LECTURE NOTES ON SYMPLECTIC TOPOLOGY

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These notes are based on lectures on symplectic topology at USTC by Jun Zhang. Assume that M is a smooth manifold of dimension $2n$.

1. symplectic structures and examples

Symplctic topology and symplectic geometry basically means the same thing. The reason for this will be explained later.

Definition 1.1. A **symplectic structure** on M is a closed 2-form $\omega \in \Omega^2(M)$ which is non-degenerate, in the sense that $\omega^n = \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is a volume form. A symplectic manifold is such a pair (M, ω).

Remark 1.2. There are several equivalent conditions to describe non-degeneracy of ω. For example, $TM \to T^*M, X \mapsto \iota_X \omega$ is a bundle isomorphism.

Example 1.3. There are lots of examples for symplectic manifolds.

(1) Area forms on a smooth surface automatically satisfy the condition for symplectic structure. In particular, closed surfaces $(\Sigma_g, \omega_{\text{area}})$ are symplectic manifolds.

(2) On $\mathbb{R}^{2n}$ with coordinate $(x^1, y^1, \cdots, x^n, y^n)$, there exists a standard symplectic structure

$$\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i.$$

Alternatively, it can be expressed as $d\lambda$, where

$$\lambda = \frac{1}{2} \sum_{i=1}^n (x^i dy^i - y^i dx^i),$$

hence ω_0 is also an exact form.

(3) Let Q be a smooth manifold. Its cotangent bundle T^*Q admits a canonical symplectic structure $\omega = d\lambda$. Under local coordinate $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ the 1-form is expressed as

$$\lambda = -p_i dq^i,$$

then $\omega = dq^i \wedge dp_i$. To show that λ is well-defined, notice that

$$\lambda_{(q,p)}(Y) = p_q((d\pi)_{(q,p)}(Y)), Y \in T_{(q,p)}T^*Q.$$

For some reasons in notation, it is a good habit to keep writting the base manifold as Q.

(4) Given two symplectic manifolds (M, ω_M) and (N, ω_N), we can cook up a new symplectic manifold by taking products. Consider $\omega_{\text{prod}} = \pi_M^* \omega_M + \pi_N^* \omega_N \in \Omega^2(M \times N)$. Since pullback commutes with exterior derivative, ω_{prod} is also closed. By computation,

$$\omega_{\text{prod}}^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} \pi_M^* \omega_M^k \pi_N^* \omega_N^{m+n-k} = \binom{m+n}{m} \pi_M^* \omega^m \pi_N^* \omega_N^n.$$

As the nonzero scaling of a symplectic structure is still symplectic, $t\pi_M^*\omega_M + s\pi_N^*\omega_N, ts \neq 0$ are all symplectic structures on $M \times N$. In this way we get plenty of different symplectic structures on product manifolds. For example, $\mathbb{T}^{2n}, \mathbb{S}^2 \times \mathbb{S}^2$ are symplectic manifolds.

(5) The complex projective spaces $\mathbb{CP}^n$ admit a special class of symplectic structures ω_{FS}, called Fubini–Study forms. It can be described as follows. Let $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ be the standard open cover

and

$$\begin{aligned} \varphi_i \colon U_i &\longrightarrow \mathbb{C}^n \\ [z_0, \cdots, z_n] &\longmapsto \left(\frac{z_0}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_n}{z_i} \right). \end{aligned}$$

Under φ_i we define

$$\omega_{\mathrm{FS}}|_{U_i} = \frac{i}{2\pi} \partial\bar{\partial} \log \left(\sum_{l=0}^{n} \left| \frac{z_l}{z_i} \right|^2 \right)$$

A straightforward computation yields

$$\partial\bar{\partial} \log \left(\sum_{i=1}^{n} |w_i|^2 + 1 \right) = \frac{1}{(1 + \sum |w_i|^2)^2} \sum h_{ij} dw_i \wedge d\bar{w}_j$$

with $h_{ij} = (1 + \sum |w_j|^2)\delta_{ij} - \bar{w}_i w_j$.

ω_{FS} are though to be special because they are Kähler forms, i.e., they are induced by the complex structures. Also note that $\mathbb{CP}^1 = \mathbb{S}^2$ has volume 1 under ω_{FS} while its area is 2π under ω_{area}.

