NOTES ON ALGEBRAIC TOPOLOGY

Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.

Michael Atiyah

1. A CRASH INTRODUCTION TO CATEGORY THEORY

Category theory permeates algebraic topology and is really put to good use, rather than being a fancy attire that dresses up and obscures some simple theory, as often is the case.

Definition 1.1. A category $\mathcal{C}$ consists of the following data:
1. A class of objects: $\mathrm{Ob}(\mathcal{C})$. We will write both $X \in \mathcal{C}$ and $X \in \mathrm{Ob}(\mathcal{C})$.
2. A set of morphisms: $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for each $X, Y \in \mathrm{Ob}(\mathcal{C})$. An element $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is denoted by $X \xrightarrow{f} Y$ or $f \colon X \to Y$.
3. Composition operations

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) \\ (f, g) &\longmapsto g \circ f, \end{aligned}$$

for each $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, which satsifies
   (i) Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ holds and will be denoted by $h \circ g \circ f$. The composition of morphisms can be illustrated by commutative diagram in the usual way.
   (ii) Identity: For each $A \in \mathrm{Ob}(\mathcal{C})$, there exists $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $f \circ 1_X = f = 1_Y \circ f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Such 1_X is unique by its definition, called the identity element.

In this note, we shall ignore some discussion due to problems in set theory.

Definition 1.2. A subcategory $\mathcal{C}'$ of $\mathcal{C}$, denoted by $\mathcal{C}' \subset \mathcal{C}$, is a category such that
1. $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$.
2. $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathcal{C}'$.
3. Compositions in $\mathcal{C}'$ coincide with that in $\mathcal{C}$ under the above inclusion.

$\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$ if $\mathrm{Hom}_{\mathcal{C}'}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for all $X, Y \in \mathrm{Ob}(\mathcal{C}')$.

Definition 1.3. A morphism $f \colon X \to Y$ is called an isomorphism if there exists $g \colon Y \to X$ such that $f \circ g = 1_Y, g \circ f = 1_X$. Two objects X, Y are said to be isomorphic if there exists an isomorphism in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Example 1.4. Here we present some categories that we will come across.
1. **Set**, the category of sets.
2. **Grp**, the category of groups. It has a full subcategory **Ab**, the category of abelian groups.
3. **Ring**, the category of rings with identity. Again, it has a full subcategory **CRing**, the category of commutative rings.
4. **Top**, the category of topological spaces. It is of particular interest in this note.

Example 1.5. Given two categories $\mathcal{C}, \mathcal{D}$, we can construct their product $\mathcal{C} \times \mathcal{D}$ as follows.
- $\mathrm{Ob}(\mathcal{C} \times \mathcal{D})$ consists of pairs (X, Y) with $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$.
- $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((X_1, Y_1), (X_2, Y_2)) = \{(f, g) \mid \mathrm{Hom}_{\mathcal{C}}(X_1, X_2), g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2)\}$.
- Compositions are componentwise.

Definition 1.6. Let $\mathcal{C}$ be a category. Let $\simeq$ be an equivalence relation on every $\mathrm{Hom}_{\mathcal{C}}(X, Y), X, Y \in \mathrm{Ob}(\mathcal{C})$, such that $(g_1 \circ f_1) \simeq (g_2 \circ f_2)$ if $f_1 \simeq f_2, g_1 \simeq g_2$. In this case we say $\simeq$ defines an equivalence relation on $\mathcal{C}$. The quotient category $\mathcal{C}/\simeq$ is the category who inherit the same objects with $\mathcal{C}$ with $\mathrm{Hom}_{\mathcal{C}/\simeq}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\simeq$.

One of the most important equivlance relation in algebraic topology is the homotopy.

Definition 1.7. Tow morphisms $f_0, f_1 \colon X \to Y$ in **Top** are said to be homotopic, denoted by $f_0 \simeq f_1$, if there exists $F \in \mathrm{Hom}_{\mathbf{Top}}(X \times I, Y)$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

Proposition 1.8. Homotopy indeed gives an equivalence relation on **Top**.

Definition 1.9. We define homotopy class

$$[f] = \{g \in \mathrm{Hom}_{\mathbf{Top}}(X,Y) \mid g \simeq f\}$$

and denote $[X,Y] = \mathrm{Hom}_{\mathbf{Top}}(X,Y)/\simeq$. The quotient category of **Top** under homotopy is called homotopy category, denoted by h**Top**.

Example 1.10. To similiplifly statement in algebraic topology, sometimes we may use the following categories.
1. $\mathbf{Top}_\bullet$, the category of pointed topological spaces. Namely, its objects are (X, x_0) for spaces X and $x_0 \in X$. Morphism $f \colon (X,x_0) \to (Y,y_0)$ means a continuous map $f \colon X \to Y$ with $f(x_0) = y_0$.

   Homotopy between $f, g \colon (X,x_0) \to (Y,y_0)$ is a ,homotopy F between f,g such that $F(x_0,t) = y_0$ for all $t \in I$. The quotient category is denoted by $\mathrm{h}\mathbf{Top}_\bullet$.
2. **TopP**, the category of paris of spaces. Namely, its objects are pairs (X,A) with space X and subspace $A \subset X$. A morphism $f \colon (X,A) \to (Y,B)$ means a continuous map $f \colon X \to Y$ with $f(A) \subset B$.

   Homotopy between $f,g \colon (X,A) \to (Y,B)$ is a homotopy F between f,g such that $F(x,t) \subset B$ for all $x \in A, t \in I$. The quotient category is denoted by h**Top**.

Definition 1.11. Two spaces X,Y are said to have the same homotopy type (or homotopy equivalent) if they are isomorphic in h**Top**.

Definition 1.12. $\mathbb{R}^n$ are homotopy equivalent to each other for every n, but not homeomorphic.

Sometimes we may consider a relative version of homotopy.

Definition 1.13. Let $A \subset X$ be a subspace. We say $f_0, f_1 \in \mathrm{Hom}_{\mathbf{Top}}(X,Y)$ are homotopic relative to A, denoted by $f_0 \simeq f_1 \operatorname{rel} A$ if there exists homotopy $F \colon X \times I \to Y$ such that $F|_{A \times I} = f_0|_A$.

Definition 1.14. Let $\mathcal{C}, \mathcal{D}$ be two categories. A covariant functor $F \colon \mathcal{C} \to \mathcal{D}$ consists of:
1. $F \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$.
2. $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ for all $X, Y \in \mathrm{Ob}(\mathcal{C})$

$$F(f) \colon F(X) \longrightarrow F(Y)$$

   satisfying
   (i) $F(g \circ f) = F(g) \circ F(f)$.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "F(g)\circ F(f)"'] & F(Y) \arrow[d, "F(g)"] \\ & F(Z) \end{tikzcd}$$

   (ii) $F(1_X) = 1_{F(X)}$.

Reversing all arrows in the diagram, we get contravariant functor.

Example 1.15. For any $X \in \mathrm{Ob}(\mathcal{C})$,

$$\begin{aligned} \mathrm{Hom}(X,-) \colon \mathrm{Ob}(\mathcal{C}) &\longrightarrow \mathbf{Set} \\ A &\longmapsto \mathrm{Hom}(X,A), \end{aligned}$$

defines a covariant functor and

$$\begin{aligned} \mathrm{Hom}(-,X) \colon \mathrm{Ob}(\mathcal{C}) &\longrightarrow \mathbf{Set} \\ A &\longmapsto \mathrm{Hom}(A,X), \end{aligned}$$

defines a contravariant functor.

Functors of these two types are said to be representable (by X).

Definition 1.16. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be faithful if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ are all injective. Similarly, a functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be full if $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ are all surjective.

Example 1.17. The forgetful functor **Grp** $\to$ **Set**, assigning every group to its underlying set, is representable by the free group with one generator. This functor is full, but obviously not faithful.

Example 1.18. Let A be an abelian group and X be a space. We will study its n-th cohomology $H^n(X;A)$ with coefficients in A. It defines a contravariant functor

$$H^n(-,A)\colon \mathrm{h}\mathbf{Top} \longrightarrow \mathbf{Grp}.$$

We will see that it is representable by the Eilenberg–Mac Lane space $K(G,n)$ if we work with the subcategory of CW-complexes.

Example 1.19. We define a contravariant functor

$$\begin{aligned} \mathcal{O}\colon \mathbf{Top} &\longrightarrow \mathbf{Ring} \\ X &\longmapsto \mathcal{O}_X = \mathrm{Hom}_{\mathbf{Top}}(X,\mathbb{C}). \end{aligned}$$

By a classic theorem of Gelfand–Kolmogorov, two compact Hausdorff spaces X, Y are homeomorphic if and only if $\mathcal{O}_X, \mathcal{O}_Y$ are isomorphic in **Ring**.

Proposition 1.20. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. If $f\colon A \to B$ is an isomorphism in $\mathcal{C}$, then $F(f)\colon F(A) \to F(B)$ is an isomorphism in $\mathcal{D}$.

Proof. Obvious. $\square$

Definition 1.21. Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\tau\colon F \to G$ consists of morphisms

$$\tau_X\colon F(X) \longrightarrow G(X), X \in \mathrm{Ob}(\mathcal{C}),$$

such that

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\tau_X"] & F(Y) \arrow[d, "\tau_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Here we assume F, G are covariant. For contravariant cases, we just need to reverse the horizontal arrows.

τ is called a natural isomorphism if τ_X is an isomorphism for all $X \in \mathrm{Ob}(\mathcal{C})$ and we write $F \simeq G$.

Definition 1.22. We consider two functors $\mathrm{GL}(n,-), (-)^\times\colon$ **CRing** $\to$ **Grp**. For a commutative ring R, $\mathrm{GL}(n,R)$ is the group of invertible $n \times n$ matrices with entries in R, while $R^\times$ is the multiplicative group of invertible elements in R. We identify $(-)^\times = \mathrm{GL}(1,-)$.

The determinant defines a natural transformation

$$\det\colon \mathrm{GL}(n,-) \longrightarrow (-)^\times.$$

The naturality of det is rooted in the fact that determinant is the same for any coefficient ring.

Example 1.23. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. We have
- A natural transformation

$$f_*\colon \mathrm{Hom}_{\mathcal{C}}(-,X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-,Y).$$

- A natural transformation

$$f^*\colon \mathrm{Hom}_{\mathcal{C}}(Y,-) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,-).$$

Example 1.24. The above example can be generalized into the following construction. Let $X \in \mathcal{C}$.
- Let $F\colon \mathcal{C} \to$ **Set** be a contravariant functor. Then any $\varphi \in F(X)$ induced a natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(-,X) \longrightarrow F$$

by assigning $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ to $F(f)(\varphi) \in F(Y)$.

- Let $G\colon \mathcal{C} \to \mathbf{Set}$ be a covariant functor. Then any $\varphi \in G(X)$ induces a natural transformation

$$f^*\colon \operatorname{Hom}_{\mathcal{C}}(X,-) \longrightarrow G$$

by assigning $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)$ to $G(f)(\varphi) \in G(Y)$.

Definition 1.25. Let $F,G,H\colon \mathcal{C} \to \mathcal{D}$ be two functors and $\theta\colon F \to G, \tau\colon G \to H$ be two natural transformations. The composition $\tau\colon \theta$ is still a natural transformation from F to H defined by

$$(\tau \circ \theta)_X = \tau_{G(X)} \circ \theta_X$$

Definition 1.26. Two categories $\mathcal{C},\mathcal{D}$ are said to be isomorphic if there exists functors $F\colon \mathcal{C} \to \mathcal{D}, G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G = 1_{\mathcal{D}}, G \circ F = 1_{\mathcal{C}}$.

$\mathcal{C},\mathcal{D}$ are said to be equivalent if there exists functors $F\colon \mathcal{C} \to \mathcal{D}, G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G \simeq 1_{\mathcal{D}}, G \circ F \simeq 1_{\mathcal{C}}$.

In applications, category isomorphism is a too strong condition to expect for most cases. Category equivalence is more realistic and equally good essentially.

Proposition 1.27. If $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, then F is fully faithful.

Proof. By definition, we can find functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $\tau\colon GF \simeq \mathrm{id}_{\mathcal{D}}$ and $\theta\colon FG \simeq \mathrm{id}_{\mathcal{C}}$. Hence for any $Z \in \operatorname{Ob}(\mathcal{D})$ we have $\theta_Z\colon FG(Z) \to Z$. Hence F is essentially surjective, i.e., F maps to any object in $\mathcal{D}$. Similarly G is essentially surjective. Notice that

$$\begin{array}{ccccccc}
\operatorname{Hom}_{\mathcal{C}}(X,Y) & \xrightarrow{F} & \operatorname{Hom}_{\mathcal{D}}(F(X),F(Y)) & \xrightarrow{G} & \operatorname{Hom}_{\mathcal{C}}(GF(X),GF(Y)) & \xrightarrow{\simeq} & \operatorname{Hom}_{\mathcal{C}}(X,Y) \\
 & & & & & & \\
f & \longmapsto & F(f) & \longmapsto & GF(f) & \longmapsto & \tau_Y \circ (GF(f)) \circ \varphi_X^{-1}
\end{array}$$

is identity map. Similarly, we can let $X = G(A)$ and $Y = G(B)$ and have diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}_{\mathcal{D}}(A,B) & \xrightarrow{G} & \operatorname{Hom}_{\mathcal{C}}(X,Y) & \xrightarrow{F} & \operatorname{Hom}_{\mathcal{D}}(F(X),F(Y)) & \xrightarrow{\simeq} & \operatorname{Hom}_{\mathcal{D}}(A,B) \\
 & & & & & & \\
g & \longmapsto & G(g) & \longmapsto & FG(g) & \longmapsto & \theta_X \circ (FG(g)) \circ \tau_Y^{-1}
\end{array}$$

Thus F,G are both invertible. $\square$

Definition 1.28. A category $\mathcal{C}$ is said to be small if $\operatorname{Ob}(\mathcal{C})$ is a set.

For a small category $\mathcal{C}$ and a category $\mathcal{D}$, we define the functor category $\operatorname{Fun}(\mathcal{C},\mathcal{D})$ as follows:
- Objects: Functors from $\mathcal{C}$ to $\mathcal{D}$.
- Morphisms: Natural transformations between two functors.

Theorem 1.29 (Yoneda lemma)**.** Let $\mathcal{C}$ be a small category and $A \in \operatorname{Ob}(\mathcal{C})$. Denote the two functors in Example 1.14

$$h_A = \operatorname{Hom}_{\mathcal{C}}(-,A), h^A = \operatorname{Hom}_{\mathcal{C}}(A,-).$$

1. Contravariant version: Let $F\colon \mathcal{C} \to \mathbf{Set}$ be a contravariant version. Then there is an isomorphism of sets

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathbf{Set})}(h_A, F) \simeq F(A).$$

This isomorphism is functorial in A.
2. Covariant version: Let $G\colon \mathcal{C} \to \mathbf{Set}$ be a covariant functor. Then there is an isomorphism of sets

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathbf{Set})}(h^A, G) \simeq G(A).$$

This isomorphism is functorial in A.

The precise meaning of functorial property in A is that we have isomorphisms of functors $\mathcal{C} \to \mathbf{Set}$

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathbf{Set})}(h_{(-)}, F) \simeq F(-), \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathbf{Set})}(h^{(-)}, G) \simeq G(-).$$

There are many dual descriptions in category theory. Roughly speaking, taking dual is reversing arrows and the order of compositions.

Definition 1.30. Given a category $\mathcal{C}$, we can define its opposite category $\mathcal{C}^{\mathrm{op}}$ by declaring
- $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$.
- $f\colon X \to Y$ is a morphism in $\mathcal{C}^{\mathrm{op}}$ if and only if $f \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$.
- The composition of morphisms $g \circ f$ in $\mathcal{C}^{\mathrm{op}}$ is the same as the composite $f \circ g$ in $\mathcal{C}$.

A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

We will often, especially in topology, consider the lifting problem by finding $F \in \mathrm{Hom}_{\mathcal{C}}(X, E)$ such that

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"'] & B \end{tikzcd}$$

The dual problem is the extension problem by finding $G \in \mathrm{Hom}_{\mathcal{C}}(E, X)$ such that

$$\begin{tikzcd} & E \arrow[dl, dashed, "G"'] \\ X & B \arrow[l, "g"] \arrow[u, "j"'] \end{tikzcd}$$

Let $L\colon \mathcal{C} \to \mathcal{D}, R\colon \mathcal{D} \to \mathcal{C}$ be two covariant functors. The rules

$$(X, Y) \longmapsto \mathrm{Hom}_{\mathcal{D}}(L(X), Y), (X, Y) \longmapsto \mathrm{Hom}_{\mathcal{C}}(X, R(Y))$$

define two functors

$$\mathrm{Hom}_{\mathcal{D}}(L(-), -), \mathrm{Hom}_{\mathcal{C}}(-, R(-))\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathbf{Set}.$$

We say L, R are adjoint to each other (more precisely, L is the left adjoint, R is the right adjoint), if there is a natural isomorphism

$$\tau\colon\ \mathrm{Hom}_{\mathcal{D}}(L(-), -) \simeq \mathrm{Hom}_{\mathcal{C}}(-, R(-)).$$

We sometimes write adjoint functors as

$$L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R.$$

Example 1.31. Let X be a set and $F(X)$ be the free abelian group generated by X. This defines a functor

$$\begin{aligned} F\colon \mathbf{Set} &\longrightarrow \mathbf{Ab} \\ X &\longmapsto F(X). \end{aligned}$$

The forget functor $G\colon \mathbf{Ab} \to \mathbf{Set}$ is the right adjoint of F. In fact, many free constructions in mathematics are left adjoint to certain forgetful functors.

2. FUNDAMENTAL GROUPOID

This section includes some elementary applications of category theory in algebraic topology.

Definition 2.1. Let X be a space. Path connectedness defines an equivalence relation on X, which induces path components

$$\pi_0(X) = X/\sim.$$

Theorem 2.2. $\pi_0\colon \mathrm{h}\mathbf{Top} \to \mathbf{Set}$ defines a covariant functor.

Proof. Let $f \in \mathrm{Hom}_{\mathbf{Top}}(X, Y)$. Then $f\colon \pi_0(X) \to \pi_0(Y), x \mapsto f(x)$ is well-defined. $\square$

Corollary 2.3. If X, Y are homotopy equivalent, then $\pi_0(X) \simeq \pi_0(Y)$.

Definition 2.4. Let $\gamma\colon I \to X$ be a path. We define the path class of γ by

$$[\gamma] = \{\sigma \in \mathrm{Hom}_{\mathbf{Top}}(I, X) \mid \sigma \simeq \gamma \ \mathrm{rel}\, \partial I\}.$$

For $\gamma_1, \gamma_2 \colon I \to X$ such that $\gamma_1(1) = \gamma_2(0)$, we define the composite path

$$\begin{aligned} \gamma_2 * \gamma_1 \colon I &\longrightarrow X \\ t &\longmapsto \begin{cases} \gamma_1(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \end{aligned}$$

Proposition 2.5. Let f_1, f_2, g_1, g_2 be paths in X such that $[f_1] = [f_2], [g_1] = [g_2]$ and $f_i(1) = g_i(0)$. Then $[g_2 * f_2] = [g_1 * f_1]$.

Proof. Let $F \colon f_1 \simeq f_2, G \colon g_1 \simeq g_2$. Then

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

gives the homotopy $g_1 * f_1 \simeq g_2 * f_2 \operatorname{rel} \partial I$. □

Proposition 2.6. The composition of paths is associative with identity elements as identity paths. Moreover, every path $\gamma \colon I \to X$ admits inverse $\gamma^{-1}(t) = \gamma(1-t)$.

Definition 2.7. Let X be a space. We define the category $\Pi_1(X)$ as follows:
- $\mathrm{Ob}(\Pi_1(X)) = X$.
- $\mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ is the path clases from x_0 to x_1.
- $1_{x_0} = i_{x_0}$.

$\Pi_1(X)$ is called the fundamental groupoid, or path category of X.

Definition 2.8. A category is called a groupoid if all its morphisms are isomorphisms. All groupoids form a category **Groupoid** as follows:
- Objects are groupoids.
- Morphisms are functors between groupoids.

Example 2.9. A group G is the same as a groupoid with a single object $*$, such that $\mathrm{Hom}(*, *) = G$ and composition is group operation.

In this way we have a fully faithful functor **Grp** $\to$ **Groupoid**.

Definition 2.10. Let $\mathcal{C}$ be a groupoid. Let $A \in \mathrm{Ob}(\mathcal{C})$, we define its automorphism group by

$$\mathrm{Aut}_{\mathcal{C}}(A) = \mathrm{Hom}_{\mathcal{C}}(A, A).$$

Any $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ induces a group isomorphism

$$\begin{aligned} \mathrm{Ad}_f \colon \mathrm{Aut}_{\mathcal{C}}(A) &\longrightarrow \mathrm{Aut}_{\mathcal{C}}(B) \\ g &\longmapsto f \circ g \circ f^{-1}. \end{aligned}$$

This further induces a functor

$$\mathcal{C} \longrightarrow \mathbf{Grp}$$

by assigning $A \to \mathrm{Aut}_{\mathcal{C}}(A)$ and $f \longmapsto \mathrm{Ad}_f$.

If we specialize this to spaces, we obtain a functor

$$\Pi_1(X) \longrightarrow \mathbf{Grp}.$$

Definition 2.11. Let X be a space and take $x_0 \in X$. The group $\pi_1(X, x_0) = \mathrm{Aut}_{\Pi_1(X)}(x_0)$ is called the fundamental group of the pointed space (X, x_0).

Remark 2.12. One should not be confused by the formalism above, but focus on the geometry picture of this definition.

Proposition 2.13. If X is path connected, then for $x_0, x_1 \in X$, we have a group isomorphism $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$.

Proof. If X is path connected, then x_0, x_1 are isomorphic in $\Pi_1(X)$. □

In the path connected case, we will simply denote by $\pi_1(X)$ the fundamental group without mentioning the base point.

Proposition 2.14. Let $f \in \operatorname{Hom}_{\mathbf{Top}}(X, Y)$. It defines a functor $\Pi_1(f) \colon \Pi_1(X) \to \Pi_1(Y)$ by assigning $x \mapsto f(x)$ and $[\gamma] \mapsto [f \circ \gamma]$. $\Pi_1 \colon \mathbf{Top} \to \mathbf{Groupoid}$ defines a functor that sends X to $\Pi_1(X)$.

Proof. Suppose γ connects x_0, x_1 and σ connects x_1, x_2. Then

$$\Pi_1(f)(\sigma * \gamma) = [f \circ (\sigma * \gamma)] = [(f \circ \sigma) * (f \circ \gamma)]$$

and $\Pi_1(f)(i_{x_0}) = [f \circ i_{x_0}] = [i_{f(x_0)}]$. Suppose $\varphi \colon X \to Y, \psi \colon Y \to Z$ are continuous. Then

$$\Pi_1(\psi \circ \varphi)(X) = Z = \Pi_1(\psi)(Y) = \Pi_1(\psi) \circ \Pi_1(\varphi)(X)$$

and

$$\Pi_1(\psi \circ \varphi)(\gamma) = [(\psi \circ \varphi) \circ \gamma]\Pi_1(\psi)[(\varphi \circ \gamma)] = \Pi_1(\psi) \circ \Pi_1(\varphi)[\gamma].$$

Finally $\Pi_1(\mathrm{id}_X)(X) = X$ and $\Pi_1(\mathrm{id}_X)[\gamma] = [\mathrm{id}_X \circ \gamma] = [\gamma]$. $\square$

Proposition 2.15. Let $f, g \colon X \to Y$ be maps which are homotopic by $F \colon X \times I \to Y$. Define path class

$$\tau_x = [F|_{\{x\} \times I}] \in \operatorname{Hom}_{\Pi_1(Y)}(f(x), g(x))$$

Then τ defines a natural transformation

$$\tau_F \colon \Pi_1(f) \longrightarrow \Pi_1(g).$$

Proof. Let $r \in \operatorname{Hom}_{\Pi_1(X)}(x_0, x_1)$. We only need to show that the following diagram commutes

$$\begin{array}{ccc}
\Pi_1(f)(x_0) = f(x_0) & \xrightarrow{f \circ r} & \Pi_1(f)(x_1) = f(x_1) \\
\big\downarrow{\scriptstyle \tau_{x_0}} & & \big\downarrow{\scriptstyle \tau_{x_1}} \\
\Pi_1(g)(x_0) = g(x_0) & \xrightarrow{g \circ r} & \Pi_1(g)(x_1) = g(x_1)
\end{array}$$

$(g \circ r) * \tau_{x_0} = F \circ \tau_1$ and $\tau_{x_1} \circ (f \circ r) = F \circ \tau_0$, where

$$\tau_1(t) = \begin{cases} (x_0, 2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ (r(2t-1), 1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}, \tau_0(t) = \begin{cases} (r(2t), 0) & 0 \leqslant t \leqslant \frac{1}{2} \\ (x_1, 2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}.$$

In particular, τ_1, τ_0 are homotopic via

$$G(t, s) = \begin{cases} (x_0, 2ts) & 0 \leqslant t \leqslant \frac{1}{2} \\ (r(2t-1), s) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}.$$

Hence $[g \circ r] * [\tau_{x_0}] = [\tau_{x_1}] * [f \circ r]$. $\square$

Theorem 2.16. Let $f \colon X \to Y$ be a homotopy equivalence. Then

$$\Pi_1(f) \colon \Pi_1(X) \longrightarrow \Pi_1(Y)$$

is an isomorphism of categories. In particular, it induces a group isomorphism

$$\pi_1(X, x_0) \simeq \pi_1(Y, f(x_0)).$$

Proof. Let $g \colon Y \to X$ represent the inverse of f in h$\mathbf{Top}$. For every path class $\operatorname{Hom}_{\Pi_1(X)}(x_1, x_2)$ and a path class $[\gamma]$ in it,

$$\Pi_1(g) \circ \Pi_1(f)[\gamma] = \Pi_1(g \circ f)[\gamma] = [(g \circ f) \circ \gamma] = [\gamma].$$

Hence $\Pi_1(g) \circ \Pi_1(f) = 1_{\Pi_1(X)}$. Similarly we can show that $\Pi_1(f) \circ \Pi_1(g) = 1_{\Pi_1(Y)}$.

As for the group isomorphism, recall that equivalence of categories is fully faithful. Hence

$$\pi_1(X, x_0) = \operatorname{Aut}_{\Pi_1(X)}(x_0) \simeq \operatorname{Aut}_{\Pi_1(Y)}(\Pi_1(f)(x_0)) = \pi_1(Y, f(x_0)).$$

$\square$

Proposition 2.17. Let X, Y be two spaces. Then we have a canonical isomorphism of categories

$$\Pi_1(X \times Y) \simeq \Pi_1(X) \times \Pi_1(Y).$$

In particular for any $x_0 \in X, y_0 \in Y$, we have a group isomorphism

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Example 2.18. For a point, $\pi_1(\mathrm{pt}) = 0$. Since $\mathbb{R}^n$ is homotopy equivalent to pt, it follows that $\pi_1(\mathbb{R}^n) = 0$.

Example 2.19. $\pi_1(\mathbb{T}^n) = \pi_1(\mathbb{S}^1)^n$. Later we will see that $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

3. Covering and fibration

Recall that in the last section, we mentioned the lifting problem: Given $p\colon E \to B, f \circ X \to B$, we hope to find $F\colon X \to E$ such that

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "F"] \arrow[r, "f"'] & B \end{tikzcd}$$

As one can image, it is almost impossible to find a satisfying answer for general case. Hopefully, we still can say lots of things when $p\colon E \to B$ is good enough.

