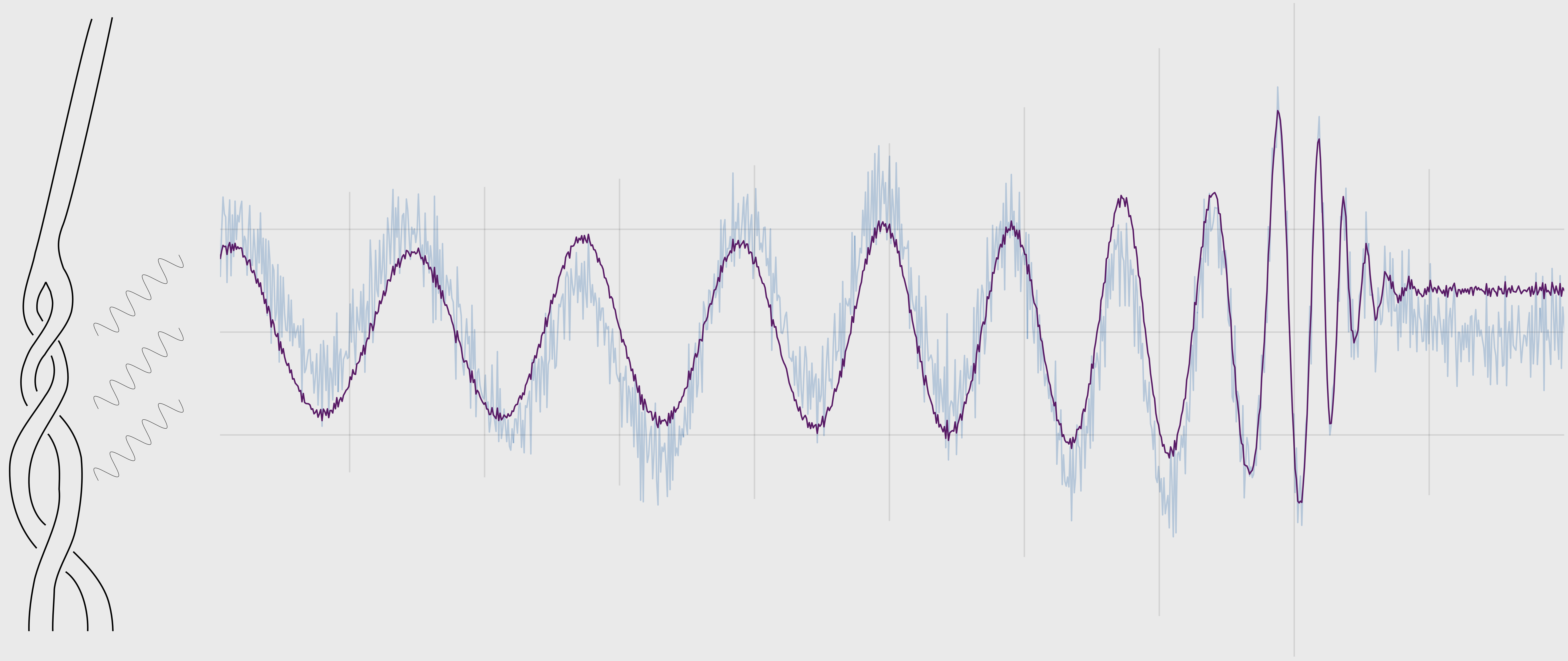
Introduction to Cauchy-characteristic Evolution (CCE)



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SXSCon, August 7th, 2024

Caltech

SIMULATING eXTREME SPACETIMES

Black holes, neutron stars, and beyond...

Questions about Astrophysics and Gravity

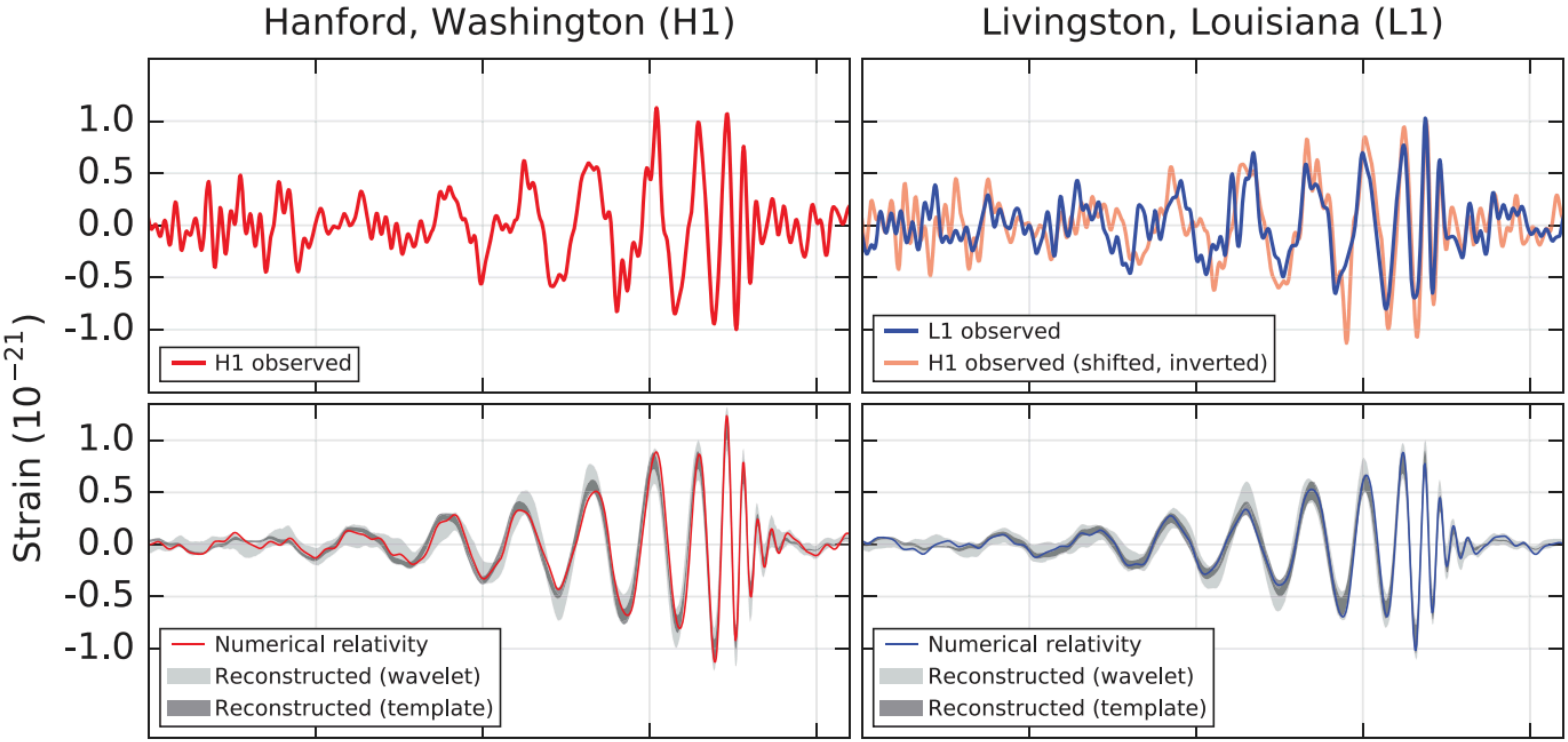
Astrophysics:

- ➡ Black Hole / Neutron Star / Merger Rates and Formation Channels
- ➡ Galaxy Merger Rate (via low frequency GWs from SMBHs)
- ➡ Neutron Star Equation of State
- ➡ Mechanism of Supernovae
- ➡ Dark Matter / Exotica

Gravity

- ➡ Is it GR?
- ➡ Is it Quantum?
- ➡ Black Holes
  - singularities, no-hair theorem, information paradox
- ➡ Origin of Dark Energy / Hubble Constant Tension
- ➡ Inflation (via stochastic background)

Learning about the Universe with Gravitational Wave Observations

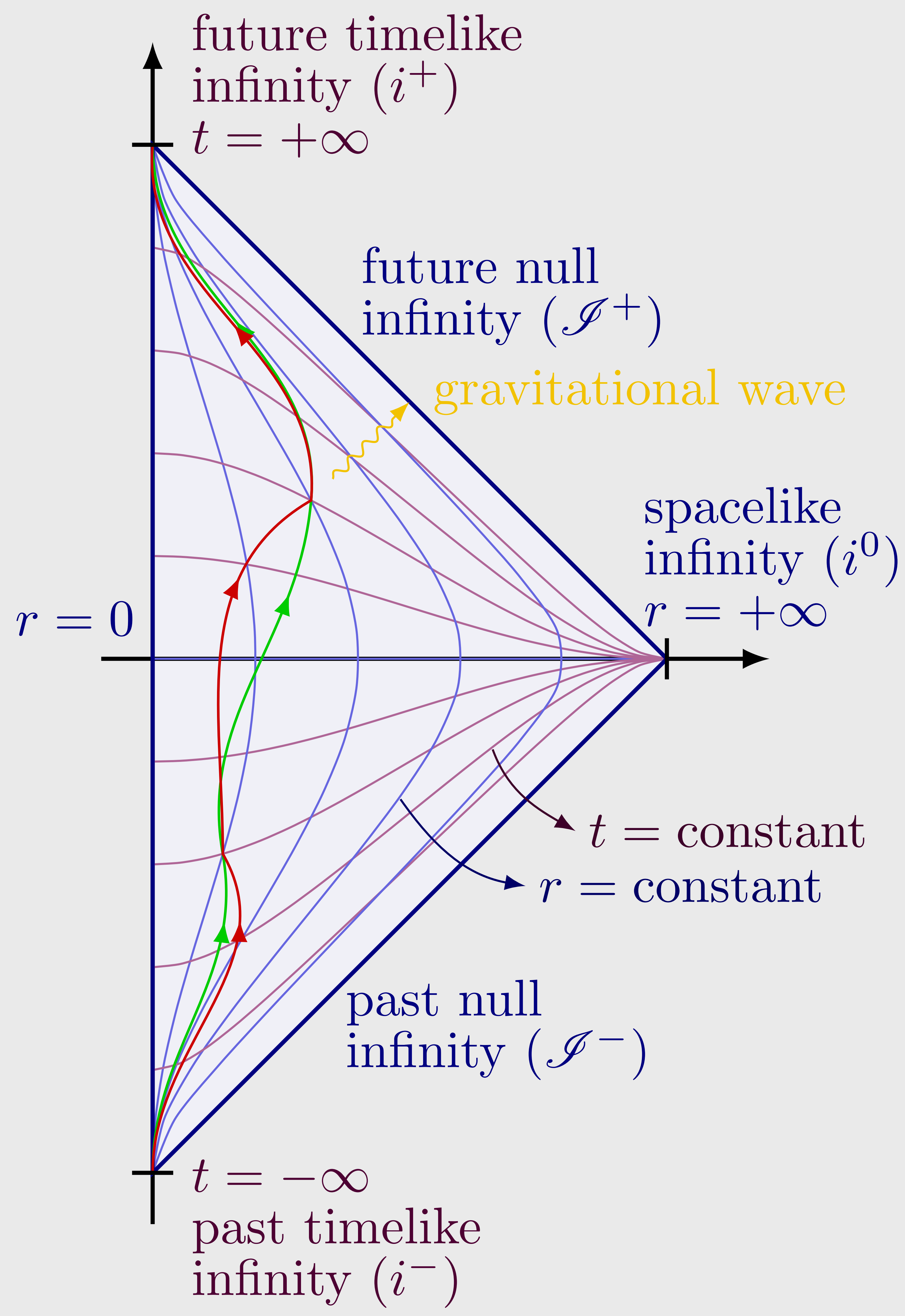


Cauchy Evolution

- Evolve Einstein's equations $R_{\mu\nu} - g_{\mu\nu}R/2 = 0$ using a Cauchy slice foliation over some finite spacetime volume

