A NOTE ON AMALGAMS OF SYMMETRIC GROUPS AND $\mathfrak{H}_n$-EXTENSIONS

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Abstract. By amalgamating copies of symmetric group S_n we construct a central extension $S_n(G)$ of a large subgroup of the wreath product $G \wr S_n$. We then establish a connection between this group and the theory of $\mathfrak{U}(G)$ and $\mathfrak{H}_n(G)$-extensions developed by K. Dennis and U. Rehmann in 1970's. We also give a topological application of this construction by describing the fundamental group of the fiber of the "alternating" map $BG^{2n} \to \mathbb{Z}[BG]$.

1. Introduction

$\mathfrak{H}_n$-extensions are certain extensions of a certain subgroup of the direct product G^n. For the first time they have appeared in U. Rehmann's paper [10], where they are used as a noncommutative analogues in the proof of the noncommutative version of Matsumoto's theorem.

Discovery of noncommutative Steinberg symbols in the proof of the latter led to the definition of noncommutative tensor and exterior products in [4].

For a pointed simplicial set (X, x_0) one denotes by $\mathbb{Z}[X]$ the associated free simplicial abelian group. Define the map of simplicial sets $h_n \colon X^{2n} \to \mathbb{Z}[X]$ by the identity:

$$h_n(x_1, x_2, \ldots x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Denote by X^∞ the colimit of X^n with respect to the natural embeddings $X^n \hookrightarrow X^{n+1}$ Taking colimit of h_n we get the map $h_\infty = \mathrm{colim}_n(h_n) \colon X^\infty \to \mathbb{Z}[X]$.

Based on a recent result of S. Podkorytov one can deduce that the natural map $h_\infty^{-1}(0) \to F_{h_\infty}$ between the fiber and the homotopy fiber of h_∞ over 0 is a weak equivalence provided X is fibrant and connected, see Lemma 4.1. For finite n this no longer remains true, however one may expect that the homotopy groups of $h_n^{-1}(0)$ and F_{h_n} are still isomorphic in small degrees. S. Podkorytov conjectured that the natural map $h_n^{-1}(0) \to F_{h_n}$ is c_n-connected where c_n tends to infinity as $n \to \infty$. The

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main goal of this note is to show the following result which confirms this conjecture in one special case.

thm:main **Theorem 1.** *If $X = BG$ is the classifying space of a group G then for $n \geq 3$ the natural map $h_n^{-1}(0) \to F_{h_n}$ induces an isomorphism of fundamental groups $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$.*

The proof of Theorem 1 is interesting on its own and goes as follows. Using Quillen theorem A and van Kampen theorem we reduce the original question to some purely group-theoretic problem of the description of the extension $S_n(G)$, which is defined as a certain amalgamated product of copies of the symmetric group S_n. It turns out that this extension can be naturally described in terms of the theory of $\mathfrak{U}(G)$ and $\mathfrak{H}_n(G)$-extensions developed by K. Dennis and U. Rehmann in 1970's.

In the course of the proof we also find a presentation of the universal $\mathfrak{H}_n(G)$-extension that is simpler than the original presentation introduced by Rehmann in [10], see Proposition 3.1.

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2. The extension $S_n(G)$

sec:QnG-def

2.1. **The definition of $S_n(G)$.** Let G be a group. Recall that the *wreath product* is, by definition, the semidirect product $G^n \rtimes S_n$, in which S_n acts on G^n on the right by permuting its factors.

Consider the family $\{{S_n}^{(g)}\}_{g\in G^n}$ of isomorphic copies of S_n indexed by elements of G^n and let F be the free product of groups from this family. For $s \in S_n$ and $g \in G^n$ we denote by s_g the image of s in F under the canonical map $S_n^{(g)} \to F$.

Definition 2.1. We define the group $S_n(G)$ to be the quotient of F modulo the following single family of relations:

{eq:main_rel}

$$s_g = s_h, \text{ where } s \text{ fixes } hg^{-1}. \tag{2.1}$$

We continue to denote by s_g the image of the element $s_g \in F$ under the canonical map $F \to S_n(G)$.

Remark 2.2. Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1}g^{(s^{-1})}, s).$$

therefore the equality $s^g = s^h$ holds in $G \wr S_n$ iff $g^{-1}g^{(s^{-1})} = h^{-1}h^{(s^{-1})}$, or what is the same, iff s fixes hg^{-1}.

The last statement immediately implies that the map $\mu\colon S_n(G) \to G \wr S_n$ given by $\mu(s_g) = s^g = (g^{-1}g^{(s^{-1})}, s)$ is well-defined. We will see shortly that μ is, in fact, a *crossed module* in the sense of [2, § 2.2].

Observe from the definition of $S_n(G)$ that there is a split exact sequence.

{eq:ex-seq} (2.2)
$$1 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow S_n(G) \underset{\iota(1)}{\overset{\pi}{\rightleftarrows}} S_n \longrightarrow 1,$$

Here the map $\pi = \pi_{S_n} \circ \mu$ removes subscript g from each s_g and the section $\iota(1)$ sends s to s_1, where 1 is the identity element of G^n. Thus, if we denote $\mathrm{Ker}(\pi)$ by $HS_n(G)$ we get a decomposition $S_n(G) = HS_n(G) \rtimes S_n$.

