A NOTE ON AMALGAMS OF SYMMETRIC GROUPS AND NONABELIAN TENSOR PRODUCTS

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Abstract. By amalgamating copies of symmetric group S_n we construct a central extension of the wreath product $G \wr S_n$. We then relate this group with extensions of type $\mathfrak{H}_n(G)$ introduced by U. Rehmann in 1970's. We also describe a topological application of this construction.

1. Introduction

The aim of this note is to establish a surprising connection between symmetric groups and (certain quotients of) nonabelian tensors squares.

Let G be a group. Denote by F the free product of copies of the symmetric group S_n in which each copy is indexed by an n-tuple $(g_1, \ldots, g_n) \in G^n$. Denote by $S_n(G)$ the quotient of F modulo relations $s_{(g)} = s_{(h)}$ in which $g, h \in G^n$ and $s \in S_n$ are such that equality $s^g = s^h$ holds in the wreath product $G \wr S_n = G^n \rtimes S_n$ (here s_g and s_h denote images of $s \in S_n$ under respective embeddings $S_n \to F$). It is not hard to check that there is a well defined map $\mu_n \colon S_n(G) \to G \wr S_n$ given by $s_{(g)} \mapsto s^g$.

Our first main result is the following theorem.

Theorem 1. *For $n \geq 3$ the map μ_n is a crossed module whose cokernel and kernel are isomorphic to $H_1(G, \mathbb{Z})$ and $H_2(G, \mathbb{Z})$ respectively. Moreover, the group $S_n(G)$ is isomorphic to $S_n \ltimes H_n^{\wedge}(G)$.*

Here $H_n^{\wedge}(G)$ denotes the quotient of the so-called "universal extension of type $\mathfrak{H}_n(G)$" (which is denoted $H_n(G)$) modulo all "symbols" of the form $\{x, x\} = 1$, $x \in G$.

Extensions of type $\mathfrak{H}_\mathfrak{n}$ were defined by U. Rehmann in [11] and played a key role in his proof of generalized Matsumoto theorem. The groups $H_n(G)$, $H_n^{\wedge}(G)$ are directly related with nonabelian tensor products (which appear under the name of "extensions of type $\mathfrak{U}$" in [11]). More formally, $G \widetilde{\otimes} G$ (resp. $G \wedge G$) are precisely the subgroups of $H_n(G)$ (resp. $H_n^{\wedge}(G)$) generated by symbols $\{x, y\}$ (we refer the reader to Section 3 for more details).

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Theorem 1 and its proof (which relies solely on basic combinatorial group theory) may be of interest for the following reasons:

- Although this is vacuous formally, the group $\mathbb{F}_1$-analogue of the well known fact that The presentation is similar in spirit to results of Brown—Loday and Kassel—Reutenauer on the presentation of the linear Steinberg group in the form of an amalgam of braid groups.
- A presentation of the extension $H_n(G)$ that is simpler than its original presentation is obtained in the process (4 instead of 5 relations that are also shorter), see Proposition 3.1.
- Theorem 1 is motivated by (and provides partial answer to) some purely topological question, which we describe below in more detail.

For a pointed simplicial set (X, x_0) denote by $\mathbb{Z}[X]$ the associated free simplicial abelian group. Define the "alternating" map of simplicial sets $h_n\colon X^{2n} \to \mathbb{Z}[X]$ by the following identity:

$$h_n(x_1, x_2, \ldots x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Define the "stabilization" map $X^{2n} \hookrightarrow X^{2(n+1)}$ by $x \mapsto (x, x_0, x_0)$ and denote by X^∞ the colimit of X^n.

Recently S. Podkorytov showed that the limit map $h_\infty = \operatorname{colim}_n(h_n)\colon X^\infty \to \mathbb{Z}[X]$ is a *quasifibration* provided X is fibrant and connected. By definition, this means that the natural map $h_\infty^{-1}(b) \to F_{h_\infty}(b)$ between the fiber and the homotopy fiber of h_∞ over any point b is a weak (hence homotopy) equivalence. This assertion is essentially proved in [10, Lemma 9.1], however some technical work is required to formulate it this way, see Lemma A.4 below.

One may wonder whether the maps h_n satisfy some weaker analogue of this property, i. e. one might expect that the map $h_n^{-1}(0) \to F_{h_n}(0)$ is $c(n)$-connected for some $c(n)$ which depends only on n and tends to infinity as $n \to \infty$.

The following result which is a consequence of Theorem 1 asserts that this map is at least 1-connected in one important special case.

Theorem 2. *If $X = BG$ is the classifying space of a group G then for $n \geq 3$ the natural map $h_n^{-1}(0) \to F_{h_n}(0)$ induces an isomorphism of fundamental groups $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$.*

Theorem 2 reduces to Theorem 1 by means of Quillen Theorem A and van Kampen theorem. It also involves some explicit computations with simplicial fundamental groups.

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2. Basic facts about $S_n(G)$

2.1. **The definition of $S_n(G)$.** Recall that we defined the group $S_n(G)$ as the quotient of the free product F of copies of S_n indexed by G^n modulo the following single family of relations:

$$s_g = s_h, \text{ if } s^g = s^h \text{ holds in } G \wr S_n. \tag{2.1}$$

We denote by s_g the image of the element $s_g \in F$ under the canonical map $F \to S_n(G)$.

Remark 2.1. Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1}g^{(s^{-1})}, s).$$

therefore the equality $s^g = s^h$ holds in $G \wr S_n$ iff $g^{-1}g^{(s^{-1})} = h^{-1}h^{(s^{-1})}$, or what is the same, iff s fixes hg^{-1}.

The last statement immediately implies that the map $\mu\colon S_n(G) \to G \wr S_n$ given by $\mu(s_g) = s^g = (g^{-1}g^{(s^{-1})}, s)$ is well-defined.

