SEMINAR NOTES: BASIC DE RHAM THEORY

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These notes is based on author's presentation on the first meeting of sminar on GTM 82, at USTC. The first meeting covers section 1- section 3 in chapter 1.

1. Introduction

Cohomology, homology and homotopy groups are three fundamental tools deveoloped in algebraic topology. The can be considered as the following functors:

$$H^*(-;A)\colon \mathbf{Top} \longrightarrow \mathbf{Ring},$$
$$H_*(-;A)\colon \mathbf{Top} \longrightarrow \mathbf{Mod}_R,$$
$$\pi_n(-)\colon \mathbf{Top}_* \longrightarrow \mathbf{Grp},$$

where A is an R-module. These tools are effective in studying and distinguishes spaces, but in return they are usually difficult to describe and compute.

The fundamental idea in GTM 82 is roughly the following: Invariants in algebraic topology can be defined and represented in an easier way on manifolds with the help of specific tools for manifolds.

We will focus on the subcategory $\mathbf{Man}^\infty$, the category of smooth manifolds, whose morphisms are smooth maps. Note that functors above actually factor through h$\mathbf{Top}$ and h$\mathbf{Top}_*$, but considering $\mathbf{Man}^\infty$ is safe, due to the Whitney approximation which claims that every continuous map of smooth manifolds is homotopic to a smooth one.

2. review of differential forms

The main ingredients for de Rham theory are differential forms on smooth manifolds.

Let M be a smooth manifold. A k-form on M is a smooth section

$$\omega\colon M \longrightarrow \bigwedge^k T^*M.$$

Collection of k-forms is denoted by $\Omega^k(M)$. In local coordinates $(x^1, \cdots, x^m)$, a k-form can be written as

$$\omega = \omega_I dx^I,$$

where I runs over ordered indices with length k. We can define the extorior derivative

$$d\omega = \frac{\partial \omega_I}{\partial x^j} dx^j \wedge dx^I.$$

Let $\varphi\colon M \to N$ be a smooth map. Then we have the natural pullback $\varphi^*\colon \Omega^k(N) \to \Omega^k(M)$. The following result is important throughout GTM 82:

Proposition 2.1. Suppose $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$ and $\varphi\colon M \to N$ is smooth.
(1) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$.
(2) $d^2 = 0$.
(3) $\varphi^*(d\omega) = d(\varphi^*\omega)$.

3. orientation and integral on manifolds

Manifolds are generalized objects of Euclidean spaces, on which we can still perform differential and integral.

Recall that if $\varphi\colon V \to U \subset M$ is a local chart of M, then we can define the local integral as

$$\int_U \omega = \int_V \varphi^*\omega, \omega \in \Omega_c^m(M),$$

where the integral on the right hand side is Lebesgue integral. However, for a global definition, we encounter some obstruction. If $\varphi_i, \varphi_j\colon V \to U$ are both local charts, then

$$\int_V \varphi_i^* \omega = \int_V (\varphi_i \circ \varphi_j^{-1})^* \varphi_j^* \omega = \int_V \det \mathrm{J}(\varphi_i \circ \varphi_j^{-1}) \varphi_j^* \omega.$$

In other words, we must restrict to those atlas satisfying $\det \mathrm{J}(\varphi_i \circ \varphi_j^{-1}) > 0$. Not all manifolds admit such atlas, and those who do are said to be orientable. In fact, not all manifolds are orientable, for example, $\mathbb{RP}^2$ and the Klein bottle.

Proposition 3.1. A manifold M is orientable if and only there exists a non-vanishing smooth top form μ, called the volume form.

As a simple corollary, any orientable manifold has exactly two orientations. Fixing an atlas $\{(U_i, \varphi_i)\}$ on M which gives an orientation, we can pick a P.O.U. $\{\rho_i\}$ with respect to $\{(U_i, \varphi_i)\}$, then

$$\int_M \omega = \sum_i \int_{U_i} \rho_i \omega, \omega \in \Omega_c^m(M).$$

Fundamental theorem of calculus finds its generalization on manifolds.