2.2. **A crossed module structure on $S_n(G)$.** Recall from [2, § 2.2] that a *crossed module* is a morphism of groups $\mu\colon M \to N$ together with a right action of N on M compatible with the conjugation action of N on itself, i. e.

eq:precrossed} (CM1)
$$\mu(m^n) = \mu(m)^n \text{ for all } n \in N,\ m \in M,$$

which also satisfies the following identity called *Peiffer identity*:

{eq:Peiffer} (CM2)
$$m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M.$$

In our situation, we let $G \wr S_n$ act on $S_n(G)$ by

{eq:action} (2.3)
$$(s_g)^{(h,t)} = s^t{}_{(gh)^t}, \text{ for } s, t \in S_n \text{ and } g, h \in G^n.$$

The goal of this subsection is to prove the following.

thm:cms **Proposition 2.3.** *For $n \geq 3$ the map $\mu\colon S_n(G) \to G \wr S_n$ is a crossed module.*

From the fact that μ is a crossed module one can deduce that $\mathrm{Ker}(\mu)$ is a central subgroup of $S_n(G)$ and $\mathrm{Im}(\mu)$ is a normal subgroup of $G \wr S_n$. It is not hard to show that the group $S_2(G)$ is isomorphic to a free product of copies of S_2 (whose center is always trivial). Therefore, the requirement $n \geq 3$ in the statement of Proposition 2.3 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of $G \wr S_n$ on $S_n(G)$ that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of $S_n(G)$, for which it takes the form:

q:Peiffer-gen} (2.4)
$$t_h^{-1} s_g t_h = s^t{}_{(gh^{-1})^t \cdot h} \text{ for all } s, t \in S_n,\ g, h \in G^n.$$

If we act on both sides of the above formula by $(h^{-1}, 1) \in G \wr S_n$ we obtain the equality $t_1^{-1} s_{gh^{-1}} t_1 = s^t{}_{(gh^{-1})^t}$. Thus, to prove (2.4) it suffices to show the following simpler relation:

eiffer-simple} (2.5)
$$t_1^{-1} s_g t_1 = s^t{}_{g^t} \text{ for all } s, t \in S_n,\ g \in G^n.$$

The key step in the proof is the following lemma.

em:transp-deff **Lemma 2.4.** *The relation* (2.5) *holds in the special case when* $s = (ij)$ *and* $t = (kl)$ *are two nonequal transpositions.*

Proof. First of all, we immediately check that (2.5) holds in the special case when t fixes $g' \in G^n$. Indeed, by (2.1) we have $t_1 = t_{g'}$, hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that $l \neq i$ and $l \neq j$. Denote by g' the vector which differs from g only at l-th position, for which we set $g'_l = g_k$. Since the only nontrivial component of $g'g^{-1}$ (resp. $g'g^{-t}$) is located at l-th (resp. k-th) position, it is fixed by s (resp. s^t), hence from (2.1) we conclude that $s_g = s_{g'}$ (resp. $(s^t)_{g'} = (s^t)_{g^t}$). Finally, since g' is fixed by t, we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g'^t} = s^t_{g'} = (s^t)_{g^t}. \quad \square$$

Proof of Proposition 2.3. Let us show that (2.5) holds for arbitrary transpositions $s, t \in S_n$. It suffices to consider the case $s = t = (ij)$. After choosing some $k \neq i, j$ and presenting (ij) as $(kj)(ik)(kj)$ we use the lemma:

$$(ij)_1^{-1}(ij)_g(ij)_1 = (ij)_1^{-1}(kj)_g(ik)_g(kj)_g(ij)_1 = (ki)_{g^{(ij)}}(jk)_{g^{(ij)}}(ki)_{g^{(ij)}} = (ij)_{g^{(ij)}}.$$

Proposition now follows by induction on the length of permutations s, t. $\square$

2.3. **An explicit presentation of $S_n(G)$.** In this section we obtain an explicit presentation of $S_n(G)$ similar in appearance to the presentation of parametrized braid groups from [5]. This presentation turns out to be much more economical than the original definition given in section 2.1. An explicit presentation for the subgroup $HS_n(G)$ is derived from this presentation via Reidemeister-Schreier method.

We start with the following simple lemma which is, in essence, a variant of the standard presentation of a Weyl group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

lm:Snpres **Lemma 2.5.** *For $n \geq 3$ The symmetric group S_n admits presentation with transpositions as the set of generators and the following list of defining relations (in every formula distinct letters denote distinct indices):*

{Sym1} (S1) $$(ij)^2 = 1,$$

{Sym2} (S2) $$(ij)^{(jk)} = (ik),$$

{Sym3} (S3) $$[(ij), (kl)] = 1.$$

{Sym0} (S4) $$(ij) = (ji),$$

Now we are ready to formulate the main result of this subsection.

prop:Q-pres **Proposition 2.6.** *For $n \geq 3$ and arbitrary group G the group $S_n(G)$ admits presentation with the set of generators $\{(ij)_a \mid i \neq j, 1 \leq i, j, \leq n, a \in G\}$ and the following list of relations (as before, distinct letters denote distinct indices):*

{Q1} (SG1) $$(ij)_a^2 = 1,$$

{Q2} (SG2) $$(ij)_a^{(jk)_b} = (ik)_{ab},$$

{Q3} (SG3) $$[(ij)_a, (kl)_b] = 1.$$

{Q4} (SG4) $$(ij)_a = (ji)_{a^{-1}}$$

Proof. Denote by S' the group from the statement of the proposition. For $1 \leq i \leq n$ and $x \in G$ denote by $x[i]$ the element of G^n whose only nontrivial component equals x and is located in the i-th position.