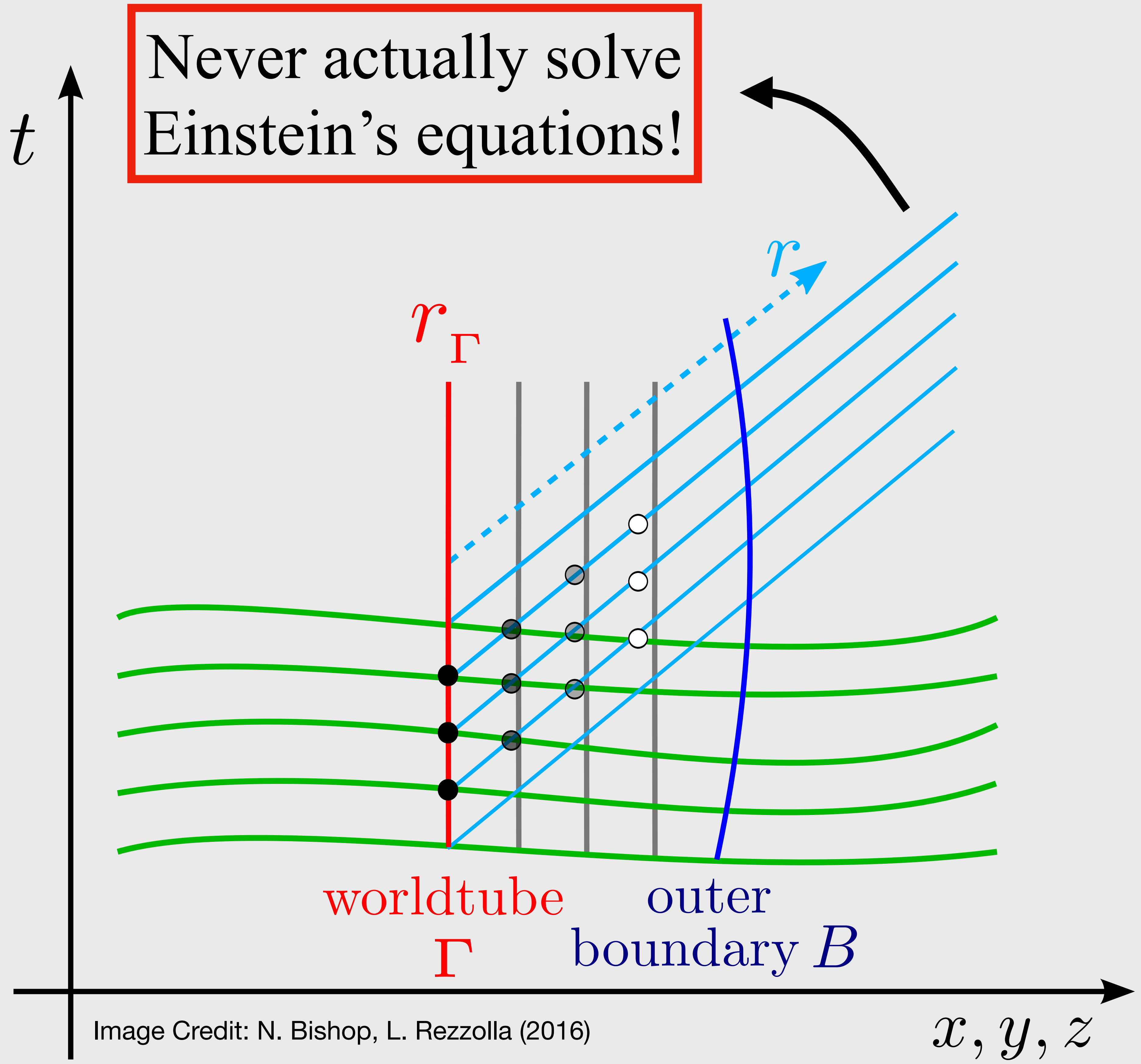
Extracting Gravitational Waves

- Can compute the correction to the Minkowski space metric at any position without the simulated volume… not generalizable!
- $$ds^2 = -Ue^{2\beta}du^2 - 2e^{2\beta}du\,dr + r^2\gamma_{AB}\left(d\theta^A - \mathcal{U}^A du\right)\left(d\theta^B - \mathcal{U}^B du\right);$$
  $$\gamma_{AB} = h_{AB} + \frac{1}{r}C_{AB} + \frac{1}{r^2}D_{AB} + \cdots$$
  "Gravitational wave strain"
  "$\mathcal{O}(10^{-20})$ correction (for LVK events)"
- Compute the waveform at future null infinity instead! Generalizable!



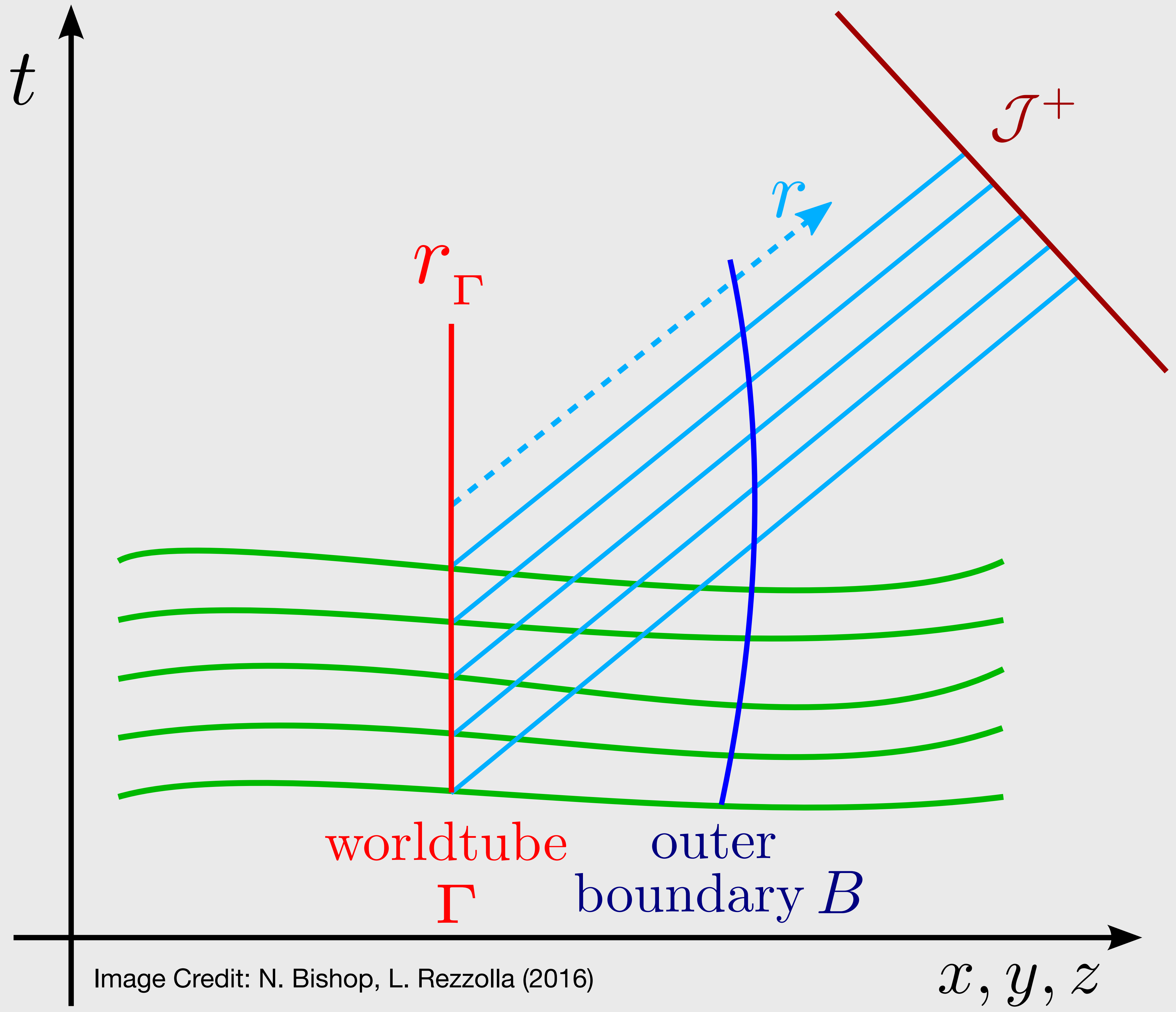
Extracting Waveforms in Numerical Relativity: Extrapolation

- **Extrapolation** (see, e.g., Boyle *et al.*, arXiv:1904.04831)
  - Want the waveform at "future null infinity", i.e., large $r \to \infty$ limit, i.e., a spin-weight -2 complex scalar
$$h(u,\theta,\phi) = \frac{1}{2}\bar{q}^A\bar{q}^B C_{AB} \quad \left[q^A = -(1, i\sin\theta)\right]$$
$$= \sum_{\ell>2,|m|\leq\ell} h_{(\ell,m)}(u)\,{}_{-2}Y_{(\ell,m)}(\theta,\phi)$$
  - Compute the metric and its derivatives at various (~24) , evenly spaced in $1/r$ areal radius points between one GW wavelength away from the origin and the outer boundary
  - Fit polynomials in $1/r$ to the metric data:
$$\hat{h}^{(\ell,m)}(u,r) \approx \sum_{k=0}^{N} \hat{h}^{(\ell,m)}_{(k)}/r^{k+1}$$
, where N is the "extrapolation order"; then, $h^{(\ell,m)}_{(0)}(u) = h^{(\ell,m)}(u)$.