Observe from the definition of $S_n(G)$ that there is a split exact sequence.

$$1 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow S_n(G) \underset{\iota(1)}{\overset{\pi}{\rightleftarrows}} S_n \longrightarrow 1, \tag{2.2}$$

Here the map $\pi = \pi_{S_n} \circ \mu$ removes subscript g from each s_g and the section $\iota(1)$ maps s to s_1, where 1 is the identity element of G^n. Thus, if we denote $\mathrm{Ker}(\pi)$ by $HS_n(G)$ we get a decomposition $S_n(G) = HS_n(G) \rtimes S_n$.

2.2. **A crossed module structure on $S_n(G)$.** Recall from [2, § 2.2] that a *crossed module* is a morphism of groups $\mu\colon M \to N$ together with a right action of N on M compatible with the conjugation action of N on itself, i. e.

$$\mu(m^n) = \mu(m)^n \text{ for all } n \in N,\ m \in M, \tag{CM1}$$

which also satisfies the following identity called *Peiffer identity*:

$$m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M. \tag{CM2}$$

In our situation, we let $G \wr S_n$ act on $S_n(G)$ by

$$(s_g)^{(h,t)} = s^t{}_{(gh)^t}, \text{ for } s, t \in S_n \text{ and } g, h \in G^n. \tag{2.3}$$

The goal of this subsection is to prove the following.

Proposition 2.2. *For $n \geq 3$ the map $\mu\colon S_n(G) \to G \wr S_n$ is a crossed module.*

From the fact that μ is a crossed module one can deduce that $\mathrm{Ker}(\mu)$ is a central subgroup of $S_n(G)$ and $\mathrm{Im}(\mu)$ is a normal subgroup of $G \wr S_n$. It is not hard to show that the group $S_2(G)$ is isomorphic to a free product of copies of S_2 (whose center is always trivial). Therefore, the requirement $n \geq 3$ in the statement of Proposition 2.2 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of $G \wr S_n$ on $S_n(G)$ that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of $S_n(G)$, for which it takes the form:

$$t_h^{-1} s_g t_h = {s^t}_{(gh^{-1})^t \cdot h} \text{ for all } s, t \in S_n,\ g, h \in G^n. \tag{2.4}$$

If we act on both sides of the above formula by $(h^{-1}, 1) \in G \wr S_n$ we obtain the equality $t_1^{-1} s_{gh^{-1}} t_1 = {s^t}_{(gh^{-1})^t}$. Thus, to prove (2.4) it suffices to show the following simpler relation:

$$t_1^{-1} s_g t_1 = {s^t}_{g^t} \text{ for all } s, t \in S_n,\ g \in G^n. \tag{2.5}$$

The key step in the proof is the following lemma.

Lemma 2.3. *The relation* (2.5) *holds in the special case when* $s = (ij)$ *and* $t = (kl)$ *are two nonequal transpositions.*

Proof. First of all, we immediately check that (2.5) holds in the special case when t fixes $g' \in G^n$. Indeed, by (2.1) we have $t_1 = t_{g'}$, hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that $l \neq i$ and $l \neq j$. Denote by g' the vector which differs from g only at l-th position, for which we set $g'_l = g_k$. Since the only nontrivial component of $g'g^{-1}$ (resp. $g'g^{-t}$) is located at l-th (resp. k-th) position, it is fixed by s (resp. s^t), hence from (2.1) we conclude that $s_g = s_{g'}$ (resp. $(s^t)_{g'} = (s^t)_{g^t}$). Finally, since g' is fixed by t, we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g'^t} = s^t_{g'} = (s^t)_{g^t}. \quad \square$$

Proof of Proposition 2.2. Let us show that (2.5) holds for arbitrary transpositions $s, t \in S_n$. It suffices to consider the case $s = t = (ij)$. After choosing some $k \neq i, j$ and presenting (ij) as $(kj)(ik)(kj)$ we use Lemma 2.3:

$$(ij)_1^{-1}(ij)_g(ij)_1 = (ij)_1^{-1}(kj)_g(ik)_g(kj)_g(ij)_1 = (ki)_{g^{(ij)}}(jk)_{g^{(ij)}}(ki)_{g^{(ij)}} = (ij)_{g^{(ij)}}.$$

The proposition now follows by induction on the length of permutations s, t. $\square$

2.3. **An explicit presentation of $S_n(G)$.** This presentation turns out to be much more economical than the original definition given in section 2.1. An explicit presentation for the subgroup $HS_n(G)$ is derived from this presentation via Reidemeister-Schreier method.

We start with the following simple lemma which is, in essence, a variant of the standard presentation of a Weyl group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

Lemma 2.4. *For $n \geq 3$ The symmetric group S_n admits presentation with transpositions as the set of generators and the following list of defining relations (in every formula distinct letters denote distinct indices):*

$$(ij)^2 = 1, \tag{S1}$$

$$(ij)^{(jk)} = (ik), \tag{S2}$$

$$[(ij), (kl)] = 1. \tag{S3}$$

$$(ij) = (ji), \tag{S4}$$

Now we are ready to formulate the main result of this subsection.

Proposition 2.5. *For $n \geq 3$ and arbitrary group G the group $S_n(G)$ admits presentation with the set of generators $\{(ij)_a \mid i \neq j, 1 \leq i, j, \leq n, a \in G\}$ and the following list of relations (as before, distinct letters denote distinct indices):*

$$(ij)_a^2 = 1, \tag{SG1}$$

$$(ij)_a^{(jk)_b} = (ik)_{ab}, \tag{SG2}$$

$$[(ij)_a, (kl)_b] = 1. \tag{SG3}$$

$$(ij)_a = (ji)_{a^{-1}} \tag{SG4}$$

Proof. Denote by S' the group from the statement of the proposition. For $1 \leq i \leq n$ and $x \in G$ denote by $x[i]$ the element of G^n whose only nontrivial component equals x and is located in the i-th position.

It is not hard to deduce from the definition of $S_n(G)$ and Lemma 2.3 that the formula $(ij)_a \mapsto (ij)_{a[j]}$ gives a well-defined map $\varphi\colon S' \to S_n(G)$.