It is not hard to deduce from the definition of $S_n(G)$ and Lemma 2.4 that the formula $(ij)_a \mapsto (ij)_{a[j]}$ gives a well-defined map $\varphi\colon S' \to S_n(G)$.

Now we are going to construct the map $\psi\colon S_n(G) \to S'$ in the opposite direction. Using the presentation of S_n given by Lemma 2.5 we define for a fixed $g \in G^n$ the map $\psi_g\colon S_n \to S'$ by $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$. It is obvious that ψ_g preserves the defining relations (S1)–(S4) of S_n. It remains to show that the equation $\psi_g(s) = \psi_h(s)$ holds whenever $g, h \in G^n$ and $s \in S_n$ satisfy the requirement of (2.1).

Indeed, if hg^{-1} is fixed by s then for every $1 \leq i \leq n$ we have $(hg^{-1})_i = (hg^{-1})_{s(i)}$, or equivalently $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$. For example, if s is a cycle of length p, i. e. $s = (i_1, i_2, \ldots i_p)$ with $i_{k+1} = s(i_k)$ we get that

$$\psi_g(s) = \psi_g\left(\prod_{k=1}^{p-1}(i_k, i_{k+1})\right) = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{g_{i_k}^{-1}g_{i_{k+1}}} = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{h_{i_k}^{-1}h_{i_{k+1}}} = \psi_h(s).$$

The proof for a general s is almost the same. Verification of the fact that ψ and φ are mutually inverse is also immediate. $\square$

We now obtain a presentation of the subgroup $HS_n(G)$. For every $i \neq j$ and $a \in G$ we define the element $h_{ij}(a) \in HS_n(G)$ as follows:

{eq:h-def} (2.6) $$h_{ij}(a) = (ij)_a \cdot (ij)_1.$$

It is not hard to show that $h_{ij}(a)$ form a generating set for $HS_n(G)$. In fact, there is an explicit formula how an element of $HS_n(G)$ originally expressed through $(ij)_a$'s can be rewritten in terms of $h_{ij}(a)$. Indeed, if h lies in $HS_n(G)$ and is written as $\prod_{k=1}^{N}(i_kj_k)_{a_k}$ for some $i_k \neq j_k$ and $a_k \in G$ then it can be rewritten as follows:

{eq:rp} (τ) $$h = \prod_{k=1}^{N} h_{\sigma_k(i_k), \sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1}(i_sj_s) \in S_n.$$

We briefly recall the notion of a *rewriting process* given in [6, § 2.3]. If G is a group presented by generators a_ν and relations $R_\mu(a_\nu)$ and H is its subgroup with a generating set $J_i(a_\nu)$ then a *rewriting process for* H is a function which maps every word u in alphabet a_ν to a word v in alphabet s_i such that u and $v[s_i := J_i]$ define the same element of G whenever u represents an element of H.

With this terminology, the mapping (τ) defined above is a rewriting process for the subgroup $HS_n(G)$. Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [6, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [6, § 2.3]):

- if U and U^* are freely equal words in $(ij)_a$ then $\tau(U)$ and $\tau(U^*)$ are also freely equal words in $h_{ij}(a)$;
- if U_1 and U_2 are two words in $(ij)_a$ which define elements of $HS_n(G)$ then the words $\tau(U_1U_2)$ and $\tau(U_1)\tau(U_2)$ are equal.

Using these two properties and repeating the arguments used in the proof of [6, Theorem 2.8] one can simplify the generic presentation of $HS_n(G)$ given by [6, Theorem 2.6] and obtain the following.

lm:h-gen **Lemma 2.7.** *For $n \geq 3$ the group $HS_n(G)$ admits presentation on the generators $h_{ij}(a)$ with the following two families of defining relations:*

{eq:tau1} (2.7)
$$h_{ij}(a) = \tau\left((ij)_a \cdot (ij)_1\right);$$

{eq:tau2} (2.8)
$$\tau(KRK^{-1}) = 1,$$

where R varies over relations of Proposition 2.6 and K is any word in $(ij)_1$, $i \neq j$.

From (2.7) one immediately obtains the equality $h_{ij}(1) = 1$, $i \neq j$. Thus, if we denote by σ the permutation corresponding to a word K and let S_n act on $h_{ij}(a)$'s in the natural way we will get that the word $\tau(KRK^{-1})$ is equivalent to ${}^{\sigma}\tau(R)$ (modulo relations $h_{ij}(1) = 1$).

