# Gradient tree-boosted mixture models and their applications in insurance loss prediction

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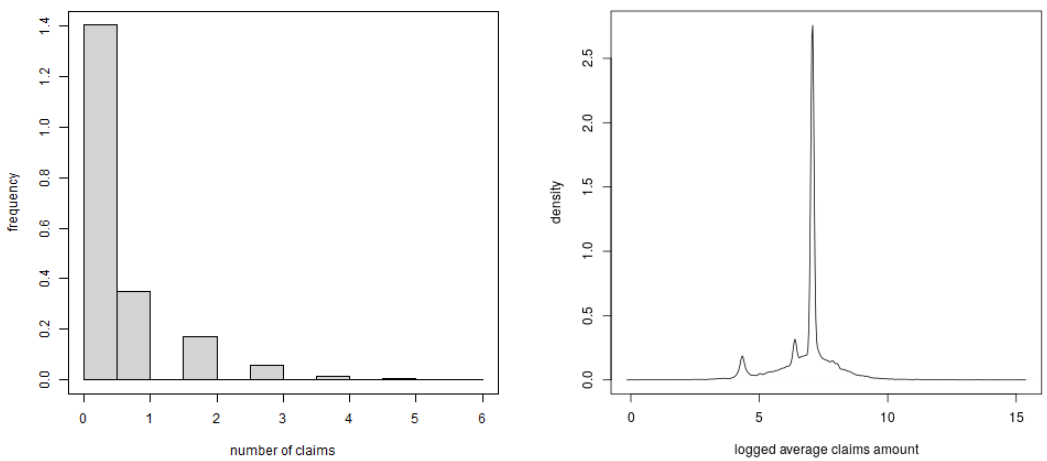
# Insurance loss

Insurance loss data sometimes cannot be well modeled by a single distribution.

- For claims counts data, there may be an excess of zero claims, so a Poisson distribution is not the best choice.
- For claims amount data, there may be a heavy tail, so a gamma distribution is not enough to describe the entire data.




One solution is to apply mixture of distributions.

# Insurance loss

- For claims counts data, we can utilize zero-modified Poisson distribution.
- For claims amount data, we can utilize mixture of gamma distribution and heavy-tailed distribution such as Pareto distribution.

When the individual risk factors are available, we can extend mixture of distributions to mixture of regressions to address risk heterogeneity in the portfolio.

# Challenges with mixture models

- Parameter estimation in mixture models is challenging: each component distribution/regression parameters are related to each other.
- Variable selection in mixture models is also challenging: we need to perform variable selection for each component regression.

# Mixture of distributions

Suppose a random variable $y$ follows the $k$-th component distribution from

$$\{f_1(y;\mu_1,\phi_1),\ldots,f_K(y;\mu_K,\phi_K)\}$$

with mixing probability $p_k, k = 1,\ldots,K$, then the probability density function for $y$ is

$$f(y) = \sum_{k=1}^{K} p_k f_k(y;\mu_k,\phi_k).$$

# Mixture of regressions

If individual features $\boldsymbol{x}_i$ are available and they have systematic effects on the distribution of $y_i$, then we establish a mixture of regressions:

$$f(y|\boldsymbol{x}) = \sum_{k=1}^{K} p_k(\boldsymbol{x}) f_k(y; \mu_k(\boldsymbol{x}), \phi_k(\boldsymbol{x})).$$

Above is the most general form of mixing. In applications, we always put some constraints on the mixing structure.

# Mixing structures

- Mixing probabilities are not related to $\boldsymbol{x}$

$$f(y) = \sum_{k=1}^{K} p_k f_k(y; \mu_k(\boldsymbol{x}), \phi_k(\boldsymbol{x})).$$

- Both mixing probabilities and dispersions are not related to $\boldsymbol{x}$

$$f(y) = \sum_{k=1}^{K} p_k f_k(y; \mu_k(\boldsymbol{x}), \phi_k),$$

# Mixing structures

- If component distributions are from the same distribution family, we might assume different component distributions have the same dispersion

$$f(y) = \sum_{k=1}^{K} p_k f_k(y; \mu_k(\boldsymbol{x}), \phi).$$

- Covariates $\boldsymbol{x}$ are only related to the mixing probabilities:

$$f(y) = \sum_{k=1}^{K} p_k(\boldsymbol{x}) f_k(y; \mu_k, \phi_k).$$

We need to determine the mixture structure according to the data and our aim. Imposing suitable constraints on mixture, we can accelerate model fitting without compromising predictive performance.

