

# PARAMETRIZED SYMMETRIC GROUPS AND NONABELIAN TENSOR PRODUCTS

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ABSTRACT. By amalgamating copies of the symmetric group  $S_n$  we construct a central extension of the wreath product  $G \wr S_n$ . We establish a connection between this group and extensions of type  $\mathfrak{H}_n(G)$  introduced by U. Rehmann in [13]. We also describe a topological application of this construction.

## 1. INTRODUCTION

The aim of this note is to describe a rather surprising connection between symmetric groups and nonabelian tensors products which apparently has not been studied in detail previously.

Let  $G$  be a (possibly noncommutative) group. Denote by  $S_n(G)$  the group generated by elements  $s_i(a)$ , where  $i$  is an integer  $1 \leq i \leq n-1$  and  $a$  is an element of  $G$ , subject to the following relations:

$$\boxed{\{\text{Cox1}\}} \quad (1.1) \quad s_i(a)^2 = 1, \quad a \in G,$$

$$\boxed{\{\text{Cox2}\}} \quad (1.2) \quad s_i(a)s_{i+1}(b)s_i(c) = s_{i+1}(a^{-1}cb)s_i(a)s_{i+1}(b), \quad a, b, c \in G,$$

$$\boxed{\{\text{Cox3}\}} \quad (1.3) \quad s_i(a)s_j(b) = s_j(b)s_i(a), \quad |i-j| \geq 2, a, b \in G.$$

Notice that in the special case  $G = 1$  the above presentation boils down to the usual Coxeter presentation of the symmetric group  $S_n$ , i. e.  $S_n(1) = S_n$ .

The group defined above bears some similarity with the group defined by Kassel—Reutenauer in [7] (cf. also with the group defined by *parametrized braid group* defined by Loday—Stein in [8]). By analogy, we call the group  $S_n(G)$  a *parametrized symmetric group*. Our first main result is the following theorem, which provides a concrete description of  $S_n(G)$  and is thus a combinatorial analogue of [7, Theorem 1].

Denote by  $G \wr S_n$  the wreath product of  $G$  and  $S_n$ .

**thm:summary**

**Theorem 1.** *For  $n \geq 3$  there is a map  $\mu_n: S_n(G) \rightarrow G \wr S_n$  which is a crossed module. The cokernel and the kernel of  $\mu_n$  are isomorphic to  $H_1(G, \mathbb{Z})$  and  $H_2(G, \mathbb{Z})$  respectively.*

To prove the above theorem we express  $S_n(G)$  in terms of so-called extensions of type  $\mathfrak{H}_n$ , which were defined by U. Rehmann in [13], see Corollary 3.6. Originally discovered in attempt to prove noncommutative Matsumoto theorem, these extensions are directly related with nonabelian tensor products. For example, the nonabelian antisymmetric square  $G \otimes G$  is a subgroup of the group  $H_n(G)$ , the "universal" extension of type  $\mathfrak{H}_n(G)$  (we refer the reader to Section 3 for more details).

Apart from the fact that Theorem 1 establishes a connection between two classical objects: symmetric groups and the second homology, it is also beneficial for the study of extensions of type

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*Date:* May 17, 2017.

*Key words and phrases.* Extensions of type  $\mathfrak{H}_n(G)$ , amalgamated products, van Kampen theorem, *Mathematical Subject Classification (2010)*: 55Q05, 20B30, 20E06, 20J06, 20F05, 20F55.

$\mathfrak{H}_n(G)$ . For example, in course of the proof of Theorem 1 we obtain a presentation of the group  $H_n(G)$  that is simpler than its original presentation given in [13] (4 instead of 5 relations that are also shorter), see Proposition 3.3.

The study of parametrized symmetric groups was originally motivated by some purely topological question, which we describe below in more detail.

For a pointed simplicial set  $(X, x_0)$  denote by  $\mathbb{Z}[X]$  the associated free simplicial abelian group. Define the "alternating" map of simplicial sets  $h_n: X^{2n} \rightarrow \mathbb{Z}[X]$  by the following identity:

$$h_n(x_1, x_2, \dots, x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Define the "stabilization" map  $X^{2n} \hookrightarrow X^{2(n+1)}$  by  $x \mapsto (x, x_0, x_0)$  and denote by  $X^\infty$  the colimit of  $X^n$ .

Motivated by the study of homotopy invariants, S. Podkorytov showed in [12] that the limit map  $h_\infty = \text{colim}_n(h_n): X^\infty \rightarrow \mathbb{Z}[X]$  is a *quasifibration* provided  $X$  is fibrant and connected. By definition, this means that the natural map  $h_\infty^{-1}(b) \rightarrow F_{h_\infty}(b)$  between the fiber and the homotopy fiber of  $h_\infty$  over any point  $b$  is a weak (hence homotopy) equivalence. This assertion is essentially proved in [12, Lemma 9.1], however some technical work is required to formulate it this way, see Appendix A below.

One may wonder whether the maps  $h_n$  satisfy some weaker property. For example, one might expect that the map  $h_n^{-1}(0) \rightarrow F_{h_n}(0)$  is  $c(n)$ -connected for some  $c(n)$  which depends only on  $n$  and tends to infinity as  $n \rightarrow \infty$ .

The following result which is a consequence of Theorem 1 asserts that this map is at least 1-connected in one important special case.

**thm:main**

**Theorem 2.** *If  $X = BG$  is the classifying space of a group  $G$  then for  $n \geq 3$  the natural map  $h_n^{-1}(0) \rightarrow F_{h_n}(0)$  induces an isomorphism of fundamental groups  $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$ .*

Apart from Theorem 1 the proof of Theorem 2 involves some explicit computations with simplicial fundamental groups using Quillen Theorem A and van Kampen theorem.

**1.1. Acknowledgements.** I wish to thank Semyon Podkorytov for suggesting the problem and also for his kind permission to use some of his unpublished results. I am also grateful to S. O. Ivanov and V. Isaev for their numerous helpful comments and interest in this work. This work received financial support from the Russian Science Foundation grant 14-21-00035.

## 2. DEFINITION OF $S_n(G)$ AND ITS BASIC PROPERTIES

**sec:QnG-def**

The aim of this section is to prove the first part of Theorem 1, namely the fact that  $S_n(G)$  is a crossed module over  $G \wr S_n$ . It will be more convenient for us to start with another definition of the group  $S_n(G)$  given in terms of amalgams of symmetric groups. In the end of the section we show that this alternative definition coincides with the one given in the introduction.

Let  $G$  be a group. Recall that the *wreath product* is, by definition, the semidirect product  $G^n \rtimes S_n$ , in which  $S_n$  acts on  $G^n$  on the right by permuting its factors.

Consider the family  $\{S_n^{(g)}\}_{g \in G^n}$  of isomorphic copies of  $S_n$  indexed by elements of  $G^n$  and let  $F$  be the free product of groups from this family. For  $s \in S_n$  and  $g \in G^n$  we denote by  $s_g$  the image of  $s$  in  $F$  under the canonical map  $S_n^{(g)} \rightarrow F$ . Define the group  $S_n(G)$  as the quotient of  $F$  modulo the following single family of relations:

**{eq:main\_rel}**

$$(2.1) \quad s_g = s_h, \text{ if } s^g = s^h \text{ holds in } G \wr S_n.$$

We denote by  $s_g$  the image of the element  $s_g \in F$  under the canonical map  $F \rightarrow S_n(G)$ .