By definition, if $U \subset (M, \omega)$ is a non-empty open subset, then $(U, \omega|_U)$ is also a symplectic manifold. A fundamental theorem by Darboux shows that, symplectic manifolds of the same dimensional locally look the same, so there is no local geometry. This a significant difference from differential geometry, and that is why the lecture is named symplectic topology.

One may wonder is there any obstruction for the existence for symplectic structures on an even-dimensional manifold.

Proposition 1.4. If M is closed, i.e., compact and without boundary, then $[\omega]^k$ is nonzero class in $H^{2k}(M; \mathbb{R})$ for every $1 \leqslant k \leqslant n$.

Proof. By contradiction, say $[\omega]^k = 0$ for some k. Since $[\omega]^k = [\omega^k]$, $\omega^k = d\alpha$ for some $\alpha \in \Omega^{2k-1}(M)$. By Stokes' formula,

$$\int_M \omega^n = \int_M \omega^k \wedge \omega^{n-k} = \int_M d\alpha \wedge \omega^{n-k} = \int_M d(\alpha \wedge \omega^{n-k}) = 0.$$

This is a contradiction. □

Corollary 1.5. Among $\mathbb{S}^{2n}, \mathbb{S}^{2p} \times \mathbb{S}^{2q}$, symplectic structures exist if and only if $n = 1$ and $p, q = 1$.

2. SYMPLECTOMORPHISMS

Let (M, ω) be a symplectic manifold.

Definition 2.1. $\varphi \in \mathrm{Diff}(M)$ is called a **symplectomorphism** if it preserves the symplectic structure, i.e., $\varphi^*\omega = \omega$. Symplectomorphisms form a subgroup of $\mathrm{Diff}(M)$, denoted by $\mathrm{Symp}(M, \omega)$ with $\mathrm{Symp}_0(M, \omega)$ representing the connected component of id.

$\mathrm{Symp}(M, \omega)$ is a huge and mysterious topological group, much of whose nature remains unknown for mathematicians. Note that $\varphi^*(\omega^n) = (\varphi^*\omega)^n = \omega^n$, so at least $\mathrm{Symp}(M, \omega)$ would be a subgroup of $\mathrm{Diff}(M)_{\mathrm{Vol}}$, the diffeomorphisms preserving volumes. In general $\mathrm{Symp}(M, \omega) \subsetneq \mathrm{Diff}(M)_{\mathrm{Vol}}$, which seems very reasonable but is not easy to proof. This result is due to Gromov with his non-squeezing theorem. Roughly speaking, although symplectic geometry has no local behaviour, it possesses certain rigidity in the global sense.

Once we focus on linear isomorphism on $(\mathbb{R}^{2n}, \omega_0)$, things get much easier.

Definition 2.2. The **symplectic group** is $\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^*\omega_0 = \omega_0\}$.

Symplectic groups are Lie groups with nice property.

Theorem 2.3. $\mathrm{Sp}(2n)$ deform retracts to $\mathrm{U}(n)$. In particular, it has fundamental group $\mathbb{Z}$.

This result is essential in defining another important invariant for symplectic manifolds.

Proposition 2.4. The first Chern class for (M, ω), $\mathrm{c}_1(TM) \in H^2(M; \mathbb{Z})$ is well-defined.

Proof. Since $\mathrm{Sp}(2n)$ deform retracts to $\mathrm{U}(n)$, TM is equivalent to a complex vector bundle over M. □

There are two natural questions about symplectic geometry:
(1) How to construct a symplectomorphism?
(2) How to tell if a map is a symplectomorphism?

To answer these questions, we need to introduce to elements in symplectic geometry respectively:
(1) Hamiltonian dynamics.
(2) Lagrangian submanifolds.

3. COMPLEXITY OF SYMPLECTIC STRUCTURES

Let (M,ω) be a symplectic manifold.

In algebraic topology, characteristic classes measure the complexity of vector bundles over a space. To describe the complexity of a symplectic strucuture, we can also make use of some special tools and invariants.

