

AMALGAMS OF SYMMETRIC GROUPS AND NONABELIAN TENSOR PRODUCTS

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ABSTRACT. By amalgamating copies of symmetric group S_n we construct a central extension $S_n(G)$ of a large subgroup of the wreath product $G \wr S_n$. We then establish a connection between this group and the theory of $\mathfrak{U}(G)$ and $\mathfrak{H}_n(G)$ -extensions developed by K. Dennis and U. Rehmann in 1970's. We also give a topological application of this construction by describing homotopy invariants of blah-blah-blah...

1. INTRODUCTION

The aim of this note is to establish a surprising connection between symmetric groups and (certain quotients of) nonabelian tensors squares.

Let G be a group. Consider the family of copies of the symmetric group S_n in which each copy is indexed by an n -tuple $(g_1, \dots, g_n) \in G^n$. Denote by $S_n(G)$ the free product of groups from this family amalgamated over the single family of relations $s_{(g)} = s_{(h)}$, where $s \in S_n$, $g, h \in G^n$ are such that hg^{-1} is fixed by s (here $s_{(g)}$ denotes a permutation $s \in S_n$ embedded into the copy with index g).

Consider the wreath product $G \wr S_n = G^n \rtimes S_n$. By the definition of $S_n(G)$ there is a well defined map $\mu_n: S_n(G) \rightarrow G \wr S_n$ given by $s_{(g)} \mapsto s^g = (g^{-1} s g^{s^{-1}}, s)$.

Our first main result is the following theorem.

thm:summary

Theorem 1. *For $n \geq 3$ the map μ_n is a crossed module whose cokernel and kernel are isomorphic to $H_1(G, \mathbb{Z})$ and $H_2(G, \mathbb{Z})$ respectively. Moreover, the group $S_n(G)$ is isomorphic to $S_n \rtimes H_n^\wedge(G)$.*

Here $H_n^\wedge(G)$ denotes the quotient of the universal extension of type $\mathfrak{H}_n(G)$ modulo all symbols of the form $\{x, x\} = 1$, $x \in G$. Theory of extensions of type \mathfrak{H}_n was developed by U. Rehmann in [11] for the purpose of generalization of Matsumoto theorem to skew-fields (we refer the reader to Section 2.4 for more details).

Theorem 1 and its proof (which itself involves little more than some basic combinatorial group theory) may be of interest for the following reasons.

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- The presentation is similar in spirit to results of Brown—Loday and Kassel—Reutenauer on the presentation of the linear Steinberg group in the form of an amalgam of braid groups.
- It gives a presentation of the universal extension $H_n(G)$ with generators and relations which is simpler than the original presentation given in [11], see Proposition 2.9. Moreover, our presentation explicitly obtained from the presentation of S_n formulated in terms of transpositions.

Theorem 1 was proved in an attempt to answer some purely topological question, which we describe in more detail below.

For a pointed simplicial set (X, x_0) denote by $\mathbb{Z}[X]$ the associated free simplicial abelian group. Define the "alternating" map of simplicial sets $h_n: X^{2n} \rightarrow \mathbb{Z}[X]$ by the identity:

$$h_n(x_1, x_2, \dots, x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Denote by X^∞ the colimit of X^n with respect to the natural embeddings $X^n \hookrightarrow X^{n+1}$. Taking colimit of h_n we get the map $h_\infty = \text{colim}_n(h_n): X^\infty \rightarrow \mathbb{Z}[X]$.

Based on a recent result of S. Podkorytov one can deduce that this map is a quasifibration provided X is fibrant and connected. In particular, the natural map $h_\infty^{-1}(0) \rightarrow F_{h_\infty}$ between the fiber and the homotopy fiber of h_∞ over 0 is a weak equivalence, see Lemma A.4.

thm:main

Theorem 2. *If $X = BG$ is the classifying space of a group G then for $n \geq 3$ the natural map $h_n^{-1}(0) \rightarrow F_{h_n}$ induces an isomorphism of fundamental groups $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$.*

The proof of Theorem 2 goes as follows. ???

In the course of the proof we also find a presentation of the universal $\mathfrak{H}_n(G)$ -extension that is simpler than the original presentation introduced by Rehmann in [11], see Proposition 2.9.

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2. THE EXTENSION $S_n(G)$

sec:QnG-def

2.1. The definition of $S_n(G)$. Let G be a group. Recall that the *wreath product* is, by definition, the semidirect product $G^n \rtimes S_n$, in which S_n acts on G^n on the right by permuting its factors.

Consider the family $\{S_n^{(g)}\}_{g \in G^n}$ of isomorphic copies of S_n indexed by elements of G^n and let F be the free product of groups from this family. For $s \in S_n$ and $g \in G^n$ we denote by s_g the image of s in F under the canonical map $S_n^{(g)} \rightarrow F$.

Definition 2.1. We define the group $S_n(G)$ to be the quotient of F modulo the following single family of relations:

$$(2.1) \quad s_g = s_h, \text{ where } s \text{ fixes } hg^{-1}.$$

We continue to denote by s_g the image of the element $s_g \in F$ under the canonical map $F \rightarrow S_n(G)$.

Remark 2.2. Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1}g^{(s^{-1})}, s).$$

therefore the equality $s^g = s^h$ holds in $G \wr S_n$ iff $g^{-1}g^{(s^{-1})} = h^{-1}h^{(s^{-1})}$, or what is the same, iff s fixes hg^{-1} .

The last statement immediately implies that the map $\mu: S_n(G) \rightarrow G \wr S_n$ given by $\mu(s_g) = s^g = (g^{-1}g^{(s^{-1})}, s)$ is well-defined. We will see shortly that μ is, in fact, a *crossed module* in the sense of [2, § 2.2].

Observe from the definition of $S_n(G)$ that there is a split exact sequence.

$$(2.2) \quad 1 \longrightarrow \text{Ker}(\pi) \longrightarrow S_n(G) \xrightleftharpoons[\iota(1)]{\pi} S_n \longrightarrow 1,$$

Here the map $\pi = \pi_{S_n} \circ \mu$ removes subscript g from each s_g and the section $\iota(1)$ sends s to s_1 , where 1 is the identity element of G^n . Thus, if we denote $\text{Ker}(\pi)$ by $HS_n(G)$ we get a decomposition $S_n(G) = HS_n(G) \rtimes S_n$.