Definition 3.1. A fiber bundle consists of the following data:
(1) A space B, called the base.
(2) A space F, called the fiber.
(3) A space E, called the bundle.
(4) A surjective continuous map $p\colon E \to B$.
Their relationships are described by the following: There exists an open cover $\{U_\alpha\}$ of B such that for each U_α, there is a homeomorphism φ_α making the following diagram commute

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\pi_\alpha"] \\ & U_\alpha & \end{tikzcd}$$

We denote it by

$$F \longrightarrow E \longrightarrow B$$

when there is no ambiguity from the context. If we can find a trivialization over B, then E is homeomorphic to $B \times F$ and we say p is a trivial bundle.

Example 3.2. The projection map $\mathbb{R}^{n+m} \to \mathbb{R}^n, (x_1, \cdots, x_{n+m}) \mapsto (x_1, \cdots, x_n)$ is a trivial bundle with fiber $\mathbb{R}^m$.

Example 3.3. A real vector bundle of rank n over a smooth manifold is a fiber bundle with fiber $\mathbb{R}^n$.

Example 3.4. We identify $\mathbb{S}^{2n+1}$ with the unit sphere in $\mathbb{C}^{n+1}$ parametrized by

$$\mathbb{S}^{2n+1} = \left\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = 1\right\}.$$

There is a natural $\mathbb{S}^1$-action on $\mathbb{S}^{2n+1}$ given by

$$e^{i\theta} \cdot (z_0, \cdots, z_1) = (e^{i\theta} z_0, \cdots, e^{i\theta} z_n).$$

The action is free, so the orbit space can be identified with $\mathbb{CP}^n$

$$\mathbb{S}^{2n+1}/\mathbb{S}^1 \cong (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^* = \mathbb{CP}^n.$$

Then the projection map $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$ is a fiber bundle with fiber $\mathbb{S}^1$. Actually, this bundle is not trivial.

The case $n = 1$ gives the Hopf fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2 = \mathbb{CP}^1,$$

which is particularly interesting. In this case, the projection sends $(z_0, z_1) \in \mathbb{S}^3 \subset \mathbb{C}^2$ to $\frac{z_0}{z_1} \in \mathbb{S}^2$. In polar coordinates, we have $z_j = r_j e^{i\theta_j}$ for $r_0^2 + r_1^2 = 1$ and $p(z_0, z_1) = \frac{r_0}{r_1} e^{i(\theta_0 - \theta_1)}$. For a fix $\rho = \frac{r_0}{r_1}$, we obtain a torus T_ρ with coordinates $\{\theta_0, \theta_1\}$ in $\mathbb{S}^3$.

Fiber bundle is important in topology, but we will focus on a simplier case in this section.

Definition 3.5. A covering map is a fiber bundle with discrete fiber F. If we would like to specify the fiber, we call it a F-covering. If the fiber has n points, we also call it an n-fold covering.

Example 3.6. The map $\mathbb{C} \to \mathbb{C}^*, z \mapsto e^{2\pi i z}$ is a $\mathbb{Z}$-covering, so its restriction $\mathbb{R} \to \mathbb{S}^1, t \mapsto e^{2\pi i t}$ is also a $\mathbb{Z}$-covering.

Example 3.7. For $n \in \mathbb{Z} \setminus \{0\}$, the map $\mathbb{S}^1 \to \mathbb{S}^1, z \mapsto z^n$ is an $|n|$-fold covering. $\mathbb{C} \to \mathbb{C}, z \mapsto z^n$ is not a covering, but $\mathbb{C}^* \to \mathbb{C}^*, z \mapsto z^n$ is an $|n|$-fold covering.

Example 3.8. Recall that

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^* = \mathbb{S}^n/\mathbb{Z}_2.$$

Then there is a natural double cover $\mathbb{S}^n \to \mathbb{RP}^n$.

Definition 3.9. Let $p\colon E \to B, f\colon X \to B$. A lifting of f along p is a map $F\colon X \to E$ such that

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "F"] \arrow[r, "f"'] & B \end{tikzcd}$$

Lemma 3.10. Let $p\colon E \to B$ be a covering. Let

$$D = \{(x, x) \in E \times E \mid x \in E\}, Z = \{(x, y) \in E \times E \mid p(x) = p(y)\}.$$

Then $D \subset Z$ is both open and closed.

Proof. Take $(x, x) \in D$. We can take an open set $p(x) \in U$ such that there eixst open sets $x \in V, y \in W$ with V, W homeomorphic to U. Hence $(x, x) \in (V \times W) \cap Z \subset D$. On the other hand, it suffices to show that $Z \setminus D$ is open in Z. Take $(x_1, x_2) \in Z$ such that $p(x_1) \neq p(x_2)$. We may pick open sets $p(x_1) \in U_1, p(x_2) \in U_2$ such that there exist open sets $x_1 \in V_1, x_2 \in V_2$ homeomorphic to U_1, U_2 respectively. In particular, we may assume $V_1 \cap V_2 = \varnothing$. Thus $(x_1, x_2) \in (V_1 \times V_2) \cap Z \subset Z \setminus D$. $\square$

Theorem 3.11. Let $p\colon E \to B$ be a covering. Let $F_0, F_1\colon X \to E$ be two liftings of f. If X is connected and F_0, F_1 agree somewhere, then $F_0 = F_1$.

Proof. Let D, Z be sets defined in Lemma 3.10. Consider the map $\tilde{F} = (F_0, F_1)\colon X \to Z$. By assumption, we have $\tilde{F}(X) \cap D \neq \varnothing$. Moreover, $\tilde{F}^{-1}(D)$ is both open and closed, which implies $F^{-1}(D) = X$. $\square$

Definition 3.12. A map $p\colon E \to B$ is said to have the homotopy lifting property for X if for any maps $\tilde{f}\colon X \to E$ and $F\colon X \times I \to B$ such that $p \circ \tilde{f} = F|_{X \times \{0\}}$, there exists a lifting $\tilde{F}$ of F along p such that $\tilde{F}|_{X \times \{0\}} = \tilde{f}$. This can be illustrated through a diagram

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & B \end{tikzcd}$$

Definition 3.13. A map $p\colon E \to B$ is called a (Hurwitz) fibration if p has homotopy lifting property for any space.

Theorem 3.14. A covering is a fibration.

Proof. Set $p\colon E \to B, f\colon X \to B, \tilde{f}\colon X \to E, F\colon X \times I \to B$ as in Definition 3.12. We only need to show that for every $x \in X$, $\tilde{F}$ exists on some neighborhood of x. To see this, we can extend $\tilde{F}$ for every x. By the uniqueness of lifting, these extensions agree everywhere and glue to the required lifting $\tilde{F}$.

Since I is commpact metric space, given $x \in X$ we can find a neighborhood N_x and a partition

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

such that p admits a trivialization over U_i with $F(N_x \times [t_i, t_{i+1}]) \subset U_i$. Now extend F on $N_x \times [t_0, t_k]$ by induction on k.

- For $k=1$, we set $\tilde{F}_x = p^{-1}|_{U_1} \circ F$.
- Suppose we have extended F for some k. Now we can construct $\tilde{F} = p^{-1}|_{U_k} \circ F$ on $N_x \times [t_k, t_{k+1}]$. Then the new part glue with the given $\tilde{F}$. This finishes the inductive step.

□

Corollary 3.15. Let $p\colon E \to B$ be a covering. Then for any path $\gamma\colon I \to B$ and $e \in E$ such that $p(e) = \gamma(0)$, there exists a unique path $\tilde{\gamma}\colon I \to E$ which lifts γ and starts at e.

Proof. Applying homotopy lifting property to the case $X = \mathrm{pt}$. □

Corollary 3.16. If $p\colon E \to B$ is a covering, then $\Pi_1(E) \to \Pi_1(B), x \mapsto p(x), [\gamma] \mapsto [p\colon \gamma]$ is a faithful functor. In particular, the induced map $\pi_1(E,e) \to \pi_1(B,b)$ is injective.

Proof. Let $[\sigma_i] \in \mathrm{Hom}_{\Pi_1(E)}(e_1, e_2)$ and $\gamma_i = p \circ \sigma_i$. Suppose $[\gamma_1] = [\gamma_2]$ and we need to show that $[\sigma_1] = [\sigma_2]$.

Let $F\colon \gamma_1 \simeq \gamma_2$ be a homotopy. Apply homotopy lifting property to the diagram

$$\begin{tikzcd} I \times \{0\} \arrow[r, "\sigma_1"] \arrow[d] & E \arrow[d, "d"] \\ I \times I \arrow[r, "F"'] & B \end{tikzcd}$$

By the uniqueness of lifting, we have $\tilde{F}|_{I\times\{1\}} = \sigma_2$ and $\tilde{F}(0,s) = e_1, \tilde{F}(1,s) = e_2$. □

Let $p\colon E \to B$ be a covering. Let $\gamma\colon I \to B$ be a path in B from b_1 to b_2. It defines a map

$$\begin{aligned} T_\gamma\colon p^{-1}(b_1) &\longrightarrow p^{-1}(b_2) \\ e_1 &\longmapsto \tilde{\gamma}(1) \end{aligned}$$

where $\tilde{\gamma}$ is the unique lifting of γ starting at e_1. By the homotopy lifting property, we find a well-defined map

$$\begin{aligned} T\colon \mathrm{Hom}_{\Pi_1(B)}(b_1, b_2) &\longrightarrow \mathrm{Hom}_{\mathbf{Set}}(p^{-1}(b_1), p^{-1}(b_2)) \\ [\gamma] &\longmapsto T_{[\gamma]}. \end{aligned}$$

Proposition 3.17.

$$T\colon \Pi_1(B) \longrightarrow \mathbf{Set}$$

which assigns b to $p^{-1}(b)$ and $[\gamma]$ to $T_{[\gamma]}$ is a functor.

Proposition 3.18. Obviously T sends constant path $[1_b]$ to $e_1 \mapsto e_1$, so $T([1_b]) = \mathrm{id}_{p^{-1}(b)}$. Take $[\gamma_1] \in \mathrm{Hom}_{\Pi_1(B)}(b_1, b_2)$ and $[\gamma_2] \in \mathrm{Hom}_{\Pi_1(B)}(b_2, b_3)$. Then $T([\gamma_2 * \gamma_1])$ is a map sending $e_1 \in p^{-1}(b_1)$ to $\widetilde{\gamma_2 * \gamma_1}(1)$. By the uniqueness of lifting, $\widetilde{\gamma_2 * \gamma_1}(1) = T_{[\gamma_2]}(\tilde{\gamma}_1(1)) = T_{[\gamma_2]} \circ T_{[\gamma_1]}(e_1)$.

Definition 3.19. Functor $T\colon \Pi_1(B) \to \mathbf{Set}$ is called the transport functor. In particular, it induces a well-defined map

$$\pi_1(B,b) = \mathrm{Aut}_{\Pi_1(B)}(b) \longrightarrow \mathrm{Aut}_{\mathbf{Set}}(p^{-1}(b)).$$

Example 3.20. Consider the covering map

$$\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{S}^1.$$

We have a well-defined map $\pi_1(\mathbb{S}^1, 1) \to \mathrm{Aut}_{\mathbf{Set}}(\mathbb{Z})$. Consider $\gamma_n\colon I \to \mathbb{S}^1, t \mapsto e^{2\pi i n t}$. Then the unique lifting $\tilde{\gamma}_n$ starting at $m \in \mathbb{Z}$ is $\tilde{\gamma}_n(t) = m + nt$, so $T_{[\gamma_n]}(m) = m + n$.

Proposition 3.21. Let $p\colon E \to B$ be a covering with E path connected. Let $e \in E, b = p(e) \in B$. Then the action of $\pi_1(B,b)$ on $p^{-1}(b)$ is transitive, whose stabilizer at e is $\pi_1(E,e)$. In other words, we have the short exact sequence

$$1 \longrightarrow \pi_1(E,e) \xrightarrow{p_*} \pi_1(B,b) \xrightarrow{\partial_e} p^{-1}(b) \longrightarrow 1$$

$$[\gamma] \longmapsto T_{[\gamma]}(e)$$

Proof. For any point $e' \in p^{-1}(b)$, we can take a path class $[\tilde{\gamma}]$ connecting e, e' and set $\gamma = p \circ \tilde{\gamma}$. Then $e' = \partial_e([\gamma])$. This shows that ∂_e is surjective.

We already see that p_* is injective, hence we can view $\pi_1(E, e)$ as a subgroup of $\pi_1(B, b)$. By definition, for $[\sigma] \in \pi_1(E, e)$, we have $\partial_e([p \circ \sigma]) = \sigma(1) = e$, so $\pi_1(E, e) \subset \operatorname{Ker} \partial_e = \operatorname{Stab}_e(\pi_1(B, b))$. On the other hand, if $T_{[\gamma]}(e) = e$, then its lifting $\tilde{\gamma}$ is a loop. Therefore $[\gamma] = p_*[\tilde{\gamma}]$. □

Corollary 3.22. $p^{-1}(b) \simeq \pi_1(B, b)/\pi_1(E, e)$ as a coset space.

Theorem 3.23 (lifting criterion)**.** Let $p\colon (E, e_0) \to (B, b_0)$ be a covering. Suppose $f\colon (X, x_0) \to (B, b_0)$ is continuous with X path connected and locally path connected. Then there exists a lifting F such that

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (X, x_0) \arrow[ur, dashed, "F"] \arrow[r, "f"'] & (B, b_0) \end{tikzcd}$$

if and only if $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$.

Proof. If such F exists, then $f_*(\pi_1(X, x_0)) = (p_* \circ F_*)(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$.

Conversely, let

$$\tilde{E} = \{(x, e) \in X \times E \mid f(x) = p(e)\} \subset X \times E$$

and take a base point $\tilde{e}_0 = (x_0, e_0) \in \tilde{E}$. Since we have the following diagram

$$\begin{tikzcd} (\tilde{E}, \tilde{e}_0) \arrow[r, "\tilde{q}"] \arrow[d, "\tilde{p}"] & (E, e_0) \arrow[d, "p"] \\ (X, x_0) \arrow[r, "f"'] & (B, b_0) \end{tikzcd}$$

where $\tilde{p}, \tilde{q}$ are component projections. Then $\tilde{p}$ is also a covering: Indeed, if p is trivialized over $\{U_i\}$ with fiber F, then $\tilde{p}$ is trivialized over $\left\{f^{-1}(U_i)\right\}$ with fiber F.

Now we show that the condition $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(E, e_0))$ implies that $\pi_1(X, x_0)$ stabilizes $\tilde{e}_0$. To see this, we can take $[\gamma] \in \pi_1(X, x_0)$ and lift it to $[\tilde{\gamma}]$ in $\tilde{E}$ with $\tilde{\gamma}(0) = \tilde{e}_0, \tilde{\gamma}(1) = (x_1, e_1)$. Then $f_*[\gamma] = p_*[\sigma]$ for some $\sigma \in \pi_1(B, b_0)$. As $(p_* \circ \tilde{q}_*)[\tilde{\gamma}] = (f_* \circ \tilde{p}_*)(\tilde{\gamma}) = p_*[\sigma]$, we see that $\tilde{q}_*[\tilde{\gamma}] = [\sigma]$. In particular, $e_1 = e_0$, which implies $T_{[\gamma]}(\tilde{e}_0) = \tilde{e}_0$. By Proposition 3.21, what we actually get is an exact sequence

$$1 \longrightarrow \pi_1(\tilde{E}, \tilde{e}_0) \xrightarrow{\tilde{p}_*} \pi_1(X, x_0) \xrightarrow{\partial_e} 1$$

Hence $\tilde{p}_*\colon \pi_1(\tilde{E}, \tilde{e}_0) \simeq \pi_1(X, x_0)$. As X is locally path connected, so is $\tilde{E}$. Then path components and connected components of $\tilde{E}$ coincide. Let C be the path component of $\tilde{E}$ containing $\tilde{e}_0$. Since $\pi_1(\tilde{E}, \tilde{e}_1) \simeq \pi_1(X, x_0)$, $\tilde{p}\colon C \to X$ is a covering with single point, hence a homeomorphism. The inverse of $p|_C$ defines a continuous map $X \to \tilde{E}$ whose composition with $\tilde{q}$ gives F. □

Definition 3.24. Let G be a discrete group, i.e., a group endowed with the discrete topology. A continuous action $G \times X \to X$ is said to be properly discontinuous if for any $x \in X$, there exists an open neighborhood U of x such that $g(U) \cap U = \varnothing$ if $g \neq 1$.

We define the orbit space $X/G = X/\sim$ where $x \sim g(x), g \in G$.

Proposition 3.25. If G acts properly discontinuously on X, then the quotient map $X \to X/G$ is a covering.

Proof. For any $x \in X$, let U be an open neighborhood satisfying $g(U) \cap U, \forall g \neq 1$. Then

$$p^{-1}(p(U)) = \bigcup_{g \in G} gU.$$

□

Definition 3.26. A G-principal covering is a covering $p\colon E \to B$ with a properly discontinuous G-action on E over B, i.e. $p(g \cdot x) = g \cdot p(x)$, such that the induced map $E/G \to B$ is a homeomorphism.

Example 3.27. $\mathbb{R} \to \mathbb{S}^1, t \mapsto e^{2\pi i t}$ is a $\mathbb{Z}$-principal covering for the action $(\mathbb{Z}, +) \subset \mathbb{R}$.

Example 3.28. $\mathbb{S}^n \to \mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2$ is a $\mathbb{Z}_2$-principal covering.

Proposition 3.29. Let $p\colon E \to B$ be a G-principal covering. Then the transport is G-equivalent, i.e.,

$$T_{[\gamma]}(g \cdot e) = g \cdot T_{[\gamma]}(e).$$

Proof. Let γ be a path from b_0 to b_1 and $e_0 \in p^{-1}(b_0)$. Then $\tilde{\gamma}$ be the lifting of γ starting at e_0. Then $e_1 = \tilde{\gamma}(1) = T_{[\gamma]}(e_0)$. If we apply the transformation g to the path $\tilde{\gamma}$, we find another lifting of γ starting at $g \cdot e_0$. Therefore

$$T_{[\gamma]}(g \cdot e_0) = g \cdot e_1 = g \cdot T_{[\gamma]}(e_1).$$

□

Theorem 3.30. Let $p\colon (E, e) \to (B, b)$ be a G-principal covering with E path connected. Then we have an exact sequence

$$1 \longrightarrow \pi_1(E, e) \xrightarrow{p_*} \pi_1(B, b) \xrightarrow{\varphi} G \longrightarrow 1.$$

In other words, $\pi_1(E, e)$ is a normal subgroup of $\pi_1(B, b)$ with $G = \pi_1(B, b)/\pi_1(E, e)$.

Proof. Take $F = p^{-1}(b)$. Since transport is G-equivalent, $\pi_1(B, b)$-action and G-action on F commute. It induces a $\pi_1(B, b) \times G$-action on F. Consider the two component projections

$$\begin{tikzcd} & \mathrm{Stab}_e(\pi_1(B, b) \times G) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \pi_1(B, b) & & G \end{tikzcd}$$

Since $E/G \to B$ is a homeomorphism, for every $[\gamma] \in \pi_1(B, b)$, we can find $g \in G$ such that $g \cdot T_{[\gamma]}(e) = e$. Therefore p_1 is an isomorphism. On the other hand, $\mathrm{Ker}\, p_2 = \mathrm{Stab}_e(\pi_1(B, b)) = \pi_1(E, e)$, so we can take $\varphi = p_2 \circ p_1^{-1}$. □

Example 3.31. Apply Theorem 3.30 to the covering $\mathbb{R} \to \mathbb{S}^1, t \mapsto e^{2\pi i t}$, we find a group isomorphism

$$\deg\colon \pi_1(\mathbb{S}^1) \simeq \mathbb{Z},$$

which is called the degree map. We see that path homotopy class of $1 \in \mathbb{S}^1$ is classified by degree, whose representatives are $t \mapsto e^{2\pi i n t}$.

Note that any loop γ is equivalent to a map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$. In this seeting, $\deg f = F(x + 1) - F(x)$ for any lifting F of f.

Definition 3.32. Let $i\colon A \subset X$ be an inclusion. A continuous map $r\colon X \to A$ is called a retraction if $r \circ i = 1_A$. Furthermore, i is called a deformation retraction if we have a homotopy $i \circ r \simeq 1_X$ rel A. We say A is a deformation retract of X if such i exists.

Proposition 3.33. If $i\colon A \to X$ is a deformation retract, then $r_*\colon \pi_1(A) \to \pi_1(X)$ is injective.

Proof. $(i \circ r)_*\colon \pi_1(A) \to \pi_1(A)$ is identity map because of the homotopy. □

Corollary 3.34. Let D^2 be the unit disk in $\mathbb{R}^2$, then $\mathbb{S}^1 = \partial D^2$ is not a deformation retract of D^2.

Proof. Since D^2 is contractible, $\pi_1(D^2) = 1$. □

Using the above fact, we can prove a toy version of the famous Brouwer fixed point theorem.

Theorem 3.35. Any continuous map $f\colon D^2 \to D^2$ admits a fixed point.

Proof. By contradiction, suppose f has no fixed point. Let I_x be the ray starting from from $f(x)$ pointing toward x. Then $D^2 \to \mathbb{S}^1, x \mapsto I_x \cap \mathbb{S}^1$ is a deformatino retraction to $\mathbb{S}^1$. □

Theorem 3.36 (fundamental theorem of algebra)**.** Let $f(z) = z^n + a_{n_1} z^{n-1} + \cdots + z_0 \in \mathbb{C}[z]$ with $n > 0$. Then f admits a root.

Proof. Assume that f has no root in $\mathbb{C}$. Define a homotopy

$$\begin{aligned} F\colon \mathbb{S}^1 \times I &\longrightarrow \mathbb{S}^1 \\ (z,t) &\longmapsto \frac{f(\tan(\frac{\pi}{2}t)z)}{\left|f(\tan(\frac{\pi}{2}t)z)\right|}. \end{aligned}$$

On one hand, $\deg(F(-,0)) = 0$. On the other hand, $\deg(F(-,1)) = \deg(z \mapsto z^n) = n$. $\square$

Proposition 3.37. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map such that $f(-x) = -f(x)$. Then $\deg f$ is odd. In particular, f is not null homotopic.

Proof. Let $\sigma\colon \mathbb{S}^1 \to \mathbb{S}^1, z \mapsto -z$. Then $\deg \sigma = -1$. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a lifting of f. Then $F(x+1) - F(x) = \deg f$ and $F(x+\frac{1}{2}) - F(x) = m + \frac{1}{2}$ for some $m \in \mathbb{Z}$. Hence $\deg f = 2m+1$ is odd. $\square$

The following thoerem is a toy version of Borsuk–Ulam theorem.

Theorem 3.38. Any continuous map $f\colon \mathbb{S}^2 \to \mathbb{R}^2$ admits a point $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$.

Proof. Assume $f(x) \neq f(-x)$. Define

$$\begin{aligned} \rho\colon \mathbb{S}^2 &\longrightarrow \mathbb{S}^1 \\ x &\longmapsto \frac{f(x) - f(-x)}{|f(x) - f(-x)|}. \end{aligned}$$

Let D^2 be the upper hemisphere. It defines a homotopy between id_{D^2} and $\rho|_{\partial D^2}\colon \mathbb{S}^1 \to \mathbb{S}^1$, so $\deg \rho|_{\partial D^2} = 0$. However, $\rho|_{\partial D^2}$ is antipode and thus has odd degree. Contradiction. $\square$

Corollary 3.39 (ham sandwich theorem)**.** Let A_1, A_2 be two bounded measurable regions in $\mathbb{R}^2$. Then there exists a line which cuts each A_i into half of equal regions.

Proof. We may assume A_1, A_2 both have positive area. We identify $\mathbb{R}^2$ with $z = 1$ in $\mathbb{R}^3$. Given $u \in \mathbb{S}^2$, we let P_u be the orthogonal complement of u. Let $A_i(u) = \{p \in A_i \mid p \cdot u \leqslant 0\}$. Define

$$\begin{aligned} f\colon \mathbb{S}^2 &\longrightarrow \mathbb{R}^2 \\ u &\longmapsto (\mathrm{Area}(A_1(u)), \mathrm{Area}(A_2(u))). \end{aligned}$$

By the Borsuk–Ulam theorem, $f(x) = f(-x)$ for some $x \in \mathbb{S}^2$. The intersection $(\mathbb{R}^2 \times \{1\}) \cap P_u$ gives the required line. $\square$

4. Tools from homological algebra

Let $\mathcal{I}$ be a small category and $\mathcal{C}$ be a category. Recall that we have a functor $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$, which is also written as $\mathcal{C}^{\mathcal{I}}$.

Definition 4.1. We define the diagonal functor

$$\Delta\colon \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{I}, \mathcal{C}),$$

which assigns $X \in \mathrm{Ob}(\mathcal{C})$ to the functor $\Delta(X)\colon \mathcal{I} \to \mathcal{C}$ that sends all objects to X and all morphisms to 1_X.

Let $\mathcal{I}$ be a diagram consisting of vertices and arrows. We can define a category, still denoted by $\mathcal{I}$:

- Objects are vertices in $\mathcal{I}$.
- Morphisms are composites of all arrows. The identities are additional identity arrows.

Example 4.2. The following diagram

$$\begin{tikzcd} \bullet \arrow[r] & \circ \arrow[d] \\ & * \end{tikzcd}$$

defines a category with three objects $\bullet, \circ, *$. We have $\mathrm{Hom}(\bullet, \circ) = \{\bullet \to \circ\}, \mathrm{Hom}(\circ, *) = \{\circ \to *\}$ and $\mathrm{Hom}(\bullet, *) = \{\bullet \to \circ \to *\}$.

Example 4.3. The following diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \circ \arrow[d] \\ & * \end{tikzcd}$$

defines a category with three objects $\bullet, \circ, *$. We have $\mathrm{Hom}(\bullet, \circ) = \{\bullet \to \circ\}, \mathrm{Hom}(\circ, *) = \{\circ \to *\}$ and $\mathrm{Hom}(\bullet, *) = \{\bullet \to \circ \to *, \bullet \to *\}$.

For a diagram $\mathcal{I}$, a functor $F \colon \mathcal{I} \to \mathcal{C}$ assigns vertices and arrows to the corresponding objects and morphisms in $\mathcal{C}$.

We make formalism above for one simple purpose: To capture types of commutative diagram.

Definition 4.4. A limit for a functor $F \colon \mathcal{I} \to \mathcal{C}$ is an object P in $\mathcal{C}$ with a natural transformation

$$\tau \colon \Delta(P) \longrightarrow F$$

such that for every object Q of $\mathcal{C}$ and every natural transformation $\eta \colon \Delta(Q) \to F$, there exists a unique morphism $f \colon Q \to P$ such that $\tau \circ \Delta(f) = \eta$, i.e.

$$\begin{tikzcd} \Delta(Q) \arrow[r, dashed, "\exists! \Delta(f)"] \arrow[dr, "\eta"'] & \Delta(P) \arrow[d, "\tau"] \\ & F \end{tikzcd}$$

This defintion may be a abstract nonsense by the first glance, and some explanations are added here. For a morphism $\circ \to \bullet$, a functor F assigns it to a morphism $\varphi \colon X \to Y$, while $\Delta(P)$ assigns it to $\mathrm{id}_P \colon P = P$. Then the natural transformation gives

$$\begin{tikzcd} P \arrow[r, equal, "\mathrm{id}_P"] \arrow[d] & P \arrow[d] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

Hence a natural transformation $\Delta(P) \to F$ amounts to given a family of morphisms $P \to F(A)$ commutative with morphisms assgined by F. If furthermore, there exists another object Q admits such property, then we can find a unique morphism $Q \to P$ commutative with morphisms above. Also note that, those familiar with algebra may realize that this setting characterizes the so called universal property.