# Likelihood function given full information

Suppose we know which component distribution each sample is from. That is we know the full information $(Y, Z, \boldsymbol{x})$ where

$$Z = (Z_1, \ldots, Z_K)^\top = (\mathbb{1}_1(k), \ldots, \mathbb{1}_K(k))^\top$$

is the one-hot encoding of component indicator variable.
The joint distribution function for full information (one sample) is given by

$$f(y, z|\boldsymbol{x}) = \prod_{k=1}^{K} [p_k(\boldsymbol{x}) f_k(y; \mu_k(\boldsymbol{x}), \phi_k(\boldsymbol{x}))]^{z_k}$$

The log-likelihood function is given by

$$l(p, \mu, \phi|y, z, \boldsymbol{x}) = \sum_{k=1}^{K} z_k \left[\log p_k(\boldsymbol{x}) + \log f_k(y; \mu_k(\boldsymbol{x}), \phi_k(\boldsymbol{x}))\right] \qquad (1)$$

# Likelihood function given full information

The parameters in $p, \mu, \phi$ can be estimated by $K+1$ independent optimizations

$$\hat{\theta}_p = \arg\max_{\theta_p} \sum_{i=1}^{n} \sum_{k=1}^{K} z_{i,k} \log p_k(\boldsymbol{x}_i; \theta_p) \tag{2}$$

$$\left(\hat{\theta}_\mu^{(k)}, \hat{\theta}_\phi^{(k)}\right) = \arg\max_{\theta_\mu^{(k)}, \theta_\phi^{(k)}} \sum_{i=1}^{n} \sum_{k=1}^{K} z_{i,k} \log f_k\left(y_i; \mu_k\left(\boldsymbol{x}_i; \theta_\mu^{(k)}\right), \phi_k\left(\boldsymbol{x}_i; \theta_\phi^{(k)}\right)\right) \tag{3}$$

# Likelihood function given full information

Those optimizations are corresponding to a multinomial logistic classification and $K$ regressions.

The multinomial logistic classification are fitted to all samples, while $K$ regressions are fitted to partial samples with $\{i : z_{i,k} = 1\}$.

# Expectation step

With iterated $\hat{p}, \hat{\mu}, \hat{\phi}$, calculate the conditional expectation of $z$:

$$\hat{z}_{i,k} = \hat{z}_k(\boldsymbol{x}_i) = \frac{\hat{p}_{i,k} f_k(y_i; \hat{\mu}_{i,k}, \hat{\phi}_{i,k})}{\sum_{l=1}^{K} \hat{p}_{i,l} f_l(y_i; \hat{\mu}_{i,l}, \hat{\phi}_{i,l})},$$

where $\hat{p}_{i,k} = \hat{p}_k(\boldsymbol{x}_i), \hat{\mu}_{i,k} = \hat{\mu}_k(\boldsymbol{x}_i), \hat{\phi}_{i,k} = \hat{\phi}_k(\boldsymbol{x}_i)$.

# Maximization step

Based on the following likelihood function for full information, calculate the MLE of parameters.

$$
\begin{aligned}
& l(p, \mu, \phi | y, \hat{z}, \boldsymbol{x}) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{K} \hat{z}_{i,k} \left[ \log p_k(\boldsymbol{x}_i) + \log f_k(y_i; \mu_k(\boldsymbol{x}_i), \phi_k(\boldsymbol{x}_i)) \right] \\
&= \sum_{i=1}^{n} \sum_{k=1}^{K} \hat{z}_{i,k} \log p_k(\boldsymbol{x}_i) + \sum_{i=1}^{n} \sum_{k=1}^{K} \hat{z}_{i,k} \log f_k(y_i; \mu_k(\boldsymbol{x}_i), \phi_k(\boldsymbol{x}_i))
\end{aligned}
\tag{4}
$$

This step is similar to (2) and (3). However, now we have $\hat{z}_{i,k} \in (0, 1)$, so we need to consider

- A multinomial logistic classification with fractional response.
- $K$ weighted-regressions fitted to all samples.

# Our proposal

We replace the maximization step in the EM algorithm by a boosting step.

The boosting step increases the likelihood, but overfitting-sensitively.

The boosting step follows a generic functional gradient descent algorithm.

# General setting

Suppose non-parametric regression function as $F : \mathbb{R}^P \to \mathbb{R}$. Our purpose is to estimate $F$ to minimize the expected loss

$$\hat{F} = \arg\min_F \mathbb{E}\left[C(Y, F(\boldsymbol{x}))\right],$$

where $C : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is the loss function.

We always replace the expected loss by sample average loss:

$$\hat{F} = \arg\min_F \frac{1}{n} \sum_{i=1}^{n} C(y_i, F(\boldsymbol{x}_i))$$

# Link functions and loss functions

Commonly used link function and loss functions:

- AdaBoost:

$$C(y, F) = \exp(yF), \quad y \in \{-1, 1\}$$

$$F(\boldsymbol{x}) = \frac{1}{2} \log \left( \frac{\Pr[Y = 1|\boldsymbol{x}]}{\Pr[Y = -1|\boldsymbol{x}]} \right)$$

- LogitBoost:

$$C(y, F) = \log_2(1 + \exp(-2yF)), \quad y \in \{-1, 1\}$$

$$F(\boldsymbol{x}) = \frac{1}{2} \log \left( \frac{\Pr[Y = 1|\boldsymbol{x}]}{\Pr[Y = -1|\boldsymbol{x}]} \right)$$

- $L_2$Boost:

$$C(y, F) = (y - F)^2/2, \quad y \in \mathbb{R}$$

$$F(\boldsymbol{x}) = \mathbb{E}(Y|\boldsymbol{x})$$

# Loss function for insurance data

We choose the negative log-likelihood function as the loss function. Hence, minimizing the loss function is equivalent to maximizing the likelihood function.

