PARAMETRIZED SYMMETRIC GROUPS AND NONABELIAN TENSOR PRODUCTS

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Abstract. By amalgamating copies of symmetric group S_n we construct a central extension of the wreath product $G \wr S_n$. We then relate this group with extensions of type $\mathfrak{H}_n(G)$ introduced by U. Rehmann in 1970's. We also describe a topological application of this construction.

1. Introduction

The aim of this note is to describe a rather surprising connection between symmetric groups and nonabelian tensors products which apparently has not been studied in detail previously.

Let G be a (possibly noncommutative) group. Denote by $S_n(G)$ the group generated by elements $s_i(a)$, where i is an integer $1 \leq i \leq n-1$ and a is an element of G, subject to the relations

$$s_i(a)^2 = 1, \qquad a \in G, \tag{1.1}$$

$$s_i(a)s_{i+1}(b)s_i(c) = s_{i+1}(a^{-1}cb)s_i(a)s_{i+1}(b), \qquad a, b, c \in G, \tag{1.2}$$

$$s_i(a)s_j(b) = s_j(b)s_i(a). \qquad |i-j| \geq 2,\ a, b \in G. \tag{1.3}$$

Notice that in the special case $G = 1$ the above presentation boils down to the usual Coxeter presentation of the symmetric group S_n, i. e. $S_n(1) = S_n$.

The group defined above bears some similarity with the *parametrized braid group* defined by Loday—Stein in [8] (cf. also with Kassel—Reutenauer, [7]). By analogy, we call the group $S_n(G)$ a *parametrized symmetric group*. Our first main result is the following theorem, which gives a concrete description of the group $S_n(G)$.

Theorem 1. *For $n \geq 3$ there is a map $\mu_n \colon S_n(G) \to G \wr S_n$ which is a crossed module. The cokernel and the kernel of μ_n are isomorphic to $\mathrm{H}_1(G, \mathbb{Z})$ and $\mathrm{H}_2(G, \mathbb{Z})$ respectively. Moreover, the group $S_n(G)$ is isomorphic to $S_n \ltimes H_n^{\wedge}(G)$.*

Here $G \wr S_n$ stands for the wreath product of G and S_n and $H_n^{\wedge}(G)$ denotes the quotient of the so-called "universal extension of type $\mathfrak{H}_n(G)$" (which is denoted $H_n(G)$) modulo all "symbols" of the form $\{x, x\} = 1$, $x \in G$.

Extensions of type $\mathfrak{H}_\mathfrak{n}$ defined by U. Rehmann in [13] played a key role in his proof of generalized Matsumoto theorem. The groups $H_n(G)$, $H_n^{\wedge}(G)$ are directly related with nonabelian tensor products and their quotients. More formally, $G \widetilde{\otimes} G$ (resp. $G \wedge G$) are precisely the subgroups of $H_n(G)$ (resp. $H_n^{\wedge}(G)$) generated by symbols $\{x, y\}$ (we refer the reader to Section 3 for more details).

Theorem 1 and its proof (which relies solely on basic combinatorial group theory) may be of interest for the following reasons:

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- It establishes a connection between two classical objects: symmetric groups and the second homology group.
- It helps to better understand the nature of extensions $H_n(G)$ appearing in the proof of noncommutative Matsumoto theorem in [13]. For example we obtain a presentation for $H_n(G)$ that is simpler than its original presentation given in [13] (4 instead of 5 relations that are also shorter), see Proposition 3.3.
- Theorem 1 is motivated by (and provides partial answer to) some purely topological question, which we describe below in more detail.

For a pointed simplicial set (X, x_0) denote by $\mathbb{Z}[X]$ the associated free simplicial abelian group. Define the "alternating" map of simplicial sets $h_n\colon X^{2n} \to \mathbb{Z}[X]$ by the following identity:

$$h_n(x_1, x_2, \dots x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Define the "stabilization" map $X^{2n} \hookrightarrow X^{2(n+1)}$ by $x \mapsto (x, x_0, x_0)$ and denote by X^∞ the colimit of X^n.

Recently S. Podkorytov showed that the limit map $h_\infty = \operatorname{colim}_n(h_n)\colon X^\infty \to \mathbb{Z}[X]$ is a *quasi-fibration* provided X is fibrant and connected. By definition, this means that the natural map $h_\infty^{-1}(b) \to F_{h_\infty}(b)$ between the fiber and the homotopy fiber of h_∞ over any point b is a weak (hence homotopy) equivalence. This assertion is essentially proved in [12, Lemma 9.1], however some technical work is required to formulate it this way, see **??** below.

One may wonder whether the maps h_n satisfy some weaker analogue of this property, i. e. one might expect that the map $h_n^{-1}(0) \to F_{h_n}(0)$ is $c(n)$-connected for some $c(n)$ which depends only on n and tends to infinity as $n \to \infty$.

The following result which is a consequence of Theorem 1 asserts that this map is at least 1-connected in one important special case.

Theorem 2. *If $X = BG$ is the classifying space of a group G then for $n \geq 3$ the natural map $h_n^{-1}(0) \to F_{h_n}(0)$ induces an isomorphism of fundamental groups $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$.*

Theorem 2 reduces to Theorem 1 by means of Quillen Theorem A and van Kampen theorem. It also involves some explicit computations with simplicial fundamental groups.

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2. Definition of $S_n(G)$ and its basic properties

The aim of this section is to prove the first part of Theorem 1, namely the fact that $S_n(G)$ is a crossed module over $G \wr S_n$. It will be more convenient for us to start with another definition of the group $S_n(G)$ given in terms of amalgams of symmetric groups. In the end of the section we show that this alternative definition coincides with the one given in the introduction.

Let G be a group. Recall that the *wreath product* is, by definition, the semidirect product $G^n \rtimes S_n$, in which S_n acts on G^n on the right by permuting its factors.

Consider the family $\{{S_n}^{(g)}\}_{g \in G^n}$ of isomorphic copies of S_n indexed by elements of G^n and let F be the free product of groups from this family. For $s \in S_n$ and $g \in G^n$ we denote by s_g the image of

s in F under the canonical map $S_n^{(g)} \to F$.

$$s_g = s_h, \text{ if } s^g = s^h \text{ holds in } G \wr S_n. \tag{2.1}$$

We denote by s_g the image of the element $s_g \in F$ under the canonical map $F \to S_n(G)$.