Now we are going to construct the map $\psi\colon S_n(G) \to S'$ in the opposite direction. Using the presentation of S_n given by Lemma 2.4 we define for a fixed $g \in G^n$ the map $\psi_g\colon S_n \to S'$ by $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$. It is obvious that ψ_g preserves the defining relations (S1)–(S4) of S_n. It remains to show that the equation $\psi_g(s) = \psi_h(s)$ holds whenever $g, h \in G^n$ and $s \in S_n$ satisfy the requirement of (2.1).

Indeed, if hg^{-1} is fixed by s then for every $1 \leq i \leq n$ we have $(hg^{-1})_i = (hg^{-1})_{s(i)}$, or equivalently $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$. For example, if s is a cycle of length p, i. e.

$s = (i_1, i_2, \dots i_p)$ with $i_{k+1} = s(i_k)$ we get that

$$\psi_g(s) = \psi_g\left(\prod_{k=1}^{p-1}(i_k, i_{k+1})\right) = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{g_{i_k}^{-1} g_{i_{k+1}}} = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{h_{i_k}^{-1} h_{i_{k+1}}} = \psi_h(s).$$

The proof for a general s is almost the same. Verification of the fact that ψ and φ are mutually inverse is also immediate. □

We now obtain a presentation of the subgroup $HS_n(G)$. For every $i \neq j$ and $a \in G$ we define the element $h_{ij}(a) \in HS_n(G)$ as follows:

$$h_{ij}(a) = (ij)_a \cdot (ij)_1. \tag{2.6}$$

It is not hard to show that $h_{ij}(a)$ form a generating set for $HS_n(G)$. In fact, there is an explicit formula how an element of $HS_n(G)$ originally expressed through $(ij)_a$'s can be rewritten in terms of $h_{ij}(a)$. Indeed, if h lies in $HS_n(G)$ and is written as $\prod_{k=1}^{N}(i_k j_k)_{a_k}$ for some $i_k \neq j_k$ and $a_k \in G$ then it can be rewritten as follows:

$$h = \prod_{k=1}^{N} h_{\sigma_k(i_k),\sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1}(i_s j_s) \in S_n. \tag{τ}$$

We briefly recall the notion of a *rewriting process* given in [7, § 2.3]. If G is a group presented by generators a_ν and relations $R_\mu(a_\nu)$ and H is its subgroup with a generating set $J_i(a_\nu)$ then a *rewriting process for* H is a function which maps every word u in alphabet a_ν to a word v in alphabet s_i such that u and $v[s_i := J_i]$ define the same element of G whenever u represents an element of H.

With this terminology, the mapping (τ) defined above is a rewriting process for the subgroup $HS_n(G)$. Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [7, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [7, § 2.3]):

- if U and U^* are freely equal words in $(ij)_a$ then $\tau(U)$ and $\tau(U^*)$ are also freely equal words in $h_{ij}(a)$;
- if U_1 and U_2 are two words in $(ij)_a$ which define elements of $HS_n(G)$ then the words $\tau(U_1U_2)$ and $\tau(U_1)\tau(U_2)$ are equal.

Using these two properties and repeating the arguments used in the proof of [7, Theorem 2.8] one can simplify the generic presentation given by [7, Theorem 2.6] and obtain the following.

Lemma 2.6. *For $n \geq 3$ the group $HS_n(G)$ admits presentation on the generators $h_{ij}(a)$ with the following two families of defining relations:*

$$h_{ij}(a) = \tau\left((ij)_a \cdot (ij)_1\right); \tag{2.7}$$

$$\tau(KRK^{-1}) = 1, \tag{2.8}$$

where R varies over relations of Proposition 2.5 and K is any word in $(ij)_1$, $i \neq j$.

From (2.7) one immediately obtains the equality $h_{ij}(1) = 1$, $i \neq j$. Thus, if we denote by σ the permutation corresponding to a word K and let S_n act on $h_{ij}(a)$'s in the natural way we will get that the word $\tau(KRK^{-1})$ is equivalent to ${}^{\sigma}\tau(R)$ (modulo relations $h_{ij}(1) = 1$).

Since the relations from Proposition 2.5 are respected by the action of S_n, only relations of the form $\tau(R) = 1$ are, in fact, needed for the presentation of $HS_n(G)$. Writing down what $\tau(R)$ is for each of (SG1)–(SG4) we get the following.

Proposition 2.7. *For $n \geq 3$ the group $HS_n(G)$ admits presentation with generators $h_{ij}(a)$ and the following list of relations:*

$$h_{ij}(1) = 1, \tag{R0}$$

$$h_{ij}(a)h_{ji}(a) = 1, \tag{R1}$$

$$h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab), \tag{R2}$$

$$[h_{ij}(a), h_{kl}(b)] = 1, \tag{R3}$$

$$h_{ij}(a)^{-1} = h_{ij}(a^{-1}). \tag{R4}$$

3. Comparison with extensions of type $\mathfrak{H}_n(G)$

We start by briefly reviewing the material of § 1–3 of [11]. Let $n \geq 3$ be a natural number. By definition, the group $H_n(G)$ is given by generators $h_{ij}(u)$, $u \in G$, $i \neq j$ and the following relations (cf. [11, H1–H5 of § 2]):

$$h_{ij}(u)h_{ji}(u) = 1, \tag{H1}$$

$$h_{ij}(u)h_{ki}(u)h_{jk}(u) = 1, \tag{H2}$$

$$h_{ij}(u)h_{ik}(v)h_{ij}(u)^{-1} = h_{ik}(uv)h_{ik}(u)^{-1}, \quad \text{for } j \neq k \tag{H3}$$

$$h_{ij}(u)h_{kj}(v)h_{ij}(u)^{-1} = h_{kj}(vu)h_{kj}(u)^{-1}, \quad \text{for } i \neq k \tag{H4}$$

$$[h_{ij}(u), h_{kl}(v)] = 1. \tag{H5}$$

There is a surjective group homomorphism $H_n(G) \twoheadrightarrow D_n(G)$ sending $h_{ij}(g)$ to $d_{ij}(g)$. Here $d_{ij}(g)$ denotes the element of G^n whose i-th component equals g, j-th component equals g^{-1} and all other components are trivial.