2.2. A crossed module structure on $S_n(G)$. Recall from [2, § 2.2] that a *crossed module* is a morphism of groups $\mu: M \rightarrow N$ together with a right action of N on M compatible with the conjugation action of N on itself, i. e.

$$(CM1) \quad \mu(m^n) = \mu(m)^n \text{ for all } n \in N, m \in M,$$

which also satisfies the following identity called *Peiffer identity*:

$$(CM2) \quad m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M.$$

In our situation, we let $G \wr S_n$ act on $S_n(G)$ by

$$(2.3) \quad (s_g)^{(h,t)} = s_{(gh)^t}^t, \text{ for } s, t \in S_n \text{ and } g, h \in G^n.$$

The goal of this subsection is to prove the following.

Proposition 2.3. *For $n \geq 3$ the map $\mu: S_n(G) \rightarrow G \wr S_n$ is a crossed module.*

From the fact that μ is a crossed module one can deduce that $\text{Ker}(\mu)$ is a central subgroup of $S_n(G)$ and $\text{Im}(\mu)$ is a normal subgroup of $G \wr S_n$. It is not hard to show that the group $S_2(G)$ is isomorphic to a free product of copies of S_2 (whose center is always trivial). Therefore, the requirement $n \geq 3$ in the statement of Proposition 2.3 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of $G \wr S_n$ on $S_n(G)$ that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of $S_n(G)$, for which it takes the form:

q:Peiffer-gen}

$$(2.4) \quad t_h^{-1} s_g t_h = s_{(gh^{-1})^t \cdot h} \text{ for all } s, t \in S_n, g, h \in G^n.$$

If we act on both sides of the above formula by $(h^{-1}, 1) \in G \wr S_n$ we obtain the equality $t_1^{-1} s_{gh^{-1}} t_1 = s_{(gh^{-1})^t}^t$. Thus, to prove (2.4) it suffices to show the following simpler relation:

peiffer-simple}

$$(2.5) \quad t_1^{-1} s_g t_1 = s_{g^t}^t \text{ for all } s, t \in S_n, g \in G^n.$$

The key step in the proof is the following lemma.

em:transp-deff

Lemma 2.4. *The relation (2.5) holds in the special case when $s = (ij)$ and $t = (kl)$ are two nonequal transpositions.*

Proof. First of all, we immediately check that (2.5) holds in the special case when t fixes $g' \in G^n$. Indeed, by (2.1) we have $t_1 = t_{g'}$, hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that $l \neq i$ and $l \neq j$. Denote by g' the vector which differs from g only at l -th position, for which we set $g'_l = g_k$. Since the only nontrivial component of $g'g^{-1}$ (resp. $g'g^{-t}$) is located at l -th (resp. k -th) position, it is fixed by s (resp. s^t), hence from (2.1) we conclude that $s_g = s_{g'}$ (resp. $(s^t)_{g'} = (s^t)_{g'^t}$). Finally, since g' is fixed by t , we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g'^t} = s_{g'}^t = (s^t)_{g'^t}. \quad \square$$

Proof of Proposition 2.3. Let us show that (2.5) holds for arbitrary transpositions $s, t \in S_n$. It suffices to consider the case $s = t = (ij)$. After choosing some $k \neq i, j$ and presenting (ij) as $(kj)(ik)(kj)$ we use the lemma:

$$(ij)_1^{-1} (ij)_g (ij)_1 = (ij)_1^{-1} (kj)_g (ik)_g (kj)_g (ij)_1 = (ki)_{g(ij)} (jk)_{g(ij)} (ki)_{g(ij)} = (ij)_{g(ij)}.$$

Proposition now follows by induction on the length of permutations s, t . \square

2.3. An explicit presentation of $S_n(G)$. This presentation turns out to be much more economical than the original definition given in section 2.1. An explicit presentation for the subgroup $HS_n(G)$ is derived from this presentation via Reidemeister-Schreier method.

We start with the following simple lemma which is, in essence, a variant of the standard presentation of a Weyl group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

lm:Snpres

Lemma 2.5. *For $n \geq 3$ The symmetric group S_n admits presentation with transpositions as the set of generators and the following list of defining relations (in every formula distinct letters denote distinct indices):*

{Sym1}	(S1)	$(ij)^2 = 1,$
{Sym2}	(S2)	$(ij)^{(jk)} = (ik),$
{Sym3}	(S3)	$[(ij), (kl)] = 1.$
{Sym0}	(S4)	$(ij) = (ji),$

Now we are ready to formulate the main result of this subsection.

prop:Q-pres

Proposition 2.6. *For $n \geq 3$ and arbitrary group G the group $S_n(G)$ admits presentation with the set of generators $\{(ij)_a \mid i \neq j, 1 \leq i, j, \leq n, a \in G\}$ and the following list of relations (as before, distinct letters denote distinct indices):*

{Q1}	(SG1)	$(ij)_a^2 = 1,$
{Q2}	(SG2)	$(ij)_a^{(jk)_b} = (ik)_{ab},$
{Q3}	(SG3)	$[(ij)_a, (kl)_b] = 1.$
{Q4}	(SG4)	$(ij)_a = (ji)_{a^{-1}}$

Proof. Denote by S' the group from the statement of the proposition. For $1 \leq i \leq n$ and $x \in G$ denote by $x[i]$ the element of G^n whose only nontrivial component equals x and is located in the i -th position.

It is not hard to deduce from the definition of $S_n(G)$ and Lemma 2.4 that the formula $(ij)_a \mapsto (ij)_{a[j]}$ gives a well-defined map $\varphi: S' \rightarrow S_n(G)$.

Now we are going to construct the map $\psi: S_n(G) \rightarrow S'$ in the opposite direction. Using the presentation of S_n given by Lemma 2.5 we define for a fixed $g \in G^n$ the map $\psi_g: S_n \rightarrow S'$ by $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$. It is obvious that ψ_g preserves the defining relations (S1)–(S4) of S_n . It remains to show that the equation $\psi_g(s) = \psi_h(s)$ holds whenever $g, h \in G^n$ and $s \in S_n$ satisfy the requirement of (2.1).

Indeed, if hg^{-1} is fixed by s then for every $1 \leq i \leq n$ we have $(hg^{-1})_i = (hg^{-1})_{s(i)}$, or equivalently $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$. For example, if s is a cycle of length p , i.e.