Remark 2.1. Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1} g^{(s^{-1})}, s).$$

therefore the equality $s^g = s^h$ holds in $G \wr S_n$ iff $g^{-1} g^{(s^{-1})} = h^{-1} h^{(s^{-1})}$, or what is the same, iff s fixes hg^{-1}.

The last statement immediately implies that the map $\mu\colon S_n(G) \to G \wr S_n$ given by $\mu(s_g) = s^g = (g^{-1} g^{(s^{-1})}, s)$ is well-defined.

Observe from the definition of $S_n(G)$ that there is a split exact sequence.

$$1 \longrightarrow \operatorname{Ker}(\pi) \longrightarrow S_n(G) \underset{\iota(1)}{\overset{\pi}{\rightleftarrows}} S_n \longrightarrow 1, \tag{2.2}$$

Here the map $\pi = \pi_{S_n} \circ \mu$ removes subscript g from each s_g and the section $\iota(1)$ maps s to s_1, where 1 is the identity element of G^n. Clearly, $S_n(G)$ decomposes as $HS_n(G) \rtimes S_n$, where $HS_n(G) = \operatorname{Ker}(\pi)$.

2.1. **Crossed module structure on $S_n(G)$.** Recall from [2, § 2.2] that a *crossed module* is a morphism of groups $\mu\colon M \to N$ together with a right action of N on M compatible with the conjugation action of N on itself, i. e.

$$\mu(m^n) = \mu(m)^n \text{ for all } n \in N,\ m \in M, \tag{CM1}$$

which also satisfies the following identity called *Peiffer identity*:

$$m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M. \tag{CM2}$$

In our situation, we let $G \wr S_n$ act on $S_n(G)$ by

$$(s_g)^{(h,t)} = s^t{}_{(gh)^t}, \text{ for } s, t \in S_n \text{ and } g, h \in G^n. \tag{2.3}$$

The goal of this subsection is to prove the following.

Proposition 2.2. *For $n \geq 3$ the map $\mu\colon S_n(G) \to G \wr S_n$ is a crossed module.*

From the fact that μ is a crossed module one can deduce that $\operatorname{Ker}(\mu)$ is a central subgroup of $S_n(G)$ and $\operatorname{Im}(\mu)$ is a normal subgroup of $G \wr S_n$. It is not hard to show that the group $S_2(G)$ is isomorphic to a free product of copies of S_2 (whose center is always trivial). Therefore, the requirement $n \geq 3$ in the statement of Proposition 2.2 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of $G \wr S_n$ on $S_n(G)$ that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of $S_n(G)$, for which it takes the form:

$$t_h^{-1} s_g t_h = s^t{}_{(gh^{-1})^t \cdot h} \text{ for all } s, t \in S_n,\ g, h \in G^n. \tag{2.4}$$

If we act on both sides of the above formula by $(h^{-1}, 1) \in G \wr S_n$ we obtain the equality $t_1^{-1} s_{gh^{-1}} t_1 = s^t{}_{(gh^{-1})^t}$. Thus, to prove (2.4) it suffices to show the following simpler relation:

$$t_1^{-1} s_g t_1 = s^t{}_{g^t} \text{ for all } s, t \in S_n,\ g \in G^n. \tag{2.5}$$

The key step in the proof is the following lemma.

Lemma 2.3. *The relation* (2.5) *holds in the special case when* $s = (ij)$ *and* $t = (kl)$ *are two nonequal transpositions.*

Proof. First of all, we immediately check that (2.5) holds in the special case when t fixes $g' \in G^n$. Indeed, by (2.1) we have $t_1 = t_{g'}$, hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that $l \neq i$ and $l \neq j$. Denote by g' the vector which differs from g only at l-th position, for which we set $g'_l = g_k$. Since the only nontrivial component of $g'g^{-1}$ (resp. $g'g^{-t}$) is located at l-th (resp. k-th) position, it is fixed by s (resp. s^t), hence from (2.1) we conclude that $s_g = s_{g'}$ (resp. $(s^t)_{g'} = (s^t)_{g^t}$). Finally, since g' is fixed by t, we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g'^t} = s^t_{g'} = (s^t)_{g^t}. \quad \square$$

Proof of Proposition 2.2. Let us show that (2.5) holds for arbitrary transpositions $s, t \in S_n$. It suffices to consider the case $s = t = (ij)$. After choosing some $k \neq i, j$ and presenting (ij) as $(kj)(ik)(kj)$ we use Lemma 2.3:

$$(ij)_1^{-1}(ij)_g(ij)_1 = (ij)_1^{-1}(kj)_g(ik)_g(kj)_g(ij)_1 = (ki)_{g^{(ij)}}(jk)_{g^{(ij)}}(ki)_{g^{(ij)}} = (ij)_{g^{(ij)}}.$$

The proposition now follows by induction on the length of permutations s, t. $\square$

2.2. **An explicit presentation of $S_n(G)$.** In this section we obtain an explicit presentation of $S_n(G)$ by generators and relations. This presentation turns out to be much more economical than the one given in the previous subsection. From this presentation we later obtain an explicit presentation for the subgroup $HS_n(G)$ via Reidemeister-Schreier method.

We start with the following simple lemma which is, in essence, a variant of the standard presentation of the symmetric group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

Lemma 2.4. *For $n \geq 3$ The symmetric group S_n admits presentation with $\{(ij) \mid i \neq j,\ 1 \leq i, j \leq n\}$ as the set of generators and the following list of relations (in every formula distinct letters denote distinct indices):*

$$(ij)^2 = 1, \tag{2.6}$$

$$(ij)^{(jk)} = (ik), \tag{2.7}$$

$$[(ij), (kl)] = 1. \tag{2.8}$$

$$(ij) = (ji), \tag{2.9}$$

Remark 2.5. To simplify notation, up to the end of this section we adhere to the convention that in our formulae distinct letters denote distinct indices.