By definition, *an extension of type* $\mathfrak{H}_n(G)$ is an extension H of $D_n(G)$ which is also a quotient of $H_n(G)$, i. e. the one which fits into the following diagram.

$$\begin{tikzcd} H_n(G) \arrow[d, two heads] \arrow[dr, two heads] & \\ H \arrow[r, two heads] & D_n(G) \end{tikzcd}$$

Any $\mathfrak{H}_n(G)$-extension is central, see [11, Proposition 2.3].

3.1. **A simpler presentation of $H_n(G)$.** The aim of this subsection is to obtain a presentation of $H_n(G)$ that is simpler than the original presentation: 4 instead of 5 relations that are also shorter. Our main result is the following.

Proposition 3.1. *For $n \geq 3$ relations* (R0)–(R4) *imply* (H1)–(H5)*. Moreover, for $n \geq 4$ relations* (R0)–(R3) *are equivalent to* (H1)–(H5).

For $u, v \in G$ we define the following two symbols:

$$c_{ij}(u,v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, c'_{ij}(u,v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies $h_{ij}(1) = 1$ therefore $c_{ij}(u, u^{-1}) = c'_{ij}(u, u^{-1})$. Our goal is to show that (H3) can be omitted from the definition of $H_n(G)$ provided $n \geq 4$.

Lemma 3.2. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following still remain true:*

(1) *The elements $c_{ij}(u, u^{-1})$ are central in $H_n(G)$;*
(2) *One has $c'_{ij}(u,v) = [h_{ij}(u), h_{kj}(v)]$, $k \neq i, j$;*
(3) *One has ${}^{h_{ij}(w)}c'_{kj}(u,v) = c'_{kj}(u,w)^{-1}c'_{kj}(u,vw)$, $k \neq i, j$;*
(4) *One has ${}^{h_{ij}(w)}c'_{ij}(u,v) = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}$.*

Proof. First one shows using (H4) that $c_{ij}(u, u^{-1})$ centralizes $h_{kj}(v)$ (cf. with the proof of [11, Lemma 2.1(2)]). Since in any group $[a, b] = 1$ implies $[a^{-1}, b] = [a, b^{-1}] = [a^{-1}, b^{-1}] = 1$ we get that $c_{ij}(u, u^{-1})$ also centralizes $h_{kj}(v)^{-1} = h_{jk}(v)$ and $c_{ij}(u^{-1}, u)^{-1} = c_{ji}(u, u^{-1})$ centralizes both $h_{kj}(v)$ and $h_{jk}(v)$. Together with (H2) and (H5) this implies that $c_{ij}(u, u^{-1})$ centralizes all the generators of $H_n(G)$ and hence lies in the center of $H_n(G)$.

The second and third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$${}^{h_{ij}(w)}c'_{ij}(u,v) = {}^{h_{ij}(w)}c'_{kj}(v,u)^{-1} = (c'_{kj}(v,w)^{-1}c'_{kj}(v,uw))^{-1} = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}.$$

□

Lemma 3.3. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following statements are equivalent:*

(1) (H3) *holds;*
(2) *one has* $c'_{ij}(u,v)^{-1} = c'_{ij}(v,u)$*;*
(3) *symbols* $c'_{ij}(u,v)$ *do not depend on* i*;*
(4) *one has* $c'_{ij}(u,vw) = c'_{ij}(wu,v)\cdot c'_{ij}(uv,w)$*;*
(5) *one has* $c'_{ij}(u,vu^{-1}) = c'_{ij}(uv,u^{-1})$*.*

Proof. Implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are essentially contained in the proof of [11, Lemmas 2.1-2.2]. Implication (4) $\implies$ (5) is trivial.

We now prove (5) $\implies$ (1). Notice that (H4) implies ${}^{h_{ij}(u)^{-1}}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$, therefore using Lemma 3.2.(1) we get that ${}^{h_{ij}(u)}h_{ik}(v) = {}^{c_{ij}(u,u^{-1})h_{ij}(u^{-1})^{-1}}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$. Thus, (H3) is equivalent to the equality

$$h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}),$$

or what is the same

$$c'_{ik}(u^{-1},uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1},u).$$

It is clear that the last equality is an equivalent reformulation of (5). □

Proof of Proposition 3.1. The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);
- for $n \geq 4$ the third statement of the previous lemma follows from Lemma 3.2.(2).

□

Corollary 3.4. *For* $n \geq 3$ *the group* $HS_n(G)$ *is the quotient of the extension* $H_n(G)$ *by* (R4) *(in particular, is an extension of type* $\mathfrak{H}_n(G)$*). Moreover, we have the following natural isomorphism:*

$$\mathrm{Ker}(HS_n(G) \to D_n(G)) = \mathrm{Ker}(S_n(G) \to G \wr S_n) \cong \mathrm{H}_2(G,\mathbb{Z}). \tag{3.1}$$

Proof. Most of the assertions follow from Proposition 3.1 and [4, Proposition 5]. One also has to use the formula $c_{ij}(u,u) = c_{ij}(u,u^{-1})$ (see [11, p. 87]). □

4. Proof of Theorem 2

Recall that for arbitrary set X one defines the space EX as the the simplicial set whose set of k-simplices EX_k is X^{k+1} and whose faces and degeneracies are obtained by omitting and repeating components. For a group G we denote by π_G the canonical map $EG \to BG$ sending $(g,h) \in EG_1$ to $g^{-1}h \in BG_1$.

Now let N be a group acting on X. We define two simplicial sets U and V as follows:

$$U = \bigcup_{n \in N} E(\Gamma_{n.-}) \subseteq E(X \times X), \quad V = \bigcup_{x,y \in X} E(N(x \to y)) \subseteq EN.$$

Here $\Gamma_{n.-}$ is the graph of the function $(x \mapsto nx)$ and $N(x \to y)$ denotes the subset of elements $n \in N$ satisfying $nx = y$. With this notation the subset $N(x \to x)$ coincides with the stabilizer subgroup $N_x \leq N$.

