In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics.

Hermann Weyl, *"Invariants", Duke Math. Journ.*

NOTES ON HARMONIC FORMS

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1. A QUICK REVIEW TO LAST TALK

Given a Riemannian manifold (M, g) of dimension n, we can construct a canonical bundle isomorphism $\sharp\colon T^*M \simeq TM$ descriped by $g(\sharp\omega, X) = \omega(X)$. The gradient of $f \in C^\infty(M)$ is therefore defined as $\nabla f = \sharp(df)$.

We can also construct a local nonvanishing top form $\omega = \sqrt{g}dx^1 \wedge \cdots \wedge dx^n$. One can check that ω differs up to a sign under coordinate transformation. Since $\mathcal{L}_X\omega$ is also a top form, $\mathcal{L}_X\omega = \operatorname{div}(X)\omega$ for some well-defined $\operatorname{div}(X) \in C^\infty(M)$. Acoording to the Cartan's magic formula, $\mathcal{L}_X\omega = d(\iota_X\omega)$.

The Laplace–Beltrami operator is defined as $\Delta = -\operatorname{div}\nabla$. After some computation in the local coordinate, $\Delta = -\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j)$.

2. A BRIEF INTRODUCTION TO DE RHAM COHOMOLOGY

Suppose M is an n-dimensional smooth manifold and $\Omega^k(M)$ is the collection of k-forms on M, i.e., smooth sections of $\bigwedge^k T^*M$.

Definition 2.1 (de Rham complex)**.** The operators $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ satisfies $d \circ d = 0$, which is usually called the cochain condition. The family of differential forms compose the de Rham complex.

$$\cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \longrightarrow \cdots$$

Notice that $\Omega^k(M) = 0$ for $k > n$.

Definition 2.2 (closed and exact form)**.** $\omega \in \Omega^k(M)$ is said to be closed if $d\omega = 0$, and exact if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. The collections of closed and exact k-forms on M are denoted by $Z^k(M), B^k(M)$ respectively, which are vector subspaces of $\Omega^k(M)$.

$d \circ d = 0$ is equivalent to $B^k(M) \leqslant Z^k(M)$.

Definition 2.3 (de Rham cohomology)**.** The quotient space $H^k_{\mathrm{dR}}(M) = Z^k(M)/B^k(M)$ is called the k-th de Rham cohomology group of M.

Cohomology is a comprehensive techinique involving geometry, analysis, and algebra. Generally, de Rham cohomology measures the number of "holes" of different dimensions: the "holes" serve as barriers of global integrability of differential forms.

Some basic facts about de Rham cohomology are listed as below.

Proposition 2.4. (1) (de Rham theorem) $H^*_{\mathrm{dR}}(M) \simeq H^*(M)$, where $H^*(M)$ stands for the singular cohomology. Therefore de Rham cohomology does not depend on the smooth structure.
(2) If $M = \bigsqcup\limits_\alpha M_\alpha$ gives the connected components, then $H^0_{\mathrm{dR}}(M) = \bigoplus\limits_\alpha \mathbb{R}$.
(3) If M admits a finite nice cover, then $\dim H^k_{\mathrm{dR}}(M) < +\infty$.
(4) If smooth manifolds M, N are homotopy equivalent, then $H^*_{\mathrm{dR}}(M) \simeq H^*_{\mathrm{dR}}(N)$.
(5) (Poincaré duality) If M is compact and orientable, then $H^k_{\mathrm{dR}}(M) \simeq (H^{n-k}(M))^*$.

Example 2.5.

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R}, k = 0, n \\ 0, \text{else} \end{cases}$$

Example 2.6. Consider the orientable closed surface of genus g, Σ_g. Then $H^0_{\mathrm{dR}}(\Sigma_g) = \mathbb{R}, H^1_{\mathrm{dR}}(\Sigma_g) = \mathbb{R}^{2g}, H^2_{\mathrm{dR}}(\Sigma_g) = \mathbb{R}$.

3. Hodge theory

Our discussion has nothing to do with Riemannian structure yet. It is forseeable that more fruitful result would come out if Riemannian structure is taken into account. So far we have no idea how Riemannian structure interacts with smooth forms. For this problem, one has to realize that $\bigwedge^* T^*M$ is a graded algebra.

