Gauss–Bonnet–Chern theorem

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In memory of

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Let E be a vector bundle over a smooth manifold M. If E is endowed with a metric g, then structure group of E can be reduced to $\mathrm{O}(r)$. Moreover, if E is orientable, the structure group can be further reduced to $\mathrm{SO}(r)$.

1 curvature on vector bundles

E can be regarded as a sheaf over M, still denoted by E. Its local sections over open subset $U \subset M$ is the vector space of all smooth sections $s\colon U \to E$ over U.

Definition 1.1. A k-form taking values in E is a smooth section of $\Omega^k(M) \otimes E$. This vector bundle is denoted by $A^k(E)$.

Definition 1.2. A connection on E is a $\mathbb{R}$-linear sheaf morphism $\nabla\colon E \to A^1(E)$ which satisfies the Leibniz rule: for every local section s of E and local function f on M,

$$\nabla(fs) = df \otimes s + f\nabla s.$$

Proposition 1.3. If ∇, ∇' are two connections on E, then $\nabla - \nabla' \in A^1(\mathrm{End}(E))(M)$. Conversely, if $a \in A^1(\mathrm{End}(E))(M)$ and ∇ is a connection on E, then $\nabla + a$ is again a connection on E.

Proof. One checks that $(\nabla - \nabla')(fs) = f(\nabla - \nabla')s$ and $(\nabla + a)(fs) = df \otimes s + f(\nabla + a)s$. □

Recall that a connection ∇ on a vector bundle has a natural extension

$$\nabla\colon A^k(E) \longrightarrow A^{k+1}(E)$$

defined by $\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$. One check that this is well-defined, as

$$\nabla(\alpha \otimes (fs)) = fd\alpha \otimes +(-1)^k \alpha \wedge df \otimes s + (-1)^k f\alpha \wedge \nabla s = \nabla((f\alpha) \otimes s).$$

Moreover, a generalized Leibniz rule holds for this extension: for $t = \alpha \otimes s \in A^l(E)$ and $\beta \in A^k$ we have

$$\begin{aligned}\nabla(\beta \wedge t) = \nabla(\beta \wedge \alpha \otimes s) &= d\beta \wedge \alpha \otimes s + (-1)^k \beta \wedge d\alpha \otimes s + (-1)^{k+l} \beta \wedge \alpha \nabla s,\\ d\beta \wedge t + (-1)^k(\beta \wedge d\alpha \otimes s &+ (-1)^l \beta \wedge \alpha \nabla s) = d\beta \wedge t + (-1)^k \beta \wedge \nabla t.\end{aligned}$$

Definition 1.4. Suppose g is a metric on E. A connection ∇ on E is called a Riemannian connection if for local sections

$$d(g(s_1, s_2)) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2).$$

Definition 1.5. The curvature F_∇ of a connection ∇ on E is the composition

$$F_\nabla = \nabla^2\colon E \longrightarrow A^2(E).$$

Here $g(\alpha \otimes s, s') = g(s, s')\alpha$.

Lemma 1.6. $F_\nabla \in A^2(\mathrm{End}(E))(M)$.

Proof. For a local section s of E and a local function f on M one computes

$$F_\nabla(fs) = \nabla(df \otimes s + f\nabla s) = d(df) \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = fF_\nabla s.$$

□

Proposition 1.7. If ∇ is a Riemannian connection on (E, g), then F_∇ is skew-symmetric under local trivializations.

Proof. We may take a local orthonormal frame $e_1, \cdots, e_r$. Then

$$0 = dg(e_i, e_j) = g(\nabla e_i, e_j) + g(e_i, \nabla e_j).$$

We have $\nabla = d + a$ for some a skew-symmetric.

$$\begin{aligned} F_\nabla^T &= dA^T + (a \wedge a)^T = da^T - a^T \wedge a^T \\ &= -da - (-a) \wedge (-a) = -F_\nabla. \end{aligned}$$

□

Lemma 1.8 (Bianchi identity)**.** For a curvature F_∇ the induced connection on $\operatorname{End}(E)$,we have

$$0 = \nabla(F_\nabla) \in A^3(\operatorname{End}(E))(M)$$

Proof. $\nabla(F_\nabla)(s) = \nabla(F_\nabla(s)) - F_\nabla(\nabla s) = \nabla(\nabla^2 s) - \nabla^2(\nabla s) = 0.$ □

2 Chern–Weil theory

Relations of differential quantities and properties of underlying the topology of manifolds have been a central topic for differential geometry. It was shown by Chern and Weil in the late 1940's that one can in fact construct topological invariants out of curvature, whose remarkable work was summarized as Chern–Weil theory.