Proposition 3.1. Every symplectic manifold (M,ω) admits an almost complex structure J such that
(1) $\omega_p(v,Jv)>0, v\in T_pM$.
(2) $J^*\omega=\omega$, i.e., $\omega_p(Ju,Jv)=\omega_p(u,v), u,v\in T_pM$.
As a consequence, $g_J(u,v)=\omega(u,Jv)$ defines a Riemannian metric on M. Also note that

$$g_J(Ju,Jv)=\omega(Ju,-v)=\omega(v,Ju)=g_J(v,u).$$

Although it may not make sense at the moment, this show that symplectic manifolds can be quantified. In the remainder of this section, we will fix a symplectic manifold together with induced almost complex structure (M,ω,J).

Definition 3.2. A ***J*-holomorphic curve** in (M,ω,J) is a smooth map $u\colon(\Sigma,j)\to(M,\omega,J)$ such that $J\circ u_*=u_*\circ j$, where Σ is a Riemann surface and j is its complex structure.

J-holomorphic curve is an important tool in symplectic geometry, developed by Gromov in 1980's. In this lecture we will be only interested in cases where $\Sigma=\mathbb{S}^2$ or $\mathbb{S}^2\setminus\{\text{finitely many pts}\}$.

Example 3.3. A map $u\colon(\mathbb{C},\sqrt{-1})\to(\mathbb{C},\sqrt{-1})$ is a J-holomorphic curve if and only if u satisfies the Cauchy–Riemann equation, i.e., u is holomorphic.

Proposition 3.4. For any smooth curve $u\colon(\Sigma,j)\to(M,\omega,J)$, let

$$E(u)=\int_\Sigma u^*\omega.$$

We have $E(u)\leqslant\operatorname{Area}(u)$, with equality holds if u is a J-holomorphic curve.

Proof. Fix a local orthonormal frame $\{e_1,e_2\}$ on $T\Sigma$ such that $g_J(u_*e_1,u_*e_2)=0$. Then

$$(u^*\omega)(e_1,e_2)=\omega(u_*e_1,u_*e_2)=g_J(Ju_*e_1,u_*e_2)=(u^*g_J)(je_1,e_2)$$

By Cauchy–Schwarz inequaltiy, we have

$$g_J(Ju_*e_1,u_*e_2)\leqslant\sqrt{g_J(Ju_*e_1,Ju_*e_1)g_J(u_*e_2,u_*e_2)}=\sqrt{(u^*g_J)(e_1,e_1)(u^*g_J)(e_2,e_2)}$$

with equality holds if and only if $Ju_*e_1=u_*e_2$.

If u is J-holomorphic curve, we can pick $e_2=je_1$, as $g\ g_J(u_*e_1,u_*je_1)=-\omega(u_*e_1,u_*e_1)=0$. □

Corollary 3.5. A J-holomorphic curve is a minimal surface with respect to g_J.

Note that if $u\colon(\mathbb{S}^2,j)\to(M,\omega,J)$ is a J-holomorphic curve, $E(u)$ can be written as $[\omega]\cap[u]$, where $[u]\in\pi_2(M)$. In symplectic geometry, we consider the image of Hurewicz map

$$H_2^S(M)=\operatorname{Im}(h\colon\pi_2(M)\to H_2(M;\mathbb{Z})),$$

which is the collection of homology classes represented by the image of $\mathbb{S}^2$.

Example 3.6. $H_2^S(\mathbb{T}^2)=0$ because $\mathbb{T}^2$ is not homeomorphic to $\mathbb{S}^2$.
$H_2^S(\mathbb{S}^2)=\mathbb{Z}, H_2^S(\mathbb{S}^2\times\mathbb{S}^2)=\mathbb{Z}\times\mathbb{Z}$.
$H_2^S(\mathbb{CP}^n)=\mathbb{Z}$.

Two functions are defined on $H_2^S(M)$:

$$\begin{aligned}[\omega]\colon H_2^S(M) &\longrightarrow \mathbb{R}\\ A &\longmapsto [\omega]\cap A = \int_{\mathbb{S}^2} A^*\omega,\\ \mathrm{c}_1(TM)\colon H_2^S(M) &\longrightarrow \mathbb{Z}\\ A &\longmapsto \mathrm{c}_1 \cap A = \int_{\mathbb{S}^2} A^*\,\mathrm{c}_1(TM).\end{aligned}$$

The complexity of (M,ω) comes from the relation of $[\omega]$ and $\mathrm{c}_1(TM)$.