Now we are ready to formulate the main result of this subsection.

Proposition 2.6. *For $n \geq 3$ and arbitrary group G the group $S_n(G)$ admits presentation with $\{(ij)_a \mid i \neq j, 1 \leq i, j, \leq n, a \in G\}$ as the set of generators and the following list of relations.*

$$(ij)_a^2 = 1, \tag{SG1}$$

$$(ij)_a^{(jk)_b} = (ik)_{ab}, \tag{SG2}$$

$$[(ij)_a, (kl)_b] = 1. \tag{SG3}$$

$$(ij)_a = (ji)_{a^{-1}} \tag{SG4}$$

Proof. Denote by S' the group from the statement of the proposition. For $1 \leq i \leq n$ and $x \in G$ denote by $x[i]$ the element of G^n whose only nontrivial component equals x and is located in the i-th position.

It is not hard to deduce from the definition of $S_n(G)$ and Lemma 2.3 that the formula $(ij)_a \mapsto (ij)_{a[j]}$ gives a well-defined map $\varphi \colon S' \to S_n(G)$.

Now we are going to construct the map $\psi \colon S_n(G) \to S'$ in the opposite direction. Using the presentation of S_n given by Lemma 2.4 we define for a fixed $g \in G^n$ the map $\psi_g \colon S_n \to S'$ by $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$. It is obvious that ψ_g preserves the defining relations (2.6)–(2.9) of S_n. It remains to show that the equality $\psi_g(s) = \psi_h(s)$ holds whenever $g, h \in G^n$ and $s \in S_n$ are such that $s_g = s_h$.

Indeed, if hg^{-1} is fixed by s then for every $1 \leq i \leq n$ we have $(hg^{-1})_i = (hg^{-1})_{s(i)}$, or equivalently $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$. For example, if s is a cycle of length p, i.e. $s = (i_1, i_2, \dots i_p)$ with $i_{k+1} = s(i_k)$ we get that

$$\psi_g(s) \;=\; \psi_g\left(\prod_{k=1}^{p-1}(i_k, i_{k+1})\right) \;=\; \prod_{k=1}^{p-1}(i_k, i_{k+1})_{g_{i_k}^{-1}g_{i_{k+1}}} \;=\; \prod_{k=1}^{p-1}(i_k, i_{k+1})_{h_{i_k}^{-1}h_{i_{k+1}}} \;=\; \psi_h(s).$$

The proof for a general s is almost the same. Verification of the fact that ψ and φ are mutually inverse is also immediate. $\square$

2.3. **Coxeter-like presentation for $S_n(G)$.** The aim of this subsection is to show the following.

Proposition 2.7. *The group $S_n(G)$ defined in Proposition 2.6 coincides with the group defined in the introduction by means of Coxeter-like presentation (denote the latter group by $S_n^C(G)$).*

The idea of the proof is to construct a sequence of "intermediate" groups

$$S_n^C(G) \to S_n(1, G) \to \ldots \to S_n(n-1, G) \to S_n(G)$$

and then show that all the maps between these groups are isomorphisms.

Definition 2.8. For $1 \leq t \leq n-1$ consider the group $S_n(t, G)$ defined by the set of generators $\{(ij)_a \mid 1 \leq i < j \leq n,\ j - i \leq t\}$ subject to the following relations:

$$(ij)_a^2 = 1, \tag{2.10}$$

$$(ij)_a^{(jk)_b} = (jk)_{b'}^{(ij)_{a'}}, \qquad \text{provided } ab = a'b', \tag{2.11}$$

$$(ij)_a(kl)_b = (kl)_b(ij)_a, \tag{2.12}$$

$$(ij)_a^{(jk)_b} = (ik)_{ab}. \tag{2.13}$$

In the above relations indices i, j, k, l satisfy the inequalities ensuring that all the generators appearing in the formulae are well-defined.

In the case $t = 1$ the relation (2.13) becomes vacuous and the remaining relations have the same form as (1.1)—(1.3). Thus, the map $S_n^C(G) \to S_n(1, G)$ defined by $s_i(g) \mapsto (i, i+1)_g$ is an isomorphism.

On the other hand, in the case $t = n-1$ the obvious embedding of generators induces a map $S_n(n-1, G) \to S_n(G)$ (we consider the latter group in the presentation of Proposition 2.6). We claim that this map is an isomorphism. Indeed, define the map $\theta \colon S_n(G) \to S_n(n-1, G)$ in the opposite direction by the following formula:

$$\theta((ij)_a) = \begin{cases} (ij)_a & \text{if } i < j, \\ (ji)_{a^{-1}} & \text{if } j < i. \end{cases}$$

Only the fact that θ preserves the relation (SG2) is not immediately obvious. Fix three indices i, j, k satisfying $i < j < k$ and let p, q, r be any permutation of these indices. One has to check that the equality $\theta((pq)_a^{(qr)_b}) = \theta((pr)_{ab})$ holds. For example, the relation $\theta((jk)_a^{(ki)_b}) = \theta((ji)_{ab})$ can be proved as follows:

$$\theta((jk)_a^{(ki)_b}) = (jk)_a^{(ik)_{b^{-1}}} = (ik)_{b^{-1}}{}^{(jk)_a} = (ij)_{b^{-1}a^{-1}} = \theta((ji)_{ab}).$$

In the above formula the second equality holds by (2.11) and the third one holds by (2.13) (in which both sides are conjugated by $(jk)_a$). Verification of the other 5 cases is similar.

To finish the proof of Proposition 2.7 it remains to show the following lemma.