# Types of boosting algorithms

Commonly used boosting algorithms:

1. Binary classification: AdaBoost, LogitBoost (real, discrete, gentle AdaBoost), AdaBoost.M1
2. Multinomial classification: Stagewise Additive Modeling using a Multi-class Exponential loss (SAMME), SAMME.R (multi-class real AdaBoost),
3. Gradient based: gradient boosting machine/model (GBM), Newton boosting, gradient boosting decision tree (GBDT), eXtreme Gradient Boosting (XGBoost), light gradient boosting machine (LightGBM)

The last type is based on gradient of loss function, and our EB algorithm uses this version of boosting.

# Generic functional gradient descent algorithm

1. Initialization: $\hat{F}^{[0]}(\boldsymbol{x}; \hat{\theta}^{[0]})$, where $\hat{\theta}^{[0]}$ are determined by $(y_i, \boldsymbol{x}_i)_{i=1:n}$. Let $m = 0$.
2. Projection of gradient to weak learner: Calculate negative gradient

$$u_i = -\frac{\partial C(y_i, F)}{\partial F}\bigg|_{F=\hat{F}^{[m]}(\boldsymbol{x}_i)}, i = 1, \ldots, n.$$

Data $(u_i, \boldsymbol{x}_i)_{i=1:n}$ is used to calibrate a weak learner $\hat{f}^{[m+1]}(\boldsymbol{x}; \hat{\theta}^{[m+1]})$ with loss function $L_2$.

# Generic functional gradient descent algorithm

3. One-dimensional optimization: Solve an one-dimensional optimization to find expansion coefficient $\hat{\omega}^{[m+1]}$:

$$\hat{\omega}^{[m+1]} = \arg\min_{\omega} \sum_{i=1}^{n} C(y_i, \hat{F}^{[m]}(\boldsymbol{x}_i) + \omega \hat{f}^{[m+1]}(\boldsymbol{x}_i))$$

   Update

$$\hat{F}^{[m+1]} = \hat{F}^{[m]} + s\hat{\omega}^{[m+1]} \hat{f}^{[m+1]},$$

   where $s$ is shrinkage factor or learning rate.

4. Iteration: Let $m$ increase by 1, and repeat steps 2-3.

# Generic functional gradient descent algorithm

- Weak learners are fitted to negative gradient $U$ rather than $Y$.
- Loss function in weak learners is always $L_2$, independently with model loss function $C$.
- If weak learners are trees, the algorithm is called gradient boosting decision tree (GBDT).
- If weak learners are trees, calibration and variable selection are performed simultaneously.
- Step 3 of one-dimensional optimization seems unnecessary given learning rate is sufficiently small according to some empirical experiments.

# Preparation

- Notations: Regression functions for $p, \mu, \phi$ are denoted by $F, G, H$.
- Key points: Link functions, cost/loss functions, negative gradient, .

# Multiple logistic link function for mixing probabilities

$$p_k(\boldsymbol{x}) = \Pr(Z_k = 1|\boldsymbol{x}) = \frac{\exp F_k(\boldsymbol{x})}{\sum_{l=1}^{K} \exp F_l(\boldsymbol{x})}, \tag{5}$$

or equivalently

$$F_k(\boldsymbol{x}) = \log p_k(\boldsymbol{x}) - \frac{1}{K} \sum_{l=1}^{K} \log p_l(\boldsymbol{x}), \quad k = 1, \ldots, K. \tag{6}$$

# Link functions in component models

- Component Gaussian model:

$$\mu_k(\boldsymbol{x}) = G_k(\boldsymbol{x})$$

$$\phi_k(\boldsymbol{x}) = \exp H_k(\boldsymbol{x})$$

- Component Poisson model:

$$\mu_k(\boldsymbol{x}) = \exp G_k(\boldsymbol{x})$$

- Component gamma model:

$$\mu_k(\boldsymbol{x}) = \exp G_k(\boldsymbol{x})$$

$$\phi_k(\boldsymbol{x}) = \exp H_k(\boldsymbol{x})$$

# Negative gradient for mixing probabilities

We use the negative log-likelihood function as the cost function

$$C_Z(Z, F(\boldsymbol{x})) = -\sum_{k=1}^{K} Z_k \log p_k(\boldsymbol{x}). \tag{7}$$

The negative gradient of the cost function w.r.t. $F_k$ is given by

$$U_k(Z, F(\boldsymbol{x})) = -\frac{\partial C_Z(Z, F(\boldsymbol{x}))}{\partial F_k(\boldsymbol{x})} = Z_k - p_k(\boldsymbol{x}), \quad k = 1, \ldots, K \tag{8}$$

# Negative gradient for component models

Similarly, we use negative log-likelihood function of each component model as its cost function:

$$C_k(Y, Z, G(\boldsymbol{x})) = -\, Z_k \log f_k(Y; \mu_k(G_k(\boldsymbol{x})), \phi_k),\ k = 1 : K. \tag{9}$$

The negative gradient of $C_k$ w.r.t. $G_k$ is denoted by $V_k(Y, Z, G(\boldsymbol{x}))$

Note that we have assumed that dispersion $\phi_k$ is fixed among samples (not related to $\boldsymbol{x}$).

