HOMOTOPY INVARIANTS, AMALGAMS OF SYMMETRIC GROUPS AND $\mathfrak{H}_n$-EXTENSIONS

SERGEY SINCHUK

Abstract. By amalgamating copies of symmetric group S_n we construct a central extension $S_n(G)$ of a large subgroup of the wreath product $G \wr S_n$. We then establish a connection between this group and the theory of $\mathfrak{U}(G)$ and $\mathfrak{H}_n(G)$-extensions developed by K. Dennis and U. Rehmann in 1970's. We also give a topological application of this construction by describing the fundamental group of the fiber of the "alternating" map $BG^{2n} \to \mathbb{Z}[BG]$.

1. Introduction

For a pointed simplicial set (X, x_0) one denotes by $\mathbb{Z}[X]$ the associated free simplicial abelian group. Define the map of simplicial sets $h_n\colon X^{2n} \to \mathbb{Z}[X]$ by the identity:

$$h_n(x_1, x_2, \ldots x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Denote by X^∞ the colimit of X^n with respect to the natural embeddings $X^n \hookrightarrow X^{n+1}$ Taking colimit of h_n we get the map $h_\infty = \mathrm{colim}_n(h_n)\colon X^\infty \to \mathbb{Z}[X]$.

Based on a recent result of S. Podkorytov one can deduce that the natural map $h_\infty^{-1}(0) \to F_{h_\infty}$ between the fiber and the homotopy fiber of h_∞ over 0 is a weak equivalence provided X is fibrant and connected, see Lemma 4.1. For finite n this no longer remains true, however one may expect that the homotopy groups of $h_n^{-1}(0)$ and F_{h_n} are still isomorphic in small degrees. S. Podkorytov conjectured that the natural map $h_n^{-1}(0) \to F_{h_n}$ is c_n-connected where c_n tends to infinity as $n \to \infty$. The main goal of this note is to show the following result which confirms this conjecture in one special case.

Theorem 1. *If $X = BG$ is the classifying space of a group G then for $n \geq 3$ the natural map $h_n^{-1}(0) \to F_{h_n}$ induces an isomorphism of fundamental groups $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$.*

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The proof of Theorem 1 is interesting on its own and goes as follows. Using Quillen theorem A and van Kampen theorem we reduce the original question to some purely group-theoretic problem of the description of the extension $S_n(G)$, which is defined as a certain amalgamated product of copies of the symmetric group S_n. It turns out that this extension can be naturally described in terms of the theory of $\mathfrak{U}(G)$ and $\mathfrak{H}_n(G)$-extensions developed by K. Dennis and U. Rehmann in 1970's.

In the course of the proof we also find a presentation of the universal $\mathfrak{H}_n(G)$-extension that is simpler than the original presentation introduced by Rehmann in [10], see Proposition 3.1.

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2. The extension $S_n(G)$

2.1. **The definition of $S_n(G)$.** Let G be a group. Recall that the *wreath product* is, by definition, the semidirect product $G^n \rtimes S_n$, in which S_n acts on G^n on the right by permuting its factors.

Consider the family $\{{S_n}^{(g)}\}_{g\in G^n}$ of isomorphic copies of S_n indexed by elements of G^n and let F be the free product of groups from this family. For $s \in S_n$ and $g \in G^n$ we denote by s_g the image of s in F under the canonical map $S_n^{(g)} \to F$.

Definition 2.1. We define the group $S_n(G)$ to be the quotient of F modulo the following single family of relations:

$$s_g = s_h, \text{ where } s \text{ fixes } hg^{-1}. \tag{2.1}$$

We continue to denote by s_g the image of the element $s_g \in F$ under the canonical map $F \to S_n(G)$.

Remark 2.2. Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1}g^{(s^{-1})}, s).$$

therefore the equality $s^g = s^h$ holds in $G \wr S_n$ iff $g^{-1}g^{(s^{-1})} = h^{-1}h^{(s^{-1})}$, or what is the same, iff s fixes hg^{-1}.

The last statement immediately implies that the map $\mu\colon S_n(G) \to G \wr S_n$ given by $\mu(s_g) = s^g = (g^{-1}g^{(s^{-1})}, s)$ is well-defined. We will see shortly that μ is, in fact, a *crossed module* in the sense of [2, § 2.2].

Observe from the definition of $S_n(G)$ that there is a split exact sequence.

$$1 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow S_n(G) \underset{\iota(1)}{\overset{\pi}{\rightleftarrows}} S_n \longrightarrow 1, \tag{2.2}$$

Here the map $\pi = \pi_{S_n} \circ \mu$ removes subscript g from each s_g and the section $\iota(1)$ sends s to s_1, where 1 is the identity element of G^n. Thus, if we denote $\mathrm{Ker}(\pi)$ by $HS_n(G)$ we get a decomposition $S_n(G) = HS_n(G) \rtimes S_n$.