Lemma 2.9. *For each $1 \leq t \leq n-2$ the map $f_t \colon S_n(t, G) \to S_n(t+1, G)$ induced by the obvious embedding of generators is an isomorphism.*

Proof. Let $(ij)_a$ be a generator of $S_n(t+1, G)$, define the map g_t in the opposite direction as follows:

$$g_t((ij)_a) = \begin{cases} (ij)_a & \text{if } j - i \leq t, \\ (ik)_a^{(kj)_1} & \text{if } j - i = t+1 \text{ for some } i < k < j. \end{cases}$$

The value of $g_t((ij)_a)$ does not depend on the choice of k. Indeed, let $k' > k$ be another index lying between i and j. By (2.12) we have the equality $(ik)_a^{(k'j)_1} = (ik)_a$ hence

$$(ik)_a^{(kj)_1} = (ik)_a^{\left((kk')_1^{(k'j)_1}\right)} = (ik)_a^{(kk')_1 (k'j)_1} = (ik')_a^{(k'j)_1}.$$

It remains to show that g_t preserves the defining relations of $S_n(t+1, G)$. Let us verify this for (2.11). Assume for a moment that $k - j \leq t$ and $j - i = t+1$ and choose some index j' between i and j, then

$$g_t((ij)_a^{(jk)_b}) = (ij')_a^{(j'j)_1 (jk)_b} = ((jk)_{b'}^{(j'j)_1})^{(ij')_{a'} (j'j)_1} = (j'j)_1^{(jk)_{b'} (ij')_{a'} (j'j)_1}$$

Verification of this fact for (2.10) and (2.13) is similar but easier and is left as an exercise to the reader. □

3. Comparison with extensions of type $\mathfrak{H}_n(G)$

We start by briefly reviewing the material of § 1–3 of [13]. Let $n \geq 3$ be a natural number. By definition, the group $H_n(G)$ is given by generators $h_{ij}(u)$, $u \in G$, $i \neq j$ and the following relations (cf. [13, H1–H5 of § 2]):

(H1) $$h_{ij}(u) h_{ji}(u) = 1,$$

(H2) $$h_{ij}(u) h_{ki}(u) h_{jk}(u) = 1,$$

(H3) $$h_{ij}(u) h_{ik}(v) h_{ij}(u)^{-1} = h_{ik}(uv) h_{ik}(u)^{-1}, \quad \text{for } j \neq k$$

(H4) $$h_{ij}(u) h_{kj}(v) h_{ij}(u)^{-1} = h_{kj}(vu) h_{kj}(u)^{-1}, \quad \text{for } i \neq k$$

(H5) $$[h_{ij}(u), h_{kl}(v)] = 1.$$

There is a surjective group homomorphism $H_n(G) \twoheadrightarrow D_n(G)$ sending $h_{ij}(g)$ to $d_{ij}(g)$. Here $d_{ij}(g)$ denotes the element of G^n whose i-th component equals g, j-th component equals g^{-1} and all other components are trivial.

By definition, *an extension of type* $\mathfrak{H}_n(G)$ is an extension H of $D_n(G)$ which is also a quotient of $H_n(G)$, i. e. the one which fits into the following diagram.

$$\begin{tikzcd} H_n(G) \arrow[d, two heads] \arrow[dr, two heads] & \\ H \arrow[r, two heads] & D_n(G) \end{tikzcd}$$

Any $\mathfrak{H}_n(G)$-extension is central, see [13, Proposition 2.3].

3.1. **Presentation for the subgroup $HS_n(G)$.** We now obtain a presentation of the subgroup $HS_n(G)$. For every $i \neq j$ and $a \in G$ we define the element $h_{ij}(a) \in HS_n(G)$ as follows:

$$h_{ij}(a) = (ij)_a \cdot (ij)_1. \tag{3.1}$$

It is not hard to show that $h_{ij}(a)$ form a generating set for $HS_n(G)$. In fact, there is an explicit formula how an element of $HS_n(G)$ originally expressed through $(ij)_a$'s can be rewritten in terms of $h_{ij}(a)$. Indeed, if h lies in $HS_n(G)$ and is written as $\prod_{k=1}^{N}(i_k j_k)_{a_k}$ for some $i_k \neq j_k$ and $a_k \in G$ then it can be rewritten as follows:

$$h = \prod_{k=1}^{N} h_{\sigma_k(i_k),\sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1}(i_s j_s) \in S_n. \tag{τ}$$

We briefly recall the notion of a *rewriting process* given in [9, § 2.3]. If G is a group presented by generators a_ν and relations $R_\mu(a_\nu)$ and H is its subgroup with a generating set $J_i(a_\nu)$ then a *rewriting process for* H is a function which maps every word u in alphabet a_ν to a word v in alphabet s_i such that u and $v[s_i := J_i]$ define the same element of G whenever u represents an element of H.

With this terminology, the mapping (τ) defined above is a rewriting process for the subgroup $HS_n(G)$. Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [9, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [9, § 2.3]):

- if U and U^* are freely equal words in $(ij)_a$ then $\tau(U)$ and $\tau(U^*)$ are also freely equal words in $h_{ij}(a)$;
- if U_1 and U_2 are two words in $(ij)_a$ which define elements of $HS_n(G)$ then the words $\tau(U_1U_2)$ and $\tau(U_1)\tau(U_2)$ are equal.

Using these two properties and repeating the arguments used in the proof of [9, Theorem 2.8] one can simplify the generic presentation given by [9, Theorem 2.6] and obtain the following.

Lemma 3.1. *For $n \geq 3$ the group $HS_n(G)$ admits presentation on the generators $h_{ij}(a)$ with the following two families of defining relations:*

$$h_{ij}(a) = \tau\left((ij)_a \cdot (ij)_1\right); \tag{3.2}$$

$$\tau(KRK^{-1}) = 1, \tag{3.3}$$

where R varies over relations of Proposition 2.6 and K is any word in $(ij)_1$, $i \neq j$.

From (3.2) one immediately obtains the equality $h_{ij}(1) = 1$, $i \neq j$. Thus, if we denote by σ the permutation corresponding to a word K and let S_n act on $h_{ij}(a)$'s in the natural way we will get that the word $\tau(KRK^{-1})$ is equivalent to ${}^{\sigma}\tau(R)$ (modulo relations $h_{ij}(1) = 1$).

Since the relations from Proposition 2.6 are respected by the action of S_n, only relations of the form $\tau(R) = 1$ are, in fact, needed for the presentation of $HS_n(G)$. Writing down what $\tau(R)$ is for each of (SG1)–(SG4) we get the following.

