PHYC90045 Introduction to Quantum Computing

Assignment 1

Name: Xinnan Shen

Student Number: 1051380

1.

(a)

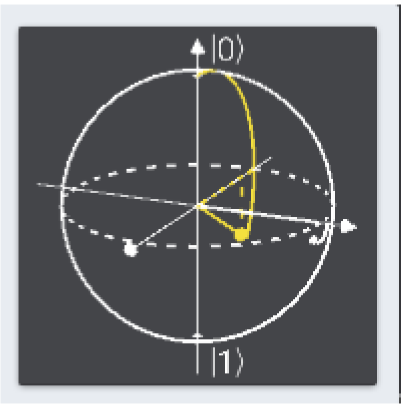
$$H\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

(b)

The operation is $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$,

such that $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$

The resulting state on the Bloch sphere is:



(c)

The operation is: $\begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2}i & 0 \end{bmatrix}$

such that $\begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2}i & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$

(d)

$$R_Z(\theta_R) = e^{i\theta_g}\left(I\cos\frac{\theta_R}{2} - iZ\sin\frac{\theta_R}{2}\right) = \begin{bmatrix} e^{i\left(\theta_g - \frac{\theta_R}{2}\right)} & 0 \\ 0 & e^{i\left(\theta_g + \frac{\theta_R}{2}\right)} \end{bmatrix}$$

This should be equal to $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta_R} \end{bmatrix}$.

Thus, $\begin{cases} \theta_g - \frac{\theta_R}{2} = 0 \\ \theta_g + \frac{\theta_R}{2} = \theta_R \end{cases}$,

So, global phase $\theta_g = \frac{\theta_R}{2}$.

2.

(a)

The sequence is $R_1R_2R_2R_1 = THTH\ HTHT\ HTHT\ THTH$.

(b)

Because the action of two H gates is actually an identity, Alice's sequence should be: $THT\ THT\ HTHT\ THTH$.

3.

$$|\psi_0\rangle = |0\rangle$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|\psi_2\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{\pi}{4}i} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}i}|1\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}i} \end{bmatrix} = \frac{1}{2}\left(1 + e^{-\frac{\pi}{4}i}\right)|0\rangle + \frac{1}{2}\left(1 - e^{-\frac{\pi}{4}i}\right)|1\rangle$$

4.

The initial state is $|00\rangle$. To get the $|\psi\rangle$ state, it first should become $\left(\frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right) \otimes |0\rangle$ after a R-gate. Then, it can become the final $|\psi\rangle$ state after a CNOT gate.

The matrix form of R-gate can be written as: $\begin{bmatrix} \frac{i}{2} & \alpha \\ -\frac{\sqrt{3}}{2} & \beta \end{bmatrix}$,(α and β can be any complex number).

Assume that the R-gate is a Z-axis rotation, then

$R_Z(\theta_R) = e^{i\theta_g}\left(I\cos\frac{\theta_R}{2} - iZ\sin\frac{\theta_R}{2}\right) = \begin{bmatrix} e^{i\left(\theta_g - \frac{\theta_R}{2}\right)} & 0 \\ 0 & e^{i\left(\theta_g + \frac{\theta_R}{2}\right)} \end{bmatrix}$, but it cannot be equal to $\begin{bmatrix} \frac{i}{2} & \alpha \\ -\frac{\sqrt{3}}{2} & \beta \end{bmatrix}$.

Assume that the R-gate is a Y-axis rotation, then

$R_Y(\theta_R) = e^{i\theta_g}\left(I\cos\frac{\theta_R}{2} - iY\sin\frac{\theta_R}{2}\right) = e^{i\theta_g}\begin{bmatrix} \cos\frac{\theta_R}{2} & -\sin\frac{\theta_R}{2} \\ -\sin\frac{\theta_R}{2} & \cos\frac{\theta_R}{2} \end{bmatrix}$, but it cannot be equal to $\begin{bmatrix} \frac{i}{2} & \alpha \\ -\frac{\sqrt{3}}{2} & \beta \end{bmatrix}$.

