It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It only gives nominal definitions for them, while the essential means of determining them appear in the form of axioms.

Bernhard Riemann, *On the Hypotheses on which Geometry is Based*

NOTES ON RIEMANNIAN METRIC

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Throughout this note M stands for a smooth manifold of dimension m.

1. Supplement to last talk

We can let image of charts be open subsets since open balls are diffeomorphic to whole spaces.

Suppose $f\colon M \to N$ is a smooth map, i.e. $\psi\circ f\circ\phi^{-1}$ is smooth for every chart ψ, ϕ. It has differential map $df_p\colon T_pM \to T_{f(p)}N$ with $df_p(X_p)(\varphi) = X_p(\varphi \circ f)$, which is $\mathbb{R}$-linear. It can be checked that d has the functorial property: $d(g \circ f)_p = dg_{f(p)} \circ df_p$ and $d(\mathrm{Id})_p = \mathrm{Id}$. Smoothness and differential are something local, so we can restrict to open neighborhoods of p.

Any open subset of $\mathbb{R}^m$ is a smooth manifold if take global chart Id. So by definition any chart $\varphi\colon U \to \mathbb{R}^m$ is diffeomorphism. Last time Ruihua showed that $T_{\varphi(p)}\mathbb{R}^m = \operatorname{span}\{\frac{\partial}{\partial x_i}\}$. By functorial property $d\varphi_p$ is an isomorphism of vector spaces (in particular we get invariance of dimension). We use $\partial_j|_p$ or ∂_j to represent basis $(d\varphi_p)^{-1}(\frac{\partial}{\partial x_j})$ in T_pM for short and convenience.

Definition 1.1 (immersion and immersed submanifold)**.** A smooth map $f\colon M \to N$ is called an immersion if df_p is injective everywhere. In particular, in this case $\dim M \leqslant \dim N$. If f is also injective, then M or $f(M)$ is called the immersed submanifold of N (with respect to f).

We have to distinguish immersed submanifolds from smooth submanifolds to be defined below. Here is a complicated/interesting example: consider the map

$$\begin{aligned} f\colon \mathbb{R} &\longrightarrow S^1 \times S^1 \\ t &\longmapsto (e^{it}, e^{i\sqrt{2}t}) \end{aligned}$$

Then f is an injective immersion, and the image $f(\mathbb{R})$ is a dense curve in torus.

Definition 1.2 (smooth submanifold)**.** Suppose $f\colon M \to N$ is an immersion. If f is an embedding, then $f(M)$ is called a smooth submanifold.

Using implicit function theorem one can show that smooth submanifold inherits charts from N. So the difference of immersed submanifold and smooth submanifold is that, the ways they get charts (and therefore topology) are different.

Definition 1.3 (partition of unity/P.O.U.)**.** Suppose $\{U_\alpha\}$ is an open cover of M. Partition of unity (P.O.U. in brief) subordinate to the cover $\{U_\alpha\}$ is a family of smooth functions $\{\rho_\alpha\}$ on M such that
(1) $0 \leqslant \rho_\alpha \leqslant 1$ for all α.
(2) $\operatorname{supp}\rho_\alpha \subset U_\alpha$ for all α.
(3) each $p \in M$ has a neighborhood intersecting finitely many $\operatorname{supp}\rho_\alpha$'s.
(4) $\sum\limits_\alpha \rho_\alpha \equiv 1$.

It's a fundamental result that, for every open cover of M there exists a P.O.U. subordinate to it. P.O.U. would be the most important technical tool to study manifolds. More specifically, we can use it to "glue" geometric/analytic objects that can be defined locally using charts.

2. Riemannian metric

Whitney embedding theorem states that M admits a (proper) smooth embedding into $\mathbb{R}^{2m+1}$. By this theorem we can realize tangent space T_pM of $p \in M$ as vector space attached to p. To be specific, suppose p has a local chart $\varphi\colon U \to \mathbb{R}^m$. The Jacobian map of φ^{-1} gives T_pM constructed in $\mathbb{R}^{2m+1}$. Futhermore, we notice that T_pM is endowed with an inner product.

