

# AMALGAMS OF SYMMETRIC GROUPS AND NONABELIAN TENSOR PRODUCTS

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ABSTRACT. By amalgamating copies of symmetric group  $S_n$  we construct a central extension  $S_n(G)$  of a large subgroup of the wreath product  $G \wr S_n$ . We then establish a connection between this group and the theory of  $\mathfrak{U}(G)$  and  $\mathfrak{H}_n(G)$ -extensions developed by K. Dennis and U. Rehmann in 1970's. We also give a topological application of this construction by describing homotopy invariants of blah-blah-blah...

## 1. INTRODUCTION

The aim of this note is to establish a surprising connection between symmetric groups and (certain quotients of) nonabelian tensors squares.

Let  $G$  be a group. Consider the family of copies of the symmetric group  $S_n$  in which each copy is indexed by an  $n$ -tuple  $(g_1, \dots, g_n) \in G^n$ . Denote by  $S_n(G)$  the free product of groups from this family amalgamated over the single family of relations  $s_{(g)} = s_{(h)}$ , where  $s \in S_n$ ,  $g, h \in G^n$  are such that  $hg^{-1}$  is fixed by  $s$  (here  $s_{(g)}$  denotes a permutation  $s \in S_n$  embedded into the copy with index  $g$ ).

Consider the wreath product  $G \wr S_n = G^n \rtimes S_n$ . By the definition of  $S_n(G)$  there is a well defined map  $\mu_n: S_n(G) \rightarrow G \wr S_n$  given by  $s_{(g)} \mapsto s^g = (g^{-1} s g^{s^{-1}}, s)$ .

Our first main result is the following theorem.

thm:summary

**Theorem 1.** *For  $n \geq 3$  the map  $\mu_n$  is a crossed module whose cokernel and kernel are isomorphic to  $H_1(G, \mathbb{Z})$  and  $H_2(G, \mathbb{Z})$  respectively. Moreover, the group  $S_n(G)$  is isomorphic to  $S_n \rtimes H_n^\wedge(G)$ .*

Here  $H_n^\wedge(G)$  denotes the quotient of the universal extension of type  $\mathfrak{H}_n(G)$  modulo all symbols of the form  $\{x, x\} = 1$ ,  $x \in G$ . Theory of extensions of type  $\mathfrak{H}_n$  was developed by U. Rehmann in [11] for the purpose of generalization of Matsumoto theorem to skew-fields (we refer the reader to Section 2.4 for more details).

Theorem 1 and its proof (which itself involves little more than some basic combinatorial group theory) may be of interest for the following reasons.

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- The presentation is similar in spirit to results of Brown—Loday and Kassel—Reutenauer on the presentation of the linear Steinberg group in the form of an amalgam of braid groups.
- It gives a presentation of the universal extension  $H_n(G)$  with generators and relations which is simpler than the original presentation given in [11], see Proposition 2.9. Moreover, our presentation explicitly obtained from the presentation of  $S_n$  formulated in terms of transpositions.

Theorem 1 was proved in an attempt to answer some purely topological question, which we describe in more detail below.

For a pointed simplicial set  $(X, x_0)$  denote by  $\mathbb{Z}[X]$  the associated free simplicial abelian group. Define the "alternating" map of simplicial sets  $h_n: X^{2n} \rightarrow \mathbb{Z}[X]$  by the identity:

$$h_n(x_1, x_2, \dots, x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Denote by  $X^\infty$  the colimit of  $X^n$  with respect to the natural embeddings  $X^n \hookrightarrow X^{n+1}$ . Taking colimit of  $h_n$  we get the map  $h_\infty = \text{colim}_n(h_n): X^\infty \rightarrow \mathbb{Z}[X]$ .

Based on a recent result of S. Podkorytov one can deduce that this map is a quasifibration provided  $X$  is fibrant and connected. In particular, the natural map  $h_\infty^{-1}(0) \rightarrow F_{h_\infty}$  between the fiber and the homotopy fiber of  $h_\infty$  over 0 is a weak equivalence, see Lemma A.4.

thm:main

**Theorem 2.** *If  $X = BG$  is the classifying space of a group  $G$  then for  $n \geq 3$  the natural map  $h_n^{-1}(0) \rightarrow F_{h_n}$  induces an isomorphism of fundamental groups  $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$ .*

The proof of Theorem 2 goes as follows. ???

In the course of the proof we also find a presentation of the universal  $\mathfrak{H}_n(G)$ -extension that is simpler than the original presentation introduced by Rehmann in [11], see Proposition 2.9.

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## 2. THE EXTENSION $S_n(G)$

sec:QnG-def

**2.1. The definition of  $S_n(G)$ .** Let  $G$  be a group. Recall that the *wreath product* is, by definition, the semidirect product  $G^n \rtimes S_n$ , in which  $S_n$  acts on  $G^n$  on the right by permuting its factors.

Consider the family  $\{S_n^{(g)}\}_{g \in G^n}$  of isomorphic copies of  $S_n$  indexed by elements of  $G^n$  and let  $F$  be the free product of groups from this family. For  $s \in S_n$  and  $g \in G^n$  we denote by  $s_g$  the image of  $s$  in  $F$  under the canonical map  $S_n^{(g)} \rightarrow F$ .

**Definition 2.1.** We define the group  $S_n(G)$  to be the quotient of  $F$  modulo the following single family of relations:

$$(2.1) \quad s_g = s_h, \text{ where } s \text{ fixes } hg^{-1}.$$

We continue to denote by  $s_g$  the image of the element  $s_g \in F$  under the canonical map  $F \rightarrow S_n(G)$ .

*Remark 2.2.* Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1}g^{(s^{-1})}, s).$$

therefore the equality  $s^g = s^h$  holds in  $G \wr S_n$  iff  $g^{-1}g^{(s^{-1})} = h^{-1}h^{(s^{-1})}$ , or what is the same, iff  $s$  fixes  $hg^{-1}$ .

The last statement immediately implies that the map  $\mu: S_n(G) \rightarrow G \wr S_n$  given by  $\mu(s_g) = s^g = (g^{-1}g^{(s^{-1})}, s)$  is well-defined. We will see shortly that  $\mu$  is, in fact, a *crossed module* in the sense of [2, § 2.2].

Observe from the definition of  $S_n(G)$  that there is a split exact sequence.

$$(2.2) \quad 1 \longrightarrow \text{Ker}(\pi) \longrightarrow S_n(G) \xrightleftharpoons[\iota(1)]{\pi} S_n \longrightarrow 1,$$

Here the map  $\pi = \pi_{S_n} \circ \mu$  removes subscript  $g$  from each  $s_g$  and the section  $\iota(1)$  sends  $s$  to  $s_1$ , where  $1$  is the identity element of  $G^n$ . Thus, if we denote  $\text{Ker}(\pi)$  by  $HS_n(G)$  we get a decomposition  $S_n(G) = HS_n(G) \rtimes S_n$ .