# Overview of EB algorithm

Step 1: Initialization of EB algorithm $\hat{p}^{[0]}, \hat{\mu}^{[0]}, \hat{\phi}^{[0]}$. Set $t = 0$.

Step 2: Calculating conditional expectation of latent variable $\hat{z}^{[t]}$ given $\hat{p}^{[t]}, \hat{\mu}^{[t]}, \hat{\phi}^{[t]}$.

Step 3: Gradient boosting mixing probabilities $\hat{p}^{[t+1]}$ and component models $\hat{\mu}^{[t+1]}, \hat{\phi}^{[t+1]}$ given latent variable $\hat{z}^{[t]}$.

Step 4: Increase $t$ by 1. Repeat Steps 2-3 until $t$ reaches to $T$.

# Step 1: Initialization of EB algorithm

1. Initialize $\hat{p}_1^{[0]}, \ldots, \hat{p}_K^{[0]}$ and $\hat{F}_1^{[0]}, \ldots, \hat{F}_K^{[0]}$:
   - $\hat{p}_k^{[0]} = \frac{1}{K}$
   - $\hat{F}_1^{[0]}, \ldots, \hat{F}_K^{[0]}$ are obtained by (6).
2. Initialize $\hat{\mu}_1^{[0]}, \ldots, \hat{\mu}_K^{[0]}$ and $\hat{G}_1^{[0]}, \ldots, \hat{G}_K^{[0]}$:
   - $\hat{\mu}_k^{[0]} = \frac{\sum_{i=1}^n Y_i}{n}$
   - $\hat{G}_k^{[0]} = \mu_k^{-1}(\hat{\mu}_k^{[0]})$
3. Initialize $\hat{\phi}_1^{[0]}, \ldots, \hat{\phi}_K^{[0]}$ as the sample variance. (Assume that dispersion is independent with covariates.)
4. Set $t = 0$.

# Step 2: Conditional expectation of latent variable

Set $\hat{z}_{i,k}^{[t]}$ as

$$\hat{z}_{i,k}^{[t]} = \frac{\hat{p}_k^{[t]}(\boldsymbol{x}_i) f_k\left(y_i; \hat{\mu}_k^{[t]}(\boldsymbol{x}_i), \phi_k^{[t]}\right)}{\sum_{l=1}^{K} \hat{p}_l^{[t]}(\boldsymbol{x}_i) f_l\left(y_i; \hat{\mu}_l^{[t]}(\boldsymbol{x}_i), \phi_l^{[t]}\right)}, \quad k = 1 : K.$$

# Step 3.1: Gradient boosting mixing probabilities

1. Initialization. Set $\hat{p}_1^{[t,0]}, \ldots, \hat{p}_K^{[t,0]}$ and $\hat{F}_1^{[t,0]}, \ldots, \hat{F}_K^{[t,0]}$ as

$$\hat{p}_k^{[t,0]} = \frac{1}{K}, \quad \hat{F}_k(\boldsymbol{x})^{[t,0]} = \log \hat{p}_k^{[t,0]} - \frac{1}{K} \sum_{l=1}^{K} \log \hat{p}_l^{[t,0]}$$

   Set $m = 0$.

2. Projection of gradient to learner. Compute the negative gradient sample $u_{1,k}^{[t,m]}, \ldots, u_{n,k}^{[t,m]}$, in which

$$u_{i,k}^{[t,m]} = U_k(\hat{z}_i^{[m]}, \hat{F}^{[t,m]}(\boldsymbol{x}_i)).$$

   Then the data $(u_{i,k}^{[t,m]}, \boldsymbol{x}_i)_{i=1:n}$ is used to calibrate a $L$-terminal node regression trees $\hat{f}_k^{[t,m+1]}\left(\boldsymbol{x}; R_{l=1:L}^{[t,m+1]}, \bar{u}_{l=1:L}^{[t,m+1]}\right)$ with $L_2$ loss, where $R_{l=1:L}^{[t,m+1]}$ is the partition of covariate space and $\bar{u}_{l=1:L}^{[t,m+1]}$ contains the average gradient in each terminal node.

   $K$ regression trees are fitted independently.