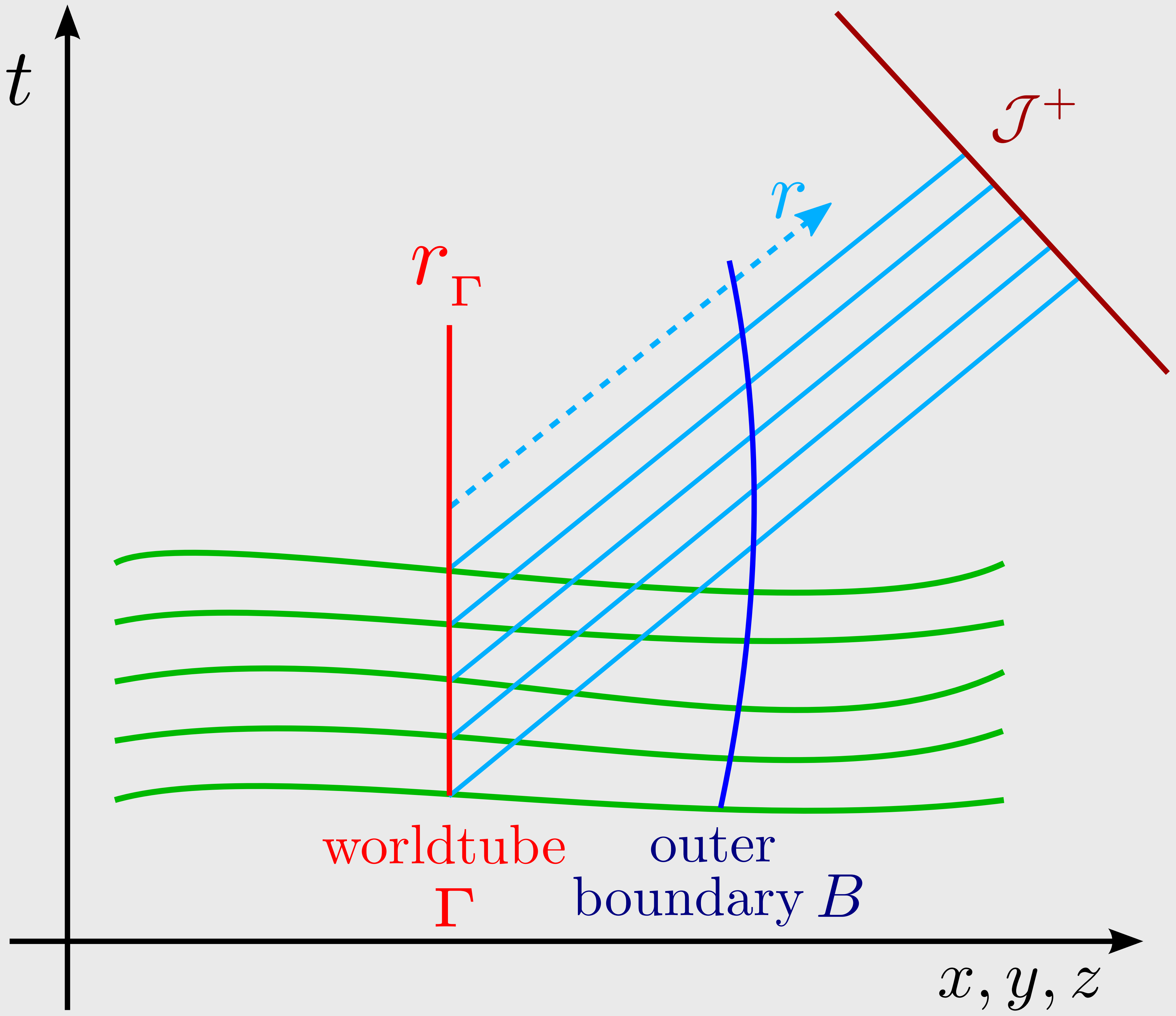


Extracting Waveforms in Numerical Relativity: CCE

- **Cauchy-characteristic Evolution (CCE)** (see Moxon *et al.*, arXiv:2007.01339 and arXiv:2110.08635)
  - Evolve Einstein's equations on null slices!
    - ➡ Can only be performed in the weak regime (caustics in the strong regime ruin the null geodesics)
  - Compactify radial coordinate to include future null infinity on computational grid
  - First formulated in arXiv:gr-qc/9705033v1 by Bishop, Gomez, Lehner, and Winicour
  - First implemented in arXiv:gr-qc/9708002 by Gomez, Marsa, and Winicour
  - First implemented in a BBH simulation in arXiv:0912.1285 by Reisswig, Bishop, Pollney, and Szilagyi
  - First implemented with spectral methods in arXiv:1406.7029 by Handler and Szilagyi



- **Cauchy-characteristic Evolution (CCE)** (see Moxon *et al.*, arXiv:2007.01339 and arXiv:2110.08635)
  - Initial Data (currently an open problem):
    - ➡ Need to specify J on initial null hypersurface
    - ➡ ZeroNonSmooth:
      - Fix $J = 0$
    - ➡ NoIncomingRadiation:
      - Fix J by requiring $\Psi_0 = 0$ (2nd-order radial ODE for J)
    - ➡ InverseCubic:
      - Fix $J = \dfrac{A}{r} + \dfrac{B}{r^3}$ and set A and B by using the values of J and $\partial_r J$ on the worldtube (no r^{-2} to avoid logarithms)
    - ★ ➡ ConformalFactor:
      - Use InverseCubic, but transform the coordinates so that the time coordinate resembles the Bondi time coordinate



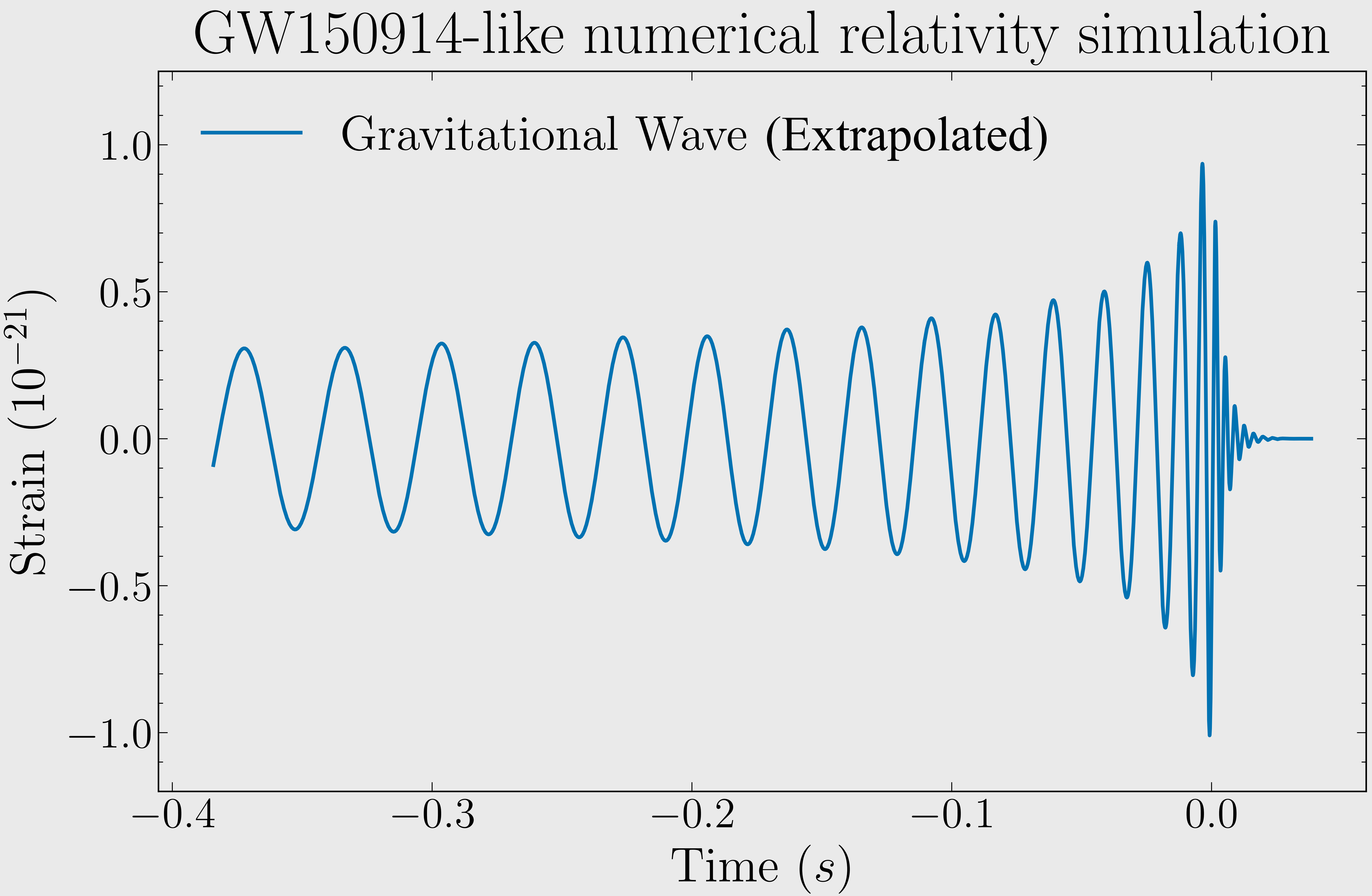
- **Cauchy-characteristic Evolution (CCE)** (see Moxon *et al.*, arXiv:2007.01339 and arXiv:2110.08635)

$$ds^2 = -Ue^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}\left(d\theta^A - \mathcal{U}^A du\right)\left(d\theta^B - \mathcal{U}^B du\right)$$

$$\mathcal{U} \equiv \mathcal{U}^A q_A$$
$$Q \equiv r^2 e^{-2\beta} q^A \gamma_{AB} \partial_r U^B$$
$$rW \equiv U - 1$$
$$J \equiv \frac{1}{2} q^A q^B \gamma_{AB}$$
$$K \equiv \frac{1}{2} q^A \bar{q}^B \gamma_{AB}$$