Let's see an example. Consider the following diagram in $\mathcal{C}$ which represents a functor $F \colon \mathcal{I} \to \mathcal{C}$

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Then its limit is an object $P \in \mathcal{C}$ that lifts into the commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Moreover for any other object Q fitting into the same commutative diagram, there exists a unique $f \colon Q \to P$ to making the following diagram commutative

$$\begin{tikzcd} Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & P \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & Z \end{tikzcd}$$

Proposition 4.5. Let $F\colon \mathcal{I} \to \mathcal{C}$ be a functor, and P_1, P_2 be two limits of F. Then there exists a unique isomorphism $P_1 \to P_2$ which makes the following diagram commute

$$
\begin{tikzcd}
\Delta(P_1) \arrow[rr] \arrow[dr, "\tau_1"] & & \Delta(P_2) \arrow[dl, "\tau_2"] \\
& F &
\end{tikzcd}
$$

where τ_1, τ_2 are natural transformtaions.

Definition 4.6. We denote the limits of $F\colon \mathcal{I} \to \mathcal{C}$ by $\lim F$ if it exists.

Example 4.7 (Pullback)**.** The limit of the diagram $X \to Y \leftarrow Z$ gives

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & Z
\end{tikzcd}
$$

which is called the pullback. The commutative diagram

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}
$$

gives its universal property.

In **Set**, the pullback exists and is given by

$$P = \{(x, y) \in X \times Y \mid f(x) = g(y)\}\,.$$

Example 4.8 (Inverse limit)**.** Given a ordered set $(I, \leqslant)$, we consider the following diagram, still denoted by I.

- Vertices are I.
- Given $i, j \in I$, $\mathrm{Hom}_I(i, j)$ has a arrow if $i \leqslant j$ and empty otherwise.

Let I^{op} be the opposite of I. The limit of a functor $F\colon I^{\mathrm{op}} \to \mathcal{C}$ is represented by the tower diagram

$$
\begin{tikzcd}
& Q \arrow[d, dashed, "\exists!"] \arrow[ddl] \arrow[ddr] & \\
& \lim X_i \arrow[dl] \arrow[dr] & \\
X_i \arrow[rr] & & X_j
\end{tikzcd}
$$

The limit of F is also called the inverse limit and written as $\varprojlim X_i$.

The notion of colimit is dual to limit.

Definition 4.9. A colimit for a functor $F\colon \mathcal{I} \to \mathcal{C}$ is an object P in $\mathcal{C}$ with a natural transformation

$$\tau\colon F \longrightarrow \Delta(P)$$

such that for every object Q of $\mathcal{C}$ and every natural transformation $\eta\colon F \to \Delta(Q)$, there exists a unique morphism $f\colon P \to Q$ such that $\Delta(f) \circ \tau = \eta$, i.e.

$$
\begin{tikzcd}
F \arrow[r, "\tau"] \arrow[dr, "\eta"'] & \Delta(P) \arrow[d, dashed, "\exists!\Delta(f)"] \\
& \Delta(P)
\end{tikzcd}
$$

The colimit, if exists, is denoted by $\operatorname{colim} F$.

Example 4.10 (Pushout)**.** The colimit of the diagram $X \leftarrow Y \rightarrow Z$ gives

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

This colimit is called the pushout. It has universal property

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \arrow[ddr, bend left] & \\ X \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists!"] & \\ & & Q \end{tikzcd}$$

Example 4.11 (Direct limit)**.** For a ordered set $(I, \leqslant)$, the colimit of a functor $F\colon I \to \mathcal{C}$ can be represented by the telescope diagram

$$\begin{tikzcd} X_i \arrow[rr] \arrow[dr] \arrow[ddr] & & X_j \arrow[dl] \arrow[ddl] \\ & \operatorname{colim} F \arrow[d, dashed, "\exists!"] & \\ & Q & \end{tikzcd}$$

The colimit of F is also called the direct limit of the telescope and denoted by $\varinjlim X_i$.

Theorem 4.12. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be adjoint functors. Assume the limit of $F\colon \mathcal{I} \to \mathcal{D}$ exists. Then the limit of $R \circ F\colon \mathcal{I} \to \mathcal{C}$ is given by

$$\lim(R \circ F) = R(\lim F).$$

Proof. Let $X \in \mathcal{C}$. Assume that we have a natural transformation $\tau\colon \Delta(X) \to R \circ F$. By adjunction, this is equivalent to a natural transformation $\Delta(L(X)) \to F$.

By the universal property, there exists a unique map $L(X) \to \lim F$ factorizing $\Delta(L(X)) \to F$. By adjunction again, this is equivalent to natural transformations $\Delta(X) \to R(\lim F) \to R \circ F$. This implies that $R(\lim F) = \lim(R \circ F)$. □

Similarly, we can show that

Theorem 4.13. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be adjoint functors. Assume the colimit of $F\colon \mathcal{I} \to \mathcal{C}$ exists. Then the limit of $L \circ F\colon \mathcal{I} \to \mathcal{D}$ is given by

$$\operatorname{colim}(L \circ F) = L(\operatorname{colim} F).$$

Corollary 4.14. The forgetful functor $\mathrm{Forget}\colon \mathbf{Top} \to \mathbf{Set}$ preserrves limit and colimit.

Proof. Forget functor has left adjoint $\mathrm{Discrete}\colon \mathbf{Set} \to \mathbf{Top}$ and right adjoint $\mathrm{Trivial}\colon \mathbf{Set} \to \mathbf{Top}$. □

We now investigate some examples in topology.

Example 4.15. Consider the following diagram in **Top**

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

We would like to understand the pullback P in **Top**. By Example 4.7 and Corollary 4.13, we know that the underlying set has to be $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$. If we endow this set with the subspace topology of the product space $X \times Y$, then P is indeed the pullback.

Pullbcak is important in topology because fibrations behave well under this operation.

Proposition 4.16. Let $p\colon E \to B$ be a fibration and $\varphi\colon P \to E, \psi\colon Q \to B$ be continuous map. Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "q"] & E \arrow[d, "p"] \\ Q \arrow[r, "\psi"] & B \end{tikzcd}$$

Then $q\colon P \to Q$ is also a fibration. In other words, the pullback of a fibration is also a fibration.

Proof. Suppowe we have continuous maps $\tilde{g}\colon X \to P$ and $G\colon X \times I \to B$ such that $q \circ \tilde{g} = F|_{X\times\{0\}}$. Then by the homotopy lifting property of $p\colon E \to B$, we can find $F\colon X \times I$ making the following diagram commutative

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\varphi\circ\tilde{g}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "\psi\circ G"] \arrow[ur, dashed, "F"] & B \end{tikzcd}$$

By the universal property of pullback, there is map $\tilde{G}\colon X \times I \to P$ such that $q \circ \tilde{G} = G$. $\square$

Example 4.17. Let $j_1\colon X_0 \to X_1, j_2\colon X_0 \to X_2$ in **Top**. Their pushout is the quotient of the disjoint union $X_1 \coprod X_2$ by identifying $j_1(y) \sim j_2(y), y \in X_0$. It glues X_1, X_2 along X_0 using j_1, j_2. For instance, $j_1, j_2\colon \partial D^2 \to D^2$ has pushout $\mathbb{S}^2$.

Actually, in **Grp**, we can also construct pushout. If $\rho_1\colon H \to G_1, \rho_2\colon H \to G_2$ are in **Grp**, then their pushout is

$$(G_1 * G_2)/N$$

where $G_1 * G_2$ is the free product and N is the normal subgroup generated by $\rho_1(h)\rho_2(h)^{-1}, h \in H$. The two constructions of pushout will be related in Seifert-Van Kampen theorem.

Definition 4.18. Let I be a set. I can be taken as a discrete category, i.e., morphisms are only identities. Let $F\colon I \to \mathcal{C}$ be a functor, with $F(i)$ denoted by A_i for $i \in I$. Then the product of $\{A_i\}$ is defined to be $\lim F$, if limit exists. We usually denote it by

$$\prod_{i\in I} A_i.$$

For a family $\{A_i\}$ of objects in $\mathcal{C}$, their product is an object $A \in \mathcal{C}$ together with $\pi_i\colon A \to A_i$ satisfying the following universal property: For any $X \in \mathcal{C}$ with $f_i\colon X \to A_i$, there exists a unique morphism $f\colon X \to A$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_i"'] & A \arrow[d, "\pi_i"] \\ & A_i \end{tikzcd}$$

Example 4.19. Product can be constructed explicitly in some common categories.
1. $\{S_i\}$ in **Set** has product as Cartesian product $\prod\limits_i S_i$.
2. $\{X_i\}$ in **Top** has product as Cartesian product with product topology $\prod\limits_i X_i$.
3. $\{G_i\}$ in **Grp** has product as Cartesian product with product group structure $\prod\limits_i G_i$.

Definition 4.20. Let I be a set. I can be taken as a discrete category, i.e., morphisms are only identities. Let $F\colon I \to \mathcal{C}$ be a functor, with $F(i)$ denoted by A_i for $i \in I$. Then the product of $\{A_i\}$ is defined to be $\operatorname{colim} F$, if colimit exists. We usually denote it by

$$\coprod_{i\in I} A_i.$$

For a family $\{A_i\}$ of objects in $\mathcal{C}$, their product is an object $A \in \mathcal{C}$ together with $j_i\colon A_i \to A$ satisfying the following universal property: For any $X \in \mathcal{C}$ with $f_i\colon A_i \to X$, there exists a unique morphism $f\colon A \to X$ such that the following diagram commutes

$$\begin{tikzcd} X & A \arrow[l, dashed, "\exists! f"'] \\ & A_i \arrow[ul, "f_i"] \arrow[u, "j_i"'] \end{tikzcd}$$

Example 4.21. Coroduct can be also constructed explicitly in some common categories.
1. $\{S_i\}$ in **Set** has product as disjoint union $\bigsqcup_i S_i$.
2. $\{X_i\}$ in **Top** has product as disjoint union $\bigsqcup_i X_i$.
3. $\{G_i\}$ in **Grp** has product as free product $\coprod_i G_i$. Namely, we first take the free group generated by G_i and then quotient the relation in G_i.

Now we consider limits and colimits in $\mathbf{Top}_\bullet$.

Given a space X, we can define a pointed space $X_+ = (X \coprod *, *)$ by adding an extra point $*$. This defines a functor

$$(-)_*\colon \mathbf{Top} \longrightarrow \mathbf{Top}_\bullet.$$

On the other hand, have a forgetful functor Forget: $\mathbf{Top}_\bullet \to \mathbf{Top}$ by forgetting the base point. They form an adjoint pair. This implies that the limit in $\mathbf{Top}_\bullet$ will be the same as the limit in $\mathbf{Top}$. In particular, the product of pointed spaces $\{(X_i, x_i)\}$ is the topological product with the base point (x_i).

In $\mathbf{Top}_\bullet$, the coproduct of two pointed space X, Y is the wedge product

$$X \vee Y = X \coprod Y/\sim,$$

where we identify the base points $x_0 \in X, y_0 \in Y$ and the identified based point is the new based point of $X \vee Y$. In general, we have

$$\bigvee_i X_i = \coprod_i X_i/\sim,$$

where $\sim$ again identifies all based points in X_i.

Example 4.22. The figure-8 can be identified with $\mathbb{S}^1 \vee \mathbb{S}^1$.

In $\mathbf{Top}_\bullet$, there is another operation, called the smash product $\wedge$. Specifically,

$$X \wedge Y = X \times Y/\sim$$

where we identify $(x, y_0) \sim (x_0, y)$ for all $x \in X, y \in Y$. The identified point is the new based point of $X \wedge Y$. Note that we can write

$$X \wedge Y = X \times Y/X \vee Y.$$

Example 4.23. There is a natrua homeomorphism $\mathbb{S}^1 \wedge \mathbb{S}^n \cong \mathbb{S}^{n+1}$. This implie that $\mathbb{S}^n \wedge \mathbb{S}^m \cong \mathbb{S}^{n+m}$. This process is a liitle bit hard to visualize. However, one can take $\mathbb{S}^1 = I/\partial I$ and then the quotient space is $I^{n+1}/\partial I^{n+1}$.

Definition 4.24. A category $\mathcal{C}$ is said to be complete if for any $F \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ with $\mathcal{I}$ small, the limit $\lim F$ exists.

$\mathcal{C}$ is said to be cocomplete if we change $\lim F$ into $\operatorname{colim} F$.

Example 4.25. $\mathbf{Set}, \mathbf{Grp}, \mathbf{Top}$ are all complete and cocomplete.

For example, in **Set**, the limit of $F\colon I \to \mathbf{Set}$ is given by

$$\lim F = \left\{ (x_i) \in \prod_{i \in I} F(i) \mid x_j = F(f)(x_i) \text{ for any } i \xrightarrow{f} j \right\}.$$

The colimit is given by

$$\operatorname{colim} F = \coprod_{i\in I} F(i) \Big/ \left\{ x_i \sim F(f)(x_i) \text{ for any } i \xrightarrow{f} j, x_i \in F(i) \right\}.$$

For another example, we consider **Top**. Since the forgetful functor **Top** $\to$ **Set** has both a left adjoint and a right adjoint, it preserves limits and colimits. Hence $\lim F$ has the same underlying set equipped with the induced topology from product and subspace. Similarly, $\operatorname{colim} F$ is equipped with the induced topology from disjoint unions and quotient.

Definition 4.26. An initial object of a category $\mathcal{C}$ is an object $*$ such that for every $X \in \mathcal{C}$, there exists exactly one morphism $* \to X$. Dually, a terminal object $*$ satisfies that for every $X \in \mathcal{C}$, there exists exactly one morphism $X \to *$. If an object is both initial and terminal, it is called the zero object.

Note that initial object and the terminal object are unique up to isomorphism if they exist.

Example 4.27. $\varnothing$ is the initial object, and the set with a single point is the terminal object in **Set**. The same is true for **Top**.

5. Compactly generated weak Hausdorff space

In algebraic topology, it would be convenient to work with a category in which we can take limits, colimits freely and enjoys nice properties. Unfortunately, **Top** does not fit this requirement. Even the subcategory of locally compact Hausdorff spaces, which is thought to be very nice in general topology, does not preserve limits and colimits in general. In order to ensure good properties and meanwhile include as many spaces as possible, we usually consider a convenient subcategory, whose spaces are compactly generated and weak Hausdorff.

Definition 5.1. A subset $Y \subset X$ is said to be compactly closed (or k-closed) if for every continuous $f\colon K \to X$ with K compact Hausdorff, $f^{-1}(Y)$ is closed in K. We define a new topology on X, denoted by kX, where close subsets subsets of kX are compactly closed subsets of X. The identity $kX \to X$ is continuous. X is said to be compactly generated if $kX = X$.

We usually denote by **CG** the full subcategory consisting of compactly generated spaces.

If a space X is compactly generated, then for any Y, $f\colon X \to Y$ is continuous if and only if the composition $K \to X \to Y$ is continuous for any continuous $K \to X$ with K compact Hausdorff. Also note that $k^2X = kX$.

Proposition 5.2. A locally compact Hausdorff space is compactly generated.

Proof. Let X be locally compact Hausdorff and Z be a k-closed subset. We need to show $Z \subset X$ is closed. Let $x \in \bar{Z}$ and then x admits a neighborhood U with $K = \bar{U}$ compact Hausdorff. Then $x \in \overline{K \cap Z}$. Since Z is k-closed, $K \cap Z$ is closed in K. Therefore Z is closed in X and $x \in Z$. □

Proposition 5.3. $k(-)\colon$ **Top** $\to$ **CG**, which is right adjoint to the embedding functor $i\colon$ **CG** $\subset$ **Top**. In other words, we have an ajoint pair $i\colon$ **CG** $\rightleftarrows$ **Top**$\colon k(-)$.

Proof. Let $X \in$ **CG**, $Y \in$ **Top**, we need to show that $f\colon X \to Y$ is continuous if and only if the same map $f\colon X \to kY$ is continuous. Assume $f\colon X \to kY$ is continuous. Then the composition $X \to kY \to Y$ is continuous. Convsersly, assume $f\colon X \to Y$ is continuous. Let $Z \subset Y$ be a k-closed subset. Then for any $g\colon K \to X$ with K compact Hausdorff,

$$g^{-1}(f^{-1}(Z)) = (f \circ g)^{-1}(Z)$$

is closed in K. It follows that $f^{-1}(Z)$ is closed in X, so $f\colon X \to kY$ is continuous. □

Proposition 5.4. Let $X \in$ **CG** and $p\colon X \to Y$ be a quotient map. Then $Y \in$ **CG**.

Proof. p factors through $X \to kY$. Since the quotient topology is the finest topology making the quotient map continuous, we find $Y = kY$. □

Theorem 5.5. **CG** is complete and cocomplete. Colimits inherit the colimits in **Top** while the limits are obtained by applying k to the limits in **Top**.

Proof. Let $F \in \mathrm{Fun}(I, \mathbf{CG})$ and $\tilde{F} = i \circ F \in \mathrm{Fun}(I, \mathbf{Top})$.

The left adjoint functor i preserves colimits. Notice that coproduct $\coprod_i F(i)$ given by the disjoint union is in $\mathbf{CG}$. Since $\operatorname{colim} \tilde{F}$ is a quotient of $\coprod_i F(i)$, it also lies in $\mathbf{CG}$. This implies that colimits in $\mathbf{CG}$ are the same as colimits in $\mathbf{Top}$.

The right adjoint functor k preserves limits. Therefore

$$\lim F = \lim(k \circ \tilde{F}) = k \lim \tilde{F}.$$

□

Corollary 5.6. Let $\{X_i\}$ be a family of objects in $\mathbf{CG}$. Then their product in $\mathbf{CG}$ is

$$k \prod_{i \in I} X_i,$$

where the product is the topological product.

To avoid confusion in notations, we can use $\times, \prod$ to denote the product in $\mathbf{CG}$ and $\times^t, \prod^t$ to denote the product in $\mathbf{Top}$.

Proposition 5.7. If X is compactly generated and Y is locally compact Hausdorff, then we have $X \times Y = X \times^t Y$.

As a consequence, we do not have to distinguish topological product and category product for $X \times I$.

Definition 5.8. Let $X, Y \in \mathbf{CG}$. We define the compactly generated topology on $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ by

$$\mathbf{Map}(X, Y) = kC(X, Y) \in \mathbf{CG}.$$

Here $C(X, Y)$ is the compact-open topology generated by $\{f \in \mathrm{Hom}_{\mathbf{Top}}(X, Y) \mid f(g(K)) \subset U\}$, where $g \colon K \to X$ with K compact Hausdorff and $U \subset Y$ open.

Note that the compact-open topology we use here is slightly different from the usual one: we ask for a map from K which is compact Hausdorff. We also use the exponential notation

$$Y^X = \mathrm{Map}(X, Y).$$

Lemma 5.9. Let $X, Y \in \mathbf{CG}$ with K compact Hausdorff, $f \colon K \to X$ continuous. Then the evaluation map

$$\begin{aligned} ev_K \colon \mathrm{Map}(X, Y) \times^t K &\longrightarrow Y \\ (g, k) &\longmapsto g(f(k)) \end{aligned}$$

is continuous. In particular, $ev_K \colon \mathrm{Map}(X, Y) \times K \to Y$ is continuous.

Proof. Let $U \subset Y$ be open, and $(g, k) \in ev_K^{-1}(U)$. Then $(g \circ f)^{-1}(U)$ is open in K containing k. Since K is compact Hausdorff, k has a neighborhood V such that $\bar{V} \subset (g \circ f)^{-1}(U)$. Then

$$\left\{h \mid h(f(\bar{V})) \subset U\right\} \times V$$

is an open neighborhood of (g, k) in $ev_K^{-1}(U)$. □

Proposition 5.10. Let $X, Y \in \mathbf{CG}$. Then the evaluation map $\mathrm{Map}(X, Y) \times X \to Y$ is continuous.

Proof. It suffices to show that $K \to \mathrm{Map}(X, Y) \times X \to Y$ is continuous for every K compact and Haudorff. But this is the same as the composition

$$K \longrightarrow \mathrm{Map}(X, Y) \times K \longrightarrow Y$$

which is continuous by the previous lemma. □

Proposition 5.11. Let $X, Y, Z \in \mathbf{CG}$ and $f \colon X \times Y \to Z$ be continuous. Then the induced map

$$\begin{aligned} \tilde{f} \colon X &\longrightarrow \mathrm{Map}(Y, Z) \\ x &\longmapsto f(x, -) \end{aligned}$$

is also continuous.

Proof. Let $h\colon K \to Y$ be a continuous map with K compact Hausdorff. Let $U \subset Z$ open and let

$$W = \{g\colon Y \to Z \mid g(h(K)) \subset U\}.$$

For every $x \in \tilde{f}^{-1}(W)$, since f is continuous, there exists an open neighborhood V of x such that $f(V, h(K)) \subset U$. Then $V \subset f^{-1}(W)$ as required. □

Theorem 5.12 (exponential law)**.** Let $X, Y, Z \in \mathbf{CG}$. Then the natural map

$$\begin{aligned} \mathrm{Map}(X \times Y, Z) &\longrightarrow \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \\ f &\longmapsto (x \mapsto f(x, -)) \end{aligned}$$

is a homeomorphism.

Proposition 5.13. Let $X, Y, Z \in \mathbf{CG}$. Then the composition $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is continuous, i.e., a morphism in $\mathbf{CG}$.

Proposition 5.14. Let $p_i\colon X_i \times Y_i, i = 1, 2$ be quotients in $\mathbf{CG}$. Then $p_1 \times p_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is a quotient.

Definition 5.15. A space X is said to be weak Hausdorff if for every compact Hausdorff K and continuous $f\colon K \to X$, the image $f(K)$ is closed in X.

Let **WH** denote the full subcategory of **Top** consisting of weak Hausdorff spaces. Let **CGWH** denote the full subcategory of **Top** consisting of compactly generated weak Hausdorff spaces.

To symplify notations, we will wirte $\mathscr{T} = \mathbf{CGWH}$ and denote the quotient category of $\mathscr{T}$ under homotopy by $\mathrm{h}\mathscr{T}$. Similarly, $\mathscr{T}_\bullet$ means the pointed category of $\mathscr{T}$. We have similar notions of $\mathscr{T}\mathbf{P}$ for pair of spaces, $\mathrm{h}\mathscr{T}\mathbf{P}$ for quotient category of $\mathscr{T}\mathbf{P}$. The pointed versions are defined similarly and denoted by $\mathbf{TopP}_\bullet$ and $\mathrm{h}\mathbf{TopP}_\bullet$. Morphisms in $\mathrm{h}\mathbf{TopP}$ and $\mathrm{h}\mathbf{TopP}_\bullet$ are denoted by

$$[(X, A), (Y, B)], [(X, A), (Y, B)]_0.$$

Example 5.16. Hausdorff spaces are weak Hausdorff since compact subsets of Hausdorff spaces are closed. Therefore any locally compact Hausdorff space is compactly generated weak Hausdorff.

Proposition 5.17. The functor $k\colon \mathbf{WH} \to \mathscr{T}$ is right adjoint to the embedding $i\colon \mathscr{T} \subset \mathbf{WH}$. In other words, we have an adjoint pair $i\colon \mathscr{T} \rightleftarrows \mathbf{WH}\colon k$.

Lemma 5.18. Let $X \in \mathbf{WH}, K$ compact Hausdorff, and $f\colon K \to X$ continuous. Then $f(K)$ is compact Hausdorff.

Proposition 5.19. Let $X \in \mathbf{CG}$. Then X is weak Hausdorff if and only if the diagonal $\Delta_X = \{(x, x) \mid x \in X\}$ is closed in $X \times X$. Recall that this product is in $\mathbf{CG}$.

Remark 5.20. This proposition says that $\mathscr{T}$ relative to $\mathbf{CG}$ is the analogue of Hausdorff spaces relative to **Top**.

Corollary 5.21. Let $\{X_i\}$ be a family of spaces in $\mathscr{T}$. Then their product $\prod_{i \in I} X_i$ in $\mathbf{CG}$ also lies in $\mathscr{T}$.

Proof. Let $X = \prod_{i \in I} X_i$ with $\pi_i\colon X \to X_i$. We need to show that the diagonal Δ_X is closed in $X \times X$. Let $\pi_i \times \pi_i\colon X \times X \to X_i \times X_i$ and $D_i = (\pi_i \times \pi_i)^{-1}(\Delta_{X_i})$. Since Δ_{X_i} is closed in $X_i \times X_i$, it follows that $\Delta_X = \bigcap_{i \in I} D_i$ is closed in $X \times X$. □

Proposition 5.22. Let $X \in \mathbf{CG}$ and $E \subset X \times X$ be an equivalence relation on X. Then the quotient space X/E by the equivalence relation E lies in $\mathscr{T}$ if and only if E is closed in $X \times X$.

Given $X \in \mathbf{CG}$, we denote by E_X the smallest equivalence relation on X, which is the intersection of all closed equivalence relations. Then the quotient X/E_X is an object in $\mathscr{T}$. The construction is functorial, so it defines a functor $h\colon \mathbf{CG} \to \mathscr{T}$.

Proposition 5.23. We have an adjoint pair $h\colon \mathbf{CG} \rightleftarrows \mathscr{T}\colon i$. Moreover, $h \circ i$ is the identity functor.

Proof. Let $X \in \mathbf{CG}, Y \in \mathscr{T}$ and $f\colon X \to Y$ continuous. We need to show that f factors through $X/E_X \to Y$. Consider

$$f \times f\colon X \times X \longrightarrow Y \times Y.$$

Since Δ_Y is closed in $Y \times Y$, $(f \times f)^{-1}(\Delta_Y)$ defines a closed equivalence relation on X. Therefore $E_X \subset (f \times f)^{-1}(\Delta_Y)$. It follows that f factors through $X \to X/E_X \to Y$. □

Theorem 5.24. *$\mathscr{T}$ is complete and cocomplete. Limits in $\mathscr{T}$ inherit the limits in $\mathbf{CG}$. THe colimits in $\mathscr{T}$ are obtained by apllying h to the colimits in $\mathbf{CG}$.*

Proof. Let $F \in \mathrm{Fun}(I, \mathscr{T})$. We need to show $\operatorname{colim} F = h(\operatorname{colim}(i \circ F)), i(\lim F) = \lim(i \circ F)$. The colimit case follows from the fact that $h \circ i$ is the identity functor and h preserves colimits: $\operatorname{colim} F = \operatorname{colim}(h \circ i \circ F) = h(\operatorname{colim}(i \circ F))$. For the limit, let

$$X = \prod_{i \in I} F(i), Y = \prod_{i \xrightarrow{f} j} F(j)$$

be the products in $\mathbf{CG}$, which also lie in $\mathscr{T}$. Consider $g_1, g_2\colon X \to Y$ where

$$g_1(\{x_i\}) = \{x_j \mid f\colon i \to j\}, g_2(\{x_i\}) = \{f(x_i) \mid f\colon i \to j\}.$$

Then

$$\lim(i \circ F) = \{x \in X \mid g_1(x) = g_2(x)\} = (g_1 \times g_2)^{-1}(\Delta_Y)$$

is a closed subspace of X, so it also lies in $\mathscr{T}$. It can be checked that this is the limit of F. □

Proposition 5.25. *Let $X, Y \in \mathscr{T}$. Then $\mathrm{Map}(X, Y) \in \mathscr{T}$.*

Theorem 5.26. *Let $X, Y, Z \in \mathscr{T}$. Then*
1. *The evaluation map $\mathrm{Map}(X, Y) \times X \to Y, (f, x) \mapsto f(x)$ is continuous.*
2. *The composition map $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ is continuous.*
3. *The exponential law holds: we have a homeomorphism*

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

Next, we simply talk about the pointed category $\mathscr{T}_\bullet$. We will write $\mathrm{Map}_\bullet(X, Y) = \mathrm{Map}((X, x_0), (Y, y_0))$. On one hand, $\mathrm{Map}_\bullet(X, Y)$ stands for the set of morphisms. On the other hand, we can view $\mathrm{Map}_\bullet(X, Y)$ as an object in $\mathscr{T}_\bullet$ whose base point is the constant map from X to the base point of Y.

