NOTES ON MORSE THEORY

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The fundamental idea in Morse theory is the following: A well chosen function encodes a lot of information about manifolds, for example, topological invariants.

1. Morse function

The notion of a "well chosen" is formalised to mean a Morse function.

Definition 1.1. A smooth function $F\colon M \to \mathbb{R}$ is called a Morse function, if the Hessian of F at crtical points are non-degenerate.

Remark 1.2. As a priori, we can define Morse functions among C^2-functions. It will turn out that Morse functions are generic, so we can always pick smooth ones.

Unlike differential, there is no way to define Hessians on smooth manifolds canonically. However, this can be done on critical points.

If M is endowed with a Riemannian metric g, we can pick the Levi-Civita connection ∇. The Hessian of F at a point x is a symmetric bilinear form, given by

$$\mathrm{Hess}(F)_p(X,Y) = Y_p(XF) - (\nabla_Y X)_p F = Y_p(XF) - (dF)_p(\nabla_Y X), X, Y \in T_pM,$$

where X is extended to a local vector field around p. In particular, if $p \in \mathrm{Crit}(F)$, $\mathrm{Hess}(F)_p(X,Y) = Y_p(XF)$ does not depend on ∇.

Definition 1.3. Let x be a critical point of a smooth function $F\colon M \to \mathbb{R}$. The Hessian of F at p is defined as

$$\mathrm{Hess}(F)_p(X,Y) = Y_p(XF), X, Y \in T_pM$$

where X is extended to a local vector field around p.

As one can see, under local ccordinates $(x_1, \cdots, x_n)$,

$$\mathrm{Hess}(F)_p(X^i\partial_i, Y^j\partial_j) = X^i(\partial_i b^j)(\partial_j F) + X^iY^j(\partial_i\partial_j F) = X^iY^j(\partial_i\partial_j F).$$

The notion of Hessian is the same as we use in calculus.

Example 1.4. On $M = \mathbb{R}^2$, $F(x,y) = x^2 - y^2$ is a Morse function, while $G(x,y) = x^3 - y^2$ is not.

Example 1.5. We will construct a Morse function on $\mathbb{CP}^n$. Consider the smooth function

$$F([z_0, \cdots, z_n]) = \frac{\sum_{j=0}^n j\,|z_j|^2}{\sum_{j=0}^n |z_j|^2}, [z_0, \cdots, z_n] \in \mathbb{CP}^n.$$

Take local charts $\mathbb{C}^n \to \mathbb{CP}^n, (w_0, \cdots, w_{n-1}) \mapsto [w_0, \cdots, w_{k-1}, 1, w_k, \cdots, w_{n-1}]$ and let F_k be the function F under the corresponding chart. Explicitly,

$$F_k(w_0, \cdots, w_{n-1}) = \frac{\sum_{j\leqslant k-1} j\,|w_j|^2 + k + \sum_{j\geqslant k}(j+1)\,|w_j|^2}{\sum_{j=0}^{n-1} |w_j|^2 + 1}.$$

Therefore,

$$\frac{\partial F_k}{\partial x_i} = \frac{2x_j(j-k-\sum_{j\geqslant k}|w_j|^2)}{(\sum_{j=0}^{n-1}|w_j|^2+1)^2}, \frac{\partial F_k}{\partial y_i} = \frac{2y_j(j-k-\sum_{j\geqslant k}|w_j|^2)}{(\sum_{j=0}^{n-1}|w_j|^2+1)^2}$$

if $i \leqslant k-1$ and

$$\frac{\partial F_k}{\partial x_i} = \frac{2x_j(j+1-k+\sum_{j\leqslant k-1}|w_j|^2)}{(\sum_{j=0}^{n-1}|w_j|^2+1)^2}, \frac{\partial F_k}{\partial y_i} = \frac{2y_j(j+1-k-\sum_{j\leqslant k-1}|w_j|^2)}{(\sum_{j=0}^{n-1}|w_j|^2+1)^2}$$