Definition 3.7. In general, there are three jumps in the complexity of a symplectic manifold.

- $[\omega] = \mathrm{c}_1(TM) \equiv 0$ on $H_2^S(M)$. Such symplectic manifolds are said to be **symplectically aspherical**. In some sense, symplectically aspherical manifolds are easy to study.
  Tori $\mathbb{T}^{2n}$ and closed surfaces $\Sigma_g(g>0)$ are examples for symplectically aspherical manifolds.
- $[\omega](A) = \kappa\,\mathrm{c}_1(TM)(A), A \in H_2^S(TM)$ for some $\kappa > 0$. Such symplectic manifolds are said to be **monotone**. By definition, a symplectically aspherical manifold is automatically monotone.
  $(\mathbb{CP}^1, \omega_{\mathrm{FS}})$ is our first example that is monotone but not symplectically aspherical. Let A be the generator of $H_2^S(\mathbb{CP}^1) = H_2(\mathbb{CP}^1;\mathbb{Z})$. Then $[\omega_{\mathrm{FS}}](A) = 1$ and $\mathrm{c}_1(A) = 2$, so $\kappa = \frac{1}{2}$. Generally, for every n, $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is monotone with $\kappa = \frac{1}{n+1}$.
- The last jump is a generalization of notion of monotone. We say (M,ω) is **semi-positive** (or **weakly monotone**) if it satisfies either one of the three conditions:
  (1) $[\omega](A) = \kappa\,\mathrm{c}_1(TM)(A), A \in H_2^S(M)$ for some $\kappa \geqslant 0$.
  (2) $\mathrm{c}_1(TM) \equiv 0$ on $H_2^S(M)$. (This condition is sometimes called Calabi–Yau).
  (3) The positive generator of $\mathrm{c}_1(TM)(H_2^S(M)) \leqslant \mathbb{Z}$ is at least $n-2$.
  The three conditions are not equivalent to each other but they can overlap on some manifolds. By definition, when $n \leqslant 3$, i.e., $\dim M \leqslant 6$, (M,ω) is automatically semi-positive.

Note that it is possible that a symplectic manifold is not semi-positive, but such examples are quite complicated and are hard to display at this moment.

Remark 3.8. Semi-positive manifolds are collections of objects people could deal with in Floer theory in 1990's, which was originally developed for monotone manifolds. Roughly speaking, almost every property holding for monotone case holds for semi-positive case. On the contrary, sometimes it is difficult to generalize results in aspherical case to monotone case.

Here comes an observation. If (M,ω) is semi-positive, then for any $A \in H_2^S(M)$ with $[\omega](A) > 0$, we must have $\mathrm{c}_1(TM)(A) \geqslant 0$ or $\mathrm{c}_1(TM)(A) \leqslant 2-n$. This turns out to be an if and only if!

Exercise 3.1. If every $A \in H_2^S(M)$ with $[\omega](A) > 0$ satisfies $\mathrm{c}_1(TM)(A) \geqslant 0$ or $\mathrm{c}_1(TM)(A) \leqslant 2-n$, then (M,ω) is semi-positive.

Proof. If $n \leqslant 3$, there is nothing to prove, so assume $n > 3$. It suffices to show that if condition (2) and (3) are violated, then (1) holds.

In this case, $\mathrm{c}_1(TM)(H_2^S(M)) = N\mathbb{Z}$ with $1 \leqslant N \leqslant n-3$. Take $A \in H_2^S(M)$ such that $\mathrm{c}_1(TM)(A) = N$, so $[\omega](A) \geqslant 0$. If $B \in H_2^S(M)$ making $\mathrm{c}_1(TM)(B) = N$, we must have $[\omega](A-B) = 0$. Indeed, otherwise say $[\omega](B-A) > 0$, then $\mathrm{c}_1(B-A) = 0$. Taking $m \in \mathbb{Z}_{\geqslant 1}$ sufficiently large, we obtain $[\omega](m(B-A)-A) > 0$ but $(m(B-A)-A) = \mathrm{c}_1(-A) \in (2-n, 0)$, which is a contradiciton.

