. It is clear that if the functor St^G satisfies \mathbb{P}^1 -glueing property for A then so does the functor K_2^G . The converse statement also holds, indeed, if $(g^+, g^-) \in \text{Ker}(\pm_A^{\text{St}})$ then inside $E^G(A[t, t^{-1}])$ we have the equality:

$$\varphi(\text{St}^G(j_+)(g^+)) = \varphi(\text{St}^G(j_-)(g^-)) \in E^G(A[t]) \cap E^G(A[t^{-1}]) = E^G(A).$$

Consequently, we can find $g_0 \in \text{St}^G(A)$ so that $(g^+g_0^{-1}, g^-g_0^{-1}) \in \text{Ker}(\pm_A^{K_2^G})$ and it remains to apply the \mathbb{P}^1 -glueing property for K_2^G .

Corollary 4.3. Let A be any commutative ring and $f \in A[t]$ be a monic polynomial. Then the map $K_2^G(A[t]) \rightarrow K_2^G(A[t]_f)$ is injective.

Proof.

\square

4.1. **The case of a field.** Throughout this section k denotes arbitrary field k .

thm:k[t]

Theorem 4. Assume that $G = G(\Phi, -)$ and Φ is irreducible of rank ≥ 2 .

satz1

(i) The subgroup $K_2^G(k[t]) \trianglelefteq \text{St}^G(k[t])$ is generated by elements of the form

$$h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}, \quad u, v \in k^*.$$

(ii) As a consequence, the canonical injection $K_2^G(k) \hookrightarrow K_2^G(k[t])$ is an isomorphism.

Proof. See [7, Satz 1] and the corollary after it. \square

cor:k[t]inj

Corollary 4.4. Let G be as in the above theorem. Then the functors St^G, K_2^G satisfy \mathbb{P}^1 -glueing property for k .

Proof. By Remark 4.2 it suffices to prove the assertion only for the functor K_2^G . By the previous theorem $K_2^G(i_+)$ and $K_2^G(i_-)$ are isomorphisms hence the morphisms $K_2^G(j_+)$ and $K_2^G(j_-)$ are split injective and $\text{Im}(j_+i_+) = \text{Im}(j_-i_-) = \text{Im}(j_+) = \text{Im}(j_-)$. \square

Corollary 4.5. Let G, k be as in Theorem 4. Then $K_2^G(k[t^{\pm 1}]) = K_2^G(k)$.

Proof. We use Theorem 2. Take $g \in K_2^G(k[t^{\pm 1}])$, then $g = x_1 y x_2$, $x_i \in \text{St}^G(k[t])$, $y \in \text{St}^G(k[t^{-1}])$. Since $E(k[t]) \cap E(k[t^{-1}]) = E(k)$, we have $y \in \text{St}^G(k) K_2^G(k[t^{-1}]) = \text{St}^G(k)$ and $x_1 x_2 \in \text{St}^G(k)$. That is, $g \in K_2^G(k)$. \square

4.2. **The case of a local ring.** For the rest of this section A denotes an arbitrary commutative local ring with the maximal ideal m and the residue field k . We denote by π the canonical projection $A \rightarrow k$. Throughout this section we will employ the following notation:

- R denotes the Laurent polynomial ring $A[t, t^{-1}]$;
- B denotes the subring $A[t] + m[t^{-1}]$ of R consisting of Laurent polynomials $f(t, t^{-1})$ whose coefficients of terms of negative degree belong to m ;
- I denotes the ideal $m[t, t^{-1}]$ of R (which can be also considered as an ideal of B).