2.2. **A crossed module structure on $S_n(G)$.** Recall from [2, § 2.2] that a *crossed module* is a morphism of groups $\mu\colon M \to N$ together with a right action of N on M compatible with the conjugation action of N on itself, i. e.

$$\mu(m^n) = \mu(m)^n \text{ for all } n \in N,\ m \in M, \tag{CM1}$$

which also satisfies the following identity called *Peiffer identity*:

$$m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M. \tag{CM2}$$

In our situation, we let $G \wr S_n$ act on $S_n(G)$ by

$$(s_g)^{(h,t)} = {s^t}_{(gh)^t}, \text{ for } s, t \in S_n \text{ and } g, h \in G^n. \tag{2.3}$$

The goal of this subsection is to prove the following.

Proposition 2.3. *For $n \geq 3$ the map $\mu\colon S_n(G) \to G \wr S_n$ is a crossed module.*

From the fact that μ is a crossed module one can deduce that $\mathrm{Ker}(\mu)$ is a central subgroup of $S_n(G)$ and $\mathrm{Im}(\mu)$ is a normal subgroup of $G \wr S_n$. It is not hard to show that the group $S_2(G)$ is isomorphic to a free product of copies of S_2 (whose center is always trivial). Therefore, the requirement $n \geq 3$ in the statement of Proposition 2.3 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of $G \wr S_n$ on $S_n(G)$ that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of $S_n(G)$, for which it takes the form:

$$t_h^{-1} s_g t_h = {s^t}_{(gh^{-1})^t \cdot h} \text{ for all } s, t \in S_n,\ g, h \in G^n. \tag{2.4}$$

If we act on both sides of the above formula by $(h^{-1}, 1) \in G \wr S_n$ we obtain the equality $t_1^{-1} s_{gh^{-1}} t_1 = {s^t}_{(gh^{-1})^t}$. Thus, to prove (2.4) it suffices to show the following simpler relation:

$$t_1^{-1} s_g t_1 = {s^t}_{g^t} \text{ for all } s, t \in S_n,\ g \in G^n. \tag{2.5}$$

The key step in the proof is the following lemma.

Lemma 2.4. *The relation* (2.5) *holds in the special case when $s = (ij)$ and $t = (kl)$ are two nonequal transpositions.*

Proof. First of all, we immediately check that (2.5) holds in the special case when t fixes $g' \in G^n$. Indeed, by (2.1) we have $t_1 = t_{g'}$, hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that $l \neq i$ and $l \neq j$. Denote by g' the vector which differs from g only at l-th position, for which we set $g'_l = g_k$. Since the only nontrivial component of $g'g^{-1}$ (resp. $g'g^{-t}$) is located at l-th (resp. k-th) position, it is fixed by s (resp. s^t), hence from (2.1) we conclude that $s_g = s_{g'}$ (resp. $(s^t)_{g'} = (s^t)_{g^t}$). Finally, since g' is fixed by t, we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g'^t} = s^t_{g'} = (s^t)_{g^t}. \quad \square$$

Proof of Proposition 2.3. Let us show that (2.5) holds for arbitrary transpositions $s, t \in S_n$. It suffices to consider the case $s = t = (ij)$. After choosing some $k \neq i, j$ and presenting (ij) as $(kj)(ik)(kj)$ we use the lemma:

$$(ij)_1^{-1}(ij)_g(ij)_1 = (ij)_1^{-1}(kj)_g(ik)_g(kj)_g(ij)_1 = (ki)_{g^{(ij)}}(jk)_{g^{(ij)}}(ki)_{g^{(ij)}} = (ij)_{g^{(ij)}}.$$

Proposition now follows by induction on the length of permutations s, t. $\square$

2.3. **An explicit presentation of $S_n(G)$.** In this section we obtain an explicit presentation of $S_n(G)$ similar in appearance to the presentation of parametrized braid groups from [5]. This presentation turns out to be much more economical than the original definition given in section 2.1. An explicit presentation for the subgroup $HS_n(G)$ is derived from this presentation via Reidemeister-Schreier method.

We start with the following simple lemma which is, in essence, a variant of the standard presentation of a Weyl group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

Lemma 2.5. *For $n \geq 3$ The symmetric group S_n admits presentation with transpositions as the set of generators and the following list of defining relations (in every formula distinct letters denote distinct indices):*

$$(ij)^2 = 1, \tag{S1}$$

$$(ij)^{(jk)} = (ik), \tag{S2}$$

$$[(ij), (kl)] = 1. \tag{S3}$$

$$(ij) = (ji), \tag{S4}$$

Now we are ready to formulate the main result of this subsection.

