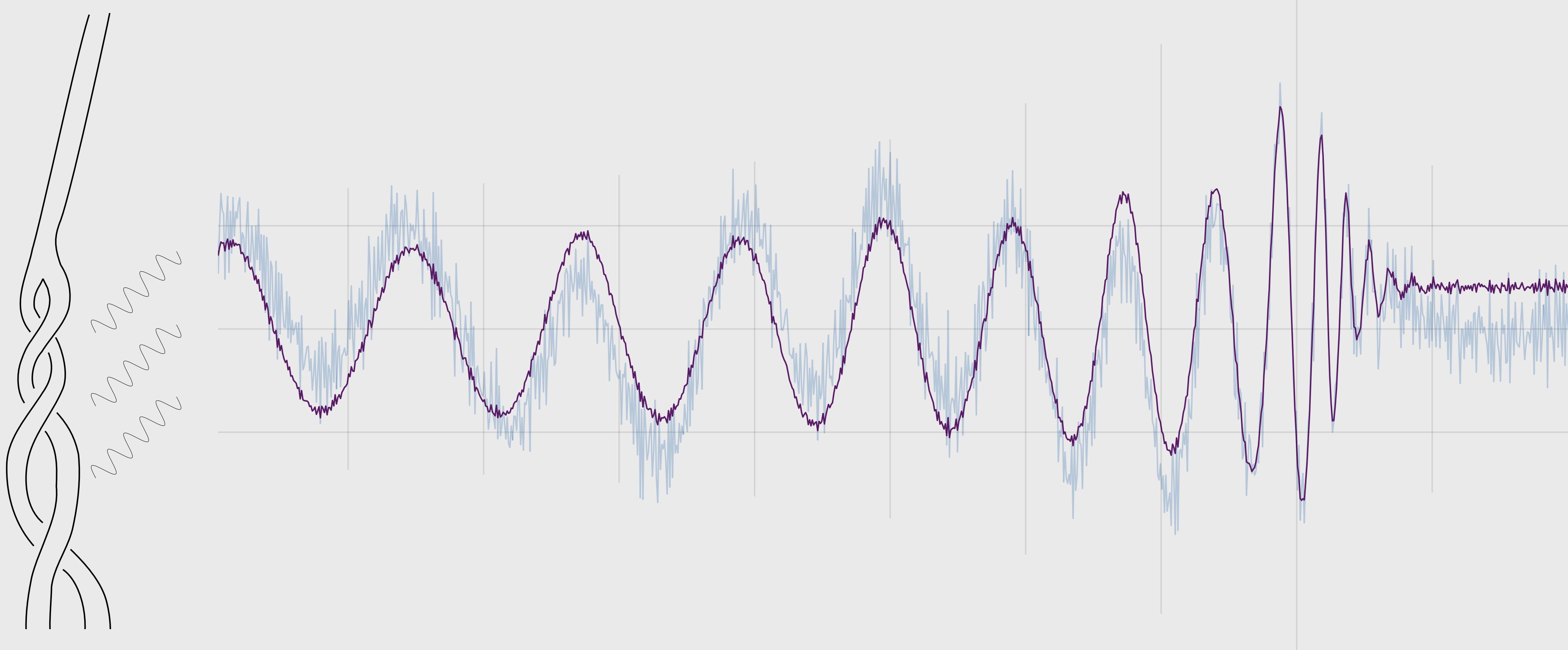


# Introduction to Cauchy-characteristic Evolution (CCE)



**Keefe Mitman,**

in collaboration with Michael Boyle, Leo C. Stein, Nils Deppe, Lawrence E. Kidder, Jordan Moxon, Kyle Nelli, Harald P. Pfeiffer, Mark A. Scheel, Saul A. Teukolsky, William Throwe, and Nils L. Vu