Suppose $(V, \langle\cdot,\cdot\rangle)$ is a Euclidean space with an orientation $e_1 \wedge \cdots \wedge e_n \in \bigwedge^n V$. We obtain an inner product on $\bigwedge^k V$ by prescribing an orthonormal basis e_I with $I = \{ i_1 < \cdots < i_k \}$. Notice that this inner product can be descriped as

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge v_k \rangle = \det(\langle v_i, w_j \rangle)$$

Lemma 3.1. We have a non-degenerate pairing

$$\begin{aligned} \bigwedge^k V \times \bigwedge^{d-k} V &\longrightarrow \bigwedge^d V \\ (\alpha, \beta) &\longmapsto \alpha \wedge \beta \end{aligned}$$

Definition 3.2 (star operator)**.** The star operator is defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle e_1 \wedge \cdots \wedge e_n$$

By definition, if $\{ 1, \cdots, d \} = I \sqcup J$,

$$*e_I = (-1)^{(I,J)} e_J$$

In particular, $*1 = e_1 \wedge \cdots \wedge e_n$ and $** = (-1)^{k(n-k)}$ on $\bigwedge^k V$.

We have seen that g carries a natural inner product on T_x^*M for every $x \in M$. Thus we can construct the star operator on $\bigwedge^k T_x^*M$. However, to extend star operator on the global section, orientation is necessary and we have to make the following adjustment:

Definition 3.3 (Hodge star operator)**.** For $\beta \in \Omega^k(M)$, there exists a unique $*\beta \in \Omega^{n-k}(M)$ satisfying $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega$ for every $\alpha \in \Omega^k(M)$. The Hodge operator is defined as

$$\begin{aligned} *\colon \Omega^k(M) &\longrightarrow \Omega^{n-k}(M) \\ \alpha &\longmapsto *\alpha \end{aligned}$$

By definition $*1 = \omega$ and $** = (-1)^{k(n-k)}$ on $\Omega^k(M)$.

Definition 3.4 (L^2-product)**.** The L^2-product of two compactly supported forms $\alpha, \beta \in \Omega^k(M)$ is defined as

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \omega = \int_M \alpha \wedge *\beta$$

which is obviously an inner product on $\Omega^k(M)$. Thus it's possible to talk about the L^2-norm of forms $\|\alpha\| = \sqrt{(\alpha, \alpha)}$.

From now on we assume M is compact.

As one expect, $\Omega^k(M)$ would not be a Hilbert space, so we can consider its completion $L^2(\Omega^k(M))$, whose elements is sections locally given by L^2 functions.

Definition 3.5. The formal adjoint of d is defined by

$$\begin{aligned} d^*\colon \Omega^k(M) &\longrightarrow \Omega^{k-1}(M) \\ \alpha &\longmapsto (-1)^{n(k+1)+1} * d * \alpha \end{aligned}$$

Proposition 3.6. For $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$, we have

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

Proof. By Stokes formula,

$$0 = \int_M d(\alpha \wedge *\beta) = \int_M (d\alpha) \wedge \beta + (-1)^{k-1} \int_M \alpha \wedge (d * \beta)$$
$$= \int_M (d\alpha) \wedge \beta + (-1)^{k-1}(-1)^{(n-k+1)(k-1)} \int_M \alpha \wedge (* * d * \beta) = (d\alpha, \beta) - (\alpha, d^*\beta)$$

□

Formally, we can define a global inner product on $\Omega^*(M)$ in which

$$\Omega^*(M) = \bigoplus_k \Omega^k(M)$$

is an orthogonal decomposition, so d^* becomes a real adjoint operator under this setting.

Definition 3.7 (Laplace–Beltrami operator)**.** the Laplace–Beltrami operator on forms is defined as $\Delta = dd^* + d^*d \colon \Omega^*(M) \to \Omega^*(M)$.

Proposition 3.8. Δ is self-adjoint. More precisely, for $\alpha, \beta \in \Omega^k(M)$, we have

$$(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$$

Proof.

$$(\Delta\alpha, \beta) = (dd * \alpha, \beta) + (d^*d\alpha, \beta) = (d^*\alpha + d^*\alpha) + (d\alpha, d\beta)$$

□

Definition 3.9 (Harmonic forms)**.** $\omega \in \Omega^k(M)$ is said to be harmonic if $\Delta\omega = 0$. The family of Harmonic k-forms is denoted by $\mathcal{H}^k(M)$.

Remark 3.10. In principle we should not expect harmonic fomrs to be smooth, but it is a marvellous result in elliptic regularity.

Lemma 3.11. Δ is a positive operator and $\Delta\alpha = 0$ if and only if $d\alpha = 0$ and $d^*\alpha = 0$.