Proposition 2.6. *For $n \geq 3$ and arbitrary group G the group $S_n(G)$ admits presentation with the set of generators $\{(ij)_a \mid i \neq j, 1 \leq i, j, \leq n, a \in G\}$ and the following* *list of relations (as before, distinct letters denote distinct indices):*

$$(ij)_a^2 = 1, \tag{SG1}$$

$$(ij)_a^{(jk)_b} = (ik)_{ab}, \tag{SG2}$$

$$[(ij)_a, (kl)_b] = 1. \tag{SG3}$$

$$(ij)_a = (ji)_{a^{-1}} \tag{SG4}$$

Proof. Denote by S' the group from the statement of the proposition. For $1 \leq i \leq n$ and $x \in G$ denote by $x[i]$ the element of G^n whose only nontrivial component equals x and is located in the i-th position.

It is not hard to deduce from the definition of $S_n(G)$ and Lemma 2.4 that the formula $(ij)_a \mapsto (ij)_{a[j]}$ gives a well-defined map $\varphi\colon S' \to S_n(G)$.

Now we are going to construct the map $\psi\colon S_n(G) \to S'$ in the opposite direction. Using the presentation of S_n given by Lemma 2.5 we define for a fixed $g \in G^n$ the map $\psi_g\colon S_n \to S'$ by $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$. It is obvious that ψ_g preserves the defining relations (S1)–(S4) of S_n. It remains to show that the equation $\psi_g(s) = \psi_h(s)$ holds whenever $g, h \in G^n$ and $s \in S_n$ satisfy the requirement of (2.1).

Indeed, if hg^{-1} is fixed by s then for every $1 \leq i \leq n$ we have $(hg^{-1})_i = (hg^{-1})_{s(i)}$, or equivalently $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$. For example, if s is a cycle of length p, i. e. $s = (i_1, i_2, \ldots i_p)$ with $i_{k+1} = s(i_k)$ we get that

$$\psi_g(s) = \psi_g\left(\prod_{k=1}^{p-1}(i_k, i_{k+1})\right) = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{g_{i_k}^{-1}g_{i_{k+1}}} = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{h_{i_k}^{-1}h_{i_{k+1}}} = \psi_h(s).$$

The proof for a general s is almost the same. Verification of the fact that ψ and φ are mutually inverse is also immediate. □

We now obtain a presentation of the subgroup $HS_n(G)$. For every $i \neq j$ and $a \in G$ we define the element $h_{ij}(a) \in HS_n(G)$ as follows:

$$h_{ij}(a) = (ij)_a \cdot (ij)_1. \tag{2.6}$$

It is not hard to show that $h_{ij}(a)$ form a generating set for $HS_n(G)$. In fact, there is an explicit formula how an element of $HS_n(G)$ originally expressed through $(ij)_a$'s can be rewritten in terms of $h_{ij}(a)$. Indeed, if h lies in $HS_n(G)$ and is written as $\prod_{k=1}^{N}(i_kj_k)_{a_k}$ for some $i_k \neq j_k$ and $a_k \in G$ then it can be rewritten as follows:

$$h = \prod_{k=1}^{N} h_{\sigma_k(i_k),\sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1}(i_sj_s) \in S_n. \tag{τ}$$

We briefly recall the notion of a *rewriting process* given in [6, § 2.3]. If G is a group presented by generators a_ν and relations $R_\mu(a_\nu)$ and H is its subgroup with a generating set $J_i(a_\nu)$ then a *rewriting process for* H is a function which maps every word u in alphabet a_ν to a word v in alphabet s_i such that u and $v[s_i := J_i]$ define the same element of G whenever u represents an element of H.

With this terminology, the mapping (τ) defined above is a rewriting process for the subgroup $HS_n(G)$. Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [6, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [6, § 2.3]):

- if U and U^* are freely equal words in $(ij)_a$ then $\tau(U)$ and $\tau(U^*)$ are also freely equal words in $h_{ij}(a)$;
- if U_1 and U_2 are two words in $(ij)_a$ which define elements of $HS_n(G)$ then the words $\tau(U_1U_2)$ and $\tau(U_1)\tau(U_2)$ are equal.

Using these two properties and repeating the arguments used in the proof of [6, Theorem 2.8] one can simplify the generic presentation of $HS_n(G)$ given by [6, Theorem 2.6] and obtain the following.

Lemma 2.7. *For $n \geq 3$ the group $HS_n(G)$ admits presentation on the generators $h_{ij}(a)$ with the following two families of defining relations:*

$$h_{ij}(a) = \tau\left((ij)_a \cdot (ij)_1\right); \tag{2.7}$$

$$\tau(KRK^{-1}) = 1, \tag{2.8}$$

where R varies over relations of Proposition 2.6 and K is any word in $(ij)_1$, $i \neq j$.

From (2.7) one immediately obtains the equality $h_{ij}(1) = 1$, $i \neq j$. Thus, if we denote by σ the permutation corresponding to a word K and let S_n act on $h_{ij}(a)$'s in the natural way we will get that the word $\tau(KRK^{-1})$ is equivalent to ${}^{\sigma}\tau(R)$ (modulo relations $h_{ij}(1) = 1$).