Assume that the R-gate is an X-axis rotation, then

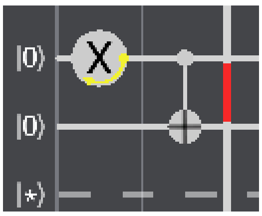
$$R_X(\theta_R) = e^{i\theta_g}\left(I\cos\frac{\theta_R}{2} - iX\sin\frac{\theta_R}{2}\right) = e^{i\theta_g}\begin{bmatrix} \cos\frac{\theta_R}{2} & -i\sin\frac{\theta_R}{2} \\ -i\sin\frac{\theta_R}{2} & \cos\frac{\theta_R}{2} \end{bmatrix}$$

This should be equal to $\begin{bmatrix} \frac{i}{2} & \alpha \\ -\frac{\sqrt{3}}{2} & \beta \end{bmatrix}$.

So $\alpha = -\frac{\sqrt{3}}{2}, \beta = \frac{i}{2}$;

Thus, $\theta_g = \frac{\pi}{2}$, $\theta_R = -\frac{2\pi}{3}$, and the R-gate is an X-axis rotation.

Therefore, the whole circuit is shown below, where the R-gate is an X-axis rotation with global phase $\theta_g = \frac{\pi}{2}$ and rotation angle $\theta_R = -\frac{2\pi}{3}$.



5.

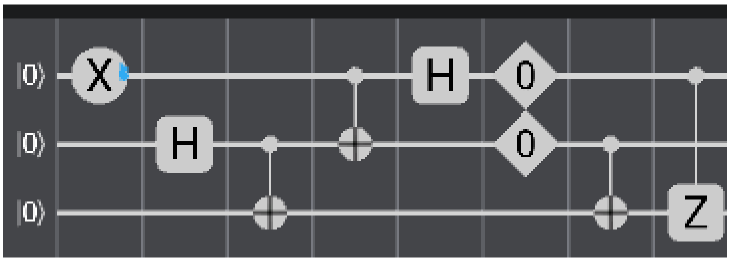
(a)

The qubits associated with Alice and Bob and the qubit state to be teleported is shown in the below circuit.

$|\psi_{teleported}\rangle$

$|\psi_{Alice}\rangle$

$|\psi_{Bob}\rangle$



$|\psi_{teleported}\rangle$

$$|\psi_1\rangle = \left(\cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle\right) \otimes |0\rangle \otimes |0\rangle = \cos\frac{\theta}{2}|000\rangle - i\sin\frac{\theta}{2}|100\rangle$$

$$|\psi_2\rangle = \left(\cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2}|000\rangle + \cos\frac{\theta}{2}|010\rangle - i\sin\frac{\theta}{2}|100\rangle - i\sin\frac{\theta}{2}|110\rangle\right)$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2}|000\rangle + \cos\frac{\theta}{2}|011\rangle - i\sin\frac{\theta}{2}|100\rangle - i\sin\frac{\theta}{2}|111\rangle\right)$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2}|000\rangle + \cos\frac{\theta}{2}|011\rangle - i\sin\frac{\theta}{2}|110\rangle - i\sin\frac{\theta}{2}|101\rangle\right)$$

$$|\psi_5\rangle = \frac{1}{2}\left(\cos\frac{\theta}{2}|000\rangle + \cos\frac{\theta}{2}|011\rangle + \cos\frac{\theta}{2}|100\rangle + \cos\frac{\theta}{2}|111\rangle - i\sin\frac{\theta}{2}|010\rangle \right.$$

$$\left. - i\sin\frac{\theta}{2}|001\rangle + i\sin\frac{\theta}{2}|110\rangle + i\sin\frac{\theta}{2}|101\rangle\right)$$

$$= \frac{|00\rangle}{2} \otimes \left(\cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle\right) + \frac{|01\rangle}{2} \otimes \left(\cos\frac{\theta}{2}|1\rangle - i\sin\frac{\theta}{2}|0\rangle\right)$$

$$+ \frac{|10\rangle}{2} \otimes \left(\cos\frac{\theta}{2}|0\rangle + i\sin\frac{\theta}{2}|1\rangle\right) + \frac{|11\rangle}{2} \otimes \left(\cos\frac{\theta}{2}|1\rangle + i\sin\frac{\theta}{2}|0\rangle\right)$$

(b)

Let $|\psi\rangle$ be the third qubit after measurement.