**SXSCon, August 7th, 2024**



# Questions about Astrophysics and Gravity

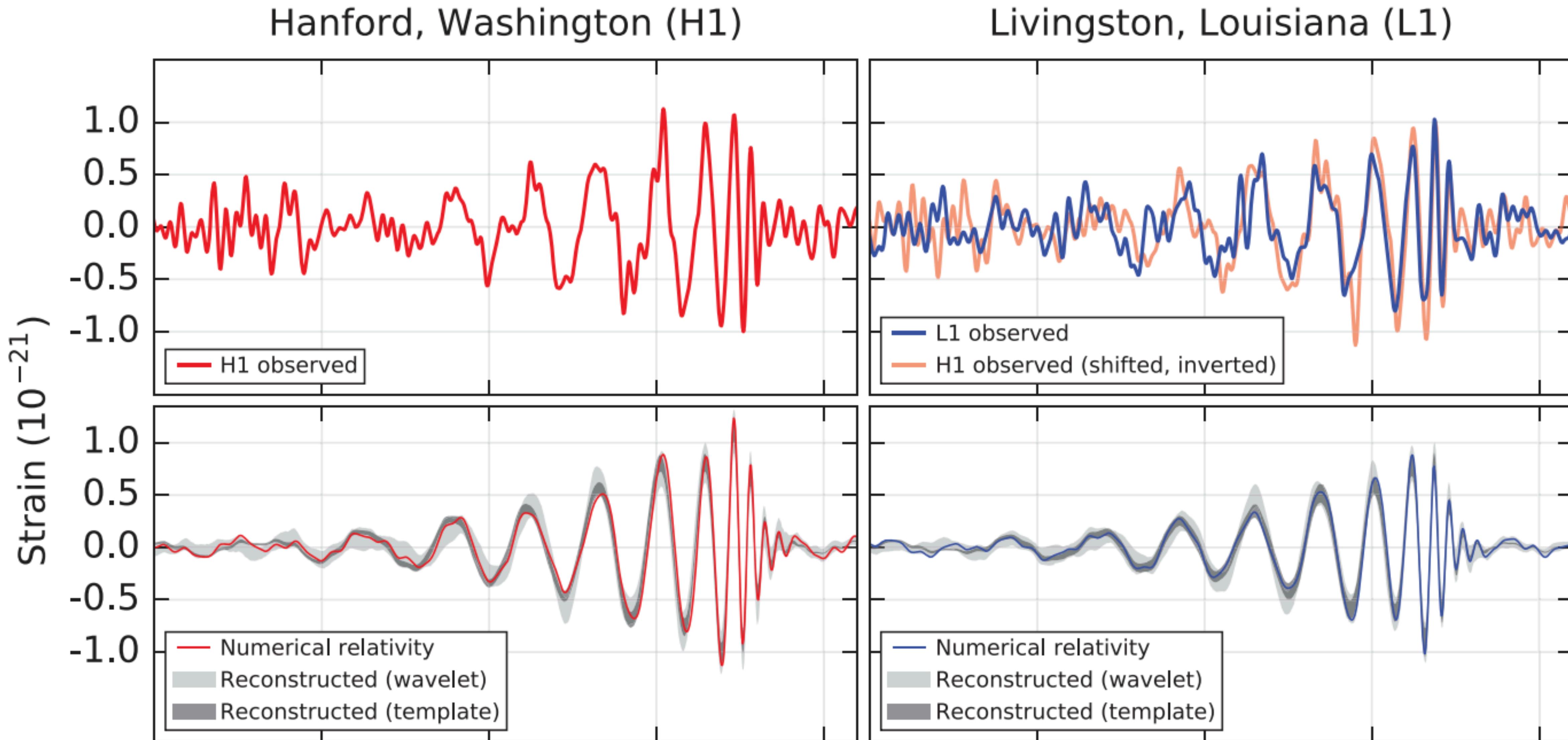
## ► Astrophysics:

- Black Hole / Neutron Star / Merger Rates and Formation Channels
- Galaxy Merger Rate (via low frequency GWs from SMBHs)
- Neutron Star Equation of State
- Mechanism of Supernovae
- Dark Matter / Exotica

## ► Gravity

- Is it GR?
- Is it Quantum?
- Black Holes
  - singularities, no-hair theorem, information paradox
- Origin of Dark Energy / Hubble Constant Tension
- Inflation (via stochastic background)

# Learning about the Universe with Gravitational Wave Observations



## ► Cauchy Evolution

- Evolve Einstein's equations  $R_{\mu\nu} - g_{\mu\nu}R/2 = 0$  using a Cauchy slice foliation over some finite spacetime volume

## ► Extracting Gravitational Waves

- Can compute the correction to the Minkowski space metric at any position without the simulated volume... not generalizable!