To have an inner product on a vector space is always excellent. However, construction above is never canonical. It highly depends on the way how we embed manifolds into Euclidean spaces. On the other hand, an abstract definition would simplify our discussion. This leads to an extra structure on smooth manifolds.

Definition 2.1 (Riemannian metric)**.** A Riemannian metric g on M is an assignment of an inner product $g_p(\cdot,\cdot) = \langle \cdot,\cdot \rangle_p$ on T_pM for each $p \in M$ that depends smoothly on p. To be specific, if X, Y are two smooth vector fields on an open subset $U \subset M$, then $f(p) = \langle X_p, Y_p \rangle_p$ is a smooth function on U.

Remark 2.2. g itself is NOT a metric, but we will see soon that g induces a natural distance on M.

One can represent the Riemannian metric g using local coordinates as follows. Let $\{ U, x_1, \cdots, x_m \}$ be a local coordinates. We denote $g_{ij}(p) = \langle \partial_i, \partial_j \rangle_p$. Then for any smooth vector fields $X = X^i \partial_i$ and $Y = Y^j \partial_j$ in U,

$$\langle X_p, Y_p \rangle_p = X^i(p) Y^j(p) \langle \partial_i, \partial_j \rangle_p = g_{ij}(p) X^i(p) Y^j(p)$$

So locally we can write $g = g_{ij} dx^i \otimes dx^j$. By definition the matrix $(g_{ij}(p))$ is the Gram matrix of $\langle \cdot,\cdot \rangle_p$ under the basis $\{ \partial_i \}$, so it's symmetric and positive definite everywhere.

This gives another description of a Riemannian metric: a Riemannian metric g is a symmetric smooth $(0,2)$-tensor field that is positive definite everywhere.

Definition 2.3 (Riemannian manifold)**.** Let g be a Riemannian metric on M. The pair (M, g) is called a Riemannian manifold.

Proposition 2.4. *M can be equipped with (lots of) Riemannian metrics. In other words, every manifold is a Riemannian manifold.*

Proof. let $\{ \rho_\alpha \}$ be a partition of unity subordinate to an atlas cover $\{ (U_\alpha, x_\alpha) \}$. For a positive definite symmetric positive definite matrix $A = (a_{ij})$, we can see that $g^A = \sum\limits_\alpha \rho_\alpha a_{ij} dx^i \otimes dx^j$ defines a Riemannian metric. $\square$

Remark 2.5. For those who may not be familiar with tensors, one can also define g^A by $g^A_p(X_p, Y_p) = \sum_\alpha \rho_\alpha a_{ij} X^i(p) Y^j(p)$ where $X = X^i \partial_i$ and $Y = Y^j \partial_j$ under given locally atlas.

Now we have a more flexible approach to construct inner structure than the previous introduction. There are also many ways to construct new Riemannian manifolds from the old ones, for example,

(1) Let $(M, g_M), (N, g_N)$ be two Riemannian manifolds, then $g_M \oplus g_N$ defined by

$$(g_M \oplus g_N)_{(p,q)}((X_p, Y_q), (X'_p, Y'_q)) = (g_M)_p(X_p, X'_p) + (g_N)_q(Y_q, Y'_q)$$

is a Riemannian metric on $M \times N$. $((M \times N), g_M \oplus g_N)$ is called the product Riemannian manifold of $(M, g_M), (N, g_N)$.

(2) Let (N, g) be a Riemannian manifold and $f\colon M \to N$ a smooth immersion. We can define the pull-back metric f^*g on M by

$$(f^*g)_p(X_p, Y_p) = g_{f(p)}(df_p(X_p), df_p(Y_p))$$

Since df_p is injective, f^*g is indeed a Riemannian metric on M, called the induced metric on M (with respect to f). In particular, when $M \subset N$ is an immersed submanifold, the inclusion map ι induces a Riemannian metric on M, which is just restriction of g onto $T_pM \subset T_pN$. In this case N is also called the Riemannian submanifold of N.

