AMALGAMS OF SYMMETRIC GROUPS AND NONABELIAN TENSOR PRODUCTS

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Abstract. By amalgamating copies of symmetric group S_n we construct a central extension $S_n(G)$ of a large subgroup of the wreath product $G \wr S_n$. We then establish a connection between this group and the theory of $\mathfrak{U}(G)$ and $\mathfrak{H}_n(G)$-extensions developed by K. Dennis and U. Rehmann in 1970's. We also give a topological application of this construction by describing homotopy invariants of blah-blah-blah...

1. Introduction

The aim of this note is to establish a surprising connection between symmetric groups and (certain quotients of) nonabelian tensors squares.

Let G be a group. Consider the family of copies of the symmetric group S_n in which each copy is indexed by an n-tuple $(g_1, \ldots, g_n) \in G^n$. Denote by $S_n(G)$ the free product of groups from this family amalgamated over the single family of relations $s_{(g)} = s_{(h)}$, where $s \in S_n$, $g, h \in G^n$ are such that hg^{-1} is fixed by s (here $s_{(g)}$ denotes a permutation $s \in S_n$ embedded into the copy with index g).

Consider the wreath product $G \wr S_n = G^n \rtimes S_n$. By the definition of $S_n(G)$ there is a well defined map $\mu_n \colon S_n(G) \to G \wr S_n$ given by $s_{(g)} \mapsto s^g = (g^{-1}g^{s^{-1}}, s)$.

Our first main result is the following theorem.

Theorem 1. *For $n \geq 3$ the map μ_n is a crossed module whose cokernel and kernel are isomorphic to $H_1(G, \mathbb{Z})$ and $H_2(G, \mathbb{Z})$ respectively. Moreover, the group $S_n(G)$ is isomorphic to $S_n \ltimes H_n^{\wedge}(G)$.*

Here $H_n^{\wedge}(G)$ denotes the quotient of the universal extension of type $\mathfrak{H}_n(G)$ modulo all symbols of the form $\{x, x\} = 1$, $x \in G$. Theory of extensions of type $\mathfrak{H}_\mathfrak{n}$ was developed by U. Rehmann in [11] for the purpose of generalization of Matsumoto theorem to skew-fields (we refer the reader to Section 2.4 for more details).

Theorem 1 and its proof (which itself involves little more than some basic combinatorial group theory) may be of interest for the following reasons.

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- The presentation is similar in spirit to results of Brown—Loday and Kassel—Reutenauer on the presentation of the linear Steinberg group in the form of an amalgam of braid groups.
- It gives a presentation of the universal extension $H_n(G)$ with generators and relations which is simpler than the original presentation given in [11], see Proposition 2.9. Moreover, our presentation explicitly obtained from the presentation of S_n formulated in terms of transpositions.

Theorem 1 was proved in an attempt to answer some purely topological question, which we describe in more detail below.

For a pointed simplicial set (X, x_0) denote by $\mathbb{Z}[X]$ the associated free simplicial abelian group. Define the "alternating" map of simplicial sets $h_n\colon X^{2n} \to \mathbb{Z}[X]$ by the identity:

$$h_n(x_1, x_2, \ldots x_{2n}) = \sum_{i=1}^{2n} (-1)^i x_i.$$

Denote by X^∞ the colimit of X^n with respect to the natural embeddings $X^n \hookrightarrow X^{n+1}$. Taking colimit of h_n we get the map $h_\infty = \mathrm{colim}_n(h_n)\colon X^\infty \to \mathbb{Z}[X]$.

Based on a recent result of S. Podkorytov one can deduce that this map is a quasifibration provided X is fibrant and connected. In particular, the natural map $h_\infty^{-1}(0) \to F_{h_\infty}$ between the fiber and the homotopy fiber of h_∞ over 0 is a weak equivalence, see Lemma A.4.

Theorem 2. *If $X = BG$ is the classifying space of a group G then for $n \geq 3$ the natural map $h_n^{-1}(0) \to F_{h_n}$ induces an isomorphism of fundamental groups $\pi_1(h_n^{-1}(0)) \cong \pi_1(F_{h_n})$.*

The proof of Theorem 2 goes as follows. ???

In the course of the proof we also find a presentation of the universal $\mathfrak{H}_n(G)$-extension that is simpler than the original presentation introduced by Rehmann in [11], see Proposition 2.9.

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2. The extension $S_n(G)$

2.1. **The definition of $S_n(G)$.** Let G be a group. Recall that the *wreath product* is, by definition, the semidirect product $G^n \rtimes S_n$, in which S_n acts on G^n on the right by permuting its factors.

Consider the family $\{{S_n}^{(g)}\}_{g\in G^n}$ of isomorphic copies of S_n indexed by elements of G^n and let F be the free product of groups from this family. For $s \in S_n$ and $g \in G^n$ we denote by s_g the image of s in F under the canonical map $S_n^{(g)} \to F$.

Definition 2.1. We define the group $S_n(G)$ to be the quotient of F modulo the following single family of relations:

$$s_g = s_h, \text{ where } s \text{ fixes } hg^{-1}. \tag{2.1}$$

We continue to denote by s_g the image of the element $s_g \in F$ under the canonical map $F \to S_n(G)$.

Remark 2.2. Expanding the definition of the semidirect product we get that

$$s^g = (1_{G^n}, s)^{(g,1)} = (g^{-1}, s)(g, 1) = (g^{-1}g^{(s^{-1})}, s).$$

therefore the equality $s^g = s^h$ holds in $G \wr S_n$ iff $g^{-1}g^{(s^{-1})} = h^{-1}h^{(s^{-1})}$, or what is the same, iff s fixes hg^{-1}.

The last statement immediately implies that the map $\mu\colon S_n(G) \to G \wr S_n$ given by $\mu(s_g) = s^g = (g^{-1}g^{(s^{-1})}, s)$ is well-defined. We will see shortly that μ is, in fact, a *crossed module* in the sense of [2, § 2.2].

Observe from the definition of $S_n(G)$ that there is a split exact sequence.

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Ker}(\pi) \arrow[r] & S_n(G) \arrow[r, shift left, "\pi"] & S_n \arrow[l, shift left, dashed, "\iota(1)"] \arrow[r] & 1, \end{tikzcd} \tag{2.2}$$

Here the map $\pi = \pi_{S_n} \circ \mu$ removes subscript g from each s_g and the section $\iota(1)$ sends s to s_1, where 1 is the identity element of G^n. Thus, if we denote $\mathrm{Ker}(\pi)$ by $HS_n(G)$ we get a decomposition $S_n(G) = HS_n(G) \rtimes S_n$.