*Remark 2.1.* Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1}g^{(s^{-1})}, s).$$

therefore the equality  $s^g = s^h$  holds in  $G \wr S_n$  iff  $g^{-1}g^{(s^{-1})} = h^{-1}h^{(s^{-1})}$ , or what is the same, iff  $s$  fixes  $hg^{-1}$ .

The last statement immediately implies that the map  $\mu: S_n(G) \rightarrow G \wr S_n$  given by  $\mu(s_g) = s^g = (g^{-1}g^{(s^{-1})}, s)$  is well-defined.

Observe from the definition of  $S_n(G)$  that there is a split exact sequence.

$$(2.2) \quad 1 \longrightarrow \text{Ker}(\pi) \longrightarrow S_n(G) \xrightarrow[\iota(1)]{\pi} S_n \longrightarrow 1,$$

Here the map  $\pi = \pi_{S_n} \circ \mu$  removes subscript  $g$  from each  $s_g$  and the section  $\iota(1)$  maps  $s$  to  $s_1$ , where  $1$  is the identity element of  $G^n$ . Clearly,  $S_n(G)$  decomposes as  $HS_n(G) \rtimes S_n$ , where  $HS_n(G) = \text{Ker}(\pi)$ .

**2.1. Crossed module structure on  $S_n(G)$ .** Recall from [2, § 2.2] that a *crossed module* is a morphism of groups  $\mu: M \rightarrow N$  together with a right action of  $N$  on  $M$  compatible with the conjugation action of  $N$  on itself, i. e.

$$(CM1) \quad \mu(m^n) = \mu(m)^n \text{ for all } n \in N, m \in M,$$

which also satisfies the following identity called *Peiffer identity*:

$$(CM2) \quad m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M.$$

In our situation, we let  $G \wr S_n$  act on  $S_n(G)$  by

$$(2.3) \quad (s_g)^{(h,t)} = s_{(gh)^t}^t, \text{ for } s, t \in S_n \text{ and } g, h \in G^n.$$

The goal of this subsection is to prove the following.

**Proposition 2.2.** *For  $n \geq 3$  the map  $\mu: S_n(G) \rightarrow G \wr S_n$  is a crossed module.*

From the fact that  $\mu$  is a crossed module one can deduce that  $\text{Ker}(\mu)$  is a central subgroup of  $S_n(G)$  and  $\text{Im}(\mu)$  is a normal subgroup of  $G \wr S_n$ . It is not hard to show that the group  $S_2(G)$  is isomorphic to a free product of copies of  $S_2$  (whose center is always trivial). Therefore, the requirement  $n \geq 3$  in the statement of Proposition 2.2 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of  $G \wr S_n$  on  $S_n(G)$  that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of  $S_n(G)$ , for which it takes the form:

$$(2.4) \quad t_h^{-1} s_g t_h = s_{(gh^{-1})^t}^t \text{ for all } s, t \in S_n, g, h \in G^n.$$

If we act on both sides of the above formula by  $(h^{-1}, 1) \in G \wr S_n$  we obtain the equality  $t_1^{-1} s_{gh^{-1}} t_1 = s_{(gh^{-1})^t}^t$ . Thus, to prove (2.4) it suffices to show the following simpler relation:

$$(2.5) \quad t_1^{-1} s_g t_1 = s_{g^t}^t \text{ for all } s, t \in S_n, g \in G^n.$$

The key step in the proof is the following lemma.

**Lemma 2.3.** *The relation (2.5) holds in the special case when  $s = (ij)$  and  $t = (kl)$  are two nonequal transpositions.*

*Proof.* First of all, we immediately check that (2.5) holds in the special case when  $t$  fixes  $g' \in G^n$ . Indeed, by (2.1) we have  $t_1 = t_{g'}$ , hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that  $l \neq i$  and  $l \neq j$ . Denote by  $g'$  the vector which differs from  $g$  only at  $l$ -th position, for which we set  $g'_l = g_k$ . Since the only nontrivial component of  $g'g^{-1}$  (resp.  $g'g^{-t}$ ) is located at  $l$ -th (resp.  $k$ -th) position, it is fixed by  $s$  (resp.  $s^t$ ), hence from (2.1) we conclude that  $s_g = s_{g'}$  (resp.  $(s^t)_{g'} = (s^t)_{g^t}$ ). Finally, since  $g'$  is fixed by  $t$ , we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g'^t} = s_{g'}^t = (s^t)_{g^t}. \quad \square$$

*Proof of Proposition 2.2.* Let us show that (2.5) holds for arbitrary transpositions  $s, t \in S_n$ . It suffices to consider the case  $s = t = (ij)$ . After choosing some  $k \neq i, j$  and presenting  $(ij)$  as  $(kj)(ik)(kj)$  we use Lemma 2.3:

$$(ij)_1^{-1} (ij)_g (ij)_1 = (ij)_1^{-1} (kj)_g (ik)_g (kj)_g (ij)_1 = (ki)_{g^{(ij)}} (jk)_{g^{(ij)}} (ki)_{g^{(ij)}} = (ij)_{g^{(ij)}}.$$

The proposition now follows by induction on the length of permutations  $s, t$ .  $\square$

**2.2. An explicit presentation of  $S_n(G)$ .** In this section we obtain an explicit presentation of  $S_n(G)$  in terms of reflections. From this presentation we later deduce both the Coxeter presentation for  $S_n(G)$  and an explicit presentation for the subgroup  $HS_n(G)$ .

We start with the following simple lemma which is, in essence, a variant of the standard presentation of the symmetric group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

lm:Snpres

**Lemma 2.4.** *For  $n \geq 3$  The symmetric group  $S_n$  admits presentation with  $\{(ij) \mid i \neq j, 1 \leq i, j \leq n\}$  as the set of generators and the following list of relations (in every formula distinct letters denote distinct indices):*

{Sym1}

$$(2.6) \quad (ij)^2 = 1,$$

{Sym2}

$$(2.7) \quad (ij)^{(jk)} = (ik),$$

{Sym3}

$$(2.8) \quad [(ij), (kl)] = 1.$$

{Sym0}

$$(2.9) \quad (ij) = (ji),$$

*Remark 2.5.* To simplify notation, up to the end of this section we adhere to the convention that in our formulae distinct letters denote distinct indices.

Now we are ready to formulate the main result of this subsection.

prop:Q-pres

**Proposition 2.6.** *For  $n \geq 3$  and arbitrary group  $G$  the group  $S_n(G)$  admits presentation with  $\{(ij)_a \mid i \neq j, 1 \leq i, j \leq n, a \in G\}$  as the set of generators and the following list of relations.*

{Q1}

$$(SG1) \quad (ij)_a^2 = 1,$$

{Q2}

$$(SG2) \quad (ij)_a^{(jk)_b} = (ik)_{ab},$$

{Q3}

$$(SG3) \quad [(ij)_a, (kl)_b] = 1.$$

{Q4}

$$(SG4) \quad (ij)_a = (ji)_{a^{-1}}$$

*Proof.* Denote by  $S'$  the group from the statement of the proposition. For  $1 \leq i \leq n$  and  $x \in G$  denote by  $x[i]$  the element of  $G^n$  whose only nontrivial component equals  $x$  and is located in the  $i$ -th position.