Since the relations from Proposition 2.6 are respected by the action of S_n, only relations of the form $\tau(R) = 1$ are, in fact, needed for the presentation of $HS_n(G)$. Writing down what $\tau(R)$ is for each of (SG1)–(SG4) we get the following.

Proposition 2.8. *For $n \geq 3$ the group $HS_n(G)$ admits presentation with generators $h_{ij}(a)$ and the following list of relations:*

$$h_{ij}(1) = 1, \tag{R0}$$

$$h_{ij}(a)h_{ji}(a) = 1, \tag{R1}$$

$$h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab), \tag{R2}$$

$$[h_{ij}(a), h_{kl}(b)] = 1, \tag{R3}$$

$$h_{ij}(a)^{-1} = h_{ij}(a^{-1}). \tag{R4}$$

3. Relationship with extensions of type $\mathfrak{H}_n(G)$

We start by briefly reviewing the material of § 1–3 of [10]. Let $n \geq 3$ be a natural number. By definition, the group $H_n(G)$ is given by generators $h_{ij}(u)$, $u \in G$, $i \neq j$ and the following relations (cf. [10, H1–H5 of § 2]):

$$h_{ij}(u)h_{ji}(u) = 1, \tag{H1}$$

$$h_{ij}(u)h_{ki}(u)h_{jk}(u) = 1, \tag{H2}$$

$$h_{ij}(u)h_{ik}(v)h_{ij}(u)^{-1} = h_{ik}(uv)h_{ik}(u)^{-1}, \qquad \text{for } j \neq k \tag{H3}$$

$$h_{ij}(u)h_{kj}(v)h_{ij}(u)^{-1} = h_{kj}(vu)h_{kj}(u)^{-1}, \qquad \text{for } i \neq k \tag{H4}$$

$$[h_{ij}(u), h_{kl}(v)] = 1. \tag{H5}$$

Denote by $D_n(G)$ the subgroup of G^n consisting of vectors $(g_1, \ldots, g_n)$ for which the product $g_1 \cdot \ldots \cdot g_n$ lies in $[G, G]$. There is a surjective group homomorphism $H_n(G) \twoheadrightarrow D_n(G)$ sending $h_{ij}(g)$ to $d_{ij}(g)$. Here $d_{ij}(g)$ stands for the element of G^n whose i-th component equals g, j-th component equals g^{-1} and all other components are trivial.

By definition, *an extension of type* $\mathfrak{H}_n(G)$ is an extension H of $D_n(G)$ that is also a quotient of $H_n(G)$, i. e. the extension that fits into the following diagram.

$$\begin{tikzcd} H_n(G) \arrow[d, two heads] \arrow[dr, two heads] & \\ H \arrow[r, two heads] & D_n(G) \end{tikzcd}$$

We now recall the notion of *an extension of type* $\mathfrak{U}(G)$ introduced by Rehmann in [10, § 1]. Let G be a group. Denote by $U(G)$ the group presented by generators (u, v), $u, v \in G$ and relations:

$$({}^u v, {}^u w)(u, w) = (uv, w), \tag{U1}$$

$$(u, vw)(v, wu)(w, uv) = 1. \tag{U2}$$

There is a left action of G on $U(G)$ given by ${}^x(u, v) = ({}^x u, {}^x v)$. Equivalently, $U(G)$ can be defined by the following relations, see [4, p. 8] (where $U(G)$ is denoted (G, G)).

$$(u, v)(v, u) = 1, \tag{3.1}$$

$$(u, v)\ {}^v(u, w) = (u, vw), \tag{3.2}$$

$${}^u(v, w)(w, v) = (u, [v, w]). \tag{3.3}$$

It is clear from the definition that $U(G)$ is a quotient of the nonabelian tensor square $G \otimes G$ and that the map $(u, v) \mapsto [u, v]$ defines a G-equivariant map $U(G) \twoheadrightarrow [G, G]$.

By definition, an *extension of type* $\mathfrak{U}(G)$ is simply a quotient of $U(G)$ that fits into the following commutative diagram with G-equivariant arrows.

$$\begin{tikzcd} U(G) \arrow[d, two heads] \arrow[dr, two heads] & \\ U \arrow[r, two heads] & {[G,G]} \end{tikzcd}$$

There is a correspondence between extensions of type $\mathfrak{H}_n(G)$ and $\mathfrak{U}(G)$. One can obtain a $\mathfrak{U}(G)$-extension from given $\mathfrak{H}_n(G)$-extension via restriction. Indeed, let H be an extension of type $\mathfrak{H}_n(G)$ and k be arbitrary number $1 \leq k \leq n$. Denote by ι_k the inclusion map $[G, G] \to D_n(G)$ that inserts an element $g \in [G, G]$ at the k-th position. Then the corresponding extension U_H is simply the pull-back of H with respect to ι_k.

