SEMINAR NOTES: BASIC DE RHAM THEORY

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These notes is based on author's presentation on the first meeting of sminar on GTM 82, at USTC. The first meeting covers section 1- section 3 in chapter 1.

1. Introduction

Cohomology, homology and homotopy groups are three fundamental tools deveoloped in algebraic topology. The can be considered as the following functors:

$$\begin{aligned} H^*(-;A)&\colon \mathbf{Top} \longrightarrow \mathbf{Ring},\\ H_*(-;A)&\colon \mathbf{Top} \longrightarrow \mathbf{Mod}_R,\\ \pi_n(-)&\colon \mathbf{Top}_* \longrightarrow \mathbf{Grp},\end{aligned}$$

where A is an R-module. These tools are effective in studying and distinguishes spaces, but in reture they are usually difficult to describe and compute.

The fundamental idea in GTM 82 is roughly the following: Invariants in algebraic topology can be defined and represented in an easier way on manifolds with the help of specific tools for manifolds.

We will focus on the subcategory $\mathbf{Man}^\infty$, the category of smooth manifolds, whose morphisms are smooth maps. Note that functors above actually factor through h$\mathbf{Top}$ and h$\mathbf{Top}_*$, but considering $\mathbf{Man}^\infty$ is safe, due to the Whitney approximation which claims that every continuous map of smooth manifolds is homotopic to a smooth one.

2. review of differential forms

The main ingredients for de Rham theory are differential forms on smooth manifolds.

Let M be a smooth manifold. A k-form on M is a smooth section

$$\omega\colon M \longrightarrow \bigwedge^k T^*M.$$

Collection of k-forms is denoted by $\Omega^k(M)$. In local coordinates $(x^1, \cdots, x^m)$, a k-form can be written as

$$\omega = \omega_I dx^I,$$

where I runs over ordered indices with length k.

3. orientation and integral on manifolds

Manifolds are generalized objects of Euclidean spaces, on which we can still perform differential and integral.

4. de Rham theory

Let M be a smooth manifold. We can consider chains of exterior derivatives

$$\Omega^0(M) \xrightarrow{\ d\ } \Omega^1(M) \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^m(M) \xrightarrow{\ d\ } 0.$$

Let $Z_k = \mathrm{Ker}(d\colon \Omega^k(M) \to \Omega^{k+1}(M))$ and $B_k = \mathrm{Im}(d\colon \Omega^{k-1}(M) \to \Omega^k(M))$. Then $d^2 = 0$ implies $B_k \leqslant Z_k$. The k-th de Rham cohomology of M is

$$H^k_{\mathrm{dR}}(M) = B_k/Z_k.$$

$(\Omega^*(M), d)$ is actually an example for cochain complexes.

Definition 4.1. A cochain complex is a pair (C^*, d) which consists of a chain of morphisms $d\colon C^q \to C^{q+1}$ such that $d^2 = 0$. Here C^q are modules or vector spaces over fields.

For every cochain complex (C^*, d), we can all define its cohomology

$$H^k(C^q, d) = \frac{\mathrm{Ker}(d\colon C^q \to C^{q+1})}{\mathrm{Im}(d\colon C^{q-1} \to C^q)}.$$

Definition 4.2. Let $(A^*, d_A), (B^*, d_B)$ be cochain complexes. A cochain map $f\colon A^* \to B^*$ is a sequence of morphisms $f_q\colon A^q \to B^q$ such that the following diagram is commutative

$$\begin{tikzcd} A^q \arrow[r, "d_A"] \arrow[d, "f_q"] & A^{q+1} \arrow[d, "f_{q+1}"] \\ B^q \arrow[r, "d'_B"] & B^{q+1} \end{tikzcd}$$

This can be simply expressed as $f \circ d_A = d_B \circ f$.

By definition, a cochain map $f\colon A^* \to B^*$ naturally induces maps of cohomology $f\colon H^q(A^*, d_A) \to H^q(B^*, d_B), [\alpha] \mapsto [f(\alpha)]$. This is well-defined because $f(d_A(\alpha)) = d_B(f(\alpha))$.

Definition 4.3. Cochain maps

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$$

is called a short exact sequence if for each q,

$$0 \longrightarrow A^q \xrightarrow{f} B^q \xrightarrow{g} C^q \longrightarrow 0$$

is exact.