If the result of measurement is 0,0, $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle$;

If the result of measurement is 0,1,$|\psi\rangle = \cos\frac{\theta}{2}|1\rangle - i\sin\frac{\theta}{2}|0\rangle$, after a correction of a X gate, it will be changed to $\cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle$;

If the result of measurement is 1,0,$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + i\sin\frac{\theta}{2}|1\rangle$, after a correction of a Z gate, it will be changed to $\cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle$;

If the result of measurement is 1,1,$|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + i\sin\frac{\theta}{2}|0\rangle$, after a correction of a X gate and a Z gate, it will be changed to $\cos\frac{\theta}{2}|0\rangle - i\sin\frac{\theta}{2}|1\rangle$.

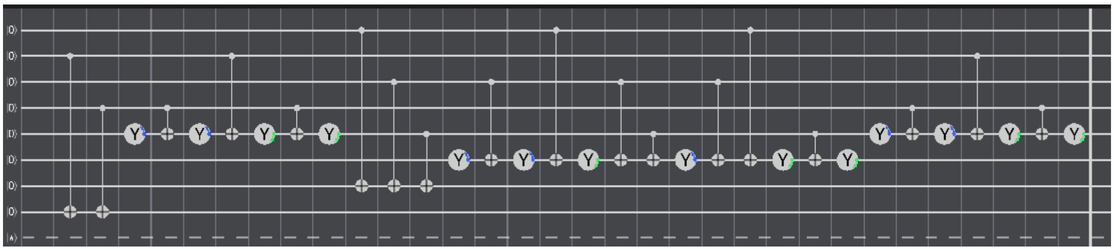
Therefore, after measurement and correction, the original state is teleported.

6.

The URL of circuit is:

https://qui.science.unimelb.edu.au/circuits/5d7c40c342584b0011d31c70.

When using the Toffoli decomposition, I have noticed that there have been some gates that can be eliminated. This is because a Y-rotation with $\theta_R = \frac{\pi}{4}$ and a Y-rotation with $\theta_R = -\frac{\pi}{4}$ is an identity, and two CNOT gates is also an identity. Therefore, the circuit can be simplified as:



7.

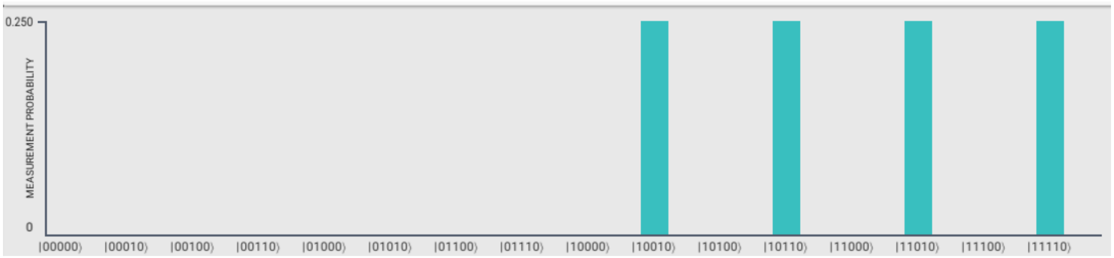
The URL of circuit is:

https://qui.science.unimelb.edu.au/circuits/5d7a17bf18b21a00ad5f0085.

In this circuit, I have used Grover's algorithm and amplitude amplification to ensure 100% probability of measuring the searching result. In this case, it is a multiple solution problem, and the number of solution M=4 (there are 4 values of x that satisfy f(x)=4, which are 2, 6, 10 and 14). As x contains 4 qubits, n=4 and N=16. Thus, $g_0 = \sqrt{\frac{M}{N}} = 0.5$. Let the number of iterations j=2, so $\theta' = \frac{\frac{\pi}{2}}{2j+1} = \frac{\pi}{10}$, and $g_0' = \sin\theta' = 0.3090$.

I use X-rotation in this circuit and set $\theta_g = \frac{\pi}{2}$, so $\theta_R = 2\sin^{-1}\left(\frac{g_0'}{g_0}\right) = 0.4241\pi$.

The result of this circuit can be seen as follows:



The value of x is the last 4 qubits. It is evident that the probability of measuring 2(0010), 6(0110), 10(1010) and 14(1110) are all 25%, while the probability of measuring the other values are all 0%, which means this circuit has achieved 100% probability of searching for the value of x.

