LECTURE NOTES ON SYMPLECTIC TOPOLOGY

SHENGXI JIN

Contents

These notes are based on lectures on symplectic topology at USTC by Jun Zhang. Assume that M is a smooth manifold of dimension $2n$.

1. symplectic structures and examples

Symplctic topology and symplectic geometry basically means the same thing. The reason for this will be explained later.

Definition 1.1. A **symplectic structure** on M is a closed 2-form $\omega \in \Omega^2(M)$ which is non-degenerate, in the sense that $\omega^n = \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is a volume form. A symplectic manifold is such a pair (M, ω).

Remark 1.2. There are several equivalent conditions to describe non-degeneracy of ω. For example, $TM \to T^*M, X \mapsto \iota_X\omega$ is a bundle isomorphism.

Example 1.3. There are lots of examples for symplectic manifolds.

(1) Area forms on a smooth surface automatically satisfy the condition for symplectic structure. In particular, closed surfaces $(\Sigma_g, \omega_{\text{area}})$ are symplectic manifolds.

(2) On $\mathbb{R}^{2n}$ with coordinate $(x^1, y^1, \cdots, x^n, y^n)$, there exists a standard symplectic structure

$$\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i.$$

Alternatively, it can be expressed as $d\lambda$, where

$$\lambda = \frac{1}{2}\sum_{i=1}^n (x^i dy^i - y^i dx^i),$$

hence ω_0 is also an exact form.

(3) Let Q be a smooth manifold. Its cotangent bundle T^*Q admits a canonical symplectic structure $\omega = d\lambda$. Under local coordinate $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ the 1-form is expressed as
$$\lambda = -p_i dq^i,$$
then $\omega = dq^i \wedge dp_i$. To show that λ is well-defined, notice that
$$\lambda_{(q,p)}(Y) = p_q((d\pi)_{(q,p)}(Y)), Y \in T_{(q,p)}T^*Q.$$
For some reasons in notation, it is a good habit to keep writting the base manifold as Q.

(4) Given two symplectic manifolds (M, ω_M) and (N, ω_N), we can cook up a new symplectic manifold by taking products. Consider $\omega_{\text{prod}} = \pi_M^*\omega_M + \pi_N^*\omega_N \in \Omega^2(M \times N)$. Since pullback commutes with exterior derivative, ω_{prod} is also closed. By computation,
$$\omega_{\text{prod}}^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} \pi_M^*\omega_M^k \pi_N^*\omega_N^{m+n-k} = \binom{m+n}{m} \pi_M^*\omega^m \pi_N^*\omega_N^n.$$
As the nonzero scaling of a symplectic structure is still symplectic, $t\pi_M^*\omega_M + s\pi_N^*\omega_N, ts \neq 0$ are all symplectic structures on $M \times N$. In this way we get plenty of different symplectic structures on product manifolds. For example, $\mathbb{T}^{2n}, \mathbb{S}^2 \times \mathbb{S}^2$ are symplectic manifolds.

(5) The complex projective spaces $\mathbb{CP}^n$ admit a special class of symplectic structures ω_{FS}, called Fubini–Study forms. It can be described as follows. Let $\mathbb{CP}^n = \bigcup_{i=0}^{n} U_i$ be the standard open cover and
$$\begin{aligned} \varphi_i \colon U_i &\longrightarrow \mathbb{C}^n \\ [z_0, \cdots, z_n] &\longmapsto \left(\frac{z_0}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_n}{z_i}\right). \end{aligned}$$
Under φ_i we define
$$\omega_{\text{FS}}|_{U_i} = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_i}\right|^2\right)$$
A straightforward computation yields
$$\partial\bar{\partial}\log\left(\sum_{i=1}^{n}|w_i|^2 + 1\right) = \frac{1}{(1+\sum|w_i|^2)^2}\sum h_{ij}dw_i \wedge d\bar{w}_j$$
with $h_{ij} = (1 + \sum|w_j|^2)\delta_{ij} - \bar{w}_i w_j$.

ω_{FS} are though to be special because they are Kähler forms, i.e., they are induced by the complex structures. Also note that $\mathbb{CP}^1 = \mathbb{S}^2$ has volume 1 under ω_{FS} while its area is 2π under ω_{area}.

Exercise 1.1. Given a smooth manifold Q and a diffeomorphism $f \colon Q \to Q$, prove that f lifts to a symplectomorphism $\varphi_f \colon T^*Q \to T^*Q$ such that $(\varphi_f)^*\lambda_{\text{can}} = \lambda_{\text{can}}$, where λ_{can} is the canonical primitive of the canonical symplectic structure $d\lambda_{\text{can}}$ on the cotangent bundle T^*Q.

Proof. Define $\varphi_f \colon T^*Q \to T^*Q, \alpha \mapsto f^*\alpha$. For every $Y \in T_{(q,p)}T^*Q$, denote $f = (q,p) = (q', p')$ and we have
$$(\varphi_f^*\lambda)_{(q,p)}(Y) = \lambda_{(q',p')}(df_{(q,p)}Y) = p'_{q'}((d\pi)_{(q',p')} \circ (df)_{(q,p)}Y) = p_q((d\pi)_{(q,p)}Y).$$
$\square$

By definition, if $U \subset (M, \omega)$ is a non-empty open subset, then $(U, \omega|_U)$ is also a symplectic manifold. A fundamental theorem by Darboux shows that, symplectic manifolds of the same dimensional locally look the same, so there is no local geometry. This a significant difference from differential geometry, and that is why the lecture is named symplectic topology.

Theorem 1.4 (Darboux)**.** Let (M, ω) be a symplectic manifold and $p \in M$. Then p admits a neighborhood U and a diffeomorphism $\varphi \colon U \to \mathbb{R}^{2n}$ such that $\varphi^*\omega_0 = \omega$.

Exercise 1.2. Let (M, ω) be a symplectic manifold. Prove that for any $p \in M$ and $0 \leqslant k \leqslant n$, the map $\bigwedge^{n-k} T_p^*M \to \bigwedge^{n+k} T_p^*M, \alpha \mapsto \alpha \wedge \omega_p^k$ is an isomorphism.

Proof. We induct on k. By Darboux theorem, we can choose coordinate x^i, y^i near p such that

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

Since $\dim \bigwedge^{n-k} T_p^* M = \dim \bigwedge^{n+k} T_p^* M = \binom{n}{n-k}$, it suffices to show that the map is either injective or surjective.

When $k = 0$, there is nothing to prove. Consider $k = 1$ and notice that $\left\{dx^I \wedge dy^J \mid |I| + |J| = n+1\right\}$ form a basis of $\bigwedge^{n-k} T_p^* M$. For every basis $dx^I \wedge dy^J$, we can find some i_0 such that $i_0 \in I \cap J$. Therefore, $dx^I \wedge dy^J = dx^{I'} \wedge dy^{J'} \wedge \omega_p$ where $I' = I \setminus i_0, J' = J \setminus \{i_0\}$.

Suppose the conclusion holds for $k = m-1$ and now consider $k = m$. Assume $\alpha \wedge \omega_p^m = 0$. Then $(\alpha \wedge \omega_p) \wedge \omega_p^{m-1} = 0$. By our induction hypothesis, $\alpha \wedge \omega_p = 0$ implies $\alpha = 0$. □

One may wonder is there any obstruction for the existence for symplectic structures on an even-dimensional manifold.

Proposition 1.5. If M is closed, i.e., compact and without boundary, then $[\omega]^k$ is nonzero class in $H^{2k}(M; \mathbb{R})$ for every $1 \leqslant k \leqslant n$.

Proof. By contradiction, say $[\omega]^k = 0$ for some k. Since $[\omega]^k = [\omega^k]$, $\omega^k = d\alpha$ for some $\alpha \in \Omega^{2k-1}(M)$. By Stokes' formula,

$$\int_M \omega^n = \int_M \omega^k \wedge \omega^{n-k} = \int_M d\alpha \wedge \omega^{n-k} = \int_M d(\alpha \wedge \omega^{n-k}) = 0.$$

This is a contradiction. □

Corollary 1.6. Among $\mathbb{S}^{2n}, \mathbb{S}^{2p} \times \mathbb{S}^{2q}$, symplectic structures exist if and only if $n = 1$ and $p, q = 1$.

2. SYMPLECTOMORPHISMS

Let (M, ω) be a symplectic manifold.

Definition 2.1. $\varphi \in \mathrm{Diff}(M)$ is called a **symplectomorphism** if it preserves the symplectic structure, i.e., $\varphi^* \omega = \omega$. Symplectomorphisms form a subgroup of $\mathrm{Diff}(M)$, denoted by $\mathrm{Symp}(M, \omega)$ with $\mathrm{Symp}_0(M, \omega)$ representing the connected component of id.

$\mathrm{Symp}(M, \omega)$ is a huge and mysterious topological group, much of whose nature remains unknown for mathematicians. Note that $\varphi^*(\omega^n) = (\varphi^* \omega)^n = \omega^n$, so at least $\mathrm{Symp}(M, \omega)$ would be a subgroup of $\mathrm{Diff}(M)_{\mathrm{Vol}}$, the diffeomorphisms preserving volumes. In general $\mathrm{Symp}(M, \omega) \subsetneq \mathrm{Diff}(M)_{\mathrm{Vol}}$, which seems very reasonable but is not easy to proof. This result is due to Gromov with his non-squeezing theorem. Roughly speaking, although symplectic geometry has no local behaviour, it possesses certain rigidity in the global sense.

Once we focus on linear isomorphism on $(\mathbb{R}^{2n}, \omega_0)$, things get much easier.

Definition 2.2. The **symplectic group** is $\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^* \omega_0 = \omega_0\}$.

Symplectic groups are Lie groups with nice property.

Theorem 2.3. $\mathrm{Sp}(2n)$ deform retracts to $\mathrm{U}(n)$. In particular, it has fundamental group $\mathbb{Z}$.

The following result is essential in defining another important invariant for symplectic manifolds.

Proposition 2.4. The first Chern class for (M, ω), $\mathrm{c}_1(TM) \in H^2(M; \mathbb{Z})$ is well-defined.

Proof. Since $\mathrm{Sp}(2n)$ deform retracts to $\mathrm{U}(n)$, TM is equivalent to a complex vector bundle over M. □

There are two natural questions about symplectic topology:
(1) **How to construct a symplectomorphism?**
(2) **How to tell if a map is a symplectomorphism?**

To answer these questions, we need to introduce two elements in symplectic geometry respectively:
(1) Hamiltonian dynamics.
(2) Lagrangian submanifolds.

3. COMPLEXITY OF SYMPLECTIC STRUCTURES

Let (M, ω) be a symplectic manifold.

In algebraic topology, characteristic classes measure the complexity of vector bundles over a space. To describe the complexity of a symplectic strucuture, we can also make use of some special tools and invariants.

Proposition 3.1. Every symplectic manifold (M, ω) admits an almost complex structure J such that
(1) $\omega_p(v, Jv) > 0, v \in T_pM$.
(2) $J^*\omega = \omega$, i.e., $\omega_p(Ju, Jv) = \omega_p(u, v), u, v \in T_pM$.
As a consequence, $g_J(u, v) = \omega(u, Jv)$ defines a Riemannian metric on M. Also note that

$$g_J(Ju, Jv) = \omega(Ju, -v) = \omega(v, Ju) = g_J(v, u).$$

Although it may not make sense at the moment, this show that symplectic manifolds can be quantified. In the remainder of this section, we will fix a symplectic manifold together with induced almost complex structure (M, ω, J).

Definition 3.2. A ***J*-holomorphic curve** in (M, ω, J) is a smooth map $u\colon (\Sigma, j) \to (M, \omega, J)$ such that $J \circ u_* = u_* \circ j$, where Σ is a Riemann surface and j is its complex structure.

J-holomorphic curve is an important tool in symplectic geometry, developed by Gromov in 1980's. In this lecture we will be only interested in cases where $\Sigma = \mathbb{S}^2$ or $\mathbb{S}^2 \setminus \{\text{finitely many pts}\}$.

Example 3.3. A map $u\colon (\mathbb{C}, \sqrt{-1}) \to (\mathbb{C}, \sqrt{-1})$ is a J-holomorphic curve if and only if u satisfies the Cauchy–Riemann equation, i.e., u is holomorphic.

Proposition 3.4. For any smooth curve $u\colon (\Sigma, j) \to (M, \omega, J)$, let

$$E(u) = \int_\Sigma u^*\omega.$$

We have $E(u) \leqslant \mathrm{Area}(u)$, with equality holds if u is a J-holomorphic curve.

Proof. Fix a local orthonormal frame $\{e_1, e_2\}$ on $T\Sigma$ such that $g_J(u_*e_1, u_*e_2) = 0$. Then

$$(u^*\omega)(e_1, e_2) = \omega(u_*e_1, u_*e_2) = g_J(Ju_*e_1, u_*e_2) = (u^*g_J)(je_1, e_2)$$

By Cauchy–Schwarz inequaltiy, we have

$$g_J(Ju_*e_1, u_*e_2) \leqslant \sqrt{g_J(Ju_*e_1, Ju_*e_1)g_J(u_*e_2, u_*e_2)} = \sqrt{(u^*g_J)(e_1, e_1)(u^*g_J)(e_2, e_2)}$$

with equality holds if and only if $Ju_*e_1 = u_*e_2$.

If u is J-holomorphic curve, we can pick $e_2 = je_1$, as g $g_J(u_*e_1, u_*je_1) = -\omega(u_*e_1, u_*e_1) = 0$. □

Corollary 3.5. A J-holomorphic curve is a minimal surface with respect to g_J.

Note that if $u\colon (\mathbb{S}^2, j) \to (M, \omega, J)$ is a J-holomorphic curve, $E(u)$ can be written as $[\omega] \cap [u]$, where $[u] \in \pi_2(M)$. In symplectic geometry, we consider the image of Hurewicz map

$$H_2^S(M) = \mathrm{im}(h\colon \pi_2(M) \to H_2(M; \mathbb{Z})),$$

which is the collection of homology classes represented by the image of $\mathbb{S}^2$.

Example 3.6. $H_2^S(\mathbb{T}^2) = 0$ because $\mathbb{T}^2$ is not homeomorphic to $\mathbb{S}^2$.
$H_2^S(\mathbb{S}^2) = \mathbb{Z}, H_2^S(\mathbb{S}^2 \times \mathbb{S}^2) = \mathbb{Z} \times \mathbb{Z}$.
$H_2^S(\mathbb{CP}^n) = \mathbb{Z}$.

Two functions are defined on $H_2^S(M)$:

$$\begin{aligned} [\omega]\colon H_2^S(M) &\longrightarrow \mathbb{R} \\ A &\longmapsto [\omega] \cap A = \int_{\mathbb{S}^2} A^*\omega, \\ \mathrm{c}_1(TM)\colon H_2^S(M) &\longrightarrow \mathbb{Z} \\ A &\longmapsto \mathrm{c}_1 \cap A = \int_{\mathbb{S}^2} A^* \mathrm{c}_1(TM). \end{aligned}$$

The complexity of (M, ω) comes from the relation of $[\omega]$ and $\mathrm{c}_1(TM)$.

Definition 3.7. In general, there are three jumps in the complexity of a symplectic manifold.

- $[\omega] = c_1(TM) \equiv 0$ on $H_2^S(M)$. Such symplectic manifolds are said to be **symplectically aspherical**. In some sense, symplectically aspherical manifolds are easy to study.
  Tori $\mathbb{T}^{2n}$ and closed surfaces $\Sigma_g(g > 0)$ are examples for symplectically aspherical manifolds.
- $[\omega](A) = \kappa\, c_1(TM)(A), A \in H_2^S(TM)$ for some $\kappa > 0$. Such symplectic manifolds are said to be **monotone**. By definition, a symplectically aspherical manifold is automatically monotone.
  $(\mathbb{CP}^1, \omega_{\mathrm{FS}})$ is our first example that is monotone but not symplectically aspherical. Let A be the generator of $H_2^S(\mathbb{CP}^1) = H_2(\mathbb{CP}^1;\mathbb{Z})$. Then $[\omega_{\mathrm{FS}}](A) = 1$ and $c_1(A) = 2$, so $\kappa = \frac{1}{2}$. Generally, for every n, $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is monotone with $\kappa = \frac{1}{n+1}$.
- The last jump is a generalization of notion of monotone. We say (M, ω) is **semi-positive** (or **weakly monotone**) if it satisfies either one of the three conditions:
  (1) $[\omega](A) = \kappa\, c_1(TM)(A), A \in H_2^S(M)$ for some $\kappa \geqslant 0$.
  (2) $c_1(TM) \equiv 0$ on $H_2^S(M)$. (This condition is sometimes called Calabi–Yau).
  (3) The positive generator of $c_1(TM)(H_2^S(M)) \leqslant \mathbb{Z}$ is at least $n-2$.
  The three conditions are not equivalent to each other but they can overlap on some manifolds. By definition, when $n \leqslant 3$, i.e., $\dim M \leqslant 6$, (M, ω) is automatically semi-positive.

Note that it is possible that a symplectic manifold is not semi-positive, but such examples are quite complicated and are hard to display at this moment.

Remark 3.8. Semi-positive manifolds are collections of objects people could deal with in Floer theory in 1990's, which was originally developed for monotone manifolds. Roughly speaking, almost every property holding for monotone case holds for semi-positive case. On the contrary, sometimes it is difficult to generalize results in aspherical case to monotone case.

Here comes an observation. If (M, ω) is semi-positive, then for any $A \in H_2^S(M)$ with $[\omega](A) > 0$, we must have $c_1(TM)(A) \geqslant 0$ or $c_1(TM)(A) \leqslant 2 - n$. This turns out to be an if and only if!

Exercise 3.1. If every $A \in H_2^S(M)$ with $[\omega](A) > 0$ satisfies $c_1(TM)(A) \geqslant 0$ or $c_1(TM)(A) \leqslant 2 - n$, then (M, ω) is semi-positive.

Proof. If $n \leqslant 3$, there is nothing to prove, so assume $n > 3$. It suffices to show that if condition (2) and (3) are violated, then (1) holds.

In this case, $c_1(TM)(H_2^S(M)) = N\mathbb{Z}$ with $1 \leqslant N \leqslant n-3$. Take $A \in H_2^S(M)$ such that $c_1(TM)(A) = N$, so $[\omega](A) \geqslant 0$. If $B \in H_2^S(M)$ making $c_1(TM)(B) = N$, we must have $[\omega](A - B) = 0$. Indeed, otherwise say $[\omega](B - A) > 0$, then $c_1(B - A) = 0$. Taking $m \in \mathbb{Z}_{\geqslant 1}$ sufficiently large, we obtain $[\omega](m(B - A) - A) > 0$ but $(m(B - A) - A) = c_1(-A) \in (2 - n, 0)$, which is a contradiciton.

Let

$$\kappa = \frac{[\omega](A)}{c_1(TM)(A)}.$$

For any $C \in H_2^S(M)$, $c_1(TM)(A) = (1 + k)N$ and

$$[\omega](C) = [\omega](C - kA) +][\omega](kA) = (k + 1)[\omega](A) = (k + 1)\kappa\, c_1(TM)(A) = \kappa\, c_1(TM)(C).$$

$\square$

Example 3.9. $(\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\mathrm{prod}})$ is monotone. If we change the symplectic structure, we obtain an example for a semi-positive manifold that is not monotone. Note that $\dim \mathbb{S}^2 \times \mathbb{S}^2 = 4 \leqslant 6$, this manifold is semi-positive with any symplectic structure. One can check that $\omega_{\mathrm{deform}} = (1+\varepsilon)\pi_1^*\omega + \pi_2^*\omega$ is non-monotone structure for $\varepsilon > 0$.

4. Quantum cohomology and Novikov rings

Cohomology is an important topological invariant for manifolds. However, it does not carry the information of symplectic structure. In symplectic geometry, we can modify cohomology in the following way.

Definition 4.1. Let $\Gamma_\omega = \mathrm{im}([\omega]\colon H_2^S(M) \to \mathbb{R}) \leqslant \mathbb{R}$. The **quantum cohomology** $QH^*(M, \omega)$ of (M, ω) with coefficient $\mathbb{K}$ is

$$QH^*(M, \omega) = H^*(M; \mathbb{K}) \otimes_{\mathbb{K}} \Lambda^{\mathbb{K}, \Gamma_\omega},$$

where $\Lambda^{\mathbb{K}, \Gamma_\omega}$ is the Novikov ring which we will introduce later.

Definition 4.2. Let $\Gamma \leqslant \mathbb{R}$ be a subgroup. Its **Novikov ring** is the formal power series in T

$$\Lambda^{\mathbb{K},\Gamma} = \left\{ \sum_{\lambda \in \Gamma} a_\lambda T^\lambda \mid a_\lambda \in \mathbb{K}, \forall c \in \mathbb{R}, \# \{a_\lambda \neq 0 \mid \lambda < c\} < +\infty \right\}.$$

"$\forall c \in \mathbb{R}, \# \{a_\lambda \neq 0 \mid \lambda < c\} < +\infty$" is called the Novikov condition.

Remark 4.3. A faster way to express the Novikov condition is $\lambda \to +\infty$, though this notion is not very appropriate for finite sums.

Example 4.4. $\mathbb{K}[[T]] \leqslant \Lambda^{\mathbb{K},\Gamma}$. On the contrary,

$$\sum_{n=1}^{\infty} T^{\frac{1}{n}}, \sum_{n=1}^{\infty} T^{-n} \notin \Lambda^{\mathbb{K},\Gamma}.$$

As a priori, $QH^*(M, \omega)$ should be considered as a module over the Novikov ring. Delightfully, it turns out that the Novikov ring is actually a field.

Lemma 4.5. $\Lambda^{\mathbb{K},\Gamma}$ is a field.

Proof. Let x be a nonzero element. By the Novikov condition, we can write

$$x = a_\mu T^\mu + \sum_{\lambda > \mu} a_\lambda T^\lambda = a_\mu T^\mu \left(1 + \sum_{\lambda > \mu} \frac{a_\lambda}{a_\mu} T^{\lambda - \mu} \right), a_\mu \neq 0.$$

Denote

$$A = - \sum_{\lambda > \mu} \frac{a_\lambda}{a_\mu} T^{\lambda - \mu}.$$

As $\lambda = \mu$ only finitely many times, $(1 - A)^{-1} = 1 + A + \cdots + A^n + \cdots$ is well-defined and

$$x^{-1} = a_\mu^{-1} T^{-\mu} (1 - A)^{-1}.$$

□

As a consequence, quantum cohomology can be considered as a $\Lambda^{\mathbb{K},\Gamma_\omega}$-vector space, and methods in linear algebra apply.

Example 4.6. We can examine what quantum cohomology likes under the three kinds of complexity mentioned last time.

- If (M, ω) is symplectically aspherical, then $\Gamma_\omega = 0$ and $\Lambda^{\mathbb{K},\Gamma_\omega}$ is simply $\mathbb{K}$. We see that the quantum cohomology on symplectically aspherical manifolds coincides with the usual cohomology.
- If (M, ω) is monotone, $\Gamma_\omega = \kappa \, \mathrm{c}_1(TM)(H_2^S(M)) \simeq \mathbb{Z}$ is a cyclic subgroup of $\mathbb{R}$. Then $\Lambda^{\mathbb{K},\Gamma_\omega} \simeq \mathbb{Z}[T]]$ can be regarded as the field of Laurant series.
- If (M, ω) is semi-positive, condition (2) and (3) have no control on Γ_ω, so it could be that Γ_ω becomes very complicated.

Here is an explicit example showing how complicated Γ_ω could be.

Example 4.7. Consider $\mathbb{S}^2 \times \mathbb{S}^2$ with $\omega_{\text{deform}} = (1 + \varepsilon)\pi_1^* \omega + \pi_2 * \omega$. Then

$$\Gamma_\omega = \mathbb{Z} \oplus \mathbb{Z}\varepsilon.$$

If ε is irrational, then Γ_ω will be a dense subgroup of $\mathbb{R}$.

Here is another enhanced version of Novikov ring.