2.2. **A crossed module structure on $S_n(G)$.** Recall from [2, § 2.2] that a *crossed module* is a morphism of groups $\mu\colon M \to N$ together with a right action of N on M compatible with the conjugation action of N on itself, i. e.

$$\mu(m^n) = \mu(m)^n \text{ for all } n \in N,\ m \in M, \tag{CM1}$$

which also satisfies the following identity called *Peiffer identity*:

$$m^{m'} = m^{\mu(m')} \text{ for all } m, m' \in M. \tag{CM2}$$

In our situation, we let $G \wr S_n$ act on $S_n(G)$ by

$$(s_g)^{(h,t)} = {s^t}_{(gh)^t}, \text{ for } s, t \in S_n \text{ and } g, h \in G^n. \tag{2.3}$$

The goal of this subsection is to prove the following.

Proposition 2.3. *For $n \geq 3$ the map $\mu\colon S_n(G) \to G \wr S_n$ is a crossed module.*

From the fact that μ is a crossed module one can deduce that $\mathrm{Ker}(\mu)$ is a central subgroup of $S_n(G)$ and $\mathrm{Im}(\mu)$ is a normal subgroup of $G \wr S_n$. It is not hard to show that the group $S_2(G)$ is isomorphic to a free product of copies of S_2 (whose center is always trivial). Therefore, the requirement $n \geq 3$ in the statement of Proposition 2.3 is essential.

Verification of the fact that formula (2.3) gives a well-defined action of $G \wr S_n$ on $S_n(G)$ that satisfies (CM1) is lengthy but straightforward. Let us show that (CM2) holds. It suffices to verify Peiffer identities only for the generators of $S_n(G)$, for which it takes the form:

$$t_h^{-1} s_g t_h = {s^t}_{(gh^{-1})^t \cdot h} \text{ for all } s, t \in S_n,\ g, h \in G^n. \tag{2.4}$$

If we act on both sides of the above formula by $(h^{-1}, 1) \in G \wr S_n$ we obtain the equality $t_1^{-1} s_{gh^{-1}} t_1 = {s^t}_{(gh^{-1})^t}$. Thus, to prove (2.4) it suffices to show the following simpler relation:

$$t_1^{-1} s_g t_1 = {s^t}_{g^t} \text{ for all } s, t \in S_n,\ g \in G^n. \tag{2.5}$$

The key step in the proof is the following lemma.

Lemma 2.4. *The relation* (2.5) *holds in the special case when* $s = (ij)$ *and* $t = (kl)$ *are two nonequal transpositions.*

Proof. First of all, we immediately check that (2.5) holds in the special case when t fixes $g' \in G^n$. Indeed, by (2.1) we have $t_1 = t_{g'}$, hence

$$t_1^{-1} s_{g'} t_1 = t_{g'}^{-1} s_{g'} t_{g'} = (s^t)_{g'} = (s^t)_{g'^t}.$$

Without loss of generality we may assume that $l \neq i$ and $l \neq j$. Denote by g' the vector which differs from g only at l-th position, for which we set $g'_l = g_k$. Since the only nontrivial component of $g' g^{-1}$ (resp. $g' g^{-t}$) is located at l-th (resp. k-th) position, it is fixed by s (resp. s^t), hence from (2.1) we conclude that $s_g = s_{g'}$ (resp. $(s^t)_{g'} = (s^t)_{g^t}$). Finally, since g' is fixed by t, we get that

$$t_1^{-1} s_g t_1 = t_1^{-1} s_{g'} t_1 = (s^t)_{g'^t} = s^t_{g'} = (s^t)_{g^t}. \quad \square$$

Proof of Proposition 2.3. Let us show that (2.5) holds for arbitrary transpositions $s, t \in S_n$. It suffices to consider the case $s = t = (ij)$. After choosing some $k \neq i, j$ and presenting (ij) as $(kj)(ik)(kj)$ we use the lemma:

$$(ij)_1^{-1}(ij)_g(ij)_1 = (ij)_1^{-1}(kj)_g(ik)_g(kj)_g(ij)_1 = (ki)_{g^{(ij)}}(jk)_{g^{(ij)}}(ki)_{g^{(ij)}} = (ij)_{g^{(ij)}}.$$

Proposition now follows by induction on the length of permutations s, t. $\square$

2.3. **An explicit presentation of $S_n(G)$.** This presentation turns out to be much more economical than the original definition given in section 2.1. An explicit presentation for the subgroup $HS_n(G)$ is derived from this presentation via Reidemeister-Schreier method.

We start with the following simple lemma which is, in essence, a variant of the standard presentation of a Weyl group in terms of reflections with some redundant generators and relations added (cf. [3, Theorem 2.4.3]).

Lemma 2.5. *For $n \geq 3$ The symmetric group S_n admits presentation with transpositions as the set of generators and the following list of defining relations (in every formula distinct letters denote distinct indices):*

$$(ij)^2 = 1, \tag{S1}$$

$$(ij)^{(jk)} = (ik), \tag{S2}$$

$$[(ij), (kl)] = 1. \tag{S3}$$

$$(ij) = (ji), \tag{S4}$$

Now we are ready to formulate the main result of this subsection.

Proposition 2.6. *For $n \geq 3$ and arbitrary group G the group $S_n(G)$ admits presentation with the set of generators $\{(ij)_a \mid i \neq j, 1 \leq i, j, \leq n, a \in G\}$ and the following list of relations (as before, distinct letters denote distinct indices):*

$$(ij)_a^2 = 1, \tag{SG1}$$

$$(ij)_a^{(jk)_b} = (ik)_{ab}, \tag{SG2}$$

$$[(ij)_a, (kl)_b] = 1. \tag{SG3}$$

$$(ij)_a = (ji)_{a^{-1}} \tag{SG4}$$

Proof. Denote by S' the group from the statement of the proposition. For $1 \leq i \leq n$ and $x \in G$ denote by $x[i]$ the element of G^n whose only nontrivial component equals x and is located in the i-th position.

It is not hard to deduce from the definition of $S_n(G)$ and Lemma 2.4 that the formula $(ij)_a \mapsto (ij)_{a[j]}$ gives a well-defined map $\varphi\colon S' \to S_n(G)$.

Now we are going to construct the map $\psi\colon S_n(G) \to S'$ in the opposite direction. Using the presentation of S_n given by Lemma 2.5 we define for a fixed $g \in G^n$ the map $\psi_g\colon S_n \to S'$ by $\psi_g((ij)) = (ij)_{g_i^{-1}g_j}$. It is obvious that ψ_g preserves the defining relations (S1)–(S4) of S_n. It remains to show that the equation $\psi_g(s) = \psi_h(s)$ holds whenever $g, h \in G^n$ and $s \in S_n$ satisfy the requirement of (2.1).

Indeed, if hg^{-1} is fixed by s then for every $1 \leq i \leq n$ we have $(hg^{-1})_i = (hg^{-1})_{s(i)}$, or equivalently $h_i^{-1}h_{s(i)} = g_i^{-1}g_{s(i)}$. For example, if s is a cycle of length p, i. e.