if $i \geqslant k$. As a consequence, F has $n+1$ critical points $p_k = [0,\cdots,1,\cdots,0]$. Moreover, near p_k we have

$$\begin{aligned} F_k(w_0,\cdots,w_{n-1}) &= (\sum_{j\leqslant k-1} j|w_j|^2 + k + \sum_{j\geqslant k}(j+1)|w_j|^2)(1-\sum_{j=0}^{n-1}|w_j|^2 + O(|w|^3)) \\ &= k + (\sum_{j\leqslant k-1}(j-k)|w_j|^2 + \sum_{j\geqslant k}(j+1-k)|w_j|^2) + O(|w|^3), \end{aligned}$$

which implies $\mathrm{Hess}(F)_{p_k}$ is non-degenerate. Also, we can read out their indices by local expansions: $\mathrm{ind}(p_k) = 2k$.

Definition 1.6. Let F be a Morse function and $x \in \mathrm{Crit}(F)$. Since $\mathrm{Hess}(F)_x$ is symmetric, it has the standard form $\mathrm{diag}(-I_r, I_s)$. r is called the index of F at x, denoted by $\mathrm{ind}_F(x)$ or simply $\mathrm{ind}(x)$.

Lemma 1.7 (Morse lemma)**.** For any Morse function F on M and a critical point p, there exists local coordinates $(x_1,\cdots,x_n)$ near p such that $x(p)=0$ and

$$F(x) = F(p) - \sum_{i=1}^{r} x_i^2 + \sum_{j=r+1}^{n} x_j^2.$$

Proof. Since the result is local, we may assume $M=\mathbb{R}^n$ and $p=0$. Moreover, we may assume $\mathrm{Hess}(F)_p$ is diagonal by taking an appropriate linear transform.

We induct on the dimension. For the case $n=1$,

$$f(x) = f(0) + \frac{1}{2}f''(0)x^2 + O(x^3),$$

so we can locally write $f(x) = f(0) + \tau g^2(x)x^2$ for some $g(0)\neq 0, \tau = \pm 1$. We set $\varphi(x) = xg(x)$, which is a local diffeomorphism at 0. By the inverse function theorem, we have

$$f\circ\varphi^{-1}(x_1) = f(x) = f(0) + \tau x_1^2.$$

Next, we write $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^{n-1}$, with coordinates (x,y). We write $f_y(x) = f(x,y)$ considering y as parameter in $\mathbb{R}^{n-1}$. The Taylor expansion gives

$$f(x,y) = f_y(0) + f_y'(0)x + \frac{1}{2}f_y''(0)x^2 + x^2\varepsilon(x,y).$$

If $f_y'(0)=0$, then we proceed as in the case $n=1$, since $f_y''(0)\neq 0$ (recall that the Hessian is diagonal at 0), which allows us to write

$$f(x,y) = f_y(0) + \tau x^2 g^2(x,y).$$

Again, $\varphi(x,y) = (xg(x),y)$ is a local diffeomorphism at 0. This gives

$$f\circ\varphi^{-1}(x_1,y) = f(0,y) + \tau x_1^2.$$

Now we show that it is possible to reduce to the case where $f_y'(0)=0$. Consider the equation

$$\frac{\partial f}{\partial x}(x,y) = 0.$$

Since $\frac{\partial^2 f}{\partial x^2}(0,0)\neq$, the implicit function theorem asserts that we can find $x=\varphi(y)$ smooth near 0 such that

$$\frac{\partial f}{\partial x}(\varphi(y),y) = 0.$$

We now use the local diffeomorphism $\Phi(x,y) = (x+\varphi(y),y)$. Then the function $f\circ\Phi$ satisfies

$$\frac{\partial}{\partial x}(f\circ\Phi)(0,y) = 0, \mathrm{Hess}(f\circ\Phi)_0 = \mathrm{Hess}(f)_0.$$

This conludes the proof of the Morse lemma. $\square$

Corollary 1.8. The critical points of a Morse function is discrete. In particular, they form a zero measure subset.