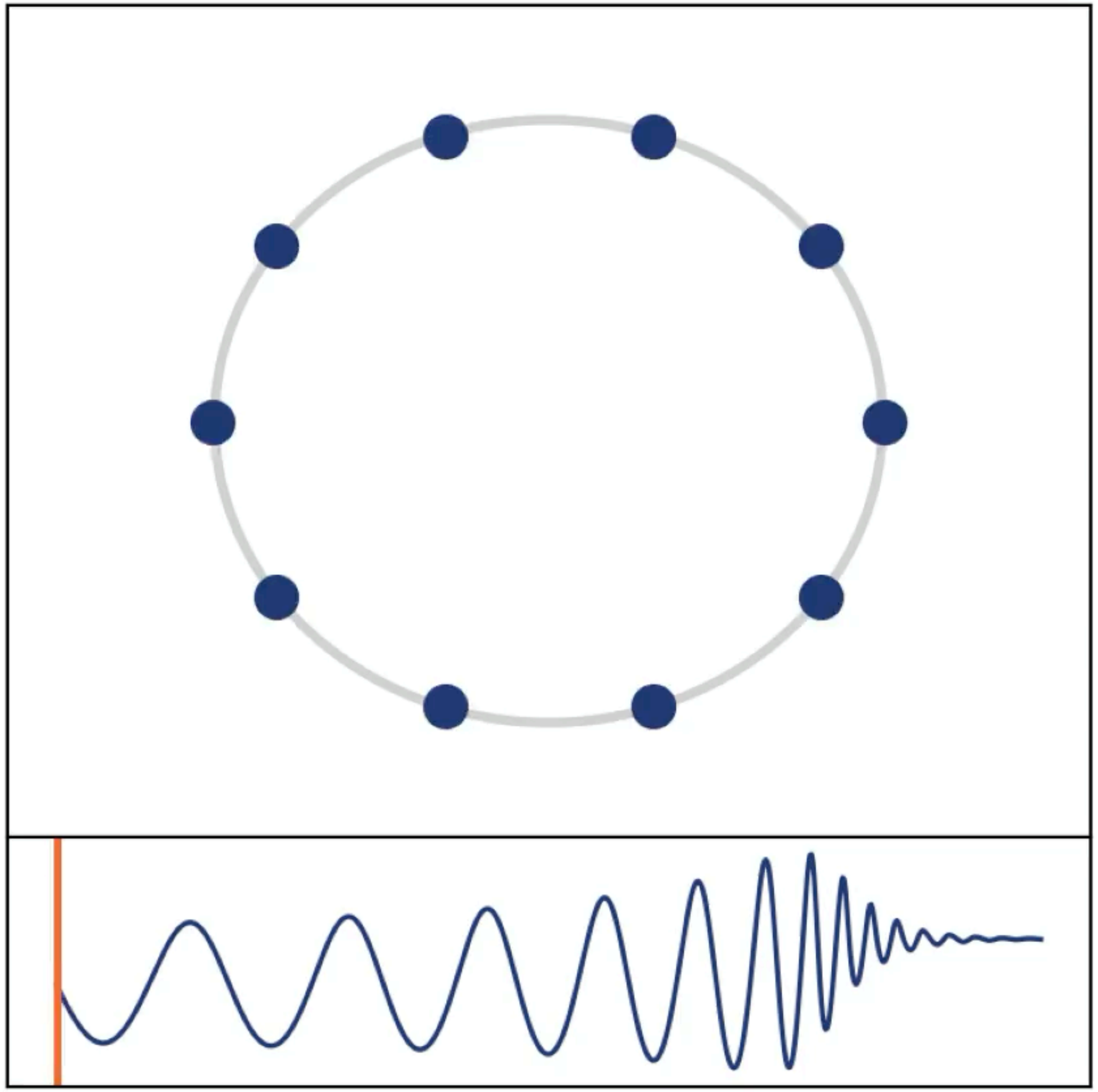
$$\partial_y \beta = S_\beta(J)$$
$$\partial_y \left(\left(1-y\right)^2 Q\right) = S_Q(J,\beta)$$
$$\partial_y \mathcal{U} = S_{\mathcal{U}}(J,\beta,Q)$$
$$\partial_y \left(\left(1-y\right)^2 W\right) = S_Q(J,\beta,\mathcal{U})$$
$$\left[\partial_y\left(\left(1-y\right)H\right) + \mathrm{Re}\left[L_H\left(J,\beta,Q,\mathcal{U},W\right)H\right]\right] = S_H\left(J,\beta,Q,\mathcal{U},W\right)$$
$$\partial_u J = H$$

Waveforms in Numerical Relativity



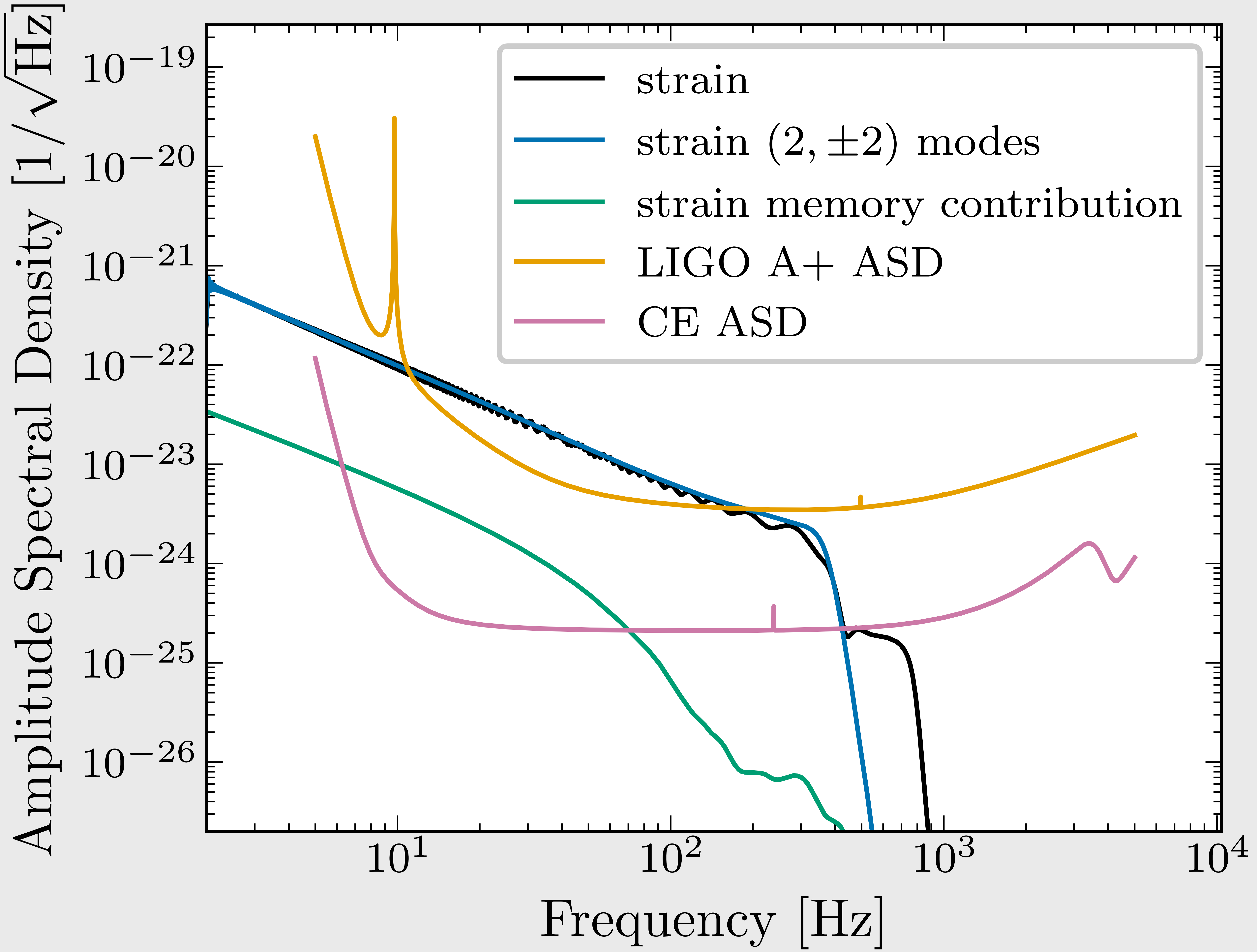
Gravitational Wave Memory Effect

- **What is it?**
  - Permanent, net displacement between two initially comoving observers
  - Nonlinear* prediction of general relativity
  - (see, e.g., Mitman *et al.*, arXiv:2405.08868)
- **Why do we care?**
  - Intimately connected to the symmetries of future null infinity
    - Think celestial holography or AdS/CFT
  - Not yet observed!
  - Intriguing and unique way to test Einstein's theory at low-frequencies