Exercise 4.1. Consider the series

$$\sum_{A \in H_2^S(M)} a_A t^A.$$

The question about this definition is, how to describe the Novikov condition? To resolve this, consider the short exact sequence

$$0 \longrightarrow \ker [\omega] \longrightarrow H_2^S(M) \xrightarrow{[\omega]} \Gamma_\omega \longrightarrow 0.$$

Γ_ω is free, so the exact sequence splits, which implies $H_2^S(M) = \ker[\omega] \oplus \Gamma_\omega, A \mapsto (B, \lambda)$. The formal series can be rewritten as

$$\sum_{\lambda\in\Gamma_\omega} \sum_{B\in\ker[\omega]} a_{(B,\lambda)} t^{(B,\lambda)} = \sum_{\lambda\in\Gamma_\omega} \left(\sum_{B\in\ker[\omega]} a_{(B,\lambda)} S^B \right) T^\lambda.$$

Prove that the following "Novikov condition"

$$\forall c \in \mathbb{R}, \# \left\{ a_{(B,\lambda)} \neq 0 \mid \lambda < c \right\} < +\infty,$$

implies that the coefficient $\sum\limits_{B\in\ker[\omega]} a_{(B,\lambda)} S^B$ lies in the group algebra $\mathbb{K}[\ker[\omega]]$.

Proof. For a fixed $\lambda_0 \in \Gamma_\omega$, we have

$$\# \left\{ a_{(B,\lambda_0)} \neq 0 \right\} \leqslant \# \left\{ a_{(B,\lambda)\neq 0} \mid \lambda < \lambda_0 + 1 \right\} < +\infty.$$

□

Before talking more about quantum cohomology, we first study some general aspects of modules over Novikov rings. Novikov admits an interesting structure, called the valuation.

Definition 4.8. We define a valuation ν on $\Lambda^{\mathbb{K},\Gamma}$ by $\nu(x) = \min \left\{ \lambda \in \Gamma \mid T^\lambda \text{ has nozero coefficient} \right\}$ and presribing $+\infty$ to 0.

Proposition 4.9. ν is indeed a valuation on $\Lambda^{\mathbb{K},\Gamma}$:
(1) $\nu(x) = +\infty$ if and only if $x = 0$.
(2) $\nu(xy) = \nu(x) + \nu(y)$.
(3) $\nu(tx) = \nu(x)$ if $0 \neq t \in \mathbb{K}$.
(4) $\nu(x + y) = \min(\nu(x) + \nu(y))$.

These properties follows immediately from definition.

With valuations, we can associate a new structure on V.

Definition 4.10. A filtration function on a $\Lambda^{\mathbb{K},\Gamma}$-vector space V is a funciton $l\colon V \to \mathbb{R} \cup \{-\infty\}$ such that:
(1) $l(x) = -\infty$ if and only if $x = 0$.
(2) $l(\alpha x) = \lambda(x) - \nu(\alpha), \alpha \in \Lambda^{\mathbb{K},\Gamma}$.
(3) $l(x + y) \leqslant \max((x), l(y))$.
Note that by definition if $l(x) \neq l(y)$, say $l(x) > l(y)$, then

$$l(y) < l(x) = l(x + y - y) \leqslant \max(l(x + y), l(y)),$$

which forces $l(x + y) = l(x) = \max(l(x), l(y))$.

A natural question is, what does linear algebra look like on (V, l)?

5. Vector spaces with filtration functions

6. Preparations for Gromov–Witten invariants

Gromov–Witten invariant is related to the central problem in symplectic geometry and algebraic geometry, counting curves. There are three major troubles we will encounter.

6.1. **The limit is bad.** Consider a family of J-holomorphic curves $\{u_n\}$. It's limit u_∞ may be complicated.

Example 6.1. Let $u_n\colon (\mathbb{S}^2, j) \to (\mathbb{CP}^2, J_0), [z_0 : z_1] \mapsto \left[z_0^2 : \frac{z_1^2}{n} : z_0 z_1 \right]$ be a sequence of J-holomorphic curves, where J_0 is the standard complex structure. $\operatorname{im} u_n = \left\{ [Z_0 : Z_1 : Z_2] \mid Z_0 Z_1 = \frac{Z_2^2}{n} \right\}$, so its limit image is

$$\operatorname{im}(u_\infty) = \{[Z_0 : Z_1 : Z_2] \mid Z_0 Z_1 = 0\} = A_0 \cup A_1,$$

where $A_0 = V(Z_0), A_1 = V(Z_1)$. Topologically, $\operatorname{im}(u_\infty)$ is two copies of $\mathbb{S}^2$ attached at $[0 : 0 : 1]$.

A question is, can $\operatorname{im}(u_\infty)$ be an image of a J-holomorphic curve? Intuitively, on $\mathbb{S}^2 \setminus \{[0 : 1]\}$ we can set $u_\infty([z_0 : z_1]) = [z_0^2 : 0 : z_0 z_1] = [z_0 : 0 : z_1]$. It seems reasonable to define $u_\infty([z_0 : z_1]) = [z_0 : 0 : z_1]$. However, this map only covers the component A_1 but misses A_0.

We can look at $[0:1]$ and see what happened to this point during the limit process. Surprisingly, $u_n([0:1]) = [0:\frac{1}{n}:0] = [0:1:0]$ is a fixed point and it is not the same point as $u_\infty([0:1]) = [0:0:1]$.

Exercise 6.1. Prove that there exsists a J-holomorphic curve $v_\infty\colon \mathbb{S}^2 \to \mathbb{CP}^2$ that covers component A_0 but misses A_1, and maps $[1:0]$ to $[0:0:1]$.

Proof. Consider a reparametrization of u_n defined by $v_n([z_0:z_1]) = u_n([z_0:nz_1]) = \left[z_0^2 : nz_1^2 : nz_0z_1\right] = \left[\frac{z_0^2}{n} : z_1^2 : z_0z_1\right]$. Then $v_\infty([z_0:z_1]) = [0:z_1:z_0]$ is the desired map. □

To resolve this puzzle, let's bravely consider $(v,u)\colon T \to \mathbb{CP}^2$, where T is two copies of $\mathbb{S}^2$ attached at $[0:0:1]$. In general, the limit of a family of J-holomorphic curves has the complicated domains represented as a tree, whose vertex are copies of $\mathbb{S}^2$ and edges are attached points. In Example 6.1 the domain is represented as two vertex attached by an edge.

6.2. **Multiply cover.**

Definition 6.2. Let $u\colon (\Sigma_g, j) \to (M,J)$ be a J-holomorphic curve. u is said to be **multiply covered** if there exists a J-holomorphic curve $u'\colon (\Sigma_{g'}, j') \to (M,J)$ and a non-constant holomorphic map $\varphi\colon \Sigma_g \to \Sigma_{g'}$ s.t.

$$\begin{tikzcd}
(\Sigma_g, j) \arrow[d, "\varphi"] \arrow[dr, "u"] & \\
(\Sigma_{g'}, j') \arrow[r, "u'"] & (M,J)
\end{tikzcd}$$

and $\deg\varphi > 1$. And we will call that u is a **multiple cover** of u'.

In general, we don't like multiple covers. By some topological constraint we can get rid of multiple covers.

Theorem 6.3 (Riemann–Hurwitz formula)**.** Let $\varphi\colon \Sigma_g \to \Sigma_{g'}$ be a non-constant holomorphic map. Then

$$2g-2 = (\deg\varphi)(2g'-2) + \sum_{p\in\Sigma_g}(e_p - 1),$$

where e_p is the ramification index at p.

In particular, if $\Sigma_g = \mathbb{S}^2$, the formula writes

$$-2 = (\deg\varphi)(2g'-2) + \sum_{p\in\Sigma_g}(e_p - 1).$$

Therefore $g' = 0$ and $\Sigma_{g'} = \mathbb{S}^2$. Note that in this case we cannot claim $\deg\varphi = 1$.

Example 6.4. Here is an example of symplectic structure ruling out multiple cover. Suppose (M,ω) is semi-positive with $\mathrm{c}_1(TM)(H_2^S(M)) = 4\mathbb{Z}$. If $u\colon (\mathbb{S}^2, j) \to (M,J)$ is a J-holomorphic curve such that $\mathrm{c}_1(TM)([u]) = 4$, then u is not a multiple cover.

To see this, by contradiction suppose u is a multiple cover of $u'\colon\ \mathbb{S}^2 \to (M,J)$ with φ. By definition, $[u] = (\deg\varphi)[u']$, so $4 = \mathrm{c}_1(TM)([u]) = (\deg\varphi)\,\mathrm{c}_1(TM)([u'])$. As $\deg\varphi > 1$, we have either $\deg\varphi = 2$ or 4, but either case would result in $0 < \mathrm{c}_1(TM)([u']) < \mathrm{c}_1(TM)([u])$.

Definition 6.5. A J-holomorphic curve is said to be **simple** if it's not multiply covered.

If a J-holomorphic curve u is not simple, then can always find a simple u' such that u is a multiple cover of u.

The following is a criterion for determing whether a J-holomorphic curve is simple.

Theorem 6.6. A J-holomorphic curve $u\colon (\Sigma_g, j) \to (M,J)$ is simple if and only if there exists $z \in \Sigma_g$ such that $(du)_z\colon T_z\Sigma_g \to T_{u(z)}M$ is injective and $u^{-1}(u(z)) = \{z\}$.

Altough many textbooks claim this criterion to be useful, Jun said he never uses it.

6.3. **Reparametrization.** If $u\colon (\mathbb{S}^2, j) \to (M, J)$ is a J-holomorphic curve and $\varphi \in \mathrm{Aut}(\mathbb{S}^2, j)$, then geometrically $u' = u \circ \varphi^{-1}$ should be considered to be the same curve.

Recall that

$$\mathrm{Aut}(\mathbb{S}^2, j) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid ad - bc \neq 0 \right\} = \mathrm{PGL}(2, \mathbb{C})$$

is a connected Lie group with real dimension 6. Let $\mathcal{C}(M, J)$ be the collection of J-holomorphic curves $u\colon (\mathbb{S}^2, j) \to (M, J)$. $\mathrm{PGL}(2, \mathbb{C})$ acts on $\mathcal{C}(M, J)$ via $\varphi \cdot u = u \circ \varphi^{-1}$. Counting J-holomorphic curves should be essentially on the moduli space

$$\mathcal{C}(M, J)/\mathrm{PGL}(2, \mathbb{C}).$$

An enhanced version of this is considering

$$(u, (z_1, \cdots, z_k)) \in \mathcal{C}(M, J) \times \left(\prod^k \mathbb{S}^2 \setminus \Delta \right),$$

where Δ is the fat diagonal, i.e., $(z_1, \cdots, z_k)$ are k distinct points. Similarly $\mathrm{PGL}(2, \mathbb{C})$ acts on this set via $\varphi \cdot (u, (z_1, \cdots, z_k)) = (u \circ \varphi^{-1}, (\varphi(z_1), \cdots, \varphi(z_k)))$ and we can consider the moduli space

$$\mathcal{C}(M, J) \times \left(\prod^k \mathbb{S}^2 \setminus \Delta \right) /\mathrm{PGL}(2, \mathbb{C}).$$

Remark 6.7. $(z_1, \cdots, z_k)$ is to formulate constraints on the J-holomorphic curves.

Definition 6.8. Let (M, J) be an almost complex manifold. A **stable map** is given by the following data.

(1) Its domain T, which is a tree-type configuration containing finitely many $(\mathbb{S}^2, j)$ attached to each other in the same way. The vertex $v(T)$ are copies of $(\mathbb{S}^2, j)$ and the edges $e(T)$ are nodal points. The attached points are usually called nodal points and the configuration T is called the nodal curve.
(2) The restriction of u on each $(\mathbb{S}^2, j)$ is J-holomorphic.
(3) If the restriction of u on $\mathbb{S}^2$-component in $v(T)$ is constant, then there exists at least three nodal points on it.

We say u is multiply covered if at least one $\mathbb{S}^2$-component is multiply covered.

Remark 6.9. Condition (3) is extracted from analysis on J-holomorphic curves. More details can be found at Theorem 5.5 at *J-holomorphic Curves in Symplectic Topology* by Jerome Dominique Wettstein.

7. MODULI SPACE OF J-HOLOMORPHIC CURVES

Let (M, ω) be a symplectic manifold and $A \in H_2^S(M)$. Take a $J \in \mathcal{J}(M, \omega)$.

Definition 7.1. The **moduli space** of J-holomorphic curves $\mathcal{M}^*(A, J)$, or $\mathcal{M}^*(M, A, J)$ is defined as

$$\left\{ \text{simple } J\text{-holomorphic curves } u\colon (\mathbb{S}^2, j) \to (M, J) \mid [u] = A \right\} /\mathrm{PSL}(2, \mathbb{C}).$$

Similarly we can define

$$\mathcal{M}_k^*(A, J) = \left(\mathcal{M}^*(A, J) \times (\mathbb{S}^2)^k \setminus \Delta \right) /\mathrm{PSL}(2, \mathbb{C}),$$

whose elements are $[(u, z_1, \cdots, z_k)]$ with simple u such that $[u] = A$ and k distinct points on $\mathbb{S}^2$. $\overline{\mathcal{M}_k(A, J)}$ is defined as

$$\left\{ \text{stable maps } u\colon (T, j) \to (M, J) \mid [u] = \sum_{\mathbb{S}^2 \in v(T)} [u|_{\mathbb{S}^2}] = A \right\}.$$

In definitions above, we can vary A, J, k to obtain different moduli spaces. For (A, J, k) fixed, their relations are as follows:

$$\begin{array}{ccc} \mathcal{M}_k^*(A, J) & \subset & \mathcal{M}_k(A, J) \\ \cap & & \cap \\ \overline{\mathcal{M}_k^*(A, J)} & \subset & \overline{\mathcal{M}_k(A, J)} \end{array}$$

For a generic J, $\mathcal{M}_k^*(A,J)$ would have a manifold structure with $\overline{\mathcal{M}_k(A,J)}$ being its compactification, due to the famous Gromov compactness theorem.

Theorem 7.2 (Gromov)**.** Let $A \in H_2^S(M)$. Then there exists a topology on $\mathcal{M}_k^*(A,J)$, called the Gromov sequential topology, such that its compactification is $\overline{\mathcal{M}_k(A,J)}$ and $\mathcal{M}_k^*(A,J)$ is an open dense subset of $\overline{\mathcal{M}_k(A,J)}$. Moreover, for a generic J, $\mathcal{M}_k^*(A,J)$ is a manifold of real dimension $\dim M + 2\,\mathrm{c}_1(TM)(A) + 2k - 6$.

Remark 7.3. Similar result holds for $\mathcal{M}_{k,T}^*(A,J)$ and

$$\dim \mathcal{M}_{k,T}^*(A,J) = \dim M + 2\,\mathrm{c}_1(TM)(A) + 2k - 6 - 2\#e(T).$$

Moreover, we have the decomposition

$$\overline{\mathcal{M}_k(A,J)} = \bigcup_T \mathcal{M}_{k,T}^*(A,J),$$

where T runs over all the possible tree domains in the limit process.

Example 7.4. Let's consider the toy version when $(M,\omega,J) = (\mathbb{CP}^2, \omega_{\mathrm{FS}}, J_0)$ where J_0 is the complex structure.

Let $A = 2L$, where $L \in H_2(\mathbb{CP}^2;\mathbb{Z})$ is the line class. We want to study the moduli space $\overline{\mathcal{M}_0(2L,J_0)}$.

According to Theorem 7.2, its real dimension is

$$\dim_{\mathbb{R}} \mathbb{CP}^2 + 2\,\mathrm{c}_1(T\mathbb{CP}^2)(2L) + 0 - 6 = 10.$$

Actually, a curve representing class $[2L]$ would be a conic in $\mathbb{CP}^2$ so $\overline{\mathcal{M}_0(2L,J_0)}$ can be considered as collections of conics. As conics are solutions to equations

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0,$$

intuitively, the space has complex dimension 5, which is consistent with Theorem 7.2.

Let's further consider the moduli space for deg 2 multiply covered curves representing L. By Riemann-Hurwitz formula,

$$-2 = 2 \times (-2) + \sum (e_p - 1),$$

thus such a curve have exactly two ramified points (they may coincide). Suppose u is a multiply cover of v. As $\operatorname{im} v$ acts as a line in $\mathbb{CP}^2$, which can be parametrized by two complex numbers, the moduli space has freedom in four complex parameters and is of real dimension 8.

Let's further consider moduli space of curves with tree domain such that $e(T) = 1, v(T) = 2$. It has complex dimension 4 and therefore real dimension 8.

Example 7.4 provides us with some basic intuition about moduli space of curves. $\overline{\mathcal{M}_k(A,J)}$ can be imagined as a manifold with corners. $\mathcal{M}_k^*(A,J)$ is a 0-codimensional open statrum while degenerating curves occupy lower dimensional parts.

8. Back to quantum cohomology

Let (M,ω) be a symplectic manifold. $QM^*(M,\omega)$ is viewed as a finite dimensional vector space over $\Lambda^{\mathbb{K},\Gamma_\omega}$. In algebraic topology, $H^*(M;\mathbb{K})$ is a graded ring under the cup product $\cup$. Let $e_1, \cdots, e_n$ be a $\mathbb{K}$-basis for $H^*(M;\mathbb{K})$ and

$$e_i \cup e_j = c_{ij}^k e_k$$

with c_{ij}^k. calle structure constants for $\cup$. On $QM^*(M,\omega)$, we have a similar operation $*$, which satisfies

$$e_i * e_j = e_i \cup e_j + \text{ deformed part }.$$

$*$ is defined based on Gromov–Witten invariant, so Zhang cannot explain this operator for now. However, we can see what extra structure can be defined on quantum coholomogy with $*$ operator.

Example 8.1. Let $(M,\omega) = (\mathbb{CP}^n, \omega_{\mathrm{FS}})$. According to algebraic topology,

$$H^*(\mathbb{CP};\mathbb{K}) = \mathbb{K}[S]/(S^{n+1}),$$

where $S \in H^2(\mathbb{CP}^n;\mathbb{K})$ is the Poincaré dual of a complex hyperplane. As for quantum cohomology,

$$QH^*(\mathbb{CP}^n, \omega_{\mathrm{FS}}) = \mathbb{K}[S,T]/(S^{n+1} - T).$$

Here $S^{n+1} = S \cup \cdots \cup S +$ deformed parts, so T represents the extra terms.

Remark 8.2. One advantage of working with quantum cohomology is that many elements become invertible. For example, in $QH^*(\mathbb{CP}^n, \omega_{\text{FS}})$, $S^{-1} = T^{-1}S^n$. Also, $QH^*(\mathbb{CP}^n, \omega_{\text{FS}})$ is an integral domain, for $T - S^{n+1}$ is irreducible.

Example 8.3. Consider $(\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\text{prod}})$.

$$H^*(\mathbb{S}^2 \times \mathbb{S}^2; \mathbb{K}) = \mathbb{K} \oplus \mathbb{K}a \oplus \mathbb{K}b \oplus \mathbb{K}a \cup b,$$

where $a = \text{PD}(\mathbb{S}^2 \times \text{pt}), b \in \text{PD}(\text{pt} \times \mathbb{S}^2)$. If we consider quantum cohomology, we have $a*a = b*b = T$, which already gives us new information. Moreover, recall the another version of Novikov rings. In fact

$$a * a = T^{[\text{pt} \times \mathbb{S}^2]}, b * b = T^{[\mathbb{S}^2 \times \text{pt}]},$$

so we can further distinguish the two elements. This shows another advantage of working with quantum cohomology.

Further assume that M is compact. Then there is a pairing structure

$$\langle x, y \rangle = \int_M x \cup y = x(\text{PD}(y))$$

for $x, y \in H^*(M; \mathbb{K})$ with $\deg x + \deg y = \dim M$. This can be extended to $QH^*(M, \omega)$ by taking

$$\langle \lambda x, \mu y \rangle = \lambda\mu \int_M x \cup y = x(\text{PD}(y)) \in \mathbb{K}, \lambda, \mu \in \mathbb{K}.$$

The following is somewhat the "final" theorem for quantum cohomology.

Theorem 8.4. The triple $(QM^*(M, \omega), *, \langle -, - \rangle)$ is an associative Frobenius algebra, i.e., $\langle -, - \rangle$ is non-degenerate bilinear form and $\langle a * b, c \rangle = \langle a, b * c \rangle$.

Remark 8.5. Just like $a \cup b = (-1)^{\deg a \deg b} b \cup a$, we also have $a * b = (-1)^{\deg a \deg b} b * a$. Here we must explain what's the degree on $QH^*(M, \omega)$. Consider the monomial xT^A with $x \in H^2(M; \mathbb{K}), A \in H_2^S(M)$ and we have $\deg xT^A = \deg x + 2\,\mathrm{c}_1(TM)(A)$.

There are some interesting properties for semi-simple quantum cohomology.

Proposition 8.6 (Entov–Polterovich, Berkovich)**.** If $QH^*(M, \omega)$ is semi-simple, then there exists $c > 0$ such that for any x in a field factor of $QH^*(M, \omega)$, we have

$$l(x) + l(-x) \geqslant -c.$$

Remark 8.7. This inequality has strong geometric meaning. It serves as an important ingredient in Shelukhin's proof of Hofer–Zehnder conjecture.

9. Gromov–Witten invariant

Let (M, ω) be a symplectic manifold and $A \in H_2^S(M)$. Take a generic $J \in \mathcal{J}(M, \omega)$.

Definition 9.1. For $[(u, z_1, \cdots, z_k)] \in \overline{\mathcal{M}_k(A, J)}$, define the map

$$\text{ev}([(u, z_1, \cdots, z_k)]) = (u(z_1), \cdots u(z_k)) \in M^k.$$

Similarly we define $\text{ev}_i([(u, z_1, \cdots, z_k)]) = u(z_i)$.

Given cycles $X_1, \cdots, X_k \in H_*(M; \mathbb{K})$, we want to conut $u \in \overline{\mathcal{M}_k(A, J)}$ such that $[\operatorname{im} u]$ intersects $X_1, \cdots, X_k$ at the same time.

Example 9.2. Let $(M, \omega) = (\mathbb{CP}^2, \omega_{\text{FS}})$ and consider a line class $A = L$. Let X_1, X_2 be two point classes in $H_0(\mathbb{CP}^2; \mathbb{Z})$. Then such a u is a curve passing through X_1, X_2.

Since the curves may degenerate and position of homology classes vary, this question is a little bit complicated. Therefore we introduce two improvements to make our lives easier:

(1) We only work on $\mathcal{M}_k^*(A, J)$.
(2) $X_1, \cdots, X_k$ are in a generic position. That is, $X_1, \cdots X_k$ are represented by submanifolds and X_i, X_j pairwise intersect transversally.

Example 9.3. On $(\mathbb{CP}^2, \omega_{\text{FS}})$, consider point classes $X_i \in H_0(M; \mathbb{Z})$ in a generic position. Let $A = L$ be a line class. By intuition, there is exactly one curve representing L and passing through X_1, X_2. Similarly, with some computation one sees that there is exactly one curve representing $2L$ (i.e. it is a conic) and passing through $X_1, \cdots, X_5$.

Definition 9.4. We define the **Gromov–Witten invariant** as follows. For k cohomology classes $a_1, \cdots, a_k \in H^*(M;\mathbb{K})$ such that $\deg a_1 + \cdots + \deg a_k = \dim \mathcal{M}_k^*(A, J)$, their Gromov–Witten invariant is

$$\mathrm{GW}_{A,k}^M(a_1, \cdots, a_k) = \int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}_1^*(a_1) \cup \cdots \cup \mathrm{ev}_k^*(a_k).$$

Let's see what the formula above means. Suppose $a_i = \mathrm{PD}(X_i)$. Then

$$\int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}_1^*(a_1) \cup \cdots \cup \mathrm{ev}_k^*(a_k) = \int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}^*(\pi_1^*(a_1) \cup \cdots \cup \pi_k^*(a_k))$$
$$= \pi_1^*(a_1) \cup \cdots \cup \pi_k^*(a_k)(\mathrm{ev}_*([\mathcal{M}_k^*(A,J)])) = \mathrm{PD}(\pi_1^*(X_1) \cap \cdots \cap \pi_k^*(X_k))(\mathrm{ev}_*([\mathcal{M}_k^*(A,J)])).$$

The only issue in the identity is that we don't know if the fundamental class $[\mathcal{M}_k^*(A, J)]$ exists. Fortunately we don't need to worry about that if we only work with semi-positive symplectic manifolds. This issue, which relates to the foundations of symplectic geometry, is a fundamental question.