It is not hard to deduce from the definition of  $S_n(G)$  and Lemma 2.3 that the formula  $(ij)_a \mapsto (ij)_{a[j]}$  gives a well-defined map  $\varphi: S' \rightarrow S_n(G)$ .

Now we are going to construct the map  $\psi: S_n(G) \rightarrow S'$  in the opposite direction. Using the presentation of  $S_n$  given by Lemma 2.4 we define for a fixed  $g \in G^n$  the map  $\psi_g: S_n \rightarrow S'$  by  $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$ . It is obvious that  $\psi_g$  preserves the defining relations (2.6)–(2.9) of  $S_n$ . It remains to show that the equality  $\psi_g(s) = \psi_h(s)$  holds whenever  $g, h \in G^n$  and  $s \in S_n$  are such that  $s_g = s_h$ .

Indeed, if  $hg^{-1}$  is fixed by  $s$  then for every  $1 \leq i \leq n$  we have  $(hg^{-1})_i = (hg^{-1})_{s(i)}$ , or equivalently  $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$ . For example, if  $s$  is a cycle of length  $p$ , i.e.  $s = (i_1, i_2, \dots, i_p)$  with  $i_{k+1} = s(i_k)$  we get that

$$\psi_g(s) = \psi_g\left(\prod_{k=1}^{p-1} (i_k, i_{k+1})\right) = \prod_{k=1}^{p-1} (i_k, i_{k+1})_{g_{i_k}^{-1}g_{i_{k+1}}} = \prod_{k=1}^{p-1} (i_k, i_{k+1})_{h_{i_k}^{-1}h_{i_{k+1}}} = \psi_h(s).$$

The proof for a general  $s$  is almost the same. Verification of the fact that  $\psi$  and  $\varphi$  are mutually inverse is also immediate.  $\square$

**2.3. Coxeter-like presentation for  $S_n(G)$ .** The aim of this subsection is to show the following.

**Proposition 2.7.** *The group  $S_n(G)$  defined in Proposition 2.6 coincides with the group defined in the introduction by means of Coxeter-like presentation (denote the latter group by  $S_n^C(G)$ ).*

The idea of the proof is to construct a sequence of "intermediate" groups

$$S_n^C(G) \rightarrow S_n(1, G) \rightarrow \dots \rightarrow S_n(n-1, G) \rightarrow S_n(G)$$

and then show that all the maps between these groups are isomorphisms.

**Definition 2.8.** For  $1 \leq t \leq n-1$  consider the group  $S_n(t, G)$  defined by the set of generators  $\{(ij)_a \mid 1 \leq i < j \leq n, j-i \leq t\}$  subject to the following relations:

$$\boxed{\{\text{CoxMix1}\}} \quad (2.10) \quad (ij)_a^2 = 1,$$

$$\boxed{\{\text{CoxMix2}\}} \quad (2.11) \quad (ij)_a^{(jk)_b} = (jk)_{b'}^{(ij)_{a'}}, \quad \text{provided } ab = a'b',$$

$$\boxed{\{\text{CoxMix3}\}} \quad (2.12) \quad (ij)_a(kl)_b = (kl)_b(ij)_a,$$

$$\boxed{\{\text{CoxMix4}\}} \quad (2.13) \quad (ij)_a^{(jk)_b} = (ik)_{ab}.$$

In the above relations indices  $i, j, k, l$  satisfy the inequalities ensuring that all the generators appearing in the formulae are well-defined.

In the case  $t = 1$  the relation (2.13) becomes vacuous and the remaining relations have the same form as (1.1)–(1.3). Thus, the map  $S_n^C(G) \rightarrow S_n(1, G)$  defined by  $s_i(g) \mapsto (i, i+1)_g$  is an isomorphism.

On the other hand, in the case  $t = n-1$  the obvious embedding of generators induces a map  $S_n(n-1, G) \rightarrow S_n(G)$  (we consider the latter group in the presentation of Proposition 2.6). We claim that this map is an isomorphism. Indeed, define the map  $\theta: S_n(G) \rightarrow S_n(n-1, G)$  in the opposite direction by the following formula:

$$\theta((ij)_a) = \begin{cases} (ij)_a & \text{if } i < j, \\ (ji)_{a^{-1}} & \text{if } j < i. \end{cases}$$

Only the fact that  $\theta$  preserves the relation (SG2) is not immediately obvious. Fix three indices  $i, j, k$  satisfying  $i < j < k$  and let  $p, q, r$  be any permutation of these indices. One has to check that the equality  $\theta((pq)_a^{(qr)^b}) = \theta((pr)_{ab})$  holds. For example, the relation  $\theta((jk)_a^{(ki)^b}) = \theta((ji)_{ab})$  can be proved as follows:

$$\theta((jk)_a^{(ki)^b}) = (jk)_a^{(ik)^{b-1}} = (ik)_{b-1}^{(jk)^a} = (ij)_{b-1}^{a-1} = \theta((ji)_{ab}).$$

In the above formula the second equality holds by (2.11) and the third one holds by (2.13) (in which both sides are conjugated by  $(jk)_a$ ). Verification of the other 5 equalities is similar and is left as an exercise.

To finish the proof of Proposition 2.7 it remains to show the following lemma.

an:coxeter-amalgam

**Lemma 2.9.** *For each  $1 \leq t \leq n-2$  the map  $f_t: S_n(t, G) \rightarrow S_n(t+1, G)$  induced by the obvious embedding of generators is an isomorphism.*

*Proof.* Let  $(ij)_a$  be a generator of  $S_n(t+1, G)$ , define the map  $g_t$  inverse to  $f_t$  as follows:

$$g_t((ij)_a) = \begin{cases} (ij)_a & \text{if } j-i \leq t, \\ (ik)_a^{(kj)_1} & \text{if } j-i = t+1 \text{ for some } i < k < j. \end{cases}$$

The value of  $g_t((ij)_a)$  does not depend on the choice of  $k$ . Indeed, let  $k' > k$  be another index lying between  $i$  and  $j$ . By (2.12) we have the equality  $(ik)_a^{(k'j)_1} = (ik)_a$  hence

$$(ik)_a^{(kj)_1} = (ik)_a^{((kk')_1^{(k'j)_1})} = (ik)_a^{(kk')_1(k'j)_1} = (ik')_a^{(k'j)_1}.$$

Let us show that  $g_t$  is well defined (i.e. preserves the defining relations of  $S_n(t+1, G)$ ). Below we outline the proof of this fact for relation (2.11) and leave the verification for other relations as an exercise.