$s = (i_1, i_2, \dots, i_p)$ with $i_{k+1} = s(i_k)$ we get that

$$\psi_g(s) = \psi_g \left(\prod_{k=1}^{p-1} (i_k, i_{k+1}) \right) = \prod_{k=1}^{p-1} (i_k, i_{k+1})_{g_{i_k}^{-1} g_{i_{k+1}}} = \prod_{k=1}^{p-1} (i_k, i_{k+1})_{h_{i_k}^{-1} h_{i_{k+1}}} = \psi_h(s).$$

The proof for a general s is almost the same. Verification of the fact that ψ and φ are mutually inverse is also immediate. \square

We now obtain a presentation of the subgroup $HS_n(G)$. For every $i \neq j$ and $a \in G$ we define the element $h_{ij}(a) \in HS_n(G)$ as follows:

{eq:h-def}

$$(2.6) \quad h_{ij}(a) = (ij)_a \cdot (ij)_1.$$

It is not hard to show that $h_{ij}(a)$ form a generating set for $HS_n(G)$. In fact, there is an explicit formula how an element of $HS_n(G)$ originally expressed through $(ij)_a$'s can be rewritten in terms of $h_{ij}(a)$. Indeed, if h lies in $HS_n(G)$ and is written as $\prod_{k=1}^N (i_k j_k)_{a_k}$ for some $i_k \neq j_k$ and $a_k \in G$ then it can be rewritten as follows:

{eq:rp}

$$(2.7) \quad h = \prod_{k=1}^N h_{\sigma_k(i_k), \sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1} (i_s j_s) \in S_n.$$

We briefly recall the notion of a *rewriting process* given in [7, § 2.3]. If G is a group presented by generators a_ν and relations $R_\mu(a_\nu)$ and H is its subgroup with a generating set $J_i(a_\nu)$ then a *rewriting process for H* is a function which maps every word u in alphabet a_ν to a word v in alphabet s_i such that u and $v[s_i := J_i]$ define the same element of G whenever u represents an element of H .

With this terminology, the mapping (τ) defined above is a rewriting process for the subgroup $HS_n(G)$. Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [7, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [7, § 2.3]):

- if U and U^* are freely equal words in $(ij)_a$ then $\tau(U)$ and $\tau(U^*)$ are also freely equal words in $h_{ij}(a)$;
- if U_1 and U_2 are two words in $(ij)_a$ which define elements of $HS_n(G)$ then the words $\tau(U_1 U_2)$ and $\tau(U_1) \tau(U_2)$ are equal.

Using these two properties and repeating the arguments used in the proof of [7, Theorem 2.8] one can simplify the generic presentation of $HS_n(G)$ given by [7, Theorem 2.6] and obtain the following.

lm:h-gen

Lemma 2.7. *For $n \geq 3$ the group $HS_n(G)$ admits presentation on the generators $h_{ij}(a)$ with the following two families of defining relations:*

{eq:tau1}

$$(2.7) \quad h_{ij}(a) = \tau((ij)_a \cdot (ij)_1);$$

{eq:tau2}

$$(2.8) \quad \tau(KRK^{-1}) = 1,$$

where R varies over relations of Proposition 2.6 and K is any word in $(ij)_1$, $i \neq j$.

From (2.7) one immediately obtains the equality $h_{ij}(1) = 1$, $i \neq j$. Thus, if we denote by σ the permutation corresponding to a word K and let S_n act on $h_{ij}(a)$'s in the natural way we will get that the word $\tau(KRK^{-1})$ is equivalent to ${}^\sigma\tau(R)$ (modulo relations $h_{ij}(1) = 1$).

Since the relations from Proposition 2.6 are respected by the action of S_n , only relations of the form $\tau(R) = 1$ are, in fact, needed for the presentation of $HS_n(G)$. Writing down what $\tau(R)$ is for each of (SG1)–(SG4) we get the following.

prop:HSpres

Proposition 2.8. *For $n \geq 3$ the group $HS_n(G)$ admits presentation with generators $h_{ij}(a)$ and the following list of relations:*

{H0}

$$(R0) \quad h_{ij}(1) = 1,$$

{H1}

$$(R1) \quad h_{ij}(a)h_{ji}(a) = 1,$$

{H2}

$$(R2) \quad h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab),$$

{H3}

$$(R3) \quad [h_{ij}(a), h_{kl}(b)] = 1,$$

{H4}

$$(R4) \quad h_{ij}(a)^{-1} = h_{ij}(a^{-1}).$$

c:Hnextensions

2.4. Extensions of type $\mathfrak{H}_n(G)$. We start by briefly reviewing the material of § 1–3 of [11]. Let $n \geq 3$ be a natural number. By definition, the group $H_n(G)$ is given by generators $h_{ij}(u)$, $u \in G$, $i \neq j$ and the following relations (cf. [11, H1–H5 of § 2]):

{RH1}

$$(H1) \quad h_{ij}(u)h_{ji}(u) = 1,$$

{RH2}

$$(H2) \quad h_{ij}(u)h_{ki}(u)h_{jk}(u) = 1,$$

{RH3}

$$(H3) \quad h_{ij}(u)h_{ik}(v)h_{ij}(u)^{-1} = h_{ik}(uv)h_{ik}(u)^{-1}, \quad \text{for } j \neq k$$

{RH4}

$$(H4) \quad h_{ij}(u)h_{kj}(v)h_{ij}(u)^{-1} = h_{kj}(vu)h_{kj}(u)^{-1}, \quad \text{for } i \neq k$$

{RH5}

$$(H5) \quad [h_{ij}(u), h_{kl}(v)] = 1.$$

There is a surjective group homomorphism $H_n(G) \twoheadrightarrow D_n(G)$ sending $h_{ij}(g)$ to $d_{ij}(g)$. Here $d_{ij}(g)$ stands for the element of G^n whose i -th component equals g , j -th component equals g^{-1} and all other components are trivial.

By definition, an *extension of type $\mathfrak{H}_n(G)$* is an extension H of $D_n(G)$ which is also a quotient of $H_n(G)$, i. e. the extension that fits into the following diagram.

$$\begin{array}{ccc} H_n(G) & & \\ \downarrow & \searrow & \\ H & \twoheadrightarrow & D_n(G) \end{array}$$

Any $\mathfrak{H}_n(G)$ -extension is central, see [11, Proposition 2.3].