$$\begin{tikzcd} U_H \arrow[r, two heads, "\pi_U"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[G,G]} \arrow[d, "\iota_k"] \\ H \arrow[r, two heads, "\pi_H"] & D_n(G). \end{tikzcd}$$

In other words, U_H is simply the subgroup of H generated by symbols $c_{kj}(u, v) = h_{kj}(u) \cdot h_{kj}(v) \cdot h_{kj}(vu)^{-1}$ (here j is any index not equal k, $c_{kj}(u, v)$ does not depend on j, see [10, Proposition 2.1]). It can be shown that U is an extension of type $\mathfrak{U}(G)$ with kernel $\mathrm{Ker}(\pi_U)$ isomorphic to $\mathrm{Ker}(\pi_H)$, see [10, Propositions 2.2–2.3]. In [10, § 3] Rehmann also describes the inverse construction which assign to every $\mathfrak{U}(G)$-extension $U \twoheadrightarrow [G, G]$ and a number $n \geq 3$ some extension H_U of type $\mathfrak{H}_n(G)$ that fits into the pull-back diagram above.

3.1. **A simpler presentation of $H_n(G)$.** The aim of this subsection is to obtain a presentation of $H_n(G)$ that is simpler than the original presentation: 4 instead of 5 relations that are also shorter. Our main result is the following.

Proposition 3.1. *For $n \geq 3$ relations* (R0)–(R4) *imply* (H1)–(H5)*. Moreover, for $n \geq 4$ relations* (R0)–(R3) *are equivalent to* (H1)–(H5).

For $u, v \in G$ we define the following two symbols:

$$c_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, c'_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies $h_{ij}(1) = 1$ therefore $c_{ij}(u, u^{-1}) = c'_{ij}(u, u^{-1})$. Our goal is to show that (H3) can be omitted from the definition of $H_n(G)$ provided $n \geq 4$.

Lemma 3.2. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following still remain true:*

(1) *The elements $c_{ij}(u, u^{-1})$ are central in $H_n(G)$;*

(2) *One has* $c'_{ij}(u,v) = [h_{ij}(u), h_{kj}(v)]$, $k \neq i,j$;
(3) *One has* ${}^{h_{ij}(w)}c'_{kj}(u,v) = c'_{kj}(u,w)^{-1}c'_{kj}(u,vw)$, $k \neq i,j$;
(4) *One has* ${}^{h_{ij}(w)}c'_{ij}(u,v) = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}$.

Proof. First one shows using (H4) that $c_{ij}(u,u^{-1})$ centralizes $h_{kj}(v)$ (cf. with the proof of [10, Lemma 2.1(2)]). Since in any group $[a,b]=1$ implies $[a^{-1},b] = [a,b^{-1}] = [a^{-1},b^{-1}] = 1$ we get that $c_{ij}(u,u^{-1})$ also centralizes $h_{kj}(v)^{-1} = h_{jk}(v)$ and $c_{ij}(u^{-1},u)^{-1} = c_{ji}(u,u^{-1})$ centralizes both $h_{kj}(v)$ and $h_{jk}(v)$. Together with (H2) and (H5) this implies that $c_{ij}(u,u^{-1})$ centralizes all the generators of $H_n(G)$ and hence lies in the center of $H_n(G)$.

The second and third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$$ {}^{h_{ij}(w)}c'_{ij}(u,v) = {}^{h_{ij}(w)}c'_{kj}(v,u)^{-1} = (c'_{kj}(v,w)^{-1}c'_{kj}(v,uw))^{-1} = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}. $$

□

Lemma 3.3. *If one excludes relation* (H3) *from the list of defining relations for the group* $H_n(G)$, $n \geq 3$, *the following statements are equivalent:*

(1) (H3) *holds;*
(2) one has $c'_{ij}(u,v)^{-1} = c'_{ij}(v,u)$;
(3) symbols $c'_{ij}(u,v)$ *do not depend on* i;
(4) one has $c'_{ij}(u,vw) = c'_{ij}(wu,v) \cdot c'_{ij}(uv,w)$;
(5) one has $c'_{ij}(u,vu^{-1}) = c'_{ij}(uv,u^{-1})$.

Proof. Implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are essentially contained in the proof of [10, Lemmas 2.1-2.2]. Implication (4) $\Longrightarrow$ (5) is trivial.