Proposition 3.2. *For $n \geq 3$ the group $HS_n(G)$ admits presentation with generators $h_{ij}(a)$ and the following list of relations:*

$$h_{ij}(1) = 1, \tag{R0}$$

$$h_{ij}(a)h_{ji}(a) = 1, \tag{R1}$$

$$h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab), \tag{R2}$$

$$[h_{ij}(a), h_{kl}(b)] = 1, \tag{R3}$$

$$h_{ij}(a)^{-1} = h_{ij}(a^{-1}). \tag{R4}$$

3.2. **A simpler presentation of $H_n(G)$.** The aim of this subsection is to obtain a presentation of $H_n(G)$ that is simpler than the original presentation: 4 instead of 5 relations that are also shorter. Our main result is the following.

Proposition 3.3. *For $n \geq 3$ relations* (R0)–(R4) *imply* (H1)–(H5)*. Moreover, for $n \geq 4$ relations* (R0)–(R3) *are equivalent to* (H1)–(H5).

For $u, v \in G$ we define the following two symbols:

$$c_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, c'_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies $h_{ij}(1) = 1$ therefore $c_{ij}(u, u^{-1}) = c'_{ij}(u, u^{-1})$. Our goal is to show that (H3) can be omitted from the definition of $H_n(G)$ provided $n \geq 4$.

Lemma 3.4. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following still remain true:*

(1) *The elements $c_{ij}(u, u^{-1})$ are central in $H_n(G)$;*
(2) *One has $c'_{ij}(u, v) = [h_{ij}(u), h_{kj}(v)]$, $k \neq i, j$;*
(3) *One has ${}^{h_{ij}(w)}c'_{kj}(u, v) = c'_{kj}(u, w)^{-1}c'_{kj}(u, vw)$, $k \neq i, j$;*
(4) *One has ${}^{h_{ij}(w)}c'_{ij}(u, v) = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}$.*

Proof. First one shows using (H4) that $c_{ij}(u, u^{-1})$ centralizes $h_{kj}(v)$ (cf. with the proof of [13, Lemma 2.1(2)]). Since in any group $[a, b] = 1$ implies $[a^{-1}, b] = [a, b^{-1}] = [a^{-1}, b^{-1}] = 1$ we get that $c_{ij}(u, u^{-1})$ also centralizes $h_{kj}(v)^{-1} = h_{jk}(v)$ and $c_{ij}(u^{-1}, u)^{-1} = c_{ji}(u, u^{-1})$ centralizes both $h_{kj}(v)$ and $h_{jk}(v)$. Together with (H2) and (H5) this implies that $c_{ij}(u, u^{-1})$ centralizes all the generators of $H_n(G)$ and hence lies in the center of $H_n(G)$.

The second and third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$${}^{h_{ij}(w)}c'_{ij}(u, v) = {}^{h_{ij}(w)}c'_{kj}(v, u)^{-1} = (c'_{kj}(v, w)^{-1}c'_{kj}(v, uw))^{-1} = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}.$$

□

Lemma 3.5. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following statements are equivalent:*

(1) (H3) *holds;*
(2) *one has* $c'_{ij}(u,v)^{-1} = c'_{ij}(v,u)$*;*
(3) *symbols* $c'_{ij}(u,v)$ *do not depend on* i*;*
(4) *one has* $c'_{ij}(u,vw) = c'_{ij}(wu,v)\cdot c'_{ij}(uv,w)$*;*
(5) *one has* $c'_{ij}(u,vu^{-1}) = c'_{ij}(uv,u^{-1})$.

Proof. Implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are essentially contained in the proof of [13, Lemmas 2.1-2.2]. Implication (4) $\implies$ (5) is trivial.

We now prove (5) $\implies$ (1). Notice that (H4) implies ${}^{h_{ij}(u)^{-1}}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$, therefore using Lemma 3.4.(1) we get that ${}^{h_{ij}(u)}h_{ik}(v) = {}^{c_{ij}(u,u^{-1})h_{ij}(u^{-1})^{-1}}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$. Thus, (H3) is equivalent to the equality

$$h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}),$$

or what is the same

$$c'_{ik}(u^{-1},uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1},u).$$

It is clear that the last equality is an equivalent reformulation of (5). □

Proof of Proposition 3.3. The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);
- for $n \geq 4$ the third statement of the previous lemma follows from Lemma 3.4.(2).

□

Corollary 3.6. *For* $n \geq 3$ *the group* $HS_n(G)$ *is the quotient of the extension* $H_n(G)$ *by* (R4) *(in particular, is an extension of type* $\mathfrak{H}_n(G)$*). Moreover, we have the following natural isomorphism:*

$$\mathrm{Ker}(HS_n(G) \to D_n(G)) = \mathrm{Ker}(S_n(G) \to G \wr S_n) \cong \mathrm{H}_2(G,\mathbb{Z}). \tag{3.4}$$

Proof. Most of the assertions follow from Proposition 3.3 and [4, Proposition 5]. One also has to use the formula $c_{ij}(u,u) = c_{ij}(u,u^{-1})$ (see [13, p. 87]). □

4. Proof of Theorem 2

Recall that for arbitrary set X one defines the space EX as the the simplicial set whose set of k-simplices EX_k is X^{k+1} and whose faces and degeneracies are obtained by omitting and repeating components. For a group G we denote by π_G the canonical map $EG \to BG$ sending $(g,h) \in EG_1$ to $g^{-1}h \in BG_1$.

Now let N be a group acting on X. We define two simplicial sets U and V as follows:

$$U = \bigcup_{n\in N} E(\Gamma_{n.-}) \subseteq E(X\times X), \quad V = \bigcup_{x,y\in X} E(N(x\to y)) \subseteq EN.$$

Here $\Gamma_{n.-}$ is the graph of the function $(x \mapsto nx)$ and $N(x \to y)$ denotes the subset of elements $n \in N$ satisfying $nx = y$. With this notation the subset $N(x \to x)$ coincides with the stabilizer subgroup $N_x \leq N$.

Lemma 4.1. *The simplicial sets* U *and* V *are homotopy equivalent.*

Proof. First, we define yet another simplicial set W as follows. Its k-simplices W_k are matrices $\left(\begin{smallmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{smallmatrix}\right)$, where $x_i \in X$ are $n_i \in N$ are such that all n_i's act each x_j in the same way, i.e. $n_i x_j = n_{i'} x_j$ for $0 \le i, i', j \le k$. The faces and degeneracies of W are the maps of omission and repetition of columns.