# Step 3.1: Gradient boosting mixing probabilities

3. The one-dimensional optimization for expansion coefficient leads to the following update (Friedman, 2001):

$$\hat{F}_k^{[t,m+1]}(\boldsymbol{x}_i) = \hat{F}_k^{[t,m]}(\boldsymbol{x}_i) + s \sum_{l=1}^{L} \gamma_l^{[t,m+1]} \mathbb{1}(\boldsymbol{x} \in R_l^{[t,m+1]}),\ k = 1, \ldots, K$$

where

$$\gamma_l^{[t,m+1]} = \frac{K-1}{K} \frac{\sum_{\boldsymbol{x}_i \in R_l^{[t,m+1]}} u_{i,k}^{[t,m]}}{\sum_{\boldsymbol{x}_i \in R_l^{[t,m+1]}} |u_{i,k}^{[t,m]}|(1 - |u_{i,k}^{[t,m]}|)}$$

We then back updated mixing probability $\hat{p}_{k=1:K}^{[t,m+1]}$ using (5).
We replace $\bar{u}_{l=1:L}^{[t,m+1]}$ by $\gamma_{l=1:L}^{[t,m+1]}$.

# Step 3.2: Gradient boosting component models

1. Initialization. Set $\hat{\mu}_1^{[t,0]}, \ldots, \hat{\mu}_K^{[t,0]}$ and $\hat{G}_1^{[t,0]}, \ldots, \hat{G}_K^{[t,0]}$:

$$\hat{\mu}_k^{[t,0]} = \frac{\sum_{i=1}^n Y_i}{n}, \hat{G}_k^{[t,0]} = \mu_k^{-1}(\hat{\mu}_k^{[t,0]})$$

   Set $m = 0$.

2. Projection of gradient to learner. Compute the negative gradient samples $v_{1,k}^{[t,m]}, \ldots, v_{n,k}^{[t,m]}$, in which

$$v_{i,k}^{[t,m]} = V_k(y_i, \hat{z}_i, \hat{G}^{[t,m]}(\boldsymbol{x}_i)),$$

   Then the data $(v_{i,k}^{[t,m]}, \boldsymbol{x}_i)_{i=1:n}$ is used to calibrate a $L$-terminal node regression trees $\hat{g}_k^{[t,m+1]}\left(\boldsymbol{x}; S_{l=1:L}^{[t,m+1]}, \bar{v}_{l=1:L}^{[t,m+1]}\right)$ with $L_2$ loss, where $S_{l=1:L}^{[t,m+1]}$ is the partition of covariate space and $\bar{v}_{l=1:L}^{[t,m+1]}$ contains the average gradient in each terminal node.
   *K* regression trees are fitted independently.

# Step 3.2: Gradient boosting component models

3. Conduct the following $K$ independent one-dimensional optimizations to find the best expansion coefficients.

$$\hat{w}_k^{[t,m+1]} = \arg\min_w \sum_{i=1}^{n} C_k(y_i, \hat{z}_i, \hat{G}_k^{[t,m]}(\boldsymbol{x}_i) + w\hat{g}_k^{[t,m+1]}(\boldsymbol{x}_i)).$$

4. Compute the updates

$$\hat{G}_k^{[t,m+1]}(\boldsymbol{x}_i) = \hat{G}_k^{[t,m]}(\boldsymbol{x}_i) + s\hat{w}_k^{[t,m+1]}\hat{g}_k^{[t,m+1]}(\boldsymbol{x}_i),$$

$$\hat{\mu}_k^{[t,m+1]}(\boldsymbol{x}_i) = \mu_k(\hat{G}_k^{[t,m+1]}(\boldsymbol{x}_i)).$$

# Step 3.3: Boosting iteration

Increase $m$ by 1, and repeat steps 3.1-3.2 until $m$ reaches to the pre-determined $M$. Set

$$\hat{F}_k^{[t+1]} = \hat{F}_k^{[t,M]}, \quad \hat{p}_k^{[t+1]} = \hat{p}_k^{[t,M]},$$

$$\hat{G}_k^{[t+1]} = \hat{G}_k^{[t,M]}, \quad \hat{\mu}_k^{[t+1]} = \hat{\mu}_k^{[t,M]}.$$

# Step 4: EB iteration

1. For parameters not related to covariate $\boldsymbol{x}$, compute the MLE $\hat{\phi}_k^{[t+1]}$ given all the other parameters $\hat{p}_k^{[t+1]}, \hat{\mu}_k^{[t+1]}$.
2. Increase $t$ by 1, and back to step 2 until $t$ reaches to the pre-determined $T$.

# Independent boosting v.s. forward boosting

- Independent boosting: In initialization of boosting, we initialize parameters independently with the previously boosting estimates $\hat{p}_k^{[t-1]}, \hat{\mu}_k^{[t-1]}$.
- Forward boosting: In contrast, we might initialize parameters as the previously boosting estimates $\hat{p}_k^{[t-1]}, \hat{\mu}_k^{[t-1]}$. This would lead a smaller required boosting iterations $M$ (or a earlier stop on validation loss).

# Independent boosting v.s. forward boosting

Issues with forward boosting: difficult to predict for new data due to the iterative initialization.

An additional boosting with default initialization needs to be conducted given the last expected hidden variables $\hat{z}_i^{[T]}$.

# Tuning parameters

- Number of EB iterations $T$. Determined by the trace plot of loss.
- Number of iterations in boosting $M$. We can specify a sufficiently large $M$ and use early stop according to validation loss.
- Learning rate $s$. Smaller learning rate tends to lead to a better fitting and predictive performance but with more iterations.
- Tuning parameters in base learner tree: complexity parameter, maximum depth, number of terminal nodes.

