LECTURE NOTES ON SYMPLECTIC TOPOLOGY

SHENGXI JIN

Contents

These notes are based on lectures on symplectic topology at USTC by Jun Zhang. Assume that M is a smooth manifold of dimension $2n$.

1. symplectic structures and examples

Symplctic topology and symplectic geometry basically means the same thing. The reason for this will be explained later.

Definition 1.1. A **symplectic structure** on M is a closed 2-form $\omega \in \Omega^2(M)$ which is non-degenerate, in the sense that $\omega^n = \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is a volume form. A symplectic manifold is such a pair (M, ω).

Remark 1.2. There are several equivalent conditions to describe non-degeneracy of ω. For example, $TM \to T^*M, X \mapsto \iota_X\omega$ is a bundle isomorphism.

Exercise 1.1. Let (M, ω) be a symplectic manifold. Prove that for any $p \in M$ and $0 \leqslant k \leqslant n$, the map $\bigwedge^{n-k} T_p^*M \to \bigwedge^{n+k} T_p^*M, \alpha \mapsto \alpha \wedge \omega_p^k$ is an isomorphism.

Example 1.3. There are lots of examples for symplectic manifolds.

(1) Area forms on a smooth surface automatically satisfy the condition for symplectic structure. In particular, closed surfaces $(\Sigma_g, \omega_{\text{area}})$ are symplectic manifolds.

(2) On $\mathbb{R}^{2n}$ with coordinate $(x^1, y^1, \cdots, x^n, y^n)$, there exists a standard symplectic structure

$$\omega_0 = \sum_{i=1}^n dx^i \wedge dy^i.$$

Alternatively, it can be expressed as $d\lambda$, where

$$\lambda = \frac{1}{2}\sum_{i=1}^n (x^i dy^i - y^i dx^i),$$

hence ω_0 is also an exact form.

(3) Let Q be a smooth manifold. Its cotangent bundle T^*Q admits a canonical symplectic structure $\omega = d\lambda$. Under local coordinate $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ the 1-form is expressed as
$$\lambda = -p_i dq^i,$$
then $\omega = dq^i \wedge dp_i$. To show that λ is well-defined, notice that
$$\lambda_{(q,p)}(Y) = p_q((d\pi)_{(q,p)}(Y)), Y \in T_{(q,p)}T^*Q.$$
For some reasons in notation, it is a good habit to keep writting the base manifold as Q.

(4) Given two symplectic manifolds (M, ω_M) and (N, ω_N), we can cook up a new symplectic manifold by taking products. Consider $\omega_{\text{prod}} = \pi_M^*\omega_M + \pi_N^*\omega_N \in \Omega^2(M \times N)$. Since pullback commutes with exterior derivative, ω_{prod} is also closed. By computation,
$$\omega_{\text{prod}}^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} \pi_M^*\omega_M^k \pi_N^*\omega_N^{m+n-k} = \binom{m+n}{m} \pi_M^*\omega^m \pi_N^*\omega_N^n.$$
As the nonzero scaling of a symplectic structure is still symplectic, $t\pi_M^*\omega_M + s\pi_N^*\omega_N, ts \neq 0$ are all symplectic structures on $M \times N$. In this way we get plenty of different symplectic structures on product manifolds. For example, $\mathbb{T}^{2n}, \mathbb{S}^2 \times \mathbb{S}^2$ are symplectic manifolds.

(5) The complex projective spaces $\mathbb{CP}^n$ admit a special class of symplectic structures ω_{FS}, called Fubini–Study forms. It can be described as follows. Let $\mathbb{CP}^n = \bigcup_{i=0}^{n} U_i$ be the standard open cover and
$$\begin{aligned} \varphi_i \colon U_i &\longrightarrow \mathbb{C}^n \\ [z_0, \cdots, z_n] &\longmapsto \left(\frac{z_0}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_n}{z_i} \right). \end{aligned}$$
Under φ_i we define
$$\omega_{\text{FS}}|_{U_i} = \frac{i}{2\pi} \partial\bar{\partial} \log \left(\sum_{l=0}^{n} \left| \frac{z_l}{z_i} \right|^2 \right)$$
A straightforward computation yields
$$\partial\bar{\partial} \log \left(\sum_{i=1}^{n} |w_i|^2 + 1 \right) = \frac{1}{(1 + \sum |w_i|^2)^2} \sum h_{ij} dw_i \wedge d\bar{w}_j$$
with $h_{ij} = (1 + \sum |w_j|^2)\delta_{ij} - \bar{w}_i w_j$.

ω_{FS} are though to be special because they are Kähler forms, i.e., they are induced by the complex structures. Also note that $\mathbb{CP}^1 = \mathbb{S}^2$ has volume 1 under ω_{FS} while its area is 2π under ω_{area}.

Exercise 1.2. Given a smooth manifold Q and a diffeomorphism $f \colon Q \to Q$, prove that f lifts to a symplectomorphism $\varphi_f \colon T^*Q \to T^*Q$ such that $(\varphi_f)^*\lambda_{\text{can}} = \lambda_{\text{can}}$, where λ_{can} is the canonical primitive of the canonical symplectic structure $d\lambda_{\text{can}}$ on the cotangent bundle T^*Q.

By definition, if $U \subset (M, \omega)$ is a non-empty open subset, then $(U, \omega|_U)$ is also a symplectic manifold. A fundamental theorem by Darboux shows that, symplectic manifolds of the same dimensional locally look the same, so there is no local geometry. This a significant difference from differential geometry, and that is why the lecture is named symplectic topology.

One may wonder is there any obstruction for the existence for symplectic structures on an even-dimensional manifold.

Proposition 1.4. If M is closed, i.e., compact and without boundary, then $[\omega]^k$ is nonzero class in $H^{2k}(M; \mathbb{R})$ for every $1 \leqslant k \leqslant n$.

Proof. By contradiction, say $[\omega]^k = 0$ for some k. Since $[\omega]^k = [\omega^k]$, $\omega^k = d\alpha$ for some $\alpha \in \Omega^{2k-1}(M)$. By Stokes' formula,
$$\int_M \omega^n = \int_M \omega^k \wedge \omega^{n-k} = \int_M d\alpha \wedge \omega^{n-k} = \int_M d(\alpha \wedge \omega^{n-k}) = 0.$$
This is a contradiction. □

Corollary 1.5. Among $\mathbb{S}^{2n}, \mathbb{S}^{2p} \times \mathbb{S}^{2q}$, symplectic structures exist if and only if $n = 1$ and $p, q = 1$.

2. SYMPLECTOMORPHISMS

Let (M, ω) be a symplectic manifold.

Definition 2.1. $\varphi \in \mathrm{Diff}(M)$ is called a **symplectomorphism** if it preserves the symplectic structure, i.e., $\varphi^*\omega = \omega$. Symplectomorphisms form a subgroup of $\mathrm{Diff}(M)$, denoted by $\mathrm{Symp}(M, \omega)$ with $\mathrm{Symp}_0(M, \omega)$ representing the connected component of id.

$\mathrm{Symp}(M, \omega)$ is a huge and mysterious topological group, much of whose nature remains unknown for mathematicians. Note that $\varphi^*(\omega^n) = (\varphi^*\omega)^n = \omega^n$, so at least $\mathrm{Symp}(M, \omega)$ would be a subgroup of $\mathrm{Diff}(M)_{\mathrm{Vol}}$, the diffeomorphisms preserving volumes. In general $\mathrm{Symp}(M, \omega) \subsetneq \mathrm{Diff}(M)_{\mathrm{Vol}}$, which seems very reasonable but is not easy to proof. This result is due to Gromov with his non-squeezing theorem. Roughly speaking, although symplectic geometry has no local behaviour, it possesses certain rigidity in the global sense.

Once we focus on linear isomorphism on $(\mathbb{R}^{2n}, \omega_0)$, things get much easier.

Definition 2.2. The **symplectic group** is $\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^*\omega_0 = \omega_0\}$.

Symplectic groups are Lie groups with nice property.

Theorem 2.3. $\mathrm{Sp}(2n)$ deform retracts to $\mathrm{U}(n)$. In particular, it has fundamental group $\mathbb{Z}$.

This result is essential in defining another important invariant for symplectic manifolds.

Proposition 2.4. The first Chern class for (M, ω), $\mathrm{c}_1(TM) \in H^2(M; \mathbb{Z})$ is well-defined.

Proof. Since $\mathrm{Sp}(2n)$ deform retracts to $\mathrm{U}(n)$, TM is equivalent to a complex vector bundle over M. $\square$

There are two natural questions about symplectic topology:

(1) **How to construct a symplectomorphism?**
(2) **How to tell if a map is a symplectomorphism?**

To answer these questions, we need to introduce to elements in symplectic geometry respectively:

(1) Hamiltonian dynamics.
(2) Lagrangian submanifolds.

3. COMPLEXITY OF SYMPLECTIC STRUCTURES

Let (M, ω) be a symplectic manifold.

In algebraic topology, characteristic classes measure the complexity of vector bundles over a space. To describe the complexity of a symplectic strucuture, we can also make use of some special tools and invariants.

Proposition 3.1. Every symplectic manifold (M, ω) admits an almost complex structure J such that

(1) $\omega_p(v, Jv) > 0, v \in T_pM$.
(2) $J^*\omega = \omega$, i.e., $\omega_p(Ju, Jv) = \omega_p(u, v), u, v \in T_pM$.

As a consequence, $g_J(u, v) = \omega(u, Jv)$ defines a Riemannian metric on M. Also note that

$$g_J(Ju, Jv) = \omega(Ju, -v) = \omega(v, Ju) = g_J(v, u).$$

Although it may not make sense at the moment, this show that symplectic manifolds can be quantified. In the remainder of this section, we will fix a symplectic manifold together with induced almost complex structure (M, ω, J).

Definition 3.2. A ***J*-holomorphic curve** in (M, ω, J) is a smooth map $u\colon (\Sigma, j) \to (M, \omega, J)$ such that $J \circ u_* = u_* \circ j$, where Σ is a Riemann surface and j is its complex structure.

J-holomorphic curve is an important tool in symplectic geometry, developed by Gromov in 1980's. In this lecture we will be only interested in cases where $\Sigma = \mathbb{S}^2$ or $\mathbb{S}^2 \setminus \{\text{finitely many pts}\}$.