We now recall the notion of an *extension of type $\mathfrak{U}(G)$* introduced by Rehmann in [11, § 1]. Let G be a group. Denote by $U(G)$ the group presented by generators (u, v) , $u, v \in G$ and relations:

$$\boxed{\{U1\}} \quad (U1) \quad ({}^u v, {}^u w)(u, w) = (uv, w),$$

$$\boxed{\{U2\}} \quad (U2) \quad (u, vv)(v, wu)(w, uv) = 1.$$

There is a left action of G on $U(G)$ given by ${}^x(u, v) = (xu, xv)$. Equivalently, $U(G)$ can be defined by the following relations, see [4, p. 8].

$$\boxed{\{U4\}} \quad (2.9) \quad (u, v)(v, u) = 1,$$

$$\boxed{\{U6\}} \quad (2.10) \quad (u, v) {}^v(u, w) = (u, vw),$$

$$\boxed{\{U9\}} \quad (2.11) \quad {}^u(v, w)(w, v) = (u, [v, w]).$$

It is clear from the definition that $U(G)$ is a quotient of the nonabelian tensor square $G \otimes G$ and that the map $(u, v) \mapsto [u, v]$ defines a G -equivariant map $U(G) \twoheadrightarrow [G, G]$. Notice that in the literature one can find alternative notations for $U(G)$ which emphasize its connection with nonabelian tensor products e. g. (G, G) or $G \widetilde{\wedge} G$.

By definition, an *extension of type $\mathfrak{U}(G)$* is simply a quotient of $U(G)$ that fits into the following commutative diagram with G -equivariant arrows.

$$\begin{array}{ccc} U(G) & & \\ \downarrow & \searrow & \\ U & \twoheadrightarrow & [G, G] \end{array}$$

There is a correspondence between extensions of type $\mathfrak{H}_n(G)$ and $\mathfrak{U}(G)$. One can obtain a $\mathfrak{U}(G)$ -extension from given $\mathfrak{H}_n(G)$ -extension via restriction. Indeed, let H be an extension of type $\mathfrak{H}_n(G)$ and k be arbitrary number $1 \leq k \leq n$. Denote by ι_k the inclusion map $[G, G] \rightarrow D_n(G)$ that inserts an element $g \in [G, G]$ at the k -th position. Then the corresponding extension U_H is simply the pull-back of H with respect to ι_k .

In other words, U_H is simply the subgroup of H generated by symbols $c_{kj}(u, v) = h_{kj}(u) \cdot h_{kj}(v) \cdot h_{kj}(vu)^{-1}$ (here j is any index not equal k , $c_{kj}(u, v)$ does not depend

on j , see [11, Proposition 2.1]). It can be shown that U is an extension of type $\mathfrak{U}(G)$ with kernel $\text{Ker}(\pi_U)$ isomorphic to $\text{Ker}(\pi_H)$, see [11, Propositions 2.2–2.3]. In [11, § 3] Rehmann also describes the inverse construction which assign to every $\mathfrak{U}(G)$ -extension $U \twoheadrightarrow [G, G]$ and a number $n \geq 3$ some extension H_U of type $\mathfrak{H}_n(G)$ that fits into the pull-back diagram above.

2.5. A simpler presentation of $H_n(G)$. The aim of this subsection is to obtain a presentation of $H_n(G)$ that is simpler than the original presentation: 4 instead of 5 relations that are also shorter. Our main result is the following.

Proposition 2.9. *For $n \geq 3$ relations (R0)–(R4) imply (H1)–(H5). Moreover, for $n \geq 4$ relations (R0)–(R3) are equivalent to (H1)–(H5).*

For $u, v \in G$ we define the following two symbols:

$$c_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, c'_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies $h_{ij}(1) = 1$ therefore $c_{ij}(u, u^{-1}) = c'_{ij}(u, u^{-1})$. Our goal is to show that (H3) can be omitted from the definition of $H_n(G)$ provided $n \geq 4$.

Lemma 2.10. *If one excludes relation (H3) from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following still remain true:*

- (1) *The elements $c_{ij}(u, u^{-1})$ are central in $H_n(G)$;*
- (2) *One has $c'_{ij}(u, v) = [h_{ij}(u), h_{kj}(v)]$, $k \neq i, j$;*
- (3) *One has $^{h_{ij}(w)}c'_{kj}(u, v) = c'_{kj}(u, w)^{-1}c'_{kj}(u, vw)$, $k \neq i, j$;*
- (4) *One has $^{h_{ij}(w)}c'_{ij}(u, v) = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}$.*

Proof. First one shows using (H4) that $c_{ij}(u, u^{-1})$ centralizes $h_{kj}(v)$ (cf. with the proof of [11, Lemma 2.1(2)]). Since in any group $[a, b] = 1$ implies $[a^{-1}, b] = [a, b^{-1}] = [a^{-1}, b^{-1}] = 1$ we get that $c_{ij}(u, u^{-1})$ also centralizes $h_{kj}(v)^{-1} = h_{jk}(v)$ and $c_{ij}(u^{-1}, u)^{-1} = c_{ji}(u, u^{-1})$ centralizes both $h_{kj}(v)$ and $h_{jk}(v)$. Together with (H2) and (H5) this implies that $c_{ij}(u, u^{-1})$ centralizes all the generators of $H_n(G)$ and hence lies in the center of $H_n(G)$.

The second and third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$$^{h_{ij}(w)}c'_{ij}(u, v) = ^{h_{ij}(w)}c'_{kj}(v, u)^{-1} = (c'_{kj}(v, w)^{-1}c'_{kj}(v, uw))^{-1} = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}.$$

□

Lemma 2.11. *If one excludes relation (H3) from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following statements are equivalent:*

- (1) (H3) holds;
- (2) one has $c'_{ij}(u, v)^{-1} = c'_{ij}(v, u)$;

- | | |
|-------|--|
| item5 | (3) symbols $c'_{ij}(u, v)$ do not depend on i ; |
| item5 | (4) one has $c'_{ij}(u, vw) = c'_{ij}(wu, v) \cdot c'_{ij}(uv, w)$; |
| item6 | (5) one has $c'_{ij}(u, vu^{-1}) = c'_{ij}(uv, u^{-1})$. |

Proof. Implications (1) \implies (2) \implies (3) \implies (4) are essentially contained in the proof of [11, Lemmas 2.1-2.2]. Implication (4) \implies (5) is trivial.