Now there are two simplicial maps $f \colon W \to U$, $g \colon W \to V$ whose action on 0-simplicies is given by $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = (x_0, n_0 x_0)$, $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = n_0$. To prove the lemma it suffices to show that f and g are homotopy equivalences. The proof for f and g is similar, let us show, for example, that g is a homotopy equivalence.

In view of Quillen theorem A (cf.[15, ex. IV.3.11]) it suffices to show that for each p-simplex $d \colon \Delta^p \to V$ the pullback $g/(p,d)$ of d and g is contractible. The simplicial set $g/(p,d)$ can be interpreted as the subset of $\Delta^p \times E(X)$ whose set of k-simplices consists of pairs $(\alpha \colon \underline{k} \to \underline{p}, (x_0, \dots, x_k) \in X(\alpha, d)^{k+1})$. Here $X(\alpha, d)$ is the subset of X consisting of all x for which $d_{\alpha(i)} x = d_{\alpha(j)} x$ for $0 \le i, j \le k$. Notice that the set $X_d := X(id_{\underline{p}}, d)$ is nonempty and is contained in every $X(\alpha, d)$ (it even equals $X(\alpha, d)$ for surjective α). Now choose a point $\widetilde{x} \in X_d$ and consider the simplicial homotopy

$$H \colon \Delta^p \times EX \times \Delta^1 \to \Delta^p \times EX$$

between the identity map of $\Delta^p \times EX$ and the map $\Delta^p \times c_{\widetilde{x}}$, where $c_{\widetilde{x}}$ is the constant map. More concretely, H sends each triple $(\alpha \colon \underline{k} \to \underline{p}, (x_0, \dots, x_k), \beta \colon \underline{k} \to \underline{1})$ to $(\alpha, (x_0, \dots, x_{i-1}, \widetilde{x}, \dots, \widetilde{x}))$, where i is the minimal number such that $\beta(i) = 1$. By the choice of $\widetilde{x}$ the image of H restricted to $g/(p,d) \times \Delta^1$ is contained in $g/(p,d)$, hence $g/(p,d)$ is contractible. $\square$

Now suppose that $X = H$ is also a group upon which N acts on the left.

Corollary 4.2. *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

There is an isomorphism $\theta \colon \operatorname{Ker}(\pi_1(S) \to N) \cong \operatorname{Ker}(\pi_1(T) \to H \times H)$*. Moreover, the higher homotopy groups of* S *and* T *are isomorphic.*

Proof. Consider the following two pull-back squares:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EN \arrow[d, "\pi_N"] \\ S \arrow[r, hook] & BN \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(H \times H) \arrow[d, "\pi_{H \times H}"] \\ T \arrow[r, hook] & B(H \times H) \end{tikzcd}$$

The required isomorphism can be obtained from the homotopy long exact sequence applied to the left arrows of these diagrams. $\square$

Now let G be a group. Set $N = G \wr S_n$, $H = G^n$ and consider the left action of N on H given by $(g, s) \cdot h = g h^{s^{-1}}$, $g, h \in G^n$, $s \in S_n$. If one reorders the components of BG^{2n} accordingly, the simplicial subset $T \subset BG^{2n}$ from the above corollary becomes precisely the preimage of 0 under $h_n \colon BG^{2n} \to \mathbb{Z}[BG]$.

It is also easy to compute the map $\pi_1(S) \to N$. Indeed, van Kampen theorem [10, Theorem 2.7] asserts that $\pi_1(S)$ is isomorphic to the free product of stabilizer subgroups $N_h \le N$ amalgamated over pairwise intersections $N_h \cap N_{h'}$, $h, h \in H$. For $h \in G^n$ the subgroup N_h consists of elements $(g, s) \in N$ satisfying $g h^{s^{-1}} = h$, i.e. elements of the form $(h h^{-s^{-1}}, s)$. Thus, $N_h \cong S_n$, $\pi_1(S)$ is isomorphic to the group $S_n(G)$ and the map $\pi_1(S) \to N$ coincides with the map μ defined in section **??**. In particular, for an abelian G its kernel is generated by symbols $c_{ij}(u,v)$.

The following lemma gives a more concrete description of the isomorphism θ from Corollary 4.2.

Lemma 4.3. *If G is abelian then every generator $c_{ij}(u,v) \in \mathrm{Ker}(\pi_1(S) \to N)$ is mapped under θ to the homotopy class of the loop $f_{ij}(u,v)$, which is defined as follows:*

$$f_{ij}(u,v) = p_{ij}(u) \circ p_{ij}(v) \circ p_{ij}(u^{-1}v^{-1}), \ \textit{where } p_{ij}(x) = (x[i], x[i]) \circ (x^{-1}[i], x^{-1}[j]).$$

Proof. Fix a vector $g \in G^n$ and denote by $\gamma_{ij}(g,x)$ the path of length 4 in W which connects the following five points of W_0:

$$\begin{pmatrix} e \\ (g;e) \end{pmatrix}, \begin{pmatrix} x[i] \\ (g;e) \end{pmatrix}, \begin{pmatrix} x[i] \\ (d_{ij}(x)g,(ij)) \end{pmatrix}, \begin{pmatrix} e \\ (d_{ij}(x)g,(ij)) \end{pmatrix}, \begin{pmatrix} e \\ (d_{ij}(x)g,e) \end{pmatrix}.$$

One can easily check that the image of $\gamma_{ij}(g,x)$ in S under $\pi_N g$ is precisely the element $h_{ij}(x)$, while $\pi_{H\times H} f$ sends $\gamma_{ij}(g,x)$ to the loop $p_{ij}(x)$ (here f, g are the maps from the proof of Lemma 4.1). $\square$

Lemma 4.4. *The only natural endotransformations of the functor* $\mathrm{H}_2(-,\mathbb{Z})\colon$ **Groups** $\to$ **Ab** *are morphisms of multiplication by $n \in \mathbb{Z}$.*

Proof. Denote by η an endotransformation $\mathrm{H}_2(-,\mathbb{Z}) \to \mathrm{H}_2(-,\mathbb{Z})$. When restricted to the subcategory of free finitely-generated abelian groups $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$ the second homology functor coincides with the second exterior power functor $A \mapsto \wedge^2 A$.