What follows from Riemannian metric is that we can consider distance in tangent space. Since tangent space is linear approximation of manifold geometrically, we wish to extend it to global manifold.

3. Riemannian distance

For the remainder of this note we suppose M is a conncected smooth manifold equipped with a Riemannian metric g.

Let $\gamma\colon [a,b] \to M$ be a smooth curve in M, so $\dot\gamma(t) = (d\gamma)_t(\frac{\mathrm{d}}{\mathrm{d}t})$ is a tangent vector in $T_{\gamma(t)}M$. If X is a vector field on M, we write $\|X\|$ for $\sqrt{\langle X, X\rangle}$, which is a continuous function on M.

Definition 3.1 (length of curve)**.** The length of γ is

$$L(\gamma) = \int_a^b \|\dot\gamma(t)\| \mathrm{d}t$$

Lemma 3.2. $L(\gamma)$ is independent of the choices of regular parametrizations.

The proof is a direct application of the change of variable formula.

Discussions above can be easily extended to piecewise smooth curves in M. Now we are ready define a distance on M. For any $p, q \in M$, let $\mathcal{C}_{pq} = \{\, \gamma\colon [a,b] \to M \mid \gamma \text{ is piecewise smooth and connects } p, q \,\}$. $\mathcal{C}_{pq}$ is nonempty since M is connected.

Definition 3.3 (Riemannian distance)**.** We define the Riemannian distance between p and q on (M, g) to be $d(p,q) = \inf \{\, L(\gamma) \mid \gamma \in C_{p,q} \,\}$.

Theorem 3.4. The Riemannian distance d makes M into a metric space, and the metric topology conincides with the manifold topology.

Proof. We first show that d is indeed a distance. Symmetric and triangle inequality are obvious by definition, so we only have to show the positivity. If $p \neq q$, let $x\colon U \to \mathbb{R}^m$ be a chart around p. By Hausdorff property there exists $\varepsilon > 0$ such that $q \notin x^{-1}(D_\varepsilon(x(p)))$. We can consider the Rayleigh quotient

$$\frac{v^T g_{ij}(p) v}{v^T v}, v \in \mathbb{R}^m \setminus \{\, 0 \,\}$$

Linear algebra tells us that its minimum is the minimum of eigen values of $(g_{ij}(p))$, say $\lambda(p) > 0$. Since $D_\varepsilon(x(p))$ is compact, $v^T g_{ij}(p) v \geqslant \lambda \|v\|^2$ for some $\lambda > 0$. Therefore, for any curve $\gamma \in C_{p,q}$

$$L(\gamma) \geqslant \sqrt{\lambda}\varepsilon > 0$$

By taking infimum we have $d(p,q) \geqslant \sqrt{\lambda}\varepsilon > 0$.

It remains to show that the two topology on M are the same. Since $x\colon U \to \mathbb{R}^m$ is a homemomorphism, it suffices to check that the induced distance $d'(x,y) = d(x^{-1}(p), x^{-1}(q))$ is equivalent with standard distance in $\mathbb{R}^m$. We have shown that $d'(x,y) \geqslant \sqrt{\lambda}\|x-y\|$. Notice that x, y can connected by line segament, which inudces a smooth curve γ' connceting p, q. Similiarly we have $v^T g_{ij}(p) v \leqslant \lambda \|v\|^2 \leqslant \mu \|v\|^2$ for some $\mu > 0$ in a neighborhood of the line segament. Therefore we have $d'(x,y) \leqslant L(\gamma') \leqslant \sqrt{\mu}\|x-y\|$, which implies the two distances are equivalent. ☐

4. geodesic and exponential map

For $p, q \in M$, we wish to find the curve $\gamma \in C_{pq}$ that minimize $L(\gamma)$. We observe that if a curve $\gamma_0\colon [a,b] \to M$ minimize $L(\gamma)$, then for $a_0 < a_1 < b_1 < b_0$ and $\gamma_0(a_1) = p_1, \gamma_0(b_1) = q_1$, $\gamma_0|_[a_1, b_1]$ also minimize $L(\gamma)$ in $C_{p_1 q_1}$. So we may localize the problem in one chart $U \subset \mathbb{R}^m$.