Theorem 3.2 (Stokes)**.** Let M be an oriented smooth manifold boundary, and its boundary ∂M is endowed with the induced orientation. Then for any $\omega \in \Omega_c^{m-1}(M)$,

$$\int_M d\omega = \int_{\partial M} \omega|_{\partial M}.$$

Later we will see that, integrals on an oriented submanifolds can be expressed as

$$\int_S \omega|_S = \eta_S \cup \omega,$$

where $\eta_S \in \Omega_c^r(M)$ is the Poincaré dual of S and $\cup$ is the cup product. Under this notation, the Stokes' formula says that

$$\eta_M \cup d\omega = \eta_{\partial M} \cup \omega,$$

so in some sense ∂, d are adjoint operators.

4. de Rham theory

Let M be a smooth manifold. We can consider chains of exterior derivatives

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0.$$

Let $Z_k = \mathrm{Ker}(d\colon \Omega^k(M) \to \Omega^{k+1}(M))$ and $B_k = \mathrm{Im}(d\colon \Omega^{k-1}(M) \to \Omega^k(M))$. Then $d^2 = 0$ implies $B_k \leqslant Z_k$. The k-th de Rham cohomology of M is

$$H^k_{\mathrm{dR}}(M) = B_k / Z_k.$$

$(\Omega^*(M), d)$ is actually an example for cochain complexes.

Definition 4.1. A cochain complex is a pair (C^*, d) which consists of a chain of morphisms $d\colon C^q \to C^{q+1}$ such that $d^2 = 0$. Here C^q are modules or vector spaces over fields.

For every cochain complex (C^*, d), we can all define its cohomology

$$H^k(C^q, d) = \frac{\mathrm{Ker}(d\colon C^q \to C^{q+1})}{\mathrm{Im}(d\colon C^{q-1} \to C^q)}.$$

Definition 4.2. Let $(A^*, d_A), (B^*, d_B)$ be cochain complexes. A cochain map $f\colon A^* \to B^*$ is a sequence of morphisms $f_q\colon A^q \to B^q$ such that the following diagram is commutative

$$\begin{array}{ccc} A^q & \xrightarrow{d_A} & A^{q+1} \\ \downarrow{\scriptstyle f_q} & & \downarrow{\scriptstyle f_{q+1}} \\ B^q & \xrightarrow{d'_B} & B^{q+1} \end{array}$$

This can be simply expressed as $f \circ d_A = d_B \circ f$.

By definition, a cochain map $f\colon A^* \to B^*$ naturally induces maps of cohomology $f\colon H^q(A^*, d_A) \to H^q(B^*, d_B), [\alpha] \mapsto [f(\alpha)]$. This is well-defined because $f(d_A(\alpha)) = d_B(f(\alpha))$.

Let $f\colon M \to N$ be a smooth map of manifolds. Then f induces the pullback of differential forms $f^*\colon \Omega^*(N) \to \Omega^*(M)$. Moreover, this is indeed a cochain map because d commutes with the pullback. If f is a diffeomorphism, then $(f \circ f^{-1})^* = (f^{-1})^* \circ f^* = \mathbb{1}$ and $(f^{-1} \circ f)^* = f^* \circ (f^{-1})^* = \mathbb{1}$. In particular, diffeomorphism manifolds have isomorphism de Rham cohomologies.

Example 4.3. Let's consider the de Rham cohomology of $X = \mathbb{R}^2 \setminus \{P, Q\}$, where P, Q are distinct points. Without loss of generosity, assume $P = (-1, 0)$ and $Q = (1, 0)$. The cochain complex writes

$$C^\infty(X) \xrightarrow{d} \{f dx + g dy \mid f, g \in C^\infty(X)\} \xrightarrow{d} \{\varphi dx \wedge dy \mid \varphi \in C^\infty\}.$$

$H^0_{\mathrm{dR}}(X) = \mathrm{Ker}(d\colon C^\infty \to \Omega^1(X))$. $df = 0$ implies that f is constant, so $H^0_{\mathrm{dR}}(X) = \mathbb{R}$.