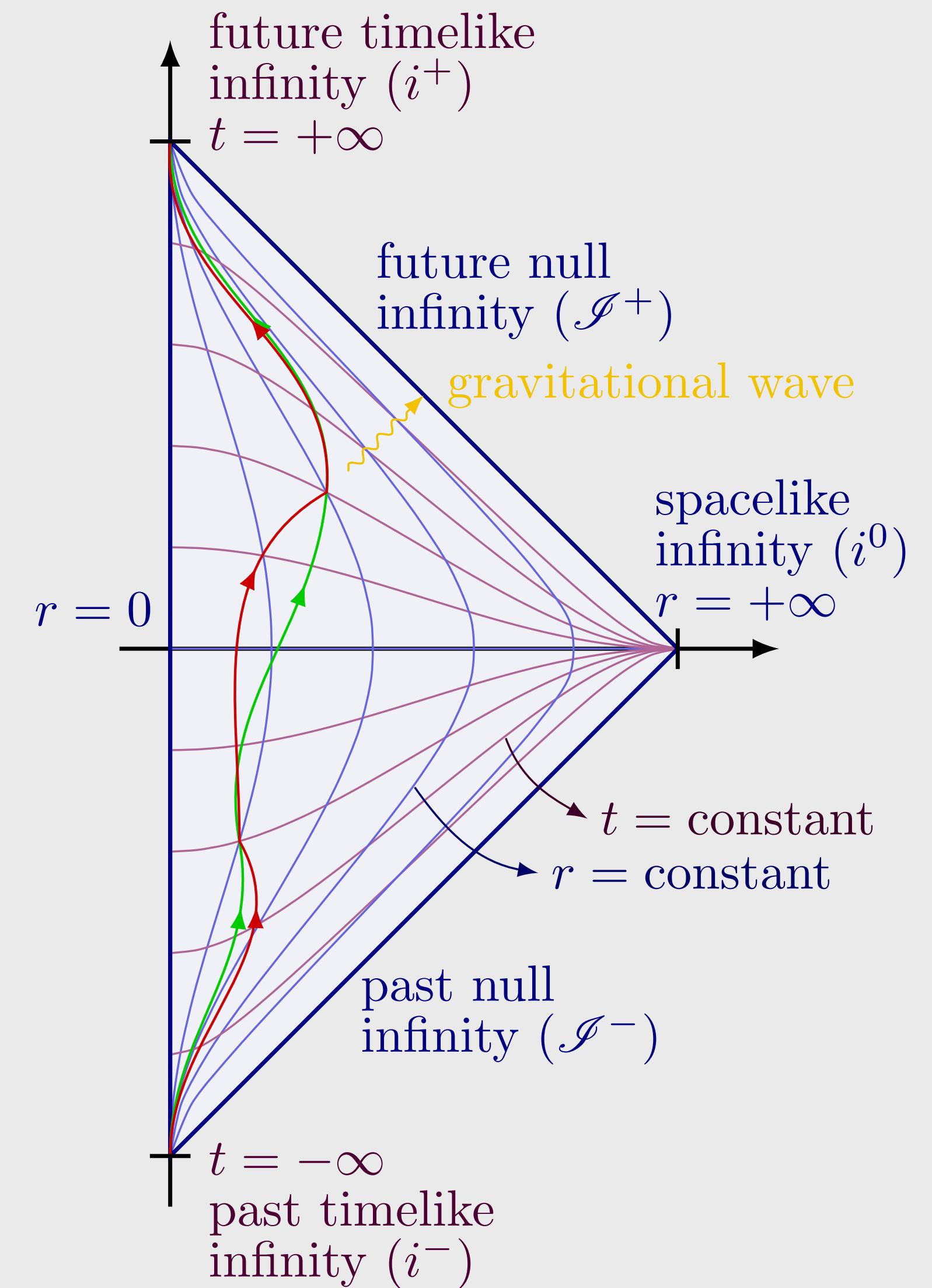
$$ds^2 = -Ue^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(d\theta^A - \mathcal{U}^A du)(d\theta^B - \mathcal{U}^B du);$$

$$\gamma_{AB} = h_{AB} + \frac{1}{r}C_{AB} + \frac{1}{r^2}D_{AB} + \dots$$

“Gravitational wave strain”

“ $\mathcal{O}(10^{-20})$  correction (for LVK events)”

- Compute the waveform at future null infinity instead! Generalizable!