# Underlying model

$$Y_i = Z_i Y_{i,1} + (1 - Z_i) Y_{i,2}$$

$$Y_{i,1} | \boldsymbol{x}_i \sim N(\mu_1(\boldsymbol{x}_i), 0.9^2)$$

$$Y_{i,2} | \boldsymbol{x}_i \sim N(\mu_2(\boldsymbol{x}_i), 0.5^2)$$

$$Z_i \sim Bernuli(p(\boldsymbol{x}_i))$$

$$\boldsymbol{x}_i = (x_{i,1}, x_{i,2}, x_{i,3})^\top$$

# Underlying model

$$\mu_1(\boldsymbol{x}_i) = 2 + 2 \times x_{i,1} + \log x_{i,2}$$

$$\mu_2(\boldsymbol{x}_i) = 0.5 - 0.5 \times x_{i,1} + x_{i,3} - x_{i,1} \times x_{i,2}$$

$$p(\boldsymbol{x}_i) = \frac{\exp \eta(\boldsymbol{x}_i)}{1 + \exp \eta(\boldsymbol{x}_i)}$$

$$\eta(\boldsymbol{x}_i) = 1 + 0.1 \times x_1 + \log x_2 + x_3 \times x_1$$

$$x_{1,i} \sim N(2, 1^2)$$

$$x_{i,2} \sim Exp(2)$$

$$x_{i,3} \sim Bernuli(0.5)$$

$$n = 5,000$$

# Data distribution

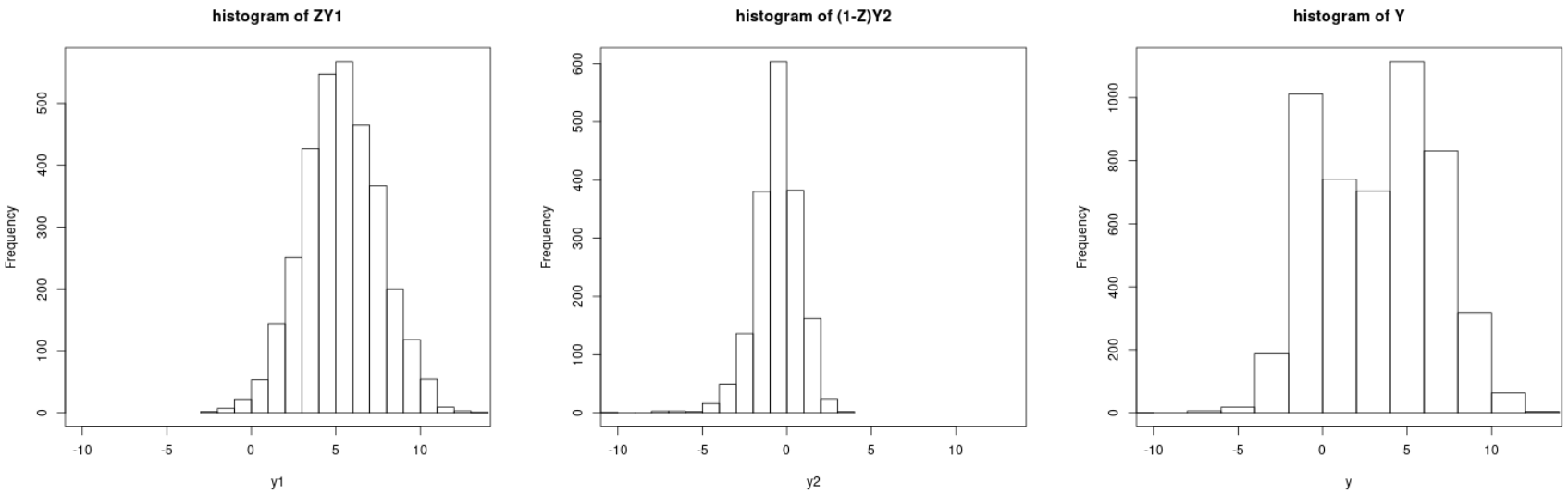



Figure 1: Distribution of $ZY$, $(1-Z)Y$, $Y$.

# Models fitted

Mixture of distributions (homo):

$$f(y; p, \mu, \sigma) = p_1 f_N(y; \mu_1, \sigma_1^2) + (1 - p_1) f_N(y; \mu_2, \sigma_2^2)$$

Mixture of linear regressions or mixture of boosting with constant mixing probabilities (glm_cp or boosting_cp)

$$f(y; p, \mu, \sigma) = p_1 f_N(y; \mu_1(\boldsymbol{x}), \sigma_1^2) + (1 - p_1) f_N(y; \mu_2(\boldsymbol{x}), \sigma_2^2)$$

Mixture of linear regressions or mixture of boosting with varying mixing probabilities (glm_vp or boosting_vp)

$$f(y; p, \mu, \sigma) = p_1(\boldsymbol{x}) f_N(y; \mu_1(\boldsymbol{x}), \sigma_1^2) + (1 - p_1(\boldsymbol{x})) f_N(y; \mu_2(\boldsymbol{x}), \sigma_2^2)$$