$s = (i_1, i_2, \dots i_p)$ with $i_{k+1} = s(i_k)$ we get that

$$\psi_g(s) = \psi_g\left(\prod_{k=1}^{p-1}(i_k, i_{k+1})\right) = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{g_{i_k}^{-1} g_{i_{k+1}}} = \prod_{k=1}^{p-1}(i_k, i_{k+1})_{h_{i_k}^{-1} h_{i_{k+1}}} = \psi_h(s).$$

The proof for a general s is almost the same. Verification of the fact that ψ and φ are mutually inverse is also immediate. $\square$

We now obtain a presentation of the subgroup $HS_n(G)$. For every $i \neq j$ and $a \in G$ we define the element $h_{ij}(a) \in HS_n(G)$ as follows:

$$h_{ij}(a) = (ij)_a \cdot (ij)_1. \tag{2.6}$$

It is not hard to show that $h_{ij}(a)$ form a generating set for $HS_n(G)$. In fact, there is an explicit formula how an element of $HS_n(G)$ originally expressed through $(ij)_a$'s can be rewritten in terms of $h_{ij}(a)$. Indeed, if h lies in $HS_n(G)$ and is written as $\prod_{k=1}^{N}(i_k j_k)_{a_k}$ for some $i_k \neq j_k$ and $a_k \in G$ then it can be rewritten as follows:

$$h = \prod_{k=1}^{N} h_{\sigma_k(i_k), \sigma_k(j_k)}(a_k), \text{ where } \sigma_k = \prod_{s=1}^{k-1}(i_s j_s) \in S_n. \tag{τ}$$

We briefly recall the notion of a *rewriting process* given in [7, § 2.3]. If G is a group presented by generators a_ν and relations $R_\mu(a_\nu)$ and H is its subgroup with a generating set $J_i(a_\nu)$ then a *rewriting process for* H is a function which maps every word u in alphabet a_ν to a word v in alphabet s_i such that u and $v[s_i := J_i]$ define the same element of G whenever u represents an element of H.

With this terminology, the mapping (τ) defined above is a rewriting process for the subgroup $HS_n(G)$. Since it does not arise as a rewriting process corresponding to a coset representative function, this process is *not* a Reidemeister rewriting process in the sense of [7, § 2.3]. However, it still satisfies the following two key properties of a Reidemeister rewriting process (cf. with (v) and (vi) of [7, § 2.3]):

- if U and U^* are freely equal words in $(ij)_a$ then $\tau(U)$ and $\tau(U^*)$ are also freely equal words in $h_{ij}(a)$;
- if U_1 and U_2 are two words in $(ij)_a$ which define elements of $HS_n(G)$ then the words $\tau(U_1U_2)$ and $\tau(U_1)\tau(U_2)$ are equal.

Using these two properties and repeating the arguments used in the proof of [7, Theorem 2.8] one can simplify the generic presentation of $HS_n(G)$ given by [7, Theorem 2.6] and obtain the following.

Lemma 2.7. *For $n \geq 3$ the group $HS_n(G)$ admits presentation on the generators $h_{ij}(a)$ with the following two families of defining relations:*

$$h_{ij}(a) = \tau\left((ij)_a \cdot (ij)_1\right); \tag{2.7}$$

$$\tau(KRK^{-1}) = 1, \tag{2.8}$$

where R varies over relations of Proposition 2.6 and K is any word in $(ij)_1$, $i \neq j$.

From (2.7) one immediately obtains the equality $h_{ij}(1) = 1$, $i \neq j$. Thus, if we denote by σ the permutation corresponding to a word K and let S_n act on $h_{ij}(a)$'s in the natural way we will get that the word $\tau(KRK^{-1})$ is equivalent to ${}^{\sigma}\tau(R)$ (modulo relations $h_{ij}(1) = 1$).

Since the relations from Proposition 2.6 are respected by the action of S_n, only relations of the form $\tau(R) = 1$ are, in fact, needed for the presentation of $HS_n(G)$. Writing down what $\tau(R)$ is for each of (SG1)–(SG4) we get the following.

Proposition 2.8. *For $n \geq 3$ the group $HS_n(G)$ admits presentation with generators $h_{ij}(a)$ and the following list of relations:*

$$h_{ij}(1) = 1, \tag{R0}$$

$$h_{ij}(a)h_{ji}(a) = 1, \tag{R1}$$

$$h_{jk}(b)h_{ik}(a)h_{ij}(b) = h_{ik}(ab), \tag{R2}$$

$$[h_{ij}(a), h_{kl}(b)] = 1, \tag{R3}$$

$$h_{ij}(a)^{-1} = h_{ij}(a^{-1}). \tag{R4}$$

2.4. **Extensions of type $\mathfrak{H}_n(G)$.** We start by briefly reviewing the material of § 1–3 of [11]. Let $n \geq 3$ be a natural number. By definition, the group $H_n(G)$ is given by generators $h_{ij}(u)$, $u \in G$, $i \neq j$ and the following relations (cf. [11, H1–H5 of § 2]):

$$h_{ij}(u)h_{ji}(u) = 1, \tag{H1}$$

$$h_{ij}(u)h_{ki}(u)h_{jk}(u) = 1, \tag{H2}$$

$$h_{ij}(u)h_{ik}(v)h_{ij}(u)^{-1} = h_{ik}(uv)h_{ik}(u)^{-1}, \qquad \text{for } j \neq k \tag{H3}$$

$$h_{ij}(u)h_{kj}(v)h_{ij}(u)^{-1} = h_{kj}(vu)h_{kj}(u)^{-1}, \qquad \text{for } i \neq k \tag{H4}$$

$$[h_{ij}(u), h_{kl}(v)] = 1. \tag{H5}$$

There is a surjective group homomorphism $H_n(G) \twoheadrightarrow D_n(G)$ sending $h_{ij}(g)$ to $d_{ij}(g)$. Here $d_{ij}(g)$ stands for the element of G^n whose i-th component equals g, j-th component equals g^{-1} and all other components are trivial.

By definition, *an extension of type* $\mathfrak{H}_n(G)$ is an extension H of $D_n(G)$ which is also a quotient of $H_n(G)$, i. e. the extension that fits into the following diagram.

$$\begin{tikzcd} H_n(G) \arrow[d] \arrow[dr] & \\ H \arrow[r, two heads] & D_n(G) \end{tikzcd}$$

Any $\mathfrak{H}_n(G)$-extension is central, see [11, Proposition 2.3].