We next prove that morse functions exist, and in fact, there are many of them.

Proposition 1.9. By Whitney embedding, suppose $M \subset \mathbb{R}^N$ as a submanifold. For $p \in \mathbb{R}^N$ almost everywhere, $f_p \colon M \to \mathbb{R}, x \mapsto |x-p|^2$ is a Morse function.

Proof. We have $(df_p)_x(v) = 2\langle x-p, v\rangle$, so the critical points occur exactly when T_xM is normal to $x-p$. We choose local coordinates $(V, u_1, \cdots, u_d)$ for M. In these coordinates, we have

$$\frac{\partial f}{\partial u_i} = \left\langle x-p, \frac{\partial x}{\partial u_i}\right\rangle, \frac{\partial^2 f}{\partial u_i \partial u_j} = 2\left(\left\langle \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right\rangle + \left\langle x-p, \frac{\partial^2 x}{\partial u_i \partial u_j}\right\rangle\right).$$

Therefore x is a non-degenerate critical point if and only if $x-p$ is normal to T_xV and the matrix on the right is non-degenerate.

To this end, consider the local normal bundle

$$N = \{(x,v) \in V \times \mathbb{R}^n \mid v \perp T_xV\}.$$

Define the map

$$\begin{aligned} \varphi \colon N &\longrightarrow \mathbb{R}^n \\ (x,v) &\longmapsto x+v. \end{aligned}$$

$p = x+v$ is a critical point if and only if

$$2\left(\left\langle \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right\rangle + \left\langle v, \frac{\partial^2 x}{\partial u_i \partial u_j}\right\rangle\right)$$

is singular. To see this, N admits local chart

$$(u_1, \cdots, u_d, t_1, \cdots, t_{n-d}) \longmapsto (x(u_1, \cdots, u_d), \sum^{n-d} t_i v_i(u_1, \cdots, u_d)),$$

where $v_1, \cdots, v_{n-d}$ form a local orthonormal frame. In these coordinates, the partial derivatives of E are

$$\frac{\partial E}{\partial u_i} = \frac{\partial x}{\partial u_i} + \sum_{i=1}^{n-d} t_k \frac{\partial v_k}{\partial u_i}, \frac{\partial E}{\partial t_j} = v_j.$$

It is a littile bit difficult to see the rank from the Jacobian, so we compute the inner products of these vectors with basis

$$\frac{\partial x}{\partial u_1}, \cdots, \frac{\partial x}{\partial u_d}, v_1, \cdots, v_{n-d}.$$

The coefficient matrix writes

$$\begin{pmatrix} \left(\left\langle \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right\rangle + \left\langle \sum_k t_k \frac{\partial v_k}{\partial u_i}, \frac{\partial x}{\partial u_j}\right\rangle\right) & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

Since v_k is orthonormal to $\frac{\partial x}{\partial x_j}$,

$$0 = \frac{\partial}{\partial u_i}\left\langle v_k, \frac{\partial x}{\partial u_j}\right\rangle = \left\langle \frac{\partial v_k}{\partial u_i}, \frac{\partial x}{\partial u_j}\right\rangle + \left\langle v_k, \frac{\partial^2 x}{\partial u_i \partial u_j}\right\rangle.$$

It follows from the Sard theorem that those p which do not meet requirements form a measure zero subset of M. $\square$

Proposition 1.10. Morse functions is dense in $C^\infty(M)$ under the Whitney C^k-topology.

2. Topology from critical points

Definition 2.1. Let $F\colon M \to \mathbb{R}$ be a smooth function. Define $M^a = \{x \in M \mid F(x) \leqslant a\}$. This is a submanifold with boundary.

Theorem 2.2. Let $F\colon M \to \mathbb{R}$ be a smooth function. Suppose $F^{-1}([a,b])$ is compact with no critical points inside $F^{-1}([a,b])$. Then M^a is diffeomorphic to M^b.