Example 3.3. A map $u\colon (\mathbb{C}, \sqrt{-1}) \to (\mathbb{C}, \sqrt{-1})$ is a J-holomorphic curve if and only if u satisfies the Cauchy–Riemann equation, i.e., u is holomorphic.

Proposition 3.4. For any smooth curve $u\colon (\Sigma, j) \to (M, \omega, J)$, let

$$E(u) = \int_\Sigma u^*\omega.$$

We have $E(u) \leqslant \mathrm{Area}(u)$, with equality holds if u is a J-holomorphic curve.

Proof. Fix a local orthonormal frame $\{e_1, e_2\}$ on $T\Sigma$ such that $g_J(u_*e_1, u_*e_2) = 0$. Then
$$(u^*\omega)(e_1, e_2) = \omega(u_*e_1, u_*e_2) = g_J(Ju_*e_1, u_*e_2) = (u^*g_J)(je_1, e_2)$$
By Cauchy–Schwarz inequaltiy, we have
$$g_J(Ju_*e_1, u_*e_2) \leqslant \sqrt{g_J(Ju_*e_1, Ju_*e_1)g_J(u_*e_2, u_*e_2)} = \sqrt{(u^*g_J)(e_1, e_1)(u^*g_J)(e_2, e_2)}$$
with equality holds if and only if $Ju_*e_1 = u_*e_2$.

If u is J-holomorphic curve, we can pick $e_2 = je_1$, as g $g_J(u_*e_1, u_*je_1) = -\omega(u_*e_1, u_*e_1) = 0$. □

Corollary 3.5. A J-holomorphic curve is a minimal surface with respect to g_J.

Note that if $u\colon (\mathbb{S}^2, j) \to (M, \omega, J)$ is a J-holomorphic curve, $E(u)$ can be written as $[\omega] \cap [u]$, where $[u] \in \pi_2(M)$. In symplectic geometry, we consider the image of Hurewicz map
$$H_2^S(M) = \mathrm{Im}(h\colon \pi_2(M) \to H_2(M; \mathbb{Z})),$$
which is the collection of homology classes represented by the image of $\mathbb{S}^2$.

Example 3.6. $H_2^S(\mathbb{T}^2) = 0$ because $\mathbb{T}^2$ is not homeomorphic to $\mathbb{S}^2$.
$H_2^S(\mathbb{S}^2) = \mathbb{Z}, H_2^S(\mathbb{S}^2 \times \mathbb{S}^2) = \mathbb{Z} \times \mathbb{Z}$.
$H_2^S(\mathbb{CP}^n) = \mathbb{Z}$.

Two functions are defined on $H_2^S(M)$:
$$\begin{aligned}[\omega]\colon H_2^S(M) &\longrightarrow \mathbb{R}\\ A &\longmapsto [\omega] \cap A = \int_{\mathbb{S}^2} A^*\omega,\\ \mathrm{c}_1(TM)\colon H_2^S(M) &\longrightarrow \mathbb{Z}\\ A &\longmapsto \mathrm{c}_1 \cap A = \int_{\mathbb{S}^2} A^*\,\mathrm{c}_1(TM).\end{aligned}$$
The complexity of (M, ω) comes from the relation of $[\omega]$ and $\mathrm{c}_1(TM)$.

Definition 3.7. In general, there are three jumps in the complexity of a symplectic manifold.

- $[\omega] = \mathrm{c}_1(TM) \equiv 0$ on $H_2^S(M)$. Such symplectic manifolds are said to be **symplectically aspherical**. In some sense, symplectically aspherical manifolds are easy to study.
  Tori $\mathbb{T}^{2n}$ and closed surfaces $\Sigma_g (g > 0)$ are examples for symplectically aspherical manifolds.
- $[\omega](A) = \kappa\,\mathrm{c}_1(TM)(A), A \in H_2^S(TM)$ for some $\kappa > 0$. Such symplectic manifolds are said to be **monotone**. By definition, a symplectically aspherical manifold is automatically monotone.
  $(\mathbb{CP}^1, \omega_{\mathrm{FS}})$ is our first example that is monotone but not symplectically aspherical. Let A be the generator of $H_2^S(\mathbb{CP}^1) = H_2(\mathbb{CP}^1; \mathbb{Z})$. Then $[\omega_{\mathrm{FS}}](A) = 1$ and $\mathrm{c}_1(A) = 2$, so $\kappa = \frac{1}{2}$. Generally, for every n, $(\mathbb{CP}^n, \omega_{\mathrm{FS}})$ is monotone with $\kappa = \frac{1}{n+1}$.
- The last jump is a generalization of notion of monotone. We say (M, ω) is **semi-positive** (or **weakly monotone**) if it satisfies either one of the three conditions:
  (1) $[\omega](A) = \kappa\,\mathrm{c}_1(TM)(A), A \in H_2^S(M)$ for some $\kappa \geqslant 0$.
  (2) $\mathrm{c}_1(TM) \equiv 0$ on $H_2^S(M)$. (This condition is sometimes called Calabi–Yau).
  (3) The positive generator of $\mathrm{c}_1(TM)(H_2^S(M)) \leqslant \mathbb{Z}$ is at least $n - 2$.
  The three conditions are not equivalent to each other but they can overlap on some manifolds. By definition, when $n \leqslant 3$, i.e., $\dim M \leqslant 6$, (M, ω) is automatically semi-positive.

Note that it is possible that a symplectic manifold is not semi-positive, but such examples are quite complicated and are hard to display at this moment.

Remark 3.8. Semi-positive manifolds are collections of objects people could deal with in Floer theory in 1990's, which was originally developed for monotone manifolds. Roughly speaking, almost every property holding for monotone case holds for semi-positive case. On the contrary, sometimes it is difficult to generalize results in aspherical case to monotone case.

Here comes an observation. If (M, ω) is semi-positive, then for any $A \in H_2^S(M)$ with $[\omega](A) > 0$, we must have $\mathrm{c}_1(TM)(A) \geqslant 0$ or $\mathrm{c}_1(TM)(A) \leqslant 2 - n$. This turns out to be an if and only if!

Exercise 3.1. If every $A \in H_2^S(M)$ with $[\omega](A) > 0$ satisfies $\mathrm{c}_1(TM)(A) \geqslant 0$ or $\mathrm{c}_1(TM)(A) \leqslant 2 - n$, then (M, ω) is semi-positive.

Proof. If $n \leqslant 3$, there is nothing to prove, so assume $n > 3$. It suffices to show that if condition (2) and (3) are violated, then (1) holds.

In this case, $c_1(TM)(H_2^S(M)) = N\mathbb{Z}$ with $1 \leqslant N \leqslant n-3$. Take $A \in H_2^S(M)$ such that $c_1(TM)(A) = N$, so $[\omega](A) \geqslant 0$. If $B \in H_2^S(M)$ making $c_1(TM)(B) = N$, we must have $[\omega](A-B) = 0$. Indeed, otherwise say $[\omega](B-A) > 0$, then $c_1(B-A) = 0$. Taking $m \in \mathbb{Z}_{\geqslant 1}$ sufficiently large, we obtain $[\omega](m(B-A)-A) > 0$ but $(m(B-A)-A) = c_1(-A) \in (2-n, 0)$, which is a contradiciton.

Let

$$\kappa = \frac{[\omega](A)}{c_1(TM)(A)}.$$

For any $C \in H_2^S(M)$, $c_1(TM)(A) = (1+k)N$ and

$$[\omega](C) = [\omega](C-kA)+][\omega](kA) = (k+1)[\omega](A) = (k+1)\kappa\, c_1(TM)(A) = \kappa\, c_1(TM)(C).$$

$\square$

Example 3.9. $(\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\text{prod}})$ is monotone. If we change the symplectic structure, we obtain an example for a semi-positive manifold that is not monotone. Note that $\dim \mathbb{S}^2 \times \mathbb{S}^2 = 4 \leqslant 6$, this manifold is semi-positive with any symplectic structure. One can check that $\omega_{\text{deform}} = (1+\varepsilon)\pi_1^*\omega + \pi_2 * \omega$ is non-monotone structure for $\varepsilon > 0$.

4. Quantum cohomology and Novikov rings

Cohomology is an important topological invariant for manifolds. However, it does not carry the information of symplectic structure. In symplectic geometry, we can modify cohomology in the following way.

Definition 4.1. Let $\Gamma_\omega = \operatorname{Im}([\omega] \colon H_2^S(M) \to \mathbb{R}) \leqslant \mathbb{R}$. The **quantum cohomology** $QH^*(M,\omega)$ of (M,ω) with coefficient $\mathbb{K}$ is

$$QH^*(M,\omega) = H^*(M;\mathbb{K}) \otimes_{\mathbb{K}} \Lambda^{\mathbb{K},\Gamma_\omega},$$

where $\Lambda^{\mathbb{K},\Gamma_\omega}$ is the Novikov ring which we will introduce later.

Definition 4.2. Let $\Gamma \leqslant \mathbb{R}$ be a subgroup. Its **Novikov ring** is the formal power series in T

$$\Lambda^{\mathbb{K},\Gamma} = \left\{ \sum_{\lambda \in \Gamma} a_\lambda T^\lambda \mid a_\lambda \in \mathbb{K}, \forall c \in \mathbb{R}, \# \{a_\lambda \neq 0 \mid \lambda < c\} < +\infty \right\}.$$

$\forall c \in \mathbb{R}, \# \{a_\lambda \neq 0 \mid \lambda < c\} < +\infty$ is called the Novikov condition.

Remark 4.3. A faster way to express the Novikov condition is $\lambda \to +\infty$, though this notion is not very appropriate for finite sums.

Example 4.4. $\mathbb{K}[[T]] \leqslant \Lambda^{\mathbb{K},\Gamma}$. On the contrary,

$$\sum_{n=1}^{\infty} T^{\frac{1}{n}}, \sum_{n=1}^{\infty} T^{-n} \notin \Lambda^{\mathbb{K},\Gamma}.$$

As a priori, $QH^*(M,\omega)$ should be considered as a module over the Novikov ring. Delightfully, it turns out that the Novikov ring is actually a field.

Lemma 4.5. $\Lambda^{\mathbb{K},\Gamma}$ is a field.