Our first result is analogous to [11, Lemma 3.1(e)] (cf. also with [4, § 2.3A]).

cor:bruhat

Lemma 4.6. The Steinberg group $\text{St}^G(A)$ admits the following analogue of the Bruhat decomposition:

$$\text{St}^G(A) = U(\Phi^+, A) \cdot W(\Phi, A) \cdot U(\Phi^+, A) \cdot \text{Im}(\text{St}^G(A, m) \rightarrow \text{St}^G(A)).$$

Proof. First of all, notice that $\text{St}^G(k)$ admits Bruhat decomposition for arbitrary field k :

$$\text{St}^G(k) = U(\Phi^+, k) \cdot W(\Phi, k) \cdot U(\Phi^+, k).$$

Indeed, this is an immediate consequence of the classical Bruhat decomposition for the linear groups (see e.g. [10, Theorem 4]) and the fact that $K_2^G(k)$ is central in $\text{St}^G(k)$ and is generated by symbols $h_\alpha(u)$ lying in $W(\Phi, k)$ (cf. Theorem 4.(i)).

As a consequence, the first three factors in the right hand side of the decomposition are mapped epimorphically onto $\text{St}^G(k)$ and the last factor coincides with $\text{Ker}(\pi_*)$ from which the assertion of the lemma follows. \square

`lem:tulinj`

Lemma 4.7. Assume that G satisfies Tulenbaev lifting property 2.2. Then the map i in the following commutative diagram of groups is injective.

`{diag:cs}`

(4.1)

$$\begin{array}{ccccccc}
 C_B & \hookrightarrow & \mathrm{St}^G(B, I) & \xrightarrow{\mu_B} & \mathrm{St}^G(B) & \xrightarrow{\pi_B} \twoheadrightarrow & \mathrm{St}^G(k[t]) \\
 \downarrow k & & \downarrow j & \nearrow \varphi & \downarrow i & & \downarrow \\
 C_R & \hookrightarrow & \mathrm{St}^G(R, I) & \xrightarrow{\mu_R} & \mathrm{St}^G(R) & \xrightarrow{\pi_R} \twoheadrightarrow & \mathrm{St}^G(k[t, t^{-1}])
 \end{array}$$

Proof. First of all, notice that by Corollary 4.4 the vertical map in the right-hand side of the diagram is injective. Invoking Tulenbaev's property 2.2 we also find a lifting map φ in the central square of the diagram.

Let $g \in \mathrm{St}^G(B)$ be an element of $\mathrm{Ker}(i)$. Since g also lies in $\mathrm{Ker}(\pi_B)$ it comes from some $\tilde{g} \in \mathrm{St}^G(B, I)$ via μ_B . But $j(\tilde{g})$ lies in C_R , hence, by Proposition 4.8 below it comes from some $\hat{g} \in C_B$ via k . Finally, $g = \varphi(j(\tilde{g})) = \varphi(k(\hat{g})) = \mu_B(\hat{g}) = 1$, as claimed. \square

The following result is analogous to [11, Proposition 4.1].

`prop:kersurj`

Proposition 4.8. Under the assumptions of Lemma 4.7 the map k in the diagram (4.1) is surjective.

Proof. \square

The following result is analogous to [11, Proposition 4.3].

`prop:p1g`

Proposition 4.9. The functors St^G , K_2^G satisfy \mathbb{P}^1 -glueing property for arbitrary local ring A .

Proof. \square

Appendices

A. LINEAR STEINBERG GROUP IN RANK 3

The main goal of this subsection is to show that Tulenbaev's [11, Lemma 2.3] remains valid for the linear Steinberg group of rank ≥ 3 . In order to do this we will need yet another presentation for the relative linear Steinberg group (cf. [8, Definitions 3.3 and 3.7]).

Definition A.1. The relative Steinberg group $\mathrm{St}^*(n, R, I)$ is the group defined by the following two families generators and four families of relations.

- Generators:

- (1) $X^1(u, v)$, where $u \in \mathrm{E}(n, R) \cdot e_1$, $v \in I^n$ such that $v^t \cdot u = 0$;
- (2) $X^2(u, v)$, where $u \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$ such that $v^t \cdot u = 0$.

Notice that ϕ maps both $X^1(u, v)$ and $X^2(u, v)$ to $T(u, v) = e + u \cdot v^t \in \mathrm{E}(n, R, I)$.