We now recall the notion of *an extension of type* $\mathfrak{U}(G)$ introduced by Rehmann in [11, § 1]. Let G be a group. Denote by $U(G)$ the group presented by generators (u, v), $u, v \in G$ and relations:

$$({}^u v, {}^u w)(u, w) = (uv, w), \tag{U1}$$

$$(u, vw)(v, wu)(w, uv) = 1. \tag{U2}$$

There is a left action of G on $U(G)$ given by ${}^x(u, v) = ({}^x u, {}^x v)$. Equivalently, $U(G)$ can be defined by the following relations, see [4, p. 8].

$$(u, v)(v, u) = 1, \tag{2.9}$$

$$(u, v)\ {}^v(u, w) = (u, vw), \tag{2.10}$$

$${}^u(v, w)(w, v) = (u, [v, w]). \tag{2.11}$$

It is clear from the definition that $U(G)$ is a quotient of the nonabelian tensor square $G \otimes G$ and that the map $(u, v) \mapsto [u, v]$ defines a G-equivariant map $U(G) \twoheadrightarrow [G, G]$. Notice that in the literature one can find alternative notations for $U(G)$ which emphasize its connection with nonabelian tensor products e. g. (G, G) or $G \mathbin{\widetilde{\wedge}} G$.

By definition, an *extension of type* $\mathfrak{U}(G)$ is simply a quotient of $U(G)$ that fits into the following commutative diagram with G-equivariant arrows.

$$\begin{tikzcd} U(G) \arrow[d] \arrow[dr] & \\ U \arrow[r, two heads] & {[G, G]} \end{tikzcd}$$

There is a correspondence between extensions of type $\mathfrak{H}_n(G)$ and $\mathfrak{U}(G)$. One can obtain a $\mathfrak{U}(G)$-extension from given $\mathfrak{H}_n(G)$-extension via restriction. Indeed, let H be an extension of type $\mathfrak{H}_n(G)$ and k be arbitrary number $1 \leq k \leq n$. Denote by ι_k the inclusion map $[G, G] \to D_n(G)$ that inserts an element $g \in [G, G]$ at the k-th position. Then the corresponding extension U_H is simply the pull-back of H with respect to ι_k.

In other words, U_H is simply the subgroup of H generated by symbols $c_{kj}(u, v) = h_{kj}(u) \cdot h_{kj}(v) \cdot h_{kj}(vu)^{-1}$ (here j is any index not equal k, $c_{kj}(u, v)$ does not depend on j, see [11, Proposition 2.1]). It can be shown that U is an extension of type $\mathfrak{U}(G)$ with kernel $\mathrm{Ker}(\pi_U)$ isomorphic to $\mathrm{Ker}(\pi_H)$, see [11, Propositions 2.2–2.3]. In [11, § 3] Rehmann also describes the inverse construction which assign to every $\mathfrak{U}(G)$-extension $U \twoheadrightarrow [G,G]$ and a number $n \geq 3$ some extension H_U of type $\mathfrak{H}_n(G)$ that fits into the pull-back diagram above.

2.5. **A simpler presentation of $H_n(G)$.** The aim of this subsection is to obtain a presentation of $H_n(G)$ that is simpler than the original presentation: 4 instead of 5 relations that are also shorter. Our main result is the following.

Proposition 2.9. *For $n \geq 3$ relations* (R0)–(R4) *imply* (H1)–(H5). *Moreover, for $n \geq 4$ relations* (R0)–(R3) *are equivalent to* (H1)–(H5).

For $u, v \in G$ we define the following two symbols:

$$c_{ij}(u,v) = h_{ij}(u)h_{ij}(v)h_{ij}(vu)^{-1}, c'_{ij}(u,v) = h_{ij}(u)h_{ij}(v)h_{ij}(uv)^{-1}.$$

Notice that (H4) implies $h_{ij}(1) = 1$ therefore $c_{ij}(u,u^{-1}) = c'_{ij}(u,u^{-1})$. Our goal is to show that (H3) can be omitted from the definition of $H_n(G)$ provided $n \geq 4$.

Lemma 2.10. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following still remain true:*

1. *The elements $c_{ij}(u,u^{-1})$ are central in $H_n(G)$;*
2. *One has $c'_{ij}(u,v) = [h_{ij}(u), h_{kj}(v)]$, $k \neq i,j$;*
3. *One has ${}^{h_{ij}(w)}c'_{kj}(u,v) = c'_{kj}(u,w)^{-1}c'_{kj}(u,vw)$, $k \neq i,j$;*
4. *One has ${}^{h_{ij}(w)}c'_{ij}(u,v) = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}$.*

Proof. First one shows using (H4) that $c_{ij}(u,u^{-1})$ centralizes $h_{kj}(v)$ (cf. with the proof of [11, Lemma 2.1(2)]). Since in any group $[a,b] = 1$ implies $[a^{-1},b] = [a,b^{-1}] = [a^{-1},b^{-1}] = 1$ we get that $c_{ij}(u,u^{-1})$ also centralizes $h_{kj}(v)^{-1} = h_{jk}(v)$ and $c_{ij}(u^{-1},u)^{-1} = c_{ji}(u,u^{-1})$ centralizes both $h_{kj}(v)$ and $h_{jk}(v)$. Together with (H2) and (H5) this implies that $c_{ij}(u,u^{-1})$ centralizes all the generators of $H_n(G)$ and hence lies in the center of $H_n(G)$.

The second and third assertions are straightforward corollaries of (H4). The fourth assertion follows from the third one:

$${}^{h_{ij}(w)}c'_{ij}(u,v) = {}^{h_{ij}(w)}c'_{kj}(v,u)^{-1} = (c'_{kj}(v,w)^{-1}c'_{kj}(v,uw))^{-1} = c'_{ij}(uw,v)c'_{ij}(w,v)^{-1}.$$

$\square$

Lemma 2.11. *If one excludes relation* (H3) *from the list of defining relations for the group $H_n(G)$, $n \geq 3$, the following statements are equivalent:*

1. (H3) *holds;*
2. *one has $c'_{ij}(u,v)^{-1} = c'_{ij}(v,u)$;*

(3) symbols $c'_{ij}(u,v)$ *do not depend on* i*;*
(4) one has $c'_{ij}(u,vw) = c'_{ij}(wu,v)\cdot c'_{ij}(uv,w)$*;*
(5) one has $c'_{ij}(u,vu^{-1}) = c'_{ij}(uv,u^{-1})$*.*

Proof. Implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are essentially contained in the proof of [11, Lemmas 2.1-2.2]. Implication (4) $\implies$ (5) is trivial.