Proof. Let x be a nonzero element. By the Novikov condition, we can write

$$x = a_\mu T^\mu + \sum_{\lambda > \mu} a_\lambda T^\lambda = a_\mu T^\mu \left(1 + \sum_{\lambda > \mu} \frac{a_\lambda}{a_\mu} T^{\lambda - \mu} \right), a_\mu \neq 0.$$

Denote

$$A = -\sum_{\lambda > \mu} \frac{a_\lambda}{a_\mu} T^{\lambda - \mu}.$$

As $\lambda = \mu$ only finitely many times, $(1-A)^{-1} = 1 + A + \cdots + A^n + \cdots$ is well-defined and

$$x^{-1} = a_\mu^{-1} T^{-\mu} (1-A)^{-1}.$$

$\square$

As a consequence, quantum cohomology can be considered as a $\Lambda^{\mathbb{K},\Gamma_\omega}$-vector space, and methods in linear algebra apply.

Example 4.6. We can examine what quantum cohomology likes under the three kinds of complexity mentioned last time.

- If (M,ω) is symplectically aspherical, then $\Gamma_\omega = 0$ and $\Lambda^{\mathbb{K},\Gamma_\omega}$ is simply $\mathbb{K}$. We see that the quantum cohomology on symplectically aspherical manifolds coincides with the usual cohomology.
- If (M,ω) is monotone, $\Gamma_\omega = \kappa\, c_1(TM)(H_2^S(M)) \simeq \mathbb{Z}$ is a cyclic subgroup of $\mathbb{R}$. Then $\Lambda^{\mathbb{K},\Gamma_\omega} \simeq \mathbb{Z}[T]]$ can be regarded as the field of Laurant series.
- If (M,ω) is semi-positive, condition (2) and (3) have no control on Γ_ω, so it could be that Γ_ω becomes very complicated.

Here is an explicit example showing how complicated Γ_ω could be.

Example 4.7. Consider $\mathbb{S}^2 \times \mathbb{S}^2$ with $\omega_{\text{deform}} = (1+\varepsilon)\pi_1^*\omega + \pi_2 * \omega$. Then

$$\Gamma_\omega = \mathbb{Z} \oplus \mathbb{Z}\varepsilon.$$

If ε is irrational, then Γ_ω will be a dense subgroup of $\mathbb{R}$.

Here is another enhanced version of Novikov ring. We can consider the series

$$\sum_{A \in H_2^S(M)} a_A t^A.$$

The question about this definition is, how to describe the Novikov condition? To resolve this, consider the short exact sequence

$$0 \longrightarrow \operatorname{Ker}[\omega] \longrightarrow H_2^S \xrightarrow{[\omega]} \Gamma_\omega \longrightarrow 0.$$

Γ_ω is free, so the exact sequence splits, which implies $H_2^S(M) = \operatorname{Ker}[\omega] \oplus \Gamma_\omega, A \mapsto (B,\lambda)$. The formal series can be rewritten as

$$\sum_{\lambda \in \Gamma_\omega} \sum_{B \in \operatorname{Ker}[\omega]} a_{(B,\lambda)} t^{(B,\lambda)} = \sum_{\lambda \in \Gamma_\omega} \left(\sum_{B \in \operatorname{Ker}[\omega]} a_{(B,\lambda)} S^B \right) T^\lambda.$$

The Novikov condition for this setting should be

$$\forall c \in \mathbb{R}, \# \left\{ a_{(B,\lambda)} \neq 0 \mid \lambda < c \right\} < +\infty,$$

which implies that the coefficient

$$\sum_{B \in \operatorname{Ker}[\omega]} a_{(B,\lambda)} S^B$$

lies in the group algebra $\mathbb{K}[\operatorname{Ker}[\omega]]$.

Before talking more about quantum cohomology, we first study some general aspects of modules over Novikov rings. Novikov admits an interesting structure, called the valuation.

Definition 4.8. We define a valuation ν on $\Lambda^{\mathbb{K},\Gamma}$ by $\nu(x) = \min \left\{ \lambda \in \Gamma \mid T^\lambda \text{ has nozero coefficient} \right\}$ and presribing $+\infty$ to 0.

Proposition 4.9. ν is indeed a valuation on $\Lambda^{\mathbb{K},\Gamma}$:
(1) $\nu(x) = +\infty$ if and only if $x = 0$.
(2) $\nu(xy) = \nu(x) + \nu(y)$.
(3) $\nu(tx) = \nu(x)$ if $0 \neq t \in \mathbb{K}$.
(4) $\nu(x+y) = \min(\nu(x) + \nu(y))$.

These properties follows immediately from definition.

With valuations, we can associate a new structure on V.

Definition 4.10. A filtration function on a $\Lambda^{\mathbb{K},\Gamma}$-vector space V is a funciton $l \colon V \to \mathbb{R} \cup \{-\infty\}$ such that:
(1) $l(x) = -\infty$ if and only if $x = 0$.
(2) $l(\alpha x) = \lambda(x) - \nu(\alpha), \alpha \in \Lambda^{\mathbb{K},\Gamma}$.
(3) $l(x+y) \leqslant \max((x), l(y))$.

Note that by definition if $l(x) \neq l(y)$, say $l(x) > l(y)$, then

$$l(y) < l(x) = l(x + y - y) \leqslant \max(l(x + y), l(y)),$$

which forces $l(x + y) = l(x) = \max(l(x), l(y))$.

A natural question is, what does linear algebra look like on (V, l)?

5. Vector spaces with filtration functions

6. Preparations for Gromov–Witten invariants

Gromov–Witten invariant is related to the central problem in symplectic geometry and algebraic geometry, counting curves. There are three major troubles we will encounter.

6.1. The limit is bad.

Consider a family of J-holomorphic curves $\{u_n\}$. It's limit u_∞ may be complicated.

Example 6.1. Let $u_n\colon (\mathbb{S}^2, j) \to (\mathbb{CP}^2, J_0), [z_0 : z_1] \mapsto \left[z_0^2 : \frac{z_1^2}{n} : z_0 z_1\right]$ be a sequence of J-holomorphic curves, where J_0 is the standard complex structure. $\operatorname{Im} u_n = \left\{[Z_0 : Z_1 : Z_2] \mid Z_0 Z_1 = \frac{Z_2^2}{n}\right\}$, so its limit image is

$$\operatorname{Im}(u_\infty) = \{[Z_0 : Z_1 : Z_2] \mid Z_0 Z_1 = 0\} = A_0 \cup A_1,$$

where $A_0 = V(Z_0), A_1 = V(Z_1)$. Topologically, $\operatorname{Im}(u_\infty)$ is two copies of $\mathbb{S}^2$ attached at $[0 : 0 : 1]$.

A question is, can $\operatorname{Im}(u_\infty)$ be an image of a J-holomorphic curve? Intuitively, on $\mathbb{S}^2 \setminus \{[0 : 1]\}$ we can set $u_\infty([z_0 : z_1]) = [z_0^2 : 0 : z_0 z_1] = [z_0 : 0 : z_1]$. It seems reasonable to define $u_\infty([z_0 : z_1]) = [z_0 : 0 : z_1]$. However, this map only covers the component A_1 but misses A_0.

We can look at $[0 : 1]$ and see what happened to this point during the limit process. Surprisingly, $u_n([0 : 1]) = [0 : \frac{1}{n} : 0] = [0 : 1 : 0]$ is a fixed point and it is not the same point as $u_\infty([0 : 1]) = [0 : 0 : 1]$.

Exercise 6.1. Prove that there exsists a J-holomorphic curve $v_\infty\colon \mathbb{S}^2 \to \mathbb{CP}^2$ that covers component A_0 but misses A_1, and maps $[1 : 0]$ to $[0 : 0 : 1]$.

Proof. Consider a reparametrization of u_n defined by $v_n([z_0 : z_1]) = u_n([z_0 : n z_1]) = \left[z_0^2 : n z_1^2 : n z_0 z_1\right] = \left[\frac{z_0^2}{n} : z_1^2 : z_0 z_1\right]$. Then $v_\infty([z_0 : z_1]) = [0 : z_1 : z_0]$ is the desired map. □

To resolve this puzzle, let's bravely consider $(v, u)\colon T \to \mathbb{CP}^2$, where T is two copies of $\mathbb{S}^2$ attached at $[0 : 0 : 1]$. In general, the limit of a family of J-holomorphic curves has the complicated domains represented as a tree, whose vertex are copies of $\mathbb{S}^2$ and edges are attached points. In Example 6.1 the domain is represented as two vertex attached by an edge.

6.2. Multiply cover.

Definition 6.2. Let $u\colon (\Sigma_g, j) \to (M, J)$ be a J-holomorphic curve. u is said to be **multiply covered** if there exists a J-holomorphic curve $u'\colon (\Sigma_{g'}, j') \to (M, J)$ and a non-constant holomorphic map $\varphi\colon \Sigma_g \to \Sigma_{g'}$ s.t.

$$\begin{tikzcd}
(\Sigma_g, j) \arrow[d, "\varphi"] \arrow[dr, "u"] & \\
(\Sigma_{g'}, j') \arrow[r, "u'"] & (M, J)
\end{tikzcd}$$

and $\deg \varphi > 1$. And we will call that u is a **multiple cover** of u'.

In general, we don't like multiple covers. By some topological constraint we can get rid of multiple covers.

Theorem 6.3 (Riemann–Hurwitz formula)**.** Let $\varphi\colon \Sigma_g \to \Sigma_{g'}$ be a non-constant holomorphic map. Then

$$2g - 2 = (\deg \varphi)(2g' - 2) + \sum_{p \in \Sigma_g} (e_p - 1),$$

where e_p is the ramification index at p.

In particular, if $\Sigma_g = \mathbb{S}^2$, the formula writes

$$-2 = (\deg \varphi)(2g' - 2) + \sum_{p \in \Sigma_g} (e_p - 1).$$

Therefore $g' = 0$ and $\Sigma_{g'} = \mathbb{S}^2$. Note that in this case we cannot claim $\deg \varphi = 1$.

Example 6.4. Here is an example of symplectic structure ruling out multiple cover. Suppose (M, ω) is semi-positive with $c_1(TM)(H_2^S(M)) = 4\mathbb{Z}$. If $u\colon (\mathbb{S}^2, j) \to (M, J)$ is a J-holomorphic curve such that $c_1(TM)([u]) = 4$, then u is not a multiple cover.