Since the relations from Proposition 2.6 are respected by the action of S_n, only relations of the form $\tau(R) = 1$ are, in fact, needed for the presentation of $HS_n(G)$. Writing down what $\tau(R)$ is for each of (SG1)–(SG4) we get the following.

prop:HSpres **Proposition 2.8.** *For $n \geq 3$ the group $HS_n(G)$ admits presentation with generators $h_{ij}(a)$ and the following list of relations:*

{H0} (R0) $$h_{ij}(1) = 1,$$
{H1} (R1) $$h_{ij}(a)h_{ji}(a) = 1,$$
{H2} (R2) $$h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab),$$
{H3} (R3) $$[h_{ij}(a), h_{kl}(b)] = 1,$$
{H4} (R4) $$h_{ij}(a)^{-1} = h_{ij}(a^{-1}).$$

c:Hnextensions

3. Relationship with extensions of type $\mathfrak{H}_n(G)$

We start by briefly reviewing the material of § 1–3 of [10]. Let $n \geq 3$ be a natural number. By definition, the group $H_n(G)$ is given by generators $h_{ij}(u)$, $u \in G$, $i \neq j$ and the following relations (cf. [10, H1–H5 of § 2]):

{RH1} (H1) $$h_{ij}(u)h_{ji}(u) = 1,$$
{RH2} (H2) $$h_{ij}(u)h_{ki}(u)h_{jk}(u) = 1,$$
{RH3} (H3) $$h_{ij}(u)h_{ik}(v)h_{ij}(u)^{-1} = h_{ik}(uv)h_{ik}(u)^{-1}, \qquad \text{for } j \neq k$$
{RH4} (H4) $$h_{ij}(u)h_{kj}(v)h_{ij}(u)^{-1} = h_{kj}(vu)h_{kj}(u)^{-1}, \qquad \text{for } i \neq k$$
{RH5} (H5) $$[h_{ij}(u), h_{kl}(v)] = 1.$$

Denote by $D_n(G)$ the subgroup of G^n consisting of vectors $(g_1, \ldots, g_n)$ for which the product $g_1 \cdot \ldots \cdot g_n$ lies in $[G, G]$. There is a surjective group homomorphism $H_n(G) \twoheadrightarrow D_n(G)$ sending $h_{ij}(g)$ to $d_{ij}(g)$. Here $d_{ij}(g)$ stands for the element of G^n whose i-th component equals g, j-th component equals g^{-1} and all other components are trivial.

By definition, *an extension of type $\mathfrak{H}_n(G)$* is an extension H of $D_n(G)$ that is also a quotient of $H_n(G)$, i. e. the extension that fits into the following diagram.

$$\begin{tikzcd}
H_n(G) \arrow[d, two heads] \arrow[dr, two heads] & \\
H \arrow[r, two heads] & D_n(G)
\end{tikzcd}$$

Any $\mathfrak{H}_n(G)$-extension is central, see [10, Proposition 2.3].

We now recall the notion of *an extension of type* $\mathfrak{U}(G)$ introduced by Rehmann in [10, § 1]. Let G be a group. Denote by $U(G)$ the group presented by generators

(u,v), $u,v\in G$ and relations:

{U1} (U1)
$$({}^u v, {}^u w)(u,w) = (uv, w),$$

{U2} (U2)
$$(u, vw)(v, wu)(w, uv) = 1.$$

There is a left action of G on $U(G)$ given by ${}^x(u,v) = ({}^x u, {}^x v)$. Equivalently, $U(G)$ can be defined by the following relations, see [4, p. 8].

{U4} (3.1)
$$(u,v)(v,u) = 1,$$

{U6} (3.2)
$$(u,v)\ {}^v(u,w) = (u, vw),$$

{U9} (3.3)
$${}^u(v,w)(w,v) = (u,[v,w]).$$

It is clear from the definition that $U(G)$ is a quotient of the nonabelian tensor square $G\otimes G$ and that the map $(u,v)\mapsto[u,v]$ defines a G-equivariant map $U(G)\twoheadrightarrow[G,G]$. Notice that in the literature one can find alternative notations for $U(G)$ which emphasize its connection with nonabelian tensor products e. g. (G,G) or $G\,\widetilde{\wedge}\,G$.

By definition, an *extension of type* $\mathfrak{U}(G)$ is simply a quotient of $U(G)$ that fits into the following commutative diagram with G-equivariant arrows.

$$\begin{tikzcd}
U(G) \arrow[d, two heads] \arrow[dr, two heads] & \\
U \arrow[r, two heads] & {[G,G]}
\end{tikzcd}$$

There is a correspondence between extensions of type $\mathfrak{H}_n(G)$ and $\mathfrak{U}(G)$. One can obtain a $\mathfrak{U}(G)$-extension from given $\mathfrak{H}_n(G)$-extension via restriction. Indeed, let H be an extension of type $\mathfrak{H}_n(G)$ and k be arbitrary number $1\le k\le n$. Denote by ι_k the inclusion map $[G,G]\to D_n(G)$ that inserts an element $g\in[G,G]$ at the k-th position. Then the corresponding extension U_H is simply the pull-back of H with respect to ι_k.