**2.2. A crossed module structure on  $S_n(G)$ .** Recall from [2, § 2.2] that a *crossed module* is a morphism of groups  $\mu: M \rightarrow N$  together with a right action of  $N$  on  $M$  compatible with the conjugation action of  $N$  on itself, i. e.

$$(CM1) \quad \mu(m^n) = \mu(m)^n \text{ for all } n \in N, m \in M,$$

which also satisfies the following identity called *Peiffer identity*:

$$(CM2) \quad m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M.$$

In our situation, we let  $G \wr S_n$  act on  $S_n(G)$  by

$$(2.3) \quad (s_g)^{(h,t)} = s_{(gh)^t}^t, \text{ for } s, t \in S_n \text{ and } g, h \in G^n.$$

The goal of this subsection is to prove the following.

**Proposition 2.3.** *For  $n \geq 3$  the map  $\mu: S_n(G) \rightarrow G \wr S_n$  is a crossed module.*

From the fact that  $\mu$  is a crossed module one can deduce that  $\text{Ker}(\mu)$  is a central subgroup of  $S_n(G)$  and  $\text{Im}(\mu)$  is a normal subgroup of  $G \wr S_n$ . It is not hard to show that the group  $S_2(G)$  is isomorphic to a free product of copies of  $S_2$  (whose center is always trivial). Therefore, the requirement  $n \geq 3$  in the statement of Proposition 2.3 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of  $G \wr S_n$  on  $S_n(G)$  that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of  $S_n(G)$ , for which it takes the form:

**q:Peiffer-gen}**

$$(2.4) \quad t_h^{-1} s_g t_h = s_{(gh^{-1})^t \cdot h} \text{ for all } s, t \in S_n, g, h \in G^n.$$

If we act on both sides of the above formula by  $(h^{-1}, 1) \in G \wr S_n$  we obtain the equality  $t_1^{-1} s_{gh^{-1}} t_1 = s_{(gh^{-1})^t}^t$ . Thus, to prove (2.4) it suffices to show the following simpler relation:

**peiffer-simple}**

$$(2.5) \quad t_1^{-1} s_g t_1 = s_{g^t}^t \text{ for all } s, t \in S_n, g \in G^n.$$

The key step in the proof is the following lemma.

**em:transp-deff**

**Lemma 2.4.** *The relation (2.5) holds in the special case when  $s = (ij)$  and  $t = (kl)$  are two nonequal transpositions.*

*Proof.* First of all, we immediately check that (2.5) holds in the special case when  $t$  fixes  $g' \in G^n$ . Indeed, by (2.1) we have  $t_1 = t_{g'}$ , hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that  $l \neq i$  and  $l \neq j$ . Denote by  $g'$  the vector which differs from  $g$  only at  $l$ -th position, for which we set  $g'_l = g_k$ . Since the only nontrivial component of  $g'g^{-1}$  (resp.  $g'g^{-t}$ ) is located at  $l$ -th (resp.  $k$ -th) position, it is fixed by  $s$  (resp.  $s^t$ ), hence from (2.1) we conclude that  $s_g = s_{g'}$  (resp.  $(s^t)_{g'} = (s^t)_{g^t}$ ). Finally, since  $g'$  is fixed by  $t$ , we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g^t} = s_{g'}^t = (s^t)_{g^t}. \quad \square$$

*Proof of Proposition 2.3.* Let us show that (2.5) holds for arbitrary transpositions  $s, t \in S_n$ . It suffices to consider the case  $s = t = (ij)$ . After choosing some  $k \neq i, j$  and presenting  $(ij)$  as  $(kj)(ik)(kj)$  we use the lemma:

$$(ij)_1^{-1} (ij)_g (ij)_1 = (ij)_1^{-1} (kj)_g (ik)_g (kj)_g (ij)_1 = (ki)_{g(ij)} (jk)_{g(ij)} (ki)_{g(ij)} = (ij)_{g(ij)}.$$

Proposition now follows by induction on the length of permutations  $s, t$ .  $\square$

**2.3. An explicit presentation of  $S_n(G)$ .** This presentation turns out to be much more economical than the original definition given in section 2.1. An explicit presentation for the subgroup  $HS_n(G)$  is derived from this presentation via Reidemeister-Schreier method.

We start with the following simple lemma which is, in essence, a variant of the standard presentation of a Weyl group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

lm:Snpres

**Lemma 2.5.** *For  $n \geq 3$  The symmetric group  $S_n$  admits presentation with transpositions as the set of generators and the following list of defining relations (in every formula distinct letters denote distinct indices):*

{Sym1}	(S1)	$(ij)^2 = 1,$
{Sym2}	(S2)	$(ij)^{(jk)} = (ik),$
{Sym3}	(S3)	$[(ij), (kl)] = 1.$
{Sym0}	(S4)	$(ij) = (ji),$

Now we are ready to formulate the main result of this subsection.

prop:Q-pres

**Proposition 2.6.** *For  $n \geq 3$  and arbitrary group  $G$  the group  $S_n(G)$  admits presentation with the set of generators  $\{(ij)_a \mid i \neq j, 1 \leq i, j, \leq n, a \in G\}$  and the following list of relations (as before, distinct letters denote distinct indices):*

{Q1}	(SG1)	$(ij)_a^2 = 1,$
{Q2}	(SG2)	$(ij)_a^{(jk)_b} = (ik)_{ab},$
{Q3}	(SG3)	$[(ij)_a, (kl)_b] = 1.$
{Q4}	(SG4)	$(ij)_a = (ji)_{a^{-1}}$

*Proof.* Denote by  $S'$  the group from the statement of the proposition. For  $1 \leq i \leq n$  and  $x \in G$  denote by  $x[i]$  the element of  $G^n$  whose only nontrivial component equals  $x$  and is located in the  $i$ -th position.

It is not hard to deduce from the definition of  $S_n(G)$  and Lemma 2.4 that the formula  $(ij)_a \mapsto (ij)_{a[j]}$  gives a well-defined map  $\varphi: S' \rightarrow S_n(G)$ .

Now we are going to construct the map  $\psi: S_n(G) \rightarrow S'$  in the opposite direction. Using the presentation of  $S_n$  given by Lemma 2.5 we define for a fixed  $g \in G^n$  the map  $\psi_g: S_n \rightarrow S'$  by  $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$ . It is obvious that  $\psi_g$  preserves the defining relations (S1)–(S4) of  $S_n$ . It remains to show that the equation  $\psi_g(s) = \psi_h(s)$  holds whenever  $g, h \in G^n$  and  $s \in S_n$  satisfy the requirement of (2.1).

Indeed, if  $hg^{-1}$  is fixed by  $s$  then for every  $1 \leq i \leq n$  we have  $(hg^{-1})_i = (hg^{-1})_{s(i)}$ , or equivalently  $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$ . For example, if  $s$  is a cycle of length  $p$ , i.e.