To see this, by contradiction suppose u is a multiple cover of $u'\colon\ \mathbb{S}^2 \to (M, J)$ with φ. By definition, $[u] = (\deg \varphi)[u']$, so $4 = c_1(TM)([u]) = (\deg \varphi)\, c_1(TM)([u'])$. As $\deg \varphi > 1$, we have either $\deg \varphi = 2$ or 4, but either case would result in $0 < c_1(TM)([u']) < c_1(TM)([u])$.

Definition 6.5. A J-holomorphic curve is said to be **simple** if it's not multiply covered.

If a J-holomorphic curve u is not simple, then can always find a simple u' such that u is a multiple cover of u.

The following is a criterion for determing whether a J-holomorphic curve is simple.

Theorem 6.6. A J-holomorphic curve $u\colon (\Sigma_g, j) \to (M, J)$ is simple if and only if there exists $z \in \Sigma_g$ such that $(du)_z\colon T_z\Sigma_g \to T_{u(z)}M$ is injective and $u^{-1}(u(z)) = \{z\}$.

Altough many textbooks claim this criterion to be useful, Jun said he never uses it.

6.3. **Reparametrization.** If $u\colon (\mathbb{S}^2, j) \to (M, J)$ is a J-holomorphic curve and $\varphi \in \mathrm{Aut}(\mathbb{S}^2, j)$, then geometrically $u' = u \circ \varphi^{-1}$ should be considered to be the same curve.

Recall that

$$\mathrm{Aut}(\mathbb{S}^2, j) = \left\{ z \mapsto \frac{az + b}{cz + d} \mid ad - bc \neq 0 \right\} = \mathrm{PGL}(2, \mathbb{C})$$

is a connected Lie group with real dimension 6. Let $\mathcal{C}(M, J)$ be the collection of J-holomorphic curves $u\colon (\mathbb{S}^2, j) \to (M, J)$. $\mathrm{PGL}(2, \mathbb{C})$ acts on $\mathcal{C}(M, J)$ via $\varphi \cdot u = u \circ \varphi^{-1}$. Counting J-holomorphic curves should be essentially on the moduli space

$$\mathcal{C}(M, J)/\mathrm{PGL}(2, \mathbb{C}).$$

An enhanced version of this is considering

$$(u, (z_1, \cdots, z_k)) \in \mathcal{C}(M, J) \times \left(\prod^k \mathbb{S}^2 \setminus \Delta \right),$$

where Δ is the fat diagonal, i.e., $(z_1, \cdots, z_k)$ are k distinct points. Similarly $\mathrm{PGL}(2, \mathbb{C})$ acts on this set via $\varphi \cdot (u, (z_1, \cdots, z_k)) = (u \circ \varphi^{-1}, (\varphi(z_1), \cdots, \varphi(z_k)))$ and we can consider the moduli space

$$\mathcal{C}(M, J) \times \left(\prod^k \mathbb{S}^2 \setminus \Delta \right) / \mathrm{PGL}(2, \mathbb{C}).$$

Remark 6.7. $(z_1, \cdots, z_k)$ is to formulate constraints on the J-holomorphic curves.

Definition 6.8. Let (M, J) be an almost complex manifold. A stable map is given by the following data.

(1) Its domain T, which is a tree-type configuration containing finitely many $(\mathbb{S}^2, j)$ attached to each other in the same way. The vertex $v(T)$ are copies of $(\mathbb{S}^2, j)$ and the edges $e(T)$ are nodal points. The attached points are usually called nodal points and the configuration T is called the nodal curve.
(2) The restriction of u on each $(\mathbb{S}^2, j)$ is J-holomorphic.
(3) If the restriction of u on $\mathbb{S}^2$-component in $v(T)$ is constant, then there exists at least three nodal points on it.

We say u is multiply covered if at least one $\mathbb{S}^2$-component is multiply covered.

Remark 6.9. Condition (3) is extracted from analysis on J-holomorphic curves. More details can be found at Theorem 5.5 at *J-holomorphic Curves in Symplectic Topology* by Jerome Dominique Wettstein.

7. Moduli space of J-holomorphic curves

Let (M,ω) be a symplectic manifold and $A \in H_2^S(M)$. Take a $J \in \mathcal{J}(M,\omega)$.

Definition 7.1. $\mathcal{M}^*(A,J)$, or $\mathcal{M}^*(M,A,J)$ is defined as

$$\left\{\text{simple } J\text{-holomorphic curves } u\colon (\mathbb{S}^2,j) \to (M,J) \mid [u]=A\right\}/\mathrm{PSL}(2,\mathbb{C}).$$

Similarly we can define

$$\mathcal{M}_k^*(A,J) = \left(\mathcal{M}^*(A,J)\times(\mathbb{S}^2)^k \setminus \Delta\right)/\mathrm{PSL}(2,\mathbb{C}),$$

whose elements are $[(u,z_1,\cdots,z_k)]$ with simple u such that $[u]=A$ and k distinct points on $\mathbb{S}^2$. $\overline{\mathcal{M}_k(A,J)}$ is defined as

$$\left\{\text{stable maps } u\colon (T,j)\to(M,J) \mid [u] = \sum_{\mathbb{S}^2\in v(T)}[u|_{\mathbb{S}^2}] = A\right\}.$$

In definitions above, we can vary A,J,k to obtain different moduli spaces. For (A,J,k) fixed, their relations are as follows:

$$\begin{array}{ccc} \mathcal{M}_k^*(A,J) & \subset & \mathcal{M}_k(A,J) \\ \cap & & \cap \\ \overline{\mathcal{M}_k^*(A,J)} & \subset & \overline{\mathcal{M}_k(A,J)} \end{array}$$

For a generic J, $\mathcal{M}_k^*(A,J)$ would have a manifold structure with $\overline{\mathcal{M}_k(A,J)}$ being its compactification, due to the famous Gromov compactness theorem.

Theorem 7.2 (Gromov)**.** Let $A \in H_2^S(M)$. Then there exists a topology on $\mathcal{M}_k^*(A,J)$, called the Gromov sequential topology, such that its compactification is $\overline{\mathcal{M}_k(A,J)}$ and $\mathcal{M}_k^*(A,J)$ is an open dense subset of $\overline{\mathcal{M}_k(A,J)}$. Moreover, for a generic J, $\mathcal{M}_k^*(A,J)$ is a manifold of real dimension $\dim M + 2\,\mathrm{c}_1(TM)(A) + 2k - 6$.

Remark 7.3. Similar result holds for $\mathcal{M}_{k,T}^*(A,J)$ and

$$\dim \mathcal{M}_{k,T}^*(A,J) = \dim M + 2\,\mathrm{c}_1(TM)(A) + 2k - 6 - 2\#e(T).$$

Moreover, we have the decomposition

$$\overline{\mathcal{M}_k(A,J)} = \bigcup_T \mathcal{M}_{k,T}^*(A,J),$$

where T runs over all the possible tree domains in the limit process.

Example 7.4. Let's consider the toy version when $(M,\omega,J) = (\mathbb{CP}^2,\omega_{\mathrm{FS}},J_0)$ where J_0 is the complex structure.

Let $A = 2L$, where $L \in H_2(\mathbb{CP}^2;\mathbb{Z})$ is the line class. We want to study the moduli space $\overline{\mathcal{M}_0(2L,J_0)}$.

According to Theorem 7.2, its real dimension is

$$\dim_{\mathbb{R}} \mathbb{CP}^2 + 2\,\mathrm{c}_1(T\mathbb{CP}^2)(2L) + 0 - 6 = 10.$$

Actually, a curve representing class $[2L]$ would be a conic in $\mathbb{CP}^2$ so $\overline{\mathcal{M}_0(2L,J_0)}$ can be considered as collections of conics. As conics are solutions to equations

$$aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ = 0,$$

intuitively, the space has complex dimension 5, which is consistent with Theorem 7.2.

Let's further consider the moduli space for deg 2 multiply covered curves representing L. By Riemann-Hurwitz formula,

$$-2 = 2\times(-2) + \sum(e_p - 1),$$

thus such a curve have exactly two ramified points (they may coincide). Suppose u is a multiply cover of v. As $\mathrm{Im}\, v$ acts as a line in $\mathbb{CP}^2$, which can be parametrized by two complex numbers, the moduli space havs freedom in four complex parameters and is of real dimension 8.

Let's further consider moduli space of curves with tree domain such that $e(T)=1, v(T)=2$. It has complex dimension 4 and therefore real dimension 8.

Example 7.4 provides us with some basic intuition about moduli space of curves. $\overline{\mathcal{M}_k(A,J)}$ can be imagined as a manifold with corners. $\mathcal{M}_k^*(A,J)$ is a 0-codimensional open statrum while degenerating curves occupy lower dimensional parts.

8. Back to quantum cohomology

Let (M,ω) be a symplectic manifold. $QM^*(M,\omega)$ is viewed as a finite dimensional vector space over $\Lambda^{\mathbb{K},\Gamma_\omega}$. In algebraic topology, $H^*(M;\mathbb{K})$ is a graded ring under the cup product $\cup$. Let $e_1,\cdots,e_n$ be a $\mathbb{K}$-basis for $H^*(M;\mathbb{K})$ and

$$e_i\cup e_j=c_{ij}^k e_k$$

with c_{ij}^k. calle structure constants for $\cup$. On $QM^*(M,\omega)$, we have a similar operation $*$, which satisfies

$$e_i * e_j = e_i\cup e_j + \text{ deformed part }.$$

$*$ is defined based on Gromov–Witten invariant, so Zhang cannot explain this operator for now. However, we can see what extra structure can be defined on quantum coholomogy with $*$ operator.

Example 8.1. Let $(M,\omega)=(\mathbb{CP}^n,\omega_{\mathrm{FS}})$. According to algebraic topology,

$$H^*(\mathbb{CP};\mathbb{K})=\mathbb{K}[S]/(S^{n+1}),$$

where $S\in H^2(\mathbb{CP}^n;\mathbb{K})$ is the Poincaré dual of a complex hyperplane. As for quantum cohomology,

$$QH^*(\mathbb{CP}^n,\omega_{\mathrm{FS}})=\mathbb{K}[S,T]/(S^{n+1}-T).$$

Here $S^{n+1}=S\cup\cdots\cup S+$ deformed parts, so T represents the extra terms.