We now prove (5) \implies (1). Notice that (H4) implies $^{h_{ij}(u)^{-1}}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$, therefore using Lemma 2.10.(1) we get that $^{h_{ij}(u)}h_{ik}(v) = c_{ij}(u, u^{-1})^{h_{ij}(u^{-1})^{-1}}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$. Thus, (H3) is equivalent to the equality

$$h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}),$$

or what is the same

$$c'_{ik}(u^{-1}, uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1}, u).$$

It is clear that the last equality is an equivalent reformulation of (5). \square

Proof of Proposition 2.9. The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);
- for $n \geq 4$ the third statement of the previous lemma follows from Lemma 2.10.(2).

\square

cor:main

Corollary 2.12. *For $n \geq 3$ the group $HS_n(G)$ is the quotient of the extension $H_n(G)$ by (R4) and, in particular, is an $\mathfrak{H}_n(G)$ -extension. The associated $\mathfrak{U}(G)$ extension $U_{HS_n(G)}$ is isomorphic to the quotient of $U(G)$ modulo single family of relations*

$$(u, u) = 1, \quad u \in G.$$

In particular, we have the following natural isomorphism:

$$(2.12) \quad \text{Ker}(HS_n(G) \rightarrow D_n(G)) = \text{Ker}(S_n(G) \rightarrow G \wr S_n) \cong H_2(G, \mathbb{Z}).$$

Proof. Most of the assertions follow from Proposition 2.9 and [4, Proposition 5]. One also has to use the formula $c_{ij}(u, u) = c_{ij}(u, u^{-1})$ (see [11, p. 87]). \square

sec:main

3. TOPOLOGICAL APPLICATIONS

In this section we prove topological claims made in the introduction.

3.1. Proof of main results. Recall that for arbitrary set X one defines the space EX as the simplicial set whose set of k -simplices EX_k is X^{k+1} and whose faces and degeneracies are obtained by omitting and repeating components. For a group G we denote by π_G the canonical map $EG \rightarrow BG$ sending $(g, h) \in EG_1$ to $g^{-1}h \in BG_1$.

Now let N be a group acting on X . We define two simplicial sets U and V as follows:

$$U = \bigcup_{n \in N} E(\Gamma_{n,-}) \subseteq E(X \times X), \quad V = \bigcup_{x, y \in X} E(N(x \rightarrow y)) \subseteq EN.$$

Here $\Gamma_{n,-}$ is the graph of the function $(x \mapsto nx)$ and $N(x \rightarrow y)$ denotes the subset of elements $n \in N$ satisfying $nx = y$. With this notation the subset $N(x \rightarrow x)$ coincides with the stabilizer subgroup $N_x \leq N$.

lm:quillen-a

Lemma 3.1. *The simplicial sets U and V are homotopy equivalent.*

Proof. First, we define yet another simplicial set W as follows. Its k -simplices W_k are matrices $\begin{pmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{pmatrix}$, where $x_i \in X$ are $n_i \in N$ are such that all n_i 's act each x_j in the same way, i.e. $n_i x_j = n_{i'} x_j$ for $0 \leq i, i', j \leq k$. The faces and degeneracies of W are the maps of omission and repetition of columns.

Now there are two simplicial maps $f: W \rightarrow U$, $g: W \rightarrow V$ whose action on 0-simplices is given by $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = (x_0, n_0 x_0)$, $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = n_0$. To prove the lemma it suffices to show that f and g are homotopy equivalences. The proof for f and g is similar, let us show, for example, that g is a homotopy equivalence.

In view of Quillen theorem A (cf.[13, ex. IV.3.11]) it suffices to show that for each p -simplex $d: \Delta^p \rightarrow V$ the pullback $g/(p, d)$ of d and g is contractible. The simplicial set $g/(p, d)$ can be interpreted as the subset of $\Delta^p \times E(X)$ whose set of k -simplices consists of pairs $(\alpha: \underline{k} \rightarrow \underline{p}, (x_0, \dots, x_k) \in X(\alpha, d)^{k+1})$. Here $X(\alpha, d)$ is the subset of X consisting of all x for which $d_{\alpha(i)}x = d_{\alpha(j)}x$ for $0 \leq i, j \leq k$. Notice that the set $X_d := X(id_{\underline{p}}, d)$ is nonempty and is contained in every $X(\alpha, d)$ (it even equals $X(\alpha, d)$ for surjective α). Now choose a point $\tilde{x} \in X_d$ and consider the simplicial homotopy

$$H: \Delta^p \times EX \times \Delta^1 \rightarrow \Delta^p \times EX$$

between the identity map of $\Delta^p \times EX$ and the map $\Delta^p \times c_{\tilde{x}}$, where $c_{\tilde{x}}$ is the constant map. More concretely, H sends each triple $(\alpha: \underline{k} \rightarrow \underline{p}, (x_0, \dots, x_k), \beta: \underline{k} \rightarrow \underline{1})$ to $(\alpha, (x_0, \dots, x_{i-1}, \tilde{x}, \dots, \tilde{x}))$, where i is the minimal number such that $\beta(i) = 1$. By the choice of \tilde{x} the image of H restricted to $g/(p, d) \times \Delta^1$ is contained in $g/(p, d)$, hence $g/(p, d)$ is contractible. \square

Now suppose that $X = H$ is also a group upon which N acts on the left.

cor:ker-iso**Corollary 3.2.** *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

There is an isomorphism $\text{Ker}(\pi_1(S) \rightarrow N) \cong \text{Ker}(\pi_1(T) \rightarrow H \times H)$. Moreover, the higher homotopy groups of S and T are isomorphic.

Proof. Consider the following two pull-back squares:

$$\begin{array}{ccc} V \hookrightarrow EN & & U \hookrightarrow E(H \times H) \\ \downarrow \lrcorner & \downarrow \pi_N & \downarrow \lrcorner \\ S \hookrightarrow BN & & T \hookrightarrow B(H \times H) \end{array}$$

The required isomorphism can be obtained from the homotopy long exact sequence applied to the left arrows of these diagrams. \square

Now let G be a group. Set $N = G \wr S_n$, $H = G^n$ and consider the left action of N on H given by $(g, s) \cdot h = gh^{s^{-1}}$, $g, h \in G^n$, $s \in S_n$. If one reorders the components of BG^{2n} accordingly, the simplicial subset $T \subset BG^{2n}$ from the above corollary becomes precisely the preimage of 0 under $h_n: BG^{2n} \rightarrow \mathbb{Z}[BG]$.