Corollary 2.3 (Reeb)**.** Suppose M is a closed manifold. If M admits a Morse function with exactly two critical points. Then M is homeomorphic to a sphere.

Remark 2.4. Note also that the final conclusion is only true up to homeomorphism, since gluing two disks can result in exotic spheres.

Theorem 2.5. Let $F\colon M \to \mathbb{R}$ be a smooth function. Suppose $F^{-1}([a,b])$ is compact and F has exactly one critical point α in $F^{-1}([a,b])$ of index k. Then M^a is homotopy equivalent to M^b with a k-cell attached.

3. Morse homology modulo 2

Fix a Riemannian metric g on M. For a Morse function $F\colon M \to \mathbb{R}$, consider the following ODE:

$$\dot{u}(t) = -(\nabla F)(u(t)),$$

where $u\colon \mathbb{R} \to M$ is a smooth curve. Any solution is simply the negative gradient flow of F and the set is called the negative gradient flowline. We wish to consider the stable case: $u(-\infty) = x, u(\infty) = y$ for $x, y \in \mathrm{Crit}(f)$. This requirement is reasonable because $\nabla(F)(u(t))$ tends to 0 if one expects $u(-\infty), u(\infty)$ to converge.

Definition 3.1. For $x, y \in \mathrm{Crit}(f)$, we define the moduli space

$$\widetilde{\mathcal{M}}_F(x,y) = \{u\colon \mathbb{R} \to M \mid u(-\infty) = x, u(\infty) = y\} \subset \{\text{solutions of } \dot{u}(t) = -(\nabla F)(u(t))\}.$$

The very first observation is that there exists an $\mathbb{R}$-action on $\widetilde{\mathcal{M}}_F(x,y)$ by $s \cdot u(t) = u(s+t)$. Therefore it's more efficient to consider the following quotient moduli space

$$\mathcal{M}_F(x,y) = \widetilde{\mathcal{M}}_F(x,y)/\mathbb{R}.$$

By the uniqueness of solutions in ODE, two flowline differs by a translating reparametrization if their graphs coincide. Thus by taking quotient we forget the information of parameter, but focus on the geometric nature of them.

By definition, a flowlinie goes in the direction in which F descends. Hence Morse lemma implies that the index of a crticial point somewhat measures how many unstable flowlines are driving away.

Proposition 3.2. $\mathcal{M}_F(x,x)$ contains only the constant flow at point x.

Proof. We have

$$g_{u(t)}(X, \dot{u}(t)) = -X_{u(t)}F.$$

In particular,

$$\frac{d}{dt}F(u(t)) = \dot{u}(t)F = -g_{u(t)}(\dot{u}(t), \dot{u}(t)) \leqslant 0.$$

In particular, $F(u(-\infty)) = F(u(\infty))$, which forces $\dot{u}(t) \equiv 0$. □

Example 3.3. Let $X = S^1 \subset \mathbb{R}^2$ as the standard circle. Consider the Morse function $h\colon S^1, (\cos t, \sin t) \mapsto \sin t$. The critical points are poles $p_0 = (0,-1), p_1 = (0,1)$.

Near p_0, we have $h(t) = -1 + t^2 + o(t^2)$, which implies $\mathrm{ind}(p_0) = 0$. Near p_1, we have $h(t) = 1 - t^2 + o(t^2)$, which implies $\mathrm{ind}(p_1) = 1$.

Take the standard metric g_0 on $\mathbb{R}^2$ and restrict it on S^1. The moduli space $\widetilde{\mathcal{M}}_h(p_1, p_0)$ consists of two $\mathbb{R}$-parametrized families of flow lines from p_1, p_0, with directions clockwise and counterclockwise, respectively. So the quotient space $\mathcal{M}_h(p_1, p_0)$ contains exactly two elements.