Let

$$\kappa = \frac{[\omega](A)}{\mathrm{c}_1(TM)(A)}.$$

For any $C \in H_2^S(M)$, $\mathrm{c}_1(TM)(A) = (1+k)N$ and

$$[\omega](C) = [\omega](C-kA) +][\omega](kA) = (k+1)[\omega](A) = (k+1)\kappa\,\mathrm{c}_1(TM)(A) = \kappa\,\mathrm{c}_1(TM)(C).$$

□

Example 3.9. $(\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\mathrm{prod}})$ is monotone. If we change the symplectic structure, we obtain an example for a semi-positive manifold that is not monotone. Note that $\dim \mathbb{S}^2 \times \mathbb{S}^2 = 4 \leqslant 6$, this manifold is semi-positive with any symplectic structure. One can check that $\omega_{\mathrm{deform}} = (1+\varepsilon)\pi_1^*\omega + \pi_2 * \omega$ is non-monotone structure for $\varepsilon > 0$.

4. QUANTUM COHOMOLOGY AND NOVIKOV RINGS

Cohomology is an important topological invariant for manifolds. However, it does not carry the information of symplectic structure. In symplectic geometry, we can modify cohomology in the following way.

Definition 4.1. Let $\Gamma_\omega = \mathrm{Im}([\omega]\colon H_2^S(M) \to \mathbb{R}) \leqslant \mathbb{R}$. The **quantum cohomology** $QH^*(M,\omega)$ of (M,ω) with coefficient $\mathbb{K}$ is

$$QH^*(M,\omega) = H^*(M;\mathbb{K}) \otimes_{\mathbb{K}} \Lambda^{\mathbb{K},\Gamma_\omega},$$

where $\Lambda^{\mathbb{K},\Gamma_\omega}$ is the Novikov ring which we will introduce later.

Definition 4.2. Let $\Gamma \leqslant \mathbb{R}$ be a subgroup. Its **Novikov ring** is the formal power series in T

$$\Lambda^{\mathbb{K},\Gamma} = \left\{ \sum_{\lambda \in \Gamma} a_\lambda T^\lambda \mid a_\lambda \in \mathbb{K}, \forall c \in \mathbb{R}, \# \{a_\lambda \neq 0 \mid \lambda < c\} < +\infty \right\}.$$

$\forall c \in \mathbb{R}, \# \{a_\lambda \neq 0 \mid \lambda < c\} < +\infty$ is called the Novikov condition.

Remark 4.3. A faster way to express the Novikov condition is $\lambda \to +\infty$, though this notion is not very appropriate for finite sums.

Example 4.4. $\mathbb{K}[[T]] \leqslant \Lambda^{\mathbb{K},\Gamma}$. On the contrary,

$$\sum_{n=1}^\infty T^{\frac{1}{n}}, \sum_{n=1}^\infty T^{-n} \notin \Lambda^{\mathbb{K},\Gamma}.$$

As a priori, $QH^*(M,\omega)$ should be considered as a module over the Novikov ring. Delightfully, it turns out that the Novikov ring is actually a field.

Lemma 4.5. $\Lambda^{\mathbb{K},\Gamma}$ is a field.

Proof. Let x be a nonzero element. By the Novikov condition, we can write

$$x = a_\mu T^\mu + \sum_{\lambda > \mu} a_\lambda T^\lambda = a_\mu T^\mu \left(1 + \sum_{\lambda > \mu} \frac{a_\lambda}{a_\mu} T^{\lambda - \mu} \right), a_\mu \neq 0.$$

Denote

$$A = -\sum_{\lambda > \mu} \frac{a_\lambda}{a_\mu} T^{\lambda - \mu}.$$

As $\lambda = \mu$ only finitely many times, $(1-A)^{-1} = 1 + A + \cdots + A^n + \cdots$ is well-defined and

$$x^{-1} = a_\mu^{-1} T^{-\mu} (1-A)^{-1}.$$

□

As a consequence, quantum cohomology can be considered as a $\Lambda^{\mathbb{K},\Gamma_\omega}$-vector space, and methods in linear algebra apply.

Example 4.6. We can examine what quantum cohomology likes under the three kinds of complexity mentioned last time.