Lemma 2.1. Suppose V is an $\mathbb{R}$-vector space with basis $e_1, \cdots, e_n$. Let $S^k(V)$ be the space of symmetric k-multilinear form on V and $I^k(V)$ be the subspace of homogeneous polynomials of degree k in $\mathbb{R}[x_1, \cdots, x_n]$. Then we have $\mathbb{R}$-linear isomorphism

$$\begin{aligned} S^k(V) &\longrightarrow I^k(V) \\ P &\longmapsto (v \mapsto P(v, \cdots, v)) \end{aligned}$$

where $V \simeq \mathbb{R}^n$ via a fixed basis $e_1, \cdots, e_n$.

Proof. Obviously this map is linear and we will construct its inverse. Suppose $f^1, \cdots, f^n$ is a dual basis of $e_1, \cdots, e_n$. Then for any $f \in I^k(V)$,

$$f = a_{i_1 \cdots i_k} f^{i_1} \otimes \cdots \otimes f^{i_k}.$$

Then we may take

$$P(v_1, \cdots, v_n) = a_{i_1 \cdots i_k} f^{i_1}(v_1) \cdots f^{i_k}(v_k).$$

□

Definition 2.2. A polynomial $f \in \mathbb{R}[x]$ is said to be $\mathrm{SO}(r)$-invariant if for all $C \in \mathrm{SO}(r)$ and $B \in \mathrm{Sk}(r, \mathbb{R})$, $P(CBC^{-1}) = P(B)$.

Invariant polynomials would be the main ingredients to define Euler class. By Lemma 2.1, to every homogeneous $\mathrm{SO}(r)$-invariant polynomial f of degree k we can associate a k-multilinear symmetric form P.

Definition 2.3. A symmetric k-multilinear form

$$P\colon \mathrm{Sk}(r, \mathbb{R}) \times \cdots \times \mathrm{Sk}(r, \mathbb{R}) \longrightarrow \mathbb{R}$$

is said to be $\mathrm{SO}(r)$-invariant if for all $C \in \mathrm{SO}(r)$ and $B_i \in \mathrm{Sk}(r, \mathbb{R})$,

$$P(CB_1C^{-1}, \cdots, CB_kC^{-1}) = P(B_1, \cdots, B_k).$$

By the construction in Lemma 2.1, we see that homogeneous invariant polynomials of degree k and symmetric k-multilinear forms on $\mathrm{Sk}(r, \mathbb{R})$ are in bijection.

Lemma 2.4. If P is $\mathrm{SO}(r)$-invariant, then for all $B, B_1, \cdots, B_k \in \mathrm{Sk}(r, \mathbb{R})$,

$$\sum_{j=1}^{k} P(B_1, \cdots, B_{j-1}, [B, B_j], B_{j+1}, \cdots, B_k) = 0.$$

Proof. Take $C = \exp(B)$ and we can diffierentiate the invariance equation at $t = 0$. $\square$

Proposition 2.5. Let P be an invariant k-multilinear symmetric form on $\mathrm{Sk}(r, \mathbb{R})$. Then for any partition $m = i_1 + \cdots + i_k$ there exists a naturally induced k-multilinear map defined by

$$P(\alpha_1 \otimes t_1, \cdots, \alpha_k \otimes t_k) = \alpha_1 \wedge \cdots \wedge \alpha_k P(t_1, \cdots, t_k),$$

where $t_j \in \mathrm{Sk}(r, \mathbb{R})$ in a localization of $\mathrm{End}(E)$.

Proof. The transition map of E is $\psi\colon U \to \mathrm{SO}(r)$, then under different local trivializations

$$\begin{aligned} P(\alpha_1 \otimes t_1', \cdots, \alpha_k \otimes t_k') &= \alpha_1 \wedge \cdots \wedge \alpha_k P(t_1', \cdots, t_k') \\ = \alpha_1 \wedge \cdots \wedge \alpha_k P(\psi t_1 \psi^{-1}, \cdots, \psi t_k \psi^{-1}) &= P(\alpha_1 \otimes t_1, \cdots, \alpha_k \otimes t_k). \end{aligned}$$

This is well-defined. $\square$

Corollary 2.6. Let F_∇ be the curvature of a connection ∇ on a vector bundle E of rank r. Then for every $\mathrm{SO}(r)$-invariant polynomial f the induced $2k$-form $f(F_\nabla)$ is closed.