First consider the case  $k-j \leq t$  and  $j-i = t+1$ . Choose some index  $j'$  between  $i$  and  $j$ . Using the defining relations of  $S_n(t, G)$  we get the following chain of equalities:

{CoxMix2New}

$$\begin{aligned} (2.14) \quad g_t((ij)_a^{(jk)^b}) &= (ij')_a^{(j'j)_1(jk)^b} = (ij')_a^{(jk)_b(j'j)_1(jk)^b} = (ij')_a^{(jk)_1(j'j)_b(jk)_1} = \\ &= (ij')_a^{(j'j)_b(jk)_1} = (j'j)_{b'}^{(ij')_a(jk)_1} = (j'j)_{b'}^{(jk)_1(ij')_{a'}} = \\ &= (jk)_{b'}^{(j'j)_1(ij')_{a'}} = (jk)_{b'}^{(ij')_{a'}(j'j)_1(ij')_{a'}} = (jk)_{b'}^{(j'j)_1(ij')_{a'}(j'j)_1} = g_t((jk)_{b'}^{(ij)_{a'}}). \end{aligned}$$

The proof in the case  $k-j = t+1$  is similar:

$$g_t((ij)_a^{(jk)^b}) = (ij)_a^{(k'k)_1(jk')_b(k'k)_1} = (ij)_a^{(jk')_b(k'k)_1} = (jk')_{b'}^{(ij)_{a'}(k'k)_1} = (jk')_{b'}^{(k'k)_1(ij)_{a'}} = g_t((jk)_{b'}^{(ij)_{a'}}).$$

Here  $k'$  is some index lying between  $j$  and  $k$ . In the central equality we use either (2.11) or (2.14) depending on whether  $j-i \leq t$  or  $j-i = t+1$ .  $\square$

sec:Hnextensions

### 3. COMPARISON WITH EXTENSIONS OF TYPE $\mathfrak{H}_n(G)$

The aim of this section is to express the group  $S_n(G)$  in terms of Rehmann's extensions of type  $\mathfrak{H}_n$  and thus finish the proof of Theorem 1.

**3.1. Presentation for the subgroup  $HS_n(G)$ .** Our first goal is to obtain an explicit presentation of the subgroup  $HS_n(G)$  of  $S_n(G)$ .

For every  $i \neq j$  and  $a \in G$  we define the element  $h_{ij}(a) \in HS_n(G)$  as follows:

$$\boxed{\text{eq:h-def}} \quad (3.1) \quad h_{ij}(a) = (ij)_a \cdot (ij)_1.$$

It is not hard to show that  $h_{ij}(a)$  form a generating set for  $HS_n(G)$ . In fact, there is an explicit formula how an element of  $HS_n(G)$  originally expressed through  $(ij)_a$ 's can be rewritten in terms of  $h_{ij}(a)$ . Indeed, if  $h$  lies in  $HS_n(G)$  and is written as  $\prod_{k=1}^N (i_k j_k)_{a_k}$  for some  $i_k \neq j_k$  and  $a_k \in G$  then it can be rewritten as follows:

$$\boxed{\text{eq:rp}} \quad (\tau) \quad h = \prod_{k=1}^N h_{\sigma_k(i_k), \sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1} (i_s j_s) \in S_n.$$

We briefly recall the notion of a *rewriting process* given in [9, § 2.3]. If  $G$  is a group presented by generators  $a_\nu$  and relations  $R_\mu(a_\nu)$  and  $H$  is its subgroup with a generating set  $J_i(a_\nu)$  then a *rewriting process for  $H$*  is a function which maps every word  $u$  in alphabet  $a_\nu$  to a word  $v$  in alphabet  $s_i$  such that  $u$  and  $v[s_i := J_i]$  define the same element of  $G$  whenever  $u$  represents an element of  $H$ .

With this terminology, the mapping  $(\tau)$  defined above is a rewriting process for the subgroup  $HS_n(G)$ . Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [9, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [9, § 2.3]):

- if  $U$  and  $U^*$  are freely equal words in  $(ij)_a$  then  $\tau(U)$  and  $\tau(U^*)$  are also freely equal words in  $h_{ij}(a)$ ;
- if  $U_1$  and  $U_2$  are two words in  $(ij)_a$  defining certain elements of  $HS_n(G)$  then the words  $\tau(U_1 U_2)$  and  $\tau(U_1) \tau(U_2)$  are equal.

Using these two properties and repeating the arguments used in the proof of [9, Theorem 2.8] one can simplify the generic presentation of  $HS_n(G)$  from [9, Theorem 2.6] and obtain the following.

$\boxed{\text{lm:h-gen}}$  **Lemma 3.1.** *For  $n \geq 3$  the group  $HS_n(G)$  admits presentation on the generators  $h_{ij}(a)$  with the following two families of defining relations:*

$$\boxed{\text{eq:tau1}} \quad (3.2) \quad h_{ij}(a) = \tau((ij)_a \cdot (ij)_1);$$

$$\boxed{\text{eq:tau2}} \quad (3.3) \quad \tau(KRK^{-1}) = 1,$$

where  $R$  varies over relations of Proposition 2.6 and  $K$  is any word in  $(ij)_1$ ,  $i \neq j$ .

From (3.2) one immediately obtains the equality  $h_{ij}(1) = 1$ ,  $i \neq j$ . Thus, if we denote by  $\sigma$  the permutation corresponding to a word  $K$  and let  $S_n$  act on  $h_{ij}(a)$ 's in the natural way we get that the word  $\tau(KRK^{-1})$  is equivalent to  ${}^\sigma \tau(R)$  (modulo relations  $h_{ij}(1) = 1$ ).

Since the image of every relator from Proposition 2.6 under the action of  $S_n$  is still a relator of the same form, only relations of the form  $\tau(R) = 1$  are, in fact, needed for the presentation of  $HS_n(G)$ . Writing down what  $\tau(R)$  is for each of (SG1)–(SG4) we get the following.

**prop:HSpres**

**Proposition 3.2.** For  $n \geq 3$  the group  $HS_n(G)$  admits presentation with generators  $h_{ij}(a)$  and the following list of relations:

- |             |      |   |
|-------------|------|---|
| <b>{H0}</b> | (R0) | $h_{ij}(1) = 1,$                            |
| <b>{H1}</b> | (R1) | $h_{ij}(a)h_{ji}(a) = 1,$                   |
| <b>{H2}</b> | (R2) | $h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab),$ |
| <b>{H3}</b> | (R3) | $[h_{ij}(a), h_{kl}(b)] = 1,$               |
| <b>{H4}</b> | (R4) | $h_{ij}(a)^{-1} = h_{ij}(a^{-1}).$          |

In the next subsection we put the group  $HS_n(G)$  into the context of Rehmann's extensions of type  $\mathfrak{H}_n(G)$ .

**3.2. Extensions of type  $\mathfrak{H}_n$ .** We start with a brief review of the material of § 1–3 of [13]. For  $n \geq 3$  denote by  $D_n(G)$  the subgroup of  $G^n$  consisting of vectors  $(g_1, \dots, g_n)$  for which the product  $g_1 g_2 \dots g_n$  lies in the derived subgroup  $[G, G]$ . Clearly,  $D_n(G)$  is a normal subgroup of  $G^n$  and the quotient  $G^n/D_n(G)$  is isomorphic to  $G^{\text{ab}} \cong H_1(G, \mathbb{Z})$ . The subgroup  $D_n(G)$  is generated by elements  $d_{ij}(g)$ , where  $d_{ij}(g)$  denotes the element of  $G^n$  whose  $i$ -th component equals  $g$ ,  $j$ -th component equals  $g^{-1}$  and all other components are trivial.

The following definition is copied from [13, § 2]. Consider the group  $H_n(G)$  defined by generators  $h_{ij}(u)$ ,  $u \in G$ ,  $i \neq j$  subject to the following relations:

- |              |      |  |                |
|--------------|------|--|----------------|
| <b>{RH1}</b> | (H1) | $h_{ij}(u)h_{ji}(u) = 1,$                                      |                |
| <b>{RH2}</b> | (H2) | $h_{ij}(u)h_{ki}(u)h_{jk}(u) = 1,$                             |                |
| <b>{RH3}</b> | (H3) | $h_{ij}(u)h_{ik}(v)h_{ij}(u)^{-1} = h_{ik}(uv)h_{ik}(u)^{-1},$ | for $j \neq k$ |
| <b>{RH4}</b> | (H4) | $h_{ij}(u)h_{kj}(v)h_{ij}(u)^{-1} = h_{kj}(vu)h_{kj}(u)^{-1},$ | for $i \neq k$ |
| <b>{RH5}</b> | (H5) | $[h_{ij}(u), h_{kl}(v)] = 1.$                                  |                |

There is a surjective group homomorphism  $H_n(G) \twoheadrightarrow D_n(G)$  sending  $h_{ij}(u)$  to  $d_{ij}(u)$ .