# Extracting Waveforms in Numerical Relativity: Extrapolation

## ► Extrapolation (see, e.g., Boyle *et al.*, arXiv:1904.04831)

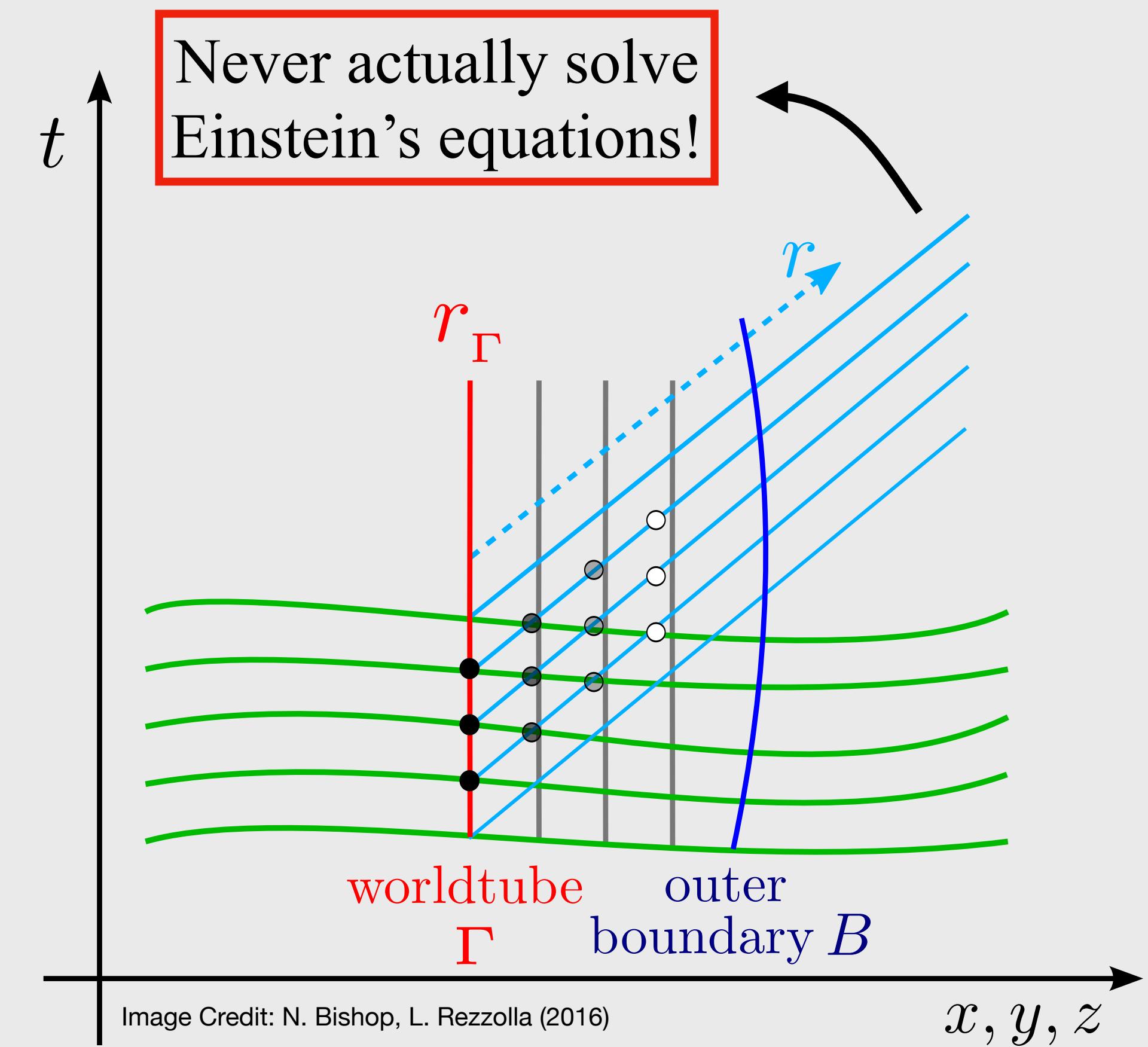
- Want the waveform at “future null infinity”, i.e., large  $r \rightarrow \infty$  limit, i.e., a spin-weight -2 complex scalar