$$\begin{tikzcd}
U_H \arrow[r, two heads, "\pi_U"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[G,G]} \arrow[d, "\iota_k"] \\
H \arrow[r, two heads, "\pi_H"] & D_n(G).
\end{tikzcd}$$

In other words, U_H is simply the subgroup of H generated by symbols $c_{kj}(u,v) = h_{kj}(u)\cdot h_{kj}(v)\cdot h_{kj}(vu)^{-1}$ (here j is any index not equal k, $c_{kj}(u,v)$ does not depend on j, see [10, Proposition 2.1]). It can be shown that U is an extension of type $\mathfrak{U}(G)$ with kernel $\mathrm{Ker}(\pi_U)$ isomorphic to $\mathrm{Ker}(\pi_H)$, see [10, Propositions 2.2–2.3]. In [10, § 3] Rehmann also describes the inverse construction which assign to every $\mathfrak{U}(G)$-extension $U\twoheadrightarrow[G,G]$ and a number $n\ge 3$ some extension H_U of type $\mathfrak{H}_n(G)$ that fits into the pull-back diagram above.

3.1. **A simpler presentation of $H_n(G)$.** The aim of this subsection is to obtain a presentation of $H_n(G)$ that is simpler than the original presentation: 4 instead of 5 relations that are also shorter. Our main result is the following.

prop:simpler **Proposition 3.1.** *For $n \geq 3$ relations* (R0)–(R4) *imply* (H1)–(H5). *Moreover, for $n \geq 4$ relations* (R0)–(R3) *are equivalent to* (H1)–(H5).

For $u, v \in G$ we define the following two symbols:

$$c_{ij}(u,v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, c'_{ij}(u,v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies $h_{ij}(1) = 1$ therefore $c_{ij}(u,u^{-1}) = c'_{ij}(u,u^{-1})$. Our goal is to show that (H3) can be omitted from the definition of $H_n(G)$ provided $n \geq 4$.

item-lem33 **Lemma 3.2.** *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following still remain true:*

tem-lem33-cntr (1) *The elements $c_{ij}(u,u^{-1})$ are central in $H_n(G)$;*
tem-lem33-comm (2) *One has $c'_{ij}(u,v) = [h_{ij}(u), h_{kj}(v)]$, $k \neq i,j$;*
tem-lem33-conj (3) *One has ${}^{h_{ij}(w)}c'_{kj}(u,v) = c'_{kj}(u,w)^{-1}c'_{kj}(u,vw)$, $k \neq i,j$;*
em-lem33-conj2 (4) *One has ${}^{h_{ij}(w)}c'_{ij}(u,v) = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}$.*

Proof. First one shows using (H4) that $c_{ij}(u,u^{-1})$ centralizes $h_{kj}(v)$ (cf. with the proof of [10, Lemma 2.1(2)]). Since in any group $[a,b] = 1$ implies $[a^{-1},b] = [a,b^{-1}] = [a^{-1},b^{-1}] = 1$ we get that $c_{ij}(u,u^{-1})$ also centralizes $h_{kj}(v)^{-1} = h_{jk}(v)$ and $c_{ij}(u^{-1},u)^{-1} = c_{ji}(u,u^{-1})$ centralizes both $h_{kj}(v)$ and $h_{jk}(v)$. Together with (H2) and (H5) this implies that $c_{ij}(u,u^{-1})$ centralizes all the generators of $H_n(G)$ and hence lies in the center of $H_n(G)$.

The second and third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$${}^{h_{ij}(w)}c'_{ij}(u,v) = {}^{h_{ij}(w)}c'_{kj}(v,u)^{-1} = (c'_{kj}(v,w)^{-1}c'_{kj}(v,uw))^{-1} = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}.$$

□

Lemma 3.3. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following statements are equivalent:*

item1 *(1)* (H3) *holds;*
item4 *(2) one has $c'_{ij}(u,v)^{-1} = c'_{ij}(v,u)$;*
item5 *(3) symbols $c'_{ij}(u,v)$ do not depend on i;*
item5 *(4) one has $c'_{ij}(u,vw) = c'_{ij}(wu,v) \cdot c'_{ij}(uv,w)$;*
item6 *(5) one has $c'_{ij}(u,vu^{-1}) = c'_{ij}(uv,u^{-1})$.*

Proof. Implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are essentially contained in the proof of [10, Lemmas 2.1-2.2]. Implication (4) $\implies$ (5) is trivial.

We now prove (5) $\implies$ (1). Notice that (H4) implies ${}^{h_{ij}(u)^{-1}}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$, therefore using Lemma 3.2.(1) we get that ${}^{h_{ij}(u)}h_{ik}(v) = {}^{c_{ij}(u,u^{-1})h_{ij}(u^{-1})^{-1}}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$. Thus, (H3) is equivalent to the equality

$$h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}),$$

or what is the same

$$c'_{ik}(u^{-1}, uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1}, u).$$

It is clear that the last equality is an equivalent reformulation of (5). □

Proof of Proposition 3.1. The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);
- for $n \geq 4$ the third statement of the previous lemma follows from Lemma 3.2.(2).