Lemma 4.1. *The simplicial sets U and V are homotopy equivalent.*

Proof. First, we define yet another simplicial set W as follows. Its k-simplices W_k are matrices $\left(\begin{smallmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{smallmatrix} \right)$, where $x_i \in X$ are $n_i \in N$ are such that all n_i's act each x_j in the same way, i. e. $n_i x_j = n_{i'} x_j$ for $0 \leq i, i', j \leq k$. The faces and degeneracies of W are the maps of omission and repetition of columns.

Now there are two simplicial maps $f\colon W \to U$, $g\colon W \to V$ whose action on 0-simplicies is given by $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix} \right) = (x_0, n_0 x_0)$, $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix} \right) = n_0$. To prove the lemma it suffices to show that f and g are homotopy equivalences. The proof for f and g is similar, let us show, for example, that g is a homotopy equivalence.

In view of Quillen theorem A (cf.[13, ex. IV.3.11]) it suffices to show that for each p-simplex $d\colon \Delta^p \to V$ the pullback $g/(p,d)$ of d and g is contractible. The simplicial set $g/(p,d)$ can be interpreted as the subset of $\Delta^p \times E(X)$ whose set of k-simplices consists of pairs $(\alpha\colon \underline{k} \to \underline{p}, (x_0, \dots, x_k) \in X(\alpha, d)^{k+1})$. Here $X(\alpha, d)$ is the subset of X consisting of all x for which $d_{\alpha(i)} x = d_{\alpha(j)} x$ for $0 \leq i, j \leq k$. Notice that the set $X_d := X(id_{\underline{p}}, d)$ is nonempty and is contained in every $X(\alpha, d)$ (it even equals $X(\alpha, d)$ for surjective α). Now choose a point $\widetilde{x} \in X_d$ and consider the simplicial homotopy

$$H\colon \Delta^p \times EX \times \Delta^1 \to \Delta^p \times EX$$

between the identity map of $\Delta^p \times EX$ and the map $\Delta^p \times c_{\widetilde{x}}$, where $c_{\widetilde{x}}$ is the constant map. More concretely, H sends each triple $(\alpha\colon \underline{k} \to \underline{p}, (x_0, \dots, x_k), \beta\colon \underline{k} \to \underline{1})$ to $(\alpha, (x_0, \dots, x_{i-1}, \widetilde{x}, \dots, \widetilde{x}))$, where i is the minimal number such that $\beta(i) = 1$. By the choice of $\widetilde{x}$ the image of H restricted to $g/(p,d) \times \Delta^1$ is contained in $g/(p,d)$, hence $g/(p,d)$ is contractible. □

Now suppose that $X = H$ is also a group upon which N acts on the left.

Corollary 4.2. *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

There is an isomorphism $\theta\colon \operatorname{Ker}(\pi_1(S) \to N) \cong \operatorname{Ker}(\pi_1(T) \to H \times H)$. Moreover, the higher homotopy groups of S and T are isomorphic.

Proof. Consider the following two pull-back squares:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EN \arrow[d, "\pi_N"] \\ S \arrow[r, hook] & BN \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(H\times H) \arrow[d, "\pi_{H\times H}"] \\ T \arrow[r, hook] & B(H\times H) \end{tikzcd}$$

The required isomorphism can be obtained from the homotopy long exact sequence applied to the left arrows of these diagrams. ☐

Now let G be a group. Set $N = G \wr S_n$, $H = G^n$ and consider the left action of N on H given by $(g, s)\cdot h = gh^{s^{-1}}$, $g, h \in G^n$, $s \in S_n$. If one reorders the components of BG^{2n} accordingly, the simplicial subset $T \subset BG^{2n}$ from the above corollary becomes precisely the preimage of 0 under $h_n \colon BG^{2n} \to \mathbb{Z}[BG]$.

It is also easy to compute the map $\pi_1(S) \to N$. Indeed, van Kampen theorem [8, Theorem 2.7] asserts that $\pi_1(S)$ is isomorphic to the free product of stabilizer subgroups $N_h \leq N$ amalgamated over pairwise intersections $N_h \cap N_{h'}$, $h, h \in H$. For $h \in G^n$ the subgroup N_h consists of elements $(g, s) \in N$ satisfying $gh^{s^{-1}} = h$, i.e. elements of the form $(hh^{-s^{-1}}, s)$. Thus, $N_h \cong S_n$, $\pi_1(S)$ is isomorphic to the group $S_n(G)$ and the map $\pi_1(S) \to N$ coincides with the map μ defined in section 2.1. In particular, for an abelian G its kernel is generated by symbols $c_{ij}(u, v)$.

The following lemma gives a more concrete description of the isomorphism θ from Corollary 4.2.

Lemma 4.3. *If G is abelian then every generator $c_{ij}(u, v) \in \mathrm{Ker}(\pi_1(S) \to N)$ is mapped under θ to the homotopy class of the loop $f_{ij}(u, v)$, which is defined as follows:*

$$f_{ij}(u, v) = p_{ij}(u) \circ p_{ij}(v) \circ p_{ij}(u^{-1}v^{-1}),\ \text{ where } p_{ij}(x) = (x[i], x[i]) \circ (x^{-1}[i], x^{-1}[j]).$$

Proof. Fix a vector $g \in G^n$ and denote by $\gamma_{ij}(g, x)$ the path of length 4 in W which connects the following five points of W_0:

$$\begin{pmatrix} e \\ (g;e) \end{pmatrix},\ \begin{pmatrix} x[i] \\ (g;e) \end{pmatrix},\ \begin{pmatrix} x[i] \\ (d_{ij}(x)g,(ij)) \end{pmatrix},\ \begin{pmatrix} e \\ (d_{ij}(x)g,(ij)) \end{pmatrix},\ \begin{pmatrix} e \\ (d_{ij}(x)g,e) \end{pmatrix}.$$