Theorem 4.4. A short exact sequence

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$$

induces a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^n(A^*) \arrow[r, "f"] & H^n(B^*) \arrow[r, "g"] & H^n(C^*) \arrow[dll, "\delta"'] & \\ & H^{n+1}(A^*) \arrow[r, "f"] & H^{n+1}(B^*) \arrow[r, "g"] & H^{n+1}(C^*) \arrow[r] & \cdots \end{tikzcd}$$

δ is called the connecting map.

Proof. We simply construct the connecting map, while the rest is easy to check directly. Given a class $[\alpha] \in H^n(C^*)$. Since $g\colon B^n \to C^n$ is surjective, $\alpha = g(\beta)$ for some $\beta \in B^n$. Consider $d\beta$. Since $g(d\beta) = d(g(\beta)) = d\alpha = 0$, there exists a unique element $\gamma \in A^{n+1}$ such that $f(\gamma) = d\beta$. Since $f(d\gamma) = d(f(\gamma)) = d^2\beta = 0$, we have $d\gamma = 0$, which implies $\gamma \in H^{n+1}(B^*)$. This can be illustrated by the following diagram

$$\begin{tikzcd} & \beta \arrow[r] \arrow[d] & \alpha \arrow[d] \\ \gamma \arrow[r] \arrow[d] & d\beta \arrow[r] \arrow[d] & 0 \\ d\gamma \arrow[r] & 0 & \end{tikzcd}$$

□

Let $f\colon M \to N$ be a smooth map of manifolds. Then f induces the pullback of differential forms $f^*\colon \Omega^*(N) \to \Omega^*(M)$. Moreover, this is indeed a cochain map because d commutes with the pullback. If f is a diffeomorphism, then $(f \circ f^{-1})^* = (f^{-1})^* \circ f^* = \mathbb{1}$ and $(f^{-1} \circ f)^* = f^* \circ (f^{-1})^* = \mathbb{1}$. In particular, diffeomorphism manifolds have isomorphism de Rham cohomologies.

Example 4.5. Let's consider the de Rham cohomology of $X = \mathbb{R}^2 \setminus \{P, Q\}$, where P, Q are distinct points. Without loss of generosity, assume $P = (-1, 0)$ and $Q = (1, 0)$. The cochain complex writes

$$C^\infty(X) \xrightarrow{d} \{f dx + g dy \mid f, g \in C^\infty(X)\} \xrightarrow{d} \{\varphi dx \wedge dy \mid \varphi \in C^\infty\}.$$

$H^0_{\mathrm{dR}}(X) = \mathrm{Ker}(d \colon C^\infty \to \Omega^1(X))$. $df = 0$ implies that f is constant, so $H^0_{\mathrm{dR}}(X) = \mathbb{R}$.

Next consider $H^1_{\mathrm{dR}}(X)$. Let $\omega = f dx + g dy$ and we want to see that how far it is from an exact form. First

$$d\omega = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx \wedge dy = 0,$$

so $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$. If $\omega = d\varphi$, then

$$\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} \omega,$$

where the integral is taken along an arbitary smooth path joining x_0, x in X. Conversely, if the integral of ω is indeed independent of the path, by multivariable calculus, φ is a well-defined smooth function such that $d\varphi = \omega$. As $d\omega = 0$, by Stokes' formula the integral is invariant up to free homotopy, so the problem lies in the two integrals

$$\lambda_1 \int_{C_1} \omega, \lambda_2 = \int_{C_2} \omega,$$

where C_1, C_2 are small circle around P, Q respectively. To get rid to this, we can consider forms

$$\eta_1 = \frac{1}{2\pi} \frac{(x+1)dy - ydx}{(x+1)^2 + y^2}, \eta_2 = \frac{1}{2\pi} \frac{(x-1)dy - ydx}{(x-1)^2 + y^2}.$$

These are all closed forms with

$$\int_{C_i} \eta_j = \delta_{ij}.$$

Thus we can modify ω by considering

$$\omega - \lambda_1 \eta_1 - \lambda_2 \eta_2.$$

This time the integral of $\omega - \lambda_1 \eta_1 - \lambda_2 \eta_2$ along arbitary paths is well-defined, so it is exact. In other words,

$$[\omega] = \lambda_1 [\eta_1] + \lambda_2 [\eta_2].$$

It follows that $H^1_{\mathrm{dR}}(X) = \mathbb{R}[\eta_1] \oplus \mathbb{R}[\eta_2] \simeq \mathbb{R}^2$.

It is discouraging that we found difficulty computing the last cohomology $H^3_{\mathrm{dR}}(X)$.

5. Mayer–Vietoris sequences

Computation for de Rham cohomology of plane punctured two points suggest we need to develop more techniques.

6. compact supported de Rham cohomology