□

cor:main **Corollary 3.4.** *For $n \geq 3$ the group $HS_n(G)$ is the quotient of the extension $H_n(G)$ by* (R4) *and, in particular, is an $\mathfrak{H}_n(G)$-extension. The associated $\mathfrak{U}(G)$ extension $U_{HS_n(G)}$ is isomophic to the quotient of $U(G)$ modulo single family of relations*

$$(u, u) = 1, \;\; u \in G.$$

In particular, we have the following natural isomorphism:

$$\operatorname{Ker}(HS_n(G) \to D_n(G)) = \operatorname{Ker}(S_n(G) \to G \wr S_n) \cong \mathrm{H}_2(G, \mathbb{Z}). \tag{3.4}$$

Proof. Most of the assertions follow from Proposition 3.1 and [4, Proposition 5]. One also has to use the formula $c_{ij}(u, u) = c_{ij}(u, u^{-1})$ (see [10, p. 87]). □

sec:main

4. Topological applications

lm:weak-equiv **Lemma 4.1.** *If X is fibrant and connected then F_{h_∞} and $h^{-1}(0)$ are weakly equivalent.*

Proof. **???** □

From the proof of [9, Lemma 10.1] it follows that the following square of abelian groups is cocartesian in the case Y is fibrant and connected.

orytov-square}

$$\begin{tikzcd} \mathbb{Z}[\mathbf{sSets}(X, Y)] \arrow[r] \arrow[d, two heads, "\alpha"'] & \mathbf{sSets}(X, A(Y)) \arrow[d, two heads, "\beta"'] \\ \mathbb{Z}[\mathbf{hsSets}(X, Y)] \arrow[r] & \mathbf{hsSets}(X, A(Y)) \end{tikzcd} \tag{4.1}$$

Here both α, β map each morphism of simplicial sets to the corresponding class in the homotopy category. The fact that the above square is cocartesian is equivalent to the surjectivity of the map $\mathrm{Ker}(\alpha) \twoheadrightarrow \mathrm{Ker}(\beta)$.

$$
\begin{tikzcd}
 & X \arrow[d, "\gamma"] & \\
h^{-1}(0) \arrow[r, hook] & Y^{\infty} \arrow[r, "h_{\infty}"] & A(Y)
\end{tikzcd}
\tag{4.2}
$$

Recall that for arbitrary set X one defines the space EX as the the simplicial set whose set of k-simplices EX_k is X^{k+1} and whose faces and degeneracies are obtained by omitting and repeating components. For a group G we denote by π_G the canonical map $EG \to BG$ sending $(g, h) \in EG_1$ to $g^{-1}h \in BG_1$.

Now let N be a group acting on X. We define two simplicial sets U and V as follows:

$$U = \bigcup_{n \in N} E(\Gamma_{n.-}) \subseteq E(X \times X), \quad V = \bigcup_{x,y \in X} E(N(x \to y)) \subseteq EN.$$

Here $\Gamma_{n.-}$ is the graph of the function $(x \mapsto nx)$ and $N(x \to y)$ denotes the subset of elements $n \in N$ satisfying $nx = y$. With this notation the subset $N(x \to x)$ coincides with the stabilizer subgroup $N_x \leq N$.

`lm:quillen-a` **Lemma 4.2.** *The simplicial sets U and V are homotopy equivalent.*

Proof. First, we define yet another simplicial set W as follows. Its k-simplices W_k are matrices $\left(\begin{smallmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{smallmatrix}\right)$, where $x_i \in X$ are $n_i \in N$ are such that all n_i's act each x_j in the same way, i. e. $n_i x_j = n_{i'} x_j$ for $0 \leq i, i', j \leq k$. The faces and degeneracies of W are the maps of omission and repetition of columns.

Now there are two simplicial maps $f \colon W \to U$, $g \colon W \to V$ whose action on 0-simplicies is given by $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = (x_0, n_0 x_0)$, $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = n_0$. To prove the lemma it suffices to show that f and g are homotopy equivalences. The proof for f and g is similar, let us show, for example, that g is a homotopy equivalence.

In view of Quillen theorem A (cf.[11, ex. IV.3.11]) it suffices to show that for each p-simplex $d \colon \Delta^p \to V$ the pullback $g/(p, d)$ of d and g is contractible. The simplicial set $g/(p, d)$ can be interpreted as the subset of $\Delta^p \times E(X)$ whose set of k-simplices consists of pairs $(\alpha \colon \underline{k} \to \underline{p}, (x_0, \dots, x_k) \in X(\alpha, d)^{k+1})$. Here $X(\alpha, d)$ is the subset of X consisting of all x for which $d_{\alpha(i)} x = d_{\alpha(j)} x$ for $0 \leq i, j \leq k$. Notice that the set $X_d := X(id_{\underline{p}}, d)$ is nonempty and is contained in every $X(\alpha, d)$ (it even equals $X(\alpha, d)$ for surjective α). Now choose a point $\widetilde{x} \in X_d$ and consider the simplicial homotopy

$$H \colon \Delta^p \times EX \times \Delta^1 \to \Delta^p \times EX$$

between the identity map of $\Delta^p \times EX$ and the map $\Delta^p \times c_{\widetilde{x}}$, where $c_{\widetilde{x}}$ is the constant map. More concretely, H sends each triple $(\alpha\colon \underline{k} \to \underline{p}, (x_0, \ldots, x_k), \beta\colon \underline{k} \to \underline{1})$ to $(\alpha, (x_0, \ldots, x_{i-1}, \widetilde{x}, \ldots, \widetilde{x}))$, where i is the minimal number such that $\beta(i) = 1$. By the choice of $\widetilde{x}$ the image of H restricted to $g/(p, d) \times \Delta^1$ is contained in $g/(p, d)$, hence $g/(p, d)$ is contractible. $\square$