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic $\mathbb{Z}$-modules (see Definition 6.13.5 ibid.) The functor $A \mapsto \wedge^2 A$ is clearly quadratic and corresponds to the quadratic $\mathbb{Z}$-module $0 \to \mathbb{Z} \to 0$ under this equivalence. Thus, we get that η restricted to $\mathbf{Add}(\mathbb{Z})$ coincides with the the morphism of multiplication by $n \in \mathbb{Z}$.

Consider the group $\Gamma_k = \langle x_1, y_1, \dots x_k, y_k \mid [x_1,y_1] \cdot \ldots \cdot [x_k,y_k] \rangle$ (the fundamental group of a sphere with k handles). It is clear that the abelianization map $\Gamma_k \to \mathbb{Z}^{2k}$ induces an injection $\mathrm{H}_2(\Gamma_k,\mathbb{Z}) \cong \mathbb{Z} \to \wedge^2\mathbb{Z}^{2k}$. Consider the following diagrams.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[d, "\eta_{\Gamma_k}"'] & \wedge^2\mathbb{Z}^{2k} \arrow[d, "n\cdot"] \\ \mathbb{Z} \arrow[r, hook] & \wedge^2\mathbb{Z}^{2k} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, "\chi"] \arrow[d, "\eta_{\Gamma_k}"'] & \mathrm{H}_2(G,\mathbb{Z}) \arrow[d, "\eta_G"] \\ \mathbb{Z} \arrow[r, "\chi"] & \mathrm{H}_2(G,\mathbb{Z}) \end{tikzcd}$$

From the left diagram it follows that η_{Γ_k} is also the morphism of multiplication by n. For every element $x \in \mathrm{H}_2(G,\mathbb{Z})$ there exist an integer k and a map $\chi\colon \Gamma_k \to G$ sending the generator of $\mathrm{H}_2(\Gamma_k,\mathbb{Z})$ to x. From the right square we conclude that $\eta_G(x) = nx$, as claimed. $\square$

Proof of Theorem 2. Factor h_n as a composition of a trivial cofibration followed by a fibration $BG^{2n} \to E_{h_n} \to \mathbb{Z}[BG]$ and write down the starting portion of the long homotopy exact sequence for the latter. If we denote by K the kernel of the map $\nu\colon \pi_1(T) \to G^{2n}$ induced by the embedding $h_n^{-1}(0) = T \subseteq BG^{2n}$ we obtain the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "\beta"'] \arrow[d, "\psi"'] & \pi_1(T) \arrow[r, "\nu"'] \arrow[d, "\varphi"'] & G^{2n} \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}_1(G,\mathbb{Z}) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{H}_2(G,\mathbb{Z}) \arrow[r, "\alpha"'] & \pi_1(F_{h_n}) \arrow[r] & \pi_1(E_{h_n}) \arrow[r] & \mathrm{H}_1(G,\mathbb{Z}) \arrow[r] & 1. \end{tikzcd}$$

We already know by Corollaries 3.6 and 4.2 that K is naturally isomorphic to $\mathrm{H}_2(G,\mathbb{Z})$ provided $n \geq 3$. Thus, it suffices to show that ψ is an isomorphism. In view of Lemma 4.4 we only need to consider the case when G is abelian.

We are going to compute the image of each generator $c_{ij}(x,y) = x \wedge y \in \mathrm{H}_2(G,\mathbb{Z})$ under ψ. In order to do this we need to obtain a more explicit description of the set of 1-simplices of F_{h_n}. Since both BG^{2n} and $\mathbb{Z}[BG]$ are fibrant E_{h_n} and F_{h_n} can be constructed using the path space construction, i. e. $E_{h_n} = BG^{2n} \times_{\mathbb{Z}[BG]} \mathbb{Z}[BG]^I$. In this case the set of 1-simplices of F_{h_n} can be identified with the set of triples $(g,t,t') \in G^{2n} \times \mathbb{Z}[BG]_2 \times \mathbb{Z}[BG]_2$ satisfying the following identites:

$$d_2(t) = h_n(g), \quad d_1(t) = d_1(t'), \quad d_0(t') = 0.$$

It will be convenient for us to perform all calculations inside the subset $F' \subset (F_{h_n})_1$ consisting of those triples for which $t' = 0$ and $d_1(t) = d_0(t) = 0$. For shortness we write down an element of F' as (g,t). It is easy to check that the image of the loop $p_{ij}(x)$ under the natural map $T \to F_{h_n}$ corresponds to the pair $((e, d_{ij}(x)), t(x))$, where $t(x) = -(x,x^{-1}) + (e,e) - (x^{-1},e) + (e,x^{-1})$.

We want to devise a concrete formula for the element of F' homotopic to the concatenation of two elements $(g_1,t_1),(g_2,t_2) \in F'$. It is easy to check that (g_1g_2,t), where t is the 1-st face of the filler f for the 3-horn $(t_2, ,t_1, h_n(g_1,g_2))$, is the desired element. By Moore's theorem one can find a filler for any horn $(p_0, ,p_2,p_3)$ in a simplicial abelian group via the formula $f = s_0p_0 - s_0s_1d_2p_0 - s_0s_0d_1p_0 + s_0s_0d_2p_0 + s_1p_2 + s_2p_3 - s_1p_3$. Substituting concrete values of p_i into this formula and using the fact that $d_1t_2 = 0$ we get the following expression for t:

$$\begin{aligned} t = d_1 f = p_0 - s_1d_2p_0 - s_0d_1p_0 + s_0d_2p_0 + p_2 - p_3 + s_1d_1p_3 = \\ = t_1 + t_2 - h_n(g_1,g_2) - s_1h_n(g_2) + s_0h_n(g_2) + s_1h_n(g_1g_2). \end{aligned}$$