We now prove (5) $\Longrightarrow$ (1). Notice that (H4) implies ${}^{h_{ij}(u)^{-1}}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$, therefore using Lemma 3.2.(1) we get that ${}^{h_{ij}(u)}h_{ik}(v) = {}^{c_{ij}(u,u^{-1})h_{ij}(u^{-1})^{-1}}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$. Thus, (H3) is equivalent to the equality

$$ h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}), $$

or what is the same

$$ c'_{ik}(u^{-1},uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1},u). $$

It is clear that the last equality is an equivalent reformulation of (5). □

Proof of Proposition 3.1. The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);

- for $n \geq 4$ the third statement of the previous lemma follows from Lemma 3.2.(2).

$\square$

Corollary 3.4. *For $n \geq 3$ the group $HS_n(G)$ is the quotient of the extension $H_n(G)$ by* (R4) *and, in particular, is an $\mathfrak{H}_n(G)$-extension. The associated $\mathfrak{U}(G)$ extension $U_{HS_n(G)}$ is isomophic to the quotient of $U(G)$ modulo single family of relations*

$$(u, u) = 1, \ u \in G.$$

In particular, we have the following natural isomorphism:

$$\operatorname{Ker}(HS_n(G) \to D_n(G)) = \operatorname{Ker}(S_n(G) \to G \wr S_n) \cong \mathrm{H}_2(G, \mathbb{Z}). \tag{3.4}$$

Proof. Most of the assertions follow from Proposition 3.1 and [4, Proposition 5]. One also has to use the formula $c_{ij}(u, u) = c_{ij}(u, u^{-1})$ (see [10, p. 87]). $\square$

4. Proof of Theorem 1

Lemma 4.1. *If X is fibrant and connected then F_{h_∞} and $h^{-1}(0)$ are weakly equivalent.*

Proof. **???** $\square$

Recall that for arbitrary set X one defines the space EX as the the simplicial set whose set of k-simplices EX_k is X^{k+1} and whose faces and degeneracies are obtained by omitting and repeating components. For a group G we denote by π_G the canonical map $EG \to BG$ sending $(g, h) \in EG_1$ to $g^{-1}h \in BG_1$.

Now let N be a group acting on X. We define two simplicial sets U and V as follows:

$$U = \bigcup_{n \in N} E(\Gamma_{n.-}) \subseteq E(X \times X), \quad V = \bigcup_{x,y \in X} E(N(x \to y)) \subseteq EN.$$

Here $\Gamma_{n.-}$ is the graph of the function $(x \mapsto nx)$ and $N(x \to y)$ denotes the subset of elements $n \in N$ satisfying $nx = y$. With this notation the subset $N(x \to x)$ coincides with the stabilizer subgroup $N_x \leq N$.

Lemma 4.2. *The simplicial sets U and V are homotopy equivalent.*

Proof. First, we define yet another simplicial set W as follows. Its k-simplices W_k are matrices $\left(\begin{smallmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{smallmatrix}\right)$, where $x_i \in X$ are $n_i \in N$ are such that all n_i's act each x_j in the same way, i. e. $n_i x_j = n_{i'} x_j$ for $0 \leq i, i', j \leq k$. The faces and degeneracies of W are the maps of omission and repetition of columns.

Now there are two simplicial maps $f \colon W \to U$, $g \colon W \to V$ whose action on 0-simplicies is given by $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = (x_0, n_0 x_0)$, $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = n_0$. To prove the lemma it suffices to show that f and g are homotopy equivalences. The proof for f and g is similar, let us show, for example, that g is a homotopy equivalence.

In view of Quillen theorem A (cf.[11, ex. IV.3.11]) it suffices to show that for each p-simplex $d\colon \Delta^p \to V$ the pullback $g/(p,d)$ of d and g is contractible. The simplicial set $g/(p,d)$ can be interpreted as the subset of $\Delta^p \times E(X)$ whose set of k-simplices consists of pairs $(\alpha\colon \underline{k} \to \underline{p}, (x_0,\ldots,x_k) \in X(\alpha,d)^{k+1})$. Here $X(\alpha,d)$ is the subset of X consisting of all x for which $d_{\alpha(i)}x = d_{\alpha(j)}x$ for $0 \le i,j \le k$. Notice that the set $X_d := X(id_{\underline{p}}, d)$ is nonempty and is contained in every $X(\alpha,d)$ (it even equals $X(\alpha,d)$ for surjective α). Now choose a point $\widetilde{x} \in X_d$ and consider the simplicial homotopy

$$H\colon \Delta^p \times EX \times \Delta^1 \to \Delta^p \times EX$$

between the identity map of $\Delta^p \times EX$ and the map $\Delta^p \times c_{\widetilde{x}}$, where $c_{\widetilde{x}}$ is the constant map. More concretely, H sends each triple $(\alpha\colon \underline{k} \to \underline{p}, (x_0,\ldots,x_k), \beta\colon \underline{k} \to \underline{1})$ to $(\alpha, (x_0,\ldots,x_{i-1},\widetilde{x},\ldots,\widetilde{x}))$, where i is the minimal number such that $\beta(i) = 1$. By the choice of $\widetilde{x}$ the image of H restricted to $g/(p,d) \times \Delta^1$ is contained in $g/(p,d)$, hence $g/(p,d)$ is contractible. □

Now suppose that $X = H$ is also a group upon which N acts on the left. It is clear that an element $n \in N$ induces a simplicial map $E(n.-)\colon EH \to EH$. We consider its graph $\Gamma_{E(n.-)}$ as a subset of $E(H \times H) = EH \times EH$.