- Relations:

- (1) $X^1(u, v) \cdot X^1(u, w) = X^1(u, v + w)$, $u \in \mathrm{E}(n, R) \cdot e_1$, $v, w \in I^n$;
- (2) $X^2(u, v) \cdot X^2(w, v) = X^2(u + w, v)$, $u, w \in I^n$, $v \in \mathrm{E}(n, R) \cdot e_1$;
- (3) $X^{\sigma(u^2, v^2)} X^{\tau}(u^1, v^1) = X^{\tau}(T(u^2, v^2) \cdot u^1, T(v^2, u^2)^{-1} \cdot v^1)$, $\sigma, \tau = 1, 2$;

- (4) $X^1(g \cdot e_1, g^* \cdot be_2) = X^2(g \cdot be_1, g^* \cdot e_2)$ where $b \in I$ and $g^* = g^{t-1}$ denotes the contragradient matrix.

Lemma A.2. *The groups $\text{St}^*(n, R, I)$ and $\text{St}(n, R, I)$ are isomorphic.*

Proof. **TODO:** □

The next step of the proof is to construct certain elements in $\text{St}(n, R)$ similar to Tulenbaev's elements $X_{u,v}(a)$ see [11, § 1].

Let $v \in R^n$ be a column. Denote by $O(v)$ the submodule of R^n consisting of all columns w such that $w^t \cdot v = 0$. A column $w \in R^n$ is called *v-decomposable* if it can be presented as a finite sum $w = \sum_{i=1}^p w^i$ such that each w^i has at least two zero entries and $v^t \cdot w^i = 0$. Denote by $D(v)$ the submodule of $O(v)$ consisting of all *v-decomposable* columns. For a column $v \in R^n$ denote by $I(v)$ the ideal of R spanned by its entries v_1, \dots, v_n .

Let $u, v, w \in R^n$ be columns such that $w^t v = 0$. It is easy to check (cf. [6, Lemma 3.2]) that

$$(uv) \cdot w = \sum_{i < j} w_{ij}, \text{ where } w_{ij} = (w_i u_j - w_j u_i)(v_j e_i - v_i e_j) \in A^n.$$

The above decomposition is called the *canonical* decomposition of $(uv) \cdot w$. In particular, this shows that the column $a \cdot w$ is always *v-decomposable* for $a \in I(v)$, $w \in O(v)$, i.e. $I(v) \cdot O(v) \subseteq D(v)$. It is also straightforward to check that $D(v) \subseteq D(bv)$, $b \cdot D(v) \subseteq D(v)$ for $b \in R$.

Denote by B^1 the subset of $R^n \times R^n \times R$ consisting of triples (u, v, a) such that $v^t \cdot u = 0$, $v \in D(u)$, $a \in I(u)$. Denote by B^2 the set consisting of triples (v, u, a) such that $(u, v, a) \in B^1$.

lem:Zfacts

Lemma A.3. *Assume that $n \geq 4$. One can define two families of elements $Z^\tau(u, v, a)$, $\tau = 1, 2$ of the group $\text{St}(n, R)$ parametrized by $(u, v, a) \in B^\tau$ satisfying the following properties:*

- (1) $\phi(Z^\tau(u, v, a)) = e + uav^t \in E(n, R)$, $(u, v, a) \in B^\tau$;
- (2) $Z^1(u, v + w, a) = Z^1(u, v, a) \cdot Z^1(u, w, a)$;
- (3) $Z^2(v + w, u, a) = Z^2(v, u, a) \cdot Z^2(w, u, a)$;
- (4) for $\tau = 1, 2$ and $b \in R$ if $(u, vb, a), (ub, v, a) \in B^\tau$ then one has

$$Z^\tau(u, vb, a) = Z^\tau(u, v, ab) = Z^\tau(ub, v, a);$$

- (5) ${}^g Z^\tau(u, v, a) = Z^\tau(\phi(g) \cdot u, \phi(g)^* \cdot v, a)$, $\tau = 1, 2$, $g \in \text{St}^G(n, R)$.