# Results and comparisons

$$e_{\mu 1} = \frac{\sum_{i=1}^{n}(\mu_{i,1} - \hat{\mu}_{i,1})^2}{n}$$

$$e_{\mu 2} = \frac{\sum_{i=1}^{n}(\mu_{i,2} - \hat{\mu}_{i,2})^2}{n}$$

$$e_{\eta} = \frac{\sum_{i=1}^{n}(\eta_i - \hat{\eta}_i)^2}{n}, \eta_i = \log\frac{p_i}{1-p_i}$$

Table 1: Summary of test losses for different models.

| model | neg LL | $e_{\mu 1}$ | $e_{\mu 2}$ | $e_{\eta}$ |
|---|---|---|---|---|
| homo | 2.5868 | 6.2657 | 2.6688 | 3.3304 |
| glm_cp | 1.8978 | 0.7775 | 0.3233 | 3.2440 |
| glm_vp | 1.7698 | 0.8028 | 0.3260 | 1.2667 |
| boosting_cp | 1.7492 | 0.2461 | **0.1062** | 3.2242 |
| boosting_vp | **1.6407** | **0.2334** | 0.1105 | **0.8435** |

# Distribution of average claim amount

Claims amount data `freMTPL2sev` from R package **CASdatasets**. Sample size $n = 24,938$. Three peaks and heavy tail.

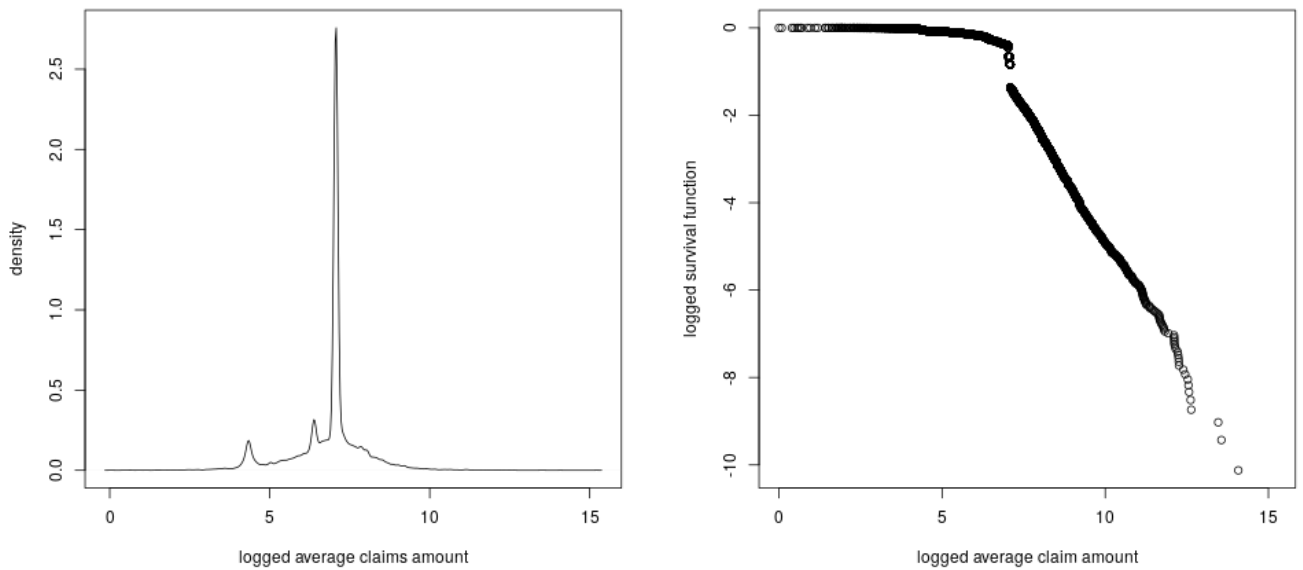



Figure 2: Histogram and logged survival function of logged average claims.

# Mixture of distributions

Three gamma distributions for three peaks. One gamma distribution for resting non-tail part. One Pareto distribution for tail.

$$f(y) = \sum_{k=1}^{4} p_k f_{gamma}(y; \mu_k, \phi_k) + p_5 f_{pareto}(y; \alpha, M)$$

where $\mu, \phi$ are mean and dispersion parameter, and $\alpha, M$ are tail index and threshold. The threshold is pre-determined as 8158.13 according to Hill plot.

# Initialization of hidden variable

Figure 2 indicates a way to initialize the hidden variable:

$$\begin{aligned}\hat{z}_i^{[0]} &= (\hat{z}_{i,1}^{[0]}, \hat{z}_{i,2}^{[0]}, \hat{z}_{i,3}^{[0]}, \hat{z}_{i,4}^{[0]}, \hat{z}_{i,5}^{[0]})^\top \\ &= (\mathbb{1}_{(0,500]} y_i, \mathbb{1}_{(500,1000]} y_i, \mathbb{1}_{(1000,1200]} y_i, \mathbb{1}_{(1200,8158.13]} y_i, \mathbb{1}_{(8158.13,\infty)} y_i)^\top \end{aligned} \tag{10}$$

Other parameters can be initialized as the MLE based on the full likelihood function.