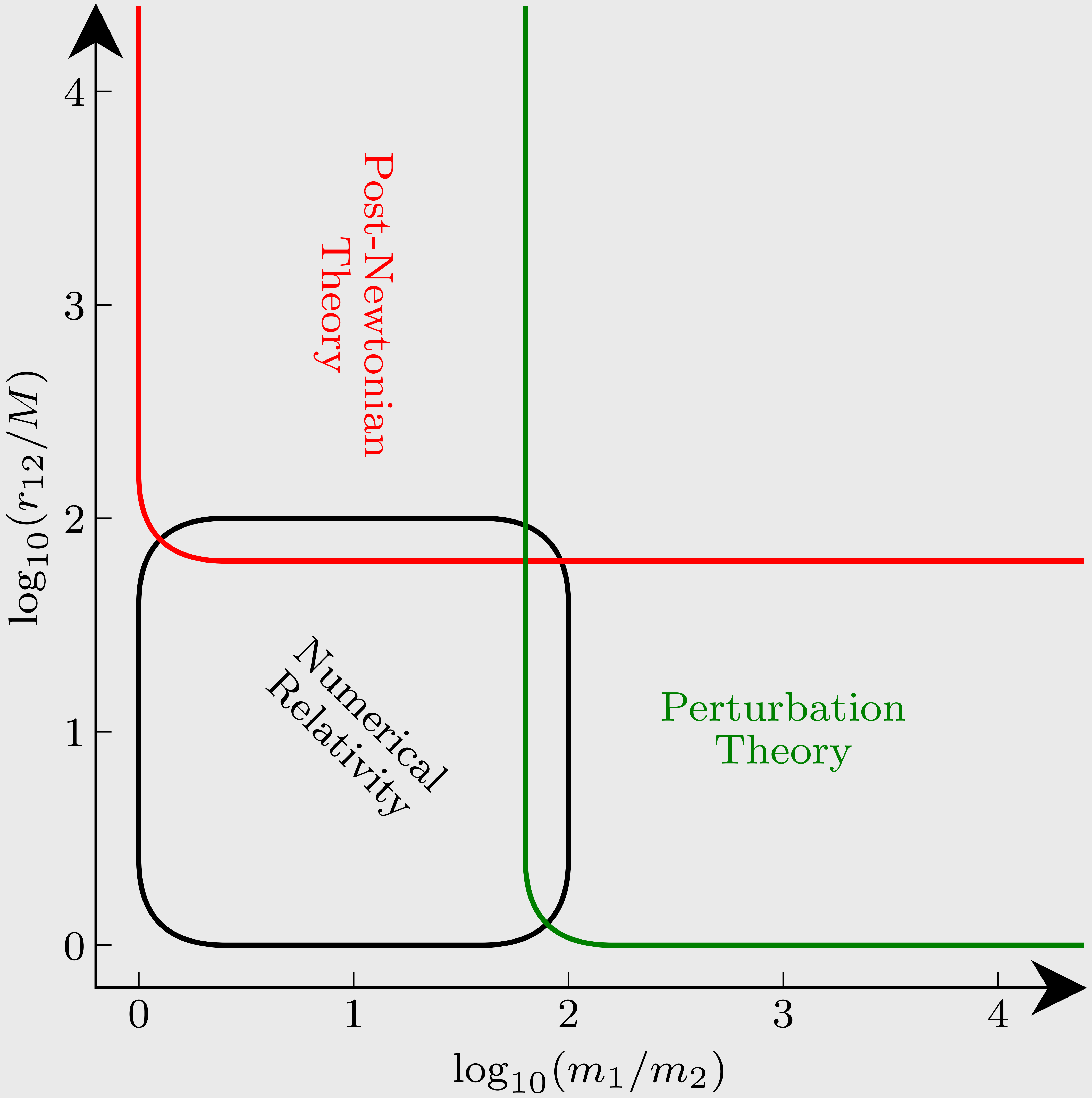
Detecting Gravitational Memory

- **Ground-based:**
  - LIGO:
    $\mathcal{O}$(2,000) events*
    arXiv:1911.12496
    arXiv:2105.02879
    arXiv:2210.16266
    arXiv:2404.11919
  - CE/ET:
    $\mathcal{O}$(1) event per year
    arXiv:2210.16266
- **LISA:**
  - $\mathcal{O}$(10) event per 4 years!
    arXiv:1906.11936

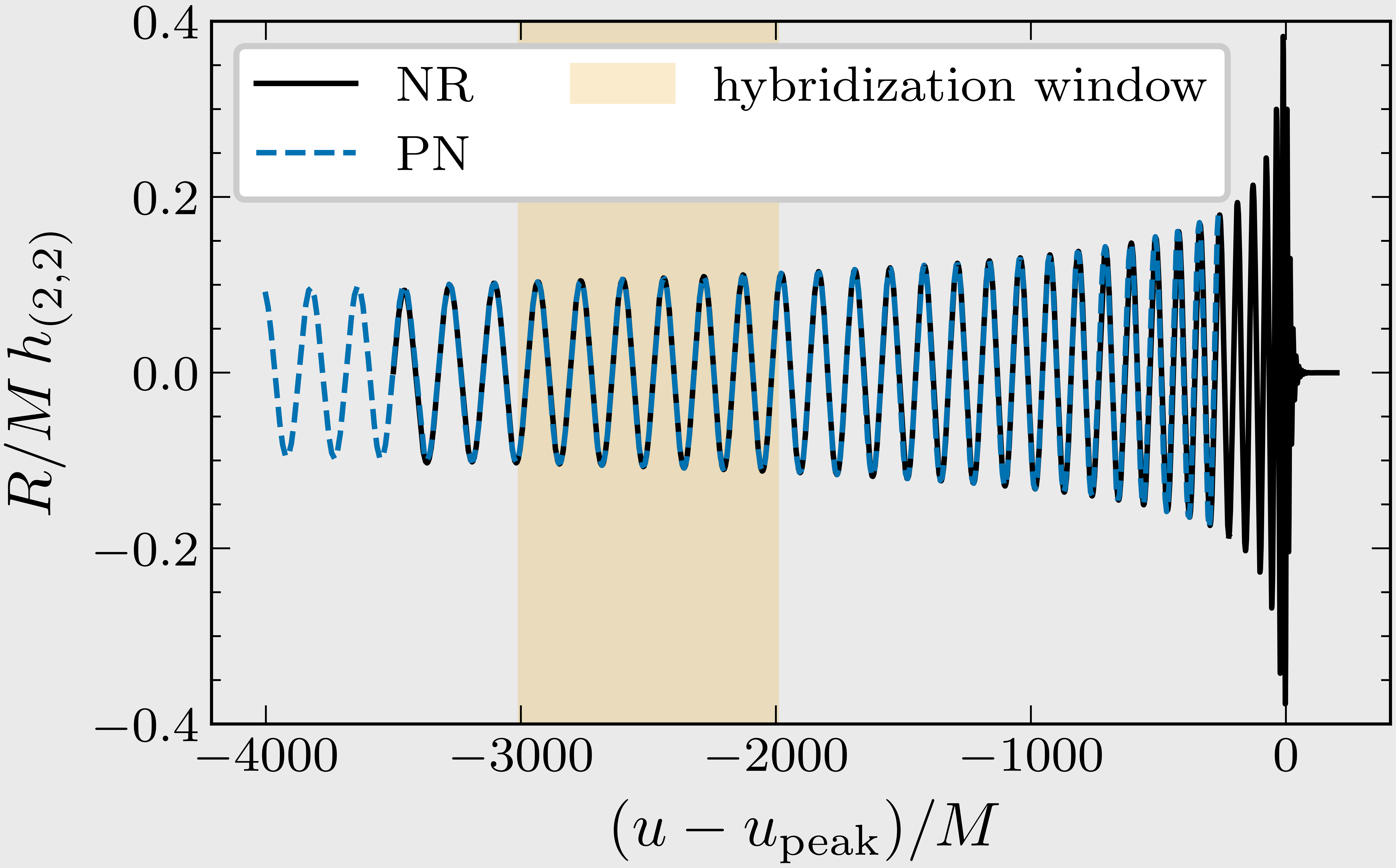


Modeling Binary Black Hole Waveforms

- **What's needed for waveform modeling?**
  - Need infinitely long waveforms for accurate frequency domain spectra
  - NR waveforms are finite in length
  - Can be extended by "hybridizing" with analytic models, like post-Newtonian (PN) waveforms
  - Crucial for building robust waveform models for GW detectors



Comparing NR Waveforms to post-Newtonian Waveforms



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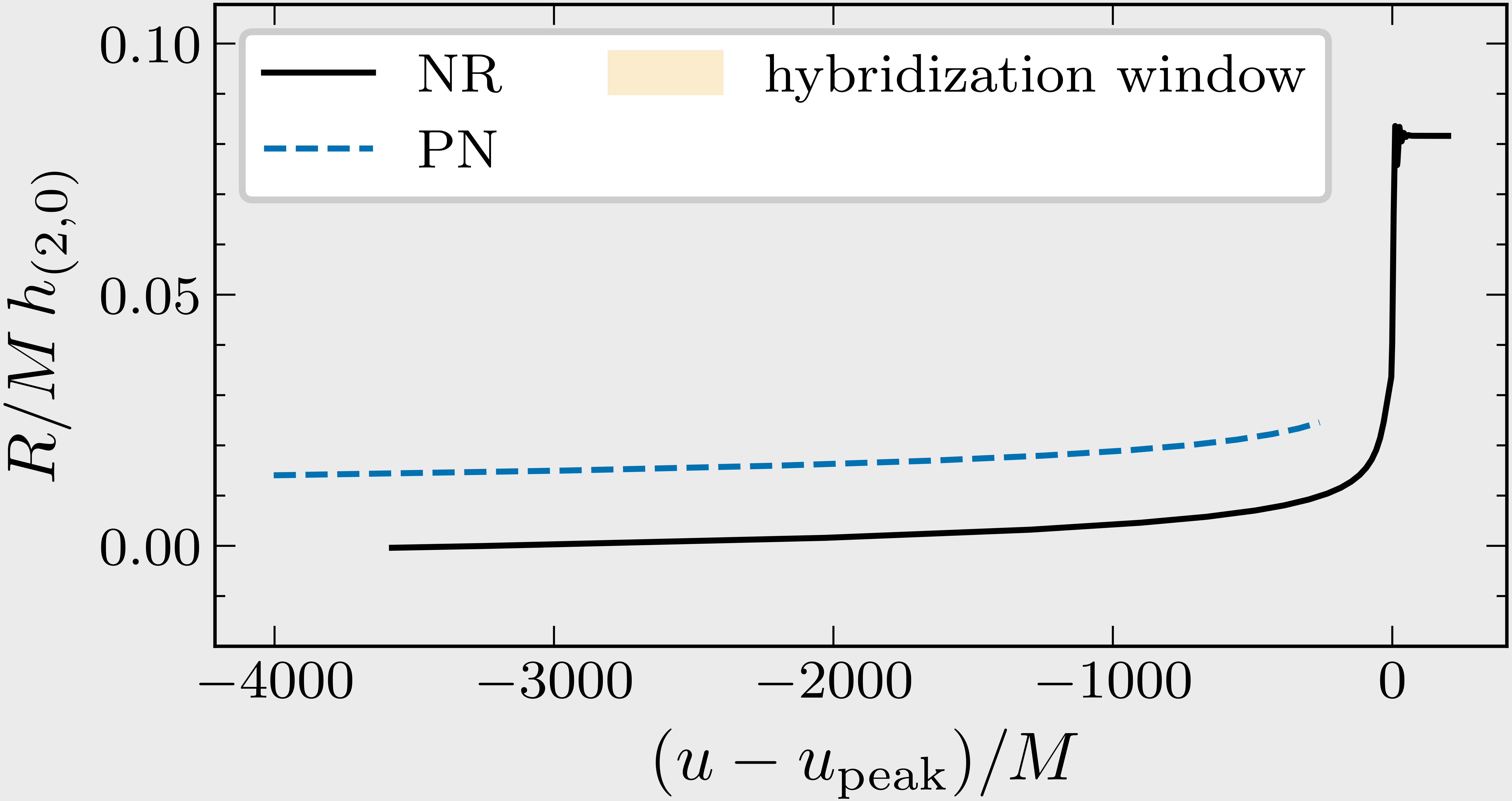