- If (M,ω) is symplectically aspherical, then $\Gamma_\omega = 0$ and $\Lambda^{\mathbb{K},\Gamma_\omega}$ is simply $\mathbb{K}$. We see that the quantum cohomology on symplectically aspherical manifolds coincides with the usual cohomology.
- If (M,ω) is monotone, $\Gamma_\omega = \kappa \, c_1(TM)(H_2^S(M)) \simeq \mathbb{Z}$ is a cyclic subgroup of $\mathbb{R}$. Then $\Lambda^{\mathbb{K},\Gamma_\omega} \simeq \mathbb{Z}[T]]$ can be regarded as the field of Laurant series.
- If (M,ω) is semi-positive, condition (2) and (3) have no control on Γ_ω, so it could be that Γ_ω becomes very complicated.

Here is an explicit example showing how complicated Γ_ω could be.

Example 4.7. Consider $\mathbb{S}^2 \times \mathbb{S}^2$ with $\omega_{\text{deform}} = (1+\varepsilon)\pi_1^*\omega + \pi_2 * \omega$. Then

$$\Gamma_\omega = \mathbb{Z} \oplus \mathbb{Z}\varepsilon.$$

If ε is irrational, then Γ_ω will be a dense subgroup of $\mathbb{R}$.

Here is another enhanced version of Novikov ring. We can consider the series

$$\sum_{A\in H_2^S(M)} a_A t^A.$$

The question about this definition is, how to describe the Novikov condition? To resolve this, consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker}[\omega] \longrightarrow H_2^S \xrightarrow{[\omega]} \Gamma_\omega \longrightarrow 0.$$

Γ_ω is free, so the exact sequence splits, which implies $H_2^S(M) = \operatorname{Ker}[\omega] \oplus \Gamma_\omega, A \mapsto (B, \lambda)$. The formal series can be rewritten as

$$\sum_{\lambda\in\Gamma_\omega}\sum_{B\in\operatorname{Ker}[\omega]} a_{(B,\lambda)} t^{(B,\lambda)} = \sum_{\lambda\in\Gamma_\omega}\left(\sum_{B\in\operatorname{Ker}[\omega]} a_{(B,\lambda)} S^B\right) T^\lambda.$$

The Novikov condition for this setting should be

$$\forall c\in\mathbb{R}, \#\left\{a_{(B,\lambda)} \neq 0 \mid \lambda < c\right\} < +\infty,$$

which implies that the coefficient

$$\sum_{B\in\operatorname{Ker}[\omega]} a_{(B,\lambda)} S^B$$

lies in the group algebra $\mathbb{K}[\operatorname{Ker}[\omega]]$.

Before talking more about quantum cohomology, we first study some general aspects of modules over Novikov rings. Novikov admits an interesting structure, called the valuation.

Definition 4.8. We define a valuation ν on $\Lambda^{\mathbb{K},\Gamma}$ by $\nu(x) = \min\left\{\lambda\in\Gamma \mid T^\lambda \text{ has nozero coefficient}\right\}$ and presribing $+\infty$ to 0.

Proposition 4.9. ν is indeed a valuation on $\Lambda^{\mathbb{K},\Gamma}$:
(1) $\nu(x) = +\infty$ if and only if $x = 0$.
(2) $\nu(xy) = \nu(x) + \nu(y)$.
(3) $\nu(tx) = \nu(x)$ if $0 \neq t \in \mathbb{K}$.
(4) $\nu(x + y) = \min(\nu(x) + \nu(y))$.

These properties follows immediately from definition.

With valuations, we can associate a new structure on V.

Definition 4.10. A filtration function on a $\Lambda^{\mathbb{K},\Gamma}$-vector space V is a funciton $l\colon V \to \mathbb{R}\cup\{-\infty\}$ such that:
(1) $l(x) = -\infty$ if and only if $x = 0$.
(2) $l(\alpha x) = \lambda(x) - \nu(\alpha), \alpha \in \Lambda^{\mathbb{K},\Gamma}$.
(3) $l(x + y) \leqslant \max((x), l(y))$.
Note that by definition if $l(x) \neq l(y)$, say $l(x) > l(y)$, then

$$l(y) < l(x) = l(x + y - y) \leqslant \max(l(x + y), l(y)),$$

which forces $l(x + y) = l(x) = \max(l(x), l(y))$.

A natural question is, what does linear algebra look like on (V, l)?