Remark 8.2. One advantage of working with quantum cohomology is that many elements become invertible. For example, in $QH^*(\mathbb{CP}^n,\omega_{\mathrm{FS}})$, $S^{-1}=T^{-1}S^n$. Also, $QH^*(\mathbb{CP}^n,\omega_{\mathrm{FS}})$ is an integral domain, for $T-S^{n+1}$ is irreducible.

Example 8.3. Consider $(\mathbb{S}^2\times\mathbb{S}^2,\omega_{\mathrm{prod}})$.

$$H^*(\mathbb{S}^2\times\mathbb{S}^2;\mathbb{K})=\mathbb{K}\oplus\mathbb{K}a\oplus\mathbb{K}b\oplus\mathbb{K}a\cup b,$$

where $a=\mathrm{PD}(\mathbb{S}^2\times\mathrm{pt}), b\in\mathrm{PD}(\mathrm{pt}\times\mathbb{S}^2)$. If we consider quantum cohomology, we have $a*a=b*b=T$, which already gives us new information. Moreover, recall the another version of Novikov rings. In fact

$$a*a=T^{[\mathrm{pt}\times\mathbb{S}^2]}, b*b=T^{[\mathbb{S}^2\times\mathrm{pt}]},$$

so we can further distinguish the two elements. This shows another advantage of working with quantum cohomology.

Further assume that M is compact. Then there is a pairing structure

$$\langle x,y\rangle=\int_M x\cup y=x(\mathrm{PD}(y))$$

for $x,y\in H^*(M;\mathbb{K})$ with $\deg x+\deg y=\dim M$. This can be extended to $QH^*(M,\omega)$ by taking

$$\langle\lambda x,\mu y\rangle=\lambda\mu\int_M x\cup y=x(\mathrm{PD}(y))\in\mathbb{K},\lambda,\mu\in\mathbb{K}.$$

The following is somewhat the ”final” theorem for quantum cohomology.

Theorem 8.4. The triple $(QM^*(M,\omega),*,\langle-,-\rangle)$ is an associative Frobenius algebra, i.e., $\langle-,-\rangle$ is non-degenerate bilinear form and $\langle a*b,c\rangle=\langle a,b*c\rangle$.

Remark 8.5. Just like $a\cup b=(-1)^{\deg a\deg b}b\cup a$, we also have $a*b=(-1)^{\deg a\deg b}b*a$. Here we must explain what's the degree on $QH^*(M,\omega)$. Consider the monomial xT^A with $x\in H^2(M;\mathbb{K}), A\in H_2^S(M)$ and we have $\deg xT^A=\deg x+2\,\mathrm{c}_1(TM)(A)$.

There are some interesting properties for semi-simple quantum cohomology.

Proposition 8.6 (Entov–Polterovich, Berkovich)**.** If $QH^*(M,\omega)$ is semi-simple, then there exists $c>0$ such that for any x in a field factor of $QH^*(M,\omega)$, we have

$$l(x)+l(-x)\geqslant -c.$$

Remark 8.7. This inequality has strong geometric meaning. It serves as an important ingredient in Shelukhin's proof of Hofer–Zehnder conjecture.

9. Gromov–Witten invariant

Let (M, ω) be a symplectic manifold and $A \in H_2^S(M)$. Take a generic $J \in \mathcal{J}(M, \omega)$.

Definition 9.1. For $[(u, z_1, \cdots, z_k)] \in \overline{\mathcal{M}_k(A, J)}$, define the map

$$\mathrm{ev}([(u, z_1, \cdots, z_k)]) = (u(z_1), \cdots u(z_k)) \in M^k.$$

Similarly we define $\mathrm{ev}_i([(u, z_1, \cdots, z_k)]) = u(z_i)$.

Given cycles $X_1, \cdots, X_k \in H_*(M; \mathbb{K})$, we want to conut $u \in \overline{\mathcal{M}_k(A, J)}$ such that $[\mathrm{Im}\, u]$ intersects $X_1, \cdots, X_k$ at the same time.

Example 9.2. Let $(M, \omega) = (\mathbb{CP}^2, \omega_{\mathrm{FS}})$ and consider a line class $A = L$. Let X_1, X_2 be two point classes in $H_0(\mathbb{CP}^2; \mathbb{Z})$. Then such a u is a curve passing through X_1, X_2.

Since the curves may degenerate and position of homology classes vary, this question is a little bit complicated. Therefore we introduce two improvements to make our lives easier:
(1) We only work on $\mathcal{M}_k^*(A, J)$.
(2) $X_1, \cdots, X_k$ are in a generic position. That is, $X_1, \cdots X_k$ are represented by submanifolds and X_i, X_j pairwise intersect transversally.

Example 9.3. On $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$, consider point classes $X_i \in H_0(M; \mathbb{Z})$ in a generic position. Let $A = L$ be a line class. By intuition, there is exactly one curve representing L and passing through X_1, X_2. Similarly, with some computation one sees that there is exactly one curve representing $2L$ (i.e. it is a conic) and passing through $X_1, \cdots, X_5$.

Definition 9.4. We define the Gromov–Witten invariant as follows. For k cohomology classes $a_1, \cdots, a_k \in H^*(M; \mathbb{K})$ such that $\deg a_1 + \cdots + \deg a_k = \dim \mathcal{M}_k^*(A, J)$, their Gromov–Witten invariant is

$$\mathrm{GW}_{A,k}^M(a_1, \cdots, a_k) = \int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}_1^*(a_1) \cup \cdots \cup \mathrm{ev}_k^*(a_k).$$

Let's see what the formula above means. Suppose $a_i = \mathrm{PD}(X_i)$. Then

$$\begin{aligned} &\int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}_1^*(a_1) \cup \cdots \cup \mathrm{ev}_k^*(a_k) = \int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}^*(\pi_1^*(a_1) \cup \cdots \cup \pi_k^*(a_k)) \\ = \;& \pi_1^*(a_1) \cup \cdots \cup \pi_k^*(a_k)(\mathrm{ev}_*([\mathcal{M}_k^*(A, J)])) = \mathrm{PD}(\pi_1^*(X_1) \cap \cdots \cap \pi_k^*(X_k))(\mathrm{ev}_*([\mathcal{M}_k^*(A, J)])). \end{aligned}$$

The only issue in the identity is that we don't know if the fundamental class $[\mathcal{M}_k^*(A, J)]$ exists. Fortunately we don't need to worry about that if we only work with semi-positive symplectic manifolds. This issue, which relates to the foundations of symplectic geometry, is a fundamental question.

Example 9.5. Consider $(\mathbb{CP}^2, \omega_{\mathrm{FS}}, J_0)$ and let L be a line class. Set $A = dL$ and consider $\mathrm{GW}_{A,3d-1}^{\mathbb{CP}^2}$. What is the possible choice for $a_1, \cdots, a_{3d-1} \in H^*(\mathbb{CP}^2; \mathbb{Z})$ so that we can consider its Gromov–Witten invariant? By definition,

$$\sum_{i=1}^{3d-1} \deg a_i = 4 + 2\,\mathrm{c}_1(T\mathbb{CP}^2)(dL) + 2(3d-1) - 6 = 4(3d-1),$$

so $\deg a_i = 4$ and a_i is a Poincaré dual of a point. Therefore $\mathrm{GW}_{A,3d-1}^{\mathbb{CP}^2}$ counts degree d curves in $\mathbb{CP}^2$ passing through $3d-1$ points. One may ask what this number is. This famous number is denoted by N_d and we have seen that $N_1 = N_2 = 1, N_3 = 12$. We will see how to compute it later.

10. computing Gromov–Witten invariant

Today we consider the Gromov–Witten invariant under some special cases.

Proposition 10.1.

$$\mathrm{GW}_{0,k}^M(a_1, \cdots, a_k) = \begin{cases} \displaystyle\int_M a_1 \cup a_2 \cup a_3 & k = 3, \\ 0 & k \neq 3. \end{cases}$$

Proof. We have

$$\sum_{i=1}^{k} \deg a_i = \dim M + 2\,\mathrm{c}_1(TM)(A) + 2k - 6 = \dim M + 2k - 6.$$

Since any curve u has $[u] = 0$, u is a constant map. Let $N_i = \mathrm{PD}(a_i)$. Then $N_1 \cap \cdots \cap N_k \neq \varnothing$, and

$$\dim M \geqslant \operatorname{codim}(\bigcap_{i=1}^{k} N_i) = \sum_{i=1}^{k} \operatorname{codim} N_i = \sum_{i=1}^{k} \deg a_i.$$

Therefore $k \leqslant 3$. Moreover, as we count stable maps, there is at least 3 marked points so $k \geqslant 3$. The only possibility is $k = 3$ and

$$\mathrm{GW}_{0,3}^{M} = \int_{\mathcal{M}_k^*(A,J)} \mathrm{ev}_1^*(a_1) \cup \mathrm{ev}_2^*(a_2) \cup \mathrm{ev}_3^*(a_3) = \int_M a_1 \cup a_2 \cup a_3.$$

□

The second thing to mention is that if we arrange, index then the Gromov–Witten invariant may differ by a sign.

Proposition 10.2. We have

$$\mathrm{GW}_{A,k}^{M}(a_1, \cdots, a_{k-1}, 1) = 0$$

if $(A, k) \neq (0, 3)$.

Proof. $1 \in H^0(M; \mathbb{K})$ has Poincaré dual $[M]$, so geometrically

$$\mathrm{GW}_{A,k}^{M}(a_1, \cdots, a_{k-1}, 1) = \mathrm{GW}_{A,k-1}^{M}(a_1, \cdots, a_{k-1}).$$

On the other hand,

$$\sum_{i=1}^{k-1} \deg a_i = \dim M + 2\,\mathrm{c}_1(TM)(A) + 2k - 6.$$

Hence the identity cannot hold unless they are all zero. □

Remark 10.3. $\mathrm{GW}_{A,2}^{M}$ could be nonzero.