Proof. One constructs the elements $Z^1(u, v, a)$ in exactly the same way as Tulenbaev constructs his elements $X_{u,v}(a)$ (see definitions preceding [11, Lemma 1.2]). Indeed, set

$$(A.1) \quad Z^1(v, w, a) = \prod_{k=1}^p X(v, a \cdot w^k), \quad Z^2(w, v, a) = \prod_{k=1}^p X(a \cdot w^k, v).$$

where $X(u, v)$ denotes the elements defined by Tulenbaev before [11, Lemma 1.1].

The correctness of this definition and all the assertions of the lemma (with the exception of the last one in the case $n = 4$) can be proved by the same token as in [11, Lemma 1.3]. □

For the rest of this section a denotes a nonnilpotent element of R and $\lambda_a: R \rightarrow R_a$ is the morphism of principal localization at a .

lem:rk3rels

Lemma A.4. *For any $g \in E(n, R_a)$ there exist $u, v \in R^n$ and sufficiently large natural numbers k, m such that the following facts hold:*

- (1) $\lambda_a(u) = g \cdot a^k e_1$, $\lambda_a(v) = g^* \cdot a^k e_2$ and $u^t \cdot v = 0$;
- (2) $(u, v, a^m) \in B^1 \cap B^2$;
- (3) for $b \in R$ divisible by some sufficiently large power of a one has

$$Z^1(u, b \cdot v, a^m) = Z^2(b \cdot u, v, a^m).$$

Proof. It is straightforward to choose u and v satisfying the first requirement of the lemma. We can even choose u, v in such a way that $u \in D(v)$ and $v \in D(u)$. Indeed, notice that $I(u) = a^{k_1}$, $I(v) = a^{k_2}$ for some natural k_1, k_2 hence for $u' = a^{k_2} \cdot u$ and $v' = a^{k_1} \cdot v$ one has

$$u' \in I(v) \cdot O(v) \subseteq D(v) \subseteq D(v'), \quad v' \in I(u) \cdot O(u) \subseteq D(u) \subseteq D(u'),$$

as required.

In fact, we can also choose two extra columns $x, y \in R^n$ and a large natural p in such a way that vectors u, v, x, y additionally satisfy the following properties

$$\begin{aligned} \lambda_a(x) &= g^* \cdot a^k e_3, \quad \lambda_a(y) = g \cdot a^k e_3, \quad y^t \cdot x = a^p \in R, \\ u^t \cdot x &= 0, \quad u^t \cdot v = 0, \quad y^t \cdot v = 0, \\ (u, x, a^m) &\in B^1, \quad (y, v, a^m) \in B^2. \end{aligned}$$

Now direct computation using Lemma A.3 shows that

$$\begin{aligned} Z^2(a^{m+p}b \cdot u, v, a^m) &= Z^2(b \cdot (e + a^m \cdot ux^t)y, (e - a^m \cdot xu^t)v, a^m) \cdot Z^2(-by, v, a^m) = \\ &= [Z^1(u, x, a^m), Z^2(b \cdot y, v, a^m)] = \\ &= Z^1(u, x, a^m) \cdot Z^1((e + a^m b \cdot yv^t)u, -(e - a^m b \cdot vy^t)x, a^m) = Z^1(u, a^{m+p}b \cdot v, a^m), \quad \square \end{aligned}$$

hence the third assertion of the lemma follows.

cor:tulmap

Corollary A.5. *[Tulenbaev's lemma] For $n \geq 4$ there is a map T_n so that the following diagram commutes.*

Proof. Follows from Lemma A.4 by the same token as in [11, Lemma 2.3]. \square

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DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA

E-mail address: `sinchukss@gmail.com`

DEPARTMENT OF MATHEMATICS AND MECHANICS, ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA

E-mail address: `anastasia.stavrova@gmail.com`

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN, THERESIENSTR. 39, D-80333 MÜNCHEN

E-mail address: `avlavrenov@gmail.com`