Next consider $H^1_{\mathrm{dR}}(X)$. Let $\omega = f dx + g dy$ and we want to see that how far it is from an exact form. First

$$d\omega = (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx \wedge dy = 0,$$

so $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$. If $\omega = d\varphi$, then

$$\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} \omega,$$

where the integral is taken along an arbitary smooth path joining x_0, x in X. Conversely, if the integral of ω is indeed independent of the path, by multivariable calculus, φ is a well-defined smooth function such that $d\varphi = \omega$. As $d\omega = 0$, by Stokes' formula the integral is invariant up to free homotopy, so the problem lies in the two integrals

$$\lambda_1 \int_{C_1} \omega, \lambda_2 = \int_{C_2} \omega,$$

where C_1, C_2 are small circle around P, Q respectively. To get rid to this, we can consider forms

$$\eta_1 = \frac{1}{2\pi} \frac{(x+1)dy - ydx}{(x+1)^2 + y^2}, \eta_2 = \frac{1}{2\pi} \frac{(x-1)dy - ydx}{(x-1)^2 + y^2}.$$

These are all closed forms with

$$\int_{C_i} \eta_j = \delta_{ij}.$$

Thus we can modify ω by considering

$$\omega - \lambda_1 \eta_1 - \lambda_2 \eta_2.$$

This time the integral of $\omega - \lambda_1\eta_1 - \lambda_2\eta_2$ along arbitary paths is well-defined, so it is exact. In other words,

$$[\omega] = \lambda_1[\eta_1] + \lambda_2[\eta_2].$$

It follows that $H^1_{\mathrm{dR}}(X) = \mathbb{R}[\eta_1] \oplus \mathbb{R}[\eta_2] \simeq \mathbb{R}^2$.

It is discouraging that we found difficulty computing the last cohomology $H^3_{\mathrm{dR}}(X)$.

In some sense, de Rham cohomology is solving the equation $d\eta = \omega$ for a given ω. As $d^2 = 0$, $d\omega$ is a necessary condition for a solution to exist. Then de Rham cohomology tells us how many such solutions on M fail to admit solutions, and they detect "holes" of certain dimensions on manifolds .

5. Mayer–Vietoris sequences

Computation for de Rham cohomology of plane punctured two points suggest we need to develop more techniques.

The basic idea for Mayer–Vietoris sequences is reconstructing cohomology of the whole M via cohomologies on two open sets that covers M. For this purpose we need to make use of tools from homological algebra.

Definition 5.1. Cochain maps

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$$

is called a short exact sequence if for each q,

$$0 \longrightarrow A^q \xrightarrow{f} B^q \xrightarrow{g} C^q \longrightarrow 0$$

is exact.

Theorem 5.2. A short exact sequence

$$0 \longrightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \longrightarrow 0$$

induces a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^n(A^*) \arrow[r, "f"] & H^n(B^*) \arrow[r, "g"] & H^n(C^*) \arrow[dll, "\delta"] \\ & H^{n+1}(A^*) \arrow[r, "f"] & H^{n+1}(B^*) \arrow[r, "g"] & H^{n+1}(C^*) \arrow[r] & \cdots \end{tikzcd}$$

δ is called the connecting map.

Proof. We simply construct the connecting map, while the rest is easy to check directly. Given a class $[\alpha] \in H^n(C^*)$. Since $g\colon B^n \to C^n$ is surjective, $\alpha = g(\beta)$ for some $\beta \in B^n$. Consider $d\beta$. Since $g(d\beta) = d(g(\beta)) = d\alpha = 0$, there exists a unique element $\gamma \in A^{n+1}$ such that $f(\gamma) = d\beta$. Since $f(d\gamma) = d(f(\gamma)) = d^2\beta = 0$, we have $d\gamma = 0$, which implies $\gamma \in H^{n+1}(B^*)$. This can be illustrated by the following diagram

$$\begin{tikzcd} & \beta \arrow[r] \arrow[d] & \alpha \arrow[d] \\ \gamma \arrow[r] \arrow[d] & d\beta \arrow[r] \arrow[d] & 0 \\ d\gamma \arrow[r] & 0 \end{tikzcd}$$