Example 9.5. Consider $(\mathbb{CP}^2, \omega_{\mathrm{FS}}, J_0)$ and let L be a line class. Set $A = dL$ and consider $\mathrm{GW}_{A,3d-1}^{\mathbb{CP}^2}$. What is the possible choice for $a_1, \cdots, a_{3d-1} \in H^*(\mathbb{CP}^2;\mathbb{Z})$ so that we can consider its Gromov–Witten invariant? By definition,

$$\sum_{i=1}^{3d-1} \deg a_i = 4 + 2\,\mathrm{c}_1(T\mathbb{CP}^2)(dL) + 2(3d-1) - 6 = 4(3d-1),$$

so $\deg a_i = 4$ and a_i is a Poincaré dual of a point. Therefore $\mathrm{GW}_{A,3d-1}^{\mathbb{CP}^2}$ counts degree d curves in $\mathbb{CP}^2$ passing through $3d-1$ points. One may ask what this number is. This famous number is denoted by N_d and we have seen that $N_1 = N_2 = 1, N_3 = 12$. We will see how to compute it later.

10. COMPUTING GROMOV–WITTEN INVARIANT

Today we consider the Gromov–Witten invariant under some special cases.

Proposition 10.1.

$$\mathrm{GW}_{0,k}^M(a_1, \cdots, a_k) = \begin{cases} \int_M a_1 \cup a_2 \cup a_3 & k = 3, \\ 0 & k \neq 3. \end{cases}$$

Proof. We have

$$\sum_{i=1}^{k} \deg a_i = \dim M + 2\,\mathrm{c}_1(TM)(A) + 2k - 6 = \dim M + 2k - 6.$$

Since any curve u has $[u] = 0$, u is a constant map. Let $N_i = \mathrm{PD}(a_i)$. Then $N_1 \cap \cdots \cap N_k \neq \varnothing$, and

$$\dim M \geqslant \mathrm{codim}(\bigcap_{i=1}^k N_i) = \sum_{i=1}^k \mathrm{codim}\, N_i = \sum_{i=1}^k \deg a_i.$$

Therefore $k \leqslant 3$. Moreover, as we count stable maps, there is at least 3 marked points so $k \geqslant 3$. The only possibility is $k = 3$ and

$$\mathrm{GW}_{0,3}^M = \int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}_1^*(a_1) \cup \mathrm{ev}_2^*(a_2) \cup \mathrm{ev}_3^*(a_3) = \int_M a_1 \cup a_2 \cup a_3.$$

□

The second thing to mention is that if we arrange, index then the Gromov–Witten invariant may differ by a sign.

Proposition 10.2. We have

$$\mathrm{GW}_{A,k}^M(a_1, \cdots, a_{k-1}, 1) = 0$$

if $(A, k) \neq (0, 3)$.

Proof. $1 \in H^0(M;\mathbb{K})$ has Poincaré dual $[M]$, so geometrically

$$\mathrm{GW}^M_{A,k}(a_1,\cdots,a_{k-1},1)=\mathrm{GW}^M_{A,k-1}(a_1,\cdots,a_{k-1}).$$

On the other hand,

$$\sum_{i=1}^{k-1}\deg a_i=\dim M+2\,\mathrm{c}_1(TM)(A)+2k-6.$$

Hence the identity cannot hold unless they are all zero. □

Remark 10.3. $\mathrm{GW}^M_{A,2}$ could be nonzero.

Proposition 10.4. If $\deg a_1=2$, then

$$\mathrm{GW}^M_{A,k}(a_1,\cdots,a_k)=\mathrm{GW}^M_{A,k-1}(a_2,\cdots,a_k)\cdot\int_A a_1.$$

Exercise 10.1. Compute the **Gromov–Witten potential** of $(\mathbb{CP}^1,\omega_{\mathrm{FS}})$ with respect to the standard complex structure J_0. Please provide all the details.

Definition 10.5. Suppose $m+1=\dim H^*(M;\mathbb{Z})$ with basis $1,e_1,\cdots,e_d$. The Gromov–Witten potential of (M,ω) is defined as the exponential generating function

$$\Phi(t_0,\cdots,t_m)=\sum_{A\in H_2^S(M)}\left(\sum_{k\geqslant 0}\frac{1}{k!}\mathrm{GW}^M_{A,k}\left(\sum_{i=1}^m t_ie_i,\cdots,\sum_{i=1}^m t_ie_i\right)\right)e^A.$$

By definition, the key to compute Gromov–Witten potential lies in calculating the term

$$\frac{1}{k!}\mathrm{GW}^M_{A,k}\left(\sum_{i=1}^m t_ie_i,\cdots,\sum_{i=1}^m t_ie_i\right),$$

which by the multilinear property can be expressed as

$$\frac{1}{k!}\sum_{i_1\leqslant\cdots\leqslant i_k}c(i)(-1)^{s(i)}\mathrm{GW}^M_{A,k}(e_{i_1},\cdots,e_{i_k})t_{i_1}\cdots t_{i_k},$$

where $c(i),s(i)$ are to be determined. However, this expression turns out not convenient for calculations. We can write the term in another way:

$$\begin{aligned}&\frac{1}{k!}\mathrm{GW}^M_{A,k}\left(\sum_{i=1}^m t_ie_i,\cdots,\sum_{i=1}^m t_ie_i\right)\\&=\sum_{\alpha_0+\cdots+\alpha_m=k}\frac{1}{k!}(-1)^{s(\alpha)}\mathrm{GW}^M_{A,k}(e_0^{\alpha}0,\cdots,e_m^{\alpha_m})\frac{k!}{\alpha_0!\cdots\alpha_d!}t_0^{\alpha_0}\cdots t_m^{\alpha_m}\\&=\sum_\alpha(-1)^{s(\alpha)}\frac{\mathrm{GW}^M_{A,k}(e^\alpha)}{\alpha!}t^\alpha.\end{aligned}$$

Moreover, if e_i all have even degrees, then the sign $(-1)^{s(\alpha)}$ is always 1.

Example 10.6. Let's calculate the Gromov–Witten potential Φ of $(M,\omega)=(\mathbb{CP}^2,\omega_{\mathrm{FS}})$. Recall that $H^*(\mathbb{CP}^2;\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}c\oplus c^2$, where c is the Poincaré dual for the line class L. There are two main cases.
(1) $A=0$. All terms vanishe except for

$$\mathrm{GW}^M_{0,3}(a_1,a_2,a_3)=\int_{\mathbb{CP}^2}a_1\cup a_2\cup a_3$$

with $(\alpha_1,\alpha_2,\alpha_3)=(1,2,0),(2,0,1)$. Since

$$\int_{\mathbb{CP}^2}1\cup c\cup c=\int_{\mathbb{CP}^2}1\cup 1\cup c^2=1,$$

we see that

$$\Phi|_{A=0}=\left(\frac{1}{2!0!1!}t_0^2t_2+\frac{1}{1!2!0!}t_0t_1^2\right)e^0=\frac{1}{2}t_0^2t_2+\frac{1}{2}t_0t_1^2.$$

(2) $A = dL$ with $d \neq 0$. By definition,

$$\sum_{i=1}^{k} \deg a_i = \dim \mathbb{CP}^2 + 2\,\mathrm{c}_1(T\mathbb{CP}^2)(dL) + 2k - 6 = 6d + 2k - 2.$$

By Proposition 10.2, if $a_i = 1$ for some i then this term vanishes. It suffices to consider $(a_1, \cdots, a_k) = (c, \cdots, c, c^2, \cdots, c^2)$, so

$$\begin{aligned} \alpha_1 + \alpha_2 &= k, \\ 2\alpha_1 + 4\alpha_2 &= 6d + 2k - 2. \end{aligned}$$

Solving the two linear equations, we obtain $\alpha_1 = k - 3d + 1, \alpha_2 = 3d - 1$. Note that the number $3d - 1$ appears naturally. By Proposition 10.4, we see that

$$\mathrm{GW}^{\mathbb{CP}^2}_{dL,k}(c, \cdots, c, c^2, \cdots, c^2) = \mathrm{GW}^{\mathbb{CP}^2}_{dL,3d-1}(c^2, \cdots, c^2)\left(\int_{dL} c\right)^{k-3d+1} = N_d \cdot d^{k-3d+1}.$$

As a consequence,

$$\Phi|_{dL} = \sum_{k \geqslant 3d-1} \left(\frac{N_d}{(3d-1)!} t_2^{3d-1} \frac{1}{(k-3d+1)!} t_1^{k-3d+1} d^{k-3d+1} \right) e^{dL} = \frac{N_d}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL}.$$

The full expression of the Gromov–Witten potential is

$$\Phi(t_0, t_1, t_2) = \frac{1}{2} t_0^2 t_2 + \frac{1}{2} t_0 t_1^2 + \sum_{d \geqslant 1} \frac{N_d}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL}.$$

The Gromov–Witten potential of $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ contains the mysterious coefficients we discussed before. If we can compute Φ, then we get to know a lot of information about N_d. Surprisingly, it is possible to compute Φ for every (M, ω) because they satisfy the famous WDVV equations.

11. WDVV equations

Let M be a compact symplectic manifold. Fix a basis $e_0, \cdots, e_m$ of $H^*(M; \mathbb{K})$ and denote $g = (g_{ij})$, where

$$g_{ij} = \int_M e_i \cup e_j.$$

By Poincaré duality, g is non-degenerate, so $(g^{ij}) = g^{-1}$ exists.

Theorem 11.1 (Witten–Dijkgraaf–Verlinde–Verlinde)**.** The Gromov–Witten potential satisfies the following PDE: for $0 \leqslant a, b, c, d \leqslant m$,

$$\sum_{i,j} \Phi_{iba} g^{ij} \Phi_{dcj} = (-1)^{\deg e_a (\deg e_b + \deg e_c)} \sum_{i,j} \Phi_{icb} g^{ij} \Phi_{daj},$$

where

$$\Phi_{iba} = \frac{\partial^3}{\partial t_i \partial t_b \partial t_a} \Phi.$$

This theorem is usually described as **WDVV equations**.

Example 11.2. $(M, \omega) = (\mathbb{CP}^1, \omega_{\mathrm{FS}})$. By Exercise 10.1, its Gromov–Witten potential is

$$\Phi = \frac{1}{2} t_0^2 t_1 + e^{t_1} e^L,$$

where L is the line class.

Exercise 11.1. Prove for $(M, \omega) = (\mathbb{CP}^2, \omega_{\mathrm{FS}})$, the WDVV equations are equivalent to the single equation

$$\Phi_{111}\Phi_{122} + \Phi_{222} = (\Phi_{112})^2.$$

Proof. $H^*(\mathbb{CP}^2;\mathbb{K}) = \mathbb{K} \oplus \mathbb{K}c \oplus \mathbb{K}c^2$, where c is the Poincaré dual of the line class. As

$$\int_{\mathbb{CP}^2} c \cup c = \int_{\mathbb{CP}^2} 1 \cup c^2 = 1,$$

we have

$$g = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = g^{-1}.$$

Therefore,

$$\sum_{i,j} \Phi_{iba} g^{ij} \Phi_{dcj} = \Phi_{0ba}\Phi_{dc2} + \Phi_{1ba}\Phi_{dc1} + \Phi_{2ba}\Phi_{dc0}.$$

Remember that in Example 10.6 we have obtained the Gromov–Witten potential of $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$

$$\Phi(t_0,t_1,t_2) = \frac{1}{2}t_0^2 t_2 + \frac{1}{2}t_0 t_1^2 + \sum_{d\geqslant 1} \frac{N_d}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL}.$$

Direct calculations imply

$$\begin{aligned}
\Phi_{002} &= \Phi_{011} = 1, \\
\Phi_{111} &= \sum_{d\geqslant 1} \frac{d^3 N_d}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL}, \\
\Phi_{112} &= \sum_{d\geqslant 1} \frac{d^2 N_d}{(3d-2)!} t_2^{3d-2} e^{dt_1} e^{dL}, \\
\Phi_{122} &= \sum_{d\geqslant 1} \frac{d N_d}{(3d-3)!} t_2^{3d-3} e^{dt_1} e^{dL}, \\
\Phi_{222} &= \sum_{d\geqslant 1} \frac{N_{d+1}}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL},
\end{aligned}$$

while other terms vanish. If we vary a, b, we see that

$$\begin{aligned}
\sum_{i,j} \Phi_{i00} g^{ij} \Phi_{dcj} &= \Phi_{000}\Phi_{dc2} + \Phi_{100}\Phi_{dc1} + \Phi_{200}\Phi_{dc0} = 0, \\
\sum_{i,j} \Phi_{i01} g^{ij} \Phi_{dcj} &= \Phi_{001}\Phi_{dc2} + \Phi_{101}\Phi_{dc1} + \Phi_{201}\Phi_{dc0} = \Phi_{dc1} = \Phi_{00c}\Phi_{d12} + \Phi_{10c}\Phi_{d11} + \Phi_{20c}\Phi_{d10}, \\
\sum_{i,j} \Phi_{i02} g^{ij} \Phi_{dcj} &= \Phi_{002}\Phi_{dc2} + \Phi_{102}\Phi_{dc1} + \Phi_{202}\Phi_{dc0} = \Phi_{dc2}, \\
\sum_{i,j} \Phi_{i11} g^{ij} \Phi_{dcj} &= \Phi_{011}\Phi_{dc2} + \Phi_{111}\Phi_{dc1} + \Phi_{211}\Phi_{dc0} = \Phi_{dc2} + \Phi_{111}\Phi_{dc1} + \Phi_{211}\Phi_{dc0}, \\
\sum_{i,j} \Phi_{i12} g^{ij} \Phi_{dcj} &= \Phi_{012}\Phi_{dc2} + \Phi_{112}\Phi_{dc1} + \Phi_{212}\Phi_{dc0} = \Phi_{112}\Phi_{dc1} + \Phi_{212}\Phi_{dc0}, \\
\sum_{i,j} \Phi_{i22} g^{ij} \Phi_{dcj} &= \Phi_{022}\Phi_{dc2} + \Phi_{122}\Phi_{dc1} + \Phi_{222}\Phi_{dc0} = \Phi_{122}\Phi_{dc1} + \Phi_{222}\Phi_{dc0}.
\end{aligned}$$

The only non-trivial equation is

$$\Phi_{222} + \Phi_{111}\Phi_{221} = \sum_{i,j} \Phi_{i11} g^{ij} \Phi_{22j} = \sum_{i,j} \Phi_{i12} g^{ij} \Phi_{21j} = \Phi_{112}^2.$$

$\square$

Example 11.3. WDVV equations help us obtain the recursive formula for N_d. By Exercise 11.1, we have

$$\Phi_{111}\Phi_{122} = \sum_{k,l\geqslant 1} N_k N_l \frac{k^3 l}{(3k-1)!(3l-3)!} t_2^{3(k+l)-4} e^{(k+l)t_1} e^{(k+l)L},$$

$$\Phi_{112}^2 = \sum_{k,l\geqslant 1} N_k N_l \frac{k^2 l^2}{(3k-2)!(3l-2)!} t_2^{3(k+l)-4} e^{(k+l)t_1} e^{(k+l)L}.$$

The WDVV equations imply

$$\frac{N_d}{(3d-4)!} + \sum_{k,l\geqslant 1, k+l=d} N_k N_l \frac{k^3 l}{(3k-1)!(3l-3)!} = \sum_{k,l\geqslant 1, k+l=d} N_k N_l \frac{k^2 l^2}{(3k-2)!(3l-2)!}.$$

It gives us the recursive formula

$$N_d = \sum_{k,l\geqslant 1, k+l=d} N_k N_l \left(k^2 l^2 \binom{3d-4}{3k-2} - k^3 l \binom{3d-4}{3k-1} \right), d \geqslant 2.$$

This formula was first found by Kontsevich. Since we know $N_1 = 1$, we theorically know the value of N_d for arbitary d. For example,

$$\begin{aligned}
N_2 &= N_1^2 \left(\binom{2}{1} - \binom{2}{2} \right) = 1, \\
N_3 &= N_1 N_2 \left(4\binom{5}{1} - 2\binom{5}{2} \right) + N_2 N_1 \left(4\binom{5}{1} - 8\binom{5}{5} \right) = 12, \\
N_4 &= N_1 N_3 \left(9\binom{8}{1} - 3\binom{8}{2} \right) + N_2^2 \left(16\binom{8}{4} - 16\binom{8}{5} \right) + N_3 N_1 \left(9\binom{8}{7} - 27\binom{8}{8} \right) = 620.
\end{aligned}$$

Similarly, $N_5 = 87304, N_6 = 26312976$ and $N_7 = 14616808192$.

12. Back to quantum products

Now we define the quantum product mentioned in section 8. Let M be a compact symplectic manifold and fix a basis $e_0, \cdots, e_m$ of $H^*(M;\mathbb{K})$.

Definition 12.1. The **quantum product** of e_i, e_j is defined via

$$(e_i * e_j)(\mathrm{PD}(e_k)) = \sum_{A\in H_2^S(M)} \mathrm{GW}_{A,3}^M(e_i, e_j, e_k) T^A.$$

By definition,

$$(e_i * e_j)(\mathrm{PD}(e_k)) = \int_M e_1 \cup e_2 \cup e_3 + \sum_{A\neq 0} \mathrm{GW}_{A,3}^M(e_i, e_j, e_k) T^A.$$

We see that the deformed parts of the quantum products comes from

$$\sum_{A\neq 0} \mathrm{GW}_{A,3}^M(e_i, e_j, e_k) T^A.$$

Example 12.2. We now explain the structure of $(QH^*(\mathbb{CP}^n, \omega_{\mathrm{FS}}), *)$ as mentioned in Example 8.1. Let's consider the basis $1, c, \cdots, c^n$ of $H^*(\mathbb{CP}^n;\mathbb{K})$ with $c = \mathrm{PD}(L)$, where L is the line class. By definition,

$$(c^i * c^j)(\mathrm{PD}(c^k)) = \int_{\mathbb{CP}^n} c^{i+j+k} + \sum_{d\geqslant 1} \mathrm{GW}_{dL,3}^{\mathbb{CP}^n}(c^i, c^j, c^k) T^{dL}.$$

If the deformed part makes sense, then we have

$$2(i+j+k) = \dim \mathbb{CP}^n + 2\,\mathrm{c}_1(T\mathbb{CP}^n)(dL) = 2n + 2d(n+1).$$

The only possibility is $d = 1$, so if $i + j + k = 2n + 1$, the deformed part is

$$\mathrm{GW}_{L,3}^{\mathbb{CP}^n}(c^i, c^j, c^k) T^L.$$

Remark 12.3. The quantum cohomology can be viewed as an explicit construction for the Floer cohomology $HF^*(M;\mathbb{K})$, the marvelous generalization of Morse theory on symplectic manifolds. In general, we have the isomorphism of graded rings

$$QH^*(M,\omega) \simeq HF^*(M;\mathbb{K}).$$

Exercise 12.1. Let setting be as in Example 12.2. Check that for the quantum product is given by

$$c^i * c^j = \begin{cases} c^{j+k}, & i+j \leqslant n \\ c^{i+j-n-1}T^L & n+1 \leqslant i+j \leqslant 2n \end{cases}.$$

Exercise 12.2. A Frobenius algebra is semi-simple if it can be decomposed into a direct sum of fields (not necessarily the same fields). There is a cirerion by Abram: a Frobenius algebra $(\mathcal{A}, \cdot, \langle -,-\rangle)$ is semi-simple if and only if the following class (called the Euler class)

$$\mathcal{E} = \sum_{\text{basis}} e_i * e_i^\vee$$

is invertible in $\mathcal{A}$, where $e_i^\vee$ is the dual of e_i with respect to $\langle -,-\rangle$.

For $(M,\omega) = (\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\text{prod}})$, denote $a = \text{PD}([\text{pt} \times \mathbb{S}^2]), b = \text{PD}([\mathbb{S}^2] \times \text{pt})$.

(1) Compute all non-trivial 3-points constraint Gromov–Witten invariants.
(2) Compute the quantum products $a * a, b * b$ and $a * b$.
(3) Check that the quantum cohomology group $QH^*(\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\text{prod}})$ (over $\mathbb{K} = \mathbb{C}$), as an algebra, is semi-simple.

13. hard stones on Lagrangian submanifolds

We set aside the study of J-holomorphic temporarily, and focus on another important element in symplectic topology. Le t (M,ω) be a symplectic manifold.

There are two phenomenons in symplectic topology. One is the rigidity, like the Gromov non-squeezing property. The other is the softness, which basically means you can move or deform homotopically in certain freedom. Here, the "hard stones" in the title represents the rigidity of Lagrangian submanifolds.

Recall that the symplectic structure ω is a closed 2-form on M that is non-degenerate. However, it is unknown whether ω is still non-degenerate on a submanifold. Let's see a toy version.

Consider the Euclidean space with the standard symplectic form $(\mathbb{R}^{2n}, \omega_0)$. Then $\omega_0 = 0$ on the subspace $\text{span}(x_1)$. The same holds for $\text{span}(x_1, x_2)$. One may ask what is the biggest possible dimension of a subspace such that ω_0 vanishes on it. After some try, we find that $\omega_0 = 0$ on the subspace $\text{span}(x_1, \cdots, x_n)$.

Definition 13.1. A submanifold $L \subset (M,\omega)$ is said to be **Lagrangian** if $\omega|_L = 0$ and $\dim L = \frac{1}{2}\dim M$.

What is interesting and strange about Lagrangian submanifolds is, although the symplectic structure cannot "see" them, they play a fundamental role in the study of symplectic topology. The reason for this will be related to the "Fukaya category".

Example 13.2. We have seen the simpliest examples for Lagrangian submanifolds in Euclidean space. Now we introduce more on different manifolds.

(1) On a closed surface $(\Sigma_g, \omega_{\text{area}})$ the Lagrangian submanifolds are of dimension 1. Obviously, any curve as a smooth submanifold will be Lagrangian for the dimension reason.
(2) Consider the cotangent bundle $(T^*Q, d\lambda_{\text{can}})$ because $\lambda_{\text{can}} = 0$ on Q. First notice that Q itself is Lagrangian. More generally, in exercise we will show that the image Γ_α of a smooth 1-form α is a Lagrangian submanifold if and only if α is closed.
(3) Let φ be a diffeomorphism of a symplectic manifold (M,ω). Consider its graph
$$\Gamma_\varphi = \{(x,\varphi(x)) \mid x \in M\}.$$
Then $\Gamma_\varphi \subset (M \times M, \omega_{\text{twist}} = \pi_1^*\omega - \pi_2^*\omega)$ is a Lagrangian submanifold if and only if $\varphi \in \text{Symp}(M,\omega)$. To see this, recall that
$$T_{(p,\varphi(p))}\Gamma_\varphi = \{(v, \varphi_* v) \mid v \in T_pM\}.$$
Then $\omega_{\text{twist}}((v,\varphi_* v),(w,\varphi_* w)) = \omega(v,w) - \omega(\varphi_* v, \varphi_* w) = (\omega - \varphi^*\omega)(v,w)$.

Exercise 13.1. (1) Let Q be a submanifold and α be a 1-form on Q. Prove that the image (or graph) of α, denoted by $\Gamma_\alpha = \{(q, \alpha(q)) \mid x \in Q\}$, is a Lagrangian submanifold in $(T^*Q, d\lambda_{\text{can}})$ if and only if α is closed.
(2) A Lagrangian submanifold $L \subset (M, d\lambda)$ is said to be exact if $\lambda|_L = df$ for some smooth function $f\colon L \to \mathbb{R}$. Prove that Γ_α defined above is an exact Lagrangian submanifold in $(T^*Q, d\lambda_{\text{can}})$ if and only if α is exact.