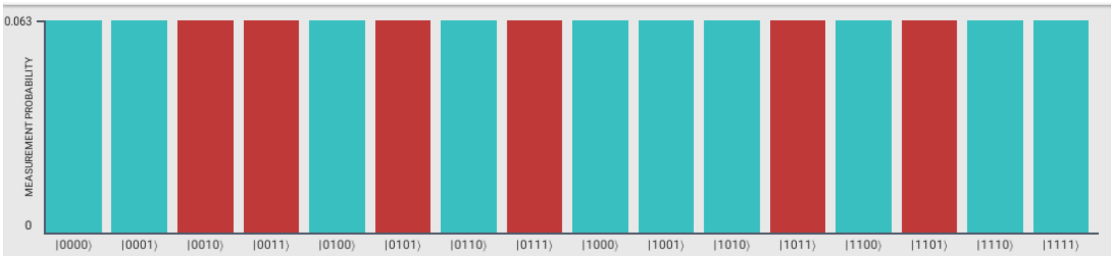
8.

(a)

The URL of circuit is:

https://qui.science.unimelb.edu.au/circuits/5d7a1f502b3f4e0053f2b3f8.

In this circuit, I have used some X gates and control-X gates to inverse the phase of prime numbers. The result of this circuit is shown below.



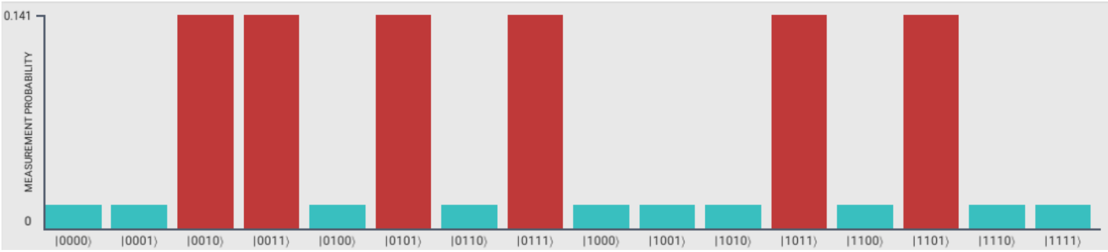
It is clear that the phase angle of prime numbers is π, and that of non-prime numbers is 0. Therefore, the prime numbers can be easily identified.

(b)

The URL of circuit is:

https://qui.science.unimelb.edu.au/circuits/5d7a20139042d1009582f69c.

In this circuit, I have implemented a single iteration of Grover's algorithm and the result is shown below.



From the result, we can see that the probability of measuring prime numbers is 84.6%.

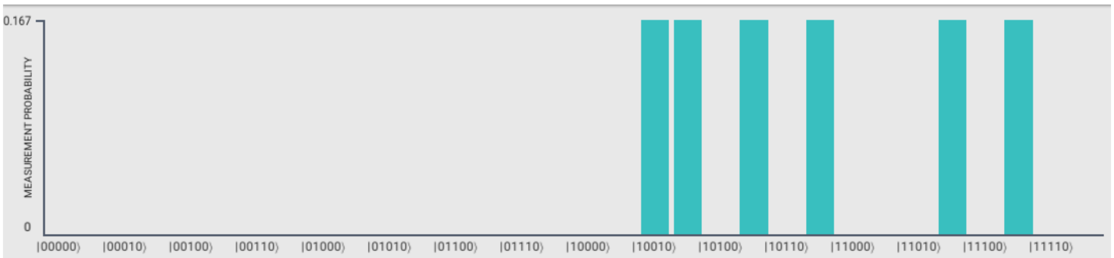
(c)

The URL of circuit is:

https://qui.science.unimelb.edu.au/circuits/5d7a256553630a0027222eff.