Now suppose that $X = H$ is also a group upon which N acts on the left. It is clear that an element $n \in N$ induces a simplicial map $E(n.-)\colon EH \to EH$. We consider its graph $\Gamma_{E(n.-)}$ as a subset of $E(H \times H) = EH \times EH$.

cor:ker-iso **Corollary 4.3.** *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

There is an isomorphism $\mathrm{Ker}(\pi_1(S) \to N) \cong \mathrm{Ker}(\pi_1(T) \to H \times H)$. *Moreover, the higher homotopy groups of S and T are isomorphic.*

Proof. The required isomorphism can be obtained from the homotopy long exact sequence applied to left arrows of the following two pull-back squares.

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EN \arrow[d, "\pi_N"] \\ S \arrow[r, hook] & BN \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(H \times H) \arrow[d, "\pi_{H \times H}"] \\ T \arrow[r, hook] & B(H \times H) \end{tikzcd}$$

$\square$

Now let G be a group. Set $N = G \wr S_n$, $H = G^n$ and consider the left action of N on H given by $(g, s) \cdot h = gh^{s^{-1}}$, $g, h \in G^n$, $s \in S_n$. If one reorders the components of BG^{2n} accordingly, the simplicial subset $T \subset BG^{2n}$ from the above corollary becomes precisely the preimage of 0 under $h_n\colon BG^{2n} \to \mathbb{Z}[BG]$.

It is also easy to compute the map $\pi_1(S) \to N$. Indeed, van Kampen theorem [7, Theorem 2.7] asserts that $\pi_1(S)$ is isomorphic to the free product of stabilizer subgroups $N_h \leq N$ amalgamated over pairwise intersections $N_h \cap N_{h'}$, $h, h \in H$. For $h \in G^n$ the subgroup N_h consists of elements $(g, s) \in N$ satisfying $gh^{s^{-1}} = h$, i.e. elements of the form $(hh^{-s^{-1}}, s)$. Thus, $N_h \cong S_n$, $\pi_1(S)$ is isomorphic to the group $S_n(G)$ and the map $\pi_1(S) \to N$ coincides with the map μ defined in section 2.1.

ex44 **Example 4.4.** We leave it as an exercise to the reader to check that an element $[h_{12}(x), h_{13}(y)] \in \pi_1(S)$ corresponds to the following element of $\pi_1(T)$ under the isomorphism of Lemma 4.2:

{eq:pathT}
$$\begin{aligned} \gamma_{x,y} = (x[1], x[1]) \circ (x^{-1}[1], x^{-1}[2]) \circ (y[1], y[1]) \circ (y^{-1}[1], y^{-1}[3]) \circ \\ \circ (x[1], x[2]) \circ (x^{-1}[1], x^{-1}[1]) \circ (y[1], y[3]) \circ (y^{-1}[1], y^{-1}[1])). \end{aligned} \tag{4.3}$$

Proof of Theorem 1. Factor h_n as a composition of a trivial cofibration followed by a fibration:

{eq:fibr-repl}
$$BG^{2n} \hookrightarrow E_{h_n} \xrightarrow{ev_1 \circ \pi_2} \mathbb{Z}[BG] \tag{4.4}$$

For example, we can define E_{h_n} and the homotopy fiber F_{h_n} via the usual path space construction (here we use the fact that BG and $\mathbb{Z}[BG]$ are fibrant).

$$\begin{tikzcd} E_{h_n} \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z}[BG]^I \arrow[d, "ev_0"] \\ BG^{2n} \arrow[r, "h_n"'] & \mathbb{Z}[BG] \end{tikzcd} \qquad \begin{tikzcd} F_{h_n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{h_n} \arrow[d, "ev_1 \circ \pi_2"] \\ pt \arrow[r, "0"'] & \mathbb{Z}[BG] \end{tikzcd}$$

Now write down the starting portion of the long homotopy exact sequence of the fibration $E_{h_n} \twoheadrightarrow \mathbb{Z}[BG]$ and denote by K the kernel of the map $\nu\colon \pi_1(T) \to G^{2n}$ induced by the embedding $h_n^{-1}(0) = T \subseteq BG^{2n}$. We come to the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "\psi"'] & \pi_1(T) \arrow[r, "\nu"'] \arrow[d, "\phi"'] & G^{2n} \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}_1(G,\mathbb{Z}) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{H}_2(G,\mathbb{Z}) \arrow[r] & \pi_1(F_{h_n}) \arrow[r] & \pi_1(E_{h_n}) \arrow[r] & \mathrm{H}_1(G,\mathbb{Z}) \arrow[r] & 1 \end{tikzcd}$$

We already know by Corollaries 3.4 and 4.3 that K is naturally isomorphic to $\mathrm{H}_2(G,\mathbb{Z})$ provided $n \geq 3$.