$s = (i_1, i_2, \dots, i_p)$  with  $i_{k+1} = s(i_k)$  we get that

$$\psi_g(s) = \psi_g \left( \prod_{k=1}^{p-1} (i_k, i_{k+1}) \right) = \prod_{k=1}^{p-1} (i_k, i_{k+1})_{g_{i_k}^{-1} g_{i_{k+1}}} = \prod_{k=1}^{p-1} (i_k, i_{k+1})_{h_{i_k}^{-1} h_{i_{k+1}}} = \psi_h(s).$$

The proof for a general  $s$  is almost the same. Verification of the fact that  $\psi$  and  $\varphi$  are mutually inverse is also immediate.  $\square$

We now obtain a presentation of the subgroup  $HS_n(G)$ . For every  $i \neq j$  and  $a \in G$  we define the element  $h_{ij}(a) \in HS_n(G)$  as follows:

{eq:h-def}

$$(2.6) \quad h_{ij}(a) = (ij)_a \cdot (ij)_1.$$

It is not hard to show that  $h_{ij}(a)$  form a generating set for  $HS_n(G)$ . In fact, there is an explicit formula how an element of  $HS_n(G)$  originally expressed through  $(ij)_a$ 's can be rewritten in terms of  $h_{ij}(a)$ . Indeed, if  $h$  lies in  $HS_n(G)$  and is written as  $\prod_{k=1}^N (i_k j_k)_{a_k}$  for some  $i_k \neq j_k$  and  $a_k \in G$  then it can be rewritten as follows:

{eq:rp}

$$(2.7) \quad h = \prod_{k=1}^N h_{\sigma_k(i_k), \sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1} (i_s j_s) \in S_n.$$

We briefly recall the notion of a *rewriting process* given in [7, § 2.3]. If  $G$  is a group presented by generators  $a_\nu$  and relations  $R_\mu(a_\nu)$  and  $H$  is its subgroup with a generating set  $J_i(a_\nu)$  then a *rewriting process for  $H$*  is a function which maps every word  $u$  in alphabet  $a_\nu$  to a word  $v$  in alphabet  $s_i$  such that  $u$  and  $v[s_i := J_i]$  define the same element of  $G$  whenever  $u$  represents an element of  $H$ .

With this terminology, the mapping  $(\tau)$  defined above is a rewriting process for the subgroup  $HS_n(G)$ . Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [7, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [7, § 2.3]):

- if  $U$  and  $U^*$  are freely equal words in  $(ij)_a$  then  $\tau(U)$  and  $\tau(U^*)$  are also freely equal words in  $h_{ij}(a)$ ;
- if  $U_1$  and  $U_2$  are two words in  $(ij)_a$  which define elements of  $HS_n(G)$  then the words  $\tau(U_1 U_2)$  and  $\tau(U_1) \tau(U_2)$  are equal.

Using these two properties and repeating the arguments used in the proof of [7, Theorem 2.8] one can simplify the generic presentation of  $HS_n(G)$  given by [7, Theorem 2.6] and obtain the following.

**lm:h-gen**

**Lemma 2.7.** *For  $n \geq 3$  the group  $HS_n(G)$  admits presentation on the generators  $h_{ij}(a)$  with the following two families of defining relations:*

**{eq:tau1}**

$$(2.7) \quad h_{ij}(a) = \tau((ij)_a \cdot (ij)_1);$$

**{eq:tau2}**

$$(2.8) \quad \tau(KRK^{-1}) = 1,$$

where  $R$  varies over relations of Proposition 2.6 and  $K$  is any word in  $(ij)_1$ ,  $i \neq j$ .

From (2.7) one immediately obtains the equality  $h_{ij}(1) = 1$ ,  $i \neq j$ . Thus, if we denote by  $\sigma$  the permutation corresponding to a word  $K$  and let  $S_n$  act on  $h_{ij}(a)$ 's in the natural way we will get that the word  $\tau(KRK^{-1})$  is equivalent to  ${}^\sigma\tau(R)$  (modulo relations  $h_{ij}(1) = 1$ ).

Since the relations from Proposition 2.6 are respected by the action of  $S_n$ , only relations of the form  $\tau(R) = 1$  are, in fact, needed for the presentation of  $HS_n(G)$ . Writing down what  $\tau(R)$  is for each of (SG1)–(SG4) we get the following.

**prop:HSpres**

**Proposition 2.8.** *For  $n \geq 3$  the group  $HS_n(G)$  admits presentation with generators  $h_{ij}(a)$  and the following list of relations:*

**{H0}**

$$(R0) \quad h_{ij}(1) = 1,$$

**{H1}**

$$(R1) \quad h_{ij}(a)h_{ji}(a) = 1,$$

**{H2}**

$$(R2) \quad h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab),$$

**{H3}**

$$(R3) \quad [h_{ij}(a), h_{kl}(b)] = 1,$$

**{H4}**

$$(R4) \quad h_{ij}(a)^{-1} = h_{ij}(a^{-1}).$$

**c:Hnextensions**

**2.4. Extensions of type  $\mathfrak{H}_n(G)$ .** We start by briefly reviewing the material of § 1–3 of [11]. Let  $n \geq 3$  be a natural number. By definition, the group  $H_n(G)$  is given by generators  $h_{ij}(u)$ ,  $u \in G$ ,  $i \neq j$  and the following relations (cf. [11, H1–H5 of § 2]):

**{RH1}**

$$(H1) \quad h_{ij}(u)h_{ji}(u) = 1,$$

**{RH2}**

$$(H2) \quad h_{ij}(u)h_{ki}(u)h_{jk}(u) = 1,$$

**{RH3}**

$$(H3) \quad h_{ij}(u)h_{ik}(v)h_{ij}(u)^{-1} = h_{ik}(uv)h_{ik}(u)^{-1}, \quad \text{for } j \neq k$$

**{RH4}**

$$(H4) \quad h_{ij}(u)h_{kj}(v)h_{ij}(u)^{-1} = h_{kj}(vu)h_{kj}(u)^{-1}, \quad \text{for } i \neq k$$

**{RH5}**

$$(H5) \quad [h_{ij}(u), h_{kl}(v)] = 1.$$

There is a surjective group homomorphism  $H_n(G) \twoheadrightarrow D_n(G)$  sending  $h_{ij}(g)$  to  $d_{ij}(g)$ . Here  $d_{ij}(g)$  stands for the element of  $G^n$  whose  $i$ -th component equals  $g$ ,  $j$ -th component equals  $g^{-1}$  and all other components are trivial.

By definition, an *extension of type  $\mathfrak{H}_n(G)$*  is an extension  $H$  of  $D_n(G)$  which is also a quotient of  $H_n(G)$ , i. e. the extension that fits into the following diagram.

$$\begin{array}{ccc} H_n(G) & & \\ \downarrow & \searrow & \\ H & \twoheadrightarrow & D_n(G) \end{array}$$

Any  $\mathfrak{H}_n(G)$ -extension is central, see [11, Proposition 2.3].