By definition, an *extension of type  $\mathfrak{H}_n(G)$*  is an extension  $H$  of  $D_n(G)$  which is also a quotient of  $H_n(G)$ , i. e. the one which fits into the following diagram.

$$\begin{array}{ccc} H_n(G) & & \\ \downarrow & \searrow & \\ H & \twoheadrightarrow & D_n(G) \end{array}$$

For  $1 \leq k \leq n$  denote by  $\iota_k$  the inclusion map which sends an element  $g$  of  $[G, G]$  to the vector of  $G^n$  whose only nontrivial component equals  $g$  and is on  $k$ -th position. The relationship between nonabelian tensor products and extensions of type  $\mathfrak{H}_n(G)$  is best explained by the following pull-back diagram.

$$\begin{array}{ccc} G \widetilde{\otimes} G & \twoheadrightarrow & [G, G] \\ \downarrow \lrcorner & & \downarrow \iota_k \\ H_n(G) & \twoheadrightarrow & D_n(G) \end{array}$$

Here  $k$  is arbitrary and  $G \widetilde{\otimes} G$  denotes the nonabelian antisymmetric square (it is denoted by  $U(G)$  in [13] and by  $(G, G)$  in [4]).



A quotient of  $G \tilde{\otimes} G$  is called an *extension of type*  $\mathfrak{U}(G)$  (cf. [13, § 1]). There is one-to-one correspondence between extensions of type  $\mathfrak{H}_n(G)$  and those of type  $\mathfrak{U}(G)$ . One obtains a  $\mathfrak{U}(G)$ -extension  $U_H$  from a given  $\mathfrak{H}_n(G)$ -extension  $H$  by taking pull-back square as in the above diagram. In other words,  $U_H$  is the subgroup of  $H$  generated by all the "symbols"  $c_{kj}(u, v)$ , where

$$c_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, \quad 1 \leq i \neq j \leq n, \quad u, v \in G.$$

(Notice that symbols  $c_{ij}(u, v)$  do not depend on  $j$ ). The inverse construction which produces an  $\mathfrak{H}_n(G)$  extension from the given  $\mathfrak{U}(G)$ -extension and a number  $n$  is described in detail in [13, § 3].

We now proceed with the proof of Theorem 1. Our main result here is the following.

prop:simpler

**Proposition 3.3.** *For  $n \geq 3$  relations (R0)–(R4) imply (H1)–(H5). Moreover, for  $n \geq 4$  relations (R0)–(R3) are equivalent to (H1)–(H5).*

For  $u, v \in G$  define the following symbol (notice the difference between  $c_{ij}$  and  $c'_{ij}$ ):

$$c'_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies  $h_{ij}(1) = 1$  therefore  $c_{ij}(u, u^{-1}) = c'_{ij}(u, u^{-1})$ . Let us show that (H3) can be omitted from the definition of  $H_n(G)$  provided  $n \geq 4$ .

item-lem33

**Lemma 3.4.** *If one excludes relation (H3) from the list of defining relations for the group  $H_n(G)$ ,  $n \geq 3$ , the following assertions still remain true:*

- (1) *The elements  $c_{ij}(u, u^{-1})$  are central in  $H_n(G)$ ;*
- (2) *One has  $c'_{ij}(u, v) = [h_{ij}(u), h_{kj}(v)]$ ,  $k \neq i, j$ ;*
- (3) *One has  $^{h_{ij}(w)}c'_{kj}(u, v) = c'_{kj}(u, w)^{-1}c'_{kj}(u, vw)$ ,  $k \neq i, j$ ;*
- (4) *One has  $^{h_{ij}(w)}c'_{ij}(u, v) = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}$ .*

*Proof.* First one shows using (H4) that  $c_{ij}(u, u^{-1})$  centralizes  $h_{kj}(v)$  (cf. with the proof of [13, Lemma 2.1(2)]). Since in any group  $[a, b] = 1$  implies  $[a^{-1}, b] = [a, b^{-1}] = [a^{-1}, b^{-1}] = 1$  we get that  $c_{ij}(u, u^{-1})$  also centralizes  $h_{kj}(v)^{-1} = h_{jk}(v)$  and  $c_{ij}(u^{-1}, u)^{-1} = c_{ji}(u, u^{-1})$  centralizes both  $h_{kj}(v)$  and  $h_{jk}(v)$ . Together with (H2) and (H5) this implies that  $c_{ij}(u, u^{-1})$  centralizes all the generators of  $H_n(G)$  and hence lies in the center of  $H_n(G)$ .

The second and the third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$$^{h_{ij}(w)}c'_{ij}(u, v) = ^{h_{ij}(w)}c'_{ij}(v, u)^{-1} = (c'_{kj}(v, w)^{-1}c'_{kj}(v, uw))^{-1} = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}.$$

□

**Lemma 3.5.** *If one excludes relation (H3) from the list of defining relations for the group  $H_n(G)$ ,  $n \geq 3$ , the following statements are equivalent:*

item1

(1) (H3) holds;

item4

(2) one has  $c'_{ij}(u, v)^{-1} = c'_{ij}(v, u)$ ;

item5

(3) symbols  $c'_{ij}(u, v)$  do not depend on  $i$ ;

item5

(4) one has  $c'_{ij}(u, vw) = c'_{ij}(wu, v) \cdot c'_{ij}(uv, w)$ ;

item6

(5) one has  $c'_{ij}(u, vu^{-1}) = c'_{ij}(uv, u^{-1})$ .

*Proof.* Implications (1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (4) are essentially contained in the proof of [13, Lemmas 2.1-2.2]. Implication (4)  $\implies$  (5) is trivial.

We now prove (5)  $\implies$  (1). Notice that (H4) implies  $h_{ij}(u)^{-1}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$ , therefore using Lemma 3.4.(1) we get that  $h_{ij}(u)h_{ik}(v) = c_{ij}(u, u^{-1})h_{ij}(u^{-1})^{-1}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$ . Thus, (H3) is equivalent to the equality

$$h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}),$$

or what is the same

$$c'_{ik}(u^{-1}, uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1}, u).$$

It is clear that the last equality is an equivalent reformulation of (5).  $\square$

*Proof of Proposition 3.3.* The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);
- for  $n \geq 4$  the third statement of the previous lemma follows from Lemma 3.4.(2).