It is also easy to compute the map $\pi_1(S) \rightarrow N$. Indeed, van Kampen theorem [8, Theorem 2.7] asserts that $\pi_1(S)$ is isomorphic to the free product of stabilizer subgroups $N_h \leq N$ amalgamated over pairwise intersections $N_h \cap N_{h'}$, $h, h' \in H$. For $h \in G^n$ the subgroup N_h consists of elements $(g, s) \in N$ satisfying $gh^{s^{-1}} = h$, i.e. elements of the form $(hh^{-s^{-1}}, s)$. Thus, $N_h \cong S_n$, $\pi_1(S)$ is isomorphic to the group $S_n(G)$ and the map $\pi_1(S) \rightarrow N$ coincides with the map μ defined in section 2.1.

We would like to have a more concrete description of the isomorphism of Corollary 3.2. Fix a vector $g \in G^n$ and denote by $\gamma_{ij}(h, x)$ the path of length 4 in W connecting the points $(\begin{smallmatrix} x_i \\ n_i \end{smallmatrix})$, $i = 0..4$ whose components are given as follows:

$$\begin{aligned} x_0 &= e, & x_1 &= x[i], & x_2 &= x[i], & x_3 &= e, & x_4 &= e; \\ n_0 &= (h; e), & n_1 &= (h, e), & n_2 &= (d_{ij}(x)h, (ij)), & n_3 &= (d_{ij}(x)h, (ij)), & n_4 &= (d_{ij}(x)h, e). \end{aligned}$$

One can easily check that the image $\gamma_{ij}(h, x)$ in S under $\pi_N g$ is precisely the element $h_{ij}(x)$ while $\pi_{H \times H} f$ sends $\gamma_{ij}(h, x)$ to the loop $p_{ij}(x) = (x[i], x[i]) \circ (x^{-1}[i], x^{-1}[j])$. Thus, we conclude that every generator $c_{ij}(u, v)$ of $\text{Ker}(\pi_1(S) \rightarrow N)$ corresponds to the loop $p_{ij}(u) \circ p_{ij}(v) \circ p_{ij}(u^{-1}v^{-1})$ in T .

Proof of Theorem 2. Factor h_n as a composition of a trivial cofibration followed by a fibration:

$$(3.1) \quad BG^{2n} \hookrightarrow E_{h_n} \xrightarrow{ev_1 \circ \pi_2} \mathbb{Z}[BG]$$

{eq:fibr-repl}

For example, we can define E_{h_n} and the homotopy fiber F_{h_n} via the usual path space construction (here we use the fact that BG and $\mathbb{Z}[BG]$ are fibrant).

$$\begin{array}{ccc} E_{h_n} & \xrightarrow{\pi_2} & \mathbb{Z}[BG]^I \\ \pi_1 \downarrow & & \downarrow ev_0 \\ BG^{2n} & \xrightarrow{h_n} & \mathbb{Z}[BG] \end{array} \quad \begin{array}{ccc} F_{h_n} & \xrightarrow{\quad} & E_{h_n} \\ \downarrow & & \downarrow ev_1 \circ \pi_2 \\ pt & \xrightarrow{0} & \mathbb{Z}[BG] \end{array}$$

Now write down the starting portion of the long homotopy exact sequence of the fibration $E_{h_n} \twoheadrightarrow \mathbb{Z}[BG]$ and denote by K the kernel of the map $\nu: \pi_1(T) \rightarrow G^{2n}$ induced by the embedding $h_n^{-1}(0) = T \subseteq BG^{2n}$. We come to the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & \pi_1(T) & \xrightarrow{\nu} & G^{2n} & \longrightarrow & H_1(G, \mathbb{Z}) \longrightarrow 1 \\ & & \psi \downarrow & & \phi \downarrow & & \cong \downarrow & & \parallel \\ 1 & \longrightarrow & H_2(G, \mathbb{Z}) & \longrightarrow & \pi_1(F_{h_n}) & \longrightarrow & \pi_1(E_{h_n}) & \longrightarrow & H_1(G, \mathbb{Z}) \longrightarrow 1 \end{array}$$

We already know by Corollaries 2.12 and 3.2 that K is naturally isomorphic to $H_2(G, \mathbb{Z})$ provided $n \geq 3$.

Let us show that ψ is an isomorphism. Assume for a moment that G is an abelian group. In this case a generator $x \wedge y$ of $H_2(G, \mathbb{Z})$ in Miller's presentation corresponds to the class of 2-cycle $c = (x, y) - (y, x)$ (cf. [9, (14), p. 582]), which, in turn, corresponds to the following normalized 2-cycle:

eq:normalized}

$$(3.2) \quad c' = c - s_0 d_0 c - s_1 d_1 c + s_1 d_0 = (x, y) - (y, x) - (x, 0) + (y, 0) + (0, x) - (0, y).$$

Since F_{h_n} and $\mathbb{Z}[BG]$ are fibrant, the elements of $\pi_1(F_{h_n}, 0)$ can be described as classes of pairs $(g, s) \in G^{2n} \times \mathbb{Z}[BG]_2$ satisfying $d_2(s) = h_n(g)$ up to natural homotopy relation. It is easy to check that the image of the loop $p_{ij}(x)$ under the natural map $T \rightarrow F_{h_n}$ corresponds to the class of $((e, d_{ij}(x)), (x, x^{-1}) + (e, e) - (x, e) - (e, x^{-1}))$.

It is not hard to describe the group structure of $\pi_1(F_{h_n}, 0)$. Indeed, if $[(g^1, s^1)]$ and $[(g^2, s^2)]$ are two elements of $\pi_1(F_{h_n}, 0)$, then their product is represented by the pair $(g^1 g^2, s)$, where s is the 1-st face of the filler f for the 3-horn $(s^2, s^1, h_n(g^1, g^2))$.