Example 3.4. Let $M = S^1 \times S^1 \subset \mathbb{R}^4$. We can consider the Morse function $F(t_1, t_2) = \sin t_1 + \sin t_2$. Then F has 4 critical points

$$p_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), p_2 = \left(\frac{\pi}{2}, \frac{\pi}{2}\right), q_1 = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right), p_0 = \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right).$$

Note that this function is the external product of the height function. Moreover, it is indeed a Morse function because

$$\left(\frac{\partial^2 F}{\partial t_i \partial t_j}\right) = \begin{pmatrix} -\sin t_1 & 0 \\ 0 & -\sin t_2 \end{pmatrix}$$

At the same time, we can read off indices of each of the critical points,

$$\operatorname{ind}(p_0) = 0, \operatorname{ind}(p_1) = \operatorname{ind}(q_1) = 1, \operatorname{ind}(p_2) = 2.$$

Take the standard metric on $\mathbb{R}^4$ and restrict it on M. Then

$$\nabla F = (\cos t_1, \cos t_2).$$

If $u = (u_1(t), u_2(t)) \colon \mathbb{R} \to M$ is a gradient flow, then the equation reads

$$\begin{cases} \dot{u}_1(t) = -\cos(u_1(t)) \\ \dot{u}_2(t) = -\cos(u_2(t)). \end{cases}$$

What's special in this case is that equations in the two directions are independent. The solution to

$$\begin{cases} \dot{x}(t) = -\cos(x(t)) \\ x(-\infty) = \frac{\pi}{2}, x(\infty) = -\frac{\pi}{2} \end{cases}$$

is given by

$$a_s(t) = 2\arctan(\tanh(\frac{1}{2}(s - t))), s \in \mathbb{R}$$

As a result, we can construct flowlines

$$u_{r,s}(t) = (a_r(t), r_s(t)) \in \widetilde{\mathcal{M}}_F(p_2, p_0).$$

By quotient the translation, we obtain

$$u_s(t) = (a_0(t), r_s(t)) \in \mathcal{M}_F(p_2, p_0).$$

In particular, we can take limits as $s \to -\infty$ and $s \to \infty$ and $u_{-\infty}, u_\infty$ turn out to split at an imtermdeiate critical points.

One can verify that there are lots of solutions in the moduli space $\mathcal{M}_F(p_2, p_0)$, where the corresponding negative flow trajectories occupy almost all the small open squares.

By Example 3.3,3.4, we have two peculiar phenomena.

1. When it happens that $\mathcal{M}_F(p, q) = \varnothing$, it seems that $\operatorname{ind}(p) \leqslant \operatorname{ind}(q)$.
2. $\mathcal{M}_F(p, q)$ may not be compact.

Definition 3.5. Let x be a critical point of a Morse function $F \colon (M, g) \to \mathbb{R}$. For $a \in M$, denote the negative graident flowline starting at a by $r_a(t)$. Define the stable manifold of x to be

$$W^s(x) = \{a \in M \mid r_a(\infty) = x\}$$

and the unstable manifold of x to be

$$W^u(x) = \{a \in M \mid r_a(-\infty) = x\}.$$

For instance, in the example of S^1, $W^s(p_0) = S^1 \setminus \{p_0\}$ and $W^u(p_1) = M \setminus \{p_1\}$. An important fact, which we don't present a proof, is that both $W^u(x), W^s(x)$ are submanifolds diffeomorphic to open disks of dimension $\operatorname{ind}(x)$.

Proposition 3.6. Given a Morse function F on a smooth manifold M, for a generic metric g, the moduli space $\widetilde{\mathcal{M}}_F(x, y)$ is a submanifold of M for any $x \neq y$, with $\dim \widetilde{\mathcal{M}}_F(x, y) = \operatorname{ind}(x) - \operatorname{ind}(y)$. In particular, $\dim \mathcal{M}_F(x, y) = \operatorname{ind}(x) - \operatorname{ind}(y) - 1$ if $x \neq y$.