Applying the above formula twice we get that the image of $c_{ij}(u,v)$ under $\varphi\beta$ equals $((e,e),t)$, where

$$\begin{aligned} t = 2(e,e) - (x^{-1},e) + (xy, x^{-1}y^{-1}) - (xy,e) - (x,x^{-1}) - (y,y^{-1}) - \\ - (e,x^{-1}y^{-1}) + (e,x^{-1}) + (x,y) + (x^{-1},y^{-1}) - (e,y) - (xyx^{-1}y^{-1},e) + (y,e). \end{aligned} \tag{4.1}$$

The map α can be described as the Dold—Kan isomorphism $\mathrm{H}_2(G,\mathbb{Z}) \cong \pi_2(\mathbb{Z}[BG])$ followed by the boundary map $\partial\colon \pi_2(\mathbb{Z}[BG]) \to \pi_1(F_{h_n})$. It is not hard to show that ∂ maps each $c \in \pi_2(\mathbb{Z}[BG])$ to $((e,e),-c)$ (see [5, p. 29]). Thus, we get that ψ maps $x \wedge y$ to $-t$, where t is given by (4.1).

On the other hand, the generator $x \wedge y$ of $\mathrm{H}_2(G,\mathbb{Z})$ corresponds to the 2-cycle $c = (x,y) - (y,x)$ (cf. [11, (14), p. 582]) and the corresponding element of $\pi_2(\mathbb{Z}[BG])$ is simply the normalized 2-cycle of c:

$$c' = c - s_0d_0c - s_1d_1c + s_1d_0 = (x,y) - (y,x) - (x,e) + (y,e) + (e,x) - (e,y).$$

It it not hard to check that $t - c'$ is the image of the following 3-chain under the differential $d_0 - d_1 + d_2 - d_3$:

$$(xy, x^{-1}, y^{-1}) - (y, x, x^{-1}) + s_1s_1(x^{-1} + xy - x - y) + s_0s_0(x^{-1} - x - x^{-1}y^{-1}).$$

Thus we obtain that $[t] = -[c']$, hence ψ is an isomorphism, as claimed. □

Appendix A. Simplicial Dold—Serre fibrations

The aim of this appendix is to show that the map $h_\infty\colon X^\infty \to \mathbb{Z}[X]$ mentioned in the introduction is a quasifibration.

We start with the definition of the simplicial analogue of the so-called "Dold—Serre fibration", i. e. a map posessing "weak covering homotopy property" (also called "delayed homotopy lifting property“), cf. [14, § 1.4.4]. We show in Lemma A.2 and **??** that the map h_∞ satisfies this property and that every map with this property is a quasifibration.

Definition A.1. Let C be a subclass of the class of acyclic cofibrations in the category of simplicial sets. We say that a map of simplicial sets $p\colon E \to B$ satisfies *weak homotopy covering property* with respect to C if for every commutative square

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ V \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd} \tag{A.1}$$

for which $i \in C$ there exists a map $\widetilde{F}$ such that the lower triangle commutes strictly and the upper one commutes up to a fiberwise homotopy H (i.e. a homotopy H such that $pH\colon U \times I \to B$ coincides with the composite $U \times I \xrightarrow{\pi} U \xrightarrow{pg} B$).

Denote by C_{pr} the class consisting of all inclusions $i\colon U \hookrightarrow V$ of finite polyhedral simplicial sets for which there exists a deformation retract (i.e. a map $r\colon V \to U$ homotopic to id_V).

Lemma A.2. *For a fibrant and connected simplicial set X the map $h_\infty\colon X^\infty \to \mathbb{Z}[X]$ satisfies the weak covering homotopy property with respect to the class C_{pr}.*

Proof. Let i_0 be a map from C_{pr} and r be the corresponding retraction. We first reduce the problem to the special case when g is the constant map at the basepoint of X^∞ (which we denote by 0). Indeed, let F, g be arbitrary maps as in (A.1). Since U is finite the image of g is contained in $X^{2N} \subset X^\infty$. If now $\widetilde{F}_0$ is a lifting in the diagram similar to (A.1) in which the top map equals 0 and the bottom one is $F_0 = F - h_\infty gr$ then $\widetilde{F} = gr \times \widetilde{F}_0$ is the desired lifting of F.

Now choose a contractible fibrant simplicial set W which maps surjectively onto X (e.g. take W to be the path space fibration). Since $p\colon W \twoheadrightarrow X$ is surjective, the associated map between free simplicial abelian groups is a Kan fibration. Since i_0 is acyclic we can choose a lifting $\widetilde{F}$ in the following diagram

$$\begin{tikzcd} U \arrow[r, "0"] \arrow[d, hook, "i_0"'] & \mathbb{Z}[W] \arrow[d] \\ V \arrow[r, "F"'] \arrow[ur, "\widetilde{F}"'] & \mathbb{Z}[X] \end{tikzcd}$$

It is clear that the image of $\widetilde{F}$ is contained in the simplicial subgroup $\mathbb{Z}[W]_0 \le \mathbb{Z}[W]$ which, by definition, consists of linear combinations $\sum_i n_i w_i$ for which $\sum_i n_i = 0$. By [12, Lemma 9.1] the canonical map $\mathbb{Z}[\underline{\mathrm{Hom}}(V, W)]_0 \to \underline{\mathrm{Hom}}(V, \mathbb{Z}[W]_0)$ is surjective, hence the map $\widetilde{F}$ can be lifted along h_W in such a way that the bottom triangle in following diagram commutes strictly.

$$\begin{tikzcd} U \arrow[r, "0"] \arrow[d, hook, "i_0"'] & W^\infty \arrow[d, "h_W"] \\ V \arrow[r, "\widetilde{F}"'] \arrow[ur, dashed, "G"] & \mathbb{Z}[W] \end{tikzcd}$$

Notice that the image of Gi_0 is contained in the fiber $h_W^{-1}(0)$. It is easy to see that $h_W^{-1}(0)$ is contractible (the contracting homotopy for W^∞ can be restricted to $h_W^{-1}(0)$). We get that Gi_0 is fiberwise homotopic to 0 hence the composite map $V \xrightarrow{G} W^\infty \to X^\infty$ is the desired lifting of F. $\square$

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