To get the 100% probability of measuring prime numbers, I used amplitude amplification in this circuit. In this case, it is a multiple solution problem, and the number of solution M=6 (there are 6 prime numbers below 16, which are 2, 3, 5, 7, 11 and 13). As x contains 4 qubits, n=4 and N=16. Thus, $g_0 = \sqrt{\frac{M}{N}} = \frac{\sqrt{6}}{4}$. Let the number of iterations j=2, so $\theta' = \frac{\frac{\pi}{2}}{2j+1} = \frac{\pi}{10}$, and $g_0' = \sin\theta' = 0.3090$. I use X-rotation in this circuit and set $\theta_g = \frac{\pi}{2}$, so $\theta_R = 2\sin^{-1}\left(\frac{g_0'}{g_0}\right) = 0.3367\pi$.

The result of this circuit can be seen as follows:



It is evident that the probability of measuring prime numbers has achieved 100%.

9.

(a)

$$U = \begin{bmatrix} \cos\frac{\pi}{3} & 0 & i\sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ i\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2}i \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & \frac{1}{2} \end{bmatrix}$$

$$U^{+}=\begin{bmatrix}\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2}i\\ 0 & 1 & 0\\ -\frac{\sqrt{3}}{2}i & 0 & \frac{1}{2}\end{bmatrix}$$

$$UU^{+}=\begin{bmatrix}\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}i\\ 0 & 1 & 0\\ \frac{\sqrt{3}}{2}i & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2}i\\ 0 & 1 & 0\\ -\frac{\sqrt{3}}{2}i & 0 & \frac{1}{2}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

So, the operation U is unitary and $UU^{+}=I$.

Let $|\psi\rangle = \mathrm{a}|0\rangle + \mathrm{b}|1\rangle + \mathrm{c}|2\rangle = \begin{bmatrix}a\\ b\\ c\end{bmatrix}$ as an arbitrary superposition over the qutrit basis states.

Then $\mathrm{U}|\psi\rangle = \begin{bmatrix}\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}i\\ 0 & 1 & 0\\ \frac{\sqrt{3}}{2}i & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix} = \begin{bmatrix}\frac{1}{2}a+\frac{\sqrt{3}}{2}ci\\ b\\ \frac{1}{2}c+\frac{\sqrt{3}}{2}ai\end{bmatrix} = \left(\frac{1}{2}a+\frac{\sqrt{3}}{2}ci\right)|0\rangle + b|1\rangle + \left(\frac{1}{2}c+\frac{\sqrt{3}}{2}ai\right)|2\rangle$

(b)

The URL of circuit is:

https://qui.science.unimelb.edu.au/circuits/5d7ae3ed8d1519008abd0672.

$\mathrm{U}|0\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}i|2\rangle$, $\mathrm{U}|1\rangle = |1\rangle$, and $\mathrm{U}|2\rangle = \frac{1}{2}|2\rangle + \frac{\sqrt{3}}{2}i|0\rangle$;

we can write it in binary form:

$\mathrm{U}|00\rangle = \frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}i|10\rangle = \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}i|1\rangle\right) \otimes |0\rangle$,

$\mathrm{U}|01\rangle = |01\rangle = |0\rangle \otimes |1\rangle$,

$\mathrm{U}|10\rangle = \frac{1}{2}|10\rangle + \frac{\sqrt{3}}{2}i|00\rangle = \left(\frac{1}{2}|1\rangle + \frac{\sqrt{3}}{2}i|0\rangle\right) \otimes |0\rangle$;

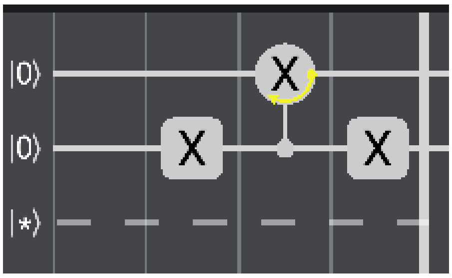
So, the operation is (1) keep the second qubit unchanged;

(2) if the second qubit is $|0\rangle$, change the first qubit (change $|0\rangle$ to $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}i|1\rangle$ and change $|1\rangle$ to $\frac{1}{2}|1\rangle + \frac{\sqrt{3}}{2}i|0\rangle$), otherwise keep the first qubit unchanged.