One can easily check that the image of $\gamma_{ij}(g, x)$ in S under $\pi_N g$ is precisely the element $h_{ij}(x)$, while $\pi_{H\times H} f$ sends $\gamma_{ij}(g, x)$ to the loop $p_{ij}(x)$ (here f, g are the maps from the proof of Lemma 4.1). ☐

Lemma 4.4. *The only natural endotransformations of the second homology functor* $\mathrm{H}_2(-, \mathbb{Z})\colon$ **Groups** $\to$ **Ab** *are morphisms of multiplication by $n \in \mathbb{Z}$.*

Proof. Denote by η an endotransformation $\mathrm{H}_2(-, \mathbb{Z}) \to \mathrm{H}_2(-, \mathbb{Z})$. When restricted to the subcategory of free finitely-generated abelian groups $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$ the second homology functor coincides with the second exterior power functor $A \mapsto \wedge^2 A$.

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic $\mathbb{Z}$-modules (see Definition 6.13.5 ibid.) The functor $A \mapsto \wedge^2 A$ is clearly quadratic and corresponds to the quadratic $\mathbb{Z}$-module $0 \to \mathbb{Z} \to 0$ under this equivalence. Thus, we get that η restricted to $\mathbf{Add}(\mathbb{Z})$ coincides with the the morphism of multiplication by $n \in \mathbb{Z}$.

Consider the group $\Gamma_k = \langle x_1, y_1, \dots x_k, y_k \mid [x_1, y_1] \cdot \ldots \cdot [x_k, y_k] \rangle$ (the fundamental group of a sphere with k handles). It is clear that the abelianization map $\Gamma_k \to \mathbb{Z}^{2k}$ induces an injective map $\mathrm{H}_2(\Gamma_k, \mathbb{Z}) \cong \mathbb{Z} \to \wedge^2 \mathbb{Z}^{2k}$. Consider the following diagrams.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[d, "\eta_{\Gamma_k}"'] & \wedge^2 \mathbb{Z}^{2k} \arrow[d, "n\cdot"] \\ \mathbb{Z} \arrow[r, hook] & \wedge^2 \mathbb{Z}^{2k} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, "\chi"] \arrow[d, "\eta_{\Gamma_k}"'] & \mathrm{H}_2(G, \mathbb{Z}) \arrow[d, "\eta_G"] \\ \mathbb{Z} \arrow[r, "\chi"] & \mathrm{H}_2(G, \mathbb{Z}) \end{tikzcd}$$

From the left diagram it follows that η_{Γ_k} is also the morphism of multiplication by n. For every element $x \in \mathrm{H}_2(G, \mathbb{Z})$ there exist an integer k and a map $\chi\colon \Gamma_k \to G$ sending the generator of $\mathrm{H}_2(\Gamma_k, \mathbb{Z})$ to x. From the right square we conclude that $\eta_G(x) = nx$, as claimed. □

Proof of Theorem 2. Factor h_n as a composition of a trivial cofibration followed by a fibration $BG^{2n} \to E_{h_n} \to \mathbb{Z}[BG]$ and write down the starting portion of the long homotopy exact sequence for the latter. If we denote by K the kernel of the map $\nu\colon \pi_1(T) \to G^{2n}$ induced by the embedding $h_n^{-1}(0) = T \subseteq BG^{2n}$ we obtain the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "\beta"'] \arrow[d, "\psi"'] & \pi_1(T) \arrow[r, "\nu"'] \arrow[d, "\varphi"'] & G^{2n} \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}_1(G, \mathbb{Z}) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{H}_2(G, \mathbb{Z}) \arrow[r, "\alpha"'] & \pi_1(F_{h_n}) \arrow[r] & \pi_1(E_{h_n}) \arrow[r] & \mathrm{H}_1(G, \mathbb{Z}) \arrow[r] & 1. \end{tikzcd}$$

We already know by Corollaries 3.4 and 4.2 that K is naturally isomorphic to $\mathrm{H}_2(G, \mathbb{Z})$ provided $n \geq 3$. Thus, it suffices to show that ψ is an isomorphism. In view of Lemma 4.4 we only need to consider the case when G is abelian.

We are going to compute the image of each generator $c_{ij}(x, y) = x \wedge y \in \mathrm{H}_2(G, \mathbb{Z})$ under ψ. In order to do this we need to obtain a more explicit description of the set of 1-simplices of F_{h_n}. Since both BG^{2n} and $\mathbb{Z}[BG]$ are fibrant E_{h_n} and F_{h_n} can be constructed using the path space construction, i. e. $E_{h_n} = BG^{2n} \times_{\mathbb{Z}[BG]} \mathbb{Z}[BG]^I$. In this case the set of 1-simplices of F_{h_n} can be identified with the set of triples $(g, t, t') \in G^{2n} \times \mathbb{Z}[BG]_2 \times \mathbb{Z}[BG]_2$ satisfying the following identites:

$$d_2(t) = h_n(g), \quad d_1(t) = d_1(t'), \quad d_0(t') = 0.$$

It will be convenient for us to perform all calculations inside the subset $F' \subset (F_{h_n})_1$ consisting of those triples for which $t' = 0$ and $d_1(t) = d_0(t) = 0$. For shortness we write down an element of F' as (g, t). It is easy to check that the image of the loop $p_{ij}(x)$ under the natural map $T \to F_{h_n}$ corresponds to the pair $((e, d_{ij}(x)), t(x))$, where $t(x) = -(x, x^{-1}) + (e, e) - (x^{-1}, e) + (e, x^{-1})$.