Case (3) of Example 13.2 provides us with a geometric way to examine whether a diffeomorphism is a symplectomorphism. The following questions are:
(1) **How do we know if a submanifold is Lagrangian ?**
(2) **How to cook up Lagrangian submanifolds?**
There are several different answers to them. Zhang's version is using the so called Poisson brackets.

Definition 13.3. Let $F, G \in C^\infty(M)$. Let X_G be the vector field on M defined via

$$\omega(X_G, -) = -dG.$$

The **Poisson bracket** of F, G is defined as

$$\{F, G\} = dF(X_G).$$

Remark 13.4. Symplectic geometers have different sign conventions. For example, some people prefer to define X_G via

$$\omega(X_G, -) = dG.$$

The vector field X_G which appears in the definition is very famous and has its name, the Hamiltonian vector field of G. Note that we only need the non-degeneracy of symplectic structure in the definition. What follows from the skew-symmetric property is the following:

Proposition 13.5. Let $F, G \in C^\infty(M)$.
(1) $\{F, G\} = -\{G, F\}$. In particular $\{F, F\} = dF(X_F) = 0$, so F remains constant along the Hamiltonian vector field of X_F.
(2) If either F or G is constant, then $\{F, G\} = 0$.
(3) The Poisson bracket satisfies the Jacobian identity, i.e., $(C^\infty(M), \{-,-\})$ is a Lie algebra.

Proposition 13.6. Let $L \subset (M, \omega)$ be a submanifold of dimension n. Then L is Lagrangian if and only if

$$I_L = \{F \in C^\infty(M) \mid F|_L = 0\}$$

is closed under the Poisson bracket.

Proof. Suppose L is Lagrangian. Tale $F, G \in I_L$ and $p \in L$. For any $v \in T_pL$,

$$\omega_p(X_G, v) = -(dG)_p(v) = 0,$$

so $X_G(p) \in T_pL$ by linear symplectic geometry (more precisely, $T_pL \subset T_pM$ is a Lagrangian subsapce). $X_F(p) \in T_pL$ for the same reason. Therefore $\{F, G\}(p) = (dF)_p(X_G) = \omega_p(X_F, X_G) = 0$.

Conversely, suppose L is not Lagrangian. Then there exists $p \in L$ such that $\omega_p \neq 0$ on T_pL. By linear symplectic geometry, we can find $v \in T_pM \setminus T_pL$ such that $\omega_p(v, -) = 0$ on T_pL. Using local coordinates, we can find $G \in I_L$ but $(dG)_p(v) \neq 0$. Hence $\omega_p(X_G, v) = (dG)_p(v) \neq 0$. This means $X_G(p) \notin T_pL$ by hypothesis that $\omega_p(v, -) = 0$ on T_pL. In the same way we can construct $F \in C^\infty(M)$ such that $F \in I_L$ but $(dF)_p(X_G) \neq 0$, which means $\{F, G\}(p) \neq 0$ and $\{F, G\} \notin I_L$. □

This property allows us to check whether a submanifold is Lagrangian or not in a way other than defintiion.

To answer the question (2) above, we point out the following theorem related to the integrable systems.

Theorem 13.7 (Arnold–Liouville)**.** Let $F_1, \cdots, F_n \in C^\infty(M)$ such that they are Poisson-commutative. Consider $\Phi\colon M \to \mathbb{R}^n, x \mapsto (F_1(x), \cdots, F_n(x))$. For any regular value $(r_1, \cdots, r_n)$ of Φ, denote $L = \Phi^{-1}(r_1, \cdots, r_n)$, which is an n-dimensional submanifold. If L is compact and connected, then L is Lagrangian torus, i.e., it is Lagrangian and topologically a torus. If L connected but is not compact, then topologically it is $\mathbb{T}^{n-k} \times \mathbb{R}^k$.

Proof. Consider the action $\mathbb{R}^n \times M \to M, (s_1, \cdots, s_n) \cdot x = \varphi_{F_1}^{s_1} \circ \cdots \circ \varphi_{F_n}^{s_n}(x)$. This will be a group action because F_i are pairwise Poisson-commutative.

$$L = \Phi^{-1}(r_1, \cdots, r_n) = \bigcap_{i=1}^{n} F_i^{-1}(r_i).$$

If $x \in L$, then $(s_1, \cdots, s_n) \cdot x$ stays in L. To see this, recall that F_i remains constant along X_{F_i} for each i. Then $L \cong \mathbb{R}^n \{x\} = \mathbb{R}^n/\mathrm{stab}(x)$. Since $\mathrm{stab}(x)$ is a discrete subgroup of $\mathbb{R}^n$ and L is compact, the stabilizer must be a lattice.

It remains to show that L is Lagrangian. On T_xL, we have a basis $X_{F_1}(x), \cdots, X_{F_n}(x)$. As

$$\omega_x(X_{F_i}, X_{F_j}) = \{F_i, F_j\}(x) = 0,$$

we see that L is indeed Lagrangian. □

Let's see some toy versions for this theorem so that we can convince ourselves.

Example 13.8. Let $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, the standard symplectic form can be written as

$$\omega_0 = \frac{\sqrt{-1}}{2} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j = \sum_{j=1}^{n} r^j dr^j \wedge d\theta^j,$$

where $z^j = x^j + \sqrt{-1}y^j = r^j \cos\theta^j + \sqrt{-1}r^j \sin\theta^j$. Consider $F_i = \frac{1}{2}\left|z^i\right|^2$. Direct calculation shows that

$$X_{F_i} = y^i \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial y^i}.$$

As a consequence, $F_1, \cdots, F_n$ are pairwise Poisson-commutative. Let $\Phi = (F_1, \cdots, F_n)$.

$$\Phi^{-1}\left(\frac{1}{2}, \cdots, \frac{1}{2}\right) = \left\{(z^1, \cdots, z^n) \in \mathbb{C}^n \mid \left|z^i\right| = 1\right\} \cong \mathbb{T}^n.$$

By the expression of ω_0 under polar coordinates, we see that the preimage of $(\frac{1}{2}, \cdots, \frac{1}{2})$ is indeed a Lagrangian torus.

Example 13.9. Consider $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{\mathrm{prod}})$ where the product is taken with respect to the Fubini–Study form.

$$F_1([z_0, z_1], [w_0, w_1]) = \frac{1}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2},$$
$$F_2([z_0, z_1], [w_0, w_1]) = \frac{1}{2} \frac{|w_0|^2}{|w_0|^2 + |w_1|^2}.$$

As the two functions factor through projections, they are Poisson-commutative. Let $\Phi = (F_1, F_2)$. Then

$$\Phi^{-1}\left(\frac{1}{4}, \frac{1}{4}\right) = \left\{([z_0, z_1], [w_0, w_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid |z_0| = |z_1|, |w_0| = |w_1|\right\} \cong \mathbb{S}^1 \times \mathbb{S}^1.$$

It is not obvious that this preimage is Lagrangian, but as we know the definition of Fubini–Study metric, it can be checked via computations.

Let's step a little bit further. Notice that the image of Φ is $[0, 1]^2$. The fiber of an interior point is a Lagrangian torus, while fiber of a boundary point may degenerate. Such manifolds are called symplectic toric manifolds.

14. Distinguishing Lagrangian submanifolds

In the last section we see that lots of Lagrangian submanifolds are topologically tori. However, are they the same in the symplectic sense? This will be the main topic for this section.

Exercise 14.1. View the sphere $\mathbb{S}^2$ as $\left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}$. Prove that the following 2-form on $\mathbb{R}^3$ gives a symplectic structure on $\mathbb{S}^2$: for $p \in \mathbb{R}^3$ and vectors $v, w \in T_p\mathbb{S}^2$, define

$$\omega_p(v, w) = \langle p, v \times w\rangle,$$

where we identify p, v, w with vectors in $\mathbb{R}^3$ and $\times$ denotes the cross product.

Proof. We have $\omega_p(w,v) = \langle p, w\times v\rangle = -\langle p, v\times w\rangle = -\omega_p(v,w)$. By linear algebra,

$$\omega_p(v,w) = \langle p, v\times w\rangle = \langle w, p\times v\rangle ,$$

so $\omega_p(v,-) = 0$ implies $p\times v = 0$. Since $v\in T_p\mathbb{S}^2$, p is perpendicular to v and then $v=0$. $\square$

Let $(\mathbb{S}^2, \omega_{\text{area}})$ be the standard sphere embedded in $\mathbb{R}^3$, where ω_{area} is given in the above exercise. Let's see some tori in $(M,\omega) = (\mathbb{S}^2\times\mathbb{S}^2, \omega_{\text{prod}})$.

14.1. **Clifford torus.** let E be the equator of $\mathbb{S}^2$. Then $E\times E\subset (M,\omega)$ is called the Clifford torus, denoted by $\mathbb{T}_{\text{Clifford}}$.

14.2. **Entov–Polterovich torus.** Let

$$\mathbb{T}_{\text{EP}} = \left\{(v,w)\in\mathbb{R}^3\times\mathbb{R}^3 \mid |v| = |w| = 1, v_1 + w_1 = 0, \langle v,w\rangle = -\frac{1}{2}\right\}.$$

"EP" stands for Entov–Polterovich, who first invented this submanifold. Zhang claims that this is a Lagrangian torus in (M,ω). To check this, define

$$F(v,w) = v_1 + w_1,$$
$$G(v,w) = \langle v,w\rangle .$$

Once we know that F,G Poisson-commute and $(0,-\frac{1}{2})$ is a critical point, the result follows from Arnold–Livoulle theorem.

Lemma 14.1. Fix $q\in\mathbb{R}^3$. Let $f\colon \mathbb{S}^2\to\mathbb{R}, v\mapsto\langle v,q\rangle$. Then $X_f(p) = p\times q$.

Proof. By definition,

$$-(df)_p(w) = \omega_p(X_f, w).$$

The left hand side is $-\langle w,q\rangle$ while the right hand side is $\langle p, X_f\times w\rangle$. Notice that

$$\langle p, (p\times q)\times w\rangle = \langle p, \langle p,w\rangle q - \langle q,w\rangle p\rangle = \langle p,q\rangle\langle p,w\rangle - \langle q,w\rangle\langle p,p\rangle = -\langle q,w\rangle ,$$

so $X_f = p\times q$ by the uniqueness. $\square$

Exercise 14.2. Check that F,G defined above indeed Poisson-commute.

Proof. F can be viewed taking inner product with vector (e_1, f_1), so by Lemma 14.1,

$$X_F(v,w) = v_3\partial_{v_2} - v_2\partial_{v_3} + w_3\partial_{w_2} - w_2\partial_{w_3}.$$

Therefore,

$$\begin{aligned}
&\{F,G\}(v,w) = (dG)_{(v,w)}(X_F)\\
&= (w_1dv_1 + w_2dv_2 + w_3dv_3 + v_1dw_1 + v_2dw_2 + v_3dw_3)(v_3\partial_{v_2} - v_2\partial_{v_3} + w_3\partial_{w_2} - w_2\partial_{w_3})\\
&= v_3w_2 - w_3v_2 - w_2v_3 + w_3v_2 = 0.
\end{aligned}$$

$\square$

Finally, it is not hard to check that $(0,-\frac{1}{2})$ is a critical point. Although we have known that $\mathbb{T}_{\text{EP}}$ is a Lagrangian torus, we are not satisfied and want to see its explicit description.

Exercise 14.3. Compute the Hamiltonian flow φ_G^t generated by G. Moreover, confirm that for the fixed point $p = \left(\left(0, -\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)\right)\in\mathbb{T}_{\text{EP}}$, the orbit of p under the flow φ_G^t is given by the following curve

$$C = \left\{\left(\left(-\frac{\sqrt{3}}{2}\sin t, -\frac{\sqrt{3}}{2}\cos t, \frac{1}{2}\right), \left(\frac{\sqrt{3}}{2}\sin t, \frac{\sqrt{3}}{2}\cos t, \frac{1}{2}\right)\right) \mid t\in\mathbb{R}/2\pi\mathbb{Z}\right\}.$$

solution. By Lemma 14.1, $X_G(v,w) = (v\times w, w\times v)$. Suppose $\varphi_G^t(v,w) = (v(t), w(t))$. It suffices to solve the ODE

$$\begin{cases}\dot v(t) = v(t)\times w(t)\\ \dot w(t) = w(t)\times v(t).\end{cases}$$

Notice that $\dot v(t) + \dot w(t) = 0$, so $v(t) + w(t) = v(0) + w(0)$, which we will denote by c. Then

$$\dot v(t) = v(t)\times(c - v(t)) = v(t)\times c.$$

This is a linear ODE, so we obtain

$$\varphi_G^t(v,w) = (v(0)e^{Ct}, c - v(0)e^{Ct}), C = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}.$$

Taking $(v,w) = p$, we get C via the above formula. $\square$

Note that C will be just one circle inside the $\mathbb{T}_{\text{EP}}$. The flow generated by F also acts on C, which creates a torus. We already know that

$$X_F(v,w) = v_3\partial_{v_2} - v_2\partial_{v_3} + w_3\partial_{w_2} - w_2\partial_{w_3},$$

The flow it generates rotates the two components simultaneously, with the rotation matrix

$$R_s = \begin{pmatrix} 1 & & \\ & \cos s & -\sin s \\ & \sin s & \cos s \end{pmatrix}, s \in \mathbb{R}/2\pi\mathbb{Z}.$$

Hence the points on $\mathbb{T}_{\text{EP}}$ are given by the action

$$s \cdot (v,w) = (R_s v, R_s w), (v,w) \in C, s \in \mathbb{R}/2\pi\mathbb{Z}.$$

14.3. **Chekanov–Schlenk torus.** We will construct a torus $\mathbb{T}_{\text{CS}}$ in $\mathbb{S}^2 \times \mathbb{S}^2$, called the Chekanov–Schlenk torus. Let $B^2(\sqrt{2})$ be the 2-dimensional open disc with the radius. We endow it with the standard plane area form ω_{area}, so $B^2(\sqrt{2})$ has area 2π.

First, pick any closed curve $\Gamma \subset B^2(\sqrt{2})$ such that it encloses a region with are $\frac{\pi}{2}$. Next, embed $(B^2(\sqrt{2}), \omega_{\text{area}})$ into $(\mathbb{S}^2, \frac{1}{2}\omega_{\text{area}})$ via the steorographic projection

$$\varphi\colon (r,\theta) \longmapsto (1 - r^2, r\cos\theta\sqrt{2-r^2}, r\sin\theta\sqrt{2-r^2}).$$

Finally, consider $\Delta(\varphi(\Gamma)) \subset \mathbb{S}^2 \times \mathbb{S}^2$ and define action

$$s \cdot (v,w) = (R_s v, R_{-s} w), (v,w) \in \Delta(\varphi(\Gamma)), s \in \mathbb{R}/2\pi\mathbb{Z}.$$

The resulting torus is denoted by $\mathbb{T}_{\text{CS}}$. What's surprising is, although $\mathbb{T}_{\text{CS}}$ is defined via a twisted action, it is still symplectomorphism to $\mathbb{T}_{\text{EP}}$!

Theorem 14.2. $\mathbb{T}_{\text{CS}}, \mathbb{T}_{\text{EP}}$ are symplectomorphic to each other.

Proof. We only give outline here.

(a) An observation from Gadbled:

$$\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\right)(R_s, R_s) = (R_s, R_{-s})\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\right).$$

In other words, the action $(R_s, R_s), (R_s, R_{-s})$ are conjugate to each other.

(b) Up to symplectomorphism, $\mathbb{T}_{\text{CS}}$ is independent of the curve $\Gamma \subset B^2(\sqrt{2})$ as long as its enclosed region has area $\frac{\pi}{2}$.

(c) The matrix in (a)

$$\left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\right)$$

acts on $\mathbb{S}^2 \times \mathbb{S}^2$ as a symplectomorphism.

$\square$

On the contrary, the Clifford torus is not symplectomorphic to the two tori above, which means Clifford torus is distinguished from the others in the symplectic sense.

Theorem 14.3. $\mathbb{T}_{\text{Clifford}}$ and $\mathbb{T}_{\text{EP}} \cong \mathbb{T}_{\text{CS}}$ are not symplectomorphic.

In order to prove this theorem, we need new machinery.

15. A_∞-ALGEBRAS AND POTENTIAL FUNCTIONS

A_∞-algebra will help us to construct a symplectic invariant, namely the potential function, which distinguishes Lagrangian submanifolds. A_∞-algebra was first defined by Stasheff and applied in the study of algebraic topology.

The basic ingredients for A_∞-algebra are a unital ring R, an R-graded algebra $C = \bigoplus_{k\in\mathbb{Z}} C^k$ and a sequence of R-linear maps $m_k\colon C^{\otimes k} \to C, k \geqslant 0$ (If $k = 0$, $m_0\colon R \to C$ is determined by $m_0(1)$). They satisfies certain relations. Let

$$B(C) = \bigoplus_{k=0}^{\infty} C^{\otimes k} = R \oplus C \oplus C^{\otimes 2} \oplus \cdots .$$

m_k can be extended to $\widehat{m}_k$ on $B(C)$ in the following way: if $n \geqslant k$, then $\widehat{m}_k$ sends $x_1 \otimes \cdots \otimes x_n$ to

$$\sum_{i=1}^{n-k+1} (-1)^{\deg x_1+\cdots+\deg x_{i-1}+i-1} x_1 \otimes \cdots \otimes x_{i-1} \otimes m_k(x_i \otimes \cdots \otimes x_{i+k-1}) \otimes x_{i+k} \otimes \cdots \otimes x_n,$$

and if $n < k$ the result is simply 0.

The above description is very complicated so let's see a simple case. If $k = 0$,

$$\widehat{m}_0(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n+1}(-1)^{\deg x_1+\cdots+\deg x_{i-1}+i-1} x_1 \otimes \cdots \otimes x_{i-1} \otimes m_0(1) \otimes x_{i+1} \otimes \cdots \otimes x_n,$$

so $\widehat{m}_0(x) = m_0(1) \otimes x + (-1)^{\deg x+1} x \otimes m_0(1)$.

Definition 15.1. $(C, \{m_k\}_{k=0}^{\infty})$ is an A_∞**-algebra** if

$$\left(\sum_{k=0}^{\infty} \widehat{m}_k\right)^2 = 0.$$

Remark 15.2. An A_∞-algebra is an enhanced version of a chain complex.

Remark 15.3. In this lecture we will only consider R with characteristic 2, so there will be no sign issues.

We can see what's the meaning of the idenity in the definition of A_∞-algebra by trying some elements. Our first choice is 1.

$$\begin{aligned}\left(\sum_{k=0}^{\infty} \widehat{m}_k\right)^2 (1) &= (\widehat{m}_0 + \widehat{m}_1 + \cdots)(\widehat{m}_0(1) + \widehat{m}_1(1) + \cdots) \\ &= (\widehat{m}_0 + \widehat{m}_1 + \cdots)(\widehat{m}_0(1)) = (\widehat{m}_0 + \widehat{m}_1 + \cdots)(m_0(1)).\end{aligned}$$

As $m_0(1) \in C$,

$$\begin{aligned}\left(\sum_{k=0}^{\infty} \widehat{m}_k\right)^2 (1) &= \widehat{m}_0(m_0(1)) + \widehat{m}_1(m_0(1)) \\ &= 2m_0(1) \otimes m_0(1) + m_1(m_0(1)) = m_1(m_0(1)).\end{aligned}$$

Therefore, $m_1(m_0(1)) = 0$. Next we try $x \in C$.

$$\begin{aligned}\left(\sum_{k=0}^{\infty} \widehat{m}_k\right)^2 (x) &= (\widehat{m}_0 + \widehat{m}_1 + \cdots)(\widehat{m}_0(x) + \widehat{m}_1(x) + \cdots) \\ &= (\widehat{m}_0 + \widehat{m}_1 + \widehat{m}_2 + \cdots)(m_0(1) \otimes x + x \otimes m_0(1) + m_1(x)).\end{aligned}$$

We have

$$\begin{aligned}
&\widehat{m}_0(m_0(1)\otimes x) = m_0(1)\otimes m_0(1)\otimes x + m_0(1)\otimes m_0(1)\otimes x + m_0(1)\otimes x\otimes m_0(1),\\
&\widehat{m}_0(x\otimes m_0(1)) = m_0(1)\otimes x\otimes m_0(1) + m_0(1)\otimes m_0(1)\otimes x + m_0(1)\otimes m_0(1)\otimes x,\\
&\widehat{m}_0(m_1(x)) = m_0(1)\otimes m_1(x) + m_1(x)\otimes m_0(1),\\
&\widehat{m}_1(x\otimes m_0(1)) = m_1(x)\otimes m_0(1) + x\otimes m_1(m_0(1)),\\
&\widehat{m}_1(m_0(1)\otimes x) = m_1(m_0(1))\otimes x + m_0(1)\otimes m_1(x),\\
&\widehat{m}_1(m_1(x)) = m_1(m_1(x)),\\
&\widehat{m}_2(m_0(1)\otimes x) = m_2(m_0(1)\otimes x),\\
&\widehat{m}_2(x\otimes m_0(1)) = m_2(x\otimes m_0(1)).
\end{aligned}$$

Since we are working under characterstic 2 case and $m_1(m_0(1))$, we see that

$$0 = \left(\sum_{k=0}^{\infty}\widehat{m}_k\right)^2(x) = m_2(m_0(1)\otimes x) + m_2(x\otimes m_0(1)) + m_1(m_1(x)).$$

In some sense, we can take m_1 as a differential operator and take m_2 as a product operator in algebra. Then $m_1(m_0(1))$ implies that $m_0(1)$ is closed. However, (C, m_1) is not necessarily a chain complex. The obstruction for this is $m_0(1)\neq 0$.

Example 15.4. Let (C, d, m) be a differential graded algebra (DGA). Then (C, d) is an A_∞-algebra by taking $m_k = 0. k \geqslant 3$ or $k = 0$, and setting $m_1 = d, m_2 = m$.

Let's check this. Indeed,

$$\widehat{m}_1(a\otimes b) = d(a)\otimes b + (-1)^{\deg a+1} a\otimes d(b)$$

and

$$\begin{aligned}
0 = (\widehat{m}_1+\widehat{m}_2)^2(a\otimes b) &= (\widehat{m}_1+\widehat{m}_2)(\widehat{m}_1(a\otimes b)+\widehat{m}_2(a\otimes b))\\
&= (\widehat{m}_1+\widehat{m}_2)(d(a)\otimes b + (-1)^{\deg a+1}a\otimes d(b) + ab)\\
&= (\widehat{m}_1+\widehat{m}_2)(d(a)\otimes b + (-1)^{\deg a+1}a\otimes d(b) + ab)\\
&= d(a)\cdot b + (-1)^{\deg a+1} a\cdot d(b) + d(ab).
\end{aligned}$$

Exercise 15.1. Let $(C, \{m_k\}_{k=0}^{\infty})$ be an A_∞-algebra with $m_k = 0$ for $k\geqslant 3$ and $k = 0$. Prove:
(1) $m_1^2 = 0$, so m_1 is the differential of C.
(2) $m_1(m_2(x\otimes y)) = m_2(m_1(x)\otimes y) + m_2(x\otimes m_1(y))$, so m_1 satisfies the Leibniz rule.
(3) $m_2(m_2(x\otimes y)\otimes z) = m_2(x\otimes m_2(y\otimes z))$, so m_2 is associative and can be viewed as a product.
In other words, (C, m_1, m_2) is a differential graded algebra. Here we simply work over characterstic 2 so we don't need to specify the sign.

Proof. (1) Taking $x\in C$ we obtain

$$0 = \left(\sum_{k=0}^{\infty}\widehat{m}_k\right)^2(x) = (\widehat{m}_1+\widehat{m}_2)(m_1(x)) = m_1(m_1(x)).$$

Hence $m_1^2 = 0$.

(2) Taking $x\otimes y$ we have

$$\begin{aligned}
0 = \left(\sum_{k=0}^{\infty}\widehat{m}_k\right)^2(x) &= (\widehat{m}_1+\widehat{m}_2)(\widehat{m}_1(x\otimes y)+\widehat{m}_2(x\otimes y))\\
=&(\widehat{m}_1+\widehat{m}_2)(m_1(x)\otimes y + x\otimes m_1(y) + m_2(x\otimes y))\\
=&m_1^2(x)\otimes y + m_1(x)\otimes m_1(y) + m_1(x)\otimes m_1(y)\\
&+ x\otimes m_1^2(y) + m_1(m_2(x\otimes y)) + m_2(m_1(x)\otimes y) + m_2(x\otimes m_1(y))\\
=&m_2(m_1(x\otimes y)) + m_2(x\otimes m_1(y)) + m_1(m_2(x\otimes y)).
\end{aligned}$$

We get the Leibniz rule.