Let us show that ψ is an isomorphism. Assume for a moment that G is an abelian group. In this case a generator $x \wedge y$ of $\mathrm{H}_2(G,\mathbb{Z})$ in Miller's presentation corresponds to the class of 2-cycle $c = (x,y) - (y,x)$ (cf. [8, (14), p. 582]), which, in turn, corresponds to the following normalized 2-cycle:

{eq:normalized}
$$c' = c - s_0 d_0 c - s_1 d_1 c + s_1 d_0 = (x,y) - (y,x) - (x,0) + (y,0) + (0,x) - (0,y). \tag{4.5}$$

Since F_{h_n} and $\mathbb{Z}[BG]$ are fibrant, any element of $\pi_1(F_{h_n}, 0)$ can be represented with some 1-simplex $(\gamma, s) \in G^{2n} \times \mathbb{Z}[BG]_2$ satisfying $d_2(s) = h_n(\gamma)$, $d_0(s) = d_1(s) = 0$. It remains to find in F_{h_n} the 1-simplex homotopic to the path $\phi(\gamma_{x,y})$ (cf. Example 4.4).

We use the following inductive procedure. Let γ_0 be an initial segment of a path γ in T for which we have already found such simplex (g^0, s^0) in F_{h_n} and let $g \in T_1$ be the next chain link. Set $s_0 = \sum(a_j, b_j) \in \mathbb{Z}[BG]_2$, $t = h_n(g^0, g) = \sum_{i=1}^{2n}(-1)^i(g_i^0, g_i)$ and compute the filler for the 3-horn $(0,\ , s_0, t)$. Denote the 1-st face of this filler by s_1. It is clear that $(g^0 g, s_1)$ is the image of $\gamma^0 \circ g$ via ϕ. The concrete formula for s_1

can be chosen e. g. as follows:

$$s_1 = \sum_{i=1}^{2n}(g_i^0 + g_i, -g_i) - \sum_j (a_j + b_j, -b_j).$$

Applying the above recipe to the path $\gamma_{x,y}$ we get an expression equal to (4.5). This shows that ψ is the identity map for an abelian group G. Since ψ is natural in G, by Lemma 4.5 below we get that ψ is an isomorphism for arbitrary G, therefore ϕ is also an isomorphism, as claimed. $\square$

lm:endotr **Lemma 4.5.** *The only natural endotransformations of the second homology functor* $\mathrm{H}_2(-,\mathbb{Z})\colon \mathbf{Groups} \to \mathbf{Ab}$ *are morphisms of multiplication by* $n \in \mathbb{Z}$.

Proof. Denote by η an endotransformation $\mathrm{H}_2(-,\mathbb{Z}) \to \mathrm{H}_2(-,\mathbb{Z})$. When restricted to the subcategory of free finitely-generated abelian groups $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$ the second homology functor coincides with the second exterior power functor $A \mapsto \wedge^2 A$.

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic $\mathbb{Z}$-modules (see Definition 6.13.5 ibid.) The functor $A \mapsto \wedge^2 A$ is clearly quadratic and corresponds to the quadratic $\mathbb{Z}$-module $0 \to \mathbb{Z} \to 0$ under this equivalence. Thus, we get that η restricted to $\mathbf{Add}(\mathbb{Z})$ coincides with the the morphism of multiplication by $n \in \mathbb{Z}$.

Consider the group $\Gamma_k = \langle x_1, y_1, \ldots x_k, y_k \mid [x_1, y_1] \cdot \ldots \cdot [x_k, y_k]\rangle$ (the fundamental group of a sphere with k handles). It is clear that the abelianization map $\Gamma_k \to \mathbb{Z}^{2k}$ induces an injective map $\mathrm{H}_2(\Gamma_k,\mathbb{Z}) \cong \mathbb{Z} \to \wedge^2\mathbb{Z}^{2k}$. Consider the following diagrams.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] \arrow[d, "\eta_{\Gamma_k}"'] & \wedge^2\mathbb{Z}^{2k} \arrow[d, "n\cdot"] \\
\mathbb{Z} \arrow[r, hook] & \wedge^2\mathbb{Z}^{2k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[r, "\chi"] \arrow[d, "\eta_{\Gamma_k}"'] & \mathrm{H}_2(G,\mathbb{Z}) \arrow[d, "\eta_G"] \\
\mathbb{Z} \arrow[r, "\chi"] & \mathrm{H}_2(G,\mathbb{Z})
\end{tikzcd}$$

From the left diagram it follows that η_{Γ_k} is also the morphism of multiplication by n. For every element $x \in \mathrm{H}_2(G,\mathbb{Z})$ there exist an integer k and a map $\chi\colon \Gamma_k \to G$ sending the generator of $\mathrm{H}_2(\Gamma_k,\mathbb{Z})$ to x. From the right square we conclude that $\eta_G(x) = nx$, as claimed. $\square$

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