5. Vector spaces with filtration functions

6. Gromov–Witten invariants

Gromov–Witten invariant is related to the central problem in symplectic geometry and algebraic geometry, counting curves. There are three major troubles we will encounter.

6.1. **The limit is bad.** Consider a family of J-holomorphic curves $\{u_n\}$. It's limit u_∞ may be complicated.

Example 6.1. Let $u_n\colon (\mathbb{S}^2, j) \to (\mathbb{CP}^2, J_0), [z_0 : z_1] \mapsto \left[z_0^2 : \frac{z_1^2}{n} : z_0 z_1\right]$ be a sequence of J-holomorphic curves, where J_0 is the standard complex structure. $\operatorname{Im} u_n = \left\{[Z_0 : Z_1 : Z_2] \mid Z_0 Z_1 = \frac{Z_2^2}{n}\right\}$, so its limit image is

$$\operatorname{Im}(u_\infty) = \{[Z_0 : Z_1 : Z_2] \mid Z_0 Z_1 = 0\} = A_0 \cup A_1,$$

where $A_0 = V(Z_0), A_1 = V(Z_1)$. Topologically, $\operatorname{Im}(u_\infty)$ is two copies of $\mathbb{S}^2$ attached at $[0 : 0 : 1]$.

A question is, can $\operatorname{Im}(u_\infty)$ be an image of a J-holomorphic curve? Intuitively, on $\mathbb{S}^2 \setminus \{[0 : 1]\}$ we can set $u_\infty([z_0 : z_1]) = [z_0^2 : 0 : z_0 z_1] = [z_0 : 0 : z_1]$. It seems reasonable to define $u_\infty([z_0 : z_1]) = [z_0 : 0 : z_1]$. However, this map only covers the component A_1 but misses A_0.

We can look at $[0 : 1]$ and see what happened to this point during the limit process. Surprisingly, $u_n([0 : 1]) = [0 : \frac{1}{n} : 0] = [0 : 1 : 0]$ is a fixed point and it is not the same point as $u_\infty([0 : 1]) = [0 : 0 : 1]$.

Exercise 6.1. Prove that there exsists a J-holomorphic curve $v_\infty\colon \mathbb{S}^2 \to \mathbb{CP}^2$ that covers component A_0 but misses A_1, and maps $[1 : 0]$ to $[0 : 0 : 1]$.

Proof. Consider a reparametrization of u_n defined by $v_n([z_0 : z_1]) = u_n([z_0 : n z_1]) = \left[z_0^2 : n z_1^2 : n z_0 z_1\right] = \left[\frac{z_0^2}{n} : z_1^2 : z_0 z_1\right]$. Then $v_\infty([z_0 : z_1]) = [0 : z_1 : z_0]$ is the desired map. $\square$

To resolve this puzzle, let's bravely consider $(v, u)\colon T \to \mathbb{CP}^2$, where T is two copies of $\mathbb{S}^2$ attached at $[0 : 0 : 1]$. In general, the limit of a family of J-holomorphic curves has the complicated domains represented as a tree, whose vertex are copies of $\mathbb{S}^2$ and edges are attached points. In Example 6.1 the domain is represented as two vertex attached by an edge.

6.2. **Multiply cover.**

Definition 6.2. Let $u\colon (\Sigma_g, j) \to (M, J)$ be a J-holomorphic curve. u is said to be **multiply covered** if there exists a J-holomorphic curve $u'\colon (\Sigma_{g'}, j') \to (M, J)$ and a non-constant holomorphic map $\varphi\colon \Sigma_g \to \Sigma_{g'}$ s.t.

$$\begin{tikzcd}
(\Sigma_g, j) \arrow[d, "\varphi"] \arrow[dr, "u"] & \\
(\Sigma_{g'}, j') \arrow[r, "u'"] & (M, J)
\end{tikzcd}$$

and $\deg \varphi > 1$. And we will call that u is a **multiple cover** of u'.

In general, we don't like multiple covers. By some topological constraint we can get rid of multiple covers.

Theorem 6.3 (Riemann–Hurwitz formula)**.** Let $\varphi\colon \Sigma_g \to \Sigma_{g'}$ be a non-constant holomorphic map. Then

$$2g - 2 = (\deg \varphi)(2g' - 2) + \sum_{p \in \Sigma_g} (e_p - 1),$$

where e_p is the ramification index at p.