We now prove (5) $\implies$ (1). Notice that (H4) implies ${}^{h_{ij}(u)^{-1}}h_{ik}(v) = h_{ik}(u)^{-1}h_{ik}(vu)$, therefore using Lemma 2.10.(1) we get that ${}^{h_{ij}(u)}h_{ik}(v) = {}^{c_{ij}(u,u^{-1})h_{ij}(u^{-1})^{-1}}h_{ik}(v) = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1})$. Thus, (H3) is equivalent to the equality

$$h_{ik}(uv)h_{ik}(u)^{-1} = h_{ik}(u^{-1})^{-1}h_{ik}(vu^{-1}),$$

or what is the same

$$c'_{ik}(u^{-1},uv) = h_{ik}(u^{-1})h_{ik}(uv) = h_{ik}(vu^{-1})h_{ik}(u) = c'_{ik}(vu^{-1},u).$$

It is clear that the last equality is an equivalent reformulation of (5). □

Proof of Proposition 2.9. The assertion of the proposition becomes evident once one notices the following:

- (H2) and (H4) are implied by (R0)–(R2);
- (R0) follows from (H4);
- (R2) follows from (H1), (H2) and (H4);
- (H4) and (H3) are equivalent in the presence of (R4);
- for $n \geq 4$ the third statement of the previous lemma follows from Lemma 2.10.(2).

□

Corollary 2.12. *For* $n \geq 3$ *the group* $HS_n(G)$ *is the quotient of the extension* $H_n(G)$ *by* (R4) *and, in particular, is an* $\mathfrak{H}_n(G)$*-extension. The associated* $\mathfrak{U}(G)$ *extension* $U_{HS_n(G)}$ *is isomophic to the quotient of* $U(G)$ *modulo single family of relations*

$$(u,u) = 1,\ u \in G.$$

In particular, we have the following natural isomorphism:

$$\operatorname{Ker}(HS_n(G) \to D_n(G)) = \operatorname{Ker}(S_n(G) \to G \wr S_n) \cong \mathrm{H}_2(G,\mathbb{Z}). \tag{2.12}$$

Proof. Most of the assertions follow from Proposition 2.9 and [4, Proposition 5]. One also has to use the formula $c_{ij}(u,u) = c_{ij}(u,u^{-1})$ (see [11, p. 87]). □

3. Proof of Theorem 2

Recall that for arbitrary set X one defines the space EX as the the simplicial set whose set of k-simplices EX_k is X^{k+1} and whose faces and degeneracies are obtained by omitting and repeating components. For a group G we denote by π_G the canonical map $EG \to BG$ sending $(g,h) \in EG_1$ to $g^{-1}h \in BG_1$.

Now let N be a group acting on X. We define two simplicial sets U and V as follows:

$$U = \bigcup_{n \in N} E(\Gamma_{n.-}) \subseteq E(X \times X), \quad V = \bigcup_{x,y \in X} E(N(x \to y)) \subseteq EN.$$

Here $\Gamma_{n.-}$ is the graph of the function $(x \mapsto nx)$ and $N(x \to y)$ denotes the subset of elements $n \in N$ satisfying $nx = y$. With this notation the subset $N(x \to x)$ coincides with the stabilizer subgroup $N_x \leq N$.

Lemma 3.1. *The simplicial sets U and V are homotopy equivalent.*

Proof. First, we define yet another simplicial set W as follows. Its k-simplices W_k are matrices $\left(\begin{smallmatrix} x_0 & x_1 & \dots & x_k \\ n_0 & n_1 & \dots & n_k \end{smallmatrix}\right)$, where $x_i \in X$ are $n_i \in N$ are such that all n_i's act each x_j in the same way, i. e. $n_i x_j = n_{i'} x_j$ for $0 \leq i, i', j \leq k$. The faces and degeneracies of W are the maps of omission and repetition of columns.

Now there are two simplicial maps $f\colon W \to U$, $g\colon W \to V$ whose action on 0-simplicies is given by $f\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = (x_0, n_0 x_0)$, $g\left(\begin{smallmatrix} x_0 \\ n_0 \end{smallmatrix}\right) = n_0$. To prove the lemma it suffices to show that f and g are homotopy equivalences. The proof for f and g is similar, let us show, for example, that g is a homotopy equivalence.

In view of Quillen theorem A (cf.[13, ex. IV.3.11]) it suffices to show that for each p-simplex $d\colon \Delta^p \to V$ the pullback $g/(p,d)$ of d and g is contractible. The simplicial set $g/(p,d)$ can be interpreted as the subset of $\Delta^p \times E(X)$ whose set of k-simplices consists of pairs $(\alpha\colon \underline{k} \to \underline{p}, (x_0, \dots, x_k) \in X(\alpha, d)^{k+1})$. Here $X(\alpha, d)$ is the subset of X consisting of all x for which $d_{\alpha(i)} x = d_{\alpha(j)} x$ for $0 \leq i, j \leq k$. Notice that the set $X_d := X(id_{\underline{p}}, d)$ is nonempty and is contained in every $X(\alpha, d)$ (it even equals $X(\alpha, d)$ for surjective α). Now choose a point $\widetilde{x} \in X_d$ and consider the simplicial homotopy

$$H\colon \Delta^p \times EX \times \Delta^1 \to \Delta^p \times EX$$

between the identity map of $\Delta^p \times EX$ and the map $\Delta^p \times c_{\widetilde{x}}$, where $c_{\widetilde{x}}$ is the constant map. More concretely, H sends each triple $(\alpha\colon \underline{k} \to \underline{p}, (x_0, \dots, x_k), \beta\colon \underline{k} \to \underline{1})$ to $(\alpha, (x_0, \dots, x_{i-1}, \widetilde{x}, \dots, \widetilde{x}))$, where i is the minimal number such that $\beta(i) = 1$. By the choice of $\widetilde{x}$ the image of H restricted to $g/(p,d) \times \Delta^1$ is contained in $g/(p,d)$, hence $g/(p,d)$ is contractible. $\square$

Now suppose that $X = H$ is also a group upon which N acts on the left.

Corollary 3.2. *Consider the following two simplicial sets:*

$$S = \bigcup_{h \in H} BN_h \subseteq BN, \quad T = \pi_N(U) \subseteq B(H \times H).$$

There is an isomorphism $\theta\colon \operatorname{Ker}(\pi_1(S) \to N) \cong \operatorname{Ker}(\pi_1(T) \to H \times H)$. Moreover, the higher homotopy groups of S and T are isomorphic.

Proof. Consider the following two pull-back squares:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EN \arrow[d, "\pi_N"] \\ S \arrow[r, hook] & BN \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E(H\times H) \arrow[d, "\pi_{H\times H}"] \\ T \arrow[r, hook] & B(H\times H) \end{tikzcd}$$

The required isomorphism can be obtained from the homotopy long exact sequence applied to the left arrows of these diagrams. □

Now let G be a group. Set $N = G \wr S_n$, $H = G^n$ and consider the left action of N on H given by $(g,s)\cdot h = gh^{s^{-1}}$, $g,h \in G^n$, $s \in S_n$. If one reorders the components of BG^{2n} accordingly, the simplicial subset $T \subset BG^{2n}$ from the above corollary becomes precisely the preimage of 0 under $h_n\colon BG^{2n} \to \mathbb{Z}[BG]$.