We can iamge parametrized curve as the motion of a particle, so physics can be a source of inpiration. A free single particle has its energy functional, which can be expressed as

$$E(\gamma) = \frac{1}{2}\int_a^b \|\dot\gamma(t)\|^2 \mathrm{d}t$$

By the Cauchy–Schwartz inequality, we know that for each smooth curve

$$L(\gamma)^2 \leqslant 2(b-a)E(\gamma)$$

with equality holds if and only $\|\dot\gamma(t)\|$ is constant. Notice that we can reparametrize γ into γ' so that $\|\dot{\gamma'}(t)\|$ is constant. Hence a curve γ minimize $E(\gamma)$ if and only if γ minimize $L(\gamma)$ and $|\dot\gamma|$ is constant.

Next let's find the critical points of $E(\gamma)$ in $\tilde{C}_{pq} = \{\, \gamma\colon [a,b] \to U \mid \gamma(a) = p, \gamma(b) = q \,\}$. We use the standard calculus of variations method. Suppose

$$\gamma_0\colon x^k = x^k(t), 1 \leqslant k \leqslant m$$

is a curve in $\tilde{C}_{pq}$ that is a critical point of $E(\gamma)$. Let $y_1, \cdots, y_m \in C^\infty[a,b]$ be smooth functions such that $y^k(a) = y^k(b) = 0$. Let

$$\gamma_\varepsilon\colon x^k(t) + \varepsilon y^k(t)$$

be a family of "nearby curves" in $\tilde{C}_{pq}$. Then we necessarily have

$$\begin{aligned} 0 = \frac{\mathrm{d}E(\gamma_\varepsilon)}{\mathrm{d}\varepsilon}(0) &= \frac{1}{2}\int_a^b \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} g_{ij}(\gamma_\varepsilon(t))(\dot{x}^i + \varepsilon\dot{y}^i)(\dot{x}^j + \varepsilon\dot{y}^j)\mathrm{d}t \\ &= \frac{1}{2}\int_a^b \left(g_{ij,k}\dot{x}^i\dot{x}^j y^k + g_{ij}\dot{y}^i\dot{x}^j + g_{ij}\dot{x}^i\dot{y}^j\right)\mathrm{d}t \end{aligned}$$

Note that

$$0 = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} g_{ij} y^i \dot{x}^j \mathrm{d}t = \int_a^b \left(g_{ij}\dot{y}^i\dot{x}^j + \frac{\mathrm{d}}{\mathrm{d}t}(g_{ij}\dot{x}^j)y^i\right)\mathrm{d}t$$

we have

$$\begin{aligned} 0 &= \frac{1}{2}\int_a^b \left(g_{ij,k}\dot{x}^i\dot{x}^j y^k - \frac{\mathrm{d}}{\mathrm{d}t}(g_{ij}\dot{x}^j)y^i - \frac{\mathrm{d}}{\mathrm{d}t}(g_{ij}\dot{x}^i)y^j\right)\mathrm{d}t \\ &= -\frac{1}{2}\int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}t}(g_{kj}\dot{x}^j) + \frac{\mathrm{d}}{\mathrm{d}t}(g_{ik}\dot{x}^i) - g_{ij,k}\dot{x}^i\dot{x}^j\right) y^k \mathrm{d}t \end{aligned}$$

By fundamental lemma of the calculus of variations, we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}(g_{kj}(x(t))\dot{x}^j + g_{ik}(x(t))\dot{x}^i) - g_{ij,k}(x(t))\dot{x}^i\dot{x}^j = 0, k = 1, \cdots, m$$

It is known as Euler–Lagrange equations in physics. We continue to calculate.