Theorem 5.27. *$\mathscr{T}_\bullet$ is complete and cocomplete. Let $X, Y, Z \mathscr{T}_\bullet$. Then*
(1) *The evaluation map $\mathrm{Map}_\bullet(X, Y) \wedge X \to Y$ is continuous.*
(2) *The composition map $\mathrm{Map}_\bullet(X, Y) \wedge \mathrm{Map}_\bullet(Y, Z) \to \mathrm{Map}_\bullet(X, Z)$ is also continuous.*
(3) *The exponential law holds: we have a homeomorphism*

$$\mathrm{Map}_\bullet(X \wedge Y, Z) \cong \mathrm{Map}_\bullet(X, \mathrm{Map}_\bullet(Y, Z)).$$

Here $\wedge$ is the smash product.

6. Group object and loop space

Definition 6.1. Let $X, Y \in \mathscr{T}_\bullet$. A based homotopy between two based maps $f_0, f_1\colon X \to Y$ is a homotopy $f_0 \simeq f_1$ relative to the base points. The homotopy classes is denoted by $[X, Y]_0$. We define the category $\mathrm{h}\mathscr{T}_\bullet$ by the quotient of $\mathscr{T}_\bullet$ where

$$\mathrm{Hom}_{\mathrm{h}\mathscr{T}_\bullet}(X, Y) = [X, Y]_0.$$

Definition 6.2. Given $(X, x_0) \in \mathscr{T}_\bullet$, we define the based loop space $\Omega_{x_0} X$ or simply ΩX by

$$\Omega X = \mathrm{Map}_\bullet(\mathbb{S}^1, X).$$

In the unpointed case, we define the free loop space

$$\mathcal{L} X = \mathrm{Map}(\mathbb{S}^1, X).$$

Our goal in this section is to explore some basic algebraic structures of based loop spaces.

Theorem 6.3. Ω defines functors $\Omega\colon \mathscr{T}_\bullet \to \mathscr{T}_\bullet, \Omega\colon \mathrm{h}\mathscr{T}_\bullet \to \mathrm{h}\mathscr{T}_\bullet$.

Definition 6.4. Suppose we have a category $\mathcal{C}$ with finite product and terminal object $*$. An object G in $\mathcal{C}$ is called a group object if it is endowed with morphisms $m\colon G \times G \to G$, $i\colon G \to G$ and $e\colon * \to G$ corresponding to multiplication, inversion, and an identity section. We require these morphisms to satisfy commutative diagrams:

- Associativity:

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id} \times m"] \arrow[d, "m \times \mathrm{id}"] & G \times G \arrow[d, "m"] \\ G \arrow[r, "m"] & G \end{tikzcd}$$

- Identity:

$$\begin{tikzcd} & * \times G \arrow[r, "e \times \mathrm{id}"] & G \times G \arrow[dr, "m"] & \\ G \arrow[ur] \arrow[rrr, "\mathrm{id}"] \arrow[dr] & & & G \\ & G \times * \arrow[r, "\mathrm{id} \times e"] & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

- Inverse:

$$\begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "i \times \mathrm{id}"] \arrow[r] \arrow[dr, "\mathrm{id} \times i"'] & * \arrow[r, "e"] & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Example 6.5. Here are some classical examples.
1. Group objects in **Set** are groups.
2. Group objects in **Top** are topological groups.
3. Group objects in h**Top** are called H-groups.

Proposition 6.6. Let $\mathcal{C}$ be a category with finite products and a terminal object. If G is a group objet, then $\mathrm{Hom}_{\mathcal{C}}(-, G)\colon \mathcal{C} \to \mathbf{Grp}$ defines a contravariant functor.

Remark 6.7. The converse is also true by Yoneda lemma.

In category $\mathscr{T}_\bullet$, products exists, which is given by $(X, x_0) \times (Y, y_0) = (X \times Y, (x_0, y_0))$.

Lemma 6.8. The quotient functor $\mathscr{T}_\bullet \to \mathrm{h}\mathscr{T}_\bullet$ preserves finite product.

Theorem 6.9. Let $X \in \mathscr{T}_\bullet$. Then ΩX is a group object in $\mathrm{h}\mathscr{T}_\bullet$.

Proof. The multiplication is the composition of paths and the inverse it the usual reverse of paths. The constant path serves as the zero object. The associativity is known. □

Corollary 6.10. Any $X \in \mathrm{h}\mathscr{T}$ defines a functor

$$[-, \Omega X]_0 = \mathrm{Hom}_{\mathrm{h}\mathscr{T}_\bullet}(-, \Omega X)\colon \mathrm{h}\mathscr{T}_\bullet \longrightarrow \mathbf{Grp}.$$

Definition 6.11. Let $(X, x_0) \in \mathscr{T}_\bullet$. We define its suspension ΣX by the quotient

$$\Sigma X = (X \times I)/(X \times \partial I \cup \{x_0\} \times I)$$

The suspension is the same as the smash product $\mathbb{S}^1 \times X$. It defines a functor $\Sigma\colon \mathrm{h}\mathscr{T}_\bullet \to \mathrm{h}\mathscr{T}_\bullet$.

Example 6.12. $\Sigma \mathbb{S}^n \cong \mathbb{S}^{n+1}$ for any $n \geqslant 0$.

Theorem 6.13. We have adjoint pairs $\Sigma\colon \mathrm{h}\mathscr{T}_\bullet \rightleftarrows \mathrm{h}\mathscr{T}_\bullet \colon \Omega$.

Definition 6.14. Let $(X, x_0) \in \mathscr{T}_\bullet$. Its n-th homotopy group is defined as

$$\pi_n(X, x_0) = [\mathbb{S}^n, X]_0.$$

Sometimes we simply denote it by $\pi_n(X)$. For $n \geqslant 1$, we know that $\pi_n(X) = [\Sigma\mathbb{S}^{n-1}, X]_0 \simeq [\mathbb{S}^{n-1}, \Omega X]_0$ is indeed a group.

In particular, we have

(1) π_0 is the path connected component.
(2) π_1 is the fundamental group.

Proposition 6.15. $\pi_n(X)$ is abelian for $n \geqslant 2$.

Proposition 6.16. Let X be a path connected space. There is a natural functor $T_n\colon \Pi_1(X) \to \mathbf{Grp}$ which sends x_0 to $\pi_n(X, x_0)$. In particular, there is a natural action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ and hence all $\pi_n(X, x_0)$ are isomorphic to each other.

Proposition 6.17. Let $f\colon X \to Y$ be a homotopy equivalence. Then

$$f_*\colon \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$$

is a group isomorphism.

7. Fiber homotopy and homotopy fiber

Definition 7.1. Given a space $X \in \mathscr{T}$ and $x \in X$, we define

- free path space: $PX = \mathrm{Map}(I, X)$.
- based path space: $P_x X = \mathrm{Map}((I, 0) \to (X, x))$.

For PX, there are two natural maps $p_0\colon PX \to X, \gamma \mapsto \gamma(0)$ and $p_1\colon PX \to X, \gamma \mapsto \gamma(1)$. It induces $p = (p_0, p_1)\colon PX \to X \times X$.

Theorem 7.2. Let $X \in \mathscr{T}$. Then

(1) $p\colon PX \to X \times X$ is a fibration.
(2) The map $p_0\colon PX \to X$ is a fibration whose fiber at x_0 is $P_{x_0}X$.
(3) The map $p_1\colon P_{x_0}X \to X$ is a fibration whose fiber at x_0 is $\Omega_{x_0}X$.
(4) $p_0\colon PX \to X$ is a homotopy equivalence. In particular, $P_{x_0}X$ is contractible.

Proof. (1): We need to prove the homotopy lifting property of the diagram

$$\begin{tikzcd} Y \times \{0\} \arrow[r] \arrow[d, hook] & PX \arrow[d, "p"] \\ Y \times I \arrow[r] & X \times X \end{tikzcd}$$

By the exponential law, $\mathrm{Map}(Y \times I, PX) \cong \mathrm{Map}(Y \times I \times I, X)$, so this is equivalent to the extension problem

$$\begin{tikzcd} Y \times \{0\} \times I \cup Y \times I \times \partial I \arrow[r] \arrow[d] & X \\ Y \times I \times I \arrow[ur, dashed] & \end{tikzcd}$$

To see this, imagine that $Y \times \{0\} \to PX$ gives familty of paths $f_{(y,0)}(t)$ for $y \in Y$, and $Y \times I \to X \times X$ gives prescribed endpoints $(f_{(y,s)}(0), f_{(y,s)}(1))$ for $(y, s) \in Y \times I$. The extension is equivalent to finding family of paths $f_{(y,s)(t)}$ with these endpoints. Thus it suffices to construct extensions from $Y \times \{0\} \times I \cup Y \times I \times \partial I$ to $Y \times I \times I$. This follows by observing that $\{0\} \times I \cup I \times \partial I$ is a deformation retract of $I \times I$.

(2): It follows from the composition of two fibrations

$$\begin{tikzcd} PX \arrow[r] \arrow[dr] & X \times X \arrow[d] \\ & X \end{tikzcd}$$

where the vertical arrow is the first component projection.

(3): Fibrations are preserved under pullback

$$\begin{tikzcd} P_{x_0}X \arrow[r] \arrow[d] & PX \arrow[d] \\ X \arrow[r, "x_0\times \mathrm{id}_X"] & X\times X \end{tikzcd}$$

(4): Let $i\colon X\to PX, x\mapsto i_x$. Then $p\circ i=\mathrm{id}_X$ and $i\circ p\colon PX\to PX$ sends any path starts at x to i_x. We consider retracting paths

$$\begin{aligned} r\colon P_X\times I &\longrightarrow PX \\ (\gamma,s) &\longmapsto (t\mapsto \gamma(st)), \end{aligned}$$

which shows that $i\circ p\simeq \mathrm{id}_{PX}$. As a consequence, by restricting the homotopy equivalence to each fiber, we see that $P_{x_0}X$ is contractible. □

Definition 7.3. Let $f\colon X\to Y$. We define the mapping path space P_f by the pullbcak diagram

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & PY \arrow[d, "p_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By definition, an element of P_f is a pair (x,γ) where γ is a path in Y that ends at $f(x)$. Let $\iota\colon X\to P_f, x\mapsto (x,1_{f(x)})$ represent the constant path and $p\colon P_f\to Y$ be the project of starting point. We have

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

Theorem 7.4. $\iota\colon X\to P_f$ is a deformation retract (and therefore a homotopy equivalence) and $p\colon P_f\to Y$ is a fibration. In particular, any map $f\colon X\to Y$ is a composition of a homotopy equivalence with a fibration.

Proof. Let $\pi\colon P_f\to X, (x,\gamma)\mapsto x$. Still, we can consider retracting paths

$$\begin{aligned} F\colon P_f\times I &\longrightarrow P_f \\ ((x,\gamma),s) &\longmapsto (x, t\mapsto \gamma(s+(1-s)t)). \end{aligned}$$

Then F gives a homotopy $\iota\circ\pi\simeq \mathrm{id}_{P_f}$. On the other hand, $\pi\circ\iota=\mathrm{id}_X$. Consider the pullback diagram

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & PY \arrow[d] \\ X\times Y \arrow[r, "f\times \mathrm{id}"] & Y\times Y \end{tikzcd}$$

This implies that $P_f\to X\times Y$ is a fibration. Since $X\times Y\to Y$ is also a fibration, so is the composition $p\colon P_f\to X\times Y\to Y$. □

This theorem says that in $\mathrm{h}\mathscr{T}$ every map is equivalent to a fibration.

Definition 7.5. Let $p_1\colon E_1\to B, p_2\colon E_2\to B$ be two fibrations. A fiber map from $p_1\to p_2$ is a map $f\colon E_1\to E_2$ such that $p_2=p_1\circ f$. To fiber maps $f_0,f_1\colon p_1\to p_2$ are said to be fiber homotopic, denoted by $f_0\simeq_B f_1$, if there exists a homotopy $F\colon E_1\times I\to E_2$ from f_0 to f_1 such that $F(-,t)$ is a fiber map for each t. $f\colon p_1\to p_2$ is called a fiber homotopic equivalence if there exists $g\colon p_2\to p_1$ such that both $f\circ g$ and $g\circ f$ are fiber homotopic to identity maps.

Proposition 7.6. Let $p_1\colon E_1\to B, p_2\colon E_2\to B$ be two fibrations and $f\colon E_1\to E_2$ be a fiber map. If $f_1\colon E_1\to E_2$ is a homotopy equivalence, then f is a fiber homotopy equivalence. In particular, $f\colon p_1^{-1}(b)\to p_2^{-1}(b)$ is a homotopy equivalence for any $b\in B$.

Proof. We only need to prove that for any fiber map f, whichi s a homotopy equivalence, there is a fiber map $g\colon E_2 \to E_1$ such that $g \circ f \simeq_B \mathrm{id}$. In fact, since such g is also a homotopy equivalence, we can find $h\colon E_1 \to E_2$ such that $h \circ g \simeq_B \mathrm{id}$. Then $f \simeq_B h \circ g \circ f \simeq_B h$, which implies $f \circ g \simeq_B \mathrm{id}$ as well.

Let $g\colon E_2 \to E_1$ represent the inverse of the homotopy class $[f]$ in $\mathrm{h}\mathscr{T}$.

First we show that we can choose g to be a fiber map, i.e., $p_1 \circ g = p_2$. To see this, observe that $p_1 \circ g = p_2 \circ f \circ g \simeq p_2$. Applying the homotopy lifting property to p_1, we obtain

$$\begin{tikzcd} E_2 \arrow[r, "g"] \arrow[d, hook] & E_1 \arrow[d, "p_1"] \\ E_2 \times I \arrow[r, "p_1 \circ g \simeq p_2"] \arrow[ur, dashed, "g \simeq g'"] & B \end{tikzcd}$$

Then g' is a fiber map, and we can replace g by g'.

Now we assume $g\colon E_2 \to E_1$ is a fiber map. We only need to prove the following statement: Let $p\colon E \to B$ be a fibration and $f\colon E \to E$ is a fiber map homotopic to id_E, then $f \simeq_B \mathrm{id}_E$.

Let F be a homotopy $f \simeq \mathrm{id}_E$ and $G = p \circ F$. Since p is a fibration, we can construct a homotopy H starting from id_E and lifting G:

$$\begin{tikzcd} E \times \{1\} \arrow[r, "\mathrm{id}_E"] \arrow[d, hook] & E \arrow[d, "p"] \\ E \times I \arrow[r, "G"] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

We can set $h(x) = H(x,0) \simeq_B \mathrm{id}_E$. Combining these two homotopies we find a homotopy $\tilde{F}$ from $h \circ f$ to id_E lifting the following homotopy

$$\begin{aligned} \tilde{G}\colon E \times I &\longrightarrow B \\ (x,t) &\longmapsto \begin{cases} G(x,2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ G(x, 2-2t) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}. \end{aligned}$$

We can construct a map $K \times E \times I \times I \to B$ that gives a homotopy between $\tilde{G}$ and the projection $p \circ \pi_E\colon E \times I \to B$:

$$\begin{aligned} K(-,u,0) &= \tilde{G}(-,u), \\ K(-,u,1) &= p(-), \\ K(-,0,t) &= p(-), \\ K(-,1,t) &= p(-). \end{aligned}$$

Since p is a fibration, we can find a lift $\tilde{K}\colon E \times I \times I \to E$ of K such that $\tilde{K}(-,u,0) = \tilde{F}(-,u)$. Then we have the following fiber homotopy

$$h \circ f = \tilde{K}(-,0,0) \simeq_B \tilde{K}(-,0,1) \simeq_B \tilde{K}(-,1,1) \simeq_B \tilde{K}(-,1,0) = \mathrm{id}_E .$$

□

Definition 7.7. Let $f\colon X \to Y$, we define its homotopy fiber over $y \in Y$ to be the fiber of $p\colon P_f \to Y$ over y.

Proposition 7.8. If Y is path connected, then all homotopy fibers of $f\colon X \to Y$ are homotopic equivalent.

Proof. Let $y_1, y_2 \in Y$ and F_1, F_2 be the homotopy fiber over y_1, y_2. Then

$$F_i = \{(x,\gamma) \mid \gamma\colon I \to Y, \gamma(0) = y_i, \gamma(1) = f(x)\}, i = 1,2$$

and composition with a path in Y from y_1 to y_2 gives a homotopy equivalence. More concretely, say σ is a path from y_2 to y_1. Then $\varphi\colon F_1 \to F_2, (x,\gamma) \mapsto (x, \sigma * \gamma), \psi\colon F_2 \to F_1, (x,\gamma) \mapsto (x, \overline{\sigma} * \gamma)$ are both continuous. Moreover, $\psi \circ \varphi, \varphi \circ \psi$ are homotopic to identity maps. □

In this case we will usually write the following diagram

$$\begin{tikzcd} F \arrow[r] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where F denotes the homotpy fiber.

Proposition 7.9. If $f\colon X \to Y$ is a fibration, then its homotopy fiber at y is homotopy equivalent to $f^{-1}(y)$.

Proof. Recall Theorem 9.4. We have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "f"'] & P_f \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where ι is a homotopy equivalence. Then ι is a fiber homotopy equivalence. ☐

Corollary 7.10. Let $f\colon X \to Y$ be a fibration with Y path connected. Then all fibers of f are homotopy equivalent.

Proof. Given $y_1, y_2 \in Y$, we know that $f^{-1}(y_1), f^{-1}(y_2)$ are homotopy equivalent to the corresponding homotopy fibers. The corollary follows since all homotopy fibers are homotopy equivalent. ☐

Recall the following theorem which gives a criterion for fibration that is every useful.

Theorem 7.11. Let $p\colon E \to B$ with B paracompact Hausdorff. Suppose there exists an open cover $\{U_\alpha\}$ of B such that $p^{-1}(U_\alpha) \to U_\alpha$ is a fibration. Then p is a fibration.

Corollary 7.12. Let $p\colon E \to B$ be a fiber bundle with B paracompact Hausdorff. Then p is a fibration.

8. Exact Puppe sequence

Definition 8.1. A sequence of maps of sets with points, i.e. in the category $\mathbf{Set}_\bullet$,

$$(A, a_0) \xrightarrow{f} (B, b_0) \xrightarrow{g} (C, c_0)$$

is said to be exact (at B) if $\operatorname{Im} f = \operatorname{Ker} g$, where $\operatorname{Im} f = f(A), \operatorname{Ker} g = g^{-1}(c_0)$. A sequence $A_{i+1} \to A_i$ is called an exact sequence if it is exact at every A_i.

Example 8.2. We view $\mathbf{Grp}$ as a subcategory of $\mathbf{Set}_\bullet$, where a group is based at the identity element. Let $H \triangleleft G$ be a normal subgroup of G. Then there is a short exact sequence in $\mathbf{Grp}$

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1.$$

Definition 8.3. A sequence of morphisms $X_{i+1} \to X_i$ in $\mathrm{h}\mathscr{T}_\bullet$ is called exact if for any $Y \in \mathrm{h}\mathscr{T}_\bullet$, the sequence $[Y, X_{i+1}]_0 \to [Y, X_i]_0$ of pointed sets is exact.

The goal of this section is to study the relationship between homotopy groups via exact sequence.

Definition 8.4. Let $f\colon (X, x_0) \to (Y, y_0)$ in $\mathscr{T}_\bullet$. We define its homotopy fiber F_f in $\mathscr{T}_\bullet$ by the pullback diagram

$$\begin{tikzcd} F_f \arrow[r] \arrow[d, "\pi"] & P_{y_0}Y \arrow[d, "p_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Namely, $F_f = \{(x, \gamma) \mid X \times PY \mid \gamma(0) = y_0, \gamma(1) = f(x)\}$.

Recall that $p_1\colon P_{y_0}Y \to Y$ is a fibration.

Lemma 8.5. $\pi\colon F_f \to X$ is a fibration.

Note that F_f is precisely the fiber of $p\colon P_f \to Y$ over y_0 (One may recall definition 9.3), so this definition agrees with our definition before.

The following lemma is the same as we did before. We restate here for convenience.

Proposition 8.6. If $X \to Y$ is a fibration, then $f^{-1}(y_0)$ is a homotopy equivalent to its homotopy fiber F_f.

For arbitary map $f\colon X \to Y$, we still have a canonical map

$$j\colon f^{-1}(y_0) \longrightarrow F_f,$$

which may not be a homotopy equivalence. The homotopy fiber can be viewed as a good replacement of fiber in homotopy category which behaves nicely for fibrations.

Lemma 8.7. The sequence

$$(F_f, (x_0, i_{y_0})) \xrightarrow{\pi} (X, x_0) \xrightarrow{f} (Y, y_0)$$

is exact at X in $\mathrm{h}\mathscr{T}_\bullet$.

Proof. $f \circ \pi$ factors through $P_{y_0}Y$ which is contractible. Therefore $f \circ \pi$ is null homotopy. Let $Z \in \mathrm{h}\mathscr{T}_\bullet$. Consider

$$[Z, F_f]_0 \xrightarrow{\pi_*} [Z, X]_0 \xrightarrow{f_*} [Z, Y]_0.$$

Since $f \circ \pi$ is null homotopy, we have $\operatorname{Im} \pi_* \subset \operatorname{Ker} f_*$.

Let $g\colon Z \to X$ such that $[g]_0 \in \operatorname{Ker} f_*$. Let G be a based homotopy of $f \circ g$ to the trivial map:

$$G\colon Z \times I \longrightarrow Y.$$

Since $G|_{Z\times\{0\}} = y_0$, it can be regarded as a map via the exponential law

$$G\colon Z \longrightarrow P_{y_0}Y$$

that fits into the following diagram

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, "G"] \arrow[ddr, "g"'] & & \\
& F_f \arrow[r] \arrow[d, "\pi"] & P_{y_0}Y \arrow[d, "p_1"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

There fore the pair (G, g) factors through F_f. This implies that $[g]_0 \in \operatorname{Im} \pi_*$, so $\operatorname{Ker} f_* \subset \operatorname{Im} \pi_*$. $\square$

Notice that the fiber of F_f over x_0 is precisely ΩY

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d] & F_f \arrow[d, "\pi"] \\
x_0 \arrow[r, hook] & X
\end{tikzcd}$$

We find the following sequence of pointed maps

$$\Omega X \xrightarrow{\Omega f} \Omega Y \longrightarrow F_f \xrightarrow{\pi} X \xrightarrow{f} Y.$$

Lemma 8.8. The sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \longrightarrow F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$

is exact in $\mathrm{h}\mathscr{T}_\bullet$.

Proof. We will construct the following commutative diagram in $\mathrm{h}\mathscr{T}_\bullet$ with all vertical arrows being homotopy equivalences

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] \arrow[d, "\tilde{j}'"] & \Omega Y \arrow[r] \arrow[d, "j"] & F_f \arrow[r, "\pi"] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
F_{\pi'} \arrow[r, "\pi''"] & F_\pi \arrow[r, "\pi'"] & F_f \arrow[r, "\pi"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

F_π is the homotopy fiber of $\pi\colon F_f \to X$, given by the pullback

$$\begin{tikzcd}
F_\pi \arrow[r] \arrow[d] & P_{x_0}X \arrow[d, "p_1"] \\
F_f \arrow[r, "\pi"] & X
\end{tikzcd}$$

More explictly, $F_\pi = \{(x, \gamma, \beta) \in X \times P_{y_0}Y \times P_{x_0}X \mid \gamma(1) = f(x), x = \beta(1)\}$. We set $\pi'\colon F_\pi \to F_f$ to be the vertical map in the above diagram. By definition, the second square in the first diagram commutes. Since $\pi\colon F_f \to X$ is a fibration with fiber ΩY over x_0, the map $j\colon \Omega \to F_\pi$ is the natural map of fiber into homotopy fiber which is a homotopy equivalence.

Explicitly, the map $j\colon \Omega Y \to F_\pi$ sends a loop γ based at y_0 to the pair $(x_0, \gamma, 1_{x_0})$. Similarly, the fiber over $(x_0, 1_{y_0}, 1_{x_0})$ of the fibration $\pi'\colon F_\pi \to F_f$ is ΩX. We find the natural map $j'\colon \Omega X \to F_{\pi'}$ into fiber homotopy, which is a homotopy equivalence. Let $(-)^{-1}\colon \Omega X \to \Omega X, \gamma \mapsto \gamma^{-1}$ be the inverse of the loop. We define

$$\tilde{j}' = j' \circ (-)^{-1}\colon \Omega X \longrightarrow F_{\pi'},$$

which is again a homotopy equivalence. Let $\pi''\colon F_{\pi'} \to F_\pi$ be the projection in the pullback diagram. We define

$$k = \pi'' \circ \tilde{j}'\colon \Omega X \longrightarrow F_\pi.$$

Here comes an important observation: $k = j \circ \Omega f$ in $\mathrm{h}\mathscr{T}_\bullet$, or equivalently,

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] \arrow[dr, "k"'] & \Omega Y \arrow[d, "j"] \\
& F_\pi
\end{tikzcd}$$

is commutative in $\mathrm{h}\mathscr{T}_\bullet$. One has to be careful because this is NOT true in $\mathscr{T}_\bullet$. To see this, let us explicitly write

$$k(\beta) = \pi''(\tilde{j}'(\gamma)) = \pi''(x_0, 1_{y_0}, \beta^{-1}),$$

$$(j \circ \Omega f)(\gamma) = j(f(\gamma)) = (x_0, f(\beta), 1_{x_0}).$$

They are homotopic via

$$F(\gamma, t) = \left(x_0, (\beta|_{[t,1]})^{-1}, f(\beta|_{[0,t]})\right).$$

Therefore the first square in the first diagram commutes. □

Lemma 8.9. Let $X_1 \to X_2 \to X_3$ be exact in $\mathrm{h}\mathscr{T}_\bullet$, then so is $\Omega X_1 \to \Omega X_2 \to \Omega X_3$.

Proof. For any $Y \in \mathrm{h}\mathscr{T}_\bullet$, applying $[\Sigma Y, -]_0$ to the exact sequence $X_1 \to X_2 \to X_3$, we get an exact sequence $[\Sigma Y, X_0] \to [\Sigma Y, X_1]_0 \to [\Sigma Y, X_2]_0$. Since Ω is right adjoint to the suspension Σ, $[\Sigma Y, X_i]_0 = [Y, \Omega X_i]_0$ and the result follows. □

Theorem 8.10 (exact puppe sequence)**.** Let $f\colon X \to Y$ in $\mathrm{h}\mathscr{T}_\bullet$. Then the following sequence is exact in $\mathrm{h}\mathscr{T}_\bullet$

$$\cdots \longrightarrow \Omega^2 Y \longrightarrow \Omega F_f \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow F_f \longrightarrow X \longrightarrow Y.$$

Proof. This is a immediate corrollary from Lemma 10.8 and Lemma 10.9. □

Theorem 8.11. Let $p\colon E \to B$ be in $\mathscr{T}_\bullet$. Assume that p is a fibration whose fiber over the base point is F. Then we have an exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(E) \longrightarrow \pi_0(B).$$

Proof. Since p is a fibration, F is homotopy equivalent to F_p. Observe that

$$[\mathbb{S}^0, \Omega^n X]_0 = [\Sigma^n \mathbb{S}^0, X]_0 = [\mathbb{S}^n, X]_0 = \pi_n(X).$$

Since the homotopy fiber F_p is homotopy equivalent to F, the theorem follows by applying $[\mathbb{S}^0, -]_0$ to the puppe exact sequence associated to p. $\square$

Example 8.12. Consider the universal covering $\exp\colon \mathbb{R} \to \mathbb{S}^1$. The associated long exact sequence implies $\pi_i(\mathbb{S}^1) = \pi_i(\mathbb{R}) = 0$ for $n \geqslant 2$.