Corollary 4.3. *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

There is an isomorphism $\mathrm{Ker}(\pi_1(S) \to N) \cong \mathrm{Ker}(\pi_1(T) \to H \times H)$*. Moreover, the higher homotopy groups of S and T are isomorphic.*

Proof. The required isomorphism can be obtained from the homotopy long exact sequence applied to left arrows of the following two pull-back squares.

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EN \arrow[d, "\pi_N"] \\ S \arrow[r, hook] & BN \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(H \times H) \arrow[d, "\pi_{H \times H}"] \\ T \arrow[r, hook] & B(H \times H) \end{tikzcd}$$

□

Now let G be a group. Set $N = G \wr S_n$, $H = G^n$ and consider the left action of N on H given by $(g,s) \cdot h = gh^{s^{-1}}$, $g,h \in G^n$, $s \in S_n$. If one reorders the components of BG^{2n} accordingly, the simplicial subset $T \subset BG^{2n}$ from the above corollary becomes precisely the preimage of 0 under $h_n\colon BG^{2n} \to \mathbb{Z}[BG]$.

It is also easy to compute the map $\pi_1(S) \to N$. Indeed, van Kampen theorem [7, Theorem 2.7] asserts that $\pi_1(S)$ is isomorphic to the free product of stabilizer subgroups $N_h \le N$ amalgamated over pairwise intersections $N_h \cap N_{h'}$, $h, h \in H$. For

$h \in G^n$ the subgroup N_h consists of elements $(g,s) \in N$ satisfying $gh^{s^{-1}} = h$, i.e. elements of the form $(hh^{-s^{-1}}, s)$. Thus, $N_h \cong S_n$, $\pi_1(S)$ is isomorphic to the group $S_n(G)$ and the map $\pi_1(S) \to N$ coincides with the map μ defined in section 2.1.

Example 4.4. We leave it as an exercise to the reader to check that an element $[h_{12}(x), h_{13}(y)] \in \pi_1(S)$ corresponds to the following element of $\pi_1(T)$ under the isomorphism of Lemma 4.2:

$$\begin{aligned}\gamma_{x,y} = {} & (x[1],x[1]) \circ (x^{-1}[1],x^{-1}[2]) \circ (y[1],y[1]) \circ (y^{-1}[1],y^{-1}[3]) \circ \\ & \circ (x[1],x[2]) \circ (x^{-1}[1],x^{-1}[1]) \circ (y[1],y[3]) \circ (y^{-1}[1],y^{-1}[1])).\end{aligned} \tag{4.1}$$

Proof of Theorem 1. Factor h_n as a composition of a trivial cofibration followed by a fibration:

$$BG^{2n} \hookrightarrow E_{h_n} \xrightarrow{ev_1 \circ \pi_2} \mathbb{Z}[BG] \tag{4.2}$$

For example, we can define E_{h_n} and the homotopy fiber F_{h_n} via the usual path space construction (here we use the fact that BG and $\mathbb{Z}[BG]$ are fibrant).

$$\begin{tikzcd} E_{h_n} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z}[BG]^I \arrow[d, "ev_0"] \\ BG^{2n} \arrow[r, "h_n"'] & \mathbb{Z}[BG] \end{tikzcd} \qquad \begin{tikzcd} F_{h_n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{h_n} \arrow[d, "ev_1 \circ \pi_2"] \\ pt \arrow[r, "0"'] & \mathbb{Z}[BG] \end{tikzcd}$$

Now write down the starting portion of the long homotopy exact sequence of the fibration $E_{h_n} \twoheadrightarrow \mathbb{Z}[BG]$. If we denote by K the kernel of the map $\nu\colon \pi_1(T) \to G^{2n}$ induced by the embedding $h_n^{-1}(0) = T \subseteq BG^{2n}$ we come to the following commutative diagram.

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "\psi"'] & \pi_1(T) \arrow[r, "\nu"'] \arrow[d, "\phi"'] & G^{2n} \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}_1(G,\mathbb{Z}) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{H}_2(G,\mathbb{Z}) \arrow[r] & \pi_1(F_{h_n}) \arrow[r] & \pi_1(E_{h_n}) \arrow[r] & \mathrm{H}_1(G,\mathbb{Z}) \arrow[r] & 1 \end{tikzcd}$$

We already know by Corollaries 3.4 and 4.3 that K is naturally isomorphic to $\mathrm{H}_2(G,\mathbb{Z})$ provided $n \geq 3$.