□

Say $\{U, V\}$ is an open cover of M. Since pullback commutes with d, we have two natural chain maps

$$\Omega^*(M) \xrightarrow{f} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{g} \Omega^*(U \cap V),$$

where $f(\omega) = (\omega|_U, \omega|_V), g(\omega_1, \omega_2) = \omega_1|_{U\cap V} - \omega_2|_{U\cap V}$. It is easy to see that this is an exact sequence. By Theorem 5.2 we obtain a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^n_{\mathrm{dR}}(M) \arrow[r, "f^*"] & H^n_{\mathrm{dR}}(U) \oplus H^n_{\mathrm{dR}}(V) \arrow[r, "g^*"] & H^n_{\mathrm{dR}}(U \cap V) \arrow[dll, "\delta"] \\ & H^{n+1}_{\mathrm{dR}}(M) \arrow[r, "f^*"'] & H^{n+1}_{\mathrm{dR}}(U) \oplus H^{n+1}_{\mathrm{dR}}(V) \arrow[r, "g^*"'] & H^{n+1}_{\mathrm{dR}}(U \cap V) \arrow[r] & \cdots \end{tikzcd}$$

One may ask what is the expression of δ. Careful calculation shows that

$$\delta[\omega] = \begin{cases} [d(\rho_V \omega)] & \text{on } V, \\ -[d(\rho_U)\omega] & \text{on } U, \end{cases}$$

where ρ_U, ρ_V is a P.O.U. with respect to $\{U, V\}$.

Example 5.3. Consider $\mathbb{S}^1 \left\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\right\}$. It admits an open cover $U = \mathbb{S}^1 \setminus \{(1,0)\}$ and $V = \mathbb{S}^1 \setminus \{(0,-1)\}$. Then U, V are diffeomorphic to $\mathbb{R}$ while $U \cap V$ is diffeomorphic to $\mathbb{R} \sqcup \mathbb{R}$. The Mayer–Vietoris sequence for this open cover writes

$$\begin{tikzcd} H^0_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R} \arrow[r] & H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) = \mathbb{R}^2 \arrow[r] & H^0_{\mathrm{dR}}(U \cap V) = \mathbb{R}^2 \arrow[dll] \\ H^1_{\mathrm{dR}}(\mathbb{S}^1) \arrow[r] & H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) = 0 & \end{tikzcd}$$

Linear algebra shows that $\dim_{\mathbb{R}} H^1_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R}$, so $H^1_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R}$.

In general, if we acknowledge the homotopy invariance of de Rham cohomology, we can compute $H^*_{\mathrm{dR}}(\mathbb{S}^n)$ for arbitary n. Actually $H^k_{\mathrm{dR}}(\mathbb{S}^n) = \mathbb{R}$ if $k = 0, n$ and $H^k_{\mathrm{dR}}(\mathbb{S}^n) = 0$ for other k. This can be done using induction. Suppose the result holds for $n-1$ and consider n. Let P, Q be the north pole and south pole of $\mathbb{S}^n$. Then $\{U = \mathbb{S}^n \setminus \{P\}, V = \mathbb{S}^n \setminus \{Q\}\}$ would be an open cover of $\mathbb{S}^n$. Since U, V are diffeomorphic to $\mathbb{R}^n$ and $U \cap V$ is homotopy equivalent to $\mathbb{S}^{n-1}$, by Mayer–Vietoris sequence we have exact sequences

$$H^{k-1}_{\mathrm{dR}}(U \cap V) = 0 \longrightarrow H^k_{\mathrm{dR}}(\mathbb{S}^n) \longrightarrow H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) = 0, 1 \leqslant k \leqslant n-1,$$

$$H^{n-1}_{\mathrm{dR}}(U) \oplus H^{n-1}_{\mathrm{dR}}(V) = 0 \longrightarrow H^{n-1}_{\mathrm{dR}}(U \cap V) = \mathbb{R} \longrightarrow H^n_{\mathrm{dR}}(\mathbb{S}^n) \longrightarrow H^n_{\mathrm{dR}}(U) \oplus H^n_{\mathrm{dR}}(V) = 0.$$

Therefore, $H^k_{\mathrm{dR}}(\mathbb{S}^n) = 0$ for $1 \leqslant k \leqslant n$ and $H^0_{\mathrm{dR}}(\mathbb{S}^n) = H^n_{\mathrm{dR}}(\mathbb{S}^n) = \mathbb{R}$.