It is also easy to compute the map $\pi_1(S) \to N$. Indeed, van Kampen theorem [8, Theorem 2.7] asserts that $\pi_1(S)$ is isomorphic to the free product of stabilizer subgroups $N_h \leq N$ amalgamated over pairwise intersections $N_h \cap N_{h'}$, $h, h \in H$. For $h \in G^n$ the subgroup N_h consists of elements $(g,s) \in N$ satisfying $gh^{s^{-1}} = h$, i.e. elements of the form $(hh^{-s^{-1}}, s)$. Thus, $N_h \cong S_n$, $\pi_1(S)$ is isomorphic to the group $S_n(G)$ and the map $\pi_1(S) \to N$ coincides with the map μ defined in section 2.1. In particular, for an abelian G its kernel is generated by symbols $c_{ij}(u,v)$.

The following lemma gives a more concrete description of the isomorphism θ from Corollary 3.2.

Lemma 3.3. *If G is abelian then every generator $c_{ij}(u,v) \in \operatorname{Ker}(\pi_1(S) \to N)$ is mapped under θ to the homotopy class of the loop $f_{ij}(u,v)$, which is defined as follows:*

$$f_{ij}(u,v) = p_{ij}(u) \circ p_{ij}(v) \circ p_{ij}(u^{-1}v^{-1}), \text{ where } p_{ij}(x) = (x[i], x[i]) \circ (x^{-1}[i], x^{-1}[j]).$$

Proof. Fix a vector $g \in G^n$ and denote by $\gamma_{ij}(g,x)$ the path of length 4 in W which connects the following five points of W_0:

$$\begin{pmatrix} e \\ (g;e) \end{pmatrix}, \begin{pmatrix} x[i] \\ (g;e) \end{pmatrix}, \begin{pmatrix} x[i] \\ (d_{ij}(x)g,(ij)) \end{pmatrix}, \begin{pmatrix} e \\ (d_{ij}(x)g,(ij)) \end{pmatrix}, \begin{pmatrix} e \\ (d_{ij}(x)g,e) \end{pmatrix}.$$

One can easily check that the image of $\gamma_{ij}(g,x)$ in S under $\pi_N g$ is precisely the element $h_{ij}(x)$, while $\pi_{H\times H} f$ sends $\gamma_{ij}(g,x)$ to the loop $p_{ij}(x)$ (here f, g are the maps from the proof of Lemma 3.1). □

Lemma 3.4. *The only natural endotransformations of the second homology functor* $\mathrm{H}_2(-,\mathbb{Z})\colon \mathbf{Groups} \to \mathbf{Ab}$ *are morphisms of multiplication by $n \in \mathbb{Z}$.*

Proof. Denote by η an endotransformation $\mathrm{H}_2(-,\mathbb{Z}) \to \mathrm{H}_2(-,\mathbb{Z})$. When restricted to the subcategory of free finitely-generated abelian groups $\mathbf{Add}(\mathbb{Z}) \subseteq \mathbf{Ab}$ the second homology functor coincides with the second exterior power functor $A \mapsto \wedge^2 A$.

Recall from [1, Theorem 6.13.12] that the category of quadratic functors is equivalent to the category of quadratic $\mathbb{Z}$-modules (see Definition 6.13.5 ibid.) The functor $A \mapsto \wedge^2 A$ is clearly quadratic and corresponds to the quadratic $\mathbb{Z}$-module $0 \to \mathbb{Z} \to 0$ under this equivalence. Thus, we get that η restricted to $\mathbf{Add}(\mathbb{Z})$ coincides with the the morphism of multiplication by $n \in \mathbb{Z}$.

Consider the group $\Gamma_k = \langle x_1, y_1, \dots x_k, y_k \mid [x_1, y_1] \cdot \dots \cdot [x_k, y_k] \rangle$ (the fundamental group of a sphere with k handles). It is clear that the abelianization map $\Gamma_k \to \mathbb{Z}^{2k}$ induces an injective map $\mathrm{H}_2(\Gamma_k, \mathbb{Z}) \cong \mathbb{Z} \to \wedge^2 \mathbb{Z}^{2k}$. Consider the following diagrams.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook] \arrow[d, "\eta_{\Gamma_k}"'] & \wedge^2 \mathbb{Z}^{2k} \arrow[d, "n\cdot"] \\ \mathbb{Z} \arrow[r, hook] & \wedge^2 \mathbb{Z}^{2k} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[r, "\chi"] \arrow[d, "\eta_{\Gamma_k}"'] & \mathrm{H}_2(G, \mathbb{Z}) \arrow[d, "\eta_G"] \\ \mathbb{Z} \arrow[r, "\chi"] & \mathrm{H}_2(G, \mathbb{Z}) \end{tikzcd}$$

From the left diagram it follows that η_{Γ_k} is also the morphism of multiplication by n. For every element $x \in \mathrm{H}_2(G, \mathbb{Z})$ there exist an integer k and a map $\chi\colon \Gamma_k \to G$ sending the generator of $\mathrm{H}_2(\Gamma_k, \mathbb{Z})$ to x. From the right square we conclude that $\eta_G(x) = nx$, as claimed. $\square$

Proof of Theorem 2. Factor h_n as a composition of a trivial cofibration followed by a fibration $BG^{2n} \to E_{h_n} \to \mathbb{Z}[BG]$ and write down the starting portion of the long homotopy exact sequence for the latter. If we denote by K the kernel of the map $\nu\colon \pi_1(T) \to G^{2n}$ induced by the embedding $h_n^{-1}(0) = T \subseteq BG^{2n}$ we obtain the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r, "\beta"'] \arrow[d, "\psi"'] & \pi_1(T) \arrow[r, "\nu"'] \arrow[d, "\varphi"'] & G^{2n} \arrow[r] \arrow[d, "\cong"'] & \mathrm{H}_1(G, \mathbb{Z}) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathrm{H}_2(G, \mathbb{Z}) \arrow[r, "\alpha"'] & \pi_1(F_{h_n}) \arrow[r] & \pi_1(E_{h_n}) \arrow[r] & \mathrm{H}_1(G, \mathbb{Z}) \arrow[r] & 1. \end{tikzcd}$$

We already know by Corollaries 2.12 and 3.2 that K is naturally isomorphic to $\mathrm{H}_2(G, \mathbb{Z})$ provided $n \geq 3$. Thus, it suffices to show that ψ is an isomorphism. In view of Lemma 3.4 we only need to consider the case when G is abelian.