Coordinate Freedom at $\mathscr{I}^+$

The Hard Way:

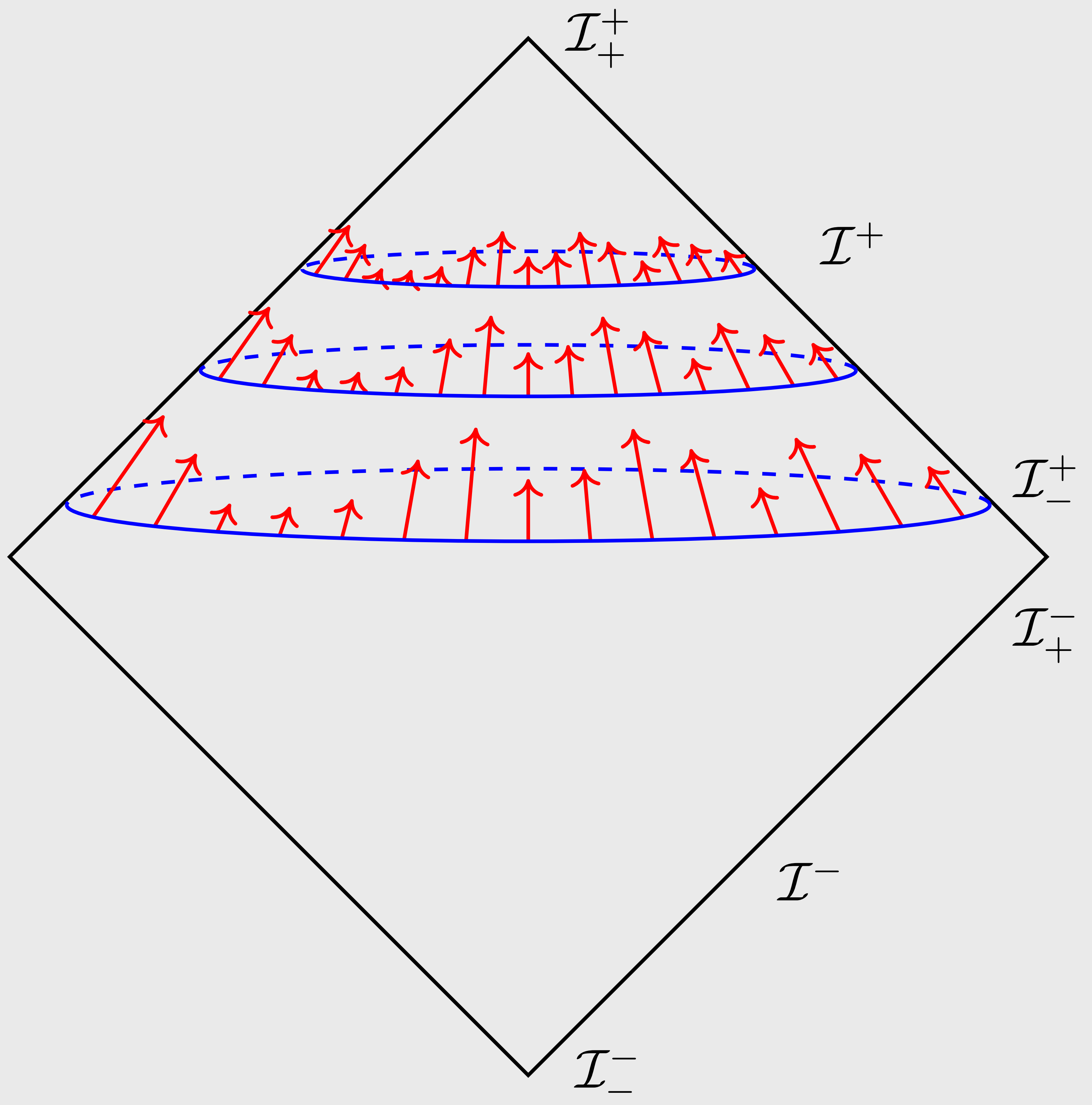
- Find an equivalence class of vector fields $\vec{\xi}$ satisfying Killing's equation $\mathscr{L}_{\vec{\xi}} g_{ab} = 0$ (approximately, i.e., w.r.t. fall-off conditions) as one approaches future null infinity $\mathscr{I}^+$

- Yields
$$\vec{\xi} = \left[\alpha\left(\theta^A\right) + \frac{1}{2} u D_A Y^A \left(\theta^B\right)\right] \partial_u + Y^A\left(\theta_B\right) \partial_A$$

where
$Y^A = D^A \chi + \epsilon^{AB} D_B \kappa =$ boost + rotation

The Easy Way:

- Consider a collection of observers on the celestial two sphere



Action of a BMS Transformation

When acted on by a BMS transformation $\left(\alpha\left(\zeta,\bar{\zeta}\right),\left(a,b,c,d\right)\right)$**...**

- $$\left(u,\zeta\right)\rightarrow\left(u',\zeta'\right)=\left(k\left(u-\alpha\right),\frac{a\zeta+b}{c\zeta+d}\right)$$

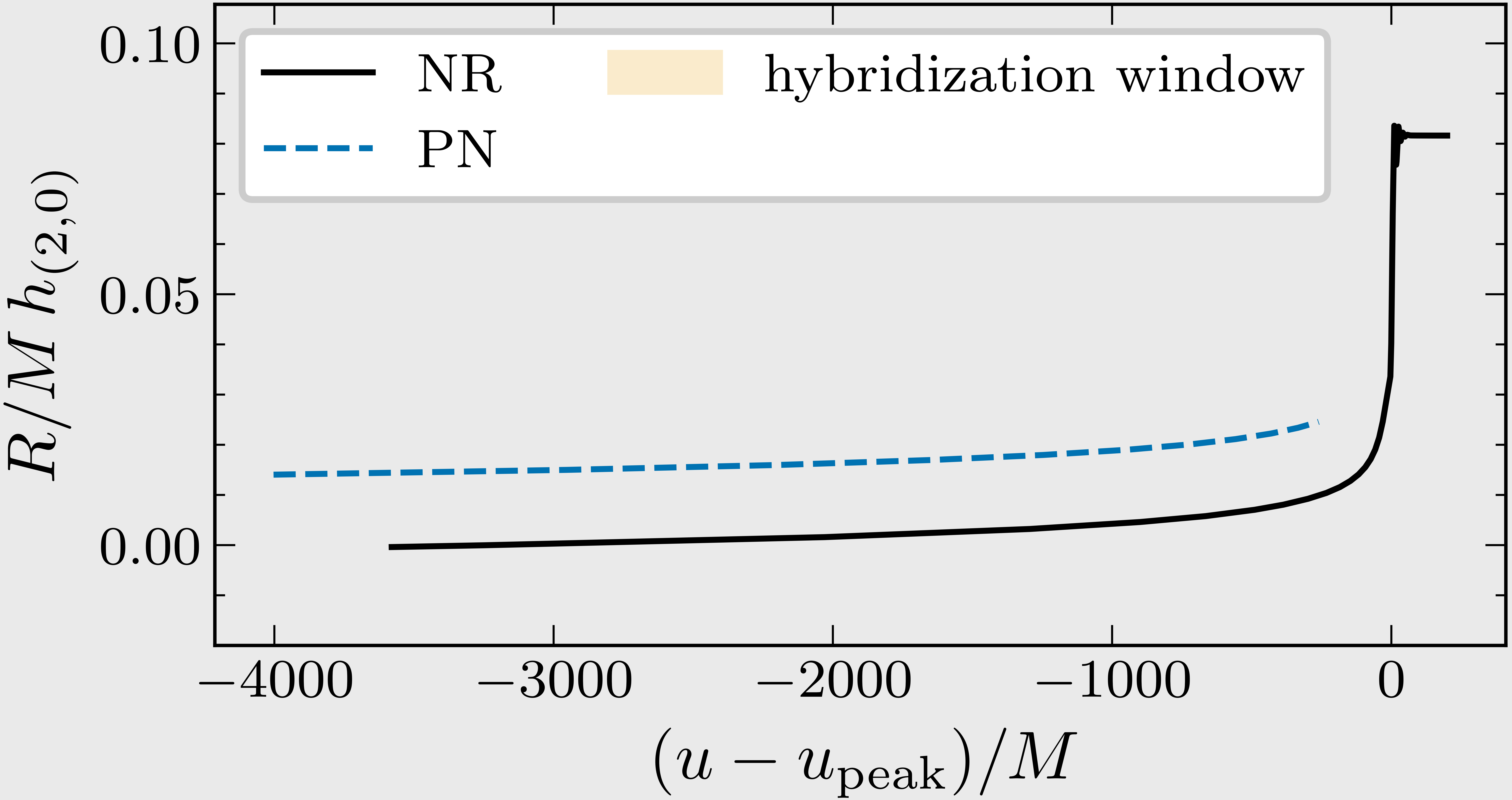
$$\sigma\left(u,\zeta,\bar{\zeta}\right)\rightarrow\sigma'\left(u',\zeta',\bar{\zeta}\right)=\sigma\left(u',\zeta,\bar{\zeta}\right)-\eth^2\alpha\left(\zeta,\bar{\zeta}\right)$$

- $$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\alpha\left(\zeta,\bar{\zeta}\right)\frac{\partial}{\partial u}\right)^n\sigma\left(u,\zeta,\bar{\zeta}\right)-\eth^2\alpha\left(\zeta,\bar{\zeta}\right)$$

Mixing between the modes

Constant change in the modes

Can a BMS Transformation Explain NR/PN Differences?