$\square$

cor:main

**Corollary 3.6.** *For  $n \geq 3$  the group  $HS_n(G)$  is the quotient of the extension  $H_n(G)$  by (R4) (in particular, it is an extension of type  $\mathfrak{H}_n(G)$ ). Moreover, there are natural isomorphisms:*

$$(3.4) \quad \text{Ker}(HS_n(G) \rightarrow D_n(G)) = \text{Ker}(S_n(G) \rightarrow G \wr S_n) \cong H_2(G, \mathbb{Z}).$$

*Proof.* Most of the assertions follow from Proposition 3.3 and [4, Proposition 5]. One also has to use the formula  $c_{ij}(u, u) = c_{ij}(u, u^{-1})$  (see [13, p. 87]).  $\square$

sec:main

#### 4. PROOF OF THEOREM 2

Recall that for arbitrary set  $X$  one defines the space  $EX$  as the the simplicial set whose set of  $k$ -simplices  $EX_k$  is  $X^{k+1}$  and whose faces and degeneracies are obtained by omitting and repeating components. For a group  $G$  we denote by  $\pi_G$  the canonical map  $EG \rightarrow BG$  sending  $(g, h) \in EG_1$  to  $g^{-1}h \in BG_1$ .

Now let  $N$  be a group acting on  $X$ . We define two simplicial sets  $U$  and  $V$  as follows:

$$U = \bigcup_{n \in N} E(\Gamma_{n,-}) \subseteq E(X \times X), \quad V = \bigcup_{x, y \in X} E(N(x \rightarrow y)) \subseteq EN.$$

Here  $\Gamma_{n,-}$  is the graph of the function  $(x \mapsto nx)$  and  $N(x \rightarrow y)$  denotes the subset of elements  $n \in N$  satisfying  $nx = y$ . With this notation the subset  $N(x \rightarrow x)$  coincides with the stabilizer subgroup  $N_x \leq N$ .

lm:quillen-a

**Lemma 4.1.** *The simplicial sets  $U$  and  $V$  are homotopy equivalent.*

*Proof.* First, we define yet another simplicial set  $W$  as follows. Its  $k$ -simplices  $W_k$  are matrices  $\begin{pmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{pmatrix}$ , where  $x_i \in X$  are  $n_i \in N$  are such that all  $n_i$ 's act each  $x_j$  in the same way, i.e.  $n_i x_j = n_{i'} x_j$  for  $0 \leq i, i', j \leq k$ . The faces and degeneracies of  $W$  are the maps of omission and repetition of columns.

Now there are two simplicial maps  $f: W \rightarrow U$ ,  $g: W \rightarrow V$  whose action on 0-simplices is given by  $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = (x_0, n_0 x_0)$ ,  $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = n_0$ . To prove the lemma it suffices to show that  $f$  and  $g$  are homotopy equivalences. The proof for  $f$  and  $g$  is similar, let us show, for example, that  $g$  is a homotopy equivalence.

In view of Quillen theorem A (cf.[15, ex. IV.3.11]) it suffices to show that for each  $p$ -simplex  $d: \Delta^p \rightarrow V$  the pullback  $g/(p, d)$  of  $d$  and  $g$  is contractible. The simplicial set  $g/(p, d)$  can be interpreted as the subset of  $\Delta^p \times E(X)$  whose set of  $k$ -simplices consists of pairs  $(\alpha: \underline{k} \rightarrow \underline{p}, (x_0, \dots, x_k) \in X(\alpha, d)^{k+1})$ . Here  $X(\alpha, d)$  is the subset of  $X$  consisting of all  $x$  for which  $d_{\alpha(i)}x = d_{\alpha(j)}x$  for  $0 \leq i, j \leq k$ . Notice that the set  $X_d := X(id_{\underline{p}}, d)$  is nonempty and is contained in every  $X(\alpha, d)$  (it even equals  $X(\alpha, d)$  for surjective  $\alpha$ ). Now choose a point  $\tilde{x} \in X_d$  and consider the simplicial homotopy

$$H: \Delta^p \times EX \times \Delta^1 \rightarrow \Delta^p \times EX$$

between the identity map of  $\Delta^p \times EX$  and the map  $\Delta^p \times c_{\tilde{x}}$ , where  $c_{\tilde{x}}$  is the constant map. More concretely,  $H$  sends each triple  $(\alpha: \underline{k} \rightarrow \underline{p}, (x_0, \dots, x_k), \beta: \underline{k} \rightarrow \underline{1})$  to  $(\alpha, (x_0, \dots, x_{i-1}, \tilde{x}, \dots, \tilde{x}))$ , where  $i$  is the minimal number such that  $\beta(i) = 1$ . By the choice of  $\tilde{x}$  the image of  $H$  restricted to  $g/(p, d) \times \Delta^1$  is contained in  $g/(p, d)$ , hence  $g/(p, d)$  is contractible.  $\square$

Now suppose that  $X = H$  is also a group upon which  $N$  acts on the left.

**cor:ker-iso**

**Corollary 4.2.** *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

*There is an isomorphism  $\theta: \text{Ker}(\pi_1(S) \rightarrow N) \cong \text{Ker}(\pi_1(T) \rightarrow H \times H)$ . Moreover, the higher homotopy groups of  $S$  and  $T$  are isomorphic.*

*Proof.* Consider the following two pull-back squares:

$$\begin{array}{ccc} V \hookrightarrow EN & & U \hookrightarrow E(H \times H) \\ \downarrow \lrcorner & \downarrow \pi_N & \downarrow \lrcorner \\ S \hookrightarrow BN & & T \hookrightarrow B(H \times H) \end{array}$$

The required isomorphism can be obtained from the homotopy long exact sequence applied to the left arrows of these diagrams.  $\square$

Now let  $G$  be a group. Set  $N = G \wr S_n$ ,  $H = G^n$  and consider the left action of  $N$  on  $H$  given by  $(g, s) \cdot h = gh^{s^{-1}}$ ,  $g, h \in G^n$ ,  $s \in S_n$ . If one reorders the components of  $BG^{2n}$  accordingly, the simplicial subset  $T \subset BG^{2n}$  from the above corollary becomes precisely the preimage of 0 under  $h_n: BG^{2n} \rightarrow \mathbb{Z}[BG]$ .

It is also easy to compute the map  $\pi_1(S) \rightarrow N$ . Indeed, van Kampen theorem [10, Theorem 2.7] asserts that  $\pi_1(S)$  is isomorphic to the free product of stabilizer subgroups  $N_h \leq N$  amalgamated over pairwise intersections  $N_h \cap N_{h'}$ ,  $h, h' \in H$ . For  $h \in G^n$  the subgroup  $N_h$  consists of elements  $(g, s) \in N$  satisfying  $gh^{s^{-1}} = h$ , i.e. elements of the form  $(hh^{-s^{-1}}, s)$ . Thus,  $N_h \cong S_n$ ,  $\pi_1(S)$  is isomorphic to the group  $S_n(G)$  and the map  $\pi_1(S) \rightarrow N$  coincides with the map  $\mu$  defined in section 2. In particular, for an abelian  $G$  its kernel is generated by symbols  $c_{ij}(u, v)$ .

The following lemma gives a more concrete description of the isomorphism  $\theta$  from Corollary 4.2.