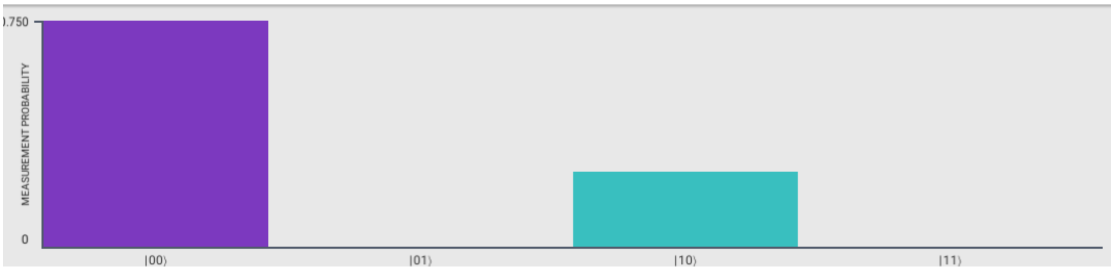
The operation of changes in first qubit can be written as: $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}$, which is a rotation around X-axis with its global phase $\theta_g = 0$, and $\theta_R = -\frac{2\pi}{3}$.

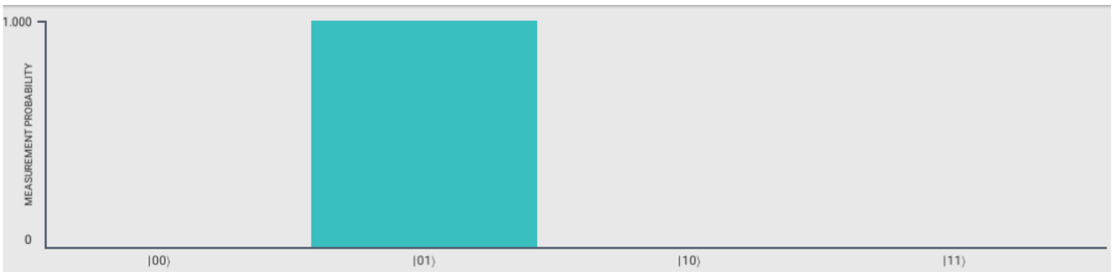
This is because

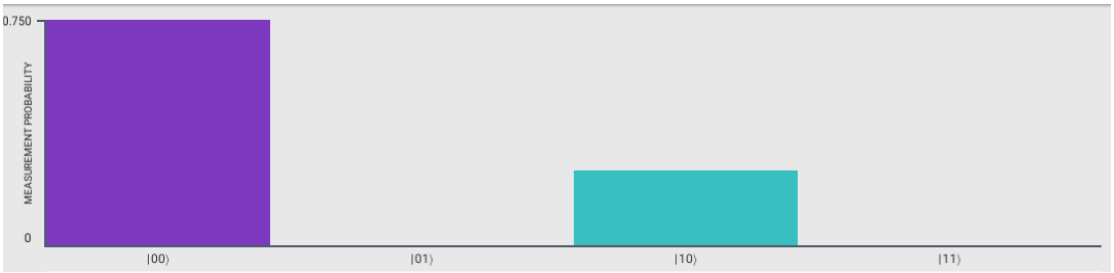
$R_X(\theta_R) = e^{i\theta_g}\left(I\cos\frac{\theta_R}{2} - iX\sin\frac{\theta_R}{2}\right) = e^{i\theta_g}\begin{bmatrix} \cos\frac{\theta_R}{2} & -i\sin\frac{\theta_R}{2} \\ -i\sin\frac{\theta_R}{2} & \cos\frac{\theta_R}{2} \end{bmatrix}$, and this matrix should equal $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}$. So, $\begin{cases} \frac{\theta_R}{2} = -\frac{\pi}{3} \\ \theta_g = 0 \end{cases} \Rightarrow \begin{cases} \theta_R = -\frac{2\pi}{3} \\ \theta_g = 0 \end{cases}$.

The circuit is 

, which only operates on the first qubit when the second qubit is $|0\rangle$. When the input is $|00\rangle$, the output is $\frac{1}{2}|00\rangle + \frac{\sqrt{3}}{2}i|10\rangle$; When the input is $|01\rangle$, the output is $|01\rangle$; When the input is $|10\rangle$, the output is $\frac{1}{2}|10\rangle + \frac{\sqrt{3}}{2}i|00\rangle$. The result can be shown below.






0.750
MEASUREMENT PROBABILITY
0
|00⟩
|01⟩
|10⟩
|11⟩