We want to devise a concrete formula for the element of F' homotopic to the concatenation of two elements $(g_1, t_1), (g_2, t_2) \in F'$. It is easy to check that (g_1g_2, t), where t is the 1-st face of the filler f for the 3-horn $(t_2, , t_1, h_n(g_1, g_2))$, is the desired element. By Moore's theorem one can find a filler for any horn $(p_0, , p_2, p_3)$ in a simplicial abelian group via the formula $f = s_0p_0 - s_0s_1d_2p_0 - s_0s_0d_1p_0 + s_0s_0d_2p_0 + s_1p_2 + s_2p_3 - s_1p_3$. Substituting concrete values of p_i into this formula and using the fact that $d_1t_2 = 0$ we get the following expression for t:

$$\begin{aligned} t = d_1f = p_0 - s_1d_2p_0 - s_0d_1p_0 + s_0d_2p_0 + p_2 - p_3 + s_1d_1p_3 = \\ = t_1 + t_2 - h_n(g_1, g_2) - s_1h_n(g_2) + s_0h_n(g_2) + s_1h_n(g_1g_2). \end{aligned}$$

Applying the above formula twice we get that the image of $c_{ij}(u, v)$ under $\varphi\beta$ equals $((e, e), t)$, where

$$\begin{aligned} t = 2(e, e) - (x^{-1}, e) + (xy, x^{-1}y^{-1}) - (xy, e) - (x, x^{-1}) - (y, y^{-1}) - \\ - (e, x^{-1}y^{-1}) + (e, x^{-1}) + (x, y) + (x^{-1}, y^{-1}) - (e, y) - (xyx^{-1}y^{-1}, e) + (y, e). \end{aligned} \tag{4.1}$$

The map α can be described as the Dold—Kan isomorphism $\mathrm{H}_2(G, \mathbb{Z}) \cong \pi_2(\mathbb{Z}[BG])$ followed by the boundary map $\partial\colon \pi_2(\mathbb{Z}[BG]) \to \pi_1(F_{h_n})$. It is not hard to show that ∂ maps each $c \in \pi_2(\mathbb{Z}[BG])$ to $((e, e), -c)$ (see [5, p. 29]). Thus, we get that ψ maps $x \wedge y$ to $-t$, where t is given by (4.1).

On the other hand, the generator $x \wedge y$ of $\mathrm{H}_2(G, \mathbb{Z})$ corresponds to the 2-cycle $c = (x, y) - (y, x)$ (cf. [9, (14), p. 582]) and the corresponding element of $\pi_2(\mathbb{Z}[BG])$ is simply the normalized 2-cycle of c:

$$c' = c - s_0d_0c - s_1d_1c + s_1d_0 = (x, y) - (y, x) - (x, e) + (y, e) + (e, x) - (e, y).$$

It it not hard to check that $t - c'$ is the image of the following 3-chain under the differential $d_0 - d_1 + d_2 - d_3$:

$$(xy, x^{-1}, y^{-1}) - (y, x, x^{-1}) + s_1s_1(x^{-1} + xy - x - y) + s_0s_0(x^{-1} - x - x^{-1}y^{-1}).$$

Thus we obtain that $[t] = -[c']$, hence ψ is an isomorphism, as claimed. □

Appendix A. Simplicial Dold—Serre fibrations

The aim of this appendix is to show that the map $h_\infty\colon X^\infty \to \mathbb{Z}[X]$ is a quasifibration under suitable assumptions on X.

We start with the definition of the simplicial analogue of the so-called "Dold—Serre fibration", i. e. a map posessing "weak covering homotopy property" (also called "delayed homotopy lifting property"), cf. [12, § 1.4.4]. We show in Lemmas A.2 and A.4 that the map h_∞ satisfies this property and that every map with this property is a quasifibration.

Definition A.1. Let C be a subclass of the class of acyclic cofibrations in the category of simplicial sets. We say that a map of simplicial sets $p\colon E \to B$ satisfies *weak homotopy covering property* with respect to C if for every commutative square

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ V \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd} \tag{A.1}$$

for which $i \in C$ there exists a map $\widetilde{F}$ such that the lower triangle commutes strictly and the upper one commutes up to a fiberwise homotopy H (i. e. a homotopy H such that $pH\colon U \times I \to B$ coincides with the composite $U \times I \xrightarrow{\pi} U \xrightarrow{pg} B$).

Denote by C_{pr} the class consisting of all inclusions $i\colon U \hookrightarrow V$ of finite polyhedral simplicial sets for which there exists a deformation retract (i. e. a map $r\colon V \to U$ homotopic to id_V).

Lemma A.2. *For a fibrant and connected simplicial set X the map $h_\infty\colon X^\infty \to \mathbb{Z}[X]$ satisfies the weak covering homotopy property with respect to the class C_{pr}.*

Proof. Let i_0 be a map from C_{pr} and r be the corresponding retraction. We first reduce the problem to the special case when g is the constant map at the basepoint of X^∞ (which we denote by 0). Indeed, let F, g be arbitrary maps as in (A.1). Since U is finite the image of g is contained in $X^{2N} \subset X^\infty$. If now $\widetilde{F}_0$ is a lifting in the diagram similar to (A.1) in which the top map equals 0 and the bottom one is $F_0 = F - h_\infty gr$ then $\widetilde{F} = gr \times \widetilde{F}_0$ is the desired lifting of F.

Now choose a contractible fibrant simplicial set W which maps surjectively onto X (e. g. take W to be the path space fibration). Since $p\colon W \twoheadrightarrow X$ is surjective, the associated map between free simplicial abelian groups is a Kan fibration. Since i_0 is acyclic we can choose a lifting $\widetilde{F}$ in the following diagram

$$\begin{tikzcd} U \arrow[r, "0"] \arrow[d, hook, "i_0"'] & \mathbb{Z}[W] \arrow[d] \\ V \arrow[r, "F"'] \arrow[ur, "\widetilde{F}"'] & \mathbb{Z}[X] \end{tikzcd}$$

It is clear that the image of $\widetilde{F}$ is contained in the simplicial subgroup $\mathbb{Z}[W]_0 \leq \mathbb{Z}[W]$ which, by definition, consists of linear combinations $\sum_i n_i w_i$ for which $\sum_i n_i = 0$. By [10, Lemma 9.1] the canonical map $\mathbb{Z}[\underline{\mathrm{Hom}}(V,W)]_0 \to \underline{\mathrm{Hom}}(V, \mathbb{Z}[W]_0)$ is surjective, hence the map $\widetilde{F}$ can be lifted along h_W in such a way that the bottom triangle in following diagram commutes strictly.