# EM algorithm

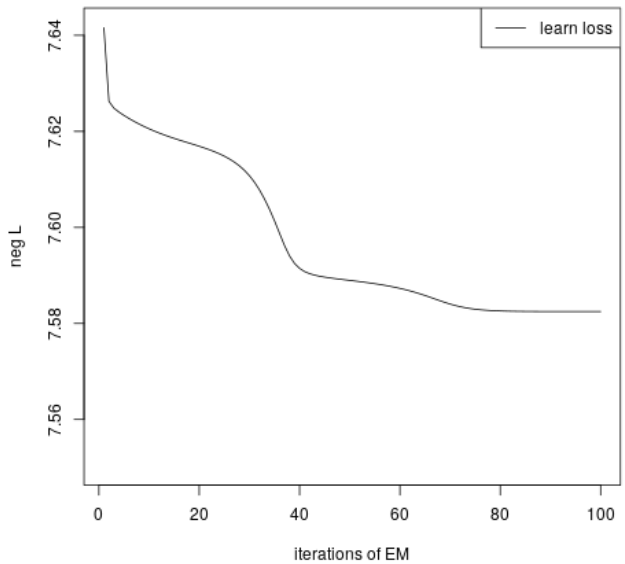



Figure 3: Learning loss of mixture of distributions.

# Estimated parameters

Table 2: MLE of four component gamma distributions. Tail index is estimated as $\hat{\alpha} = 1.0773$

| component $k$ | $\mu_k$ | shape $(1/\phi_k)$ | scale | rate |
|---|---|---|---|---|
| 1 | 76.8727 | 105.556 | 0.7283 | 1.3731 |
| 2 | 592.5909 | 653.539 | 0.9067 | 1.1029 |
| 3 | 1171.3811 | 999.9999 | 1.1714 | 0.8537 |
| 4 | 1534.5143 | 1.0377 | 1478.7768 | 7e-04 |

Those large shape parameters (small dispersion) implies the difficulties with gamma mean modeling.

# Boosting mixing probabilities

$$f(y|\boldsymbol{x}) = \sum_{k=1}^{4} p_k(\boldsymbol{x}) f_{gamma}(y; \mu_k, \phi_k) + p_5(\boldsymbol{x}) f_{pareto}(y; \alpha, M)$$

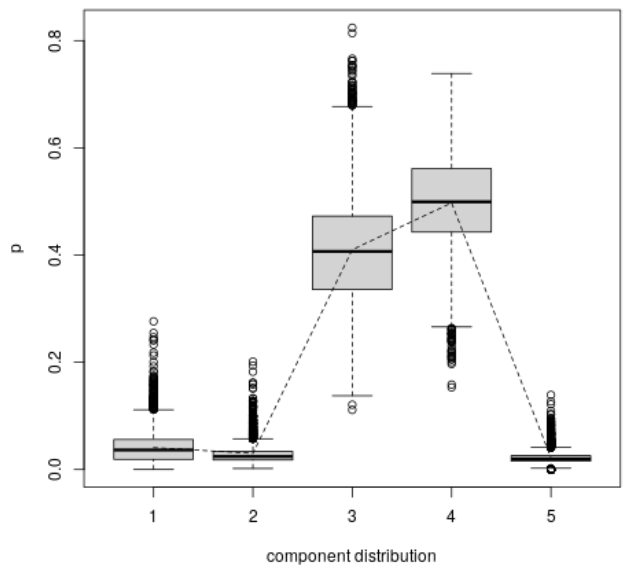



Figure 4: Boxplot of estimated mixing probabilities.

# Boosting the forth component mean

$$f(y|\boldsymbol{x}) = \sum_{k=1}^{3} p_k(\boldsymbol{x}) f_{gamma}(y; \mu_k, \phi_k) +$$
$$p_4(\boldsymbol{x}) f_{gamma}(y; \mu_4(\boldsymbol{x}), \phi_4) + p_5(\boldsymbol{x}) f_{pareto}(y; \alpha, M)$$

Test loss (negative log-likelihood)

1. Mixture of distributions: 7.5815
2. Boosting mixing probabilities: **7.5588**
3. Boosting mixing probabilities and the forth component mean: 7.5573.

# Our proposal: Expectation-Boosting algorithm

Expectation-Boosting (EB) algorithm:

- Replaces the maximization step by an overfitting-sensitive boosting step.
- The boosting step follows a generic functional gradient descent algorithm.

# Advantages

Several advantages of EB algorithm over the EM algorithm.

- No need for specifying the form of component regression functions and performing covariate transformation.
- Only need for component loss functions.
- Boosting algorithm is a flexible non-parametric regression facilitating both non-linear effects and interaction.
- Boosting algorithm is overfitting-sensitive, we can perform variable selection simultaneously during the EB algorithm.

Thank you!

Q & A