Proposition 8.13. If $i < n$, then $\pi_i(\mathbb{S}^n) = 0$.

Proof. This proof is based on differential topology. Let $f\colon \mathbb{S}^i \to \mathbb{S}^n$ be any continuous map. Then by Whiteny approximation, f is homotopic to a smooth map. Thus, we can assume that f is smooth, so f is not surjective by Sard's theorem. We see that f is null homotopic because $\mathbb{S}^n \setminus \mathrm{pt} \simeq \mathbb{R}^n$ is contractible. $\square$

Example 8.14. Consider the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ with fiber $\mathbb{S}^1$. The associated long exact sequence of homotopy groups implies $\pi_2(\mathbb{S}^2) \simeq \pi_1(\mathbb{S}^1) = \mathbb{Z}$ and $\pi_n(\mathbb{S}^3) \simeq \pi_n(\mathbb{S}^2)$ for $n \geqslant 3$.

Remark 8.15. Later we will see that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$, so $\pi_3(\mathbb{S}^2) = \pi_2(\mathbb{S}^2) = \mathbb{Z}$. This shows that $\pi_i(\mathbb{S}^n)$ can be non-trivial for $i > n$.

9. COFIBRATION

In this section, we will discuss the dual of fibration.

Definition 9.1. A map $i\colon A \to X$ is said to have the homotopy extension property with respect to Y if for any $f\colon X \to Y$ and amy homotopy $F\colon A \times I \to Y$ where $F(-, 0) = f \circ i$, there exists a homotopy $\tilde{F}\colon X \times I$ such that

$$\tilde{F}(i(a), t) = F(a, t), F(x, 0) = f(x), \forall a \in A, x \in X, t \in I.$$

This definition can be illustrated via the following diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d] & X \arrow[d] \arrow[ddr, "f"] & \\
A \times I \arrow[r] \arrow[drr, "F"'] & X \times I \arrow[dr, dashed, "\tilde{F}"] & \\
& & Y
\end{tikzcd}$$

Definition 9.2. $i\colon A \to X$ is called a cofibration if it has homotopy extension property for any spaces.

Cofibration the dual notion of fibration, though this is not obvious by definition. We consider the diagram of fibration

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\tilde{F}"] & B
\end{tikzcd}$$

If we reverse the arrows and observe that $Y \times I$ is dual to the path space Y^I via the adjoint pairs $(-) \times I$ and $(-)^I$, we reach to the diagram of homotopy extension property using the exponential law:

$$\begin{tikzcd}
Y & X \arrow[l, "f"'] \arrow[dl, dashed, "\exists \tilde{F}"'] \\
Y^I \arrow[u, "p_0"] & A \arrow[l, "F"] \arrow[u, "i"']
\end{tikzcd}$$

Definition 9.3. Let $f\colon A \to X$. We define its mapping cylinder M_f by the pushout

$$\begin{tikzcd}
A \times \{0\} \arrow[r, hook] \arrow[d, "f"] & A \times I \arrow[d] \arrow[ddr, "f\times\mathrm{id}"] & \\
X \times \{0\} \arrow[r] \arrow[drr, hook, bend right] & M_f \arrow[dr, "j"] & \\
 & & X \times I
\end{tikzcd}$$

We have already constructed pushout in **Top**:

$$M_f = (A \times I \coprod X)/((a,0) \sim f(a), a \in A).$$

There is a natural map $j\colon M_f \to X \times I$ induced by inclusion $X = X \times \{0\} \to X \times I$ and $f \times \mathrm{id}\colon A \times I \to X \times I$.

By our construction, $g\colon M_f \to Z$ is continuous if and only if g is continuous when it restrcited to X and $A \times I$.

Lemma 9.4. The homotopy extension property of $i\colon A \to X$ is equivalent to the property of filling the commutative diagram

$$\begin{tikzcd}
M_i \arrow[r] \arrow[d] & Y \\
X \times I \arrow[ur, dashed] &
\end{tikzcd}$$

Proof. This follows from the universal property of pushout. □

Proposition 9.5. Let $i\colon A \to X$ and $j\colon M_i \to X \times I$ be as defined before. Then i is a cofibration if and only if there exists $r\colon X \times I \to M_i$ such that $r \circ j = \mathrm{id}_{M_i}$.

Proof. If i is a cofibration, then we can take $Y = M_i$ in the lemma above and we get the desired r. On the other hand, if r exists, then any $f\colon M_i \to Y$ lifts to $f \circ r$. □

Proposition 9.6. Let $i\colon A \to X$ be a cofibration. Then i is an embedding. If we work in $\mathscr{T}$, then i is a closed embedding.

Proof. Consider the following commutative diagram obtained from the previous proposition

$$\begin{tikzcd}
M_i \arrow[r, "\mathrm{id}"] \arrow[d, "j"'] & M_i \\
X \times I \arrow[ur, dashed, "r"'] &
\end{tikzcd}$$

This implies that M_i is homeomorphic to its image $j(M_i)$. Consider the next commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & M_i \arrow[d, "j"] \\
X \arrow[r] & X \times I
\end{tikzcd}$$

Since $A \to M_i, M_i \to X \times I, X \to X \times I$ are all embeddings, so is i.

Assume now that $A, X \in \mathscr{T}$. Then $j\colon M_i \to X \times I$ is a closed embedding. Since $A \to M_i, X \to X \times I$ are closed embeddings too, so is $i\colon A \to X$. □

Remark 9.7. A cofibration is not closed in general. A counterexample is $X = \{a, b\}$ with the trivial topology and $A = \{a\} \subset X$.

Definition 9.8. Let A be a subspace of X. The pair (X, A) is said to be cofibered if the inclusion is a cofibration.

Proposition 9.9. Let A be a closed subspace of X. Then the inclusion map $i\colon A \to X$ is a cofibration if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof. If i is closed, then M_i is homeomorphic to the subspace $X \times \{0\} \cup A \times I$ of $X \times I$. $\square$

Remark 9.10. If $A \subset X$ is not closed, then the mapping cylinder topology of M_i and the subspace topology $X \times \{0\} \cup A \times I$ may not be the same. For example, we can choose

$$X = [0,1], A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

Consider the subspace $Z = \left\{ (\frac{1}{n}, \frac{1}{n}) \mid n \in \mathbb{N} \right\} \subset A \times I$. Then Z is closed in $A \times I$ and $Z \cap (X \times 0) = \varnothing$, so Z is closed in the mapping cylinder. However, Z is not closed in $X \times \{0\} \cup A \times I$, as

$$\left(\frac{1}{n}, \frac{1}{n} \right) \to (0,0) \notin A \times I.$$

Example 9.11. The inclusion $\mathbb{S}^{n-1} \to D^n$ is a cofibration, because $D^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$ is a retract of $D^n \times I$.

Proposition 9.12. Let $f\colon A \to X$ be any map. Then the closed inclusion $i_1\colon A \to M_f, a \mapsto (a,1)$ is a cofibration.

Proof. This is because $M_f \times \{0\} \cup A \times I$ is a retract of $M_f \times I$. We give the explicit construction here:

$$\begin{aligned} r\colon M_f \times I &\longrightarrow M_f \times \{0\} \cup A \times I \\ (x,t) &\longmapsto \begin{cases} (x,0) \in M_f \times \{0\}, & x \in X \\ ((a,s), st) \in M_f \times I, & x = (a,s) \in A \times I \subset M_f. \end{cases} \end{aligned}$$

Keep in mind that $a \in A$ is identified with $(a,1) \in M_f$. If $x = f(a)$ for some $a \in A$, then $(x,t) = ((a,0),t) \mapsto ((a,0),0) = (x,0) \in X \times \{0\}$. Hence r is well defined. Moreover, r is identity on $M_f \times \{0\} \cup A \times I$. $\square$

Example 9.13. The inclusion $A \to A \times I, a \mapsto (a,0)$ is a cofibration. In fact, we can view it as $A \to M_{\mathrm{id}_A}$.

Definition 9.14. Let A be a subspace of X. A is called a neighborhood deformation retract if there exists $u\colon X \to I$ with $A = u^{-1}(0)$ and homotopy $H\colon X \times I \to X$ such that

$$\begin{cases} H(x,0) = x, & x \in X \\ H(a,t) = a, & a \in A \\ H(x,1) \in A, & u(x) < 1. \end{cases}$$

Note that if A is a neighborhood deformation of X, then A is a deformation retract of the open subset $u^{-1}([0,1))$ of X. Also note that A has to be a closed subspace.

Theorem 9.15. Let A be a closed subspace of X. Then the following conditions are equivalent:
(1) (X, A) is a cofibered pair.
(2) A is a neighborhood deformation retract of X.
(3) $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.
(4) $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$.

Proof. We have seen (1)$\Longleftrightarrow$(3) and (4)$\Longrightarrow$(3) is obvious.

(3)$\Longrightarrow$(2): Let $r\colon X \times I \to X \times \{0\} \cup A \times I$ be a retract and $\pi_X\colon X \times I \to X, \pi_I\colon X \times I \to I$ be the projections. We obtain the data for neighborhood deformation by

$$\begin{aligned} u\colon X &\longrightarrow I \\ x &\longmapsto \sup_{t \in I} |t - \pi_I \circ r(x,t)| \end{aligned}$$

and

$$\begin{aligned} H\colon X \times I &\longrightarrow X \\ (x,t) &\longmapsto \pi_X \circ r(x,t). \end{aligned}$$

If $u(x_0) = 0$ for some $x_0 \notin A$, then $\pi_I(r(x_0,t)) = t$ for all $t \in I$. In particular, $\pi_I(r(x_0,t)) > 0$ if $t > 0$. There exists open neighborhood $x_0 \in U$ such that $U \cap A = \varnothing$. Therefore taking limits $t \to 0^+$

we have $r(x_0,t) \to r(x_0,0) = (x_0,0)$, but $r(x_0,t) \in A \times I$ for all $t > 0$. Since A is closed, we have $(x_0,0) \in A \times I$, which contradicts with $x_0 \notin A$.

On the other hand, if $H(x,1) = \pi_X \circ r(x,1) \notin A$ with $u(x) < 1$, then $\pi_I \circ r(x,1) = 0$ by construction for r. However, this results in $1 - \pi_I \circ r(x,1) = 1$, which contradicts with $u(x) < 1$.

(2)$\Longrightarrow$(3): Given the data (u,H) for neighborhood deformation retract, we define a retract r by

$$r(x,t) = \begin{cases} (x,0) & u(x) = 1 \\ (H(x, 2(1-u(x))t), 0) & \frac{1}{2} \leqslant u(x) < 1 \\ (H(x, \frac{t}{2u(x)}), 0) & 0 < u(x) \leqslant \frac{1}{2}, 0 \leqslant t \leqslant 2u(x) \\ (H(x,1), t - 2u(x)) & 0 < u(x) \leqslant \frac{1}{2}, 2u(x) \leqslant t \leqslant 1 \\ (x,t) & u(x) = 0. \end{cases}$$

It remains to show (3)$\Longrightarrow$(4): Let r be a retract. Then the following homotopy

$$\begin{aligned} F\colon X \times I \times I &\longrightarrow X \times \{0\} \cup A \times I \operatorname{rel} X \times \{0\} \cup A \times I \\ (x,t,s) &\longmapsto (\pi_X \circ r(x,(1-s)t), (1-s)\pi_I \circ r(x,t) + st) \end{aligned}$$

gives the required deformation retract. $\square$

Now we discuss some basic properties for cofibration.

Proposition 9.16. Let $i\colon A \to X$ be a cofibration and $f\colon A \to B$ be a map. Consider the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[d, "j"] \\ X \arrow[r] & Y \end{tikzcd}$$

Then $j\colon B \to Y$ is also a cofibration. In other words, the pushout of a cofibration is a cofibration.

Proof. This purely the dual of fibration version. $\square$

Proposition 9.17. Let $i\colon X \to Y, j\colon Y \to Z$ be cofibrations. Then $j \circ i\colon X \to Z$ is also a cofibration.

Proposition 9.18. If $i\colon A \to X$ is a cofibration with A contractible, then the quotient map $\pi\colon X \to X/A$ is a homotopy equivalence.

Proof. A is homotopy equivalent to pt via F. We use the homotopy extension property for i by the following diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, hook] & X \arrow[d] \arrow[ddr, "\mathrm{id}_X"] & \\ A \times I \arrow[r] \arrow[drr, "i \circ F"'] & X \times I & \\ & & X \end{tikzcd}$$

We obtain $g_t\colon X \to X$ such that $g_0 = \mathrm{id}_X$ and $g_1|_A = i \circ F$. In particular, g_1 factorizes as $q \circ \pi$ for some $q\colon X/A \to X$. Hence, $q \circ \pi \simeq \mathrm{id}_X$. It remians to show $\pi \circ q \simeq \mathrm{id}_{X/A}$. Since $g_t(A) \subset A$, $\pi \circ g_t = h_t \circ \pi$ for some $h_t\colon X/A \to X/A$. Notice that $\mathrm{id}_X \circ \pi = \pi \circ g_0 = \pi$ and $\pi \circ q \circ \pi = \pi \circ g$, so $h_0 = \mathrm{id}_X$ and $h_1 = \pi \circ q$ by the universal property of quotient. $\square$

The next proposition is very useful in contructing homotopies.

Proposition 9.19. Let $A \subset X$ and $B \subset Y$ be closed inclusions which are both cofibrations. Then the inclusion $X \times B \cup A \times Y \subset X \times Y$ is also a cofibration. As a consequence, $A \times B \to X \times Y$ is a cofibration.

Proof. Let $u\colon X\times I, H\colon X\times I\to X$ be the data of neighborhood deformation for $A\subset X$ and $v\colon Y\to I, K\colon Y\times I\to Y$ be the data of neighborhood deformation for $Y\subset Y$. Consider the following maps

$$\begin{aligned}\varphi\colon X\times Y &\longrightarrow I\\ (x,y)&\longmapsto \min\{u(x),v(y)\}\end{aligned}$$

and

$$\begin{aligned}\Sigma\colon X\times Y\times I &\longrightarrow X\times Y\\ (x,y,t)&\longmapsto \begin{cases}(x,y), & u(x)=v(y)=0\\ (H(x,t),K(y,t\frac{u(x)}{v(y)})), & u(x)\leqslant v(y)\neq 0\\ (H(x,t\frac{v(y)}{u(x)}),K(y,t)), & 0\leqslant u(x)\geqslant v(y).\end{cases}\end{aligned}$$

Then (φ,Σ) defines a neighborhood deformation retract for $X\times B\cup A\times Y\subset X\times Y$, and therefore a cofibration. As a special case, if $B=\varnothing$, then $A\times Y\to X\times Y$ is a cofibration.

Now consider the following pushout diagram

$$\begin{tikzcd} A\times B \arrow[r] \arrow[d] & A\times Y \arrow[d]\\ X\times B \arrow[r] & X\times B\cup A\times Y\end{tikzcd}$$

So all arrows in this diagram are cofibrations. It follows that $A\times B\to X\times Y$ is a cofibration. □

Let $f\colon A\to X$ be a map. Consider the diagram of mapping cylinder

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[d, "f"'] & A\times I \arrow[d]\\ X \arrow[r] & M_f\end{tikzcd}$$

There is a natural commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "f"] & \\ M_f \arrow[rr, "r"] & & X\end{tikzcd}$$

Here we can write r explicitly: $r(a,t)=f(a)$ for $a\in A$ and $r(x)=x$ for $x\in X$. It is easy to see that r is a homotopy equivalence. We have the following dual statement of Theorem 9.4.

Theorem 9.20. $r\colon M_f\to X$ is a homotopy equivalence and $i_1\colon A\to M_f$ is a cofibration. In particular, any map $f\colon A\to X$ is a composition of a cofibration with a homotopy equivalence.

Definition 9.21. Let $i\colon A\to X, j\colon A\to Y$ be cofibrations. A map $f\colon X\to Y$ is called a cofiber map if $j=f\circ i$.

A cofiber homotopy between two cofiber maps f_0,f_1 is a homotopy f_t such that each f_t is a cofiber map. Similarly, we define cofiber homotopy equivalence.

And we also have the cofibration analogue of Proposition 9.6.

Proposition 9.22. Let $i\colon A\to X, j\colon A\to Y$ be cofibrations. Let $f\colon X\to Y$ be a cofiber map. If f is a homotopy equivalence, then f is a cofiber homotopy equivalence.

Now we work in the category $\mathscr{T}_\bullet$ and $\mathrm{h}\mathscr{T}_\bullet$. All maps and diagrams are assumed to be based.

Definition 9.23. A based space (X,x_0) is said to be well-pointed, if the inclusion of the base point x_0 is a cofibration in the unbased sense.

Definition 9.24. Let $(X,x_0)\in\mathscr{T}_\bullet$. We define its (reduced cone) by

$$C_*X=X\wedge I=X\times I/(X\times\{0\}\cup\{x_0\}\times I).$$

Proposition 9.25. If (X, x_0) is well-pointed, then the embedding $i_1 \colon X \to C_*X, x \mapsto (x, 1)$ is a cofibration.

Proof. C_*X is given by the pushout

$$\begin{tikzcd} \{x_0\} \arrow[r] \arrow[d] & I \arrow[d] \\ X \arrow[r, "i_1"] & C_*X \end{tikzcd}$$

so i_1 is a cofibration. □

Definition 9.26. Let $f \colon (X, x_0) \to (Y, y_0)$ be in $\mathscr{T}_\bullet$. We define its (reduced) mapping cylinder by

$$M_{*f} = M_f/(\{x_0\} \times I).$$

If (X, x_0) is well-pointed, then the quotient $M_f \to M_{*f}$ is a homotopy equivalence.

Definition 9.27. Given $f \colon X \to Y$ in $\mathscr{T}_\bullet$, we define its reduced homotopy cofiber C_{*f} by the pushout

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "f"'] & C_*X \arrow[d] \\ Y \arrow[r, "j"] & C_{*f} \end{tikzcd}$$

If X is well-pointed, then j is also a cofibration. Note that the quotient of C_{*f} by Y is precisely ΣX. We can extend it the above maps by

$$X \longrightarrow Y \longrightarrow C_{*f} \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_{*f} \longrightarrow \Sigma^2 X \longrightarrow \cdots.$$

Definition 9.28. A sequence in $\mathrm{h}\mathscr{T}_\bullet$

$$\cdots \longrightarrow X_{i+1} \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \cdots$$

is said to be co-exact if for any $Y \in \mathrm{h}\mathscr{T}_\bullet$ the following sequence of pointed sets is exact

$$\cdots \longrightarrow [X_{i-1}, Y]_0 \longrightarrow [X_i, Y]_0 \longrightarrow [X_{i+1}, Y]_0 \longrightarrow \cdots$$

is exact.

Theorem 9.29 (co-exact puppe sequence)**.** Let $f \colon X \to Y$ in $\mathrm{h}\mathscr{T}_\bullet$ between well-pointed spaces. The following sequence is co-exact in $\mathrm{h}\mathscr{T}_\bullet$

$$X \longrightarrow Y \longrightarrow C_{*f} \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma C_{*f} \longrightarrow \Sigma^2 X \longrightarrow \cdots.$$

Lemma 9.30. Let $f \colon A \to X$ be a cofibration between well-pointed spaces. Then the natural embedding $C_*A \to C_{*f}$ is a cofibration.

Proof. This immediately follows from the pushout. □

Theorem 9.31. Let $f \colon A \to X$ be a cofibration between well-pointed spaces. Then the natural map

$$\bar{r} \colon C_{*f} \longrightarrow X/A$$

is a homotopy equivalence. In other words, the cofiber is homotopy equivalent to the homotopy cofiber.

Proof. $C_*(A) \to C_{*f}$ is a cofibration and $C_*(A)$ is contractible, so $C_{*f} \to C_{*f}/C_*(A) = X/A$ is a homotopy equivalence. □

Theorem 9.32. Let $i \colon A \to X$ be a cofibration between well-pointed spaces. The following sequence

$$A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \Sigma(X/A) \longrightarrow \Sigma^2 A \longrightarrow \cdots$$

is co-exact in $\mathrm{h}\mathscr{T}_\bullet$.

10. CW complex

Definition 10.1. A cell decomposition of a space X is a family

$$\mathcal{E} = \{e^n_\alpha \mid \alpha \in J_n\}$$

of subspaces of X such that each e^n_α is a n-cell, i.e. the interior of D^n, and we have a disjoint union of sets

$$X = \coprod e^n_\alpha.$$

The n-skeleton of X is the subsapce

$$X^n = \coprod_{\alpha \in J_m, m \leqslant n} e^m_\alpha.$$

A CW complex means a complex with C (closure finiteness) and W (weak topology).

Definition 10.2. A CW-complex is a pair $(X, \mathcal{E})$ of a Hausdorff space X with a cell decomposition:
1. Characteristic map: for each n-cell e^n_α, there is a characteristic map

$$\Phi^n_\alpha \colon D^n \longrightarrow X$$

such that the restrction of Φ^n_α to $(D^n)^\circ$ is a homeomorphism to e^n_α and $\Phi^n_\alpha(\partial D^n) \subset X^{n-1}$.
2. Closure finiteness: For any cell $e \in \mathcal{E}$ the closure $\bar{e}$ intersects only a finite number of cells in $\mathcal{E}$.
3. Weak topology: A subset $A \subset X$ is closed if and only if $A \cap \bar{e}$ is closed for each $e \in \mathcal{E}$.

We say X has dimension n if the maximal dimension of cells in $\mathcal{E}$ is n.

Note taht the Hausdorff property of X implies that $\bar{e} = \Phi_e(D^n)$ for each $e \in \mathcal{E}$. The surjective map $\Phi_e \colon D^n \to \bar{e}$ is a quotient map since D^n is compact and $\bar{e}$ is Hausdorff. The full characterstic map is denoted by

$$\Phi \colon \coprod_{e \in \mathcal{E}} \longrightarrow X.$$

Then the weak topology implies that Φ is a quotient map. As a consequence, we have the following result.

Proposition 10.3. Let $(X, \mathcal{E})$ be a CW complex. Then a map $f \colon X \to Y$ is continuous if and only if $f \circ \Phi_e \colon D^n \to Y$ is continuous for each $e \in \mathcal{E}$.

Proposition 10.4. Any compact subspace of CW complex $(X, \mathcal{E})$ meets only finitely many cells.

Proof. By contradiction, suppose a compact subspace K meets infinitely many cells. Let $x_i \in K \cap e_i$, where e_i are different cells. Consider $Z_m = \{x_k \mid k \geqslant m\}$ for $m \geqslant 1$. By the closure finiteness, Z_m intersects each $\bar{e}$ at finite points, so they are closed in $\bar{e}$. By the weak topology, Z_m is a closed subset of X, hence $Z_m \subset K$ is closed. Observe

$$\bigcap_m Z_m = \varnothing$$

but any finite intersection of Z_m is non-empty. This contradicts the compactness of K. □

Proposition 10.5. A CW complex $(X, \mathcal{E})$ is the direct limit of its skeletons $\varinjlim X^n$.

Proof. $f \colon X \to Y$ is continuous if and only if $f|_{X^n} \colon X^n \to Y$ is continuous for each n. □

Proposition 10.6. A CW complex $(X, \mathcal{E})$ is compactly generated weak Hausdorff.

Proof. X is already Hausdorff, hence also weak Hausdorff. Assume $Z \subset X$ is compactly closed. Since $\bar{e}$ is compact Hausdorff, $Z \cap \bar{e}$ is closed in $\bar{e}$. The weak topology tells us that $Z \subset X$ is closed. □

Example 10.7. Here are some classical examples for CW complex.
- $\mathbb{S}^n = e^0 \cup e^n$.
- $\mathbb{S}^n = (e^n_+ \cup e^n_-) \cup \mathbb{S}^{n-1}$, so inductively

$$\mathbb{S}^n = \bigcup_{i=0}^n (e^i_+ \cup e^i_-).$$

- $\mathbb{R}^n$ have gride decomposition into cubes $I^n \cong D^n$.
- $\mathbb{CP}^n = (\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$ and we have
$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \subset \mathbb{CP}^n \subset \cdots \subset \mathbb{CP}^\infty.$$
Moreover, $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1} = \{[z_0, \cdots, z_n] \mid z_n \neq 0\} \cong \mathbb{C}^n \simeq e^{2n}$. Thus $\mathbb{CP}^n$ has exactly one cell in every even dimension up to $2n$ with characteristic map
$$\begin{aligned}\Phi^{2n} \colon D^{2n} &\longrightarrow \mathbb{CP}^n \\ (z_0, \cdots, z_n) &\longmapsto [z_0, \cdots, z_{n-1}, (1 - \sum_{i=0}^{n-1} |z_i|^2)^{\frac{1}{2}}].\end{aligned}$$
Here $D^{2n} \subset \mathbb{C}^{n+1}$.

Definition 10.8. A subcomplex $(X', \mathcal{E}')$ of a CW complex $(X, \mathcal{E})$ is a closed subspace $X' \subset X$ with a cell decomposition $\mathcal{E}' \subset \mathcal{E}$. Equivalently, a subcomplex is described by a subset $\mathcal{E}' \subset \mathcal{E}$ such that for $e_1 \in \mathcal{E}', e_2 \in \mathcal{E}$, $e_2 \in \mathcal{E}'$ if $\bar{e}_1 \cap e_2 \neq \varnothing$.

Definition 10.9. Given $f \colon \mathbb{S}^{2n-1} \to X$. Consider the pushout
$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, hook] \\ D^n \arrow[r, "\Phi_f"] & D^n \coprod_f X \end{tikzcd}$$
We say $D^n \coprod_f X$ is obtained by attaching an n-cell to X. Φ_f is called the characterstic map of the attached n-cell. More generally, if we have a family of maps $f_\alpha \colon \mathbb{S}^{n-1} \to X$, the pushout is obtained by attaching n-cells to X:
$$\begin{tikzcd} \coprod_\alpha \mathbb{S}^{n-1} \arrow[r, "f"] \arrow[d, hook] & X \arrow[d, hook] \\ \coprod_\alpha D^n \arrow[r, "\Phi_f"] & (\coprod D^n) \coprod_f X \end{tikzcd}$$
where $f = \bigsqcup f_\alpha$.

Example 10.10. $\mathbb{S}^n$ can be obtained by attaching an n-cell to a point.

Proposition 10.11. Let $(X, \mathcal{E})$ be a CW complex and $\mathcal{E} = \coprod \mathcal{E}^n$ where $\mathcal{E}^n$ be the set of n-cells. The the diagram
$$\begin{tikzcd} \coprod_{e \in \mathcal{E}^n} \mathbb{S}^{n-1} \arrow[r, "\partial\Phi^n"] \arrow[d, hook] & X^{n-1} \arrow[d] \\ \coprod_{e \in \mathcal{E}^n} D^n \arrow[r, "\Phi^n"] & X^n \end{tikzcd}$$
is a pushout. In particular, X^n is obtained by attaching n-cells to X^{n-1}.

Proof. This follows from the fact that $X^{n-1} \subset X^n$ is closed and the weak topology. $\square$

Lemma 10.12. Let X be an $(n-1)$-dimensional CW complex and Y is obtained by attaching n-cells to X. Then Y is an n-dimensional CW complex.

Proof. Y already admits a cell decomposition. It remains to check the following properties.