We now recall the notion of an *extension of type  $\mathfrak{U}(G)$*  introduced by Rehmann in [11, § 1]. Let  $G$  be a group. Denote by  $U(G)$  the group presented by generators  $(u, v)$ ,  $u, v \in G$  and relations:

$$\boxed{\{U1\}} \quad (U1) \quad ({}^u v, {}^u w)(u, w) = (uv, w),$$

$$\boxed{\{U2\}} \quad (U2) \quad (u, vv)(v, wu)(w, uv) = 1.$$

There is a left action of  $G$  on  $U(G)$  given by  ${}^x(u, v) = (xu, xv)$ . Equivalently,  $U(G)$  can be defined by the following relations, see [4, p. 8].

$$\boxed{\{U4\}} \quad (2.9) \quad (u, v)(v, u) = 1,$$

$$\boxed{\{U6\}} \quad (2.10) \quad (u, v) {}^v(u, w) = (u, vw),$$

$$\boxed{\{U9\}} \quad (2.11) \quad {}^u(v, w)(w, v) = (u, [v, w]).$$

It is clear from the definition that  $U(G)$  is a quotient of the nonabelian tensor square  $G \otimes G$  and that the map  $(u, v) \mapsto [u, v]$  defines a  $G$ -equivariant map  $U(G) \twoheadrightarrow [G, G]$ . Notice that in the literature one can find alternative notations for  $U(G)$  which emphasize its connection with nonabelian tensor products e. g.  $(G, G)$  or  $G \tilde{\wedge} G$ .

By definition, an *extension of type  $\mathfrak{U}(G)$*  is simply a quotient of  $U(G)$  that fits into the following commutative diagram with  $G$ -equivariant arrows.

$$\begin{array}{ccc} U(G) & & \\ \downarrow & \searrow & \\ U & \twoheadrightarrow & [G, G] \end{array}$$

There is a correspondence between extensions of type  $\mathfrak{H}_n(G)$  and  $\mathfrak{U}(G)$ . One can obtain a  $\mathfrak{U}(G)$ -extension from given  $\mathfrak{H}_n(G)$ -extension via restriction. Indeed, let  $H$  be an extension of type  $\mathfrak{H}_n(G)$  and  $k$  be arbitrary number  $1 \leq k \leq n$ . Denote by  $\iota_k$  the inclusion map  $[G, G] \rightarrow D_n(G)$  that inserts an element  $g \in [G, G]$  at the  $k$ -th position. Then the corresponding extension  $U_H$  is simply the pull-back of  $H$  with respect to  $\iota_k$ .

In other words,  $U_H$  is simply the subgroup of  $H$  generated by symbols  $c_{kj}(u, v) = h_{kj}(u) \cdot h_{kj}(v) \cdot h_{kj}(vu)^{-1}$  (here  $j$  is any index not equal  $k$ ,  $c_{kj}(u, v)$  does not depend



on  $j$ , see [11, Proposition 2.1]). It can be shown that  $U$  is an extension of type  $\mathfrak{U}(G)$  with kernel  $\text{Ker}(\pi_U)$  isomorphic to  $\text{Ker}(\pi_H)$ , see [11, Propositions 2.2–2.3]. In [11, § 3] Rehmann also describes the inverse construction which assign to every  $\mathfrak{U}(G)$ -extension  $U \twoheadrightarrow [G, G]$  and a number  $n \geq 3$  some extension  $H_U$  of type  $\mathfrak{H}_n(G)$  that fits into the pull-back diagram above.

**2.5. A simpler presentation of  $H_n(G)$ .** The aim of this subsection is to obtain a presentation of  $H_n(G)$  that is simpler than the original presentation: 4 instead of 5 relations that are also shorter. Our main result is the following.

**Proposition 2.9.** *For  $n \geq 3$  relations (R0)–(R4) imply (H1)–(H5). Moreover, for  $n \geq 4$  relations (R0)–(R3) are equivalent to (H1)–(H5).*

For  $u, v \in G$  we define the following two symbols:

$$c_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, c'_{ij}(u, v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies  $h_{ij}(1) = 1$  therefore  $c_{ij}(u, u^{-1}) = c'_{ij}(u, u^{-1})$ . Our goal is to show that (H3) can be omitted from the definition of  $H_n(G)$  provided  $n \geq 4$ .

**Lemma 2.10.** *If one excludes relation (H3) from the list of defining relations for the group  $H_n(G)$ ,  $n \geq 3$ , the following still remain true:*

- (1) *The elements  $c_{ij}(u, u^{-1})$  are central in  $H_n(G)$ ;*
- (2) *One has  $c'_{ij}(u, v) = [h_{ij}(u), h_{kj}(v)]$ ,  $k \neq i, j$ ;*
- (3) *One has  $^{h_{ij}(w)}c'_{kj}(u, v) = c'_{kj}(u, w)^{-1}c'_{kj}(u, vw)$ ,  $k \neq i, j$ ;*
- (4) *One has  $^{h_{ij}(w)}c'_{ij}(u, v) = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}$ .*

*Proof.* First one shows using (H4) that  $c_{ij}(u, u^{-1})$  centralizes  $h_{kj}(v)$  (cf. with the proof of [11, Lemma 2.1(2)]). Since in any group  $[a, b] = 1$  implies  $[a^{-1}, b] = [a, b^{-1}] = [a^{-1}, b^{-1}] = 1$  we get that  $c_{ij}(u, u^{-1})$  also centralizes  $h_{kj}(v)^{-1} = h_{jk}(v)$  and  $c_{ij}(u^{-1}, u)^{-1} = c_{ji}(u, u^{-1})$  centralizes both  $h_{kj}(v)$  and  $h_{jk}(v)$ . Together with (H2) and (H5) this implies that  $c_{ij}(u, u^{-1})$  centralizes all the generators of  $H_n(G)$  and hence lies in the center of  $H_n(G)$ .

The second and third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$$^{h_{ij}(w)}c'_{ij}(u, v) = ^{h_{ij}(w)}c'_{kj}(v, u)^{-1} = (c'_{kj}(v, w)^{-1}c'_{kj}(v, uw))^{-1} = c'_{ij}(uw, v)c'_{ij}(w, v)^{-1}.$$

□

**Lemma 2.11.** *If one excludes relation (H3) from the list of defining relations for the group  $H_n(G)$ ,  $n \geq 3$ , the following statements are equivalent:*

- (1) (H3) holds;
- (2) one has  $c'_{ij}(u, v)^{-1} = c'_{ij}(v, u)$ ;

- |       |  |
|-------|--|
| item5 | (3) symbols $c'_{ij}(u, v)$ do not depend on $i$ ;                   |
| item5 | (4) one has $c'_{ij}(u, vw) = c'_{ij}(wu, v) \cdot c'_{ij}(uv, w)$ ; |
| item6 | (5) one has $c'_{ij}(u, vu^{-1}) = c'_{ij}(uv, u^{-1})$ .            |

*Proof.* Implications (1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (4) are essentially contained in the proof of [11, Lemmas 2.1-2.2]. Implication (4)  $\implies$  (5) is trivial.

We now prove (5)  $\implies$  (1). Notice that (H4) implies  $^{h_{ij}(u)^{-1}}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$ , therefore using Lemma 2.10.(1) we get that  $^{h_{ij}(u)}h_{ik}(v) = c_{ij}(u, u^{-1})^{h_{ij}(u^{-1})^{-1}}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$ . Thus, (H3) is equivalent to the equality

$$h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}),$$

or what is the same

$$c'_{ik}(u^{-1}, uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1}, u).$$

It is clear that the last equality is an equivalent reformulation of (5).  $\square$

*Proof of Proposition 2.9.* The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);
- for  $n \geq 4$  the third statement of the previous lemma follows from Lemma 2.10.(2).