$$g_{kj}\ddot{x}^j + g_{ik}\ddot{x}^i + g_{kj,l}\dot{x}^j\dot{x}^l + g_{ik,l}\dot{x}^i\dot{x}^l - g_{ij,k}(x(t))\dot{x}^i\dot{x}^j = 0$$

Renaming some indices and using the symmetry $g_{ij} = g_{ji}$, we get

$$2g_{kn}\ddot{x}^n + (g_{kj,i} + g_{ik,j} - g_{ij,k})\dot{x}^i\dot{x}^j = 0, k = 1, \cdots, m$$

We introduce notations $(g^{ij}) = (g_{ij})^{-1}$ and $\Gamma^i_{jk} = \frac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l})$ (Christoffel symbols),then the equations are

$$g^{lk}g_{kn}\ddot{x}^n + \frac{1}{2}g^{lk}(g_{kj,i} + g_{ik,j} - g_{ij,k})\dot{x}^i\dot{x}^j = 0, l = 1, \cdots, m$$

Because of $g^{lk}g_{kn} = \delta_{ln}$ we obtain

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j\dot{x}^k = 0, i = 1, \cdots, m$$

Definition 4.1 (geodesic)**.** A smooth curve is called a geodesic if it satisfies the Euler–Lagrange equations, or equivalently by our calculation,

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, i = 1, \cdots, m$$

We have proved

Theorem 4.2. If a curve minimizes $E(\gamma)$ or $L(\gamma)$ in C_{pq}, then it is a geodesic.

Remark 4.3. Euler–Lagrange equations should be independent of the choice of the local chart, in the sense that theorem 4.2 is so. In fact, the equations can be expressed by Levi-Civita connection $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$, which will be defined in the following chapters.

Proposition 4.4. Each geodesic is parametrized proportionally to arc length.

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{x}, \dot{x}\rangle = \frac{\mathrm{d}}{\mathrm{d}t}(g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)) &= g_{ij}\ddot{x}^i\dot{x}^j + g_{ij}\dot{x}^i\ddot{x}^j + g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k \\ &= -(g_{jk,l} + g_{lj,k} - g_{lk,j})\dot{x}^l\dot{x}^k\dot{x}^j + g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k = 0\end{aligned}$$

□

The Euler–Lagrange equations are a system of second order ODEs, so Picard–Lindelöf theorem implies the following theorem.

Theorem 4.5. For any $p \in M$ and $X_p \in T_pM$, there exists $\varepsilon > 0$ and a unique geodesic

$$\gamma\colon (-\varepsilon, \varepsilon) \longrightarrow M$$

such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p$. Moreover, $\gamma(t) = \gamma(t; p, X_p)$ depends smoothly on p and X_p.

Remark 4.6. Since Euler–Lagrange equations are nonlinear, one cannot hope to extend the domain of a geodesic starting at an arbitrary point in an arbitrary direction.

Definition 4.7 (expotienal map)**.** For $p \in M$, let $V = \{ (p, X_p) \in TM \mid \gamma(t; p, X_p) \text{ is defined on } [0,1] \}$. The exponential map is defined to be

$$\begin{aligned}\exp\colon V &\longrightarrow M \\ (p, X_p) &\longmapsto \gamma(1; p, x_p)\end{aligned}$$

We write $\exp_p(X_p)$ for $\exp(p, X_p)$ and $V_p = \{ X_p \in T_pM \mid \gamma(t; p, X_p) \text{ is defined on } [0,1] \}$.

By the existence theorem above V is an open neighborhood of M in TM and exp is a smooth map on V. By definition the point $\exp_p(X_p)$ is the end point of the geodesic segment that starts at p in the direction of X_p whose length equals $|X_p|$.

Note that a linear reparametrization of a geodesic is again a geodesic. So for any geodesic $\gamma(t; p, X_p)$ and any $\lambda > 0$, $\tilde{\gamma}(t) = \gamma(\lambda t; p, X_p)$ is the geodesic with $\tilde{\gamma}(0) = p$ and $\dot{\tilde{\gamma}}(0) = \lambda X_p$.