(3) Taking $x \otimes y \otimes z$ we get

$$\left(\sum_{k=0}^{\infty} \widehat{m}_k\right)^2 (x \otimes y \otimes z) = (\widehat{m}_1 + \widehat{m}_2)(\widehat{m}_1(x \otimes y \otimes z) + \widehat{m}_2(x \otimes y \otimes z))$$

$$= (\widehat{m}_1 + \widehat{m}_2)(m_1(x) \otimes y \otimes z + x \otimes m_1(y) \otimes z + x \otimes y \otimes m_1(z) + m_2(x \otimes y) \otimes z + x \otimes m_2(y \otimes z))$$

$$= m_1(x) \otimes m_1(y) \otimes z + m_1(x) \otimes m_1(y) \otimes z + x \otimes m_1(y) \otimes m_1(z) + m_1(x) \otimes y \otimes z + x \otimes m_1(y) \otimes m_1(z)$$

Let's write them out term by term.

$$\begin{aligned}
&\widehat{m}_1(m_1(x) \otimes y \otimes z + x \otimes m_1(y) \otimes z + x \otimes y \otimes m_1(z)) \\
=&m_1(x) \otimes m_1(y) \otimes z + x \otimes m_1(y) \otimes m_1(z) + m_1(x) \otimes m_1(y) \otimes z \\
&+ x \otimes m_1(y) \otimes m_1(z) + m_1(x) \otimes y \otimes m_1(z) + x \otimes m_1(y) \otimes m_1(z) = 0, \\
&\widehat{m}_1(m_2(x \otimes y) \otimes z + x \otimes m_2(y \otimes z)) \\
=&m_1(m_2(x \otimes y) \otimes z) + m_2(x \otimes y) \otimes m_1(z) + m_1(x) \otimes m_2(y \otimes z) + x \otimes m_1(m_2(y \otimes z)), \\
&\widehat{m}_2(m_1(x) \otimes y \otimes z + x \otimes m_1(y) \otimes z + x \otimes y \otimes m_1(z)) \\
=&m_2(m_1(x) \otimes y) \otimes z + m_1(x) \otimes m_2(y \otimes z) + m_2(x \otimes m_1(y)) \otimes z \\
&+ x \otimes m_2(m_1(y) \otimes z) + m_2(x \otimes y) \otimes m_1(z) + x \otimes m_2(y \otimes m_1(z)), \\
&\widehat{m}_2(m_2(x \otimes y) \otimes z + x \otimes m_2(y \otimes z)) = m_2(m_2(x \otimes y) \otimes z) + m_2(x \otimes m_2(y \otimes z)).
\end{aligned}$$

Applying the Leibniz rule, we obtain the associativity. □

To cook up a true chain complex out of an A_∞-algebra, we need to take a deformation. For the sake of simplicity, assume $m_k = 0$ for $k \geqslant 3$. Pick $b \in C$ such that $m_0(1) + m_1(b) + m_2(b \otimes b) = 0$. Deform m_k to m_k^b by

$$m_k^b(x_1 \otimes \cdots \otimes x_k) = \sum_{l_0, \cdots, l_k \geqslant 0} m_{k+\sum_{i=0}^k l_i}(b^{\otimes l_0} \otimes x_1 \otimes b^{\otimes l_1} \otimes \cdots \otimes x_n \otimes b^{\otimes l_k}).$$

Proposition 15.5. $(C, \{m_k^b\}_{k=0}^\infty)$ is an A_∞ algebra, and (C, m_1^b) is a chain complex.

Remark 15.6. For a general A_∞ algebra, such b should satisfy

$$m_0(1) + m_1(b) + \cdots + m_k(b^{\otimes k}) + \cdots = 0.$$

This equation is called the **Maure–Cartan equation**. Usually $m_k = 0$ for k sufficiently large, so this condition would be a finite sum. However, there is still gaurantee that such b would exist.

Definition 15.7. $b \in C$ satisfying the Maure–Cartan equation is called a **bounding cochain**.

More generally, if $b \in C$ satisfies the **weak Maure–Cartan equation**:

$$m_0(1) + m_1(b) + \cdots + m_k(b^{\otimes k}) + \cdots = \lambda(b) \in R,$$

then the deformation (C, m_1^b) is still a chain complex.

Definition 15.8. A $b \in C$ satisfying the weak Maure–Cartan equation is called a **weakly bounding cochain**.

We will see our first example of an A_∞-algebra coming from symplectic topology.

Example 15.9. Gromov–Witten invariants counts J-holomorphic curves $u\colon (\mathbb{S}^2, j) \to (M, J)$ via moduli space of stable maps. Given a Lagrangian submanifold $L \subset (M, \omega)$, we can define a relative version of Gromov–Witten invariants.

The relative version counts J-holomorphic maps $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2, j) \to (M, \omega, J)$ via moduli space of stable maps in $\mathcal{M}_k(\mathfrak{P}, J)$ for $\mathfrak{P} \in \pi_2(M, L)$. Fix cocyles $a_1, \cdots, a_k \in C^*(L)$, where $C^*(L)$ is the singular cochain. There Gromov–Witten invariant is

$$\mathrm{GW}_{k,\mathfrak{P}}^{M,L}(a_1, \cdots, a_k) \in \mathbb{K}.$$

Just like quantum cohomology generalizing classical cohomology, we can consider the **Lagrangian quantum cohomology** $QH^*(L)$. In particular, the promised A_∞-algebra is given by

$$(C^*(L), \{m_k\colon C^*(L)^{\otimes k} \to C^*(L)\}).$$

The map m_k is defined as follows. For $a_1, \cdots, a_k \in C^*(L)$ and $\mathfrak{P} \in \pi_2(M, L)$, we assign them to

$$m_{k,\mathfrak{P}}(a_1, \cdots, a_k) = \mathrm{PD}_L(\mathrm{ev}_*(\mathcal{M}_{k+1}(\mathfrak{P}, J)) \cap \mathrm{PD}_L(a_1) \cap \cdots \cap \mathrm{PD}_L(a_k)) \in C^*(L).$$

The m_k is defined via

$$m_k(a_1, \cdots, a_k) = \sum_{\mathfrak{P} \in \pi_2(M,L)} m_{k,\mathfrak{P}}(a_1, \cdots, a_k) T^{\omega(\mathfrak{P})}.$$

For example, if $k = 0$, m_0 is written as

$$\begin{aligned} m_0 \colon \Lambda_{\geqslant 0}^{\mathbb{K}, \Gamma_\omega} &\longrightarrow C^*(L) \\ 1 &\longmapsto \sum_{\mathfrak{P} \in \pi_2(M,L)} m_{0,\mathfrak{P}}(1) T^{\omega(\mathfrak{P})} = \sum_{\mathfrak{P} \in \pi_2(M,L)} \mathrm{PD}_L(\mathrm{ev}_*(\mathcal{M}_1(\mathfrak{P}, J))) T^{\omega(\mathfrak{P})}. \end{aligned}$$

We can also extend to the coefficient of $C^*(L)$ from $\mathbb{K}$ to $\Lambda^{\mathbb{K},\Gamma_\omega}$, where $\Gamma_\omega = \mathrm{im}(\omega \colon \pi_2(M, L) \to \mathbb{R})$. Recall that for a J-holomorphic curve $u \colon (\Sigma, j) \to (M, \omega, J)$,

$$\int_\Sigma u^*\omega = \mathrm{Area}(u) \geqslant 0,$$

so we can further restrict m_k to be a homomorphism over $\Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega}$, where

$$\Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega} = \left\{ \sum a_\lambda T^\lambda \in \Lambda^{\mathbb{K},\Gamma_\omega} \mid \lambda \geqslant 0 \right\}.$$

Remark 15.10. The theory of Lagrangian quantum cohomology is not well-established yet and waits for further development.

Theorem 15.11 (Fukaya–Oh–Ohta–Ono)**.** Let $L \subset (M, \omega)$. $(C^*(L), \{m_k\})$ defined in Example 15.9 is an A_∞-algebra over $\Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega}$. Moreover, this A_∞-algebra is invariant under symplecticomorphism.

This important theorem, sometimes abbreviated to FOOO, gives us a machinary to distinguish Lagrangian submanifolds under symplectomorphisms. However, in general to compute A_∞-algebras is almost impossible!

To resolve this problem, we may seek some invariants out of A_∞-categories. Consider the collection of weak boundary cochain in $(C^*(L), \{m_k\})$, denoted by

$$\mathcal{M}_{\mathrm{weak}}(L) = \left\{ b \in C^*(L) \mid m_0(1) + m_1(b) + \cdots + = \lambda(b) \in \Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega} \right\}.$$

We can view $\lambda(b)$ in the above expression as a function $\mathcal{M}_{\mathrm{weak}}(L) \to \Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega}$. This function is called the **potential function** of L, usually denoted by $\mathfrak{PO}_L$. Still, it is extremely difficult to compute $\mathcal{M}_{\mathrm{weak}}(L)$ and $\mathfrak{PO}_L$ for a general pair $L \subset (M, \omega)$. Fortunartely, this can be done for a special family of symplectic manifolds.

Definition 15.12. A **symplectic toric manifold** M, is a closed symplectic manifold that admits the stratification structure over a convex polytope $\Omega \subset \mathbb{R}^n_{\geqslant 0}$. In details, if

$$\Omega = \Omega^0 \cup \cdots \cup \Omega^n,$$

where Ω^i is the codimension-i facet of Ω, then there is a map $\mu \colon M \to \Omega$, called the moment map, and

$$M = M^0 \cup \cdots \cup M^n$$

with $\mu \colon M^i \to \Omega^i$ being a trivial bundle with fiber $\mathbb{T}^{n-i}$. Moreover, each fiber $\mu^{-1}(x)$ for $x \in \Omega^0$ is Lagrangian.

Usually, a symplectic toric manifold is written as X_Ω^{2n}.

There are lots of examples for symplectic toric manifolds. In fact, the notion of toric manifold is deeply connected to Liouville theorem.

Example 15.13. Consider $h \colon \mathbb{S}^2 \subset \mathbb{R}^3, (x, y, z) \mapsto z$. The $\mathbb{S}^2$ is a symplectic toric manifold over $[-1, 1]$. To see this, notice that $h^{-1}(z) \cong \mathbb{S}^1$ if $z \in (-1, 1)$ and $h^{-1}(-1) \cong h^{-1}(1) \cong \mathbb{T}^0$.

As a consequence, $\mathbb{S}^2 \times \mathbb{S}^2$ is a symplectic toric manifold over $\Omega = [0, 1]^2$.

Now we consider a symplectic toric manifold X_Ω^{2n} with the moment map μ. Take a point $x \in \Omega^0$.

Theorem 15.14 (FOOO)**.** $\mathcal{M}_{\mathrm{weak}}(\mu^{-1}(x))$ contains $H^1(\mu^{-1}(x); \Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega}) \simeq (\Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega})^n$. In particular, there exists a bounding cochain for $(C^*(\mu^{-1}), \{m_k\})$.

Now we can consider $H^*(C^*(L), m_1^b)$. This cohomology is called the Lagrangian Floer cohomology of $\mu^{-1}(x)$. We focus on the restriction of $\mathfrak{PO}_L$ on $H^*(\mu^{-1}(x); \Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega})$. We write this map in details in the following steps:

(1) Identify $H^*(\mu^{-1}(x); \Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega})$ with $(\Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega})^n$. This is uniform in x.
(2) Denote the hyperplane bounding Ω by $l_1 = 0, \cdots, l_m = 0$, where $l_i(x) = \langle x, v_i\rangle - c_i$ with c_i pointing outwards.
(3) We have

$$\mathrm{PO}_{\mu^{-1}(x)}(y_1, \cdots, y_n) = \sum_{i=1}^m e^{\langle v_i, y\rangle} T^{l_i(x)}.$$

Example 15.15. Consider $X_\Omega = \mathbb{S}^2 \times \mathbb{S}^2$ with $\Omega = [0,1]^2$. Let's compute the potential function for the Clifford torus $\mathbb{T}_{\text{Clifford}} = \mu^{-1}(\frac{1}{2}, \frac{1}{2})$. Ω is bounded by

$$l_1(x_1, x_2) = x_2, l_2(x_1, x_2) = x_1 - 1, l_3(x_1, x_2) = x_2 - 1, l_4(x_1, x_2) = x_1.$$

Therefore,

$$\mathfrak{PO}_{\mu^{-1}(\frac{1}{2},\frac{1}{2})}(y_1, y_2) = e^{-y_2}T^{\frac{1}{2}} + e^{-y_1}T^{\frac{1}{2}} + e^{y_2}T^{\frac{1}{2}} + e^{y_1}T^{\frac{1}{2}}.$$

Evaluating at 1 and write $z_i = e^{-y_i}$, we have

$$\mathfrak{PO}_{\mu^{-1}(\frac{1}{2},\frac{1}{2})}(z_1, z_2)|_{T=1} = \frac{1}{z_1} + \frac{1}{z_2} + z_1 + z_2.$$

This is a famous equation in mirror symmetry, or more precisely, in the Landau–Ginzburg model.

Exercise 15.2. Recall that $(\mathbb{CP}^2, \omega_{\text{FS}})$ is a symplectic toric manifold X_Ω with the associated convex polytope being the standard 2-simplex $\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2_{\geqslant 0} \mid x_1 + x_2 \leqslant 1\right\}$. The moment map $\mu\colon X_\Omega \to \Omega$ is

$$\mu([z_0 : z_1 : z_2]) = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\right).$$

Compute the potential function $\mathfrak{PO}_{\mu^{-1}(x)}$ for any $x \in \Omega^0$ on the domian $H^1(\mu^{-1}; \Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega}) \simeq (\Lambda_{\geqslant 0}^{\mathbb{K},\Gamma_\omega})^2$.

Proposition 15.16 (Auroux)**.** For the Entov–Polterovich torus $\mathbb{T}_{\text{EP}}$,

$$\mathfrak{PO}_{\mathbb{T}_{\text{ES}}}(z_1, z_2)|_{T=1} = z_1 + \frac{(1+z_2)^2}{z_1 z_2} = z_1 + z_2 + \frac{1}{z_1 z_2} + \frac{2}{z_1}.$$

As a consequence, $\mathbb{T}_{\text{EP}} \simeq \mathbb{T}_{\text{CS}}$ is not symplectic equivalent to $\mathbb{T}_{\text{Clifford}}$ in $\mathbb{S}^2 \times (\mathbb{S}^2, \omega_{\text{prod}})$.

16. Hamiltonian dynamics

We have seen many ingredients of Hamiltonian dynamics, e.g., Hamiltonian vector fields and Poisson brackets. In this section we will study the dynamics in more details. Let (M, ω) be a symplectic manifold.

Definition 16.1. Let $H_t\colon [0,1] \times M \to \mathbb{R}$ be a smooth map. The **Hamiltonian flow** $\{\varphi_H^t\}$ it generates is the flow generated by its Hamiltonian vector fields X_H^t. Every map φ_H^t is called a Hamiltonian diffeomorphism.

It will be useful to recall the equation of flow:

$$\frac{d}{dt}\varphi_H^t = X_H^t \circ \varphi_H^t.$$

Example 16.2. On $(\mathbb{R}^2, \omega_0)$, the Hamiltonian diffeomorphism generated by $H(x, y) = y$ is

$$\varphi_H^t(x, y) = (x + t, y).$$

Definition 16.3. The **Hamiltonian group** is defined as

$$\mathrm{Ham}(M, \omega) = \left\{\varphi \in \mathrm{Diff}(M) \mid \varphi = \varphi_H^1 \text{ for some } H_t \in C^\infty([0,1] \times M)\right\}.$$

Exercise 16.1. Prove that $\mathrm{Ham}(M, \omega)$ is a subgroup of $\mathrm{Symp}(M, \omega)$.

Proof. It suffices to show that $(\varphi_H^t)^*\omega = \omega$ for every $t \in [0,1]$. By the Cartan's magic formula,
$$\frac{d}{dt}(\varphi_H^t)^*\omega = \mathcal{L}_{X_H^t}\omega = d\iota_{X_H^t}\omega + \iota_{X_H^t}d\omega = 0,$$
so $(\varphi_H^t)^*\omega = (\varphi_H^0)^*\omega = \omega$.

Let X_t, Y_t be the Hamiltonian vector field of H_t, G_t. For every $x \in M$, we have
$$\begin{aligned}\frac{d}{dt}(\psi_t \circ \varphi_t)(x) &= \frac{d\psi}{dt}(\varphi_t(x)) + (d\psi_t)_{\varphi_t(x)}\left(\frac{d\varphi_t}{dt}(x)\right)\\ &= Y_t((\psi_t \circ \phi_t)(x)) + (d\psi_t)_{\varphi_t(x)}(X_t(\varphi_t(x)))\end{aligned}$$
Denote this vector field by Z_t. It suffices to check
$$\iota_{Z_t}\omega = -d(H_t + G_t \circ \psi_t^{-1})$$
To see this, notice that
$$Z_t((\psi_t \circ \varphi_t)(x)) = \frac{d}{dt}(\psi_t \circ \varphi_t)(x)$$
by pre-composing $(\psi_t \circ \varphi_t)^{-1}$, we have
$$Z_t(x) = Y_t(x) + (d\psi_t)_{\psi_t^{-1}(x)}\left(X_t(\psi_t^{-1}(x))\right)$$
We see that
$$\begin{aligned}\omega_x(Z_t(x), W(x)) &= \omega_x(Y_t(x), W(x)) + \omega_x((d\psi_t)_{\psi_t^{-1}(x)}\left(X_t(\psi_t^{-1}(x))\right), W(x))\\ &= dH_t(x)(W(x)) + (\psi_t^*\omega)_{\psi_t^{-1}(x)}(X_t(\psi_t^{-1}(x)), (d\psi_t)_x^{-1}(W(x)))\\ &= dH_t(W(x)) + \omega_{\psi_t^{-1}(x)}(X_t(\psi_t^{-1}(x)), (d\psi_t)_x^{-1}(W(x)))\\ &= dH_t(W(x)) + dG_{t\,\psi_t^{-1}(x)}(d\psi_t)_x^{-1}(W(x)) = d(H_t + G_t \circ \psi_t^{-1})_x(W(x))\end{aligned}$$
Here, we use the fact that $\psi_t^*\omega = \omega$ for any Hamiltonian diffeomorphism.

It remains to show that $\mathrm{Ham}(M,\omega)$ is closed under inverse. To see this, we can pick a Hamiltonian $\{\varphi_t\}$ generated by H_t. Consider the Hamiltonian flow $\{\psi_t\}$ generated by $-H_t \circ \varphi_t$. Then $\{\varphi_t \circ \psi_t\}$ is generated by
$$H_t + (-H_t \circ \varphi_t) \circ \varphi_t^{-1} = 0$$
Thus ψ_1 serves as the right inverse of φ_1. Recall that in a group right inverses coincide with inverse. □

By considering the time-1 map of Hamiltonian flow, we actually include every possible Hamiltonian diffeomorphism. To see this, complete the exercise below.

Exercise 16.2. Let $H(t,x) \in C^\infty([0,1] \times M)$ that generates a Hamiltonian flow $\{\phi_H^t\}$. Prove that for any time reparametrization $\alpha(t)\colon [0,1] \times [0,1]$ where $\alpha(0) = 0$, the flow $\left\{\phi_H^{\alpha(t)}\right\}$ is also a Hamiltonian flow. In particular, if $\phi = \phi_H^1 \in \mathrm{Ham}(M,\omega)$, then for any $t \in [0,1]$, the diffeomorphism $\phi_H^t \in \mathrm{Ham}(M,\omega)$.

Proposition 16.4. $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$. In fact, it is also a normal subgroup of $\mathrm{Diff}(M)$.

Proof. Let $\varphi = \varphi_H^1 \in \mathrm{Ham}(M,\omega)$. For any $\psi \in \mathrm{Diff}(M)$, $\psi^{-1}\varphi_H^t\psi$ is a Hamiltonian flow generated by $H_t \circ \psi \in C^\infty([0,1] \times M)$. In particular, $\mathrm{Ham}(M,\omega)$ is also normal in $\mathrm{Symp}(M,\omega)$. □

By definition, $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}_0(M,\omega)$. One may ask if the equality holds.

Theorem 16.5 (Banyaga,Ono)**.** There exists a subgroup $\Gamma \leqslant H^1(M;\mathbb{R})$, called the Flux group, such that
$$\mathrm{Symp}_0(M,\omega)/\mathrm{Ham}(M,\omega) \simeq H^1(M;\mathbb{R})/\Gamma.$$

As a corollary, $\mathrm{Ham}(\mathbb{S}^2,\omega) = \mathrm{Symp}(\mathbb{S}^2,\omega)$. On the other hand, if $H^1(M;\mathbb{R}) \neq 0$ it is possible that $\mathrm{Ham}(M,\omega) \subsetneq \mathrm{Symp}(M,\omega)$. For example, $(\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2, \omega_0)$ has a flow of symplectomorphisms $\varphi^t(\theta_1,\theta_2) = (\theta_1 + t, \theta_2)$. Later we will see that if $t \notin 2\pi\mathbb{Z}$, φ^t is not Hamiltonian.

There are several surprising facts about Hamiltonian.

(1) If (M,ω) is closed, then $\mathrm{Ham}(M,\omega)$ is simple. In particular, there is no nontrivial group homomorphism from $\mathrm{Ham}(M,\omega)$.

(2) $\mathrm{Ham}(M,\omega)$ satisfies the fragmentation property: For any open cover $\{U_i\}$ of M and for any $\varphi \in \mathrm{Ham}(M,\omega)$, we have $\varphi = \varphi_1 \cdots \varphi_n$ where φ_j is compactly supported in U_{i_j}, i.e., there exists $H_j \in C^\infty([0,1]\times M)$ with $\mathrm{supp}\, H_j \subset U_{i_j}$ and $\varphi^1_{H_j} = \varphi_j$.
(3) For any isotopy $\{\varphi^t\}$ in $\mathrm{Ham}(M,\omega)$, there exists $F \in C^\infty([0,1]\times M)$ such that $\varphi^t = \varphi^t_F$.

The magic results above were established by Banyaga in one single paper. As a consequence of fact (3), to every path $\{\varphi^t\} \subset \mathrm{Ham}(M,\omega)$ we can associate a length

$$L(\{\varphi^t\}) = \inf_{\varphi^t_F = \varphi^t} \int_0^1 (\max_M F_t - \min_M F_t) dt.$$

Note that this expression only involves C^0-values. This means that we can impose some quantites on Hamiltonian groups.

Definition 16.6. Let $\varphi \in \mathrm{Ham}(M,\omega)$. Its **Hofer norm** is defined as

$$\|\varphi\|_{\mathrm{Hofer}} = \inf_{\varphi^1 = \varphi} L(\{\varphi^t\}) = \inf_{\varphi^1_F = \varphi} \int_0^1 (\max_M F_t - \min_M F_t) dt.$$

As a priori, Hofer norm should be a pseudo-norm on $\mathrm{Ham}(M,\omega)$: $\|\varphi\psi\|_{\mathrm{Hofer}} \leqslant \|\varphi\|_{\mathrm{Hofer}} + \|\psi\|_{\mathrm{Hofer}}$.

Theorem 16.7. Let $\varphi \in \mathrm{Ham}(M,\omega)$. We have:
(1) $\|\varphi\|_{\mathrm{Hofer}} = 0$ implies $\varphi = \mathrm{id}$.
(2) $\|\theta^{-1}\varphi\theta\|_{\mathrm{Hofer}} = \|\varphi\|_{\mathrm{Hofer}}, \theta \in \mathrm{Symp}(M,\omega)$.
(3) $\|\varphi\|_{\mathrm{Hofer}} = \|\varphi^{-1}\|_{\mathrm{Hofer}}$.
In particular, $\|\cdot\|_{\mathrm{Hofer}}$ defines a conjugate-invariant norm on $\mathrm{Ham}(M,\omega)$.