$\square$

cor:main

**Corollary 2.12.** *For  $n \geq 3$  the group  $HS_n(G)$  is the quotient of the extension  $H_n(G)$  by (R4) and, in particular, is an  $\mathfrak{H}_n(G)$ -extension. The associated  $\mathfrak{U}(G)$  extension  $U_{HS_n(G)}$  is isomorphic to the quotient of  $U(G)$  modulo single family of relations*

$$(u, u) = 1, \quad u \in G.$$

*In particular, we have the following natural isomorphism:*

$$(2.12) \quad \text{Ker}(HS_n(G) \rightarrow D_n(G)) = \text{Ker}(S_n(G) \rightarrow G \wr S_n) \cong H_2(G, \mathbb{Z}).$$

*Proof.* Most of the assertions follow from Proposition 2.9 and [4, Proposition 5]. One also has to use the formula  $c_{ij}(u, u) = c_{ij}(u, u^{-1})$  (see [11, p. 87]).  $\square$

sec:main

### 3. PROOF OF THEOREM 2

Recall that for arbitrary set  $X$  one defines the space  $EX$  as the the simplicial set whose set of  $k$ -simplices  $EX_k$  is  $X^{k+1}$  and whose faces and degeneracies are obtained by omitting and repeating components. For a group  $G$  we denote by  $\pi_G$  the canonical map  $EG \rightarrow BG$  sending  $(g, h) \in EG_1$  to  $g^{-1}h \in BG_1$ .

Now let  $N$  be a group acting on  $X$ . We define two simplicial sets  $U$  and  $V$  as follows:

$$U = \bigcup_{n \in N} E(\Gamma_{n,-}) \subseteq E(X \times X), \quad V = \bigcup_{x,y \in X} E(N(x \rightarrow y)) \subseteq EN.$$

Here  $\Gamma_{n,-}$  is the graph of the function  $(x \mapsto nx)$  and  $N(x \rightarrow y)$  denotes the subset of elements  $n \in N$  satisfying  $nx = y$ . With this notation the subset  $N(x \rightarrow x)$  coincides with the stabilizer subgroup  $N_x \leq N$ .

lm:quillen-a

**Lemma 3.1.** *The simplicial sets  $U$  and  $V$  are homotopy equivalent.*

*Proof.* First, we define yet another simplicial set  $W$  as follows. Its  $k$ -simplices  $W_k$  are matrices  $\begin{pmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{pmatrix}$ , where  $x_i \in X$  and  $n_i \in N$  are such that all  $n_i$ 's act each  $x_j$  in the same way, i. e.  $n_i x_j = n_{i'} x_j$  for  $0 \leq i, i', j \leq k$ . The faces and degeneracies of  $W$  are the maps of omission and repetition of columns.

Now there are two simplicial maps  $f: W \rightarrow U$ ,  $g: W \rightarrow V$  whose action on 0-simplices is given by  $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = (x_0, n_0 x_0)$ ,  $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = n_0$ . To prove the lemma it suffices to show that  $f$  and  $g$  are homotopy equivalences. The proof for  $f$  and  $g$  is similar, let us show, for example, that  $g$  is a homotopy equivalence.

In view of Quillen theorem A (cf.[13, ex. IV.3.11]) it suffices to show that for each  $p$ -simplex  $d: \Delta^p \rightarrow V$  the pullback  $g/(p, d)$  of  $d$  and  $g$  is contractible. The simplicial set  $g/(p, d)$  can be interpreted as the subset of  $\Delta^p \times E(X)$  whose set of  $k$ -simplices consists of pairs  $(\alpha: \underline{k} \rightarrow \underline{p}, (x_0, \dots, x_k) \in X(\alpha, d)^{k+1})$ . Here  $X(\alpha, d)$  is the subset of  $X$  consisting of all  $x$  for which  $d_{\alpha(i)}x = d_{\alpha(j)}x$  for  $0 \leq i, j \leq k$ . Notice that the set  $X_d := X(id_{\underline{p}}, d)$  is nonempty and is contained in every  $X(\alpha, d)$  (it even equals  $X(\alpha, d)$  for surjective  $\alpha$ ). Now choose a point  $\tilde{x} \in X_d$  and consider the simplicial homotopy

$$H: \Delta^p \times EX \times \Delta^1 \rightarrow \Delta^p \times EX$$

between the identity map of  $\Delta^p \times EX$  and the map  $\Delta^p \times c_{\tilde{x}}$ , where  $c_{\tilde{x}}$  is the constant map. More concretely,  $H$  sends each triple  $(\alpha: \underline{k} \rightarrow \underline{p}, (x_0, \dots, x_k), \beta: \underline{k} \rightarrow \underline{1})$  to  $(\alpha, (x_0, \dots, x_{i-1}, \tilde{x}, \dots, \tilde{x}))$ , where  $i$  is the minimal number such that  $\beta(i) = 1$ . By the choice of  $\tilde{x}$  the image of  $H$  restricted to  $g/(p, d) \times \Delta^1$  is contained in  $g/(p, d)$ , hence  $g/(p, d)$  is contractible.  $\square$

Now suppose that  $X = H$  is also a group upon which  $N$  acts on the left.

cor:ker-iso

**Corollary 3.2.** *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

*There is an isomorphism  $\theta: \text{Ker}(\pi_1(S) \rightarrow N) \cong \text{Ker}(\pi_1(T) \rightarrow H \times H)$ . Moreover, the higher homotopy groups of  $S$  and  $T$  are isomorphic.*

*Proof.* Consider the following two pull-back squares:

$$\begin{array}{ccc} V \hookrightarrow EN & & U \hookrightarrow E(H \times H) \\ \downarrow \lrcorner & \downarrow \pi_N & \downarrow \lrcorner \\ S \hookrightarrow BN & & T \hookrightarrow B(H \times H) \end{array} \quad \begin{array}{ccc} & & \downarrow \pi_{H \times H} \end{array}$$

The required isomorphism can be obtained from the homotopy long exact sequence applied to the left arrows of these diagrams.  $\square$

Now let  $G$  be a group. Set  $N = G \wr S_n$ ,  $H = G^n$  and consider the left action of  $N$  on  $H$  given by  $(g, s) \cdot h = gh^{s^{-1}}$ ,  $g, h \in G^n$ ,  $s \in S_n$ . If one reorders the components of  $BG^{2n}$  accordingly, the simplicial subset  $T \subset BG^{2n}$  from the above corollary becomes precisely the preimage of 0 under  $h_n: BG^{2n} \rightarrow \mathbb{Z}[BG]$ .

It is also easy to compute the map  $\pi_1(S) \rightarrow N$ . Indeed, van Kampen theorem [8, Theorem 2.7] asserts that  $\pi_1(S)$  is isomorphic to the free product of stabilizer subgroups  $N_h \leq N$  amalgamated over pairwise intersections  $N_h \cap N_{h'}$ ,  $h, h' \in H$ . For  $h \in G^n$  the subgroup  $N_h$  consists of elements  $(g, s) \in N$  satisfying  $gh^{s^{-1}} = h$ , i.e. elements of the form  $(hh^{-s^{-1}}, s)$ . Thus,  $N_h \cong S_n$ ,  $\pi_1(S)$  is isomorphic to the group  $S_n(G)$  and the map  $\pi_1(S) \rightarrow N$  coincides with the map  $\mu$  defined in section 2.1. In particular, for an abelian  $G$  its kernel is generated by symbols  $c_{ij}(u, v)$ .