$$h(u, \theta, \phi) = \frac{1}{2} \bar{q}^A \bar{q}^B C_{AB} \quad [q^A = -(1, i \sin \theta)] \\ = \sum_{\ell > 2, |m| \leq \ell} h_{(\ell,m)}(u) {}_{-2}Y_{(\ell,m)}(\theta, \phi)$$

- Compute the metric and its derivatives at various ( $\sim 24$ ), evenly spaced in  $1/r$  areal radius points between one GW wavelength away from the origin and the outer boundary

- Fit polynomials in  $1/r$  to the metric data:

$$\hat{h}^{(\ell,m)}(u, r) \approx \sum_{k=0}^N \hat{h}_{(k)}^{(\ell,m)} / r^{k+1}, \text{ where } N \text{ is the “extrapolation order”; then, } h_{(0)}^{(\ell,m)}(u) = h^{(\ell,m)}(u).$$



# Extracting Waveforms in Numerical Relativity: CCE

## ► Cauchy-characteristic Evolution (CCE) (see Moxon *et al.*, arXiv:2007.01339 and arXiv:2110.08635)

- ─ Evolve Einstein's equations on null slices!
- Can only be performed in the weak regime  
(caustics in the strong regime ruin the null geodesics)
- ─ Compactify radial coordinate to include future null infinity on computational grid
- ─ First formulated in arXiv:gr-qc/9705033v1 by Bishop, Gomez, Lehner, and Winicour
- ─ First implemented in arXiv:gr-qc/9708002 by Gomez, Marsa, and Winicour
- ─ First implemented in a BBH simulation in arXiv:0912.1285 by Reisswig, Bishop, Pollney, and Szilagyi
- ─ First implemented with spectral methods in arXiv:1406.7029 by Handler and Szilagyi