The most non-trivial part of Theorem 16.7 is the non-degeneracy of the Hofer norm. As one may expect, this property was first studied and obtained by Hofer in the case of Euclidean spaces. Later this result was generalized to the general cases by other mathematicians.

Hofer norm naturally induces a metric on $\mathrm{Ham}(M,\omega)$.

Definition 16.8. The **Hofer distance** on $\mathrm{Ham}(M,\omega)$ is defined as

$$d_{\mathrm{Hofer}}(\varphi,\psi) = \|\varphi\psi^{-1}\|_{\mathrm{Hofer}}, \varphi,\psi \in \mathrm{Ham}(M,\omega).$$

By definition, the Hofer distance is bi-invariant: For any $\theta \in \mathrm{Ham}(M,\omega)$,

$$d_{\mathrm{Hofer}}(\theta\varphi,\theta\psi) = d_{\mathrm{Hofer}}(\varphi\theta,\psi\theta) = d_{\mathrm{Hofer}}(\varphi,\psi).$$

17. Distances between Lagrangian submanifolds at small-scale

Let $L \subset (M,\omega)$ be a subset. The Hamiltonian group $\mathrm{Ham}(M,\omega)$ naturally acts on (M,ω) and preserves the symplectic structure. The orbit of L under this action is denoted by $\mathcal{O}(L)$. In this section, we will associate a metric on $\mathcal{O}(L)$.

Definition 17.1. For $L_1, L_2 \in \mathcal{O}(L)$, we define

$$\delta_{\mathrm{CH}}(L_1,L_2) = \inf\left\{\|\varphi\|_{\mathrm{Hofer}} \mid \varphi(L_1) = L_2, \varphi \in \mathrm{Ham}(M,\omega)\right\}.$$

By definition, it is easy to see that δ_{CH} is a pseudo-norm. Moreover, it is translate-invariant: For $\theta \in \mathrm{Ham}(M,\omega)$,

$$\begin{aligned}\delta_{\mathrm{CH}}(\theta(L_1),\theta(L_2)) &= \inf\left\{\|\varphi\|_{\mathrm{Hofer}} \mid \varphi(\theta(L_1)) = \theta(L_2), \varphi \in \mathrm{Ham}(M,\omega)\right\}\\ &= \inf\left\{\|\varphi\|_{\mathrm{Hofer}} \mid (\theta^{-1}\varphi\theta)(L_1) = L_2, \varphi \in \mathrm{Ham}(M,\omega)\right\}\\ &= \inf\left\{\|\varphi'\|_{\mathrm{Hofer}} \mid \varphi'(L_1) = L_2, \varphi' \in \mathrm{Ham}(M,\omega)\right\}\\ &= \delta_{\mathrm{CH}}(L_1,L_2).\end{aligned}$$

Theorem 17.2 (Chekanov,Usher)**.** Suppose M is compact and L is a closed connected submanifold of dimension n. δ_{CH} on $\mathcal{O}(L)$ either is non-degenerate, or vanishes identically.

Remark 17.3. Actually, Chekanov proved that, if L is a closed connected submanifold of dimension n, then δ_{CH} is non-degenerate if and only if L is Lagrangian.

Definition 17.4. Let $L \subset (M, \omega)$ be a subset. Its **rigid locus** is

$$R_L = \bigcap_{\varphi \in \overline{\mathrm{Stab}(L)}} \varphi^{-1}(L),$$

where the closure is taken in $\mathrm{Ham}(M, \omega)$ with respect to the Hofer distance.

At first glance it is hard to understand this definition. The key lies in the closure. If the closure is removed in the definition, then the rigid locus is trivial, because every preimage remains the same.

Lemma 17.5.

$$\overline{\mathrm{Stab}(L)} = \{g \in \mathrm{Ham}(M, \omega) \mid \delta_{\mathrm{CH}}(g(L), L) = 0\}.$$

Proof. Consider $\overline{\mathrm{Stab}(L)}$. Recall that the closure of a subgroup in a topological group is again a subgroup. For any $g \in \overline{\mathrm{Stab}(L)}$ and $\varepsilon > 0$, we can find $h \in \mathrm{Stab}(L)$ such that $\|h^{-1}g\|_{\mathrm{Hofer}} < \varepsilon$. Then

$$\delta_{\mathrm{CH}}(g(L), L) = \delta_{\mathrm{CH}}(g(L), h(L)) = \delta_{\mathrm{CH}}((h^{-1}g)(L), L) < \varepsilon.$$

Since ε is arbitary, $\delta_{\mathrm{CH}}(g(L), L) = 0$.

Conversely, if $\delta_{\mathrm{CH}}(g(L), L) = 0$ for some $g \in \mathrm{Ham}(M, \omega)$, then for any $\varepsilon > 0$ we can find h such that

$$\|h\|_{\mathrm{Hofer}} < \varepsilon, (hg)(L_1) = L_2.$$

In particular, $hg \in \mathrm{Stab}(L)$ so $\|g(hg)^{-1}\|_{\mathrm{Hofer}} < \varepsilon$. This implise $g \in \overline{\mathrm{Stab}(L)}$. □

Corollary 17.6. δ_{CH} is non-degenerate on $\mathcal{O}(L)$ if and only if $\mathrm{Stab}(L)$ is a closed subgroup of $\mathrm{Ham}(M, \omega)$.

Proposition 17.7. Suppose L is a closed subset of (M, ω).
(1) R_L is a closed subset of L.
(2) $R_L = L$ if and only if δ_{CH} is non-degenerate.

Proof. (1) Since $\mathrm{id} \in \mathrm{Stab}(L)$, $R_L \subset L$. As $\varphi^{-1}(L)$ is closed, we have R_L closed.

(2) If $R_L = L$, then $L \subset \varphi^{-1}(L)$ for every $\varphi \in \overline{\mathrm{Stab}(L)}$. This implise $\varphi(L) = L$ and $\varphi \in \mathrm{Stab}(L)$, so $\mathrm{Stab}(L)$ is closed subgroup.

Conversely, suppose δ_{CH} is non-degenerate. Then

$$R_L = \bigcap_{\varphi \in \overline{\mathrm{Stab}(L)}} \varphi^{-1}(L) = \bigcap_{\varphi \in \mathrm{Stab}(L)} \varphi^{-1}(L) = L.$$

□

One may expect: Assume M is compact and then $R_L = \varnothing$ if and only if δ_{CH} vanishes identically. In fact, we will prove a more general property.

Definition 17.8. For an open subset $U \subset M$, we define

$$\mathrm{Ham}_c(U) = \left\{\varphi = \varphi_H^1 \in \mathrm{Ham}(M, \omega) \mid, \mathrm{supp}\, H \subset [0, 1] \times U\right\}.$$

Lemma 17.9. Assume that M is compact. $\mathrm{Ham}_c(M \setminus R_L) \subset \overline{\mathrm{Stab}(L)}$. In particular, if $R_L = \varnothing$, $\mathrm{Ham}(M, \omega) = \overline{\mathrm{Stab}(L)}$, and δ_{CH} vanishes identically.

Proof. Let $x \in M \setminus R_L$. By the definition of R_L,

$$x \in (R_L)^c = \bigcup_{\varphi \in \overline{\mathrm{Stab}(L)}} (\varphi^{-1}(L))^c,$$

so we can find $\varphi_x \in \overline{\mathrm{Stab}(L)}$ such that $\varphi_x(x) \cap L = \varnothing$. Moreover, since R_L is closed, we can pick an open neighborhood U_x of x such that $\varphi_x(U_x) \cap R_L = \varnothing$. Notice that if $\psi \in \mathrm{Ham}_c(\varphi_x(U_x))$, then we see that $\varphi_x^{-1}\psi\varphi_x \in \mathrm{Ham}_c(U_x)$, hence $\mathrm{Ham}_c(U_x) = \varphi_x^{-1}\mathrm{Ham}_c(\varphi_x(U_x))\varphi_x$. As $\mathrm{Ham}_c(\varphi_x(U_x)) \subset \overline{\mathrm{Stab}(L)}$, we have

$$\mathrm{Ham}_c(U_x) \subset \overline{\mathrm{Stab}(L)}.$$

Take $\varphi = \varphi_H^1 \in \mathrm{Ham}_c(M \setminus R_L)$. Since $\{U_x \mid x \in M \setminus R_L\}$ is an open cover of $M \setminus R_L$, by fragmentation property,

$$\varphi = \varphi_1 \cdots \varphi_n \in \overline{\mathrm{Stab}(L)}, \varphi_i \in \mathrm{Ham}_c(U_{x_i}).$$

In particular, if $R_L = \varnothing$, then $\delta_{\mathrm{CH}}(g(L), L) = 0$ for any $g \in \overline{\mathrm{Stab}(L)} = \mathrm{Ham}(M, \omega)$. □

Corollary 17.10. If $H|_{R_L} = 0$, then $\varphi_H^1 \in \overline{\mathrm{Stab}(L)}$.

Proof. Find neighborhoods $\{U_n\}$ of R_L such that $R_L = \bigcap\limits_{i=1}^{n} U_i$. For example, we may take a metric on M and consider $U_n = \left\{x \in M \mid d(x, R_L) < \frac{1}{n}\right\}$. Let ρ_n be a cutoff funciton with $\rho_n|_{U_n^c} = 1, \rho_n|_{R_L} = 0$. Then we set $F_n = F \cdot \rho_n$ and we have $F_n \to F$ in the C^0 sense. By definition, $\varphi_{F_n}^1 \in \overline{\mathrm{Stab}(L)}$ amd $d_{\mathrm{Hofer}}(\varphi_{F_n}^1, F_F^1) \to 0$ so $\varphi_F^1 \in \overline{\mathrm{Stab}(L)}$. □

Example 17.11. Let $F_1, \cdots, F_n \in C^\infty(M)$ such that $F_i|_{R_L} = 0$ and dF_i are linearly independent at some $x \in R_L$. We can define

$$\begin{aligned} \Phi\colon \mathbb{R}^n &\longrightarrow M \\ (t_i) &\longmapsto \varphi^1_{\sum t_i F_i}(x). \end{aligned}$$

As $\sum t_i F_i$ vanishes on R_L too, $\varphi^1_{\sum t_i F_i} \in \overline{\mathrm{Stab}(L)}$. Since R_L is closed, we have $\varphi^1_{\sum t_i F_i}(x) \in R_L$. Φ is a local immersion near x.

Now we can give a proof of Theorem17.2:

Proof. If $R_L = \varnothing$, then we have already known that δ_{CH} vanishes identically.

Assume $R_L \neq \varnothing$. For every $x \in R_L \subset L$, we can take a local coordinate $(x_1, \cdots, x_{2n})$ around x such that L is determined by $x_1 = \cdots = x_n = 0$. Then we may take $F_i = x_i, 1 \leqslant i \leqslant n$. By Example17.11 $\Phi\colon \mathbb{R}^n \to M$ is a local immersion, which implies $R_L \subset L$ is an open subset. Since L is connected, $R_L = L$ and δ_{CH} is non-degenerate. □

Remark 17.12. We may generalize this proof to the case when L is a k-dimensional submanifold. If $R_L = \varnothing$, then we may take a local coordinate $(x_1, \cdots, x_{2n})$ around x such that L is determined by $x_1 = \cdots = x_{2n-k} = 0$. Still we can define a local immersion $\Phi\colon \mathbb{R}^{2n-k} \to M$. This forces $2n - k \leqslant k$, so $k \geqslant n$. In other words, if $k < n$, we must have $R_L = \varnothing$.

Theorem17.2 suggests us that it is possible to replace the study of an n-dimensional submanifold by the study of its rigid locus in the symplectic sense. Although the rigid locus may admit some singularities, it enjoys other benefit.

Proposition 17.13. Let $L \subset M$ be a closed subset. Define

$$I_{R_L} = \{F \in C^\infty(M) \mid F|_{R_L} = 0\}.$$

Then I_{R_L} is closed under the Poisson bracket.

Proof. Let $F, G \in I_{R_L}$. Then by Corollary 17.10, $\varphi_F^1, \varphi_G^1 \in \overline{\mathrm{Stab}(L)}$. For any $x \in R_L$,

$$\{F, G\}(x) = -\left.\frac{d}{dt}\right|_{t=0} G(\varphi_F^t(x)) = 0.$$

The last equality holds because $\varphi_F^t(x) \in \overline{\mathrm{Stab}(L)}$ and

$$\varphi_F^t(x) \in \varphi_F^t\Big(\bigcap_{\varphi \in \mathrm{Stab}(L)} \varphi^{-1}(L)\Big) = R_L.$$

□

In symplectic geometry, we say a submanifold $S \subset M$ is **coisotropic** if $\{F \in C^\infty(M) \mid F|_S = 0\}$ is closed under the Poisson bracket. Proposition 17.13 implies that we can generalize this notation to closed subsets.

Definition 17.14. Let $L \subset M$ be a closed subset. L is said to be Hofer-coisotropic if δ_{CH} is non-degenerate on $\mathcal{O}(L)$.

Obviously, $\mathbb{S}^1 \subset (\mathbb{R}^2, \omega_0)$ is a Lagrangian submanifold, but to say a square is Lagrangian in $(\mathbb{R}^2, \omega_0)$ is nonsense . This definition makes it possible to consider Lagrangian subsets, and we can verify that a square is indeed Hofer-coisotropic.

If L is a submanifold, then L being Hofer-coisotropic implies L being coisotropic. Whether the converse holds is still an open question:

Conjecture: $L \subset M$ is a submanifold. Then L is coisotropic if and only if it is Hofer-coisotropic.

Another thing Zhang would like to mention is that we do not use the specific expression of Hofer norm in this section. Actually, we can replace the Hofer norm by any conjugate-invariant norm that depends on the C^0-information of $\mathrm{Ham}(M, \omega)$. In 2022, Viterbo constructed such a norm different from the Hofer norm.

Exercise 17.1. Consider the unit circle $L \subset (\mathbb{R}^2, \omega_0)$ and the orbit space

$$\mathcal{O}(L) = \left\{ \phi(L) \mid \phi \in \mathrm{Ham}_c(\mathbb{R}^2, \omega_0) \right\}.$$

Following the sub-problems below, conclude that the metric space $(\mathcal{O}(L), \delta_{\mathrm{CH}})$ has finite diameter.

(a) For any two different $L_1, L_2 \in \mathcal{O}(L)$, prove that for a sufficiently large $\kappa > 0$, a shift of the starting L, say $L(\kappa) = \left\{ (x, y) \in \mathbb{R}^2 \mid (x-\kappa)^2 + y^2 = 1 \right\}$ satisfies $\delta_{\mathrm{CH}}(L_1, L(\kappa)) = \delta_{\mathrm{CH}}(L_2, L_\kappa)$.

(b) Suppose $L_1 \cap L = L_2 \cap L = \varnothing$. Prove that for $L(\kappa)$ obtained from (a) above, there exists paths γ_1, γ_2 such that γ_1 lies in the unbounded component of $\mathbb{R}^2 \setminus L_1$ and γ_2 lies in the unbounded component of $\mathbb{R}^2 \setminus L_2$, and moreover both of them connects points $(1, 0) \in L$ and $(\kappa - 1, 0) \in L(\kappa)$ in $\mathbb{R}^2$.

(c) Deduce from (b) above that $\delta_{\mathrm{CH}}(L, L_i) = \delta_{\mathrm{CH}}(L(\kappa), L_i)$ for $i = 1, 2$. This implies that if two different $L_1, L_2 \in \mathcal{O}(L)$ satisfy $L_1 \cap L = L_2 \cap L = \varnothing$, then we have $\delta_{\mathrm{CH}}(L, L_1) = \delta_{\mathrm{CH}}(L, L_2)$.

(d) Prove that there exists a uniform constant $C > 0$ such that for any $L' \in \mathcal{O}(L)$, we have $\delta_{\mathrm{CH}}(L, L') \leqslant C$. Here C is independent of L'.

To end this section, here is a different perspective to view R_L. To be consistent with notations in papers, let's consider R_N for some subset $N \subset M$.

Definition 17.15. Let $A, N \subset M$ be subsets of M. The **displacement energy** with respect to Hofer norm of A away from N is defined by

$$e(A, N) = \inf \left\{ \|\varphi\|_{\mathrm{Hofer}} \mid \varphi(A) \cap N = \varnothing, \varphi \in \mathrm{Ham}_c(M, \omega) \right\}.$$

Sometimes we consider $e(A) = e(A, A)$, which measures the minimum energy to move a subset A away from itself.

Example 17.16. Let $(M, \omega) = (\mathbb{R}^2, \omega_0)$ and $A = N = \mathrm{pt}$. As one may expect, we have $e(A) = 0$. We can take a bump function ρ such that $\rho_{B(A,1)} = 1$. Then $H = \rho \cdot (\lambda x)$ has Hamiltonian flow $\varphi_H^t(x, y) = (x, y + t)$ near A. Similarly, $\varphi_{\lambda H}^t(x, y) = (x, y + \lambda t)$ near A. Therefore $e(A) \leqslant \|\lambda H\|_{\mathrm{Hofer}} \leqslant 2\pi\lambda$. As λ is arbitary, $e(A) = 0$.

The above example may not seem to be interesting because points are small intuitively.

Example 17.17. Let E be the equator of $(\mathbb{S}^2, \omega_{\mathrm{area}}) \subset \mathbb{R}^3$. Although E has symplectic volume 0, we have $e(E) = +\infty$! In other words, we can never move E away from itself by any Hamiltonian diffeomorphism. This phenomenon is called the Lagrangian version Arnold conjecture, which has been verified in many cases, including surfaces.

The conclusion holds for $M \subset (T^*M, d\lambda_{\mathrm{can}})$, where M is regarded as the zero section.

This example suggests that, when symplectic topology talks about something big or small, it is sometimes against the intuition.

One may ask what is the displace energy of a small ball inside $(\mathbb{R}^2, \omega_0)$, or a general symplectic manifold. The answer is the following fact:

Theorem 17.18 (Hofer, Lalonde–McDuff, Polterovich)**.** For any ball B which has symplectic volume ε inside any symplectic manifold (M, ω), the displacement energy $e(B)$ is a positive constant.

Proposition 17.19. Let $N \subset M$ be a closed subset. Then

$$R_N = \{ x \in N \mid \text{For any neighborhood } U \text{ of } x, e(U, N) > 0 \}.$$

Proof. Denote the set on the right hand side by E_N.

We first prove $E_N \subset R_N$. Pick $x \in E_N$. Suppose by contradiction, $x \notin R_N$. Then we can find $\varphi \in \overline{\mathrm{Stab}(N)}$ such that $x \notin \varphi^{-1}(N)$. Take an open neighborhood U of x such that $U \cap \varphi^{-1}(N) = \varnothing$. By Proposition 17.5, $\delta(\varphi(N), N) = \delta_{\mathrm{CH}}(N, \varphi^{-1}(N)) = 0$, so we can find $\varphi_n \in \overline{\mathrm{Stab}(L)}$ such that $\varphi^{-1}(N) = \varphi_n(N)$ and $\|\varphi_n\|_{\mathrm{Hofer}} \to 0$. Therefore $e(U, N) \leqslant \|\varphi_n^{-1}\|_{\mathrm{Hofer}} \to 0$, which violates the definition of E_N.

It remains to show that $R_N \subset E_N$. Let $x \in R_N$ and consider an open neighborhood U of x. If $U \subset N$, there is nothing to prove because $e(U, N) \leqslant e(U, U) > 0$. Assume $U \cap (M \setminus N) \neq \varnothing$. Then we can take $\varphi \in \mathrm{Ham}_c(U)$ such that $\varphi(x) \notin N$. In particular, $\varphi \notin \overline{\mathrm{Stab}(N)}$ and $\delta_{\mathrm{CH}}(\varphi(N), N) > 2\varepsilon_0$ for some $\varepsilon_0 > 0$. Take $\theta \in \mathrm{Ham}(M, \omega)$ with $\theta(U) \cap N$ and consider $\|\theta\|_{\mathrm{Hofer}}$. Notice that $\theta\varphi^{-1}\theta^{-1}$ is supported in $\theta(U)$, so $\varphi\theta\varphi^{-1}\theta^{-1}(N) = \varphi(N)$. Therefore, $\|\varphi\theta\varphi^{-1}\theta^{-1}\|_{\mathrm{Hofer}} > 2\varepsilon_0$. On the other hand,

$$\|\varphi\theta\varphi\theta^{-1}\|_{\mathrm{Hofer}} > 2\varepsilon_0 \leqslant \|\varphi\theta\varphi^{-1}\|_{\mathrm{Hofer}} + \|\theta^{-1}\|_{\mathrm{Hofer}} = 2\|\theta\|_{\mathrm{Hofer}},$$

which implies $e(U, N) > 0$. $\square$

Corollary 17.20. If $N_1 \subset N_2$ are closed subsets, then $R_{N_1} \subset R_{N_2}$.

18. Distances between Lagrangian submanifolds at large-scale

In the last section, we usually consider situations when δ_{CH} are small. Now we shift to the global aspect of δ_{CH}.

Let $L \subset (M, \omega)$ be a Lagrangian submanifold. Then Chekanov showed that $(\mathcal{O}(L), \delta_{\mathrm{CH}})$ is a metric space. We are interested in the diameter of this space.

Example 18.1. Let E be the equator of $(\mathbb{S}^2, \omega_{\mathrm{area}})$. In some sense, this is the simplest non-trivial model of Lagrangian submanifolds. However, it remains an open whether $(L, \delta_{\mathrm{CH}})$ is bounded or not.

Theorem 18.2 (Khanevsky, Seyfaddini)**.** Consider the unit disc $(\mathbb{D}^2, \omega_0)$ and the real line segement L. Then $(\mathcal{O}(L), \delta_{\mathrm{CH}})$ admits a large scale geometry. Explicitly, there exists some constants $A, B > 0$ and a map $\Phi\colon (C_c^\infty(0,1), |\cdot|_\infty) \mapsto (\mathcal{O}(L), \delta_{\mathrm{CH}})$ such that

$$A(|f-g|_\infty - B) \leqslant \delta_{\mathrm{CH}}(\Phi(f), \Phi(g)) \leqslant |f-g|_\infty .$$

Remark 18.3. The same conclusion holds for the unit ball B inside $\mathbb{R}^{2n}$, with

$$L = \{(x_1, y_1, \cdots, x_n, y_n) \in B \mid y_1 = \cdots = y_n = 0\} .$$

The standard tool to cook up the lower bound estimate is the quasi-morphism.

Definition 18.4. Let G be a group. A **quasi-morphism** of G is a map $q\colon G \to \mathbb{R}$ such that

$$|q(gh) - q(g) - q(h)| \leqslant C, g, h \in G$$

for some $C \geqslant 0$.

q is said to be homogeneous if $q(g^n) = nq(g), n \in \mathbb{Z}$.

Exercise 18.1. Let $q\colon G \to \mathbb{R}$ be a homogeneous quasi-morphism.
 (i) Prove that it is a homomorphism when it is restricted to any abelian subgroup of G.
 (ii) Prove that q is a class function.

Proof. (i) Let H be an abelian subgroup of G and take $a, b \in H$.

$$|q(a^n b^n) - q(a^n) - q(b^n)| = n\,|q(ab) - q(a) - q(b)| \leqslant C,$$

so taking $n \to \infty$ implies $|q(ab) - q(a) - q(b)| = 0$.

(ii) Notice that $q(1) = nq(1)$ for every $n \in \mathbb{Z}$, so we must have $q(1) = 0$. Let $x, g \in G$.

$$\left|q(gx^ng^{-1}) - q(x^n)\right| = n\left|q(gxg^{-1}) - q(x)\right|,$$

so we have

$$\left|q(gxg^{-1}) - q(x)\right| = \frac{1}{n}\left|q(gx^ng^{-1}) - q(x^n)\right| = \frac{1}{n}\left|q(gx^ng^{-1}) - q(x^n) - q(1)\right| \leqslant \frac{C}{n},$$

and the conclusion follows. $\square$

The following is an example of a quasi-morphism.