Remark 4.8. If M is a Lie group, we also have a exponential map in Lie theory. In general they are not the same. However, the two maps indeed coincide if M is a compact Lie group and g is the bi-invariant metric.

Lemma 4.9. For any $p \in M$, if we identify $T_0(T_pM)$ with T_pM, then $(d\exp_p)_o = \mathrm{Id}_{T_pM}$.

Proof.

$$(d\exp_p)_0(X_p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX_p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(1; p, tX_p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp_p(t; p, X_p) = X_p$$

□

Remark 4.10. Some explanations on identifying $T_0(T_pM)$ with T_pM may be needed. Here we take tangent space as equivalence class of smooth curves and $X_p \in T_pM$ defines a curve tX_p naturally.

As a consequence of the inverse function theorem, we immediately get

Corollary 4.11. $\exp_p$ is a local diffeomorphism at 0.

Definition 4.12 (normal coordinates)**.** The local coordinates defined by the chart $\{ d\exp_p^{-1}, U, V \}$ is called the normal coordinates at p.

This is a particularly nice coordinate system. By the identification $T_0(T_pM) = T_pM$, $\{ \partial \}_i$ is the a orthonormal basis under $\langle \cdot, \cdot \rangle_p$. Also, the parametric equation for the geodesic $\gamma(t) = d\exp_p(tX_p)$ where $X_p = x^i\partial_i$ is given by

$$x(t) = (tx^1, \cdots, tx^m)$$

Moreover, we have

Theorem 4.13. In normal coordinates at $p \in M$, we have
(1) $g_{ij}(p) = \delta_{ij}$ for all $1 \leqslant i, j \leqslant m$.

(2) $\Gamma^i_{jk} = 0$ for all $1 \leqslant i, j, k \leqslant m$.
(3) $\partial_k g_{ij}(p) = 0$ for all $1 \leqslant i, j, k \leqslant m$.

Proof. (1) is obvious under the identification.

As for (2), put the geodesic parametric equation

$$x(t) = (tx^1, \cdots, tx^m)$$

into the Euler–Lagrange equation, we get

$$0 = \dot{x}^j \dot{x}^k \Gamma^i_{jk}(\gamma(t))$$

Letting $p = 0$ we conclude that

$$x^j x^k \Gamma^i_{jk}(p) = 0$$

for all x^j, x^k. It follows that for any fixed i, all eigenvalues of the symmetric matrix (Γ^i_{jk}) are 0, so $\Gamma^i_{jk}(p) = 0$ for all i.

(3) is a consequence of (2). □

Definition 4.14 (injectivity radius)**.** For each $p \in M$, the injectivity radius of (M, g) at p is

$$\mathrm{inj}_p(M, g) = \sup \{ \, r \mid \exp_p \text{ is a diffeomorphism on } B(0, r) \subset T_pM \, \}$$

and the injectivity radius of (M, g) is

$$\mathrm{inj}(M, g) = \inf \{ \, \mathrm{inj}_p(M, g) \mid p \in M \, \}$$

Definition 4.15 (geodisic ball and geodisic sphere)**.** For any $\rho < \mathrm{inj}_p(M, g)$, we call $B(p, \rho) = \exp_p(B(0, \rho))$ the geodesic ball of radius ρ centered at p in M, and its boundary $S_\rho(p) = \partial B_(p, \rho)$ the geodesic sphere of radius ρ centered at p in M.

Now let γ be any geodesic starting at p. Then $\exp_p^{-1}(\gamma \cap B(p, \rho))$ is the line segment in $B(0, \rho) \subset T_pM$ starting at 0 in the direction $\dot{\gamma}$ whose length is $|\gamma \cap B(p, \rho)|$. As a consequence, the geodesics starting at p p of lengths less than ρ.

Corollary 4.16. Suppose $p \in M$ and $\rho < \mathrm{inj}_p(M, g)$. For any $q \in B(p, \rho)$ there is a unique geodesic connecting p, q whose length is less than ρ.