Proof. The induced connection on $\mathrm{End}(E)$ is of the form $\nabla = d + a$ with a acting as $\gamma \mapsto [a, \gamma]$. Using the usual Leibniz formula one finds

$$\begin{aligned} df(F_\nabla) &= \sum_{j=1}^{k} P(F_\nabla, \cdots, dF_\nabla, \cdots, F_\nabla) \\ &= \sum_{j=1}^{k} P(F_\nabla, \cdots, (\nabla - a)F_\nabla, \cdots, F_\nabla). \end{aligned}$$

It follows from the Bianchi identity and Lemma 2.4 that $df(F_\nabla) = 0$. $\square$

3 geometric Euler class

Definition 3.1. Let $X = (x_{ij}) \in \mathrm{Sk}(2k, \mathbb{R})$. The Pfaffian of X is

$$\mathrm{Pf}(X) = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} (-1)^\sigma \prod_{i=1}^{k} x_{\sigma(2i-1)\sigma(2i)}.$$

Lemma 3.2. For any $X \in \mathrm{Sk}(2k, \mathbb{R})$, we have $\mathrm{Pf}(CXC^T) = \det(C)\,\mathrm{Pf}(X)$ for any $C \in M(2k, \mathbb{R})$.

Proof. Direct calculations. □

Corollary 3.3. Pf is $\mathrm{SO}(2n)$-invariant.

Definition 3.4. Let E be an oriented vector bundle of rank $2n$ over an oriented manifold of dimension $2n$. The geometric Euler class of E is defined by

$$\frac{1}{(2\pi)^n}[\mathrm{Pf}(F_\nabla)] \in H^{2n}(M; \mathbb{R}),$$

where ∇ is a Riemannian connection for some metric g on E.

Proposition 3.5. Geometric Euler class is well-defined.

Proof. First, since ∇ is a Riemannian connection, F_∇ is skew-symmetric under local trivializations and $\mathrm{Pf}(F_\nabla)$ is well-defined. As E is orientable, we can reduce the structure group to $\mathrm{SO}(2n)$. Hence under different local trivializations,

$$\mathrm{Pf}(QF_\nabla Q^{-1}) = \mathrm{Pf}(QF_\nabla Q^T) = \det(Q)\,\mathrm{Pf}(F_\nabla) = \mathrm{Pf}(F_\nabla)$$

remains invariant.

Second, suppose ∇, ∇' are two Riemannian connections for a fixed metric g on E. We have $\nabla' = \nabla + a$ for some $a \in \mathrm{End}(E)(M)$. Then $\nabla + ta$ is also a Riemannian connection for every t and

$$F_{\nabla+ta} = F_\nabla + ktP(F_\nabla, \cdots, F_\nabla, \nabla(a)) + O(t^2).$$

Now by the Bianchi identity,

$$P(F_\nabla, \cdots, F_\nabla, \nabla(a)) = dP(F_\nabla, \cdots, F_\nabla, a) - \sum_{j=1}^{k} P(F_\nabla, \cdots, \nabla F_\nabla, \cdots, F_\nabla, a) = d(F_\nabla, \cdots, F_\nabla, a),$$

which implies

$$\left.\frac{d}{dt}\right|_{t=0} F_{\nabla+ta} = kd(F_\nabla, \cdots, F_\nabla, a),$$

and so

$$F_{\nabla'} - F_\nabla = \int_0^1 \frac{d}{dt} F_{\nabla+ta} ds \in \mathrm{Im}(d\colon A^{2k-1}(M, \mathbb{R}) \to A^{2k}(M, \mathbb{R})).$$

Finally, suppose g, g' are both metrics on M. We can consider a smooth family of metrics $g_t = (1-t)g + tg'$. If $\nabla' = \nabla + a$, we see that $\nabla_t = \nabla + ta$ is Riemannian metric for g_t. By the same argument we have $[\mathrm{Pf}(F_\nabla)] = [\mathrm{Pf}(F_{\nabla'})]$ □

4 Gauss–Bonnet–Chern theorem and its generalization

Theorem 4.1 (Chern)**.** If M is an oriented $2n$-dimensional compact Riemannian manifold, then the geometric Euler class $e(M)$ and the Euler characteristic $\chi(M)$ are related by

$$\chi(M) = \int_M e(M).$$

This elegant theorem has a natrual generalization, in the view of Theorem 4.4.

Theorem 4.2. Suppose E is an oriented vector bundle of rank $2n$ over a oriented $2n$-dimensional compact Riemannian manifold. Then the topological Euler class is represented by the geometric Euler class

$$e(E) = \frac{1}{(2\pi)^n}[\mathrm{Pf}(F_\nabla)]$$

Proof. This is shown in [2]. □

References

[1] Z.Q. Wang, Lecture Notes on Smooth Manifolds, 2023. http://staff.ustc.edu.cn/~wangzuoq/Courses/23F-Manifolds/index.html

[2] D. Bell, The Gauss–Bonnet Theorem for Vector Bundles, J-Geom. 85, 2006, 15-21.