Example 18.5. Consider $\widetilde{\mathrm{Diff}^+}(\mathbb{S}^1)$, the group of oreintation-preserving diffeomorphisms from $\mathbb{S}^1$ to itself. Its universal cover $\mathrm{Diff}^+(\mathbb{S}^1)$ can be identified with

$$G = (\{\varphi \in C^\infty(\mathbb{R}) \mid \varphi \text{ is strickly increasing}, \varphi(x+1) = \varphi(x) + 1\}, \circ).$$

In fact, G is a perfect group, so it has no abelian quotient. In other words, there is no non-trivial homomorphism $G \to \mathbb{R}$. However, it admits a quasi-morphism $c\colon G \to \mathbb{R}, \varphi \mapsto \varphi(0)$. Indeed,

$$\begin{aligned}\varphi \circ \psi(0) &\leqslant \varphi([\psi(0)] + 1) = \varphi(0) + [\psi(0)] + 1 \leqslant \varphi(0) + \psi(0) + 1,\\ \varphi \circ \psi(0) &\geqslant \varphi([\psi(0)]) = \varphi(0) + [\psi(0)] \geqslant \varphi(0) + \psi(0) - 1.\end{aligned}$$

Therefore q is a quasi-morphism with constant 1, but it is not homogeneous. To resolve this problem, we can its homogenization, namely

$$q(\varphi) = \lim_{n\to\infty} \frac{c(\varphi^n)}{n}.$$

To show that q is well-defined, let $a_n = c(\varphi^n) + 1$. Then

$$a_{m+n} = c(\varphi^{m+n}) + 1 \leqslant c(\varphi^m) + c(\varphi^n) + 2 = a_m + a_n.$$

Then

Remark 18.6. We have learnt that, when (M,ω) is a closed symplectic manifold, $\mathrm{Ham}(M,\omega)$ is a simple group. As a consequence, looking for interesting quasi-morphisms $\mathrm{Ham}(M,\omega) \to \mathbb{R}$ used to be a central problem.

Proposition 18.7 (Entov–Polterovich)**.** There exists a homogeneous quasi-morphism $\mu\colon \mathrm{Ham}(\mathbb{CP}^1, \omega_{\mathrm{FS}}) \to \mathbb{R}$ such that $|\mu(\varphi)| \leqslant \|\varphi\|_{\mathrm{Hofer}}$. Moreover, μ satisfies:

(1) (Calabi property) If $F_t\colon [0,1] \times \mathbb{CP}^1 \to \mathbb{R}$ has a displaceable support, then

$$\mu(\varphi) = \int_0^1 dt \int_{\mathbb{CP}^1} F_t \omega_{\mathrm{FS}}.$$

(2) (Lagrangian control property) If $F\colon [0,1]\mathbb{CP}^1 \to \mathbb{R}$ is constant r along the equator L of $\mathbb{CP}^1 = \mathbb{S}^2$, then

$$\mu(\varphi) = -r + \int_0^1 dt \int_{\mathbb{CP}^1} F_t \omega_{\mathrm{FS}}.$$

Remark 18.8. In fact Entov and Polterovich proved the existence of homogeneous quasi-morphisms on $(\mathbb{CP}^n, \omega_{\mathrm{FS}}), (\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\mathrm{prod}})$.

Remark 18.9. The Calabi property in Proposition 18.7 is named after the Calabi morphism

$$\begin{aligned}\mathrm{Cal}\colon \mathcal{P}\mathrm{Ham}(M,\omega) &\longrightarrow \mathbb{R},\\ \{\varphi_H^t\} &\longmapsto \int_0^1 dt \int_M H_t \omega^n,\end{aligned}$$

where $\mathcal{P}\mathrm{Ham}(M,\omega)$ is the collection of Hamiltonian isotopies. It is a homomorphism because

$$\mathrm{Cal}(\{\varphi^t \circ \psi^t\}) = \mathrm{Cal}(\{\varphi^t\}) + \mathrm{Cal}(\{\psi^t\}).$$

Due to Banyaga and Calabi, if (M,ω) is an exact symplectic manifold ($\omega = d\lambda$), then Cal is well-defined on $\mathrm{Ham}(M,\omega)$, as

$$\mathrm{Cal}(\{\varphi^t\}) = \frac{1}{n+1} \int_M (\varphi^1)^* \phi\lambda \wedge \lambda \wedge \omega^{n-1}.$$

Note that this construction does not contradict the simple property of Hamiltonian groups of closed symplectic manifolds, for an exact symplectic manifold is never closed.

19. HARD STONES ON LAGRANGIAN SUBMANIFOLDS REVISITED

Question: How to relative multiple Lagrangian submanifolds?

Let $\{\varphi_H^t\}$ be a Hamiltonian isotopy generated by $H \in C^\infty([0,1] \times M)$. Assume H vanishes near $t = 0, 1$. Then H can be extended to $H \in C^\infty(\mathbb{R} \times M)$ supported inside $[0,1] \times M$.

Definition 19.1. Let $L \subset (M, \omega)$ be a Lagrangian submanifold, and let $\{\varphi_H^t\}_{t\in\mathbb{R}}$ be a Hamiltonian isotopy supported in $[0,1] \times M$. The **Lagrangian suspension** of L is

$$S(L, \{\varphi_H^t\}) = \left\{(t, H(t, \varphi_H^t(x)), \varphi_H^t(x)) \mid t \in \mathbb{R}, x \in L\right\},$$

which is a submanifold of $\mathbb{R}^2 \times M$.

Proposition 19.2. Let $L \subset (M, \omega)$ be a Lagrangian submanifold and take a Hamiltonian isotopy $\{\varphi_H^t\}_{t\in\mathbb{R}}$ supported in $[0,1] \times M$. Then the Lagrangian suspension $S(L, \{\varphi_H^t\})$ of L is a Lagrangian submanifold in $\mathbb{R}^2 \times M$ with respect to the symplectic structure $\omega_0 \oplus \omega$.

Proof. Consider

$$\begin{aligned} f\colon \mathbb{R} \times L &\longrightarrow M \\ (t, x) &\longmapsto (t, H(t, \varphi_H^t(x)), \varphi_H^t(x)). \end{aligned}$$

Then f is an embedding with $S(L, \{\varphi_H^t\}) = f(\mathbb{R} \times M)$. It suffices to show that

$$\begin{gathered} \omega_0 \oplus \omega(f_*X_1, f_*X_2), X_1, X_2 \in TL, \\ \omega_0 \oplus \omega(f_*Y, f_*\partial_t) = 0, Y \in TM, \partial_t \in T\mathbb{R}. \end{gathered}$$

Indeed, $\omega_0 = dx \wedge dy$ and $f_*X_i \in \operatorname{span}(\partial_y) \oplus TM$, so

$$\omega_0 \oplus \omega(f_*X_1, f_*X_2) = \omega_0((\pi_1 \circ f)_*X_1, (\pi_1 \circ f)_*X_2) + (\varphi_H^t)^*\omega(X_1, X_2) = 0.$$

As for the second equality, since $f_*Y = (0, (\varphi_H^t)_*(Y)H, (\varphi_H^t)_*(Y))$ and $f_*\partial_t = (1, \frac{\partial H}{\partial t}(t, \varphi_H^t(x)), X_H^t)$,

$$\omega_0 \oplus \omega(f_*Y, f_*\partial_t) = (\varphi_H^t)_*(Y)H + \omega(\varphi_H^*(Y), X_H^t) = (\varphi_H^t)_*(Y)H - (\varphi_H^t)_*(Y)H = 0.$$

□

Let $S(L, \{\varphi_H^t\}) \subset (\mathbb{R}^2 \times M, \omega_0 \oplus \omega)$ be a Lagrangian suspension as set in Proposition 19.2. Let $\pi\colon S(L, \{\varphi_H^t\}) \to \mathbb{R}$ be the projection of $\mathbb{R}^2$-component, with image denoted by Π. As the Hamiltonian function H is supported in $[0,1] \times M$, we have

$$\begin{aligned} \Pi \cap \{(t, s) \mid t < 0\} &= \{(t, 0) \mid t < 0\}, \\ \Pi \cap \{(t, s) \mid t > 1\} &= \{(t, 0) \mid t > 1\}. \end{aligned}$$

The non-trivial component of Π is $\Pi \cap \{(t, s) \mid 0 \leqslant t \leqslant 1\}$. As a consequence,

$$\operatorname{Area}(\Pi) = \int_0^1 (\max_{\varphi_H^t(L)} H_t - \min_{\varphi^t(L)} H_t) dt \leqslant \int_0^1 (\max_M H_t - \min_M H_t) dt.$$

Note that the right hand side is the ingredient for Hofer norm. Two things derverse a mention. One is that L and $\varphi_H^1(L)$ are coborded by $S(L, \{\varphi_H^t\})$. The other is $\|\cdot\|_{\mathrm{Hofer}}$ obtains a geometric lower bound. Sometimes, the non-trivial part of Π is called the shadow of $S(L, \{\varphi_H^t\})$.

Example 19.3 (Lagrangian surgery)**.** We introduce a technique to resolve singularities of intersecting Lagrangian submanifolds. Let's start with a toy version.

Consider $\mathbb{C}$ and its real axis and imaginary axis, denoted by L_1, L_2. Pick a smooth curve $\gamma\colon \mathbb{R} \to \mathbb{C}, t \mapsto a(t) + \sqrt{-1}b(t)$, where

$$\begin{cases} (a(t), b(t)) = (t, 0), t \leqslant -1 \\ (a(t), b(t)) = (0, t), t \geqslant 1 \\ a'(t), b'(t) > 0, -1 < t < 1. \end{cases}$$

Then we can define

$$L_1 \#_0 L_2 = \left\{x \cdot (a(t) + \sqrt{-1}b(t)) \mid x \in \mathbb{R}, x^2 = 1, t \in \mathbb{R}\right\},$$

which consists of two connected components. In general, consider $L_1 = \mathbb{R}^n, L_2 = \sqrt{-1}\mathbb{R}^n \subset (\mathbb{C}^n, \omega_0)$. We can define

$$L_1 \#_0 L_2 = \left\{ (x_i \cdot (a(t) + \sqrt{-1}b(t))) \in \mathbb{C}^n \mid x_i \in \mathbb{R}, \sum_i x_i^2 = 1, t \in \mathbb{R} \right\}.$$

It will be an exercise to prove that $L_1 \#_0 L_2$ is a Lagrangian submanifold.

Remark 19.4. By Darboux theorem, similar operation can be done on general symplectic manifolds. Let $L_1, L_2 \subset (M, \omega)$ be Lagrangian submanifolds such that $L_1 \pitchfork L_2$ and $L_1 \cap L_2 = \{p_1, \cdots, p_n\}$. We can also define $L_1 \#_{p_1 \cdots, p_n} L_2$.

Note that in general case, the orientaion at each intersection point matters. For example, consider $L_1 = \{z \in \mathbb{C} \mid |z-1| = 2\}$ and $L_2 = \{z \in \mathbb{C} \mid |z+1| = 2\}$. Then $L_2 \#_{p_1, p_2} L_2$ can be a single $\mathbb{S}^1$ or two copies of $\mathbb{S}^1$.

Definition 19.5. Let (M, ω) is a symplectic manifold and $(L_1, \cdots, L_k), (L_1', \cdots, L_{k'}')$ be two families of Lagrangian submanifolds. A **Lagrangian cobordism** from $(L_1, \cdots, L_k)$ to $(L_1', \cdots, L_{k'}')$ is a Lagrangian submanifold $V \subset (\mathbb{C} \times M, \omega_0 \oplus \omega)$ satisfying the following property:

Let $\pi \colon \mathbb{C} \times M \to \mathbb{C}$ be the projetion of the first factor. Then there exists $R > 0$ such that

$$\pi^{-1}(B(0,R)^c) = \bigsqcup_{i=1}^{k} ([R, +\infty) \times \{i\} \times L_i) \sqcup \bigsqcup_{j=1}^{k'} \left((-\infty, -R] \times \{j\} \times L_j'\right).$$

By definition, we can see that a Lagrangian suspension $S(L, \{\varphi_H^t\})$ is a Lagrangian cobordism from $\varphi_H^1(L)$ to L.

Proposition 19.6. There exists a Lagrangian cobordism between (L_1, L_2) and $L_1 \#_0 L_2$.

There is a more abstract picture for Lagrangian cobordism. To explain this we shall introduce the cobordism category.

Definition 19.7. The **cobordism category $\mathbf{Cob}(M)$** of (M, ω) is defined in the following way. Its objects are Lagrangian submanifolds in (M, ω). Let L, L' be two Lagrangian submanifolds. $\mathrm{Hom}(L, L')$ is the collection of Lagrangian cobordisms from L to $(L_1, \cdots, L_k, L')$, where L_i are also Lagrangian submanifolds. The identity in $\mathrm{Hom}(L, L)$ is the Lagrangian suspension $S(L, \{\varphi_H^t\})$ with $H \equiv 0$, and the composition of two morphisms is concatenating the two cobordisms.

Theorem 19.8 (Biran-Cornea)**.** There exists a triangulated category $\mathcal{F}$ whose objects are Lagrangian submanifolds $L \subset (M, \omega)$ and a functor $\varphi \colon \mathbf{Cob}(M) \to \mathcal{F}$ such that $\mathcal{F}(L) = L$ and for a cobordism $V \in \mathrm{Hom}(L, L')$ from L to $(L_1, \cdots, L_k, L')$, $\varphi(V)$ is a sequence of exact triangles in the following form:

$$\begin{cases} L_1 \to 0 \to Z_1 \to L_1[1] \\ L_2 \to Z_1 \to Z_2 \to L_2[1] \\ \vdots \\ L_k \to Z_{k-1} \to Z_k \to L_k[1] \\ L' \to Z_k \to L \to L'[1]. \end{cases}$$

This is saying that L can be built up from $L_1, \cdots, L_k, L'$ at least algebraically.

Remark 19.9. Actually, $\mathcal{F}$ is called the derived Fukaya category, which Zhang did not have time to talk about in this class.

Note that $\mathbf{Cob}(M)$ itself do not have the triangulated structures, so this theorem relates the topology of Lagrangian submanifolds with algebraic structure in $\mathcal{F}$.

Theorem 19.8 may be an abstract nonsense to those who are not familiar with triangulated categories, so the following is a brief introduction. A good reference is the classical book *Sheaves on Manifolds* by Kashiwara and Schapira.

Definition 19.10. A **triangulated abelian category** consists of the following data:
(1) An abelian category $\mathcal{C}$.
(2) An equivalence $T \colon \mathcal{C} \to \mathcal{C}$. We usually write $TX = X[1]$ and $Tf = f[1]$.
(3) A family of triangles $X \to Y \to Z \to X[1]$, usually called exact triangles or distinguished triangles.

Their relations are as follows:

(1) If a triangle $X \to Y \to Z \to X[1]$ is isomorphic to an exact triangle $X' \to Y' \to Z' \to X'[1]$, i.e., there exists isomorphisms f, g, h such that

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1], \end{tikzcd}$$

then $X \to Y \to Z \to X[1]$ is also an exact triangle.

(2) For any $X \in \mathrm{Ob}(\mathcal{C})$, the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is exact.

(3) For any morphism $f\colon X \to Y$, there exists $Z \in \mathrm{Ob}(\mathcal{C})$ such that $X \xrightarrow{f} Y \to Z \to X[1]$ is an exact triangle.

(4) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is an exact triangle if and only if $Y \xrightarrow{h} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is an exact triangle.

(5) Let $X \to Y \to Z \to X[1]$ and $X' \to Y' \to Z' \to X'[1]$ be two exact triangles and $u\colon X \to X', v\colon Y \to Y'$ such that the diagram commutes

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Y \arrow[d, "v"] \\ X' \arrow[r] & Y'. \end{tikzcd}$$

Then there exists $w\colon Z \to Z'$ (not necessarily unique) such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, "w"] & X[1] \arrow[d, "u[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]. \end{tikzcd}$$

(octahedral axiom)

Remark 19.11. Z in the exact triangle $X \xrightarrow{f} Y \to Z \to X[1]$ is sometimes called the mapping cone of f and denoted by $\mathrm{Cone}(f)$. This terminology comes from the classical construction in algebraic topology, which recover the mapping cone here under reasonable circumstances.

Proposition 19.12. If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is an exact triangles in a triangle category $\mathcal{C}$, then $g \circ f = h \circ g = f[1] \circ h = 0$. In particular it is also an exact sequence.

Proof. It suffices to show that $g \circ f = 0$, and the following conclusions follow from rotation. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

It naturally induces

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}_{X[1]}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1]. \end{tikzcd}$$

□

Example 19.13. Let $L_1, L_2 \subset (M, \omega)$ be two Lagrangian submanifolds. Take $x \in L_1 \cap L_2$ and we have a Lagrangian cobordism V from $L_1 \#_x L_2$ to (L_1, L_2). By Theorem 19.8, we obtain an exact triangle (in $\mathcal{F}$)

$$L_2 \longrightarrow L_1[1] \longrightarrow L_1 \#_x L_2 \longrightarrow L_2[1].$$

As a consequence, roughly speaking, $L_1 \#_x L_2 = \mathrm{Cone}(L_2 \to L_1[1])$.

Example 19.14. Fix a field $\mathbb{K}$ and denote the category of $\mathbb{K}$-vector spaces by $\mathcal{A}$. Let $C(\mathcal{A})$ be the category of cochain complexes over $\mathcal{A}$. It has a quotient category $K(\mathcal{A})$, called the homotopy category over $\mathcal{A}$, whose morphisms are modulo by homotopy relations. In fact, $K(\mathcal{A})$ is triangulated.

Proposition 19.15. *$K(\mathcal{A})$ is triangulated.*

Proof. We only give an outline. Define $T\colon K(\mathcal{A}) \to K(\mathcal{A})$ by setting $T(C_\bullet)_k = C_{k+1}$ and $(Tf)_k = f_{k+1}$.

Next we construct the mapping cone for $f\colon C_\bullet \to D_\bullet$. Let $\mathrm{Cone}(f)_k = C_{k+1} \oplus D_k$. Then differential is given by

$$d_k = \begin{pmatrix} (d_C)_{k+1} & 0 \\ f_{k+1} & -(d_D)_k \end{pmatrix}.$$

By definition, $d_{k+1} \circ d_k = 0$, so $\mathrm{Cone}(f)$ is well-defined. Then we say a triangle is exact if they are isomorphic to some $C_\bullet \xrightarrow{f} D_\bullet \to \mathrm{Cone}(f) \to C_\bullet[1]$.

It suffices to show that $C_\bullet \xrightarrow{\mathrm{id}} C_\bullet \to 0 \to C_\bullet[1]$, i.e., $\mathrm{Cone}(\mathrm{id})$ and 0 are homotopic equivalent. We only need to show that $\mathrm{id}, 0\colon \mathrm{Cone}(\mathrm{id}) \to \mathrm{Cone}(\mathrm{id})$ are homotopic. Indeed, consider $h_k\colon C_{k+1} \oplus C_k \to C_k \oplus C_{k-1}, (x,y) \mapsto (y,0)$. Then

$$(d_{k-1}h_k + h_{k+1}d_k)(x,y) = d_{k-1}(y,0) + h_{k+1}(dx, x-dy) = (dy,y) + (x-dy,0) = (x,y).$$

□

Exercise 19.1. If $0 \to C_\bullet \xrightarrow{f} D_\bullet \to E_\bullet \to 0$ is a short exact sequence of cochain complexes in $C(\mathcal{A})$, then $E_\bullet \simeq \mathrm{Cone}(f)$ in $K(\mathcal{A})$. In particular, a short exact sequence in $C(\mathcal{A})$ induces an exact triangle $C_\bullet \xrightarrow{f} D_\bullet \to E_\bullet \to C_\bullet[1]$ in $K(\mathcal{A})$.

20. Contact geometry and neck-stretching

It is possible to embed the 2-torus into $\mathbb{C}^2$ such that its image is a Lagrangian submanifold. Is there any Lagrangian Klein bottle inside $\mathbb{C}^2$?

The answer to this question is no, due to Nemirovski.

Theorem 20.1 (Gromov)**.** *There is no compact exact Lagrangian submanifold in $\mathbb{C}^n$.*

Theorem 20.2 (Cieliebak, Mohnke)**.** *There is no exact Lagrangian torus in $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$.*

Remark 20.3. There needs some explanation for the definition of exact Lagrangian submanifolds in Theorem 20.2 because $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is not an exact symplectic manifold.

We say a Lagrangian submanifold $L \subset (\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is exact if any embedding $f\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (\mathbb{CP}^2, L)$ satisfies $f^*\omega_{\mathrm{FS}}(\mathbb{D}^2) \in \mathbb{Z}a$, where $a = \omega_{\mathrm{FS}}(L)$ and $L \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is the line class.

For example, any curve in $(\mathbb{CP}^1 = \mathbb{S}^2, \omega_{\mathrm{FS}})$ is not exact because $\omega_{\mathrm{FS}}(\mathbb{S}^2) = 1$ but $f^*\omega_{\mathrm{FS}}(\mathbb{D}^2) \in (0,1)$.

Although it is not possible to introduce the proofs to the first two results, we will present the proof of the third theorem by introducing the important technique **neck-stretching**.

Neck stretching is from **contact geometry**, the odd-dimensional analogue of symplectic geometry.

Definition 20.4. Let M be a $2n+1$-dimensional manifold. A contact structure on M is a hyperplane field $\xi = \ker\alpha$ for some 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume form.

Example 20.5. Consider $\mathbb{R}^3$ with $\alpha = dz - ydx$. Then $d\alpha = dx \wedge dy$ and $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$ is a volume form. We have $\xi = \ker\alpha = \mathrm{span}(\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$.

Similarly, let $\beta = \cos r dz + r\sin r d\theta$ where (r,θ,z) is the cylinder coordinate. Then

$$\beta \wedge d\beta = \left(\frac{\sin r\cos r}{r} + 1\right) r dr \wedge d\theta \wedge dz.$$

The corresponding hyperplane field is $\xi = \mathrm{span}(\frac{\partial}{\partial r}, r\sin r\frac{\partial}{\partial z} - \cos\frac{\partial}{\partial\theta})$.

Many concepts in symplectic geometry find their correspondence in contact geometry:

Here are some basic observations:

(1) If α is a contact form on M, then so is $\beta = e^f\alpha$ for $f \in C^\infty(M)$. Indeed, $\ker\alpha = \ker\beta$ and $d\beta = e^{nf}\alpha \wedge (d\alpha)^n$ is also a volume form.

symplectic manifold (X^{2n},ω)	contact manifold (M^{2n+1},α)
Lagrangian submanifold	Legendrian submanifold
Hamiltonian dynamic	contact Hamiltonian dynamic
Poisson bracket	contact Poisson bracket
Hofer norm	Hofer-like norm
Quantum cohomology	In progress (by Yong-Geun Oh)

(2) If α is a contact form and $\xi=\ker\alpha$, then $d\alpha|_\xi$ is non-degenerate. Take a basis $x_i, y_i \in \xi_x, x \in M$. Then we can extend x_i, y_i to a basis of T_xM by adding $z \in T_xM$. Since $\alpha\wedge(d\alpha)^n$ is a volume form,

$$0 \neq \alpha\wedge(d\alpha)^n(z, x_1, y_1, \cdots, x_n, y_n) = \alpha(z)(d\alpha)(x_1,y_1)\cdots(d\alpha)(x_n,y_n) \neq 0.$$

In particular, $(d\alpha)(x_i,y_i)\neq 0$, so $d\alpha|_\xi$ is non-degenerate. Some people would say that $d\alpha$ restricts to a symplectic form on ξ.

(3) Let α be a contact form. There exists a unique vector field R_α such that $\alpha(R_\alpha)=0$ and $\iota_{R_\alpha}d\alpha=0$. This vector field is called the **Reeb vector field** of α.