Now if (p_0, p_2, p_3) is a horn then by Moore's theorem we get the following formula for the filler: $f = s_0 p_0 - s_2 s_0 d_2 p_0 - s_1 s_0 d_1 p_0 + s_1 s_0 d_2 p_0 + s_1 p_2 + s_2 p_3 - s_1 p_3$. Substituting $p_0 + p_2 - p_3 - s_1 d_2 p_0 + s_1 d_1 p_3 + s_0 d_2 p_0$

We use the following inductive procedure. Let γ_0 be an initial segment of a path γ in T for which we have already found such simplex (g^0, s^0) in F_{h_n} and let $g \in T_1$ be the next chain link. Set $s_0 = \sum (a_j, b_j) \in \mathbb{Z}[BG]_2$, $t = h_n(g^0, g) = \sum_{i=1}^{2n} (-1)^i (g_i^0, g_i)$ and compute the filler for the 3-horn $(0, s_0, t)$. Denote the 1-st face of this filler by

s_1 . It is clear that $(g^0 g, s_1)$ is the image of $\gamma^0 \circ g$ via ϕ . The concrete formula for s_1 can be chosen e. g. as follows:

$$s_1 = \sum_{i=1}^{2n} (g_i^0 + g_i, -g_i) - \sum_j (a_j + b_j, -b_j).$$

Applying the above recipe to the path $\gamma_{x,y}$ we get an expression equal to (3.2). This shows that ψ is the identity map for an abelian group G . Since ψ is natural in G , by Lemma 3.3 below we get that ψ is an isomorphism for arbitrary G , therefore ϕ is also an isomorphism, as claimed. \square

lm:endotr

Lemma 3.3. *The only natural endotransformations of the second homology functor $H_2(-, \mathbb{Z}): \mathbf{Groups} \rightarrow \mathbf{Ab}$ are morphisms of multiplication by $n \in \mathbb{Z}$.*

Proof. Denote by η an endotransformation $H_2(-, \mathbb{Z}) \rightarrow H_2(-, \mathbb{Z})$. When restricted to the subcategory of free finitely-generated abelian groups $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$ the second homology functor coincides with the second exterior power functor $A \mapsto \wedge^2 A$.

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic \mathbb{Z} -modules (see Definition 6.13.5 *ibid.*) The functor $A \mapsto \wedge^2 A$ is clearly quadratic and corresponds to the quadratic \mathbb{Z} -module $0 \rightarrow \mathbb{Z} \rightarrow 0$ under this equivalence. Thus, we get that η restricted to $\mathbf{Add}(\mathbb{Z})$ coincides with the morphism of multiplication by $n \in \mathbb{Z}$.

Consider the group $\Gamma_k = \langle x_1, y_1, \dots, x_k, y_k \mid [x_1, y_1] \cdot \dots \cdot [x_k, y_k] \rangle$ (the fundamental group of a sphere with k handles). It is clear that the abelianization map $\Gamma_k \rightarrow \mathbb{Z}^{2k}$ induces an injective map $H_2(\Gamma_k, \mathbb{Z}) \cong \mathbb{Z} \rightarrow \wedge^2 \mathbb{Z}^{2k}$. Consider the following diagrams.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad} & \wedge^2 \mathbb{Z}^{2k} \\ \eta_{\Gamma_k} \downarrow & & \downarrow n \cdot \\ \mathbb{Z} & \xrightarrow{\quad} & \wedge^2 \mathbb{Z}^{2k} \end{array} \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\chi} & H_2(G, \mathbb{Z}) \\ \eta_{\Gamma_k} \downarrow & & \downarrow \eta_G \\ \mathbb{Z} & \xrightarrow{\chi} & H_2(G, \mathbb{Z}) \end{array}$$

From the left diagram it follows that η_{Γ_k} is also the morphism of multiplication by n . For every element $x \in H_2(G, \mathbb{Z})$ there exist an integer k and a map $\chi: \Gamma_k \rightarrow G$ sending the generator of $H_2(\Gamma_k, \mathbb{Z})$ to x . From the right square we conclude that $\eta_G(x) = nx$, as claimed. \square

APPENDIX A. SIMPLICIAL DOLD—SERRE FIBRATIONS

The aim of this appendix is to show that the map $h_\infty: X^\infty \rightarrow \mathbb{Z}[X]$ is a quasifibration under suitable assumptions on X .

We start with the definition of the simplicial analogue of the so-called "Dold—Serre fibration", i. e. a map possessing "weak covering homotopy property" (also called "delayed homotopy lifting property"), cf. [12, § 1.4.4]. We show in Lemmas A.2

and A.4 that the map h_∞ satisfies this property and that every map with this property is a quasifibration.

df:dhlp

Definition A.1. Let C be a subclass of the class of acyclic cofibrations in the category of simplicial sets. We say that a map of simplicial sets $p: E \rightarrow B$ satisfies *weak homotopy covering property* with respect to C if for every commutative square

$$(A.1) \quad \begin{array}{ccc} U & \xrightarrow{g} & E \\ i \downarrow & \tilde{F} \nearrow & \downarrow p \\ V & \xrightarrow{F} & B \end{array}$$

for which $i \in C$ there exists a map \tilde{F} such that the lower triangle commutes strictly and the upper one commutes up to a fiberwise homotopy H (i.e. a homotopy H such that $pH: U \times I \rightarrow B$ coincides with the composite $U \times I \xrightarrow{\pi} U \xrightarrow{pg} B$).

Denote by C_{pr} the class consisting of all inclusions $i: U \hookrightarrow V$ of finite polyhedral simplicial sets for which there exists a deformation retract (i.e. a map $r: V \rightarrow U$ homotopic to id_V).

lm:dhlp

Lemma A.2. For a fibrant and connected simplicial set X the map $h_\infty: X^\infty \rightarrow \mathbb{Z}[X]$ satisfies the weak covering homotopy property with respect to the class C_{pr} .

Proof. Let i_0 be a map from C_{pr} and r be the corresponding retraction. We first reduce the problem to the special case when g is the constant map at the basepoint of X^∞ (which we denote by 0). Indeed, let F, g be arbitrary maps as in (A.1). Since U is finite the image of g is contained in $X^{2N} \subset X^\infty$. If now \tilde{F}_0 is a lifting in the diagram similar to (A.1) in which the top map equals 0 and the bottom one is $F_0 = F - h_\infty gr$ then $\tilde{F} = gr \times \tilde{F}_0$ is the desired lifting of F .