Let us show that ψ is an isomorphism. Assume for a momemnt that G is an abelian group. In this case a generator $x \wedge y$ of $\mathrm{H}_2(G,\mathbb{Z})$ in Miller's presentation corresponds to the class of 2-cycle $c = (x,y) - (y,x)$ (cf. [8, (14), p. 582]), which, in turn, corresponds to the following normalized 2-cycle:

$$c' = c - s_0d_0c - s_1d_1c + s_1d_0 = (x,y) - (y,x) - (x,0) + (y,0) + (0,x) - (0,y). \tag{4.3}$$

Since F_{h_n} and $\mathbb{Z}[BG]$ are fibrant, any element of $\pi_1(F_{h_n}, 0)$ can be represented with some 1-simplex $(\gamma, s) \in G^{2n} \times \mathbb{Z}[BG]_2$ satisfying $d_2(s) = h_n(\gamma)$, $d_0(s) = d_1(s) = 0$. It remains to find in F_{h_n} the 1-simplex homotopic to the path $\phi(\gamma_{x,y})$ (cf. Example 4.4).

We use the following inductive procedure. Let γ_0 be an initial segment of a path γ in T for which we have already found such simplex (g^0, s^0) in F_{h_n} and let $g \in T_1$ be the next chain link. Set $s_0 = \sum(a_j, b_j) \in \mathbb{Z}[BG]_2$, $t = h_n(g^0, g) = \sum_{i=1}^{2n}(-1)^i(g_i^0, g_i)$ and compute the filler for the 3-horn $(0, \ , s_0, t)$. Denote the 1-st face of this filler by s_1. It is clear that $(g^0 g, s_1)$ is the image of $\gamma^0 \circ g$ via ϕ. The concrete formula for s_1 can be chosen e. g. as follows:

$$s_1 = \sum_{i=1}^{2n}(g_i^0 + g_i, -g_i) - \sum_j (a_j + b_j, -b_j).$$

Applying the above recipe to the path $\gamma_{x,y}$ we get an expression equal to (4.3). This shows that ψ is the identity map for an abelian group G. Since ψ is natural in G, by Lemma 4.5 below we get that ψ is an isomorphism for arbitrary G, therefore ϕ is also an isomorphism, as claimed. □

Lemma 4.5. *The only natural endotransformations of the second homology functor* $\mathrm{H}_2(-, \mathbb{Z})$: **Groups** $\to$ **Ab** *are morphisms of multiplication by* $n \in \mathbb{Z}$.

Proof. Denote by η an endotransformation $\mathrm{H}_2(-, \mathbb{Z}) \to \mathrm{H}_2(-, \mathbb{Z})$. When restricted to the subcategory of free finitely-generated abelian groups $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$ the second homology functor coincides with the second exterior power functor $A \mapsto \wedge^2 A$.

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic $\mathbb{Z}$-modules (see Definition 6.13.5 ibid.) The functor $A \mapsto \wedge^2 A$ is clearly quadratic and corresponds to the quadratic $\mathbb{Z}$-module $0 \to \mathbb{Z} \to 0$ under this equivalence. Thus, we get that η restricted to $\mathbf{Add}(\mathbb{Z})$ coincides with the the morphism of multiplication by $n \in \mathbb{Z}$.

Consider the group $\Gamma_k = \langle x_1, y_1, \dots x_k, y_k \mid [x_1, y_1] \cdot \ldots \cdot [x_k, y_k] \rangle$ (the fundamental group of a sphere with k handles). It is clear that the abelianization map $\Gamma_k \to \mathbb{Z}^{2k}$ induces an injective map $\mathrm{H}_2(\Gamma_k, \mathbb{Z}) \cong \mathbb{Z} \to \wedge^2 \mathbb{Z}^{2k}$. Consider the following diagrams.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[d, "\eta_{\Gamma_k}"'] & \wedge^2 \mathbb{Z}^{2k} \arrow[d, "n\cdot"] \\ \mathbb{Z} \arrow[r, hook] & \wedge^2 \mathbb{Z}^{2k} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, "\chi"] \arrow[d, "\eta_{\Gamma_k}"'] & \mathrm{H}_2(G, \mathbb{Z}) \arrow[d, "\eta_G"] \\ \mathbb{Z} \arrow[r, "\chi"] & \mathrm{H}_2(G, \mathbb{Z}) \end{tikzcd}$$

From the left diagram it follows that η_{Γ_k} is also the morphism of multiplication by n. For every element $x \in \mathrm{H}_2(G, \mathbb{Z})$ there exist an integer k and a map $\chi \colon \Gamma_k \to G$ sending the generator of $\mathrm{H}_2(\Gamma_k, \mathbb{Z})$ to x. From the right square we conclude that $\eta_G(x) = nx$, as claimed. □

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