Proposition 10.4. If $\deg a_1 = 2$, then

$$\mathrm{GW}_{A,k}^{M}(a_1, \cdots, a_k) = \mathrm{GW}_{A,k-1}^{M}(a_2, \cdots, a_k) \cdot \int_A a_1.$$

Exercise 10.1. Compute the Gromov–Witten potential of $(\mathbb{CP}^1, \omega_{\mathrm{FS}})$ with respect to the standard complex structure J_0. Please provide all the details.

Definition 10.5. Suppose $m + 1 = \dim H^*(M; \mathbb{Z})$ with basis $1, e_1, \cdots, e_d$. The Gromov–Witten potential of (M, ω) is defined as the exponential generating function

$$\Phi(t_0, \cdots, t_m) = \sum_{A \in H_2^S(M)} \left(\sum_{k \geqslant 0} \frac{1}{k!} \mathrm{GW}_{A,k}^{M} \left(\sum_{i=1}^{m} t_i e_i, \cdots, \sum_{i=1}^{m} t_i e_i \right) \right) e^A.$$

By definition, the key to compute Gromov–Witten potential lies in calculating the term

$$\frac{1}{k!} \mathrm{GW}_{A,k}^{M} \left(\sum_{i=1}^{m} t_i e_i, \cdots, \sum_{i=1}^{m} t_i e_i \right),$$

which by the multilinear property can be expressed as

$$\frac{1}{k!} \sum_{i_1 \leqslant \cdots \leqslant i_k} c(i)(-1)^{s(i)} \mathrm{GW}_{A,k}^{M}(e_{i_1}, \cdots, e_{i_k}) t_{i_1} \cdots t_{i_k},$$

where $c(i), s(i)$ are to be determined. However, this expression turns out not convenient for calculations. We can write the term in another way:

$$
\begin{aligned}
&\frac{1}{k!}\mathrm{GW}^M_{A,k}\left(\sum_{i=1}^m t_i e_i, \cdots, \sum_{i=1}^m t_i e_i\right) \\
=& \sum_{\alpha_0+\cdots+\alpha_m=k} \frac{1}{k!}(-1)^{s(\alpha)}\mathrm{GW}^M_{A,k}(e_0^{\alpha}{}_0, \cdots, e_m^{\alpha_m})\frac{k!}{\alpha_0!\cdots\alpha_d!}t_0^{\alpha_0}\cdots t_m^{\alpha_m} \\
=& \sum_{\alpha}(-1)^{s(\alpha)}\frac{\mathrm{GW}^M_{A,k}(e^\alpha)}{\alpha!}t^\alpha.
\end{aligned}
$$

Moreover, if e_i all have even degrees, then the sign $(-1)^{s(\alpha)}$ is always 1.

Example 10.6. Let's calculate the Gromov–Witten potential Φ of $(M, \omega) = (\mathbb{CP}^2, \omega_{\mathrm{FS}})$. Recall that $H^*(\mathbb{CP}^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}c \oplus c^2$, where c is the Poincaré dual for the line class L. There are two main cases.

(1) $A = 0$. All terms vanishe except for

$$\mathrm{GW}^M_{0,3}(a_1, a_2, a_3) = \int_{\mathbb{CP}^2} a_1 \cup a_2 \cup a_3$$

with $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 0), (2, 0, 1)$. Since

$$\int_{\mathbb{CP}^2} 1 \cup c \cup c = \int_{\mathbb{CP}^2} 1 \cup 1 \cup c^2 = 1,$$

we see that

$$\Phi|_{A=0} = \left(\frac{1}{2!0!1!}t_0^2 t_2 + \frac{1}{1!2!0!}t_0 t_1^2\right)e^0 = \frac{1}{2}t_0^2 t_2 + \frac{1}{2}t_0 t_1^2.$$

(2) $A = dL$ with $d \neq 0$. By definition,

$$\sum_{i=1}^k \deg a_i = \dim \mathbb{CP}^2 + 2\,\mathrm{c}_1(T\mathbb{CP}^2)(dL) + 2k - 6 = 6d + 2k - 2.$$

By Proposition 10.2, if $a_i = 1$ for some i then this term vanishes. It suffices to consider $(a_1, \cdots, a_k) = (c, \cdots, c, c^2, \cdots, c^2)$, so

$$
\begin{aligned}
\alpha_1 + \alpha_2 &= k, \\
2\alpha_1 + 4\alpha_2 &= 6d + 2k - 2.
\end{aligned}
$$

Solving the two linear equations, we obtain $\alpha_1 = k - 3d + 1, \alpha_2 = 3d - 1$. Note that the number $3d - 1$ appears naturally. By Proposition 10.4, we see that

$$\mathrm{GW}^{\mathbb{CP}^2}_{dL,k}(c, \cdots, c, c^2, \cdots, c^2) = \mathrm{GW}^{\mathbb{CP}^2}_{dL,3d-1}(c^2, \cdots, c^2)\left(\int_{dL} c\right)^{k-3d+1} = N_d \cdot d^{k-3d+1}.$$

As a consequence,

$$\Phi|_{dL} = \sum_{k \geqslant 3d-1}\left(\frac{N_d}{(3d-1)!}t_2^{3d-1}\frac{1}{(k-3d+1)!}t_1^{k-3d+1}d^{k-3d+1}\right)e^{dL} = \frac{N_d}{(3d-1)!}t_2^{3d-1}e^{dt_1}e^{dL}.$$

The full expression of the Gromov–Witten potential is

$$\Phi(t_0, t_1, t_2) = \frac{1}{2}t_0^2 t_2 + \frac{1}{2}t_0 t_1^2 + \sum_{d \geqslant 1}\frac{N_d}{(3d-1)!}t_2^{3d-1}e^{dt_1}e^{dL}.$$

The Gromov–Witten potential of $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$ contains the mysterious coefficients we discussed before. If we can compute Φ, then we get to know a lot of information about N_d. Surprisingly, it is possible to compute Φ for every (M, ω) because they satisfy the famous WDVV equations.

11. WDVV equations

Let M be a compact symplectic manifold. Fix a basis $e_0, \cdots, e_m$ of $H^*(M;\mathbb{K})$ and denote $g = (g_{ij})$, where

$$g_{ij} = \int_M e_i \cup e_j.$$

By Poincaré duality, g is non-degenerate, so $(g^{ij}) = g^{-1}$ exists.

Theorem 11.1 (Witten–Dijkgraaf–Verlinde–Verlinde)**.** The Gromov–Witten potential satisfies the following PDE: for $0 \leqslant a, b, c, d \leqslant m$,

$$\sum_{i,j} \Phi_{iba} g^{ij} \Phi_{dcj} = (-1)^{\deg e_a (\deg e_b + \deg e_c)} \sum_{i,j} \Phi_{icb} g^{ij} \Phi_{daj},$$

where

$$\Phi_{iba} = \frac{\partial^3}{\partial t_i \partial t_b \partial t_a} \Phi.$$

This theorem is usually described as WDVV equations.

Example 11.2. $(M, \omega) = (\mathbb{CP}^1, \omega_{\mathrm{FS}})$. By Exercise 10.1, its Gromov–Witten potential is

$$\Phi = \frac{1}{2} t_0^2 t_1 + e^{t_1} e^L,$$

where L is the line class.

Exercise 11.1. Prove for $(M, \omega) = (\mathbb{CP}^2, \omega_{\mathrm{FS}})$, the WDVV equations are equivalent to the single equation

$$\Phi_{111} \Phi_{122} + \Phi_{222} = (\Phi_{112})^2.$$

Proof. $H^*(\mathbb{CP}^2; \mathbb{K}) = \mathbb{K} \oplus \mathbb{K}c \oplus \mathbb{K}c^2$, where c is the Poincaré dual of the line class. As

$$\int_{\mathbb{CP}^2} c \cup c = \int_{\mathbb{CP}^2} 1 \cup c^2 = 1,$$

we have

$$g = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} = g^{-1}.$$

Therefore,

$$\sum_{i,j} \Phi_{iba} g^{ij} \Phi_{dcj} = \Phi_{0ba} \Phi_{dc2} + \Phi_{1ba} \Phi_{dc1} + \Phi_{2ba} \Phi_{dc0}.$$

Remember that in Example 10.6 we have obtained the Gromov–Witten potential of $(\mathbb{CP}^2, \omega_{\mathrm{FS}})$

$$\Phi(t_0, t_1, t_2) = \frac{1}{2} t_0^2 t_2 + \frac{1}{2} t_0 t_1^2 + \sum_{d \geqslant 1} \frac{N_d}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL}.$$

Direct calculations imply

$$\begin{aligned}
\Phi_{002} &= \Phi_{011} = 1, \\
\Phi_{111} &= \sum_{d \geqslant 1} \frac{d^3 N_d}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL}, \\
\Phi_{112} &= \sum_{d \geqslant 1} \frac{d^2 N_d}{(3d-2)!} t_2^{3d-2} e^{dt_1} e^{dL}, \\
\Phi_{122} &= \sum_{d \geqslant 1} \frac{d N_d}{(3d-3)!} t_2^{3d-3} e^{dt_1} e^{dL}, \\
\Phi_{222} &= \sum_{d \geqslant 1} \frac{N_{d+1}}{(3d-1)!} t_2^{3d-1} e^{dt_1} e^{dL},
\end{aligned}$$

while other terms vanish. If we vary a, b, we see that

$$\sum_{i,j} \Phi_{i00} g^{ij} \Phi_{dcj} = \Phi_{000}\Phi_{dc2} + \Phi_{100}\Phi_{dc1} + \Phi_{200}\Phi_{dc0} = 0,$$

$$\sum_{i,j} \Phi_{i01} g^{ij} \Phi_{dcj} = \Phi_{001}\Phi_{dc2} + \Phi_{101}\Phi_{dc1} + \Phi_{201}\Phi_{dc0} = \Phi_{dc1} = \Phi_{00c}\Phi_{d12} + \Phi_{10c}\Phi_{d11} + \Phi_{20c}\Phi_{d10},$$

$$\sum_{i,j} \Phi_{i02} g^{ij} \Phi_{dcj} = \Phi_{002}\Phi_{dc2} + \Phi_{102}\Phi_{dc1} + \Phi_{202}\Phi_{dc0} = \Phi_{dc2},$$

$$\sum_{i,j} \Phi_{i11} g^{ij} \Phi_{dcj} = \Phi_{011}\Phi_{dc2} + \Phi_{111}\Phi_{dc1} + \Phi_{211}\Phi_{dc0} = \Phi_{dc2} + \Phi_{111}\Phi_{dc1} + \Phi_{211}\Phi_{dc0},$$

$$\sum_{i,j} \Phi_{i12} g^{ij} \Phi_{dcj} = \Phi_{012}\Phi_{dc2} + \Phi_{112}\Phi_{dc1} + \Phi_{212}\Phi_{dc0} = \Phi_{112}\Phi_{dc1} + \Phi_{212}\Phi_{dc0},$$

$$\sum_{i,j} \Phi_{i22} g^{ij} \Phi_{dcj} = \Phi_{022}\Phi_{dc2} + \Phi_{122}\Phi_{dc1} + \Phi_{222}\Phi_{dc0} = \Phi_{122}\Phi_{dc1} + \Phi_{222}\Phi_{dc0}.$$