Now choose a contractible fibrant simplicial set W which maps surjectively onto X (e.g. take W to be the path space fibration). Since $p: W \rightarrow X$ is surjective, the associated map between free simplicial abelian groups is a Kan fibration. Since i_0 is acyclic we can choose a lifting \tilde{F} in the following diagram

$$\begin{array}{ccc} U & \xrightarrow{0} & \mathbb{Z}[W] \\ i_0 \downarrow & \tilde{F} \nearrow & \downarrow \\ V & \xrightarrow{F} & \mathbb{Z}[X] \end{array}$$

It is clear that the image of \tilde{F} is contained in the simplicial subgroup $\mathbb{Z}[W]_0 \leq \mathbb{Z}[W]$ which, by definition, consists of linear combinations $\sum_i n_i w_i$ for which $\sum_i n_i = 0$. By [10, Lemma 9.1] the canonical map $\mathbb{Z}[\underline{\text{Hom}}(V, W)]_0 \rightarrow \underline{\text{Hom}}(V, \mathbb{Z}[W]_0)$ is surjective,

hence the map \tilde{F} can be lifted along h_W in such a way that the bottom triangle in following diagram commutes strictly.

$$\begin{array}{ccc} U & \xrightarrow{0} & W^\infty \\ i_0 \downarrow & \nearrow G & \downarrow h_W \\ V & \xrightarrow{\tilde{F}} & \mathbb{Z}[W] \end{array}$$

Notice that the image of Gi_0 is contained in the fiber $h_W^{-1}(0)$. It is easy to see that $h_W^{-1}(0)$ is contractible (the contracting homotopy for W^∞ can be restricted to $h_W^{-1}(0)$). We get that Gi_0 is fiberwise homotopic to 0 hence the composite map $V \xrightarrow{G} W^\infty \rightarrow X^\infty$ is the desired lifting of F . \square

The assertions of the following lemma are straightforward and are given without proof.

lem:topo-facts

Lemma A.3. *Let $p: (E, e) \rightarrow (B, b)$ be a map of pointed topological spaces. Let $p^{-1}(b) \hookrightarrow F_p(b) \subseteq E \times_B B^I$ be the inclusion map of the fiber of p into the homotopy fiber. Denote k -th relative homotopy group (or set) $\pi_k(F_p(b), p^{-1}(b), e)$ by G_k .*

- (1) *Maps of triples $(D^k, S^{k-1}, pt) \rightarrow (F_p(b), p^{-1}(b), e)$ are in one-to-one correspondence with commuting diagrams of the form*

{eq:plp}

(A.2)

$$\begin{array}{ccc} D^k & \xrightarrow{\quad a \quad} & E \\ i_0 \downarrow & & \downarrow p \\ D^k \times I & \xrightarrow{\quad \pi \quad} (D^k \times I)/J \xrightarrow{\quad A \quad} & B \end{array}$$

Here J denotes $(S^{k-1} \times I) \cup (D^k \times \{1\})$. In the sequel we denote such a diagram by (a, A) .

- (2) *Two diagrams (a_0, A_0) and (a_1, A_1) represent the same element of G_k iff there exists a "diagram homotopy" that connects them. By a diagram homotopy we mean a family of maps (a_t, A_t) continuously depending on $t \in [0, 1]$ such that $pa_t = A_t\pi i_0$ holds for all t .*
- (3) *Let (a, A) be a diagram and a' be any other map homotopic to a via some homotopy H such that $H(S^{k-1} \times I) \subseteq p^{-1}(b)$. Then there exists a map A' such that (a', A') is a diagram homotopic to (a, A) .*
- (4) *Assume that in the diagram (A.2) there exists a diagonal map $\tilde{A}: D^k \times I \rightarrow E$ such that the bottom triangle is commutative and the upper is commutative up to a fiberwise homotopy. Then the element of G_k given by (a, A) is trivial.*

item:continue

item:weaker

lm:weak-equiv

Lemma A.4. *Let $p: E \rightarrow B$ be a map of simplicial sets satisfying the weak covering homotopy property with respect to the class C_{pr} . Then the geometric realization of p*

is a quasifibration, i. e. for every point $b \in B_0$ the inclusion $|p|^{-1}(b) \hookrightarrow F_{|p|}(b)$ is a weak equivalence.

Proof. We want to prove that $\pi_k(F_{|p|}(b), |p|^{-1}(b))$ are all trivial for $k \geq 1$. Clearly, it suffices to show that for every diagram (a, A) of the form (A.2) the map $|p|$ has a lifting \tilde{A} satisfying the requirements of Lemma A.3.(4).

The idea of the proof is to approximate (a, A) with a homotopic pair of geometric realizations of simplicial maps and then invoke the weak covering homotopy property. The key ingredient in the proof below is the simplicial approximation theorem [5, Theorem 4.7]. For brevity we denote the subdivision functor $\text{sd}^m \text{sd}_*(-)$ from its statement by $s^m(-)$ and the canonical natural transformation $s^m X \rightarrow X$ by δ^m . We proceed in a number of steps.

- Denote the restriction of a to S^{k-1} by a_0 . Notice that the image of a_0 is contained in $|p|^{-1}(b)$. Using the approximation theorem we find a simplicial map $a'_0: s^m(S^{k-1}) \rightarrow p^{-1}(b)$ such that $a|\delta^m| \cong |a'_0|$.
- We can find a map $a': D^k \rightarrow |E|$ extending $|a'_0|$ and homotopic to a via some homotopy H satisfying $H(S^k \times I) \subseteq p^{-1}(b)$.
- Invoking the approximation theorem once again (with the initial condition specified by $|a'_0|$) we find $q > m$ and $a'': s^q(D^k) \rightarrow E$ such that $a'|\delta^q| \cong |a''|$ rel S^k .
- Using Lemma A.3.(3) we extend the homotopy $a|\delta^q| \cong a'|\delta^q| \cong |a''|$ to a homotopy of diagrams $(a|\delta^q|, A|\delta^q|) \cong (|a''|, A')$.
- Applying the approximation theorem to A' with the initial condition on the boundary $\partial(D^k \times I) = D^k \times \{0\} \cup J$ specified by pa'' and const_b , respectively, we find an integer $r > q$ and a map $A'': s^r(D^k \times I) \rightarrow B$ such that $A'|\delta^r| \cong |A''|$ rel $\partial(D^k \times I)$.

Thus, we have obtained the following commutative diagram of simplicial sets for which $(|a''|, |A''|)$ is a diagram homotopic to (a, A) .

$$\begin{array}{ccc}
 s^r(D^k) & \xrightarrow{a''\gamma^{r-q}} & E \\
 s^r(i) \downarrow & \nearrow \tilde{A} & \downarrow p \\
 s^r(D^k \times I) & \xrightarrow{A''} & B
 \end{array}$$

By functoriality of the subdivision functor the map $s^r(i)$ possesses a deformation retract hence there exists the desired lifting \tilde{A} . \square

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