**concrete-formula**

**Lemma 4.3.** *If  $G$  is abelian then every generator  $c_{ij}(u, v) \in \text{Ker}(\pi_1(S) \rightarrow N)$  is mapped under  $\theta$  to the homotopy class of the loop  $f_{ij}(u, v)$ , which is defined as follows:*

$$f_{ij}(u, v) = p_{ij}(u) \circ p_{ij}(v) \circ p_{ij}(u^{-1}v^{-1}), \text{ where } p_{ij}(x) = (x[i], x[i]) \circ (x^{-1}[i], x^{-1}[j]).$$

*Proof.* Fix a vector  $g \in G^n$  and denote by  $\gamma_{ij}(g, x)$  the path of length 4 in  $W$  which connects the following five points of  $W_0$ :

$$\left( \begin{smallmatrix} e \\ (g; e) \end{smallmatrix} \right), \left( \begin{smallmatrix} x[i] \\ (g; e) \end{smallmatrix} \right), \left( \begin{smallmatrix} x[i] \\ (d_{ij}(x)g, (ij)) \end{smallmatrix} \right), \left( \begin{smallmatrix} e \\ (d_{ij}(x)g, (ij)) \end{smallmatrix} \right), \left( \begin{smallmatrix} e \\ (d_{ij}(x)g, e) \end{smallmatrix} \right).$$

One can easily check that the image of  $\gamma_{ij}(g, x)$  in  $S$  under  $\pi_N g$  is precisely the element  $h_{ij}(x)$ , while  $\pi_{H \times H} f$  sends  $\gamma_{ij}(g, x)$  to the loop  $p_{ij}(x)$  (here  $f, g$  are the maps from the proof of Lemma 4.1).  $\square$

**lm: endotr**

**Lemma 4.4.** *The only natural endotransformations of the functor  $H_2(-, \mathbb{Z}): \mathbf{Groups} \rightarrow \mathbf{Ab}$  are morphisms of multiplication by  $n \in \mathbb{Z}$ .*

*Proof.* Denote by  $\eta$  an endotransformation  $H_2(-, \mathbb{Z}) \rightarrow H_2(-, \mathbb{Z})$ . When restricted to the subcategory of free finitely-generated abelian groups  $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$  the second homology functor coincides with the second exterior power functor  $A \mapsto \wedge^2 A$ .

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic  $\mathbb{Z}$ -modules (see Definition 6.13.5 ibid.) The functor  $A \mapsto \wedge^2 A$  is clearly quadratic and corresponds to the quadratic  $\mathbb{Z}$ -module  $0 \rightarrow \mathbb{Z} \rightarrow 0$  under this equivalence. Thus, we get that  $\eta$  restricted to  $\mathbf{Add}(\mathbb{Z})$  coincides with the morphism of multiplication by  $n \in \mathbb{Z}$ .

Consider the group  $\Gamma_k = \langle x_1, y_1, \dots, x_k, y_k \mid [x_1, y_1] \cdots [x_k, y_k] \rangle$  (the fundamental group of a sphere with  $k$  handles). It is clear that the abelianization map  $\Gamma_k \rightarrow \mathbb{Z}^{2k}$  induces an injection  $H_2(\Gamma_k, \mathbb{Z}) \cong \mathbb{Z} \rightarrow \wedge^2 \mathbb{Z}^{2k}$ . Consider the following diagrams.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad} & \wedge^2 \mathbb{Z}^{2k} \\ \eta_{\Gamma_k} \downarrow & & \downarrow n \cdot \\ \mathbb{Z} & \xrightarrow{\quad} & \wedge^2 \mathbb{Z}^{2k} \end{array} \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\chi} & H_2(G, \mathbb{Z}) \\ \eta_{\Gamma_k} \downarrow & & \downarrow \eta_G \\ \mathbb{Z} & \xrightarrow{\chi} & H_2(G, \mathbb{Z}) \end{array}$$

From the left diagram it follows that  $\eta_{\Gamma_k}$  is also the morphism of multiplication by  $n$ . For every element  $x \in H_2(G, \mathbb{Z})$  there exist an integer  $k$  and a map  $\chi: \Gamma_k \rightarrow G$  sending the generator of  $H_2(\Gamma_k, \mathbb{Z})$  to  $x$ . From the right square we conclude that  $\eta_G(x) = nx$ , as claimed.  $\square$

*Proof of Theorem 2.* Factor  $h_n$  as a composition of a trivial cofibration followed by a fibration  $BG^{2n} \rightarrow E_{h_n} \rightarrow \mathbb{Z}[BG]$  and write down the starting portion of the long homotopy exact sequence for the latter. If we denote by  $K$  the kernel of the map  $\nu: \pi_1(T) \rightarrow G^{2n}$  induced by the embedding  $h_n^{-1}(0) = T \subseteq BG^{2n}$  we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\beta} & \pi_1(T) & \xrightarrow{\nu} & G^{2n} & \longrightarrow & H_1(G, \mathbb{Z}) & \longrightarrow & 1 \\ & & \psi \downarrow & & \varphi \downarrow & & \cong \downarrow & & \parallel & & \\ 1 & \longrightarrow & H_2(G, \mathbb{Z}) & \xrightarrow{\alpha} & \pi_1(F_{h_n}) & \longrightarrow & \pi_1(E_{h_n}) & \longrightarrow & H_1(G, \mathbb{Z}) & \longrightarrow & 1. \end{array}$$

We already know by Corollaries 3.6 and 4.2 that  $K$  is naturally isomorphic to  $H_2(G, \mathbb{Z})$  provided  $n \geq 3$ . Thus, it suffices to show that  $\psi$  is an isomorphism. In view of Lemma 4.4 we only need to consider the case when  $G$  is abelian.

We are going to compute the image of each generator  $c_{ij}(x, y) = x \wedge y \in H_2(G, \mathbb{Z})$  under  $\psi$ . In order to do this we need to obtain a more explicit description of the set of 1-simplices of  $F_{h_n}$ . Since both  $BG^{2n}$  and  $\mathbb{Z}[BG]$  are fibrant  $E_{h_n}$  and  $F_{h_n}$  can be constructed using the path space

construction, i.e.  $E_{h_n} = BG^{2n} \times_{\mathbb{Z}[BG]} \mathbb{Z}[BG]^I$ . In this case the set of 1-simplices of  $F_{h_n}$  can be identified with the set of triples  $(g, t, t') \in G^{2n} \times \mathbb{Z}[BG]_2 \times \mathbb{Z}[BG]_2$  satisfying the following identities:

$$d_2(t) = h_n(g), \quad d_1(t) = d_1(t'), \quad d_0(t') = 0.$$

It will be convenient for us to perform all calculations inside the subset  $F' \subset (F_{h_n})_1$  consisting of those triples for which  $t' = 0$  and  $d_1(t) = d_0(t) = 0$ . For shortness we write down an element of  $F'$  as  $(g, t)$ . It is easy to check that the image of the loop  $p_{ij}(x)$  under the natural map  $T \rightarrow F_{h_n}$  corresponds to the pair  $((e, d_{ij}(x)), t(x))$ , where  $t(x) = -(x, x^{-1}) + (e, e) - (x^{-1}, e) + (e, x^{-1})$ .