In particular, if $\Sigma_g = \mathbb{S}^2$, the formula writes

$$-2 = (\deg \varphi)(2g' - 2) + \sum_{p \in \Sigma_g} (e_p - 1).$$

Therefore $g' = 0$ and $\Sigma_{g'} = \mathbb{S}^2$. Note that in this case we cannot claim $\deg \varphi = 1$.

Example 6.4. Here is an example of symplectic structure ruling out multiple cover. Suppose (M, ω) is semi-positive with $\mathrm{c}_1(TM)(H_2^S(M)) = 4\mathbb{Z}$. If $u\colon (\mathbb{S}^2, j) \to (M, J)$ is a J-holomorphic curve such that $\mathrm{c}_1(TM)([u]) = 4$, then u is not a multiple cover.

To see this, by contradiction suppose u is a multiple cover of $u'\colon \mathbb{S}^2 \to (M, J)$ with φ. By definition, $[u] = (\deg \varphi)[u']$, so $4 = \mathrm{c}_1(TM)([u]) = (\deg \varphi)\, \mathrm{c}_1(TM)([u'])$. As $\deg \varphi > 1$, we have either $\deg \varphi = 2$ or 4, but either case would result in $0 < \mathrm{c}_1(TM)([u']) < \mathrm{c}_1(TM)([u])$.

Definition 6.5. A J-holomorphic curve is said to be **simple** if it's not multiply covered.

If a J-holomorphic curve u is not simple, then can always find a simple u' such that u is a multiple cover of u.

The following is a criterion for determing whether a J-holomorphic curve is simple.

Theorem 6.6. A J-holomorphic curve $u\colon (\Sigma_g, j) \to (M, J)$ is simple if and only if there exists $z \in \Sigma_g$ such that $(du)_z \colon T_z\Sigma_g \to T_{u(z)}M$ is injective and $u^{-1}(u(z)) = \{z\}$.

Altough many textbooks claim this criterion to be useful, Jun said he never uses it.

6.3. **Reparametrization.** If $u\colon (\mathbb{S}^2, j) \to (M, J)$ is a J-holomorphic curve and $\varphi \in \mathrm{Aut}(\mathbb{S}^2, j)$, then geometrically $u' = u \circ \varphi^{-1}$ should be considered to be the same curve.

Recall that

$$\mathrm{Aut}(\mathbb{S}^2, j) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid ad - bc \neq 0 \right\} = \mathrm{PGL}(2, \mathbb{C})$$

is a connected Lie group with real dimension 6. Let $\mathcal{C}(M, J)$ be the collection of J-holomorphic curves $u\colon (\mathbb{S}^2, j) \to (M, J)$. $\mathrm{PGL}(2, \mathbb{C})$ acts on $\mathcal{C}(M, J)$ via $\varphi \cdot u = u \circ \varphi^{-1}$. Counting J-holomorphic curves should be essentially on the moduli space

$$\mathcal{C}(M, J)/\mathrm{PGL}(2, \mathbb{C}).$$

An enhanced version of this is considering

$$(u, (z_1, \cdots, z_k)) \in \mathcal{C}(M, J) \times \left(\prod^k \mathbb{S}^2 \setminus \Delta \right),$$

where Δ is the fat diagonal, i.e., $(z_1, \cdots, z_k)$ are k distinct points. Similarly $\mathrm{PGL}(2, \mathbb{C})$ acts on this set via $\varphi \cdot (u, (z_1, \cdots, z_k)) = (u \circ \varphi^{-1}, (\varphi(z_1), \cdots, \varphi(z_k)))$ and we can consider the moduli space

$$\mathcal{C}(M, J) \times \left(\prod^k \mathbb{S}^2 \setminus \Delta \right) / \mathrm{PGL}(2, \mathbb{C}).$$

Remark 6.7. $(z_1, \cdots, z_k)$ is to formulate constraints on the J-holomorphic curves.

Definition 6.8. Let (M, J) be an almost complex manifold. A stable map is given by the following data.

(1) Its domain T, which is a tree-type configuration containing finitely many $(\mathbb{S}^2, j)$ attached to each other in the same way. The vertex $v(T)$ are copies of $(\mathbb{S}^2, j)$ and the edges $e(T)$ are nodal points.

(2)