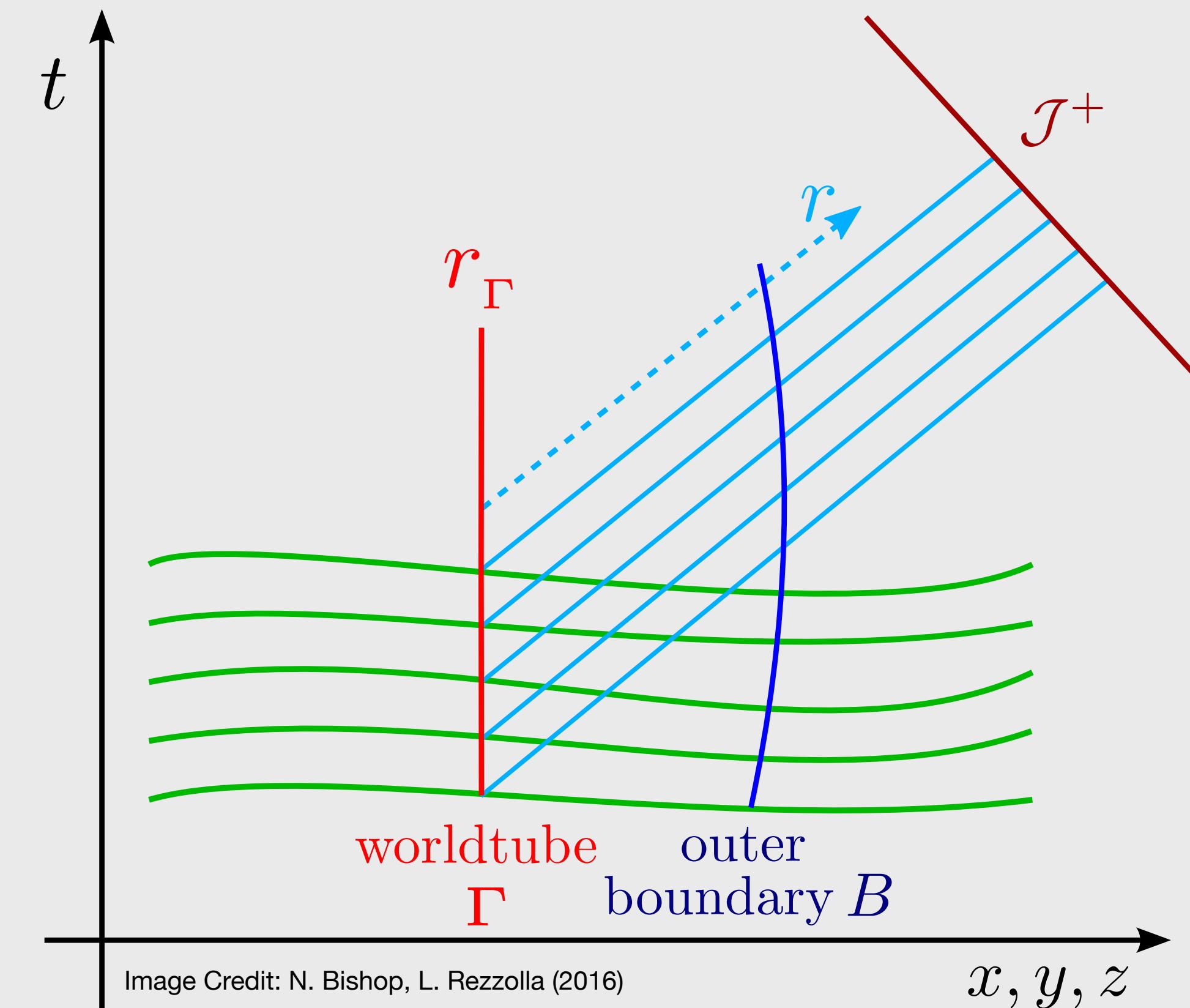
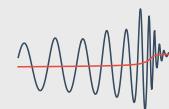


Image Credit: N. Bishop, L. Rezzolla (2016)