We want to devise a concrete formula for the element of  $F'$  homotopic to the concatenation of two elements  $(g_1, t_1), (g_2, t_2) \in F'$ . It is easy to check that  $(g_1 g_2, t)$ , where  $t$  is the 1-st face of the filler  $f$  for the 3-horn  $(t_2, t_1, h_n(g_1, g_2))$ , is the desired element. By Moore's theorem one can find a filler for any horn  $(p_0, p_2, p_3)$  in a simplicial abelian group via the formula  $f = s_0 p_0 - s_0 s_1 d_2 p_0 - s_0 s_0 d_1 p_0 + s_0 s_0 d_2 p_0 + s_1 p_2 + s_2 p_3 - s_1 p_3$ . Substituting concrete values of  $p_i$  into this formula and using the fact that  $d_1 t_2 = 0$  we get the following expression for  $t$ :

$$\begin{aligned} t = d_1 f &= p_0 - s_1 d_2 p_0 - s_0 d_1 p_0 + s_0 d_2 p_0 + p_2 - p_3 + s_1 d_1 p_3 = \\ &= t_1 + t_2 - h_n(g_1, g_2) - s_1 h_n(g_2) + s_0 h_n(g_2) + s_1 h_n(g_1 g_2). \end{aligned}$$

Applying the above formula twice we get that the image of  $c_{ij}(u, v)$  under  $\varphi\beta$  equals  $((e, e), t)$ , where

$$(4.1) \quad \begin{aligned} t &= 2(e, e) - (x^{-1}, e) + (xy, x^{-1}y^{-1}) - (xy, e) - (x, x^{-1}) - (y, y^{-1}) - \\ &\quad - (e, x^{-1}y^{-1}) + (e, x^{-1}) + (x, y) + (x^{-1}, y^{-1}) - (e, y) - (xyx^{-1}y^{-1}, e) + (y, e). \end{aligned}$$

The map  $\alpha$  can be described as the Dold—Kan isomorphism  $H_2(G, \mathbb{Z}) \cong \pi_2(\mathbb{Z}[BG])$  followed by the boundary map  $\partial: \pi_2(\mathbb{Z}[BG]) \rightarrow \pi_1(F_{h_n})$ . It is not hard to show that  $\partial$  maps each  $c \in \pi_2(\mathbb{Z}[BG])$  to  $((e, e), -c)$  (see [5, p. 29]). Thus, we get that  $\psi$  maps  $x \wedge y$  to  $-t$ , where  $t$  is given by (4.1).

On the other hand, the generator  $x \wedge y$  of  $H_2(G, \mathbb{Z})$  corresponds to the 2-cycle  $c = (x, y) - (y, x)$  (cf. [11, (14), p. 582]) and the corresponding element of  $\pi_2(\mathbb{Z}[BG])$  is simply the normalized 2-cycle of  $c$ :

$$c' = c - s_0 d_0 c - s_1 d_1 c + s_1 d_0 = (x, y) - (y, x) - (x, e) + (y, e) + (e, x) - (e, y).$$

It is not hard to check that  $t - c'$  is the image of the following 3-chain under the differential  $d_0 - d_1 + d_2 - d_3$ :

$$(xy, x^{-1}, y^{-1}) - (y, x, x^{-1}) + s_1 s_1 (x^{-1} + xy - x - y) + s_0 s_0 (x^{-1} - x - x^{-1}y^{-1}).$$

Thus we obtain that  $[t] = -[c']$ , hence  $\psi$  is an isomorphism, as claimed.  $\square$

## APPENDIX A. SIMPLICIAL DOLD—SERRE FIBRATIONS

The aim of this appendix is to show that the map  $h_\infty: X^\infty \rightarrow \mathbb{Z}[X]$  mentioned in the introduction is a quasifibration.

We start with the definition of the simplicial analogue of the so-called "Dold—Serre fibration", i.e. a map possessing "weak covering homotopy property" (also called "delayed homotopy lifting property"), cf. [14, § 1.4.4]. We show in Lemma A.2 and ?? that the map  $h_\infty$  satisfies this property and that every map with this property is a quasifibration.

**Definition A.1.** Let  $C$  be a subclass of the class of acyclic cofibrations in the category of simplicial sets. We say that a map of simplicial sets  $p: E \rightarrow B$  satisfies *weak homotopy covering property* with

respect to  $C$  if for every commutative square

**{eq:plp-def}**

(A.1)

$$\begin{array}{ccc} U & \xrightarrow{g} & E \\ i \downarrow & \nearrow \tilde{F} & \downarrow p \\ V & \xrightarrow{F} & B \end{array}$$

for which  $i \in C$  there exists a map  $\tilde{F}$  such that the lower triangle commutes strictly and the upper one commutes up to a fiberwise homotopy  $H$  (i.e. a homotopy  $H$  such that  $pH: U \times I \rightarrow B$  coincides with the composite  $U \times I \xrightarrow{\pi} U \xrightarrow{pg} B$ ).

Denote by  $C_{pr}$  the class consisting of all inclusions  $i: U \hookrightarrow V$  of finite polyhedral simplicial sets for which there exists a deformation retract (i.e. a map  $r: V \rightarrow U$  homotopic to  $\text{id}_V$ ).

**lm:dhlp**

**Lemma A.2.** *For a fibrant and connected simplicial set  $X$  the map  $h_\infty: X^\infty \rightarrow \mathbb{Z}[X]$  satisfies the weak covering homotopy property with respect to the class  $C_{pr}$ .*

*Proof.* Let  $i_0$  be a map from  $C_{pr}$  and  $r$  be the corresponding retraction. We first reduce the problem to the special case when  $g$  is the constant map at the basepoint of  $X^\infty$  (which we denote by 0). Indeed, let  $F, g$  be arbitrary maps as in (A.1). Since  $U$  is finite the image of  $g$  is contained in  $X^{2N} \subset X^\infty$ . If now  $\tilde{F}_0$  is a lifting in the diagram similar to (A.1) in which the top map equals 0 and the bottom one is  $F_0 = F - h_\infty g r$  then  $\tilde{F} = gr \times \tilde{F}_0$  is the desired lifting of  $F$ .

Now choose a contractible fibrant simplicial set  $W$  which maps surjectively onto  $X$  (e.g. take  $W$  to be the path space fibration). Since  $p: W \twoheadrightarrow X$  is surjective, the associated map between free simplicial abelian groups is a Kan fibration. Since  $i_0$  is acyclic we can choose a lifting  $\tilde{F}$  in the following diagram

$$\begin{array}{ccc} U & \xrightarrow{0} & \mathbb{Z}[W] \\ i_0 \downarrow & \nearrow \tilde{F} & \downarrow \\ V & \xrightarrow{F} & \mathbb{Z}[X] \end{array}$$

It is clear that the image of  $\tilde{F}$  is contained in the simplicial subgroup  $\mathbb{Z}[W]_0 \leq \mathbb{Z}[W]$  which, by definition, consists of linear combinations  $\sum_i n_i w_i$  for which  $\sum_i n_i = 0$ . By [12, Lemma 9.1] the canonical map  $\mathbb{Z}[\underline{\text{Hom}}(V, W)]_0 \rightarrow \underline{\text{Hom}}(V, \mathbb{Z}[W]_0)$  is surjective, hence the map  $\tilde{F}$  can be lifted along  $h_W$  in such a way that the bottom triangle in following diagram commutes strictly.

$$\begin{array}{ccc} U & \xrightarrow{0} & W^\infty \\ i_0 \downarrow & \nearrow G & \downarrow h_W \\ V & \xrightarrow{\tilde{F}} & \mathbb{Z}[W] \end{array}$$

Notice that the image of  $Gi_0$  is contained in the fiber  $h_W^{-1}(0)$ . It is easy to see that  $h_W^{-1}(0)$  is contractible (the contracting homotopy for  $W^\infty$  can be restricted to  $h_W^{-1}(0)$ ). We get that  $Gi_0$  is fiberwise homotopic to 0 hence the composite map  $V \xrightarrow{G} W^\infty \rightarrow X^\infty$  is the desired lifting of  $F$ .  $\square$

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