Proof. $(\Delta\alpha, \alpha) = (d\alpha, d\alpha) + (d^*\alpha) + (d^*\alpha) \geqslant 0$. If $\Delta\alpha = 0$, then $d\alpha = 0$ and $d^*\alpha = 0$ are forced. □

Corollary 3.12. If M is compact and connected, then $\mathcal{H}^0(M) = \mathbb{R}$.

Lemma 3.13. There is a injection

$$\begin{aligned} \mathcal{H}^k(M) &\longrightarrow H^k_{\mathrm{dR}}(M) \\ \alpha &\longmapsto [\alpha] \end{aligned}$$

Proof. If $[\alpha] = [0]$, then $\alpha = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. By Lemma 3.11 $0 = d^*\alpha = d^*d\eta$ and we obtain

$$(\alpha, \alpha) = (d\eta, d\eta) = (\eta, d^*d\eta) = 0$$

□

Theorem 3.14 (Hodge)**.**

$$\begin{aligned} \mathcal{H}^k(M) &\longrightarrow H^k_{\mathrm{dR}}(M) \\ \alpha &\longmapsto [\alpha] \end{aligned}$$

is actually an isomorphism.

Proof. It suffices to check the surjectivity. Take $[\omega_0] \in H^k_{\mathrm{dR}(M)}$ and we wish to find $\alpha \in \Omega^{k-1}(M)$ such that $\omega_0 + d\alpha$ is harmonic. To achieve this, we shall use the Dirichlet's principle. If the minimum of L^2-norm

$$\lambda = \inf \{ (\omega_0 + d\alpha, \omega_0 + d\alpha) \mid \alpha \in \Omega^{k-1}(M) \}$$

is achieved by $\beta = \omega_0 + \alpha_0 \in \Omega^{k-1}(M)$, then

$$0 = \frac{d}{dt}(\beta + td\alpha, \beta + td\alpha)\bigg|_{t=0} = 2(\beta, d\alpha) = 2(d^*\beta, \alpha)$$

for every $\alpha \in \Omega^{k-1}(M)$, so $d^*\beta = 0$. Since $d\beta = 0$ automatically, β is harmonic.

The most difficult part is showing the minimum is indeed achieved. The sketch are as below. There exists $\{\omega_n\}$ converges to the minimum, then this sequence is bounded and therefore weak sequential compact. We may assume $\omega_n \rightharpoonup \omega \in L^2(\Omega^k(M))$. By Banach–Steinhaus theorem,

$$\lambda^{\frac{1}{2}} \leqslant \|\omega\| \leqslant \limsup_{n\to\infty} \|\omega_n\| \leqslant \lambda^{\frac{1}{2}}$$

We see that $(\omega, d\beta) = 0$ for every $\beta \in \Omega^{k-1}(M)$.

Next, since $\omega_n - \omega_0 = d\alpha_n$ for some $\alpha_n \in \Omega^{k-1}(M)$,

$$(\omega_n - \omega_0, \varphi) = (d\alpha_n, \theta) = (\alpha_n, d^*\theta)$$

Therefore, if we put $\eta = \omega - \omega_0$, the functional

$$\begin{aligned} l\colon d^*(\Omega^k(M)) &\longrightarrow \mathbb{R} \\ d^*\theta &\longmapsto (\theta, \eta) \end{aligned}$$

is well-defined. By some estimation we can show that l is bounded, so by Riesz representation theorem, there exists $\alpha \in L^2(\Omega^{k+1}(M))$ such that

$$(d^*\theta, \alpha) = (\theta, \eta)$$

which implies η is harmonic in the weak sense. $\square$

We end this section showing both definitions of Δ coincide on $C^\infty(\mathbb{R})$. For every $\varphi \in C_c^\infty(M)$, we may assume φ is supported in a local coordinate $x^1, \cdots, x^n$, so

$$\int_M d^*df\varphi\omega = (d^*df, \varphi) = (df, d\varphi) = \int_M \langle df, d\varphi\rangle\omega = \int_{\mathbb{R}^n} g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \varphi}{\partial x^j}\sqrt{g}dx^1\cdots dx^n$$

Using integration by parts,

$$\int_M d^*df\varphi\omega = -\int_{\mathbb{R}}^n \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^i})\varphi\sqrt{g}dx^1\cdots dx^n = -\int_M \operatorname{div}(\nabla f)\varphi\omega$$

Thus we obtain $d^*df = -\operatorname{div}(\nabla f)$.

Remark 3.15. Using coarea theorem, we could have a coordinate-free prove.

$$\int_M d^*df\varphi\omega = \int_M \langle\nabla f, \nabla\varphi\rangle\omega = -\int_M \operatorname{div}(\nabla f)\varphi\omega$$