Proof. (Sketch) For a generic metric g and for two different critical points x, y, $W^u(x), W^s(y)$ intersect transversally. Therefore, $\widetilde{\mathcal{M}}_F(x,y)$ is a submanifold of M with dimension $\operatorname{ind}(x) - \operatorname{ind}(y)$. □

Corollary 3.7. (1) If $\operatorname{ind}(x) - \operatorname{ind}(y) \geqslant 1$, then $\mathcal{M}_F(x,y)$ is non-empty. In particular, the quotient moduli space is discrete if and only if the equality holds.
(2) If $x \neq y$ but $\operatorname{ind}(x) = \operatorname{ind}(y)$, $\widetilde{\mathcal{M}}_F(x,y) = \varnothing$.

Definition 3.8. A smooth manifold with corners M of dimension n is a second countable Hausdorff space such that for any $x \in M$, there exists a neighborhood homeomorphic to an open subset of $\mathbb{R}^{n-k} \times [0, +\infty)^k$ for some k and the transition maps are smooth.

Given a manifold with corners M, we define the codimension-k stratum of M as follows

$$M_k = \left\{ x \in M \mid x \text{ has a neighborhood } U \cong V \subset \mathbb{R}^{n-k} \times [0,\infty)^k \right\}.$$

Taking the closure inside M, we can see that

$$\overline{M_k} = M_k \cup M_{k+1} \cup \cdots \cup M_n.$$

The boundary of M, namely, $\partial M = M \setminus \operatorname{Int}(M)$ can be expressed as

$$\partial M = \overline{M_1} = M_1 \cup \cdots \cup M_n.$$

Theorem 3.9. Given a closed (i.e. compact and without boundary) manifold M and a generic pair of Morse function nad metric (F, g) on M, for any two different critical points x, y, $\mathcal{M}(x,y)$ admits a compactification $\overline{\mathcal{M}(x,y)}$ as a smooth manifold with corners. Moreover, the codimension-k stratum is

$$\overline{\mathcal{M}(x,y)}_k = \bigcup_{z_i \in \operatorname{Crit}(F)} \mathcal{M}(x,z_1) \times \mathcal{M}(z_1,z_2) \times \cdots \times \mathcal{M}(z_k,y),$$

where $x, z_1, \cdots, z_k, y$ are distinct. Let us recall that the index is decreasing along the trajectories of M, hence only those $\mathcal{M}(z_k, z_{k+1})$ with $\operatorname{ind}(z_k) > \operatorname{ind}(z_{k+1})$ contribute.

Example 3.10. $k = 1$. Any element in codimension-1 stratum $\overline{\mathcal{M}(x,y)}_1$ has only one intermediate point. Elements inside $\overline{\mathcal{M}(x,y)}_1$ look like $x \to z \to y$.

As we have already seen,

$$\overline{\overline{\mathcal{M}(x,y)}_1} = \bigcup_{k \geqslant 1} \overline{\mathcal{M}(x,y)}_k$$

gives the boundary $\partial \overline{\mathcal{M}(x,y)}$.

Example 3.11. Things become considerably easier if $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$. In this case $\overline{\mathcal{M}(x,y)}$ is empty, which implies $\partial M = \varnothing$. We already knows $\dim \mathcal{M}(x,y) = 0$. By the compactness of $\mathcal{M}(x,y) = \overline{\mathcal{M}(x,y)}$, the moduli space is a collection of finitely many points.

Example 3.12. Things get slightly more complicated if $\operatorname{ind}(x) - \operatorname{ind}(y) = 2$.

$$\overline{\mathcal{M}(x,y)}_1 = \bigcup_{z \in \operatorname{Crit}(F)} \mathcal{M}(x,z) \times \mathcal{M}(z,y).$$

As a consequence,

$$\begin{aligned} \partial \overline{\mathcal{M}(x,y)} &= \overline{\mathcal{M}(x,y)}_1 \cup \overline{\mathcal{M}(x,y)}_2 \cup \cdots \\ &= \overline{\mathcal{M}(x,y)} = \bigcup_{z \in \operatorname{Crit}(F)} \mathcal{M}(x,z) \times \mathcal{M}(z,y) \end{aligned}$$

is a compact 1-dimensional manifold (possibly with boundary). By the classification of compact 1-dimensional manifolds, the cardinatlity of the boundary is always even.