We are going to compute the image of each generator $c_{ij}(x, y) = x \wedge y \in \mathrm{H}_2(G, \mathbb{Z})$ under ψ. In order to do this we need to obtain a more explicit description of the set of 1-simplices of F_{h_n}. Since both BG^{2n} and $\mathbb{Z}[BG]$ are fibrant E_{h_n} and F_{h_n} can be constructed using the path space construction, i. e. $E_{h_n} = BG^{2n} \times_{\mathbb{Z}[BG]} \mathbb{Z}[BG]^I$. In this case the set of 1-simplices of F_{h_n} can be identified with the set of triples $(g, t, t') \in G^{2n} \times \mathbb{Z}[BG]_2 \times \mathbb{Z}[BG]_2$ satisfying the following identites:

$$d_2(t) = h_n(g), \quad d_1(t) = d_1(t'), \quad d_0(t') = 0.$$

It will be convenient for us to perform all calculations inside the subset $F' \subset (F_{h_n})_1$ consisting of those triples for which $t' = 0$ and $d_1(t) = d_0(t) = 0$. For shortness we write down an element of F' as (g, t). It is easy to check that the image of the loop $p_{ij}(x)$ under the natural map $T \to F_{h_n}$ corresponds to the pair $((e, d_{ij}(x)), t(x))$, where $t(x) = -(x, x^{-1}) + (e, e) - (x^{-1}, e) + (e, x^{-1})$.

We want to devise a concrete formula for the element of F' homotopic to the concatenation of two elements $(g_1, t_1), (g_2, t_2) \in F'$. It is easy to check that (g_1g_2, t), where t is the 1-st face of the filler f for the 3-horn $(t_2, , t_1, h_n(g_1, g_2))$, is the desired element. By Moore's theorem one can find a filler for any horn $(p_0, , p_2, p_3)$ in a simplicial abelian group via the formula $f = s_0p_0 - s_0s_1d_2p_0 - s_0s_0d_1p_0 + s_0s_0d_2p_0 + s_1p_2 + s_2p_3 - s_1p_3$. Substituting concrete values of p_i into this formula and using the fact that $d_1t_2 = 0$ we get the following expression for t:

$$t = d_1f = p_0 - s_1d_2p_0 - s_0d_1p_0 + s_0d_2p_0 + p_2 - p_3 + s_1d_1p_3 = \\ = t_1 + t_2 - h_n(g_1, g_2) - s_1h_n(g_2) + s_0h_n(g_2) + s_1h_n(g_1g_2).$$

Using the above formula we get that the image of $c_{ij}(u, v)$ via $\varphi\beta$ equals $((e, e), t)$, where

$$t = 2(e, e) - (x^{-1}, e) + (xy, x^{-1}y^{-1}) - (xy, e) - (x, x^{-1}) - (y, y^{-1})- \\ - (e, x^{-1}y^{-1}) + (e, x^{-1}) + (x, y) + (x^{-1}, y^{-1}) - (e, y) - (xyx^{-1}y^{-1}, e) + (y, e). \tag{3.1}$$

The map α can be described as the Dold—Kan isomorphism $\mathrm{H}_2(G, \mathbb{Z}) \cong \pi_2(\mathbb{Z}[BG])$ followed by the boundary map $\partial\colon \pi_2(\mathbb{Z}[BG]) \to \pi_1(F_{h_n})$. It is not hard to show that ∂ maps each $c \in \pi_2(\mathbb{Z}[BG])$ to $((e, e), -c)$ (see [5, p. 29]). Thus, we get that ψ maps $x \wedge y$ to $-t$, where t is given by (3.1).

On the other hand, the generator $x \wedge y$ of $\mathrm{H}_2(G, \mathbb{Z})$ corresponds to the 2-cycle $c = (x, y) - (y, x)$ (cf. [9, (14), p. 582]) and the corresponding element of $\pi_2(\mathbb{Z}[BG])$ is simply the normalized 2-cycle of c:

$$c' = c - s_0d_0c - s_1d_1c + s_1d_0 = (x, y) - (y, x) - (x, e) + (y, e) + (e, x) - (e, y).$$

It it not hard to check that $t - c'$ is the image of the following 3-cycle under the differential $d_0 - d_1 + d_2 - d_3$:

$$(xy, x^{-1}, y^{-1}) - (y, x, x^{-1}) + s_1s_1(x^{-1} + xy - x - y) + s_0s_0(x^{-1} - x - x^{-1}y^{-1}).$$

Thus we obtain that $[t] = -[c']$, hence ψ is an isomorphism, as claimed. □

Appendix A. Simplicial Dold—Serre fibrations

The aim of this appendix is to show that the map $h_\infty\colon X^\infty \to \mathbb{Z}[X]$ is a quasifibration under suitable assumptions on X.

We start with the definition of the simplicial analogue of the so-called "Dold—Serre fibration", i. e. a map posessing "weak covering homotopy property" (also called "delayed homotopy lifting property"), cf. [12, § 1.4.4]. We show in Lemmas A.2 and A.4 that the map h_∞ satisfies this property and that every map with this property is a quasifibration.

Definition A.1. Let C be a subclass of the class of acyclic cofibrations in the category of simplicial sets. We say that a map of simplicial sets $p\colon E \to B$ satisfies *weak homotopy covering property* with respect to C if for every commutative square

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ V \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & B \end{tikzcd} \tag{A.1}$$

for which $i \in C$ there exists a map $\widetilde{F}$ such that the lower triangle commutes strictly and the upper one commutes up to a fiberwise homotopy H (i. e. a homotopy H such that $pH\colon U \times I \to B$ coincides with the composite $U \times I \xrightarrow{\pi} U \xrightarrow{pg} B$).

Denote by C_{pr} the class consisting of all inclusions $i\colon U \hookrightarrow V$ of finite polyhedral simplicial sets for which there exists a deformation retract (i. e. a map $r\colon V \to U$ homotopic to id_V).

Lemma A.2. *For a fibrant and connected simplicial set X the map $h_\infty\colon X^\infty \to \mathbb{Z}[X]$ satisfies the weak covering homotopy property with respect to the class C_{pr}.*

Proof. Let i_0 be a map from C_{pr} and r be the corresponding retraction. We first reduce the problem to the special case when g is the constant map at the basepoint of X^∞ (which we denote by 0). Indeed, let F, g be arbitrary maps as in (A.1). Since U is finite the image of g is contained in $X^{2N} \subset X^\infty$. If now $\widetilde{F}_0$ is a lifting in the diagram similar to (A.1) in which the top map equals 0 and the bottom one is $F_0 = F - h_\infty gr$ then $\widetilde{F} = gr \times \widetilde{F}_0$ is the desired lifting of F.