The only non-trivial equation is

$$\Phi_{222} + \Phi_{111}\Phi_{221} = \sum_{i,j} \Phi_{i11} g^{ij} \Phi_{22j} = \sum_{i,j} \Phi_{i12} g^{ij} \Phi_{21j} = \Phi_{112}^2.$$

□

Example 11.3. WDVV equations help us obtain the recursive formula for N_d. By Exercise 11.1, we have

$$\Phi_{111}\Phi_{122} = \sum_{k,l\geqslant 1} N_k N_l \frac{k^3 l}{(3k-1)!(3l-3)!} t_2^{3(k+l)-4} e^{(k+l)t_1} e^{(k+l)L},$$

$$\Phi_{112}^2 = \sum_{k,l\geqslant 1} N_k N_l \frac{k^2 l^2}{(3k-2)!(3l-2)!} t_2^{3(k+l)-4} e^{(k+l)t_1} e^{(k+l)L}.$$

The WDVV equations imply

$$\frac{N_d}{(3d-4)!} + \sum_{k,l\geqslant 1, k+l=d} N_k N_l \frac{k^3 l}{(3k-1)!(3l-3)!} = \sum_{k,l\geqslant 1, k+l=d} N_k N_l \frac{k^2 l^2}{(3k-2)!(3l-2)!}.$$

It gives us the recursive formula

$$N_d = \sum_{k,l\geqslant 1, k+l=d} N_k N_l \left(k^2 l^2 \binom{3d-4}{3k-2} - k^3 l \binom{3d-4}{3k-1} \right), d \geqslant 2.$$

This formula was first found by Kontsevich. Since we know $N_1 = 1$, we theorically know the value of N_d for arbitary d. For example,

$$N_2 = N_1^2 \left(\binom{2}{1} - \binom{2}{2} \right) = 1,$$

$$N_3 = N_1 N_2 \left(4\binom{5}{1} - 2\binom{5}{2} \right) + N_2 N_1 \left(4\binom{5}{1} - 8\binom{5}{5} \right) = 12,$$

$$N_4 = N_1 N_3 \left(9\binom{8}{1} - 3\binom{8}{2} \right) + N_2^2 \left(16\binom{8}{4} - 16\binom{8}{5} \right) + N_3 N_1 \left(9\binom{8}{7} - 27\binom{8}{8} \right) = 620.$$

Similarly, $N_5 = 87304, N_6 = 26312976$ and $N_7 = 14616808192$.

12. BACK TO QUANTUM PRODUCTS

Now we define the quantum product mentioned in section 8. Let M be a compact symplectic manifold and fix a basis $e_0, \cdots, e_m$ of $H^*(M; \mathbb{K})$.

Definition 12.1. The quantum product of e_i, e_j is defined via

$$(e_i * e_j)(\mathrm{PD}(e_k)) = \sum_{A\in H_2^S(M)} \mathrm{GW}_{A,3}^M(e_i, e_j, e_k) T^A.$$

By definition,

$$(e_i * e_j)(\mathrm{PD}(e_k)) = \int_M e_1 \cup e_2 \cup e_3 + \sum_{A\neq 0} \mathrm{GW}^M_{A,3}(e_i, e_j, e_k)T^A.$$

We see that the deformed parts of the quantum products comes from

$$\sum_{A\neq 0} \mathrm{GW}^M_{A,3}(e_i, e_j, e_k)T^A.$$

Example 12.2. We now explain the structure of $(QH^*(\mathbb{CP}^n, \omega_{\mathrm{FS}}), *)$ as mentioned in Example 8.1. Let's consider the basis $1, c, \cdots, c^n$ of $H^*(\mathbb{CP}^n; \mathbb{K})$ with $c = \mathrm{PD}(L)$, where L is the line class. By definition,

$$(c^i * c^j)(\mathrm{PD}(c^k)) = \int_{\mathbb{CP}^n} c^{i+j+k} + \sum_{d\geqslant 1} \mathrm{GW}^{\mathbb{CP}^n}_{dL,3}(c^i, c^j, c^k)T^{dL}.$$

If the deformed part makes sense, then we have

$$2(i + j + k) = \dim \mathbb{CP}^n + 2\,\mathrm{c}_1(T\mathbb{CP}^n)(dL) = 2n + 2d(n + 1).$$

The only possibility is $d = 1$, so if $i + j + k = 2n + 1$, the deformed part is

$$\mathrm{GW}^{\mathbb{CP}^n}_{L,3}(c^i, c^j, c^k)T^L.$$

Remark 12.3. The quantum cohomology can be viewed as an explicit construction for the Floer cohomology $HF^*(M; \mathbb{K})$, the marvelous generalization of Morse theory on symplectic manifolds. In general, we have the isomorphism of graded rings

$$QH^*(M, \omega) \simeq HF^*(M; \mathbb{K}).$$

Exercise 12.1. Let setting be as in Example 12.2. Check that for the quantum product is given by

$$c^i * c^j = \begin{cases} c^{j+k}, & i + j \leqslant n \\ c^{i+j-n-1}T^L & n + 1 \leqslant i + j \leqslant 2n \end{cases}.$$

Exercise 12.2. A Frobenius algebra is semi-simple if it can be decomposed into a direct sum of fields (not necessarily the same fields). There is a cirerion by Abram: a Frobenius algebra $(\mathcal{A}, \cdot, \langle -, - \rangle)$ is semi-simple if and only if the following class (called the Euler class)

$$\mathcal{E} = \sum_{\text{basis}} e_i * e_i^\vee$$

is invertible in $\mathcal{A}$, where $e_i^\vee$ is the dual of e_i with respect to $\langle -, - \rangle$.

For $(M, \omega) = (\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\mathrm{prod}})$, denote $a = \mathrm{PD}([\mathrm{pt} \times \mathbb{S}^2]), b = \mathrm{PD}([\mathbb{S}^2] \times \mathrm{pt})$.

(1) Compute all non-trivial 3-points constraint Gromov–Witten invariants.
(2) Compute the quantum products $a * a, b * b$ and $a * b$.
(3) Check that the quantum cohomology group $QH^*(\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\mathrm{prod}})$ (over $\mathbb{K} = \mathbb{C}$), as an algebra, is semi-simple.

13. Hard stones on Lagrangian submanifolds

We set aside the study of J-holomorphic temporarily, and focus on another important element in symplectic topology. Le t (M, ω) be a symplectic manifold.

There are two phenomenons in symplectic topology. One is the rigidlity, like the Gromov non-squeezing property. The other is the softness, which basically means you can move or deform homotopically in certain freedom. Here, the "hard stones" in the title represents the rigidility of Lagrangian submanifolds.

Recall that the symplectic structure ω is a closed 2-form on M that is non-degenerate. However, it is unknown whether ω is still non-degenerate on a submanifold. Let's see a toy version.

Consider the Euclidean space with the standard symplectic form $(\mathbb{R}^{2n}, \omega_0)$. Then $\omega_0 = 0$ on the subspace $\mathrm{span}(x_1)$. The same holds for $\mathrm{span}(x_1, x_2)$. One may ask what is the biggest possible dimension of a subspace such that ω_0 vanishes on it. After some try, we find that $\omega_0 = 0$ on the subspace $\mathrm{span}(x_1, \cdots, x_n)$.

Definition 13.1. A submanifold $L \subset (M, \omega)$ is said to be Lagrangian if $\omega|_L = 0$ and $\dim L = \frac{1}{2} \dim M$.

What is interesting and strange about Lagrangian submanifolds is, although the symplectic structure cannot "see" them, they play a fundamental role in the study of symplectic topology. The reason for this will be related to the "Fukaya category".

Example 13.2. We have seen the simpliest examples for Lagrangian submanifolds in Euclidean space. Now we introduce more on different manifolds.

(1) On a closed surface $(\Sigma_g, \omega_{\text{area}})$ the Lagrangian submanifolds are of dimension 1. Obviously, any curve as a smooth submanifold will be Lagrangian for the dimension reason.
(2) Consider the cotangent bundle $(T^*Q, d\lambda_{\text{can}})$ because $\lambda_{\text{can}} = 0$ on Q. First notice that Q itself is Lagrangian. More generally, in exercise we will show that the image Γ_α of a smooth 1-form α is a Lagrangian submanifold if and only if α is closed.
(3) Let φ be a diffeomorphism of a symplectic manifold (M, ω). Consider its graph

$$\Gamma_\varphi = \{(x, \varphi(x)) \mid x \in M\}.$$

Then $\Gamma_\varphi \subset (M \times M, \omega_{\text{twist}} = \pi_1^*\omega - \pi_2^*\omega)$ is a Lagrangian submanifold if and only if $\varphi \in \text{Symp}(M, \omega)$. To see this, recall that

$$T_{(p,\varphi(p))}\Gamma_\varphi = \{(v, \varphi_* v) \mid v \in T_p M\}.$$

Then $\omega_{\text{twist}}((v, \varphi_* v), (w, \varphi_* w)) = \omega(v, w) - \omega(\varphi_* v, \varphi_* w) = (\omega - \varphi^*\omega)(v, w)$.