With all the ingredients introduced above, we are now ready to activate the machinary of Morse homology. We shall fix a generic pair (F, g) on a closed manifold M.

Definition 3.13. The Morse complex of (F, g) (with coefficient $\mathbb{Z}_2$) is defined as

$$CM_n(F, g; \mathbb{Z}_2) = \bigoplus_{\mathrm{ind}(x)=n} \mathbb{Z}_2.$$

Elements in $CM_n(F, g; \mathbb{Z}_2)$ are written as $\sum a_x(x)$ where $a_x \in \mathbb{Z}_2, x \in \mathrm{Crit}(F)$.

The boundary map ∂ sends $(x) \in CM_n(F, g; \mathbb{Z}_2)$ to

$$\sum_{\mathrm{ind}(y)=\mathrm{ind}(x)-1} \#_{\mathbb{Z}_2} \mathcal{M}(x, y)(y).$$

Proposition 3.14. Morse complex is indeed a complex.

Proof. For $(x) \in CM_n(F, g; \mathbb{Z}_2)$, we have

$$\begin{aligned}
\partial^2(x) =& \partial \left(\sum_{\mathrm{ind}(y)=\mathrm{ind}(x)-1} \#_{\mathbb{Z}_2} \mathcal{M}(x, y)(y) \right) \\
=& \sum_{\mathrm{ind}(y)=\mathrm{ind}(x)-1} \#_{\mathbb{Z}_2} \mathcal{M}(x, y) \sum_{\mathrm{ind}(z)=\mathrm{ind}(x)-2} \#_{\mathbb{Z}_2} \mathcal{M}(y, z)(z) \\
=& \sum_{\mathrm{ind}(z)=\mathrm{ind}(x)-2} \left(\sum_{\mathrm{ind}(y)=\mathrm{ind}(x)-1} \#_{\mathbb{Z}_2} \mathcal{M}(x, y) \cdot \#_{\mathbb{Z}_2} \mathcal{M}(y, z) \right) \\
=& \sum_{\mathrm{ind}(z)=\mathrm{ind}(x)-2} \left(\sum_{\mathrm{ind}(y)=\mathrm{ind}(x)-1} \#_{\mathbb{Z}_2} (\mathcal{M}(x, y) \times \mathcal{M}(y, z)) \right) \\
=& \sum_{\mathrm{ind}(z)=\mathrm{ind}(x)-2} \#_{\mathbb{Z}_2} \partial \overline{\mathcal{M}(x, z)} = 0.
\end{aligned}$$

□

Corollary 3.15. The non-degenerate critical points of a function are isolated.

Definition 3.16. The Morse homology of (F, g) is defined as the homology of $CM_n(F, g; \mathbb{Z}_2)$, denoted by $HM_*(F, g; \mathbb{Z}_2)$.

Theorem 3.17 (Morse)**.** Given a generic pair of Morse function and Riemannian metric (F, g) on compact manifold M, $HM_*(F, g; \mathbb{Z}_2) \simeq H_*(M; \mathbb{Z}_2)$, where the latter homology is the singular one. As a result, $HM_*(F, g; \mathbb{Z}_2) \simeq HM_*(F', g'; \mathbb{Z}_2)$ if (F', g') is another generic pair.

We will denote the Morse homology over a characteristic 2 field k by $HM(M; k)$.