The following lemma provides a more concrete description of the isomorphism  $\theta$  from Corollary 3.2.

concrete-formula

**Lemma 3.3.** *If  $G$  is abelian then every generator  $c_{ij}(u, v) \in \text{Ker}(\pi_1(S) \rightarrow N)$  is mapped under  $\theta$  to the homotopy class of the loop  $f_{ij}(u, v)$ . The latter is defined as the following concatenation of loops in  $T$ :*

$$f_{ij}(u, v) = p_{ij}(u) \circ p_{ij}(v) \circ p_{ij}(u^{-1}v^{-1}), \text{ where } p_{ij}(x) = (x[i], x[i]) \circ (x^{-1}[i], x^{-1}[j]).$$

*Proof.* Fix a vector  $g \in G^n$  and denote by  $\gamma_{ij}(h, x)$  the path of length 4 in  $W$  which connects the following five points of  $W_0$ :

$$\begin{pmatrix} e \\ (h; e) \end{pmatrix} \begin{pmatrix} x[i] \\ (h; e) \end{pmatrix} \begin{pmatrix} x[i] \\ (d_{ij}(x)h, (ij)) \end{pmatrix} \begin{pmatrix} e \\ (d_{ij}(x)h, (ij)) \end{pmatrix} \begin{pmatrix} e \\ (d_{ij}(x)h, e) \end{pmatrix}.$$

One can easily check that the image  $\gamma_{ij}(h, x)$  in  $S$  under  $\pi_N g$  is precisely the element  $h_{ij}(x)$ , while  $\pi_{H \times H} f$  sends  $\gamma_{ij}(h, x)$  to the loop  $p_{ij}(x)$  (here  $f, g$  are the maps from the proof of Lemma 3.1).  $\square$

lm:endotr

**Lemma 3.4.** *The only natural endotransformations of the second homology functor  $H_2(-, \mathbb{Z}): \mathbf{Groups} \rightarrow \mathbf{Ab}$  are morphisms of multiplication by  $n \in \mathbb{Z}$ .*

*Proof.* Denote by  $\eta$  an endotransformation  $H_2(-, \mathbb{Z}) \rightarrow H_2(-, \mathbb{Z})$ . When restricted to the subcategory of free finitely-generated abelian groups  $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$  the second homology functor coincides with the second exterior power functor  $A \mapsto \wedge^2 A$ .

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic  $\mathbb{Z}$ -modules (see Definition 6.13.5 *ibid.*) The functor  $A \mapsto \wedge^2 A$  is clearly quadratic and corresponds to the quadratic  $\mathbb{Z}$ -module  $0 \rightarrow \mathbb{Z} \rightarrow 0$  under this equivalence. Thus, we get that  $\eta$  restricted to  $\mathbf{Add}(\mathbb{Z})$  coincides with the morphism of multiplication by  $n \in \mathbb{Z}$ .

Consider the group  $\Gamma_k = \langle x_1, y_1, \dots, x_k, y_k \mid [x_1, y_1] \cdot \dots \cdot [x_k, y_k] \rangle$  (the fundamental group of a sphere with  $k$  handles). It is clear that the abelianization map  $\Gamma_k \rightarrow \mathbb{Z}^{2k}$  induces an injective map  $H_2(\Gamma_k, \mathbb{Z}) \cong \mathbb{Z} \rightarrow \wedge^2 \mathbb{Z}^{2k}$ . Consider the following diagrams.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\quad} & \wedge^2 \mathbb{Z}^{2k} \\ \eta_{\Gamma_k} \downarrow & & \downarrow n \cdot \\ \mathbb{Z} & \xrightarrow{\quad} & \wedge^2 \mathbb{Z}^{2k} \end{array} \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{\chi} & H_2(G, \mathbb{Z}) \\ \eta_{\Gamma_k} \downarrow & & \downarrow \eta_G \\ \mathbb{Z} & \xrightarrow{\chi} & H_2(G, \mathbb{Z}) \end{array}$$

From the left diagram it follows that  $\eta_{\Gamma_k}$  is also the morphism of multiplication by  $n$ . For every element  $x \in H_2(G, \mathbb{Z})$  there exist an integer  $k$  and a map  $\chi: \Gamma_k \rightarrow G$  sending the generator of  $H_2(\Gamma_k, \mathbb{Z})$  to  $x$ . From the right square we conclude that  $\eta_G(x) = nx$ , as claimed.  $\square$

*Proof of Theorem 2.* Factor  $h_n$  as a composition of a trivial cofibration followed by a fibration  $BG^{2n} \rightarrow E_{h_n} \rightarrow \mathbb{Z}[BG]$  and write down the starting portion of the long homotopy exact sequence for the latter. If we denote by  $K$  the kernel of the map  $\nu: \pi_1(T) \rightarrow G^{2n}$  induced by the embedding  $h_n^{-1}(0) = T \subseteq BG^{2n}$  we come to the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\beta} & \pi_1(T) & \xrightarrow{\nu} & G^{2n} & \longrightarrow & H_1(G, \mathbb{Z}) & \longrightarrow & 1 \\ & & \psi \downarrow & & \varphi \downarrow & & \cong \downarrow & & \parallel & & \\ 1 & \longrightarrow & H_2(G, \mathbb{Z}) & \xrightarrow{\alpha} & \pi_1(F_{h_n}) & \longrightarrow & \pi_1(E_{h_n}) & \longrightarrow & H_1(G, \mathbb{Z}) & \longrightarrow & 1. \end{array}$$

We already know by Corollaries 2.12 and 3.2 that  $K$  is naturally isomorphic to  $H_2(G, \mathbb{Z})$  provided  $n \geq 3$ . Thus, it suffices to show that  $\psi$  is an isomorphism. In view of Lemma 3.4 we only need to consider the case when  $G$  is abelian.

We are going to compute the image of each generator  $c_{ij}(x, y) = x \wedge y \in H_2(G, \mathbb{Z})$  under  $\psi$ . In order to do this we need to obtain a more explicit description of the set of 1-simplices of  $F_{h_n}$ . Since both  $BG^{2n}$  and  $\mathbb{Z}[BG]$  are fibrant  $E_{h_n}$  and  $F_{h_n}$  can be constructed using the path space construction, i.e.  $E_{h_n} = BG^{2n} \times_{\mathbb{Z}[BG]} \mathbb{Z}[BG]^I$ . In this case the set of 1-simplices of  $F_{h_n}$  can be identified with the set of triples  $(g, t, t') \in G^{2n} \times \mathbb{Z}[BG]_2 \times \mathbb{Z}[BG]_2$  satisfying the following identities:

$$d_2(t) = h_n(g), \quad d_1(t) = d_1(t'), \quad d_0(t') = 0.$$

It will be convenient for us to perform all calculations inside the subset  $F' \subset (F_{h_n})_1$  consisting of those triples for which  $t' = 0$  and  $d_1(t) = d_0(t) = 0$ . For shortness we write down an element of  $F'$  as  $(g, t)$ . It is easy to check that the image of the loop  $p_{ij}(x)$  under the natural map  $T \rightarrow F_{h_n}$  corresponds to the pair  $((e, d_{ij}(x)), t(x))$ , where  $t(x) = -(x, x^{-1}) + (e, e) - (x^{-1}, e) + (e, x^{-1})$ .