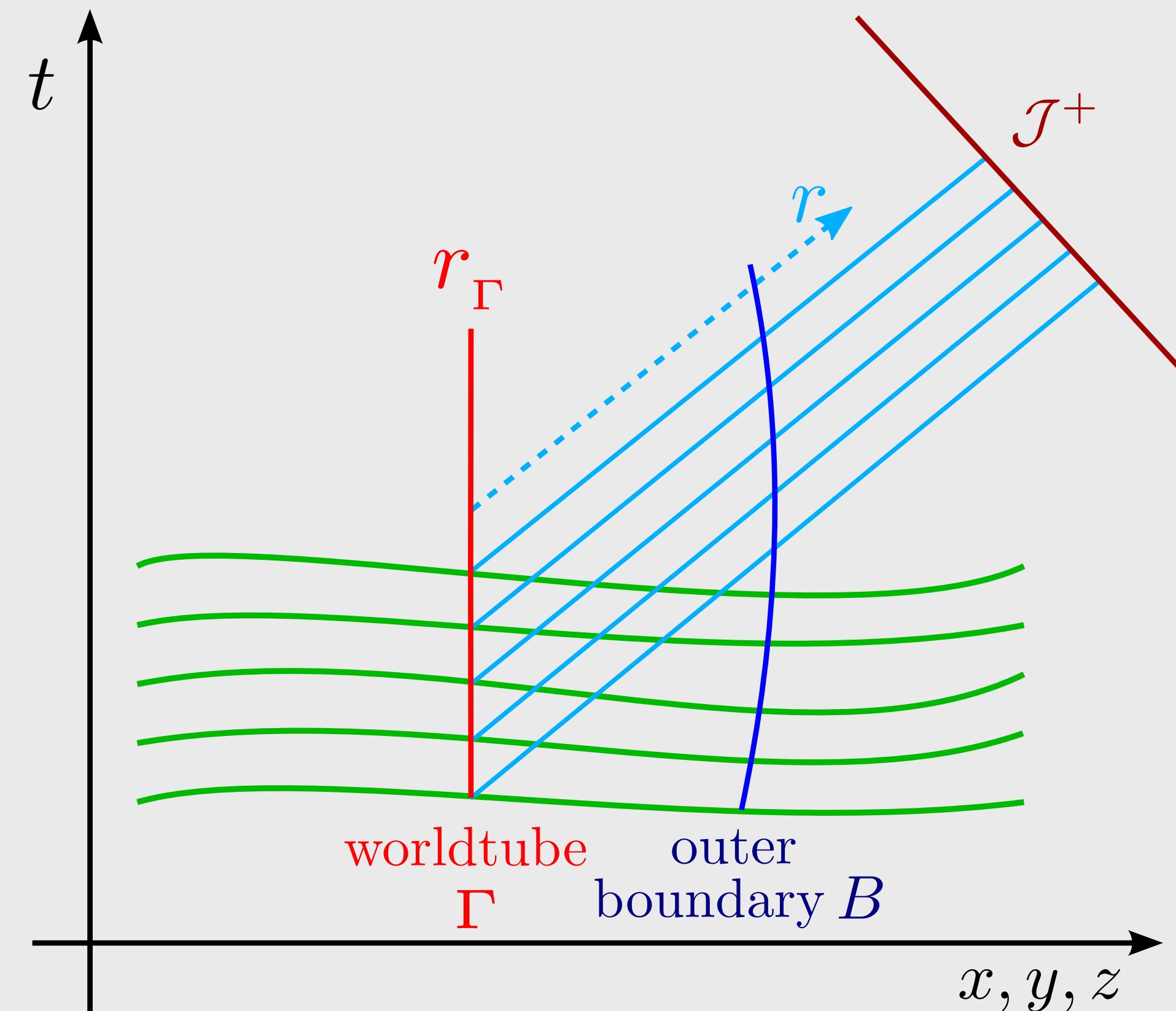
# Extracting Waveforms in Numerical Relativity: CCE

## ► Cauchy-characteristic Evolution (CCE) (see Moxon *et al.*, arXiv:2007.01339 and arXiv:2110.08635)



Initial Data (currently an open problem):

- Need to specify  $J$  on initial null hypersurface
- ZeroNonSmooth:
  - Fix  $J = 0$
- NoIncomingRadiation:
  - Fix  $J$  by requiring  $\Psi_0 = 0$  (2nd-order radial ODE for  $J$ )
- InverseCubic:
  - Fix  $J = \frac{A}{r} + \frac{B}{r^3}$  and set  $A$  and  $B$  by using the values of  $J$  and  $\partial_r J$  on the worldtube (no  $r^{-2}$  to avoid logarithms)
- ★ → ConformalFactor:
  - Use InverseCubic, but transform the coordinates so that the time coordinate resembles the Bondi time coordinate



# Extracting Waveforms in Numerical Relativity: CCE

- Cauchy-characteristic Evolution (CCE) (see Moxon *et al.*, arXiv:2007.01339 and arXiv:2110.08635)