Wald-Zoupas/Dray Charges:

$$Q_{\vec{\xi}} = \frac{1}{4\pi}\int d^2\Omega \left[\alpha m + \frac{1}{2}Y^A\hat{N}_A\right]$$

where

$$m \equiv -\,\mathrm{Re}\left[\psi_2 + \sigma\dot{\bar{\sigma}}\right]$$

$$\hat{N} \equiv -\left(\psi_1 + \sigma\eth\bar{\sigma} + \frac{1}{2}\eth\left(\sigma\bar{\sigma}\right) + u\eth m\right)$$

Balance Laws:

For u_0 a non-radiative section of $\mathscr{I}^+$,

$$\mathrm{Re}\left[\bar{\eth}^2\sigma\right] = \mathrm{Re}\left[m + \mathscr{E} + \left(\bar{\eth}^2\sigma - m\right)|^{u_0}\right]$$

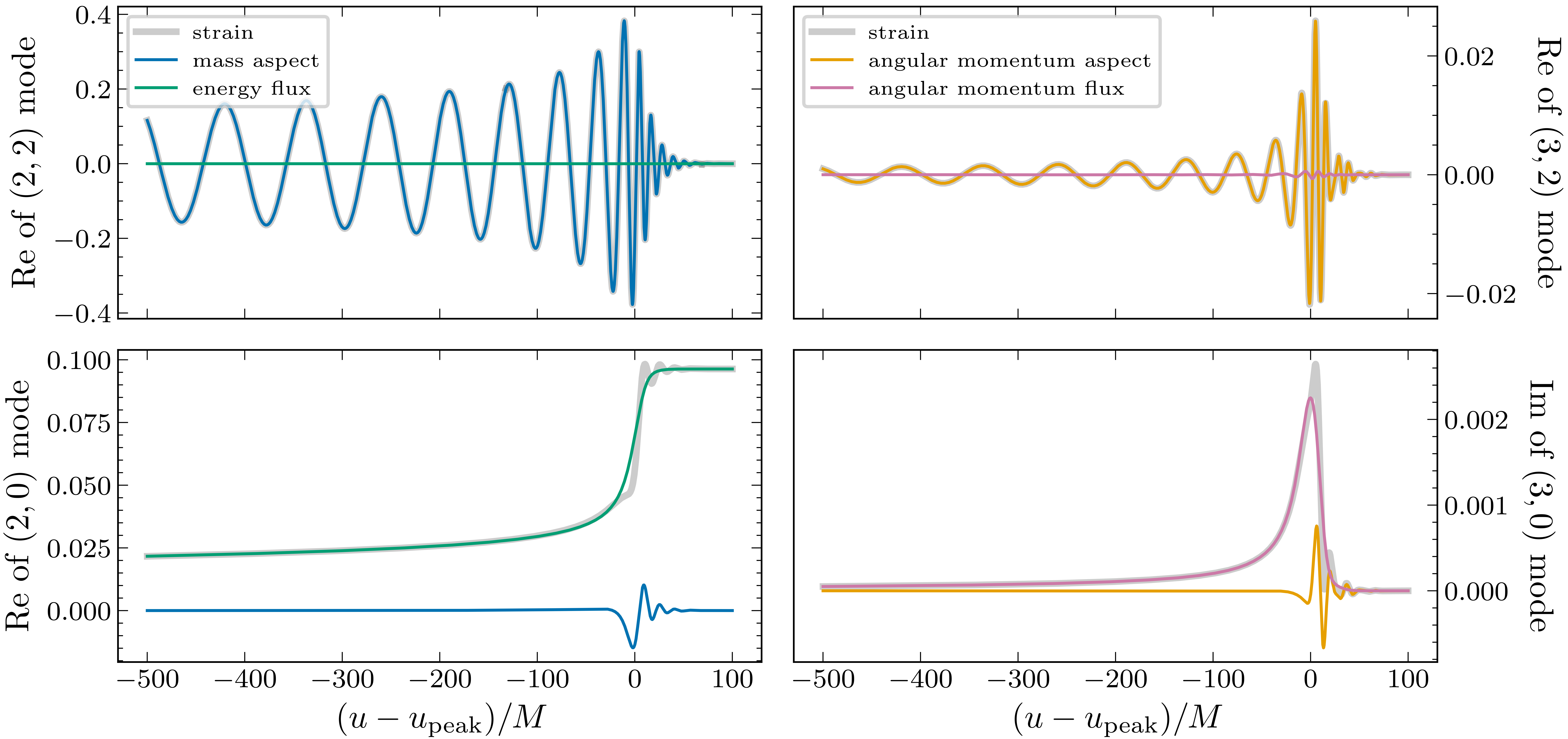
$$\mathrm{Im}\left[\bar{\eth}^2\sigma\right] = -\frac{d}{du}\left(\eth\bar{\eth}\right)^{-1}\mathrm{Im}\left[\bar{\eth}\left(\hat{N} + \mathscr{J}\right)\right]$$

where

$$\mathscr{E} \equiv \int_{u_0}^{u} |\dot{\sigma}|^2 du$$

$$\mathscr{J} \equiv \frac{1}{2}\int_{u_0}^{u}\left(3\dot{\sigma}\eth\bar{\sigma} - 3\sigma\eth\dot{\bar{\sigma}} + \bar{\sigma}\eth\dot{\sigma} - \dot{\bar{\sigma}}\eth\sigma\right) du$$

Noether's Theorem and BMS Charges



Poincaré Frame:

- translations
- boosts
- rotations

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Poincaré Charges:

$$P^a(u) = \frac{1}{4\pi}\int_{S^2} n^a m \, d\Omega;$$

$$J^a(u) = \frac{1}{4\pi}\int_{S^2} \mathrm{Im}\left[\left(\bar{\eth} n^a\right)\hat{N}\right] d\Omega;$$

$$K^a(u) = \frac{1}{4\pi}\int_{S^2} \mathrm{Re}\left[\left(\bar{\eth} n^a\right)\hat{N}\right] d\Omega$$

Fixing the Poincaré Frame

Poincaré Frame:

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Other "Charges":

Center-of-Mass Charge:

$$G^a(u) = \left(K^a + uP^a\right)/P^t$$

Angular Velocity:

$$\vec{\omega} = -\langle \vec{L}\vec{L}\rangle^{-1}\cdot\langle \vec{L}\partial_t\rangle$$

$$\langle \vec{L}\vec{L}\rangle^{ab} \equiv \sum_{\ell,m,m'} \bar{f}^{\ell,m'}\langle \ell,m'|L^{(a}L^{b)}|\ell,m\rangle f^{\ell,m}$$

$$\langle \vec{L}\partial_t\rangle^a \equiv \sum_{\ell,m,m'} \mathrm{Im}\left[\bar{f}^{\ell,m'}\langle \ell,m'|L^a|\ell,m\rangle \partial_t f^{\ell,m}\right]$$

Fixing the Poincaré Frame

Poincaré Frame:

- translations
- boosts

$\}\rightarrow$ **center-of-mass charge "rest frame"**

- rotations

$\}\rightarrow$ **rotation charge "aligned with $\hat{z}$-axis"**

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Fixing the Poincaré Frame

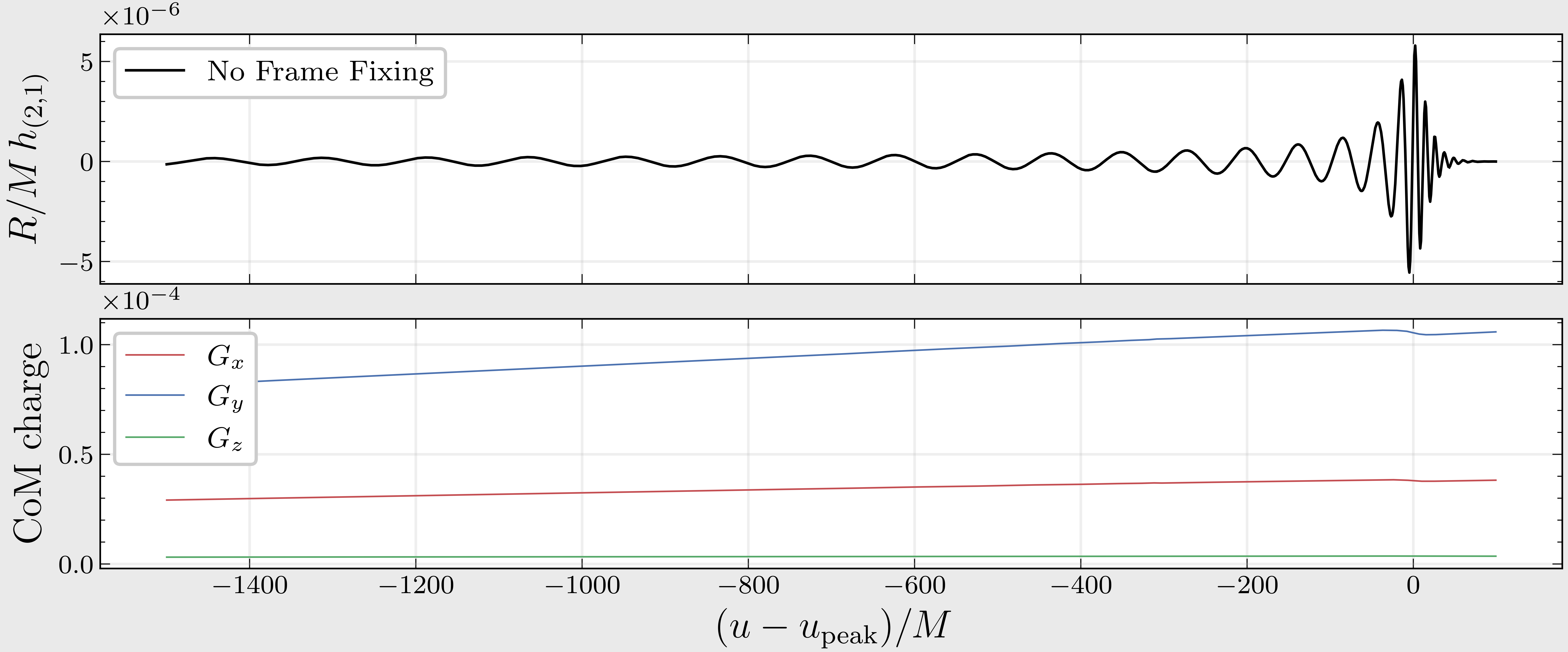
Poincaré Frame:

- translations
- boosts

} → **center-of-mass charge "rest frame"**

- rotations

} → **rotation charge "aligned with $\hat{z}$-axis"**



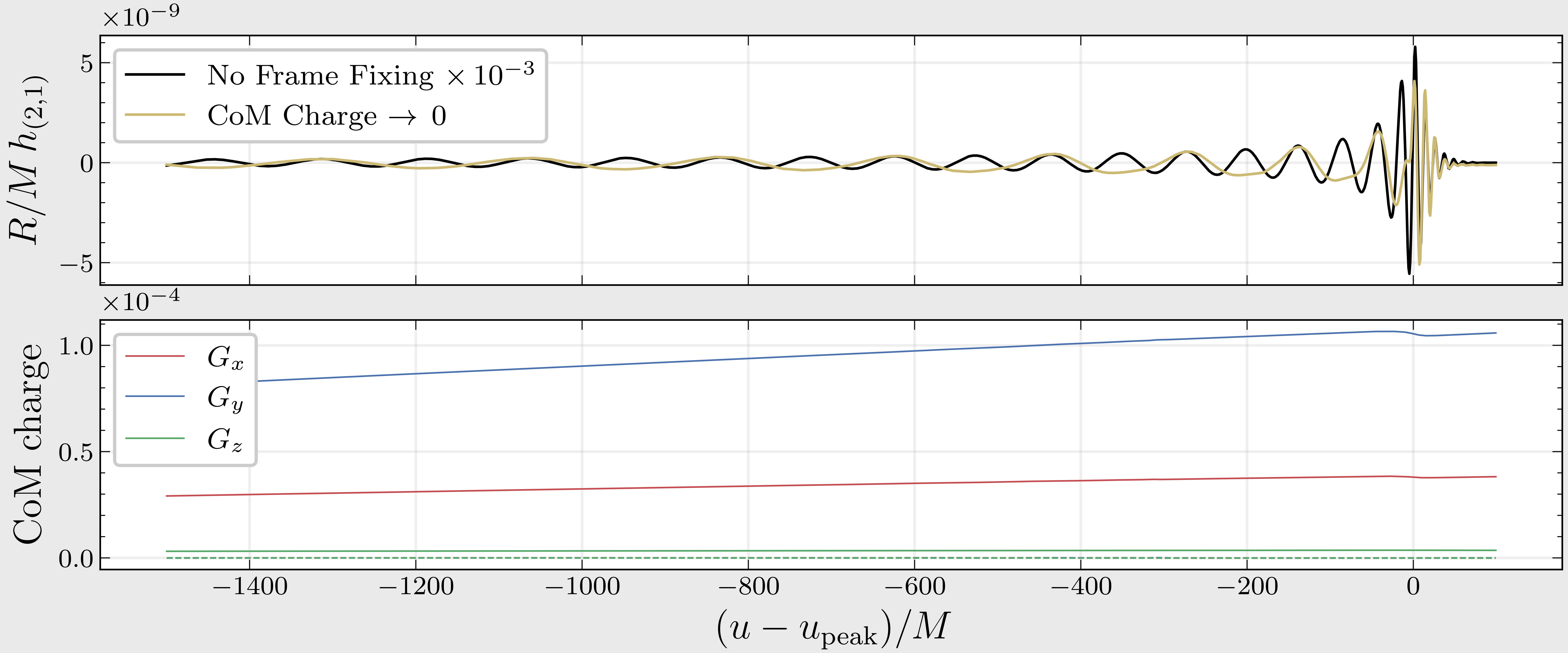
Poincaré Frame:

- translations
- boosts

→ **center-of-mass charge "rest frame"**

- rotations

→ **rotation charge "aligned with $\hat{z}$-axis"**



Fixing the BMS Frame

BMS Frame:

- supertranslations $\Big\} \rightarrow$ **?**
- translations
- boosts $\Big\} \rightarrow$ **center-of-mass charge "rest frame"**
- rotations $\Big\} \rightarrow$ **rotation charge "aligned with $\hat{z}$-axis"**

BMS Frame:

- supertranslations } → **supermomentum**
- translations
- boosts } → **center-of-mass charge "rest frame"**
- rotations } → **rotation charge "aligned with $\hat{z}$-axis"**

Fixing the BMS Frame

- **Use the supermomentum from Dray/Streubel:**
  $$\Psi_{p,q} = \Psi_2 + \sigma\dot{\bar{\sigma}} + p\left(\eth^2\bar{\sigma}\right) - q\left(\bar{\eth}^2\sigma\right)$$
  - ➡ $p = q \Rightarrow$ no supermomentum flux in Minkowski space;
  - ➡ $p + q = 1 \Rightarrow$ supermomentum is real
  - ➡ For $p = q = 1/2$, we have the Geroch supermomentum: $\Psi_{\mathrm{G}} = \Psi_2 + \sigma\dot{\bar{\sigma}} + \frac{1}{2}\left(\eth^2\bar{\sigma} - \bar{\eth}^2\sigma\right)$ (which is just the Bondi mass aspect), but this is supertranslation-covariant in non-radiative regimes of $\mathscr{I}$

- **Moreschi supermomentum ($p = 1, q = 0$):**
  $$\Psi_{\mathrm{M}} = \Psi_2 + \sigma\dot{\bar{\sigma}} + \eth^2\bar{\sigma} = \int_{-\infty}^{u} |\dot{\sigma}|^2 du - M_{\mathrm{ADM}}$$
  - ➡ Measures the "supertranslation flux" in Minkowski space, i.e., the "instantaneous null memory"
  - ➡ Enables "superrest frames" to be computed iteratively: $\eth^2\bar{\eth}^2\alpha = \Psi_{\mathrm{M}}\left(u = \alpha, \zeta, \bar{\zeta}\right) + k_{\mathrm{rest}}\left(\alpha, \zeta, \bar{\zeta}\right)^3 M_{\mathrm{B}}(\alpha)$

Fixing the BMS Frame to Match PN Waveforms

PN waveforms are in a “canonical” BMS frame

$$m\left(u \to -\infty, \zeta, \bar{\zeta}\right) = m_0 = \text{constant}$$

$$\sigma\left(u \to -\infty, \zeta, \bar{\zeta}\right) = 0$$

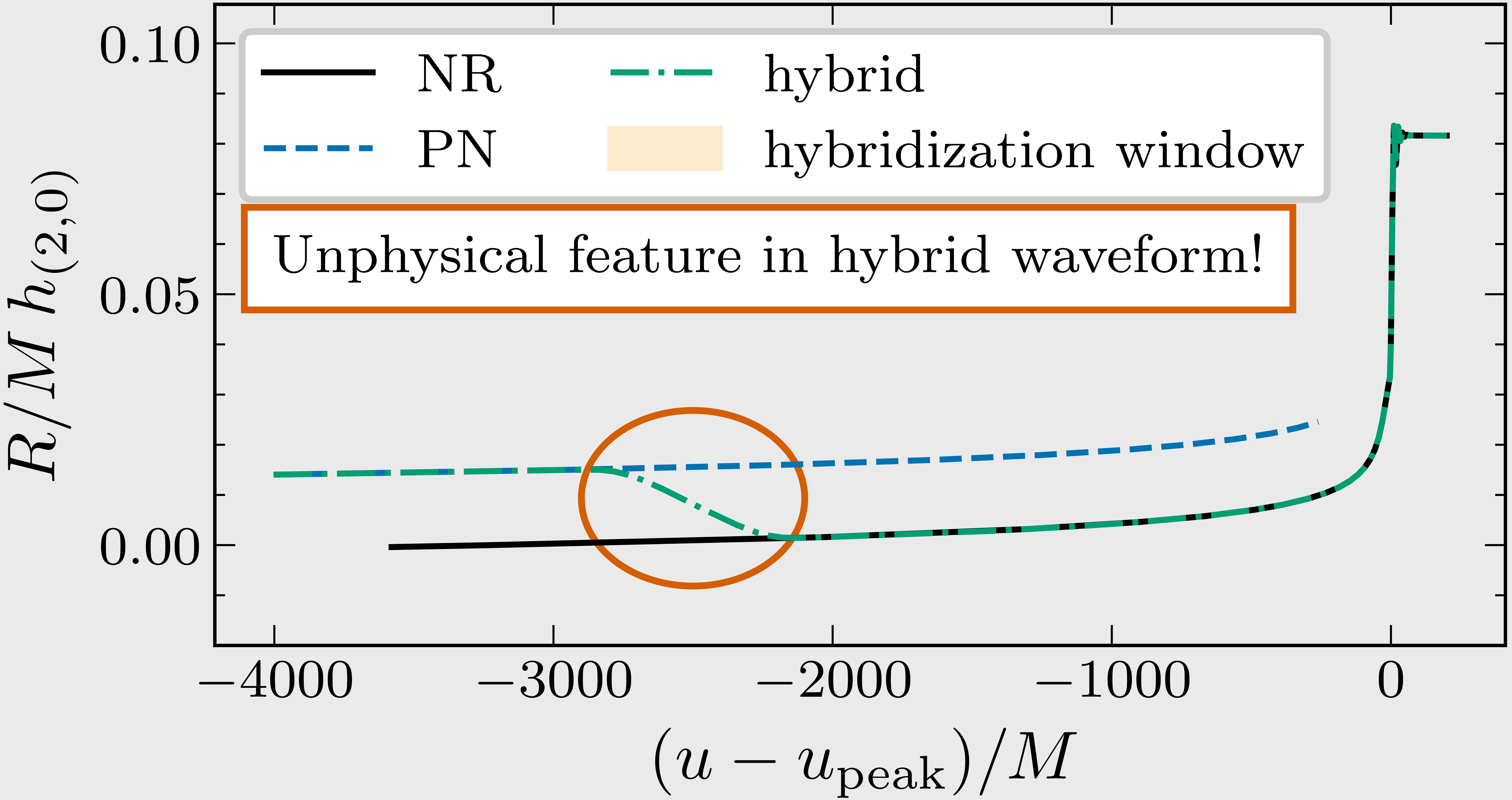
$$\hat{N}\left(u \to -\infty, \zeta, \bar{\zeta}\right) = \text{magnetic parity}, \ell = 1 \;\left(\text{only rotational}\right)$$

- Map to this frame by mapping NR Poincaré charges to PN Poincaré charges and…
- by mapping the NR supermomentum to the PN supermomentum
- Need to know the PN Moreschi supermomentum

➡ Can compute this “easily” because

$$\Psi_{\mathrm{M}}\left(u, \zeta, \bar{\zeta}\right) = \int_{-\infty}^{u} |\dot{\sigma}|^2 du - M_{\mathrm{ADM}}$$

Comparing NR Waveforms to post-Newtonian Waveforms



Thank you!