Example 20.6. On $(\mathbb{R}^3, dz-ydx)$, the Reeb vector field is $\frac{\partial}{\partial z}$. On $(\mathbb{R}^3, \cos r dz + r\sin r d\theta)$, the Reeb vector field is

$$\frac{\sin r}{r+\sin r\cos r}\frac{\partial}{\partial\theta} + \frac{\sin r + r\cos r}{r+\sin r\cos r}\frac{\partial}{\partial z}.$$

Remark 20.7. To compute the Reeb vector field is easy, but to find a closed orbit of the flow generated by a Reeb vector field (called the **Reeb flow**) is very hard.

Weinstein conjecture: Every closed contact manifold admits at least one Reeb orbit. This conjecture has been proved in dimension 3 by Taubes.

Remark 20.8. By the classical manifold theory, the contact hyperplane field ξ is completely non-integrable.

Exercise 20.1. Let (M,α) be a contact manifold with a presribed contact form. A contact Hamiltonian vector field X on (M,α) satisfies $\mathcal{L}_X\alpha = e^f\alpha$ for some $f\colon M\to\mathbb{R}$, called the conformal factor. Complete the following problems.

(1) Given a function $h\colon [0,1]\times M\to\mathbb{R}$, prove that the following system of equations

$$\begin{aligned} \iota_{X_t}d\alpha &= dh_t(R_\alpha)\alpha - dh_t, \\ \alpha(X_t) &= h_t, \end{aligned}$$

determines a contact Hamiltonian vector field X_t and calculate the corresponding conformal factor.

(2) Calculate X_t that is generated by $h_t\equiv 1$.

(3) By intergrating X_t, one obtains a flow denoted by $\{\phi_h^t\}$ called a contact isotopy generated by h_t. Prove that a contact isotopy preserves the contact structure, that is, $(\phi_h^t)^*\xi=\xi$ for any $t\in[0,1]$. Here $\xi=\ker\alpha$.

(4) Conversely, prove that any isotopy $\{\phi_t\}$ that preserves the contact structure ξ is Hamiltonian. As comparison, in symplectic geometry, not every isotopy preserving the symplectic structure is Hamiltonian.

Proof. (1) Let x_i, y_i be a basis at ξ_p such that $d\alpha(x_i,y_i)=1$. Suppose

$$X_t(p) = \sum A_i x_i + B_i y_i + h_t R_\alpha.$$

Then $d\alpha(X_t, y_j) = A_j, d\alpha(X_t, x_j) = B_j$. We can take

$$\begin{aligned} A_i &= dh_t(R_\alpha)\alpha(y_i) - dh_t(y_i), \\ B_i &= dh_t(R_\alpha)\alpha(x_i) - dh_t(x_i). \end{aligned}$$

By definition,

$$\mathcal{L}_{X_t}\alpha = d\iota_{X_t}\alpha + \iota_{X_t}d\alpha = dh_t + dh_t(R_\alpha)\alpha - dh_t = dh_t(R_\alpha)\alpha.$$

The conformal factor is $\log dh_t(R_\alpha)$.

(2) If $h_t\equiv 1$, $A_i=B_i=0$, so $X_t=R_\alpha$.

(3) Take a vector field Y inside ξ and let X_t be the contact Hamiltonian vector field generated by h_t. Then $\mathcal{L}_{X_t} Y = [X_t, Y]$. It follows that

$$\alpha([X_t, Y]) = d\alpha(X_t, Y) - X_t\alpha(Y) + Y\alpha(X_t) = dh_t(R_\alpha)\alpha(Y) - dh_t(Y) + Yh_t = 0,$$

hence $[X_t, Y]$ remains inside ξ.

(4) It suffices to check that

$$\iota_{X_t} d\alpha = dg_t(R_\alpha) - dg_t,$$

where $g_t = \alpha(X_t)$. Indeed, for any Y inside ξ,

$$d(\alpha(X_t))(R_\alpha)\alpha(Y) - d(\alpha(X_t))(Y) = -(d\iota_{X_t}\alpha)(Y) = (\iota_{X_t} d\alpha)(Y) - (\mathcal{L}_{X_t}\alpha)(Y) = (\iota_{X_t} d\alpha)(Y).$$

As for the Reeb vector field, we have $d(\alpha(X_t))(R_\alpha)\alpha(R_\alpha) - d(\alpha(X_t))(R_\alpha) = 0$. $\square$

Exercise 20.2. Given a contact manifold (M, α) with $\dim M = 3$, prove that any contact vector field X on (M, α) with the condition $\alpha(X_t) > 0$ everywhere can be realized as a Reeb vector field of some contact 1-form β.

Proof. Let $\beta = h_t^{-1}\alpha$ where $h_t = \alpha(X_t)$. Since X is a contact vector field, we can find $f > 0$ such that $\mathcal{L}_X \beta = f\beta$. Then

$$\iota_X d\beta = \mathcal{L}_X \beta - d\iota_X \beta = \mathcal{L}_X \beta = f\beta,$$

hence $f = f\beta(X) = \iota_X d\beta(X) = 0$, which imples X is the Reeb vector field of β. $\square$

Contact geometry and symplectic geometry are highly connected.

Example 20.9. Consider $(\mathbb{C}^n, \omega_0 = d\lambda_0)$. Take domain U such that it is star-shaped with respect to the origin and ∂U is smooth. Define the radial vector field $Y = \frac{1}{2}\sum x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$. Then $\partial U \pitchfork Y$ and $(\partial U, \lambda|_{\partial U})$ is a contact manifold.

Indeed, $\iota_Y \omega_0 = \lambda_0$, so

$$\lambda|_{\partial U} \wedge (d\lambda|_{\partial U})^n = \lambda \wedge (\omega)^n|_{\partial U} = (\iota_Y \omega_0 \wedge (\omega)^n)|_{\partial U} = \frac{1}{n}\iota_Y(\omega^n)|_{\partial U}.$$

Since $\partial U \pitchfork Y$, the top form above is non-degenerate.

A classical family of such contact manifolds are given by symplectic ellipsoids:

$$E(a,b) = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi |z_1|^2}{a} + \frac{\pi |z_2|^2}{b} \leqslant 1\right\}.$$

As each $E(a, b)$ is star-shaped with respect to the origin, $(\partial E(a, b), \lambda)$ is a contact manifold.

Exercise 20.3. Consider the star-shaped domain

$$P(c,d) = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid \pi |z|^1 \leqslant c, \pi |z_2|^2 \leqslant d\right\}.$$

Describe all the closed Reeb orbits on $\partial P(c, d)$, where the contact structure is given by the restriction of the standard primitive λ_0.

Example 20.10. Let (Q, g) be a Riemannian manifold. Consider

$$D_g^* Q = \{q \in T^*Q \mid |q| \leqslant 1\} \subset (T^*Q, d\lambda_{\text{can}}).$$

Under the local coordinate (q, p), we define $Y = p\frac{\partial}{\partial p}$ (which is well-defined on the global manifold, and called the Liouville vector field). Following the same argument as Example 20.9, we see that

$$S_g^* Q = \{q \in T^*Q \mid |q| = 1\}$$

is a contact manifold with $\lambda_{\text{can}}|_{S_g^*Q}$.

Exercise 20.4. The closed Reeb orbits of $(S_g^*Q, \lambda_{\text{can}}|_{S_g^*Q})$ is 1-1 corresponding to closed geodesics in (Q, g).

Example 20.11 (Symplectization)**.** Given a contact manifold (M, α), we can define its **symplecization** SM, which is a symplectic manifold. Explicitly, we set $SM = \mathbb{R} \times M$ and $\omega = d(e^r\alpha)$ where r is the coordinate on $\mathbb{R}$. By definition, $\omega = e^r dr \wedge \alpha + e^r d\alpha$ and

$$\omega^n = e^r dr \wedge \alpha \wedge (d\alpha)^n$$

is a volume form on SM.

People prefer to imagine SM as a cone whose cusp lies at $-\infty$ side. Interestingly, by Cartan's magic formula, $\mathcal{L}_{\partial_r}\omega = d\iota_{\partial_r} d(e^r\alpha) = d(e^r\alpha) = \omega$. This means that ω is rescaled exponentially along the parameter r.

On SM, there is a special almost complex structure J, satisfying:
(1) $J(\partial_r) = R_\alpha, J(R_\alpha) = -\partial_r$.
(2) J is $\mathbb{R}$-invariant, i.e. $J_{r,p} = J_{r+1,p}$.
(3) $(J|_\xi)^2 = -\operatorname{id}$ and $d\alpha|_\xi(-, J-)$ is a metric.
Let's analyze a J-holomorphic curve $u\colon (\mathbb{S}^2, j) \to (M, \omega, J)$ with J defined above. Decompose u into $(u_\mathbb{R}, u_M)$. Then the equation of J-holomorphic curve writes

$$J_{u(s,t)}\left(\frac{\partial u_\mathbb{R}}{\partial t}\frac{\partial}{\partial r}, \alpha(\frac{\partial u_M}{\partial t})R_\alpha, \frac{\partial u_M}{\partial t} - \alpha(\frac{\partial u_M}{\partial t})R_\alpha\right) + \left(\frac{\partial u_\mathbb{R}}{\partial s}\frac{\partial}{\partial r}, \alpha(\frac{\partial u_M}{\partial s})R_\alpha, \frac{\partial u_M}{\partial s} - \alpha(\frac{\partial u_M}{\partial s})R_\alpha\right) = 0.$$

Then the equation is equivalent to

$$\begin{aligned}
&\frac{\partial u_\mathbb{R}}{\partial t} + \alpha(\frac{\partial u_M}{\partial s}) = 0,\\
&\alpha(\frac{\partial u_M}{\partial t}) - \frac{\partial u_\mathbb{R}}{\partial s} = 0,\\
&J_{u(s,t)}(\frac{\partial u_M}{\partial t} - \alpha(\frac{\partial u_M}{\partial t})R_\alpha) + \frac{\partial u_M}{\partial s} - \alpha(\frac{\partial u_M}{\partial s})R_\alpha = 0.
\end{aligned}$$

One may notice that $u_\mathbb{R}\colon \mathbb{S}^2 \to \mathbb{R}$ is a subharmonic function. Indeed,

$$\frac{\partial^2}{\partial t^2}u_\mathbb{R} + \frac{\partial^2}{\partial s^2}u_\mathbb{R} = \frac{\partial}{\partial s}\alpha(\frac{\partial u_M}{\partial t}) - \frac{\partial}{\partial t}\alpha(\frac{\partial u_M}{\partial s}) = \frac{\partial}{\partial s}(u_M^*\alpha(\frac{\partial}{\partial t})) - \frac{\partial}{\partial t}(u_M^*\alpha(\frac{\partial}{\partial s})).$$

Recall that $d\beta(X, Y) = X\beta(Y) - Y\beta(X) + \beta([X, Y])$, so

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u_\mathbb{R} + \frac{\partial^2}{\partial s^2}u_\mathbb{R} &= d(u_M^*\alpha)(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}) = d\alpha(\frac{\partial u_M}{\partial s}, \frac{\partial u_M}{\partial t})\\
&= d\alpha(\alpha(\frac{\partial u_M}{\partial t})R_\alpha - J_{u(s,t)}(\frac{\partial u_M}{\partial t} - \alpha(\frac{\partial u_M}{\partial t})R_\alpha), \frac{\partial u_M}{\partial t})\\
&= d\alpha|_\xi(\frac{\partial u_M}{\partial t} - \alpha(\frac{\partial u_M}{\partial t})R_\alpha, J(\frac{\partial u_M}{\partial t} - \alpha(\frac{\partial u_M}{\partial t})R_\alpha)) \geqslant 0.
\end{aligned}$$

However, the compacteness of $\mathbb{S}^2$ shows that $u_\mathbb{R}$ must find its maximum somewhere. This shows that any J-holomorphic curves from Riemann spheres are constant on the $\mathbb{R}$-component. To avoid such phenomenon, we can consider punctured Riemann spheres. Explicitly, $[0, +\infty) \times \mathbb{S}^1$ is biholomorphic to $\mathbb{D}^2 \setminus \{0\}$ via $(s, t) \mapsto e^{-2\pi(s+\sqrt{-1}t)}$. Therefore, one can view $\mathbb{S}^2 \setminus \{p\}$ as a Riemann surface obtained from attaching a $+\infty$-cylinder to the sphere. By our analysis above, any J-holomorphic curve $u\colon (\mathbb{S}^2 \setminus \{p_1, \cdots, p_n\}, j) \to (SM, d(e^r\alpha), J)$ must have at least one positive end.

Example 20.12. Let (M, α) be a contact manifold and γ is a closed Reeb orbit, with period $\int_\gamma \alpha = T$. Consider $u\colon \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times M, (s, t) \mapsto (Ts, \gamma(Tt))$. One can check that u is a J-holomorphic cylinder:

$$\begin{cases}
\frac{\partial u_\mathbb{R}}{\partial t} + \alpha(\frac{\partial u_M}{\partial s}) = 0 + 0 = 0,\\
\alpha(\frac{\partial u_M}{\partial t}) - \frac{\partial u_\mathbb{R}}{\partial s} = \alpha(TR_\alpha) - T = 0,\\
J_{u(s,t)}(\frac{\partial u_M}{\partial t} - \alpha(\frac{\partial u_M}{\partial t})R_\alpha) + \frac{\partial u_M}{\partial s} - \alpha(\frac{\partial u_M}{\partial s})R_\alpha = J(TR_\alpha - \alpha(TR_\alpha)R_\alpha) + 0 = 0.
\end{cases}$$

The area of this holomorphic curve is

$$\begin{aligned}
\int_{\mathbb{R}\times\mathbb{S}^1} u^*d(e^r\alpha) &= \int_{\mathbb{R}\times\mathbb{S}^1} d(e^{Ts}u^*\alpha) = \int_{\mathbb{R}\times\mathbb{S}^1} Te^{Ts}u^*\alpha + \int_{\mathbb{R}\times\mathbb{S}^1} e^{Ts}du^*\alpha\\
&= T\int_{\mathbb{S}^1} u^*\alpha \int_\mathbb{R} e^T s + \int_{\mathbb{S}^1} du^*\alpha \int_\mathbb{R} e^{Ts} = T^2\int_\mathbb{R} e^{Ts} = +\infty.
\end{aligned}$$

Our computation shows that this u is a trivial cylinder.

Consider $\varphi\colon \mathbb{R} \to (-1,1)$ with $\varphi' > 0$ (for example $\varphi(r) = \frac{2}{\pi}\arctan r$). Define a new symplectic structure $\omega_\varphi = d(e^{\varphi(r)}\alpha)$. Then compute the area of u with respect to ω_φ:

$$\int_{\mathbb{R}\times\mathbb{S}^1} u^* d(e^{\varphi(r)}\alpha) = \int_{\mathbb{S}^1} u^*\alpha \int_{\mathbb{R}} e^{\varphi(Ts)} T\varphi'(Ts) + \int_{\mathbb{S}^1} du^*\alpha \int_{\mathbb{R}} e^{\varphi(Ts)} = T\int_{\varphi(-\infty)}^{\varphi(\infty)} e^s ds < +\infty.$$

In general, Hofer defined

$$E_{\mathrm{Hof}}(u) = \sup\left\{E_\varphi(u) \mid \varphi \in \mathbb{R} \to (-1,1), \varphi' > 0\right\}.$$

Our discussion above follows a famous work by Hofer.

Theorem 20.13 (Hofer)**.** Let (M,α) be a contact manifold. Suppose $u\colon (\mathbb{S}^2 \setminus \{p_1,\cdots,p_n\}, j) \to (SM, d(e^r\alpha), J)$ is a J-holomorphic curve with $E_{\mathrm{Hof}}(u) < +\infty$. Then under mild conditions on the punctures $p_1,\cdots,p_n$, u is asymptotic to closed Reeb orbits of (M,α) (i.e., under the parametrization (s,t) near a punctured point, $u(s,-)$ converges to a closed Reeb orbit when $s \to +\infty$).

Remark 20.14. This does not imply Weinstein conjecture unless people know such a J-holomorphic curve exists.

Definition 20.15. A **Liouville cobordism** is an exact symplectic manifold $(X, d\lambda)$ with boundary ∂X such that there exists a vector field Y on X with $Y \pitchfork \partial X$ and $\iota_Y d\lambda = \lambda$.

The condition $\iota_Y d\lambda = \lambda$ shows that $\mathcal{L}_Y d\lambda = \iota_Y dd\lambda + d\iota_Y d\lambda = d\lambda$. Usually, people decompose $\partial X = M^- \cup M^+$ with Y pointing inward or outward.

Remark 20.16. (1) With the same argument as Example 20.9, both $(M^-, \lambda|_{M^-})$ and $(M^+, \lambda|_{M^+})$ are contact manifolds. Note that since $d\lambda$ is non-degenerate, so there always exists some vector field Y satisfying $\iota_Y d\lambda = \lambda$, hence the non-trivial part is to guarantee that $\partial X \pitchfork Y$, which makes λ a contact structure on ∂X.
(2) If $M^- = \varnothing$, $(X, d\lambda)$ is called a **Liouville domain**. To find a Liouville domain for a given contact manifold is a well-studied area in symplectic and contact geometry, called the symplectic filling.
(3) A Liouville cobordism is a top way to relate two different contact manifolds.

Example 20.17. Suppose U_1, U_2 are two star-shaped domains in $\mathbb{C}^n$, centered at 0 with U_2 lies in the interior of U_1. Then the domain bounded by U_1, U_2 is a Liouville cobordism.

Let (Q,g) be a Riemannian manifold, $0 < c < 1$ and then we have a Liouville cobordism

$$\{q \in T^*M \mid c \leqslant |q| \leqslant 1\} \subset (T^*Q, d\lambda_{\mathrm{can}}).$$

Similarly, for any given contact manifold (M,α), the domain $[r_-, r_+] \times M$ is a Liouville cobordism with $Y = \partial_r$.

One advantage of a Liouville cobordism is that we can do completion to enlarge X into a complete non-compact manifold

$$\widehat{X} = \left((-\infty, R_-] \times M^-\right) \cup_{M^-} X \cup_{M^+} \left([R_+, +\infty) \times M^+\right).$$

For a Liouville cobordism X in $\mathbb{C}^n$, we have $\widehat{X} = \mathbb{C}^n \setminus \{0\}$. For a Liouville domain X in T^*Q, we have $\widehat{X} = T^*Q \setminus \{Q\}$.

One can apply the similar construction on a compact symplectic manifold with boundary (X,ω) (not necessarily exact). We can define a vector field Y with $Y \pitchfork \partial X$ and $\mathcal{L}_Y\omega = \omega$ in a neighborhood of ∂X. Then $(\partial X, \iota_Y\omega|_{\partial X})$ is a contact manifold. The completion $\widehat{X}$ is a non-compact symplectic manifold and exact outside a compact region.

In fact, such discussion can even be applied to closed submanifold, and this is one of the motivations of neck-stretching.

Definition 20.18. Let (X,ω) be a closed symplectic manifold and $M \subset X$ is a hypersurface. We say M is in a **contact type** if in a neighborhood of M, there exists a vector field Y such that $Y \pitchfork M$ and $\mathcal{L}_Y\omega = \omega$.

Neck-stretching is a process that thickens a hyperplane in a contact type in a symplectic way.

Example 20.19 (neck-stretching)**.** Let $M \subset (X,\omega)$ be a hyperplane in a contact type with the vector field Y in the neighborhood. Assume that $X \setminus M = X^+ \bigsqcup X^-$ consists of two components with signs suggesting inward or outward with respect to the vector field Y. Then we define

$$X_k = X^+ \cup_M ([-k, 0] \times M) \cup_M X^-.$$

X_k, X are diffeomorphic, but we set a new symplectic structure and an almost complex structure:

$$\omega_k = \begin{cases} \omega & X^+ \\ d(e^r\alpha) & [-k,0]\times M \\ e^{-k}\omega & X^-. \end{cases}$$

If we pick $J \in \mathcal{J}(X,\omega)$, we usually define J_k by setting $J_k = J$ on $X^+ \cup X^-$ and the standard almost complex structure of SM on $[-k,0]\times M$.

The limit $(X_\infty, \omega_\infty, J_\infty)$ as $k \to \infty$ is $\widehat{X}^+ \cup SM \cup \widehat{X}^-$, where

$$\widehat{X}^+ = X^+ \cup ((-\infty,0]\times M),$$
$$\widehat{X}^- = ([0,+\infty)\times M) \cup X^-.$$

Just like the limit of J-holomorphic curves, $(X_\infty, \omega_\infty, J_\infty)$ may no longer have the same structure with (X,ω), but luckily there are some compactness results controlling it.

Example 20.20. By Weinstein neighbobrhood theorem, a Lagrangian submanifold $L \subset (X,\omega)$ admits a neighborhood U symplectomorphic to a neighborhood D_g^*L of $L \subset (T^*L, d\lambda_{\text{can}})$. Then the sphere bundle S_g^*L is a contact hypersurface such that $X \setminus S_g^*L = X^+ \bigsqcup X^-$, where $X^+ = X \setminus D_g^*L$ and $X^- = D_g^*L \setminus S_g^*L$.

In this case, $X_\infty = (X\setminus L) \cup (T^*L \setminus L) \cup T^*L$.

Theorem 20.21 (Eliashberg-Givental-Hofer, Bourgeois-Eliashberg-Wysocki-Zehnder, Abass, $\cdots$)**.** Let (X,ω) be a symplectic manifold with a contact type hypersurface M. Consider the J-holomorphic curves

$$u_k \colon (\mathbb{S}^2, j) \longrightarrow (X,\omega,J) \text{ for closed } (X_k,\omega_k,J_k),$$

$$u_k \colon (\mathbb{S}^2 \setminus p_1,\cdots,p_n, j) \longrightarrow (X_k,\omega_k,J_k) \text{ for Liouville cobordism } (X, d\lambda).$$

If the energy of u_k is uniform bounded and $[u_k]$ is fixed in either $H_2(X;\mathbb{Z})$ (closed case) or $H_2(X,\partial X;\mathbb{Z})$, then there exists a subsequence converging to a u_∞ in X_∞, called the holomorphic building, in the sense that u_∞ is a union of multiple J-holomorphic curves coming from each buildings $\widehat{X}^+, \widehat{X}^-, SM$ (for multiple times) and they are connected along closed Reeb orbits.

The holomorphic building u_∞ possesses the following data:
(1) Genus of u_∞ is the genus of each u_k.
(2) $E(u_\infty) = \sum E(v_i) \leqslant \sup_k E(u_k) < +\infty$.
(3) The homology class remains: $[u_\infty] = [u_k]$.
(4) The Fredholm index of u_∞ satisfies certain relation, which Zhang doesn't have time to introduce.

Now we give the proof of Theroem 20.2. As one may expect, we will use neck-stretching and compactness theorem, and $(\mathbb{CP}^n, \omega_{\text{FS}})$ is an ideal object with many suitable curves.

Proof. Let $c \in H^2(\mathbb{CP}^n;\mathbb{Z})$ be the generator and L be the line class. Recall that

$$\mathrm{GW}_{L,3}(c^i,c^j,c^k) = \begin{cases} 1 & i+j+k = 2n+1 \\ 0 & \text{else.} \end{cases}$$

By contradiction, suppose there exists some Lagrangian torus L, whose sphere bundle in Weinstein neighborhood U is M. Fix a point inside U and consider curves passing through this point. The virtual dimension of moduli space of such curves is $\dim \mathcal{M}_{L,3} - (2n-2) = 2n+4 > 0$. There always exist curves $u_k \colon (\mathbb{S}^2, j) \to (\mathbb{CP}^n, \omega_k, J_k)$. Consider the holomorphic building u_∞ with components lying in $\widehat{X}^+ = \mathbb{CP}^n \setminus L, SM, \widehat{X}^- = T^*L$ respectively. By Exercise 20.4 closed Reeb orbits correspond to closed geodesics in L, which are non-contractible. Therefore, for each component in T^*L, it must have at least two positive aymptotic ends. Since u_∞ remains zero genus, there must be at least two components in $\mathbb{CP}^n \setminus L$. As a result, one of the components in $\mathbb{CP}^n \setminus L$ must be a disk, say D. Now by the definition of u_∞, we have $[u_\infty] = L$. However, $0 < \omega_{\text{FS}}(D) \leqslant \frac{1}{2}\omega_{\text{FS}}([u_\infty])$, which violates the exactness of L. $\square$