Now choose a contractible fibrant simplicial set W which maps surjectively onto X (e. g. take W to be the path space fibration). Since $p\colon W \twoheadrightarrow X$ is surjective, the associated map between free simplicial abelian groups is a Kan fibration. Since i_0 is acyclic we can choose a lifting $\widetilde{F}$ in the following diagram

$$\begin{tikzcd} U \arrow[r, "0"] \arrow[d, hook, "i_0"'] & \mathbb{Z}[W] \arrow[d] \\ V \arrow[r, "F"'] \arrow[ur, "\widetilde{F}"] & \mathbb{Z}[X] \end{tikzcd}$$

It is clear that the image of $\widetilde{F}$ is contained in the simplicial subgroup $\mathbb{Z}[W]_0 \leq \mathbb{Z}[W]$ which, by definition, consists of linear combinations $\sum_i n_i w_i$ for which $\sum_i n_i = 0$. By [10, Lemma 9.1] the canonical map $\mathbb{Z}[\underline{\mathrm{Hom}}(V,W)]_0 \to \underline{\mathrm{Hom}}(V, \mathbb{Z}[W]_0)$ is surjective, hence the map $\widetilde{F}$ can be lifted along h_W in such a way that the bottom triangle in following diagram commutes strictly.

$$\begin{tikzcd} U \arrow[r, "0"] \arrow[d, hook, "i_0"'] & W^\infty \arrow[d, "h_W"] \\ V \arrow[r, "\widetilde{F}"'] \arrow[ur, dashed, "G"] & \mathbb{Z}[W] \end{tikzcd}$$

Notice that the image of Gi_0 is contained in the fiber $h_W^{-1}(0)$. It is easy to see that $h_W^{-1}(0)$ is contractible (the contracting homotopy for W^∞ can be restricted to $h_W^{-1}(0)$). We get that Gi_0 is fiberwise homotopic to 0 hence the composite map $V \xrightarrow{G} W^\infty \to X^\infty$ is the desired lifting of F. $\square$

The assertions of the following lemma are straightforward and are given without proof.

Lemma A.3. *Let $p\colon (E,e) \to (B,b)$ be a map of pointed topological spaces. Let $p^{-1}(b) \hookrightarrow F_p(b) \subseteq E \times_B B^I$ be the inclusion map of the fiber of p into the homotopy fiber. Denote k-th relative homotopy group (or set) $\pi_k(F_p(b), p^{-1}(b), e))$ by G_k.*

(1) *Maps of triples $(D^k, S^{k-1}, pt) \to (F_p(b), p^{-1}(b), e)$ are in one-to-one correspondence with commuting diagrams of the form*

(A.2)
$$\begin{tikzcd} D^k \arrow[rr, "a"] \arrow[d, hook, "i_0"'] & & E \arrow[d, "p"] \\ D^k \times I \arrow[r, "\pi"] & (D^k \times I)/J \arrow[r, "A"] & B \end{tikzcd}$$

Here J denotes $(S^{k-1} \times I) \cup (D^k \times \{1\})$. In the sequel we denote such a diagram by (a, A).

(2) *Two diagrams (a_0, A_0) and (a_1, A_1) represent the same element of G_k iff there exists a "diagram homotopy" that connects them. By a diagram homotopy we mean a family of maps (a_t, A_t) continously depending on $t \in [0,1]$ such that $pa_t = A_t \pi i_0$ holds for all t.*

(3) *Let (a, A) be a diagram and a' be any other map homotopic to a via some homotopy H such that $H(S^{k-1} \times I) \subseteq p^{-1}(b)$. Then there exists a map A' such that (a', A') is a diagram homotopic to (a, A).*

(4) *Assume that in the diagram* (A.2) *there exists a diagonal map* $\widetilde{A}\colon D^k \times I \to E$ *such that the bottom triangle is commutative and the upper is commutative up to a fiberwise homotopy. Then the element of* G_k *given by* (a, A) *is trivial.*

Lemma A.4. *Let* $p\colon E \to B$ *be a map of simplicial sets satisfying the weak covering homotopy property with respect to the class* C_{pr}*. Then the geometric realization of* p *is a quasifibration, i. e. for every point* $b \in B_0$ *the inclusion* $|p|^{-1}(b) \hookrightarrow F_{|p|}(b)$ *is a homotopy equivalence.*

Proof. It suffices to show that $\pi_k(F_{|p|}(b), |p|^{-1}(b))$ are all trivial for $k \geq 1$. Consider a diagram (a, A) of the form (A.2) (with the map $|p|$ in the right hand side). In view of the previous lemma it suffices to construct a lifting $\widetilde{A}$ satisfying the requirements of Lemma A.3.(4).

The idea of the proof is to approximate (a, A) with a homotopic pair of geometric realizations of simplicial maps and then invoke the weak covering homotopy property. The key ingredient in the proof below is the simplicial approximation theorem [6, Theorem 4.7]. For brevity we denote the subdivision functor $\operatorname{sd}^m \operatorname{sd}_*(-)$ from its statement by $s^m(-)$ and the canonical natural transformation $s^m X \to X$ by δ^m.

We proceed in a number of steps. Denote the restriction of a to S^{k-1} by a_0. Notice that the image of a_0 is contained in $|p^{-1}(b)|$. Using the approximation theorem we find a simplicial map $a_0'\colon s^m(S^{k-1}) \to p^{-1}(b)$ such that $a|\delta^m| \cong |a_0'|$.

We can find a map $a'\colon D^k \to |E|$ extending $|a_0'|$ and homotopic to a via some homotopy H satisfying $H(S^k \times I) \subseteq p^{-1}(b)$.

Invoking the approximation theorem once again (with the initial condition specified by $|a_0'|$) we find $q > m$ and $a''\colon s^q(D^k) \to E$ such that $a'|\delta^q| \cong |a''|$ rel S^k. Using Lemma A.3.(3) we extend the homotopy $a|\delta^q| \cong a'|\delta^q| \cong |a''|$ to a homotopy of diagrams $(a|\delta^q|, A|\delta^q|) \cong (|a''|, A')$.

Applying the approximation theorem to A' with the initial condition on the boundary $\partial(D^k \times I) = D^k \times \{0\} \cup J$ specified by pa'' and $const_b$, respectively, we find an integer $r > q$ and a map $A''\colon s^r(D^k \times I) \to B$ such that $A'|\delta^r| \cong |A''|$ rel $\partial(D^k \times I)$.

Thus, we have obtained the following commutative diagram of simplicial sets for which $(|a''|, |A''|)$ is a diagram homotopic to (a, A).

$$\begin{tikzcd}
s^r(D^k) \arrow[r, "a''\gamma^{r-q}"] \arrow[d, hook, "s^r(i)"'] & E \arrow[d, "p"] \\
s^r(D^k \times I) \arrow[r, "A''"'] \arrow[ur, dashed, "\widetilde{A}"'] & B
\end{tikzcd}$$

By functoriality of the subdivision functor the map $s^r(i)$ posesses a deformation retract hence there exists the desired lifting $\widetilde{A}$. $\square$

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