We want to devise a concrete formula for the element of  $F'$  homotopic to the concatenation of two elements  $(g_1, t_1), (g_2, t_2) \in F'$ . It is easy to check that  $(g_1 g_2, t)$ , where  $t$  is the 1-st face of the filler  $f$  for the 3-horn  $(t_2, t_1, h_n(g_1, g_2))$ , is the desired element. By Moore's theorem one can find a filler for any horn  $(p_0, p_2, p_3)$  in a simplicial abelian group via the formula  $f = s_0 p_0 - s_0 s_1 d_2 p_0 - s_0 s_0 d_1 p_0 + s_0 s_0 d_2 p_0 + s_1 p_2 + s_2 p_3 - s_1 p_3$ . Substituting concrete values of  $p_i$  into this formula and using the fact that  $d_1 t_2 = 0$  we get the following expression for  $t$ :

$$\begin{aligned} t = d_1 f &= p_0 - s_1 d_2 p_0 - s_0 d_1 p_0 + s_0 d_2 p_0 + p_2 - p_3 + s_1 d_1 p_3 = \\ &= t_1 + t_2 - h_n(g_1, g_2) - s_1 h_n(g_2) + s_0 h_n(g_2) + s_1 h_n(g_1 g_2). \end{aligned}$$

Using the above formula we get that the image of  $c_{ij}(u, v)$  via  $\varphi\beta$  equals  $((e, e), t)$ , where

[eq:formula-t]

$$\begin{aligned} (3.1) \quad t &= 2(e, e) - (x^{-1}, e) + (xy, x^{-1}y^{-1}) - (xy, e) - (x, x^{-1}) - (y, y^{-1}) - \\ &\quad - (e, x^{-1}y^{-1}) + (e, x^{-1}) + (x, y) + (x^{-1}, y^{-1}) - (e, y) - (xyx^{-1}y^{-1}, e) + (y, e). \end{aligned}$$

The map  $\alpha$  can be described as the Dold—Kan isomorphism  $H_2(G, \mathbb{Z}) \cong \pi_2(\mathbb{Z}[BG])$  followed by the boundary map  $\partial: \pi_2(\mathbb{Z}[BG]) \rightarrow \pi_1(F_{h_n})$ . It is not hard to show that  $\partial$  maps each  $c \in \pi_2(\mathbb{Z}[BG])$  to  $((e, e), -c)$  (see [5, p. 29]). Thus, we get that  $\psi$  maps  $x \wedge y$  to  $-t$ , where  $t$  is given by (3.1).

On the other hand, the generator  $x \wedge y$  of  $H_2(G, \mathbb{Z})$  corresponds to the 2-cycle  $c = (x, y) - (y, x)$  (cf. [9, (14), p. 582]) and the corresponding element of  $\pi_2(\mathbb{Z}[BG])$  is simply the normalized 2-cycle of  $c$ :

$$c' = c - s_0 d_0 c - s_1 d_1 c + s_1 d_0 = (x, y) - (y, x) - (x, e) + (y, e) + (e, x) - (e, y).$$

It is not hard to check that  $t - c'$  is the image under the differential  $d_0 - d_1 + d_2 - d_3$  of the following 3-cycle:

$$(xy, x^{-1}, y^{-1}) - (y, x, x^{-1}) + s_1 s_1 (x^{-1} + xy - x - y) + s_0 s_0 (x^{-1} - x - x^{-1}y^{-1}).$$

Thus we obtain that  $[t] = -[c']$ , hence  $\psi$  is an isomorphism, as claimed. □

## APPENDIX A. SIMPLICIAL DOLD—SERRE FIBRATIONS

The aim of this appendix is to show that the map  $h_\infty: X^\infty \rightarrow \mathbb{Z}[X]$  is a quasifibration under suitable assumptions on  $X$ .

We start with the definition of the simplicial analogue of the so-called "Dold—Serre fibration", i. e. a map possessing "weak covering homotopy property" (also called "delayed homotopy lifting property"), cf. [12, § 1.4.4]. We show in Lemmas A.2 and A.4 that the map  $h_\infty$  satisfies this property and that every map with this property is a quasifibration.

df:dhlp

**Definition A.1.** Let  $C$  be a subclass of the class of acyclic cofibrations in the category of simplicial sets. We say that a map of simplicial sets  $p: E \rightarrow B$  satisfies *weak homotopy covering property* with respect to  $C$  if for every commutative square

$$(A.1) \quad \begin{array}{ccc} U & \xrightarrow{g} & E \\ i \downarrow & \tilde{F} \nearrow & \downarrow p \\ V & \xrightarrow{F} & B \end{array}$$

for which  $i \in C$  there exists a map  $\tilde{F}$  such that the lower triangle commutes strictly and the upper one commutes up to a fiberwise homotopy  $H$  (i. e. a homotopy  $H$  such that  $pH: U \times I \rightarrow B$  coincides with the composite  $U \times I \xrightarrow{\pi} U \xrightarrow{pg} B$ ).

Denote by  $C_{pr}$  the class consisting of all inclusions  $i: U \hookrightarrow V$  of finite polyhedral simplicial sets for which there exists a deformation retract (i. e. a map  $r: V \rightarrow U$  homotopic to  $\text{id}_V$ ).

lm:dhlp

**Lemma A.2.** *For a fibrant and connected simplicial set  $X$  the map  $h_\infty: X^\infty \rightarrow \mathbb{Z}[X]$  satisfies the weak covering homotopy property with respect to the class  $C_{pr}$ .*

*Proof.* Let  $i_0$  be a map from  $C_{pr}$  and  $r$  be the corresponding retraction. We first reduce the problem to the special case when  $g$  is the constant map at the basepoint of  $X^\infty$  (which we denote by 0). Indeed, let  $F, g$  be arbitrary maps as in (A.1). Since  $U$  is finite the image of  $g$  is contained in  $X^{2N} \subset X^\infty$ . If now  $\tilde{F}_0$  is a lifting in the diagram similar to (A.1) in which the top map equals 0 and the bottom one is  $F_0 = F - h_\infty gr$  then  $\tilde{F} = gr \times \tilde{F}_0$  is the desired lifting of  $F$ .