- (Hausdorff property) Take different points $x, y \in Y$. If either x or y lies in an n-cell, then it is easy to separate x, y. Ohterwise, take their open neighborhoods $U, V \subset X$ that separate them. Suppose $Y = (\coprod D^n) \coprod_g X$ with $g = \coprod g_\alpha$. Then $g_\alpha^{-1}(U), g_\alpha^{-1}(V)$ are open in $\mathbb{S}^{n-1}$. By definition we can take ther open neighborhoods U_α, V_α in D^n such that $U_\alpha \cap \mathbb{S}^{n-1} = g_\alpha^{-1}(U), V_\alpha \cap \mathbb{S}^{n-1} = g_\alpha^{-1}(V)$. Then x, y are separated by $U \cup (\bigcup_\alpha U_\alpha)$ and $V \cup (\bigcup_\alpha V_\alpha)$.
- (Closure finiteness) It suffices to check it property for an arbitary n-cells e. Since $\Phi_e(D^n) = \bar{e}$ and $\mathbb{S}^{n-1}$ is compact, it meets with finitely many other n-cells in Y. As for cells from X, notice that $\bar{e}$ is closed, so it meets finitely many cells in X.

- (Weak topology) Suppose $A \subset Y$. Then by the pushout construction, A is closed if and only if $p^{-1}(A) \subset (\coprod_\alpha) D^n \coprod X$ is closed, where $p\colon (\coprod_\alpha) D^n \coprod X \to Y$ is the quotient map. Since $\bar{e} = \Phi_e(D^n)$, we see that A is closed if and only if $A \cap \bar{e} \subset \bar{e}$ is closed.

$\square$

Theorem 10.13. Suppose we have a sequence of spaces $X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots$ where X^n is obtained by attaching n-cells to X^{n-1}. Let $X = \bigcup_n X^n$ be the union endowed with the weak topology with respect to X^n. Then X is a CW complex.

Definition 10.14. Let A be a subspace of X. A CW decomposition of (X, A) consists of a sequence

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$$

such that X^n is obtained from X^{n-1} by attaching n-cells and X carries the weak topology with respect to the subspace X^n. The pair (X, A) is called a relative CW complex.

We say (X, A) has realtive dimension n if the maximal dimension of cells is n.

Note that for a relative CW complex (X, A), A may not be a CW complex.

Proposition 10.15. Let (X, A) be a relative CW complex. Then $A \subset X$ is a cofibration.

Proof. $\mathbb{S}^{n-1} \to D^n$ is a cofibration, and cofibration is preserved under pushout. Hence $X^{n-1} \to X^n$ is a cofibration. The proposition follows since composition of cofibrations is still a cofibration. $\square$

Corollary 10.16. Let X be a CW complex and X' be a CW complex. THen $X' \to X$ is a cofibration.

Let $(X, \mathcal{E}), (Y, \mathcal{F})$ be two CW complexes. We can define a cellular structure on $X \times Y$ with n-skeleton

$$(X \times Y)^n = \left\{ e^k_\alpha \times f^l_\beta \mid k + l \leqslant n, e^k_\alpha \in \mathcal{E}, f^l_\beta \in \mathcal{F} \right\}$$

and characteristic maps

$$\Phi^{k,l}_{\alpha,\beta} = (\Phi^k_\alpha, \Phi^l_\beta)\colon D^{k+l} \longrightarrow X \times Y$$

Here we identify $D^{k+l} \cong D^k \times D^l$.

This natural cellular structure is closure finite. However, the product topology $X \times Y$ may not be the same as the weak topology, so it may not be a CW complex. So the right way to do this is taking categorical product in $\mathscr{T}$. Then this compactly generated product will have the weak topology and therefore becomes a CW complex.

Theorem 10.17. Let X, Y be CW complexes with Y locally compact. Then the topological product $X \times Y$ is a CW complex.

Example 10.18. CW complex structure on $\mathbb{S}^1$ induces a CW complex structure on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

Example 10.19. If X is a CW complex, then so is $X \times I$.

Definition 10.20. A CW complex X is said to be locally finite if each point in X has an open neighborhood that intersects only finite many cells.

In practical, locally finite CW complexes are good because they are locally compact and Hausdorff.

11. Cellular approximation and CW approximation

Theorem 11.1. $f\colon (X, A) \to (Y, B)$ be a morphism in $\mathrm{h}\mathscr{T}\mathbf{P}_\bullet$. Let $\bar{f} = f|_A$. Then the sequence

$$(X, A) \longrightarrow (Y, B) \longrightarrow (C_f, C_{\bar{f}}) \longrightarrow \Sigma(X, A) \longrightarrow \Sigma(C_f, C_{\bar{f}}) \longrightarrow \Sigma^2(X, A) \longrightarrow \cdots$$

is co-exact in $\mathrm{h}\mathscr{T}\mathbf{P}_\bullet$.

Definition 11.2. Let $(X, A) \in \mathscr{T}\mathbf{P}_\bullet$. We define the relative homotopy group $\pi_n(X, A)$ by

$$\pi_n(X, A) = [(D^n, \mathbb{S}^{n-1}), (X, A)]_0.$$

We shall also write $\pi_n(X, A; x_0)$ when we want to specify the base point.

Note that $(D^n, \mathbb{S}^{n-1}) \simeq \Sigma^{n-1}(D^1, \mathrm{pt})$ for $n \geqslant 2$, so $\pi_n(X, A)$ is a group for $n \geqslant 2$ due to the adjunct pair (Σ, Ω).

Lemma 11.3. $f\colon (D^n, \mathbb{S}^{n-1}) \to (X, A)$ is zero in $\pi_n(X, A)$ if and only if f is homotopic rel $\mathbb{S}^{n-1}$ to a map whose image lies in A.

Proof. Suppose $[f]_0 = 0$ in $\pi_n(X, A)$. The we can find a homotopy $F\colon f \simeq 1_{x_0} \operatorname{rel} \mathbb{S}^{n-1}$. As there is a radial projection $r\colon D^n \times \{1\} \to D^n \times I$ whose image lies in $D^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$, we can consider the homotopy $F \circ r_t\colon D^n \to X$ where $r_t = (1-t)\operatorname{id} + r$. By our construction, $F \circ r_0 = f$, $F \circ r_1(D^n) \subset A$ and $F \circ r_t|_{\mathbb{S}^{n-1}} = \operatorname{id}$ for every t.

Conversely, assume that there exists $g\colon (D^n, \mathbb{S}^{n-1}) \to (X, A)$ such that $g \simeq f \operatorname{rel} \mathbb{S}^{n-1}$. Let $F\colon D^n \times I \to D^n$ from the identity map to the trivial map. Then the homotopy $F \circ g\colon D^n \times I \to X$ shows that $[g]_0 = 0$, hence $[f]_0 = 0$ as well. $\square$

The lemma above can be illustrated by the following diagram

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\
D^n \arrow[ur, dashed, "g"] \arrow[r, bend left] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow"] & X
\end{tikzcd}$$

Theorem 11.4. Let $i\colon A \subset X$ in $\mathscr{T}_\bullet$. Then there is a long exact sequence

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(X).$$

Here the boundary map ∂ sends $\varphi \in [(D^n, \mathbb{S}^{n-1}), (X, A)]_0$ to its restrction on $\mathbb{S}^{n-1}$.

Proof. Consider

$$f\colon (\mathbb{S}^0, \{0\}) \longrightarrow (\mathbb{S}^0, \mathbb{S}^0).$$

Let $\bar{f} = f|_{\{0\}} \to \mathbb{S}^0$. It is easy to see that

$$(C_f, C_{\bar{f}}) \simeq (D^1, \mathbb{S}^0).$$

Since $\Sigma^n(\mathbb{S}^0) = \mathbb{S}^n, \Sigma(D^n, \mathbb{S}^{n-1}) = (D^{n+1}, \mathbb{S}^n)$, the co-exact Puppe sequence

$$(\mathbb{S}^0, \{0\}) \longrightarrow (\mathbb{S}^0, \mathbb{S}^0) \longrightarrow (D^1, \mathbb{S}^0) \longrightarrow (\mathbb{S}^1, \{0\}) \longrightarrow (\mathbb{S}^1, \mathbb{S}^1) \longrightarrow (D^2, \mathbb{S}^1)$$

$$\longrightarrow (\mathbb{S}^2, \{0\}) \longrightarrow \cdots$$

implies the exact sequence. $\square$

Definition 11.5. A pair (X, A) is said to be n-connected if $\pi_0(A) \to \pi_0(X)$ is surjective and $\pi_k(X, A; x_0) = 0$ for $\forall 1 \leqslant k \leqslant n, x_0 \in A$. Here n is possibly ∞.

From the long exact sequence

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \longrightarrow \cdots \longrightarrow \pi_0(X).$$

We see that (X, A) is n-connected if for any $x_0 \in A$,

$$\begin{cases} \pi_r(A, x_0) \to \pi_r(X, x_0) \text{ is bijective for } r < n \\ \pi_n(A, x_0) \to \pi_n(X, x_0) \text{ is surjective.} \end{cases}$$

Definition 11.6. A map $f\colon X \to Y$ is called an n-equivalence if for any $x_0 \in X$,

$$\begin{cases} f_*\colon \pi_r(X, x_0) \to \pi_r(Y, y_0) \text{ is bijective for } r < n \\ f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0) \text{ is surjective.} \end{cases}$$

Here n is possibly ∞.

f is called a weak homotopy equivalence if f is n-equivalence for any $n \geqslant 0$.

Example 11.7. For $n \geqslant 0$, we consider $(D^{n+1}, \mathbb{S}^n)$. Then there is a long exact sequence

$$\pi_k(\mathbb{S}^n) \longrightarrow \pi_k(D^{n+1}) = 0 \longrightarrow \pi_k(D^{n+1}, \mathbb{S}^n) \longrightarrow \pi_{k-1}(\mathbb{S}^n) \longrightarrow \cdots \longrightarrow \pi_0(D^{n+1})$$

We see that

$$\pi_k(D^{n+1}, \mathbb{S}^n) = \begin{cases} \mathbb{Z}, & k = n+1 \\ 0, & \text{else} \end{cases}$$

We also see that $(D^n, \mathbb{S}^n)$ is n-connected.

Lemma 11.8. Let X be obtained from A by attaching n-cells. Let (Y, B) be a pair such that $\pi_n(Y, B; b) = 0, \forall b \in B$ for $n \geqslant 1$ and $\pi_0(B) \to \pi_0(Y)$ is surjective. Then any map $(X, A) \to (Y, B)$ is homotopic rel A to a map from X to B.

Proof. Since $\pi_n(Y, B, b) = 0$ for any $b \in B$, we have

$$\begin{tikzcd}
& A \arrow[dr] & \\
\coprod \mathbb{S}^{n-1} \arrow[ur] \arrow[rr] \arrow[d, hook] & & B \arrow[d, hook] \\
\coprod D^n \arrow[urr, dashed] \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow"] & & Y
\end{tikzcd}$$

By the universal property of pushout, we obtain

$$\begin{tikzcd}
\coprod \mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] & A \arrow[r] \arrow[d, hook] & B \arrow[d, hook] \\
\coprod D^n \arrow[r] \arrow[urr, dashed] & X \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

□

Theorem 11.9. Let (X, A) be a relative CW complex with relative dimension n. Let (Y, B) be n-connected (possibly ∞). Then any map $(X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.

Proof. Apply the previous lemma to

$$X^{-1} = A \subset X^0 \subset X^1 \subset \cdots \subset X^n = X$$

and recall that $X^{n-1} \to X^n$ are all cofibrations. □

Proposition 11.10. Let $f\colon X \to Y$ be a weak homotopy equivalence and P be a CW complex. Then $f_*\colon [P, X] \to [P, Y]$ is a bijection.

Proof. Recall that $r\colon M_f \to Y$ is a homotopy equivalence, we may assume f is an embedding. Then (Y, X) is ∞-connected by assumption.

Surjectivity is illustrated by the diagram

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\
P \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

applying Theorem 13.9. Here $\dim X$ may be ∞ because of the weak topology.

Observe that $P \times \partial I, P \times I$ are still CW complexes. Injectivity is illustrated by the diagram

$$\begin{tikzcd}
P \times \partial I \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\
P \times I \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

$\square$

Theorem 11.11 (Whitehead theorem)**.** A map between CW complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.

Proof. Let $f\colon X \to Y$ be a weak homotopy equivalence between CW complexes. We find bijections $f_*\colon [X,X] \to [X,Y], f_*\colon [X,Y] \to [Y,Y]$. Therefore we can take $g \in [Y,X]$ such that $f_*[g] = \mathrm{id}_Y$. Then $f \circ g \simeq \mathrm{id}_Y$. On the other hand,

$$f_*[g \circ f] = [f \circ g \circ f] = [f \circ \mathrm{id}_Y] = f_*[\mathrm{id}_X],$$

which implies $g \circ f \simeq \mathrm{id}_X$. $\square$

Remark 11.12. This is basically the combination of Proposition 13.10 and Yoneda lemma.

Definition 11.13. Let X, Y be CW complexes. A map $f\colon X \to Y$ is said to be cellular if $f(X^n) \subset Y^n$. We define the category **CW** whose objects are CW complexes and morphisms are cellular maps.

Definition 11.14. A cellular homotopy between cellular maps $f, g\colon X \to Y$ is a homotopy $X \times I \to Y$ that is a cellular map. Here $X \times I$ is naturally a CW complex.

It is easy to see that cellular homotopy gives a equivalence relation on **CW**. The quotient category is denoted by h**CW**.

Lemma 11.15. Let X be obtained by attaching n-cells for $n \geqslant 1$. Then (X, A) is $(n-1)$ connected.

Proof. Let $r < n$ and consider the diagram

$$\begin{tikzcd}
\mathbb{S}^{r-1} \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\
D^r \arrow[r, "f"] & X
\end{tikzcd}$$

Since D^r is compact, $f(D^r)$ meets only finitely many attached n-cells on X, say $e_1, \cdots, e_m$. Let p_i be the center of e_i and $e_i^* = e_i \setminus \{p_i\}$. For $Y = X \setminus \{p_1, \cdots, p_m\}$, we subdivide D^r into small disks $D^r = \bigcup_\alpha D^r_\alpha$ such that $f(D^r_\alpha) \subset Y$ or $f(D^r_\alpha) \subset e_i$. For each D^r_α such that $f(D^r_\alpha) \subset e_i$ but not in Y, we use the fact that $(e_i, e_i^*) \simeq (D^n, \mathbb{S}^{n-1})$ is $(n-1)$-connected to find a homotopy rel $\mathbb{S}^{r-1} = \partial D^r$ to adjust mapping D^r_α into e_i^*. It glues together to obtain

$$\begin{tikzcd}
\mathbb{S}^{r-1} \arrow[r] \arrow[d, hook] & Y \arrow[d, hook] \\
D^r \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & X
\end{tikzcd}$$

Then we can further find a homotopy

$$\begin{tikzcd}
\mathbb{S}^{r-1} \arrow[r] \arrow[d, hook] & A \arrow[d, hook] \\
D^r \arrow[ur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

$\square$

Corollary 11.16. Let (X, A) be a relative CW complex, then for any $n \geqslant 0$, the pair (X, X^n) is n-connected.

Theorem 11.17. Let $f\colon (X, A) \to (Z, C)$ be a map between relative CW complexes which is cellular on a subcomplex (Y, B) of (X, A). Then f is homotopic rel Y to a cellular map $g\colon (X, A) \to (Z, C)$.

Proof. Assume that we have constructed $f_{n-1}\colon (X, A) \to (Z, C)$ which is homotopic to f rel Y and cellular on X^{n-1}. Consider

$$\begin{tikzcd} X^{n-1} \arrow[r] \arrow[d, hook] & Z^n \arrow[d, hook] \\ X^n \arrow[r, "f_{n-1}"] & Z \end{tikzcd}$$

Since X^n is obtained from X^{n-1} by attaching n-cells and (Z, Z^n) is n-connected,

$$\begin{tikzcd} X^{n-1} \arrow[r] \arrow[dd, hook] & Z^n \arrow[dd, hook] \\ & \\ X^n \arrow[uur, dashed] \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Z \end{tikzcd}$$

We can find a homotopy rel X^{n-1} from $f_{n-1}|_{X^n}$ to a map $X^n \to Z^n$. Since f is already cellular on Y, we can choose this homotopy rel Y by adjusting only those n-cells not in Y. This homotopy extends to a homotopy rel $X^{n-1} \cup Y$ from f_{n-1} to a map $f_n\colon X \to Z$ since $X^n \subset X$ is a cofibration. Then f_∞ works. $\square$

Theorem 11.18 (cellular approximation)**.** Any map between relative CW complexes is homotopic to a cellular map. If two cellular maps between relative CW complexes are homotopic, then they are cellular homotopic.

Proof. Apply the previous theorem to $(X, \varnothing)$ and $(X \times I, X \times \partial I)$. $\square$

This theorem shows that h**CW** is a full subcategory of h**Top**.

Definition 11.19. A CW approximation of a topological space Y is a CW complex X with a weak homotopy equivalence $f\colon X \to Y$.

Theorem 11.20. Any space has a CW approximation.

Proof. We may assume Y is path conneted. We construct the desired X by induction on the skeletons. Assume we have constructed $f_n\colon X^n \to Y$ which is an n-equivalence. We can extend f_n to a map $g_{n+1}\colon Y^{n+1} \to Y$ by attaching an $(n+1)$-cell to every generator of $\operatorname{Ker}(\pi_n(X^n) \to \pi_n(Y))$:

$$\begin{tikzcd} \coprod \mathbb{S}^n \arrow[r] \arrow[d, hook] & X^n \arrow[d] \arrow[ddr, bend left, "f_n"] & \\ \coprod D^{n+1} \arrow[r] \arrow[drr, dashed, bend right] & Y^{n+1} \arrow[dr, dashed, "g_{n+1}"'] & \\ & & Y \end{tikzcd}$$

Since (Y^{n+1}, X^n) is also n-connected, g_{n+1} is an n-equivalence. By construction and the surjectivity of $\pi_n(X^n) \to \pi_n(Y^{n+1})$, g_{n+1} defines also an isomorphism for $\pi_n(Y^{n+1}) \to \pi_n(Y)$.

Now for every generator $\mathbb{S}^{n+1}_\alpha$ of $\operatorname{Coker}(\pi_{n+1}(Y^{n+1} \to \pi_{n+1}(Y)))$, we take a wedge sum to obtain

$$X^{n+1} = Y^{n+1} \vee \Big(\bigvee \mathbb{S}^{n+1}_\alpha\Big).$$

Then the induced map $f_{n+1}\colon X^{n+1} \to Y$ extends f_n to an $(n+1)$-equivalence. Inductively, we can take $f_\infty\colon X = X^\infty \to Y$. $\square$

Theorem 11.21. Let $f\colon X \to Y$. Suppose $\Gamma X \to X, \Gamma Y \to Y$ are CW approximations. Then there exists a unique map in $[\Gamma X, \Gamma Y]$ making the following diagram commute in h**Top**

$$\begin{tikzcd} \Gamma X \arrow[r, "\Gamma f"] \arrow[d] & \Gamma Y \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Proof. Weak homotopy equivalence of $\Gamma Y \to Y$ implies the bijection $[\Gamma_X, \Gamma_Y] \to [\Gamma X, Y]$. □

12. Eilenberg–Mac Lane space

Theorem 12.1 (homotopy excision theorem)**.** Let $(A, C), (B, C)$ be relative CW complexes. Let X be the pushout

$$\begin{tikzcd} C \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

If (A, C) is m-connected and (B, C) is n-connected, then $\pi_k(A, C) \to \pi_k(X, B)$ is an isomorphism for $i < m + n$, and a surjection for $i = m + n$.

Proofs of related theorems may be found in Hatcher or May.

Corollary 12.2 (Freudenthal suspension theorem)**.** The suspension map

$$\pi_k(\mathbb{S}^n) \longrightarrow \pi_{k+1}(\mathbb{S}^n)$$

is an isomorphism for $k < 2n - 1$ and a surjection for $k = 2n - 1$.

Proof. Apply homotopy excision theorem to $(A, C) = (\mathbb{S}^n, x_0)$ and . Then $X = \mathbb{S}^{n+1}$. □

Corollary 12.3. $\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ for $n \geqslant 1$.

Proof. Freudenthal suspension theorem reduces to show that $\pi_2(\mathbb{S}^2)$, which follows from the Hopf fibration

$$\mathbb{S}^1 \longrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2.$$

□

13. singular homology

Let R be a commutative ring.

Definition 13.1. A chain complex over R is a sequence of R-module maps $\partial_n\colon C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$. When R is not specified, we mean $R = \mathbb{Z}$.

Sometimes we just write the map by ∂ and the chain complex by (C_*, ∂).

Definition 13.2. A chain map $f\colon C_* \to C'_*$ between two chain complex over R is a sequence of R-module maps $f_n \circ C_n \to C'_n$ such that the following diagram is commutative

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \end{tikzcd}$$

This can be simply expressed as $f \circ \partial = \partial' \circ f$.

Definition 13.3. We define the category $\mathbf{Ch}_*(R)$ whose objects are chain complexes over R and morphisms are chain maps. We simply write $\mathbf{Ch}_*$ when $R = \mathbb{Z}$.

Definition 13.4. Given a chain complex (C_*, ∂), we define its n-cycles Z_n and n-boundaries B_n by

$$Z_n = \mathrm{Ker}(\partial\colon C_n \to C_{n-1}), B_n = \mathrm{Im}(\partial\colon C_{n+1} \to C_n).$$

The equation $\partial^2 = 0$ implies $B_n \subset Z_n$. We define the n-th homology group by

$$H_n(C_*, \partial) = Z_n/B_n.$$

C_* is said to be acyclic if $H_n(C_*) = 0$ for any n.

Proposition 13.5. The n-the homology defines a functor

$$H_n\colon \mathbf{Ch}_* \longrightarrow \mathbf{Ab}.$$

Proof. A chain map $f\colon C_* \to C'_*$ induces a group homomorphism

$$\begin{aligned} f_*\colon H_n(C_*) &\longrightarrow H_n(C'_*) \\ \alpha &\longmapsto f(\alpha) \end{aligned}$$

This homomorphism is well defined because $f \circ \partial = \partial' \circ f$. $\square$

Definition 13.6. A chain map $f\colon C_* \to D_*$ is called a quasi-isomorphism if $f_*\colon H_n(C_*) \to H_n(D_*)$ is an isomorphism for all n.

Definition 13.7. A chain homotopy between two chain maps $f, g\colon C_* \to c'_*$, denoted by $f \simeq g$, is a sequence of homormophisms $s_n\colon C_n \to C'_{n+1}$ such that $f_n - g_n = s_{n-1} \circ \partial_n + \partial'_{n+1} \circ s_n$, or simply

$$f - g = s \circ \partial + \partial' \circ s.$$

Two complexes C_*, C'_* are said to be chain homotopy equivalent if there exists chain maps $f\colon C_* \to C'_*, g\colon C'_* \to C_*$ such that $f \circ g \simeq 1$ and $g \circ f \simeq 1$.

Proposition 13.8. Chain homotopy defines an equivalence relation on $\mathbf{Ch}_*$.

We define the quotient category h$\mathbf{Ch}_*$.

Proposition 13.9. If f, g are chain homotopic maps, then they induce the same map on homology groups. In other words, homology functor factors through h$\mathbf{Ch}_*$.

Proof. $f - g = s \circ \partial + \partial' \circ s$. Consider

$$f_* - g_*\colon H_n(C_*) \longrightarrow H_n(C'_*).$$

Let $\alpha \in C_n$ be a representative of a class $[\alpha]$ in $H_n(C_*)$. Since $\partial\alpha = 0$, we have

$$(f - g)(\alpha) = (s \circ \partial + \partial' \circ s)(\alpha) = (\partial' \circ s)(\alpha) \in B_n(C'_*).$$

$\square$

Definition 13.10. We define the standard n-simplex

$$\Delta^n = \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geqslant 0 \right\}.$$

We let $\{v_0, \cdots, v_n\}$ denote its vertices.

Definition 13.11. Let X be a space. A singular n-simplex in X is a continuous $\sigma\colon \Delta^n \to X$. For each $n \geqslant 0$, we define $S_n(X)$ to be the free abelian group generated by all singular n-simplexes in X. An element of $S_n(X)$ is called a n-chain in X.

Definition 13.12. Given a singular n-simplex $\sigma\colon \Delta^n \to X$ and $0 \leqslant i \leqslant n$, we define

$$\partial^i\sigma\colon \Delta^{n-1} \longrightarrow X$$

to be the $(n-1)$-simplex by restricting σ to the i-th fact of Δ^n given by $\{v_0, \cdots, \hat{v}_i, \cdots, v_n\}$. More precisely, $\partial^i\sigma = \sigma \circ f^i$, where f^i is the embedding of Δ^{n-1} as the i-th face.

We define the boundary map $\partial\colon S_n(X) \to S_{n-1}(X)$ to be the homomorphism generated by

$$\partial\sigma = \sum_{i=0}^n (-1)^i \partial^i\sigma.$$

Given a subset $\{v_{i_1}, \cdots, v_{i_k}\}$ of the vertices of Δ^n, we will write $\sigma|[v_{i_1}, \cdots, v_{i_k}]$ or just $[v_{i_1}, \cdots, v_{i_k}]$ for restrcting σ to the face of Δ^n spanned by $\{v_{i_1}, \cdots, v_{i_k}\}$. Then the boundary map can be expressed by

$$\partial[v_0, \cdots, v_n] = \sum_{i=0}^{n}(-1)^i[v_0, \cdots, \hat{v}_i, \cdots, v_n].$$

Proposition 13.13. $(S_*(X), \partial)$ defines a chain complex.

Proof.

$$\partial \circ \partial[v_0, \cdots, v_n] = \partial \sum_{i=0}^{n}(-1)^i[v_0, \cdots, \hat{v}_i, \cdots, v_n]$$

$$= \sum_{i<j}(-1)^{i+j-1}[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n] + \sum_{j<i}(-1)^{i+j}[v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n] = 0.$$

□

Example 13.14. Consider a 2-simplex $\sigma \colon \Delta^2 \to X$. Then

$$\partial\sigma = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

and

$$\partial^2\sigma = ([v_2] - [v_1]) - ([v_2] - [v_0]) + ([v_1] - [v_0]) = 0.$$

Let $f \colon X \to Y$ be a continuous map. Then it induces a chain map $S_*(f) \colon S_*(X) \to S_*(Y), \sigma \mapsto f \circ \sigma$. Hence we define a functor $S_* \colon \mathbf{Top} \to \mathbf{Ch}_*$. Singular homology can be viewed as the composition of functors $H_n \circ S_* \colon \mathbf{Top} \to \mathbf{Ab}$.

Proposition 13.15. Let $f, g \colon X \to Y$ be homotopic maps. Then

$$S_*(f), S_*(g) \colon S_*(f) \longrightarrow S_*(g)$$

are chain homotopic.

Proof. We only need to prove for $i_0, i_1 \colon X \to X \times I$, the induced map $S_*(i_0), S_*(i_1)$ are chain homotopic. Then $f = F \circ i_0, g = F \circ i_1$ are chain homotopic.

We define a homotopy

$$s \colon S_n(X) \longrightarrow S_{n+1}(X \times I).$$

For $\sigma \colon \Delta^n \to X$, we define $s(\sigma) = \sigma \times \mathrm{id} \colon \Delta^n \times I \to X \times I$. Here we treat $\Delta^n \times I$ as a collection of $(n+1)$ simplexes as follows. Let $\{v_1, \cdots, v_n\}$ denote the vertices of Δ^n. The vertices of $\Delta^n \times I$ contain two copies $\{v_0, \cdots, v_n\}$ and $\{w_0, \cdots, w_n\}$. Then

$$\Delta^n \times I = \sum_{i=0}^{n}(-1)^i[v_0, \cdots, v_i, w_i, w_{i+1}, \cdots, w_n].$$

Its sum defines $s(\sigma) \in S_{n+1}(X \times I)$.