Exercise 13.1. (1) Let Q be a submanifold and α be a 1-form on Q. Prove that the image (or graph) of α, denoted by $\Gamma_\alpha = \{(q, \alpha(q)) \mid x \in Q\}$, is a Lagrangian submanifold in $(T^*Q, d\lambda_{\text{can}})$ if and only if α is closed.
(2) A Lagrangian submanifold $L \subset (M, d\lambda)$ is said to be exact if $\lambda|_L = df$ for some smooth function $f: L \to \mathbb{R}$. Prove that Γ_α defined above is an exact Lagrangian submanifold in $(T^*Q, d\lambda_{\text{can}})$ if and only if α is exact.

Case (3) of Example 13.2 provides us with a geometric way to examine whether a diffeomorphism is a symplectomorphism. The following questions are:
(1) **How do we know if a submanifold is Lagrangian ?**
(2) **How to cook up Lagrangian submanifolds?**
There are several different answers to them. Zhang's version of answer is using the so called Poisson brackets.

Definition 13.3. Let $F, G \in C^\infty(M)$. Let X_G be the vector field on M defined via

$$\omega(X_G, -) = dG.$$

The Poisson bracket of F, G is defined as

$$\{F, G\} = dF(X_G).$$

Remark 13.4. Symplectic geometers have different sign conventions. For example, some people prefer to define X_G via

$$\omega(X_G, -) = -dG.$$

The vector field X_G which appears in the definition is very famous and has its name, the Hamiltonian vector field of G. Note that we only need the non-degeneracy of symplectic structure in the definition. What follows from the skew-symmetric property is the following:

Proposition 13.5. Let $F, G \in C^\infty(M)$.
(1) $\{F, G\} = -\{G, F\}$. In particular $\{F, F\} = dF(X_F) = 0$, so F remains constant along the Hamiltonian vector field of X_F.
(2) If either F or G is constant, then $\{F, G\} = 0$.
(3) The Poisson bracket satisfies the Jacobian identity, i.e., $(C^\infty(M), \{-, -\})$ is a Lie algebra.

Proposition 13.6. Let $L \subset (M, \omega)$ be a submanifold of dimension n. Then L is Lagrangian if and only if

$$I_L = \{F \in C^\infty(M) \mid F|_L = 0\}$$

is closed under the Poisson bracket.

Proof. Suppose L is Lagrangian. Tale $F, G \in I_L$ and $p \in L$. For any $v \in T_pL$,

$$\omega_p(X_G, v) = (dG)_p(v) = 0,$$

so $X_G(p) \in T_pL$ by linear symplectic geometry (more precisely, $T_pL \subset T_pM$ is a Lagrangian subsapce). $X_F(p) \in T_pL$ for the same reason. Therefore $\{F, G\}(p) = (dF)_p(X_G) = \omega_p(X_F, X_G) = 0$.

Conversely, suppose L is not Lagrangian. Then there exists $p \in L$ such that $\omega_p \neq 0$ on T_pL. By linear symplectic geometry, we can find $v \in T_pM \setminus T_pL$ such that $\omega_p(v, -) = 0$ on T_pL. Using local coordinates, we can find $G \in I_L$ but $(dG)_p(v) \neq 0$. Hence $\omega_p(X_G, v) = (dG)_p(v) \neq 0$. This means $X_G(p) \notin T_pL$ by hypothesis that $\omega_p(v, -) = 0$ on T_pL. In the same way we can construct $F \in C^\infty(M)$ such that $F \in I_L$ but $(dF)_p(X_G) \neq 0$, which means $\{F, G\}(p) \neq 0$ and $\{F, G\} \notin I_L$. □

This property allows us to check whether a submanifold is Lagrangian or not in a way other than defintiion.

To answer the question (2) above, we point out the following theorem related to the integrable systems.

Theorem 13.7 (Arnold–Liouville)**.** Let $F_1, \cdots, F_n \in C^\infty(M)$ such that they are Poisson-commutative. Consider $\Phi\colon M \to \mathbb{R}^n, x \mapsto (F_1(x), \cdots, F_n(x))$. For any regular value $(r_1, \cdots, r_n)$ of Φ, denote $L = \Phi^{-1}(r_1, \cdots, r_n)$, which is an n-dimensional submanifold. If L is compact, then L is Lagrangian torus, i.e., it is Lagrangian and topologically a torus. If L is not compact, then topologically it is $\mathbb{T}^{n-k} \times \mathbb{R}^k$.

Proof. Consider the action $\mathbb{R}^n \times M \to M, (s_1, \cdots, s_n) \cdot x = \varphi_{F_1}^{s_1} \circ \cdots \circ \varphi_{F_n}^{s_n}(x)$. This will be a group action because F_i are pairwise Poisson-commutative.

$$L = \Phi^{-1}(r_1, \cdots, r_n) = \bigcap_{i=1}^n F_i^{-1}(r_i).$$

If $x \in L$, then $(s_1, \cdots, s_n) \cdot x$ stays in L. To see this, recall that F_i remains constant along X_{F_i} for each i. Then $L \cong \mathbb{R}^n \{x\} = \mathbb{R}^n/\mathrm{stab}(x)$. $\mathrm{stab}(x)$ will be a discrete subgroup of $\mathbb{R}^n$, which implies it must be a lattice.

It remains to show that L is Lagrangian. On T_xL, we have a basis $X_{F_1}(x), \cdots, X_{F_n}(x)$. As

$$\omega_x(X_{F_i}, X_{F_j}) = \{F_i, F_j\}(x) = 0,$$

we see that L is indeed Lagrangian. □

Let's see some toy versions for this theorem so that we can convince ourselves.

Example 13.8. Let $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$. If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, the standard symplectic form can be written as

$$\omega_0 = \frac{\sqrt{-1}}{2} \sum_{j=1}^n dz^j \wedge d\bar{z}^j = \sum_{j=1}^n r^j dr^j \wedge d\theta^j,$$

where $z^j = x^j + \sqrt{-1}y^j = r^j \cos\theta^j + \sqrt{-1}r^j \sin\theta^j$. Consider $F_i = \frac{1}{2}\left|z^i\right|^2$. Direct calculation shows that

$$X_{F_i} = y^i \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial y^i}.$$

As a consequence, $F_1, \cdots, F_n$ are pairwise Poisson-commutative. Let $\Phi = (F_1, \cdots, F_n)$.

$$\Phi^{-1}\left(\frac{1}{2}, \cdots, \frac{1}{2}\right) = \left\{(z^1, \cdots, z^n) \in \mathbb{C}^n \mid \left|z^i\right| = 1\right\} \cong \mathbb{T}^n.$$

By the expression of ω_0 under polar coordinates, we see that the preimage of $(\frac{1}{2}, \cdots, \frac{1}{2})$ is indeed a Lagrangian torus.

Example 13.9. Consider $(\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{\mathrm{prod}})$ where the product is taken with respect to the Fubini–Study form.

$$F_1([z_0, z_1], [w_0, w_1]) = \frac{1}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2},$$

$$F_2([z_0, z_1], [w_0, w_1]) = \frac{1}{2} \frac{|w_0|^2}{|w_0|^2 + |w_1|^2}.$$

As the two functions factor through projections, they are Poisson-commutative. Let $\Phi = (F_1, F_2)$. Then

$$\Phi^{-1}\left(\frac{1}{4}, \frac{1}{4}\right) = \left\{([z_0, z_1], [w_0, w_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid |z_0| = |z_1|, |w_0| = |w_1|\right\} \cong \mathbb{S}^1 \times \mathbb{S}^1 .$$

It is not obvious that this preimage is Lagrangian, but as we know the definition of Fubini–Study metric, it can be checked via computations.

Let's step a little bit further. Notice that the image of Φ is $[0, \frac{1}{2}]^2$. The fiber of an interior point is a Lagrangian torus, while fiber of a boundary point may degenerate. Such manifolds are called symplectic toric manifolds.

14. distinguishing Lagrangian submanifolds

In the last section we see that lots of Lagrangian submanifolds are topologically tori. However, are they the same in the symplectic sense? This will be the main topic for this section.

Let $(\mathbb{S}^2, \omega_{\text{area}})$ be the standard sphere embedded in $\mathbb{R}^3$. Let's see some tori in $(M, \omega) = (\mathbb{S}^2 \times \mathbb{S}^2, \omega_{\text{prod}})$.

(1) let E be the equator of $\mathbb{S}^2$. Then $E \times E \subset (M, \omega)$ is called the Clifford torus.

(2) Let

$$L_{\text{EP}} = \left\{(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |v| = |w| = 1, v_1 + w_1 = 0, \langle v, w\rangle = -\frac{1}{2}\right\}.$$

"EP" stands for Entov–Polterovich, who first invented this submanifold. Zhang claims that this is a Lagrangian torus in (M, ω). To check this, define

$$F(v, w) = v_1 + w_1,$$
$$G(v, w) = \langle v, w\rangle .$$

After some check, we know that F, G Poisson-commute and $(0, -\frac{1}{2})$ is a critical point. The details are left as an exercise. Then the result follows from Arnold–Livoulle theorem.

Exercise 14.1. View the sphere $\mathbb{S}^2$ as $\left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}$.

(1) Prove that the following 2-form on $\mathbb{R}^3$ gives a symplectic structure on $\mathbb{S}^2$: for $p \in \mathbb{R}^3$ and vectors $v, w \in T_p\mathbb{S}^2$, define

$$\omega_p(v, w) = \langle p, v \times w\rangle ,$$

where we identify p, v, w with vectors in $\mathbb{R}^3$ and $\times$ denotes the cross product.

(2) For coordinate functions x, y, z, compute their Poisson brackets with repsect to the symplectic structure in (1).

Lemma 14.1. Fix $q \in \mathbb{R}^3$. Let $f\colon \mathbb{S}^2 \to \mathbb{R}, v \mapsto \langle v, q\rangle$. Then $X_f(p) = p \times q$.

Proof. By definition,

$$-(df)_p(w) = \omega_p(X_f, w).$$

The left hand side is $-\langle w, q\rangle$ while the right hand side is $\langle p, X_f \times w\rangle$. Notice that

$$\langle p, (p \times q) \times w\rangle = \langle p, \langle p, w\rangle q - \langle q, w\rangle p\rangle = \langle p, q, \langle\ ,\rangle\rangle p, w - \langle q, w, \langle\ ,\rangle\rangle pp = -\langle q, w, ,\rangle$$

so $X_f = p \times q$ by the uniqueness. ∎