Now choose a contractible fibrant simplicial set  $W$  which maps surjectively onto  $X$  (e. g. take  $W$  to be the path space fibration). Since  $p: W \rightarrow X$  is surjective, the associated map between free simplicial abelian groups is a Kan fibration. Since  $i_0$  is acyclic we can choose a lifting  $\tilde{F}$  in the following diagram

$$\begin{array}{ccc} U & \xrightarrow{0} & \mathbb{Z}[W] \\ i_0 \downarrow & \tilde{F} \nearrow & \downarrow \\ V & \xrightarrow{F} & \mathbb{Z}[X] \end{array}$$

It is clear that the image of  $\tilde{F}$  is contained in the simplicial subgroup  $\mathbb{Z}[W]_0 \leq \mathbb{Z}[W]$  which, by definition, consists of linear combinations  $\sum_i n_i w_i$  for which  $\sum_i n_i = 0$ . By [10, Lemma 9.1] the canonical map  $\mathbb{Z}[\underline{\text{Hom}}(V, W)]_0 \rightarrow \underline{\text{Hom}}(V, \mathbb{Z}[W]_0)$  is surjective, hence the map  $\tilde{F}$  can be lifted along  $h_W$  in such a way that the bottom triangle in following diagram commutes strictly.

$$\begin{array}{ccc} U & \xrightarrow{0} & W^\infty \\ \downarrow i_0 & \nearrow G & \downarrow h_W \\ V & \xrightarrow{\tilde{F}} & \mathbb{Z}[W] \end{array}$$

Notice that the image of  $Gi_0$  is contained in the fiber  $h_W^{-1}(0)$ . It is easy to see that  $h_W^{-1}(0)$  is contractible (the contracting homotopy for  $W^\infty$  can be restricted to  $h_W^{-1}(0)$ ). We get that  $Gi_0$  is fiberwise homotopic to 0 hence the composite map  $V \xrightarrow{G} W^\infty \rightarrow X^\infty$  is the desired lifting of  $F$ .  $\square$

The assertions of the following lemma are straightforward and are given without proof.

lem:topo-facts

**Lemma A.3.** *Let  $p: (E, e) \rightarrow (B, b)$  be a map of pointed topological spaces. Let  $p^{-1}(b) \hookrightarrow F_p(b) \subseteq E \times_B B^I$  be the inclusion map of the fiber of  $p$  into the homotopy fiber. Denote  $k$ -th relative homotopy group (or set)  $\pi_k(F_p(b), p^{-1}(b), e)$  by  $G_k$ .*

- (1) *Maps of triples  $(D^k, S^{k-1}, pt) \rightarrow (F_p(b), p^{-1}(b), e)$  are in one-to-one correspondence with commuting diagrams of the form*

{eq:plp}

(A.2)

$$\begin{array}{ccc} D^k & \xrightarrow{\quad a \quad} & E \\ \downarrow i_0 & & \downarrow p \\ D^k \times I & \xrightarrow{\quad \pi \quad} (D^k \times I)/J \xrightarrow{\quad A \quad} & B \end{array}$$

Here  $J$  denotes  $(S^{k-1} \times I) \cup (D^k \times \{1\})$ . In the sequel we denote such a diagram by  $(a, A)$ .

- (2) *Two diagrams  $(a_0, A_0)$  and  $(a_1, A_1)$  represent the same element of  $G_k$  iff there exists a "diagram homotopy" that connects them. By a diagram homotopy we mean a family of maps  $(a_t, A_t)$  continuously depending on  $t \in [0, 1]$  such that  $pa_t = A_t \pi i_0$  holds for all  $t$ .*
- (3) *Let  $(a, A)$  be a diagram and  $a'$  be any other map homotopic to  $a$  via some homotopy  $H$  such that  $H(S^{k-1} \times I) \subseteq p^{-1}(b)$ . Then there exists a map  $A'$  such that  $(a', A')$  is a diagram homotopic to  $(a, A)$ .*

item:continue



item:weaker

- (4) Assume that in the diagram (A.2) there exists a diagonal map  $\tilde{A}: D^k \times I \rightarrow E$  such that the bottom triangle is commutative and the upper is commutative up to a fiberwise homotopy. Then the element of  $G_k$  given by  $(a, A)$  is trivial.

lm:weak-equiv

**Lemma A.4.** Let  $p: E \rightarrow B$  be a map of simplicial sets satisfying the weak covering homotopy property with respect to the class  $C_{pr}$ . Then the geometric realization of  $p$  is a quasifibration, i. e. for every point  $b \in B_0$  the inclusion  $|p|^{-1}(b) \hookrightarrow F_{|p|}(b)$  is a homotopy equivalence.

*Proof.* It suffices to show that  $\pi_k(F_{|p|}(b), |p|^{-1}(b))$  are all trivial for  $k \geq 1$ . Consider a diagram  $(a, A)$  of the form (A.2) (with the map  $|p|$  in the right hand side). In view of the previous lemma it suffices to construct a lifting  $\tilde{A}$  satisfying the requirements of Lemma A.3.(4).

The idea of the proof is to approximate  $(a, A)$  with a homotopic pair of geometric realizations of simplicial maps and then invoke the weak covering homotopy property. The key ingredient in the proof below is the simplicial approximation theorem [6, Theorem 4.7]. For brevity we denote the subdivision functor  $\text{sd}^m \text{sd}_*(-)$  from its statement by  $s^m(-)$  and the canonical natural transformation  $s^m X \rightarrow X$  by  $\delta^m$ . We proceed in a number of steps.

- Denote the restriction of  $a$  to  $S^{k-1}$  by  $a_0$ . Notice that the image of  $a_0$  is contained in  $|p|^{-1}(b)$ . Using the approximation theorem we find a simplicial map  $a'_0: s^m(S^{k-1}) \rightarrow p^{-1}(b)$  such that  $a|\delta^m| \cong |a'_0|$ .
- We can find a map  $a': D^k \rightarrow |E|$  extending  $|a'_0|$  and homotopic to  $a$  via some homotopy  $H$  satisfying  $H(S^k \times I) \subseteq p^{-1}(b)$ .
- Invoking the approximation theorem once again (with the initial condition specified by  $|a'_0|$ ) we find  $q > m$  and  $a'': s^q(D^k) \rightarrow E$  such that  $a'|\delta^q| \cong |a''|$  rel  $S^k$ .
- Using Lemma A.3.(3) we extend the homotopy  $a|\delta^q| \cong a'|\delta^q| \cong |a''|$  to a homotopy of diagrams  $(a|\delta^q|, A|\delta^q|) \cong (|a''|, A')$ .
- Applying the approximation theorem to  $A'$  with the initial condition on the boundary  $\partial(D^k \times I) = D^k \times \{0\} \cup J$  specified by  $pa''$  and  $\text{const}_b$ , respectively, we find an integer  $r > q$  and a map  $A'': s^r(D^k \times I) \rightarrow B$  such that  $A'|\delta^r| \cong |A''|$  rel  $\partial(D^k \times I)$ .

Thus, we have obtained the following commutative diagram of simplicial sets for which  $(|a''|, |A''|)$  is a diagram homotopic to  $(a, A)$ .

$$\begin{array}{ccc}
 s^r(D^k) & \xrightarrow{a''\gamma^{r-q}} & E \\
 \downarrow s^r(i) & \nearrow \tilde{A} & \downarrow p \\
 s^r(D^k \times I) & \xrightarrow{A''} & B
 \end{array}$$

By functoriality of the subdivision functor the map  $s^r(i)$  possesses a deformation retract hence there exists the desired lifting  $\tilde{A}$ .  $\square$

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