The following intuitive formula holds

$$\partial(\Delta^n \times I) = \Delta \times \partial I - (\partial\Delta^n) \times I$$

as an equation for singular chains. This leads to the chain homotopy

$$S_*(i_1) - S_*(i_0) = \partial \circ s + s \circ \partial.$$

□

Theorem 13.16. Singular homologies are homotopy invariants.

Theorem 13.17 (dimension axiom)**.** If X is contractible, then

$$H_n(X) = \begin{cases} 0 & n > 0 \\ \mathbb{Z} & n = 0. \end{cases}$$

Proof. We can assume $X = \mathrm{pt}$. For each $n \geqslant 0$, there is only one $\sigma_n \colon \Delta^n \to X$. Therefore $S_n(X) = \langle \sigma_n \rangle$. The boundary map is

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_{n-1}.$$

The singular chain complex becomes

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

which implies the theorem. □

Definition 13.18. Chain maps

$$0 \longrightarrow C'_* \xrightarrow{i} C_* \xrightarrow{j} C''_* \longrightarrow 0$$

is called a short exact sequence if for each n,

$$0 \longrightarrow C'_n \xrightarrow{i} C_n \xrightarrow{j} C''_n \longrightarrow 0$$

is exact.

Theorem 13.19. A short exact sequence

$$0 \longrightarrow C'_* \xrightarrow{i} C_* \xrightarrow{j} C''_* \longrightarrow 0$$

induces a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(C'_*) \arrow[r, "i_*"] & H_n(C_*) \arrow[r, "j_*"] & H_n(C''_*) \arrow[dll, "\delta"'] & \\ & H_{n-1}(C'_*) \arrow[r, "i_*"] & H_{n-1}(C_*) \arrow[r, "j_*"] & H_{n-1}(C''_*) \arrow[r] & \cdots \end{tikzcd}$$

δ is called the connecting map, given by the snake lemma.

Proof. We simply construct the connecting map. Given a class $[\alpha] \in H_n(C''_*)$. Since $j \colon C_n \to C''_n$ is surjective, $\alpha = j(\beta)$ for some $\beta C''_n$. Consider $\partial\beta$. Since $j(\partial(\beta)) = \partial(j(\beta)) = \partial\alpha = 0$, there exists a unique element $\gamma \in C_{n-1}$ such that $i(\gamma) = \partial\beta$. Since $i(\partial(\gamma)) = \partial(i(\gamma)) = \partial^2\gamma = 0$, we have $\partial(\gamma) = 0$, which implies $\gamma \in H_{n-1}(C'_*)$. This can be illustrated by the following diagram

$$\begin{tikzcd} & \beta \arrow[r] \arrow[d] & \alpha \arrow[d] \\ \gamma \arrow[r] \arrow[d] & \partial\beta \arrow[r] & 0 \\ 0 & & \end{tikzcd}$$

□

Definition 13.20. Let $i \colon A \subset X$ be a subspace. It induces a natural injective chain map $S_*(A) \to S_*(X)$. The singular chain complex of X relative A is defined to be

$$S_*(X, A) = S_*(X)/S_*(A).$$

Since $\partial \circ i = i \circ \partial$, it naturally induces ∂ on $S_*(X, A)$. Its homology is called the relative homology, denoted by $H_n(X, A)$.

Theorem 13.21. For $i \colon A \subset X$, there exists a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r] & H_n(X, A) \arrow[dll, "\delta"'] & \\ & H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r] & H_{n-1}(X, A) \arrow[r] & \cdots \end{tikzcd}$$

Proof. This follows from the short exact sequence

$$0 \longrightarrow S_*(A) \longrightarrow S_*(X) \longrightarrow S_*(X,A) \longrightarrow 0.$$

□

Let us define the relative n-th cycles $Z_n(X,A) = \{\gamma \in S_n(X) \mid \partial\gamma \in S_{n-1}(A)\}$ and relative n-th boundaries $B_n = B_n(X) + S_n(A) \subset S_n(X)$. Then

$$H_n(X,A) = Z_n(X,A)/B_n(X,A).$$

The connecting map $\delta\colon H_n(X,A) \to H_{n-1}(A)$ can be understood as follows: a n-cycle in $H_n(X,A)$ is represented by an n-chain $\gamma \in S_n(X)$ such that $\partial\gamma \in S_{n-1}(A)$. Viewing $\partial(\gamma)$ as an $(n-1)$-cycle in A, then

$$\delta[\gamma] = [\partial(\gamma)].$$

Let $f\colon (X,A) \to (Y,B)$ be a map of pairs. It naturally induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S_*(A) \arrow[r] \arrow[d] & S_*(X) \arrow[r] \arrow[d] & S_*(X,A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & S_*(B) \arrow[r] & S_*(Y) \arrow[r] & S_*(Y,B) \arrow[r] & 0 \end{tikzcd}$$

which further induces compatible maps on homology groups

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots \end{tikzcd}$$

Theorem 13.22. Let $B \subset A \subset X$ be subspaces. Then there exists a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A,B) \arrow[r, "i_*"] & H_n(X,B) \arrow[r, "j_*"] & H_n(X,A) \arrow[dlll, "\delta"'] \\ H_{n-1}(A,B) \arrow[r, "i_*"] & H_{n-1}(X,B) \arrow[r, "j_*"] & H_{n-1}(X,A) \arrow[r] & \cdots \end{tikzcd}$$

Proof. This follows from the short exact sequence

$$0 \longrightarrow S_*(A,B) \longrightarrow S_*(X,B) \longrightarrow S_*(X,A) \longrightarrow 0.$$

□

Theorem 13.23 (homotopy axiom for pairs)**.** If $f\colon (X,A) \to (Y,B)$ and $f \simeq g \operatorname{rel} A$, then

$$H_*(f) = H_*(g)\colon H_*(X,A) \longrightarrow H_*(Y,B).$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d] & H_n(X,A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & \cdots \end{tikzcd}$$

and the five lemma. □

Proposition 13.24. Let X_α be path components of X, then

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$

Proof. This is because $S_*(X) = \bigoplus_\alpha S_*(X_\alpha)$. □

Proposition 13.25. If X is path connected, then $H_0(X) = \mathbb{Z}$.

Proof. Define the map

$$\epsilon\colon S_0(X) \longrightarrow \mathbb{Z}$$
$$\sum m_p(p) \longmapsto \sum m_p.$$

Obviously, $\epsilon(\sum m_p(p)) = 0$ if and only if $\sum m_p(p) = \sum(p_i - q_i)$. Since X is path connected, $\operatorname{Ker}\epsilon = B_0(X)$, which implies $\epsilon\colon H_0(X) \simeq \mathbb{Z}$. □

In general, we have a surjective map $\epsilon\colon H_0(X) \to \mathbb{Z}$.

Definition 13.26. The reduced homology of X is defined by the chain complex

$$\cdots \longrightarrow S_2(X) \longrightarrow S_1(X) \longrightarrow S_0(X) \xrightarrow{\epsilon} \mathbb{Z}.$$

The reduced homology is denoted by $\tilde{H}_*(X)$.

As exactness is local, the long exact sequence still holds of the reduced case.

Example 13.27. If X is contratible, then $\tilde{H}_n(X) = 0$ for all n.

Example 13.28. Take $x_0 \in X$. By the long exact sequence, we have

$$\cdots \longrightarrow \tilde{H}_n(x_0) = 0 \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X, x_0) \longrightarrow H_{n-1}(x_0) = 0 \longrightarrow \cdots.$$

Hence, $\tilde{H}_n(X) = H_n(X, x_0)$.

14. excision

The fundamental property of homology which makes it computable is excision.

Theorem 14.1 (excision axiom)**.** Let $U \subset A \subset X$ be subspaces such that $\overline{U} \subset \operatorname{Int}(A)$. Then the inclusion $i\colon (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms

$$i_*\colon H_n(X \setminus U, A \setminus U) \simeq H_n(X, A).$$

for all n.

Corollary 14.2. Let X_1, X_2 be subspaces of X and $X = \operatorname{Int}(X_1) \cup \operatorname{Int}(X_2)$. Then

$$H_n(X_1, X_1 \cap X_2) \simeq H_n(X, X_2)$$

for all n.

Proof. Take $U = X \setminus X_1, A = X_2$. Then $\overline{U} \subset \operatorname{Int}(A)$ and we can apply excision. □

Theorem 14.3 (Mayer–Viectoris)**.** Let $j_1\colon X_1 \subset X, j_2\colon X_2 \to X$ be subspaces of X and $X = \operatorname{Int}(X_1) \cup \operatorname{Int}(X_2)$. Denote $i_1\colon X_1 \cap X_2 \to X_1, i_2\colon X_1 \cap X_2 \to X_2$.Then there exists an exact sequence

$$\cdots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{i_{1*} \oplus i_{2*}} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_{1*} - j_{2*}} H_n(X) \longrightarrow H_{n-1}(X_1 \cap X_2) \longrightarrow \cdots$$

which is also true for reduced homology.

Theorem 14.4. Let $A \subset X$ be a closed subset. If A is a deformation retract of a neighborhood in X, i.e., $A \subset X$ is a cofibration, then the quotient $(X, A) \mapsto (X/A, A/A)$ induces an isomorphism

$$H_n(X, A) \simeq H(X/A, A/A) = \tilde{H}_n(X/A)$$

for all n.

Proof. Let $A \subset U$ be an open neighborhood of A that deformation retracts to A. Then $H_n(A) \simeq H_n(U)$, so $H_n(X, A) \simeq H_n(X, U)$ by the five lemma. Since A is closed and U is open, we can apply excision to get

$$H_n(X, A) \simeq H_n(X, U) \simeq H_n(X \setminus A, U \setminus A).$$

The same argument applied to $(X/A, A/A)$ and U/A gives

$$H_n(X/A, A/A) \simeq H_n((X/A) \setminus (A/A), (U/A) \setminus (A/A)) = H_n(X \setminus A, U \setminus A).$$

The last equalatiy holds because of the homeomorphism

$$((X/A)\setminus(A/A),(U/A)\setminus(A/A))\setminus(U/A)\cong(X\setminus A,U\setminus A).$$

□

Corollary 14.5. Let $A\subset X$ be a closed cofibration. Then $H_n(X,A)=\tilde{H}_n(X/A)$.

Let (X,x_0) be a well-pointed space. Recall that its reduced cone C_*X and reduced suspension ΣX are

$$C_*X=X\wedge I=\frac{X\times I}{X\times\{0\}\cup\{x_0\}\times I},\Sigma X=X\wedge\mathbb{S}^1=\frac{X\times I}{X\times\{0\}\cup X\times\{1\}\cup\{x_0\}\times I}.$$

Since (X,x_0) is well-pointed, we have homotopy equivalence

$$C_*X\simeq\frac{X\times I}{X\times\{0\}},\Sigma X\simeq\frac{X\times I}{X\times\{0\}\cup X\times\{1\}}.$$

Theorem 14.6. Let (X,x_0) be a well-pointed space. Then $\tilde{H}_n(\Sigma X)=\tilde{H}_{n-1}(X)$.

Proof. Let

$$Z=\frac{X\times I}{X\times\{0\}},Y=Z/X.$$

Since Z is contractible, the homology exact sequence associated to the pair $X\subset Z$ implies

$$\tilde{H}_n(Z,X)=\tilde{H}_{n-1}(X).$$

It follows that

$$\tilde{H}_n(\Sigma X)=\tilde{H}_n(Z/X)=\tilde{H}_n(Z,X)=\tilde{H}_{n-1}(X).$$

□

Proposition 14.7. The reduced homology of $\mathbb{S}^n$ is given by

$$\tilde{H}_k(\mathbb{S}^n)=\begin{cases}0 & k\neq n\\ \mathbb{Z} & k=n.\end{cases}$$

In particular, spheres of different dimensions are not homotopy equivalent.

Proof. $\mathbb{S}^n=\Sigma^n\mathbb{S}^0$. □

Corollary 14.8. If $m\neq n$, then $\mathbb{R}^m,\mathbb{R}^n$ are not homeomorphic.

Proof. By contradiction, assume $f\colon\mathbb{R}^m\to\mathbb{R}^n$ is a homeomorphism. Then f induces a homeomorphism $\mathbb{R}^m\setminus\{p\}\to\mathbb{R}^n\setminus\{f(p)\}$, which is a contradiction. □

Definition 14.9. A continuous map $f\colon\mathbb{S}^n\to\mathbb{S}^n$ has degree d, denoted by $\deg f=d$, if

$$f_*\colon\tilde{H}_n(\mathbb{S}^n)=\mathbb{Z}\longrightarrow\tilde{H}_n(\mathbb{S}^n)=\mathbb{Z}$$

is multiplication by d. Here, we have to pick the same generator on both sides.

Example 14.10. Identify $\mathbb{S}^2=\mathbb{C}\cup\{\infty\}$. Consider the map $f\colon\mathbb{S}^2\to\mathbb{S}^2,z\mapsto z^k$. Then $\deg f=k$.

Lemma 14.11. Let $f,g\colon\mathbb{S}^n\to\mathbb{S}^n$ be a continuous map.
1. $\deg(f\circ g)=\deg f\cdot\deg g$.
2. If $f\simeq g$, then $\deg f=\deg g$.
3. If f is a homotopy equivalence, then $\deg f=\pm1$.

Proof. All follow from the fact H_n is a factor from h**Top** to **Grp**. □

Proposition 14.12. Let $r\colon\mathbb{S}^n\to\mathbb{S}^n,(x_0,x_1,\cdots,x_n)\mapsto(-x_0,x_1,\cdots,x_n)$ be a reflection. Then $\deg r=-1$.

Proof. We prove by induction on n. This is true for $n=0$. Assume the proposition holds for $n-1$.

Consider the pair $(D^n, \mathbb{S}^{n-1})$. We find an isomorphism $\tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_{n-1}(\mathbb{S}^{n-1})$ by

$$\tilde{H}_n(\mathbb{S}^n) = \tilde{H}_n(D^n/\mathbb{S}^{n-1}) = \tilde{H}_n(D^n, \mathbb{S}^{n-1}) \xrightarrow{\delta} \tilde{H}_{n-1}(\mathbb{S}^{n-1}).$$

This isomorphism is compatible with r_*:

$$\begin{tikzcd}
\tilde{H}_n(\mathbb{S}^n) \arrow[r, "\delta"] \arrow[d, "r_*"] & \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[d, "r_*"] \\
\tilde{H}_n(\mathbb{S}^n) \arrow[r, "\delta"] & \tilde{H}_{n-1}(\mathbb{S}^{n-1})
\end{tikzcd}$$

This proves the case for n. □

Corollary 14.13. Let $\sigma\colon \mathbb{S}^n \to \mathbb{S}^n, (x_0, \cdots, x_n) \mapsto (-x_0, \cdots, -x_n)$ be the antipodal map. Then $\deg \sigma = (-1)^{n+1}$.

Proposition 14.14. If $f\colon \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points. Then f is homotopic to the antipodal map.

Proof. Let σ be the antipodal map. Then we can construct homotopy directly

$$\begin{aligned} F\colon \mathbb{S}^n \times I &\longrightarrow \mathbb{S}^n \\ (x,t) &\longmapsto \frac{(1-t)\sigma(x) + tf(x)}{|(1-t)\sigma(x)+tf(x)|}. \end{aligned}$$

Note that $(1-t)\sigma(x) + tf(x) = 0$ implies $f(x) = (1-\frac{1}{t})x$, which is impossible. □

Theorem 14.15 (hairy ball theorem)**.** $\mathbb{S}^n$ admits a nowhere vanishing tangent vector field if and only if n is odd.

Proof. If $n = 2m-1$ is odd, we construct $v(x_0, \cdots, x_{2m-1}) = (-x_1, x_0, \cdots, -x_{2m-1}, x_{2m-2})$. Conversely, if v is a nowhere vanishing tangent vector field on $\mathbb{S}^n$. Let f be the unit normalized tangent vector field. The map

$$\begin{aligned} f_t\colon \mathbb{S}^n &\longrightarrow \mathbb{S}^n \\ x &\longmapsto \cos(\pi t)x + \sin(\pi t)f(x) \end{aligned}$$

defines a homotopy between $f_0 = \mathrm{id}$ and $f_1 = \sigma$. It follows that $\deg \sigma = 1$. □

Theorem 14.16 (Brower's fixed point theorem)**.** Any continuous map $f\colon D^n \to D^n$ admits a fixed point.

Proof. By contradiction, suppose f has no fixed. Define $r\colon D^n \to \mathbb{S}^{n-1}$ where $r(p)$ is the intersection of $\mathbb{S}^{n-1} = \partial D^n$ and the ray starting from $f(p)$ pointing towards p. Then r defines a retract, which implies

$$\begin{tikzcd}
\tilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[r, "i_*"] \arrow[dr, "\mathrm{id}_*"'] & \tilde{H}_{n-1}(D^n) = 0 \arrow[d, "r_*"] \\
& \tilde{H}_{n-1}(\mathbb{S}^{n-1})
\end{tikzcd}$$

This gives arise to a contradiction. □

15. Cellular homology

Lemma 15.1. Let $\{(X_i, x_i)\}$ be well-pointed spaces. Then

$$\tilde{H}_n(\bigvee_i X) = \bigoplus_i \tilde{H}_n(X).$$

Proof. Let $X = \bigvee_i X_i$ $A = \coprod_i \{x_i\}$. Then $A \subset Y$ is a cofibration, so

$$\tilde{H}_n(Y) = \tilde{H}_n(Y/A) = \tilde{H}_n(Y, A) = \bigoplus_i H_n(X_i, x_i) = \bigoplus_i \tilde{H}_n(X_i).$$

□

Definition 15.2. Let (X, A) be a relative CW complex with skeletons X^n. We define the relative cellular chain complex $(CC_*(X, A), \partial)$ by $CC_n(X, A) = H_n(X^n, X^{n-1})$, where the boundary map ∂ is defined by the commutative diagram

$$\begin{tikzcd} H_n(X^n, X^{n-1}) \arrow[r, "\partial"] \arrow[d, "\delta"] & H_{n-1}(X^{n-1}, X^{n-2}) \\ H_{n-1}(X^{n-1}, A) \arrow[ur, "j_*"] & \end{tikzcd}$$

Here δ comes from the long exact sequence for $(X^{n-1}, A) \to (X^n, A)$, while j_* is the induced map in the long exact sequence for $(X^{n-2}, A) \to (X^{n-1}, A)$.

Assume X^n is obtained from X^{n-1} by attaching n-cells

$$\begin{tikzcd} \coprod_\alpha \mathbb{S}^{n-1} \arrow[r, "f"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ \coprod_\alpha D^n \arrow[r, "\Phi_f"] & X^n \end{tikzcd}$$

Since $X^{n-1} \subset X^n$ is a cofibration, we have

$$CC_n(X, A) = H_n(X^n, X^{n-1}) = \tilde{H}_n(X^n/X^{n-1}) = \tilde{H}_n(\bigvee_\alpha \mathbb{S}^n) \simeq \bigoplus_\alpha \tilde{H}_n(\mathbb{S}^n) \simeq \bigoplus_\alpha \mathbb{Z}.$$

is a free $\mathbb{Z}$-module generated by each attached $H_n(D^n, \mathbb{S}^{n-1}) = \tilde{H}_n(\mathbb{S}^n)$. Using the diagram

$$\begin{tikzcd} H_n(X^n, X^{n-1}) \arrow[d, "\delta_n"] \arrow[dr, "\partial"] & & \\ H_{n-1}(X^{n-1}, A) \arrow[r, "j_*"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d, "\delta_{n-1}"] \arrow[dr, "\partial"] & \\ & H_{n-1}(X^{n-2}, A) \arrow[r, "j_*"'] & H_{n-2}(X^{n-2}, X^{n-3}) \end{tikzcd}$$

and $\delta_{n-1} \circ j_* = 0$, we see that $\partial^2 = 0$.

Definition 15.3. Let (X, A) be a relative CW complex. We define its n-th relative cellular homology by $CH_n(X, A)$. When $A = \varnothing$, i.e., X is a CW complex, we simply denote it by $CH_n(X)$ and call it the n-th cellular homology.

Lemma 15.4. Let (X, A) be a relative CW complex. Let $0 \leqslant q < p \leqslant \infty$. Then for $n \leqslant q$ or $n > p$

$$H_n(X^p, X^q) = 0.$$

Proof. Consider the cofibrations

$$X^q \subset X^{q+1} \subset \cdots \subset X^p$$

where each quotient is a wedge of spheres $X^{k+1}/X^k = \bigvee \mathbb{S}^{k+1}$. Assume $n \leqslant q$ or $n > q$ Then

$$H_n(X^{q+1}, X^q) = \cdots = H_n(X^p, X^{p-1}) = 0.$$

Consider the triple $X^q \subset X^{q+1} \subset X^{q+2}$. The exact sequence

$$H_n(X^{q+1}, X^q) \longrightarrow H_n(X^{q+2}, X^q) \longrightarrow H_n(X^{q+2}, X^{q+1})$$

implies $H^{q+2,X^q} = 0$. The same argument applying to $X^q \subset X^{q+2} \subset X^{q+3}$. Repeating this process until arriving at $X^q \subset X^{p-1} \subset X^p$, we find $H_n(X^p, X^q) = 0$. □

Theorem 15.5. Let (X, A) be a relative CW complex. Then

$$CH_*(X, A) \simeq H_n(X, A).$$

Proof. Consider the following commutative diagram

$$
\begin{tikzcd}
 & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\delta_{n+1}"] \arrow[dr, "\partial_{n+1}"] & & H_n(X^{n-2}, A) = 0 \arrow[d] \\
H_n(X^{n-1}, A) = 0 \arrow[r] & H_n(X^n, A) \arrow[r, "j_{n+1}"] \arrow[d, "i_n"] & H_n(X^n, X^{n-1}) \arrow[r, "\delta_n"] \arrow[dr, "\partial_n"] & H_{n-1}(X^{n-1}, A) \arrow[d, "j_n"] \\
 & H_n(X^{n+1}, A) \arrow[d] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[d] \\
 & H_n(X^{n+1}, X^n) = 0 & & H_{n-1}(X^{n-2}, A) = 0
\end{tikzcd}
$$

Since j_{n+1} is injective, $\operatorname{Im} \partial_{n+1} = \operatorname{Im} \delta_{n+1} = \operatorname{Ker} i_n$. On the other hand, $\operatorname{Ker} \partial_n = \operatorname{Ker} \delta_n$ because j_n is an isomorphism. Then $\operatorname{Ker} \delta_n = \operatorname{Im} j_{n+1} = H_n(X^n, A)$, which implies

$$CH_n(X, A) = H_n(X^n, A)/\operatorname{Ker} i_n = H_n(X^{n+1}, A) = H_n(X, A).$$

The last equality holds because we have

$$H_{n+1}(X, X^{n+1}) = 0 \to H_n(X^{n+1}, A) \to H_n(X, A) \to H_n(X, X^{n+1}) = 0.$$

□

Now we analyze cellular map

$$\partial \colon H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2}).$$

For each n-cell e^n_α, we have the gluing map $f \colon \mathbb{S}^{n-1} \to X^{n-1}$. This defines a map

$$\bar{f} \colon \mathbb{S}^{n-1} \longrightarrow X^{n-1}/X^{n-2} = \bigvee_{J_{n-1}} \mathbb{S}^{n-1}$$

which induces a degree map

$$(\bar{f})_* \colon \tilde{H}_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z} \longrightarrow \bigoplus_{J_{n-1}} \tilde{H}_{n-1}(\mathbb{S}^{n-1}) = \bigoplus_{J_{n-1}} \mathbb{Z}.$$

Counting all n-cells, this generates the degree map

$$d_n \colon \bigoplus_{J_n} \mathbb{Z} \longrightarrow \bigoplus_{J_{n-1}} \mathbb{Z}.$$

Theorem 15.6. Under the identification $CC_n(X, A) = \bigoplus_{J_n} \mathbb{Z}$, cellular map coincides with the degree map

$$\partial_n = d_n.$$

Proof. This follows from the diagram chasing. □

Example 15.7. $\mathbb{CP}^n$ has a CW complex structure with a single $2n$-cell for $m \leqslant n$. Since there is no odd dimension cells, the degree map is 0. We find

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2. \cdots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Example 15.8. A closed oriented surface Σ_g of genus g has a CW complex structure with a 0-cell e_0, $2g$ 1-cells $\{a_1, b_1, \cdots, a_g, b_g\}$ and a 2-cell e_2.

The cell complex is

$$\mathbb{Z}e_2 \xrightarrow{d_2} \bigoplus_{i=1}^{g} \mathbb{Z}a_i \oplus \bigoplus_{i=1}^{g} \mathbb{Z}b_i \xrightarrow{d_1} \mathbb{Z}e_0.$$

d_2 sends e_2 to $(\sum(a_i - a_i), \sum(b_i - b_i) = 0$. Similarly $d_1(a_i) = d_1(b_i) = 0$. We find

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{2g} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k > 2. \end{cases}$$

Example 15.9. $\mathbb{RP}^n$ has a CW structure with a k-cell for each $0 \leqslant k \leqslant n$. A way to see this is using the skeletons $\mathbb{S}^0 \subset \mathbb{S}^1 \subset \cdots \subset \mathbb{S}^n$, with $\mathbb{S}^k$ is obtained from $\mathbb{S}^{k-1}$ by attaching two cells. Then this CW complex admits a natrual $\mathbb{Z}_2$-action, which induces the skeleton structure $\mathbb{RP}^0 \subset \cdots \subset \mathbb{RP}^n$.

The cell complex is

$$\mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_1} \mathbb{Z}.$$

The degree map is $d_k = 1 + (-1)^k$. If n is even,

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_2 & 0 < k \leqslant n, k \text{ odd} \\ \mathbb{Z} & k = n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 15.10. Let X be a finite CW complex of dimension n. $b_i(X) = \operatorname{rank}(H_i(X))$ is called the i-th Betti number of X. The Euler characteristic of X is defined as

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Since X is finite, $\chi(X)$ is a finite sum. Moreover, $H_i(X)$ is a finitely generated $\mathbb{Z}$-module. Also note that $\operatorname{rank}(H_i(X)) = \dim_{\mathbb{Q}} H_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 15.11. Let (M_*, ∂) be a chain complex of finitely generated $\mathbb{Z}$-modules such that $G_n = 0$ for n large enough. Then

$$\sum_i (-1)^i \operatorname{rank}(M_i) = \sum_i (-1)^i \operatorname{rank}(H_i(M_i))$$

Proof. We have $\operatorname{rank}(\operatorname{Ker} \partial_i) + \operatorname{rank}(\operatorname{Im} \partial_i) = \operatorname{rank}(M_i)$ and $\operatorname{rank}(H_i(M_i)) = \operatorname{rank}(\operatorname{Ker} \partial_i) - \operatorname{rank}(\operatorname{Im} \partial_{i-1})$, so the lemma follows. □

Example 15.12. $\chi(\mathbb{S}^n) = 1 + (-1)^n$.

Example 15.13. Let X a convex polyhedron. Then X has a natural CW structure given by vertices, edges and faces. Let v, e, f be their numbers. Then

$$\chi(X) = \chi(\mathbb{S}^2) = v - e + f.$$

Euler was the first to discover this beautiful identity.

16. COHOMOLOGY

Definition 16.1. A cochain complex over R is a sequence of R-module maps $d_n\colon C^n \to C^{n+1}$ such that $d_{n+1} \circ d_n = 0$. When R is not specified, we mean $R = \mathbb{Z}$.

We define $Z^n = \operatorname{Ker}(d\colon C^n \to C^{n+1})$ and $B^n = \operatorname{Im}(d\colon C^{n-1} \to C^n)$. The n-th cohomology group is defined by

$$H_n(C^*, d) = Z^n / B^n.$$

A cochain complex (C^*, d) is said to be acyclic if $H^n(C^*, d)$ for all n.