$$\begin{tikzcd} U \arrow[r, "0"] \arrow[d, hook, "i_0"'] & W^\infty \arrow[d, "h_W"] \\ V \arrow[r, "\widetilde{F}"'] \arrow[ur, dashed, "G"] & \mathbb{Z}[W] \end{tikzcd}$$

Notice that the image of Gi_0 is contained in the fiber $h_W^{-1}(0)$. It is easy to see that $h_W^{-1}(0)$ is contractible (the contracting homotopy for W^∞ can be restricted to $h_W^{-1}(0)$). We get that Gi_0 is fiberwise homotopic to 0 hence the composite map $V \xrightarrow{G} W^\infty \to X^\infty$ is the desired lifting of F. $\square$

The assertions of the following lemma are straightforward and are given without proof.

Lemma A.3. *Let $p\colon (E,e) \to (B,b)$ be a map of pointed topological spaces. Let $p^{-1}(b) \hookrightarrow F_p(b) \subseteq E \times_B B^I$ be the inclusion map of the fiber of p into the homotopy fiber. Denote k-th relative homotopy group (or set) $\pi_k(F_p(b), p^{-1}(b), e))$ by G_k.*

1. *Maps of triples $(D^k, S^{k-1}, pt) \to (F_p(b), p^{-1}(b), e)$ are in one-to-one correspondence with commuting diagrams of the form*

(A.2)
$$\begin{tikzcd} D^k \arrow[rr, "a"] \arrow[d, hook, "i_0"'] & & E \arrow[d, "p"] \\ D^k \times I \arrow[r, "\pi"] & (D^k \times I)/J \arrow[r, "A"] & B \end{tikzcd}$$

   Here J denotes $(S^{k-1} \times I) \cup (D^k \times \{1\})$. In the sequel we denote such a diagram by (a, A).
2. *Two diagrams (a_0, A_0) and (a_1, A_1) represent the same element of G_k iff there exists a "diagram homotopy" that connects them. By a diagram homotopy we mean a family of maps (a_t, A_t) continously depending on $t \in [0,1]$ such that $pa_t = A_t \pi i_0$ holds for all t.*
3. *Let (a, A) be a diagram and a' be any other map homotopic to a via some homotopy H such that $H(S^{k-1} \times I) \subseteq p^{-1}(b)$. Then there exists a map A' such that (a', A') is a diagram homotopic to (a, A).*

(4) *Assume that in the diagram* (A.2) *there exists a diagonal map* $\widetilde{A}\colon D^k \times I \to E$ *such that the bottom triangle is commutative and the upper is commutative up to a fiberwise homotopy. Then the element of* G_k *given by* (a, A) *is trivial.*

Lemma A.4. *Let* $p\colon E \to B$ *be a map of simplicial sets satisfying the weak covering homotopy property with respect to the class* C_{pr}*. Then the geometric realization of* p *is a quasifibration, i. e. for every point* $b \in B_0$ *the inclusion* $|p|^{-1}(b) \hookrightarrow F_{|p|}(b)$ *is a homotopy equivalence.*

Proof. It suffices to show that $\pi_k(F_{|p|}(b), |p|^{-1}(b))$ are all trivial for $k \geq 1$. Consider a diagram (a, A) of the form (A.2) (with the map $|p|$ in the right hand side). In view of the previous lemma it suffices to construct a lifting $\widetilde{A}$ satisfying the requirements of Lemma A.3.(4).

The idea of the proof is to approximate (a, A) with a homotopic pair of geometric realizations of simplicial maps and then invoke the weak covering homotopy property. The key ingredient in the proof below is the simplicial approximation theorem [6, Theorem 4.7]. For brevity we denote the subdivision functor $\operatorname{sd}^m \operatorname{sd}_*(-)$ from its statement by $s^m(-)$ and the canonical natural transformation $s^m X \to X$ by δ^m.

We proceed in a number of steps. Denote the restriction of a to S^{k-1} by a_0. Notice that the image of a_0 is contained in $|p^{-1}(b)|$. Using the approximation theorem we find a simplicial map $a_0'\colon s^m(S^{k-1}) \to p^{-1}(b)$ such that $a|\delta^m| \cong |a_0'|$.

We can find a map $a'\colon D^k \to |E|$ extending $|a_0'|$ and homotopic to a via some homotopy H satisfying $H(S^k \times I) \subseteq p^{-1}(b)$.

Invoking the approximation theorem once again (with the initial condition specified by $|a_0'|$) we find $q > m$ and $a''\colon s^q(D^k) \to E$ such that $a'|\delta^q| \cong |a''|$ rel S^k. Using Lemma A.3.(3) we extend the homotopy $a|\delta^q| \cong a'|\delta^q| \cong |a''|$ to a homotopy of diagrams $(a|\delta^q|, A|\delta^q|) \cong (|a''|, A')$.

Applying the approximation theorem to A' with the initial condition on the boundary $\partial(D^k \times I) = D^k \times \{0\} \cup J$ specified by pa'' and $const_b$, respectively, we find an integer $r > q$ and a map $A''\colon s^r(D^k \times I) \to B$ such that $A'|\delta^r| \cong |A''|$ rel $\partial(D^k \times I)$.

Thus, we have obtained the following commutative diagram of simplicial sets for which $(|a''|, |A''|)$ is a diagram homotopic to (a, A).

$$\begin{tikzcd}
s^r(D^k) \arrow[r, "a''\gamma^{r-q}"] \arrow[d, hook, "s^r(i)"'] & E \arrow[d, "p"] \\
s^r(D^k \times I) \arrow[r, "A''"'] \arrow[ur, dashed, "\widetilde{A}"'] & B
\end{tikzcd}$$

By functoriality of the subdivision functor the map $s^r(i)$ posesses a deformation retract hence there exists the desired lifting $\widetilde{A}$. $\square$

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