Example 3.18. In Example 1.5 we have seen that F have n critical points $p_k = [0, \cdots, 1, \cdots, 0]$ with indices $\mathrm{ind}(p_k) = 2k$. Therefore $CM_{2k}(F; \mathbb{Z}_2) = \mathbb{Z}_2$ and $CM_{2k-1}(F; \mathbb{Z}_2) = 0$. In particular, this forces $HM_{2k}(\mathbb{CP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ and $HM_{2k-1}(\mathbb{CP}^n; \mathbb{Z}_2) = 0$.

Note that this example does not involve Riemannian metric. In general, one has to check the so called Morse–Smale condition, i.e., $W^s(x), W^u(x)$ intersect transversally, in order to ensure $HM_*(F, g; \mathbb{Z}_2) = H_*(M; \mathbb{Z}_2)$.

As a direct application, we see that

$$\#\mathrm{Crit}(F) = \sum_i \dim_k CM_i(F, g; k) \geqslant \sum_i \dim_k HM_i(M; k) = \sum_i b_i(M; k).$$

For example, on $\mathbb{T}^n$ any Morse function must have at least 2^n critical points and on $\mathbb{CP}^n$ any Morse function must have n critical points.

4. integral Morse homology

Next we define Morse homology with coefficients in $\mathbb{Z}$. Again, let's consider a Morse function $F\colon M \to \mathbb{R}$.

The main technicality is to introduce orientations on each $\mathcal{M}(x,y)$ for $\operatorname{ind}(y) = \operatorname{ind}(x) - 1$. Since these are zero dimensional manifolds, orientations are exactly choices of sign for each point.

Since $W^s(x)$ is diffeomorphic to some open disk, they are orientable. Choose any $u \in \mathcal{M}(x,y)$. There is a short exact sequence

$$0 \longrightarrow T_u\widetilde{\mathcal{M}}(x,y) \longrightarrow T_uW^s(y) \longrightarrow N_uW^u(x) \longrightarrow 0.$$

But $N_uW^u(x)$ is canonically isomorphic to $T_xW^s(x)$, giving a short exact sequence

$$0 \longrightarrow T_u\widetilde{\mathcal{M}}(x,y) \longrightarrow T_uW^s(y) \longrightarrow T_uW^s(x) \longrightarrow 0.$$

Since the middle and right term are oriented, there is an induced orientation on $T_u\,\mathcal{M}(x,y)$. Also we have a short exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow T_u\widetilde{\mathcal{M}}(x,y) \longrightarrow T_u\,\mathcal{M}(x,y) \longrightarrow 0,$$

where R is oriented by time. This induces an orientation on $T_u\,\mathcal{M}(x,y)$ as required.

Although choices are being made when orienting the stable manifolds, reversing the orientation of a given stable manifold corresponds to multiplying the boundary map by -1, which does not change the homology.

Definition 4.1. The integral Morse complex is defined by

$$CM_k(F;\mathbb{Z}) = \bigoplus_{\operatorname{ind}(x)=n} \mathbb{Z}.$$

The boundary map ∂ sends $(x) \in CM_n(F,g;\mathbb{Z})$ to

$$\sum_{\operatorname{ind}(y)=\operatorname{ind}(x)-1} N(x,y)(y),$$

where $N(x,y)$ is the signed count of trajectories from x to (y).

5. applications of Morse homology

Theorem 5.1 (strong Morse inequalities)**.** Let $F\colon M \to \mathbb{R}$ be a Morse function. Let N_i be the number of index i critical points. Then for any $k \geqslant 0$,

$$(-1)^k \sum_{i=0}^{k} (-1)^i N_i \geqslant (-1)^k \sum_{i=0}^{n} (-1)^i b_i(M).$$

Corollary 5.2.

$$\sum_{i=0}^{n} (-1)^i N_i = \chi(M).$$

Corollary 5.3 (weak Morse inequalities)**.** Let $F\colon M \to \mathbb{R}$ be a Morse function. Let N_i be the number of index i critical points. Then for any $k \geqslant 0$,

$$N_k \geqslant b_k(M).$$