Definition 16.2. Let (C_*, ∂) be a chain complex over R and M be a R-module. We define its dual cochain complex $C^* = \operatorname{Hom}(C_*, M)$ and $(d_n(f))(c) = f(\partial_{n+1}(c))$.

Definition 16.3. Let M be a $\mathbb{Z}$-module and X be a space. For $n \geqslant 0$, we define the group of singular cochain complex of X with coefficient in A to be

$$S^n * (X; M) = \mathrm{Hom}(S_*(X), M).$$

Its cohomology is called the singular cohomology, denoted by $H^n(X; M)$. When $A = \mathbb{Z}$, we simply wrtie it as $H^n(X)$.

Theorem 16.4. $H^n(-, M)$ defines a contravariant functor $\mathbf{Top} \to \mathbf{Ab}$.

Theorem 16.5 (dimension axiom)**.** If X is contractible, then

$$H^n(X; M) = \begin{cases} M & n = 0 \\ 0 & n > 0. \end{cases}$$

Lemma 16.6. Let M be a R-module and $0 \to A_1 \to A_2 \to A_3$ be an exact sequence of R-modules. Then the following sequence is exact

$$0 \longrightarrow \mathrm{Hom}_R(A_3, M) \longrightarrow \mathrm{Hom}_R(A_2, M) \longrightarrow \mathrm{Hom}_R(A_1, M).$$

In other words, $\mathrm{Hom}_R(-, M)$ is a left exact functor. Moreover, $\mathrm{Hom}_R(-, M)$ is exact if and only if M is projective.

Definition 16.7. Let M be a $\mathbb{Z}$-module. Let $A \subset X$ be a subspace. We define the relative singular cochain complex with coefficient in M by

$$S^*(X, A; M) = \mathrm{Hom}(S_*(X, A), M).$$

Its cohomology is called the relative singular cohomology, denoted by $H^*(X, A; M)$.

Since $S_*(X, A)$ is a free abelian group, we have a short exact sequence of cochain complex

$$0 \longrightarrow S^*(X, A; M) \longrightarrow S^*(X; M) \longrightarrow S^*(A; M) \longrightarrow 0$$

which induces a long exact sequence of cohomology groups

$$\begin{tikzcd} \cdots \arrow[r] & H^n(X, A; M) \arrow[r] & H^n(X; M) \arrow[r] & H^n(A; M) \arrow[dll, "\delta"'] & \\ & H^{n+1}(X, A; M) \arrow[r] & H^{n+1}(X; M) \arrow[r] & H^{n+1}(A; M) \arrow[r] & \cdots \end{tikzcd}$$

Excision and Mayer–Viectoris sequence also holds for cohomology.

Theorem 16.8 (excision axiom)**.** Let $U \subset A \subset X$ be subspaces such that $\overline{U} \subset \mathrm{Int}(A)$. Then the inclusion $i\colon (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphism

$$i^*\colon H^*(X, A; M) \simeq H^*(X \setminus U, A \setminus U; M).$$

Theorem 16.9 (Mayer–Viectoris)**.** Let X_1, X_2 be subspaces of X and $X = \mathrm{Int}(X_1) \cup \mathrm{Int}(X_2)$. Then there is an exact sequence

$$\cdots \longrightarrow H^n(X; M) \xrightarrow{{i_1}^* \oplus {i_2}^*} H^n(X_1; M) \oplus H^n(X_2; M) \xrightarrow{{j_1}^* - {j_2}^*} H^n(X_1 \cap X_2) \longrightarrow H^{n+1}(X; M) \longrightarrow \cdots$$

Definition 16.10. Let M, N be two R-modules. Let $P_* \to M$ be a free reolution of M, i.e., an exact sequence

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with P_i free. We define the Ext group

$$\mathrm{Ext}^k_R(M, N) = H^k(\mathrm{Hom}(P_*, N))$$

and the Tor group

$$\mathrm{Tor}_k^R(M, N) = H_k(P_* \otimes_R N).$$

Ext and Tor are called the derived functors of Hom and $\otimes$. It is a classical result in homological algebra that Ext and Tor does not depend on the choice of free resolution. They are functorial with respect to both variables and Tor_k^R is symmetric in two variables:

$$\mathrm{Tor}_k^R(M,N) \simeq \mathrm{Tor}_k^R(N,M).$$

Note that $\mathrm{Ext}_R^0(M,N) = \mathrm{Hom}_R(M,N)$ and $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$.

Now we focus on the case of $R = \mathbb{Z}$. For any $\mathbb{Z}$-module, we can finish the resolution in two steps because submodule of a free module over a PID is still free. This implies that

$$\mathrm{Ext}_{\mathbb{Z}}^k(M,N) = 0, \mathrm{Tor}_k^{\mathbb{Z}}(M,N) = 0, k \geqslant 2.$$

For $\mathbb{Z}$-modules we will simply write $\mathrm{Ext}(M,N) = \mathrm{Ext}_{\mathbb{Z}}^1(M,N), \mathrm{Tor}(M,N) = \mathrm{Tor}_1^{\mathbb{Z}}(M,N)$.

Lemma 16.11. If either M is free or N is divisible, then $\mathrm{Ext}(M,N) = 0$.

Theorem 16.12 (universal coefficient theorem for cohomology)**.** Let M be a $\mathbb{Z}$-module and X be a space. Then for any $n \geqslant 0$, there exists a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), M) \longrightarrow H^n(X;M)] \longrightarrow \mathrm{Hom}(H_n(X), M) \longrightarrow 0$$

which induces isomorphisms

$$H^n(X;M) \simeq \mathrm{Hom}(H_n(X), M) \oplus \mathrm{Ext}(H_{n-1}(X), M).$$

Corollary 16.13. If $R = \mathbb{Z}$ or a field, we have $H^n(X;R) = \mathrm{Hom}(H_n(X), R)$.

Definition 16.14. Let M be a $\mathbb{Z}$-module. Let $A \subset X$ be a subspace. We define the relative singular chain complex with coefficient in M by

$$S_*(X,A;M) = S_*(X,A) \otimes \mathbb{Z}M.$$

Its homology is called the relative singular homology with coefficient in G, denoted by $H_*(X,A;M)$. When $A = \varnothing$, we simply get the singular homology $H_*(X;M)$.

Theorem 16.15 (universal coefficient theorem for homology)**.** Let M be a $\mathbb{Z}$-module and X be a space. Then for any $n \geqslant 0$, there exists a split exact sequence

$$0 \longrightarrow H_n(X) \otimes M \longrightarrow H_n(X;M) \longrightarrow \mathrm{Tor}(H_{n-1}, M) \longrightarrow 0$$

which induces isomorphisms

$$H_n(X;M) \simeq (H_n(X) \otimes_{\mathbb{Z}} M) \oplus \mathrm{Tor}(H_{n-1}(X), M).$$

17. Hurewicz theorem

Hurewicz theorem connectes homotopy groups with homology groups.

Definition 17.1. For $n \geqslant 1$, take a generator $i_n \in \tilde{H}_n(\mathbb{S}^n)$. The Hurewicz map is

$$\begin{aligned} \rho_n \colon \pi_n(X) &\longrightarrow \tilde{H}_n(X) \\ [f] &\longmapsto f_*(i_n). \end{aligned}$$

Proposition 17.2. The Hurewicz map is a group homomorphism.

Proof. Given $[f], [g] \in \pi_n(X)$, their product in $\pi_n(X)$ is represented by

$$\mathbb{S}^n \xrightarrow{\varphi} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{f \vee g} X \vee X \longrightarrow X.$$

Here φ shrinks the equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ to a point, and $\mathbb{S}^n/\mathbb{S}^{n-1} = \mathbb{S}^n \vee \mathbb{S}^n$. Apply $H_n(-)$ we get

$$\tilde{H}_n(\mathbb{S}^n) \xrightarrow{\varphi_*} \tilde{H}_n(\mathbb{S}^n) \oplus \tilde{H}_n(\mathbb{S}^n) \xrightarrow{f_* \oplus g_*} \tilde{H}_n(X) \oplus \tilde{H}_n(X) \xrightarrow{\text{sum}} \tilde{H}_n(X).$$

Observe $\varphi_* \colon \tilde{H}_n(\mathbb{S}^n) \to \tilde{H}_n(\mathbb{S}^n) \oplus \tilde{H}_n(\mathbb{S}^n)$ is the diagonal map $x \mapsto x \oplus x$. It follows that

$$\rho_n([f][g]) = f_*(i_n) + g_*(i_n) = \rho_n(f) + \rho_n(g).$$

$\square$

Proposition 17.3. Let $f\colon X \to Y$ be a weak homotopy equivalence. Then

$$f_*\colon H_n(X) \longrightarrow H_n(Y)$$

is an isomorphism for all n.

Proposition 17.4. If $Y = \bigvee \mathbb{S}^n$ for $n \geqslant 1$. Then

$$\pi_n(Y) \longrightarrow \tilde{H}_n(Y)$$

is the abelianization homomorphism.

Theorem 17.5 (Hurewicz theorem)**.** Let X be a path connected space which is $(n-1)$-connected for $n \geqslant 1$. Then the Hurewicz map $\rho_n\colon \pi_n(X) \to H_n(X)$ is an abelianization homomorphism, i.e., $\operatorname{Ker} \rho_n = [\pi_n(X), \pi_n(X)]$ is the commutator.

Explicitly, if $n = 1$, the Hurewicz map induces an isomorphism $\pi_1(X)_{\mathrm{ab}} \simeq H_1(X)$, and if $n > 1$, the Hurewicz map gives an isomorphism $\pi_n(X) \simeq H_n(X)$.

Hurewicz theorem has a relative version.

Theorem 17.6. Let (X, A) be a pair of path connected spaces. Assume (X, A) is $(n-1)$-connected for $n \geqslant 2$ and A is simply connected. Then

$$H_n(X, A) = 0, i < n$$

and the Hurewicz map

$$\pi_n(X, A) \longrightarrow H_n(X, A)$$

is an isomorphism.

Theorem 17.7 (homology Whitehead theorem)**.** Let $f\colon X \to Y$ between simply connected CW complexes. Assume $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for each n. Then f is a homotopy equivalence.

Proof. We can replace Y by mapping cylinder M_f, so we may assume X is a CW subcomplex of Y. Then $H_n(Y, X) = 0$ for all n by the long exact sequence. By Hurewicz theorem, $\pi_n(Y, X) = 0$ for all n. Therefore f is a homotopy equivalence by Whitehead theorem. $\square$

Proposition 17.8. Every simply connected and orientable closed 3 dimensional manifold is homotopy equivalent to $\mathbb{S}^3$.

Proof. Let X be a such a manifold. Then $H_0(X) = \mathbb{Z}, H_1(X) = \pi_1(X) = 0$. Since X is orientable, $H_3(X) = \mathbb{Z}$ and Poincaré duality holds. $H_2(X) = H^1(X)$. By the universal coefficient theorem, $H^1(X) = \operatorname{Hom}(H_1(X), \mathbb{Z}) = 0$. By Hurewicz theorem, $\rho_3\colon \pi_3(X) \to H_3(X)$ is an isomorphism. Let $f\colon \mathbb{S}^3 \to X$ represent a generator of $\pi_3(X) = \mathbb{Z}$. Then $f_*\colon H_*(\mathbb{S}^3) \to H_*(X)$ are isomorphisms. It follows by homology Whitehead theorem that f is a homotopy equivalence. $\square$

The famous Poincaré conjecture asks if such X is homeomorphic to $\mathbb{S}^3$.

18. Spectral sequence

19. Cup product and cap product

One of the key structure that distinguishes cohomology with homology is that cohomology carries an algebraic structure so $H^*(X)$ becomes a graded ring. This algebraic structure is called the cup product. Moreover, $H_*(X)$ will be a module of $H_*(X)$, and this module structure is called the cap product.

We have a natural cochain maps

$$S^*(X; R) \otimes_R S^*(Y; R) \longrightarrow \operatorname{Hom}(S_*(X) \otimes S_*(Y), R) \longrightarrow S^*(X \times Y; R)$$

The first map sends $(\varphi_p, \eta_q) \in S^p(X; R) \otimes_R S^q(Y; R)$ to $\varphi_p \otimes_R \eta_q$. The second map is dual to the Alexander–Whitney map

$$AW\colon S_*(X \times Y; R) \longrightarrow S_*(X; R) \otimes_R S_*(Y; R).$$

This leads to a cochain map

$$S^*(X;R) \otimes_R S^*(Y;R) \longrightarrow S^*(X \times Y;R),$$

which further induces

$$H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y;R).$$

Definition 19.1. We define the cup product on cohomology groups

$$\cup\colon H^p(X;R) \otimes H^q(X;R) \longrightarrow H^{p+q}(X;R)$$

by the composition

$$\begin{tikzcd} H^*(X;R) \otimes_R H^*(Y;R) \arrow[r] \arrow[dr, "\cup"'] & H^*(X \times X;R) \arrow[d, "\Delta^*"] \\ & H^*(X;R) \end{tikzcd}$$

Alexander–Whitney map gives an explicit product formula

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma_p)\dot{\beta}(\sigma_q), \alpha \in S^p(X;R), \beta \in S^q(X;R), \sigma \in \Delta^{p+q} \to X.$$

Theorem 19.2. $H^*(X;R)$ is a graded commutative ring with unit:
1. (Unit) Let $1 \in H^0(X;R)$ be reprensented by the cocycle which takes every singular 0-simplex to $1 \in R$. Then $1 \cup \alpha = \alpha \cup 1 = \alpha$.
2. (Associativity) $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.
3. (Graded commutativity) $\alpha \cup \beta = (-1)^{pq}\beta \cup \alpha$ for $\alpha \in H^p(X;R), \beta \in H^q(X;R)$.

Theorem 19.3. Let $f\colon X \to Y$ be a continuous map. Then

$$f^*\colon H^*(Y;R) \longrightarrow H^*(X;R)$$

is a morphism of graded commutative rings, i.e., $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$.

Theorem 19.4 (Künneth formula)**.** If R is a PID and $H_i(X;R)$ are finitely generated R-module, then there exists a split exact sequence of R-modules

$$0 \longrightarrow \bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R) \longrightarrow H^n(X \times Y;R) \longrightarrow \bigoplus_{p+q=n+1} \mathrm{Tor}_1^R(H^p(X;R), H^q(Y;R)) \longrightarrow 0.$$

In particular, if $H^*(X;R)$ or $H^*(Y;R)$ are free, we have an isomorphism of graded commutative rings

$$H^*(X \times Y;R) \simeq H^*(X;R) \otimes_R H^*(Y;R).$$

Example 19.5. $H^*(\mathbb{S}^n) = \mathbb{Z}[\eta]/\eta^2$ where $\eta \in H^n(\mathbb{S}^n)$ is a generator.

Example 19.6. Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. Then

$$H^*(\mathbb{T}^n) = \bigotimes_n H^*(\mathbb{S}^1) = \mathbb{Z}[\eta_1, \cdots, \eta_n]/(\eta_i\eta_j + \eta_j\eta_i).$$

Proposition 19.7. $H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1})$, where $x \in H^2(\mathbb{CP}^n)$ is a generator.

Proof. We prove by induction on n. Let x be a generator of $H^2(\mathbb{CP}^n)$. We only need to show that x^k is a generator of $H^{2k}(\mathbb{CP}^n)$. Using cellular chain complex, we know that $H^{2k}(\mathbb{CP}^n) \to H^{2k}(\mathbb{CP}^k)$ is an isomrophism for $k < n$. By induction, this implies that x^k is a generator of $H^{2k}(\mathbb{CP}^n)$. Poincaé duality implies that

$$H^2(\mathbb{CP}^n) \otimes_{\mathbb{Z}} H^{2n-2}(\mathbb{CP}^n) \xrightarrow{\cup} H^{2n}(\mathbb{CP}^n)$$

is an isomorphism. □

Definition 19.8. We define the evaluation map

$$\langle -, - \rangle : S^*(X;R) \times S_*(X;R) \longrightarrow R$$

as follows: We assign $\alpha \in S^p(X;R), \sigma \in S_p(X), r \in R$ to $\langle \alpha, \sigma \otimes r \rangle = \alpha(\sigma) \cdot r$.

The evalutaion map is compatible with boundary map and induces an evaluation map

$$\langle -,-\rangle : H^p(X;R) \otimes_R H^p(X;R) \longrightarrow R.$$

This generalizes to

$$H^*(X;R) \otimes_R H^*(X;R) \longrightarrow H^*(X;R) \otimes_R H^*(X;R) \otimes_R S_*(Y;R) \xrightarrow{\langle -,-\rangle \otimes 1} S_*(Y;R)$$

which induces

$$H^p(X;R) \otimes_R H_{p+q}(X \times Y;R) \longrightarrow H_q(Y;R).$$

Definition 19.9. We define the cap product

$$\cap \colon H^p(X;R) \otimes_R H_{p+q}(X;R) \longrightarrow H_q(X;R)$$

by the composition

$$\begin{tikzcd} H^p(X;R) \otimes_R H_{p+q}(X;R) \arrow[r, "1\otimes\Delta_*"] \arrow[dr, "\cap"'] & H^p(X;R) \otimes_R H_{p+q}(X \times X;R) \arrow[d] \\ & H_q(X;R) \end{tikzcd}$$

Theorem 19.10. The cap product gives $H_*(X;R)$ a structure of $H^*(X;R)$-module.

20. Poincaré duality

In this section, a manifold always means a topological manifold.

Definition 20.1. Let X be a n dimensional manifold. For $x \in X$, we can pick an open neighborhood $U \cong \mathbb{R}^n$. Applying excision to $X \backslash U \subset X \backslash \{x\} \subset X$, we obtain $H_n(X, X \backslash \{x\}) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) = \mathbb{Z}$. A generator of $H_n(X, X \setminus \{x\})$ is called a local orientation of x.

For any $x \in X$, there are two choices of local orientation at x. We obtain a double cover $\pi \colon \widetilde{X} \to X$, where $\widetilde{X} = \{(x, \mu_x) \mid \mu_x$ is a local orientation of X at $x\}$. Let $U \cong \mathbb{R}^n$ be an open neighborhood of x. Then we have an isomorphism

$$H_n(X, X \setminus U) \simeq H_n(X, X \setminus \{x\}),$$

which induces a identification $\pi^{-1}(U) \simeq U \times \mathbb{Z}_2$.

Definition 20.2. An orientation of X is a section of $\pi \colon \tilde{X} \to X$, i.e., a map $s \colon X \to \widetilde{X}$ with $\pi \circ s = \mathrm{id}_X$. If an orientation exists, we say X is orientable.

Theorem 20.3. Let X be a connected manifold. Then X is orientable if and only if $\widetilde{X}$ has two connected components. In particular, a connected orientable manifold has precisely two orientations.

Proof. Assume X is orientable. Let $s_1 \colon X \to \widetilde{X}$ be such a section. Since $\widetilde{X}$ admits a $\mathbb{Z}_2$-action, $s_2 = \tau \cdot s_1$ gives another orientation, where $\mathbb{Z}_2 = \{1, \tau\}$. The sections s_1, s_2 lead to a homeomorphism $\widetilde{X} = X \times \mathbb{Z}_2$.

Conversely, if $\widetilde{X}$ has two connected components, then each one is homeomorphic to X under the projection and so defines an orientation. □

Example 20.4. A simply connected manifold is orientable, because its covering must be trivial.

Proposition 20.5. For a manifold X, $\widetilde{X}$ is always orientable.

Proposition 20.6. We may assume X is non-orientable, so $\widetilde{X}$ has only one connected component. Since $\pi \colon \widetilde{X} \to X$ is a covering map, it is a local homeomorphism and induces an isomorphism

$$H_n(\widetilde{X}, \widetilde{X} \setminus \{\tilde{x}\}) = H_n(X, X \setminus \{x\}), x = \pi(\tilde{x}).$$

In particular, we have a canonical section $s \colon \widetilde{X} \to \widetilde{\widetilde{X}}, \tilde{x} = (x, \mu_x) \mapsto (\tilde{x}, \mu_x)$.

Theorem 20.7. Let X be a connected n dimensional manifold. For any abelian group G, we have

$$\begin{cases} H_i(X;G) = 0 & i > n \\ H_n(X;G) = 0 & X \text{ is non-compact.} \end{cases}$$

Definition 20.8. Let X be a dimension n manifold. A fundamental class of X at a subspace $A \subset X$ is an element $s \in H_n(X, X \setminus A)$ whose image under $H_n(X, X \setminus A) \to H_n(X, X \setminus \{x\})$ defines a local orientation for each $x \in A$. When $A = X$, $s \in H_n(X)$ is called a fundamental class of X.

Theorem 20.9. Let X be an oriented manifold X of dimension n, $K \subset X$ be a compact subspace. Then
(1) $H_i(X, X \setminus K) = 0$ for any $i > n$.
(2) The orientation of X defines a unique fundamental class of X at K.

Definition 20.10. Let $\mathcal{K}$ denote the set of compact subspaces of X. We define compactly supported cohomology of X by

$$H_c^k(X) = \operatorname{colim}_{K \in \mathcal{K}} H^k(X, X \setminus K)$$

where the colimit is taken with repect to induced maps of inclusions. In particular, if X is compact, $H_c^k(X) = H^k(X)$.

The functorial structure of compactly supported cohomology is with respect to the proper maps.

Example 20.11. Any smooth manifold admits an exhaustion $\{B_k\}$. Any compact subspace is contained in B_k. Therefore

$$H_c^i(X) = \operatorname{colim}_k H^i(X, X \setminus B_k).$$

If $X = \mathbb{R}^n$, we can take $B_k = \overline{B(0,k)}$. Then

$$H_c^i(\mathbb{R}^n) = \operatorname{colim}_k H^i(\mathbb{R}^n, \mathbb{R}^n \setminus B_k) = \operatorname{colim}_k \tilde{H}^{i-1}(\mathbb{R}^n \setminus B_k) = \tilde{H}^{i-1}(\mathbb{S}^{n-1}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{otherwise} \end{cases}.$$

Let X be an oriented manifold of dimension n. For each compact subset K, let $[K] \in H_n(X, X \setminus K)$ be the fundamental class. Taking the cap product we get

$$D_k \colon H^p(X, X \setminus K) \xrightarrow{\cap [K]} H_{n-p}(X).$$

This passes to the colimit and induces a map

$$D \colon H_c^p(X) \longrightarrow H_{n-p}(X).$$

Theorem 20.12 (Poincaré duality)**.** Let X be an oriented manifold of dimension n. Then for any p,

$$D \colon H_c^p(X) \longrightarrow H_{n-p}(X)$$

is an isomorphism. In particular, if X is compact then $H^p(X) \simeq H_{n-p}(X), \alpha \mapsto \alpha \cap [X]$.

21. Lefschetz fixed point theorem

In this section X will be an oriented connected compact manifold of dimension n. $[X]$ is its fundamental class.

Poincaré duality gives an isomorphism $H^p(X) \simeq H_{n-i}(X), \alpha \mapsto \alpha \cap [X]$. The cup product on cohomology has a geometric meaning under Poincaré duality as follows. Let Y, Z be oriented closed submanifold of X. Assume $\dim Y = i, \dim Z = j$ and Y intersects Z transversely so that their intersection $Y \cap Z$ is a submanifold of dimension $i + j - n$. $Y \cap Z$ has an induced orientation. Let $[Y]^* \in H^{n-i}(X)$ be the Poincaré dual of the fundamental class $[Y] \in H_i(Y) \subset H_i(X)$. Then

$$[Y]^* \cup [Z]^* = [Y \cap Z]^*.$$

Therefore the cup product can be taken as intersection under Poincaré duality.

Definition 21.1. We define the intersection pairing

$$\langle -, - \rangle : H_i(X) \times H_{n-i}(X) \longrightarrow H_0(X) = \mathbb{Z}$$

Equivalently, we have the pairing on cohomology

$$\langle -, - \rangle : H^i(X) \times H^{n-i}(X) \longrightarrow H^n(X) = \mathbb{Z}\langle [X] \rangle.$$

The intersection pairing is non-degenerate if we tensor with $\mathbb{Q}$. In particular,

$$\begin{aligned} H^i(X; \mathbb{Q}) \times H^{n-i}(X; \mathbb{Q}) &\longrightarrow \mathbb{Q} \\ (\alpha, \beta) &\longmapsto \alpha \cdot \beta = (\alpha^* \cup \beta^*) \cap [X]. \end{aligned}$$

where α^*, β^* are Poincaré dual in cohomology, is a non-degenerate pairing.

Example 21.2. $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. Let $Y_1 = \mathbb{S}^1 \times \{1\}, Y_2 = \{1\} \times \mathbb{S}^1$. $Y_1 \cap Y_2$ is a point. $H^*(\mathbb{T}^2) = \mathbb{Z}[\eta_1, \eta_2]/(\eta_1^2, \eta_1\eta_2 + \eta_2\eta_1, \eta_2^2)$, where $[Y_i]^* = \eta_i$. Then $[Y_1]^* \cup [Y_2]^* = \eta_1\eta_2 = [\mathbb{T}^2] = [\mathrm{pt}]^*$.

Consider $\Delta \subset X \times X$. Let e_i be a basis of $H_*(X; \mathbb{Q})$, consisting of elements of pure degree. Let e^j be its dual basis of $H_*(X; \mathbb{Q})$ such that

$$\left\langle e_i, e^j \right\rangle = \delta_i^j.$$

First we observe that

$$[\Delta] \in H_n(X \times X; \mathbb{Q}) \simeq \bigoplus_p H_p(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_{n-p}(X; \mathbb{Q})$$

is given by

$$[\Delta] = \sum_i (-1)^{|e_i|} e_i \otimes e^i.$$

Given a map $f \colon X \to X$, consider $\Gamma_f \subset X \times X$. Let $\alpha \in H_p(X), \beta \in H_{n-p}(X)$. From the geometry of graph, we find

$$[\Gamma_f] \cdot (\alpha \otimes \beta) = (f_* \alpha) \cdot \beta.$$

Applying this to $[\Delta]$, we find

$$[\Gamma_f] \cdot [\Delta] = \sum_i (-1)^{|e_i|} (f_* e_i) \cdot e^i = \sum_p (-1)^p \mathrm{Tr}(f_* \colon H_p(X; \mathbb{Q})) \to H_p(X; \mathbb{Q}).$$

On the other hand, if Γ_f and Δ intersects transversely, $[\Gamma_f] \cdot [\Delta] = [\Gamma_f \cap \Delta]^* \cap [X]$ gives a signed count of fxied points of the map f.

Definition 21.3. The Lefschetz number of $f \colon X \times X$ of f is defined as

$$L(f) = [\Gamma_f] \cdot [\Delta] = \sum_p (-1)^p \mathrm{Tr}(f_* \colon H_p(X; \mathbb{Q})) \to H_p(X; \mathbb{Q}).$$

Example 21.4. Let n be even. Then any map $f \colon \mathbb{CP}^n \to \mathbb{CP}^n$ has a fixed point. Since $H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1})$, $f^*(x) = kx$ for some $k \in \mathbb{Z}$. Then

$$L(f) = \sum_p (-1)^p \mathrm{Tr}(f_* \colon H_p(\mathbb{CP}^n; \mathbb{Q}) \to H_p(\mathbb{CP}^n; \mathbb{Q})) = \sum_p k^p$$

is an odd number. By Lefschetz fixed point theorem, f must admits a fixed point.

Example 21.5. The Lefschetz number of the identity map id_X is precisely the Euler characteristc:

$$L(\mathrm{id}_X) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{id} \colon H_i(X; \mathbb{Q}) \to H_i(X; \mathbb{Q})) = \sum (-1)^i b_i(X) = \chi(X).$$

Example 21.6. A compact connected Lie group G is always orientable. The left multiplication by $g \neq 1$ is a map without fixed point. Moreover, this map is homotopic to the identity map, so $\chi(G) = L(\mathrm{id}_G) = 0$.