Example 5.4. With the help of Mayer–Vietoris sequence, we can now pin down $H^2_{\mathrm{dR}}(X)$ in Example 4.3. As we can cover X with U, V such that U contains P, V contains Q and U, V both diffeomorphic to $\mathbb{R}^2 \setminus \{(0,0)\}$, $U \cap V$ diffeomorphic to $\mathbb{R}^2$, we obtain

$$H^1_{\mathrm{dR}}(U \cap V) = 0 \longrightarrow H^2_{\mathrm{dR}}(X) \longrightarrow H^2_{\mathrm{dR}}(U) \oplus H^2_{\mathrm{dR}}(V) = 0.$$

Therefore $H^2(X)_{\mathrm{dR}} = 0$.

6. COMPACT SUPPORTED DE RHAM COHOMOLOGY

Notice that we have another type of differential forms, the compactly supported ones. They are especially when we deal with integrals on non-compact manifolds. In fact, there is also a de Rham theory for them.

Definition 6.1. $(\Omega^*_c(M), d)$ is still a cochain complex, and its cohomology $H^*_c(M)$ is called the compactly supported de Rham cohomology.

If M is compact, then obviously the two cohomologies coincide. However, the following example shows that, even for Euclidean spaces, the simpliest non-compact manifolds, the two cohomologies are different.

Example 6.2. $H^k_c(\mathbb{R}^n) = 0$ if $k \neq n$ and $H^n_c(\mathbb{R}^n)$. In general, if M is connected and oriented, then we have an isomorphism

$$\begin{aligned} H^m_c(M) &\longrightarrow \mathbb{R} \\ [\omega] &\longmapsto \int_M \omega. \end{aligned}$$

Note that a smooth map $f\colon M \to N$ cannot induce a chain map of compactly supported forms if it is not proper. As a result, Mayer–Vietoris sequence in section 5 may not hold, since inclusion maps are not always proper. To remedy this problem, we can consider the extension map instead. If $U \subset M$ is open and $\omega \in \Omega_c(U)$ can be extended to a global smooth form by declaring $\omega_p = 0, p \notin U$. Therefore, we obtain a chain map

$$\Omega^*_c(U \cap V) \xrightarrow{f} \Omega^*_c(U) \oplus \Omega^*_c(V) \xrightarrow{g} \Omega^*_c(M),$$

where $f(\omega) = (\omega, \omega)$ and $g(\omega_1, \omega_2) = \omega_1 - \omega_2$. Similarly, by Theorem 5.2 we get the long exact sequence

$$\cdots \longrightarrow H_c^n(U \cap V) \xrightarrow{f^*} H_c^n(U) \oplus H_c^n(V) \xrightarrow{g^*} H_c^n(M) \xrightarrow{\delta} H_c^{n+1}(U \cap V) \xrightarrow{f^*} H_c^{n+1}(U) \oplus H_c^{n+1}(V) \xrightarrow{g^*} H_c^{n+1}(M) \longrightarrow \cdots$$

Here $\delta([\omega]) = [d(\rho_U)\omega]$ where $\{\rho_U, \rho_V\}$ is a P.O.U. of $\{U, V\}$.

7. Shortcomings of de Rham cohomology

The notion of de Rham is still not satisfactory. We list two problems:

(1) Since de Rham cohomology is $\mathbb{R}$-coefficient, it odmits the torsion component, which is also an important topological invariant.

(2) Mayer–Vietoris sequence is of limited use, because it applies for open cover with merely two open sets.

For the first problem, we can study cohomology with general coefficients and also, the homotopy groups. For the second problem, we will introcuce the Cěch cohomoology.

8. Additional topics

As the last part of my talk, I gave a mini talk on Morse theory. The reference is *Morse Theory* by Shintaro Fushida-Hardy and I submitted it to our chat group.