$$ds^2 = -Ue^{2\beta}du^2 - 2e^{2\beta}dudr$$

$$+ r^2\gamma_{AB}(d\theta^A - \mathcal{U}^A du)(d\theta^B - \mathcal{U}^B du)$$



$$\mathcal{U} \equiv \mathcal{U}^A q_A$$

$$Q \equiv r^2 e^{-2\beta} q^A \gamma_{AB} \partial_r U^B$$

$$rW \equiv U - 1$$

$$J \equiv \frac{1}{2}q^A q^B \gamma_{AB}$$

$$K \equiv \frac{1}{2}q^A \bar{q}^B \gamma_{AB}$$



$$\partial_y \beta = S_\beta(J)$$

$$\partial_y \left( (1-y)^2 Q \right) = S_Q(J, \beta)$$

$$\partial_y \mathcal{U} = S_{\mathcal{U}}(J, \beta, Q)$$

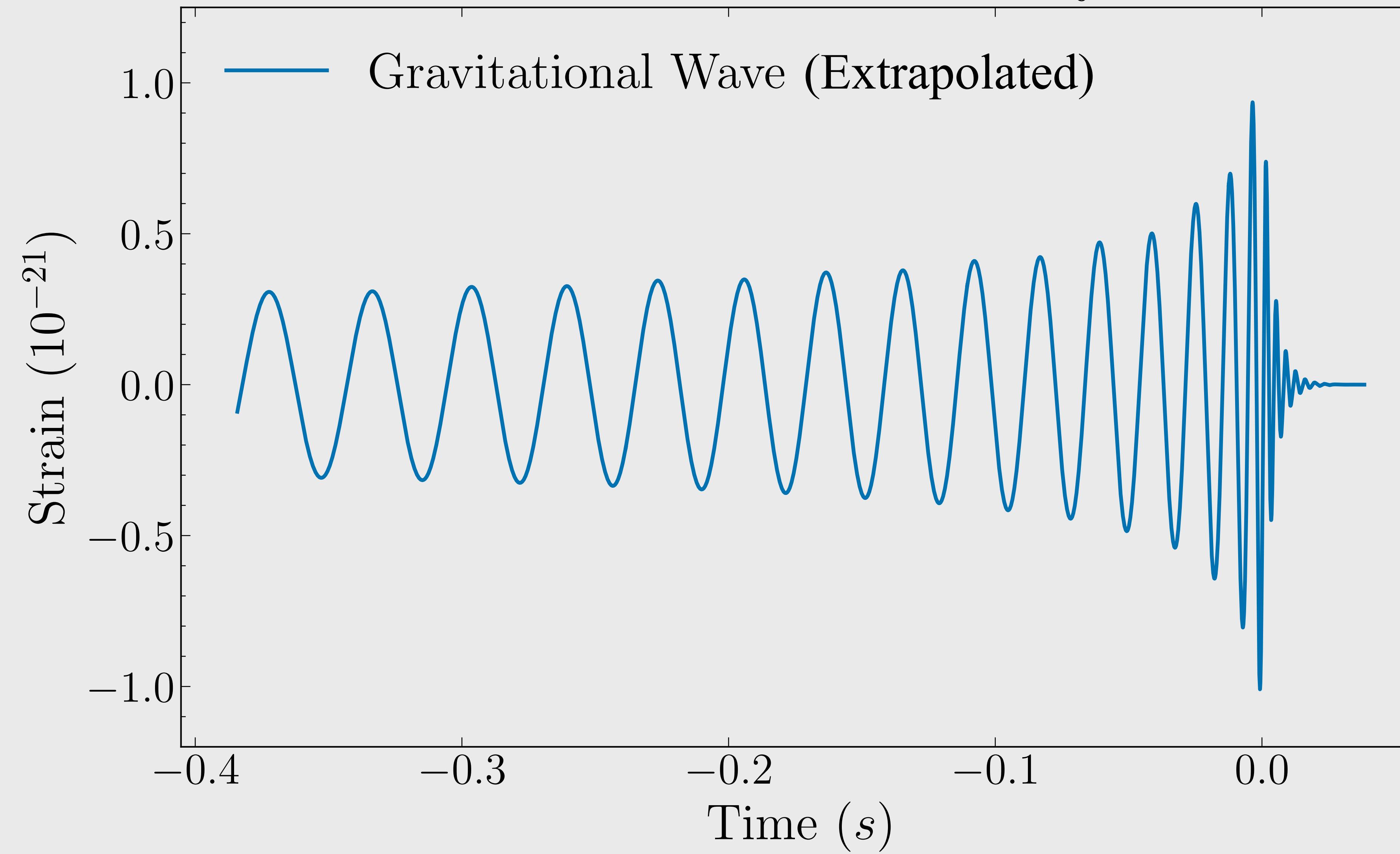
$$\partial_y \left( (1-y)^2 W \right) = S_W(J, \beta, \mathcal{U})$$

$$\left[ \partial_y \left( (1-y) H \right) + \text{Re} \left[ L_H (J, \beta, Q, \mathcal{U}, W) H \right] \right] = S_H (J, \beta, Q, \mathcal{U}, W)$$

$$\partial_u J = H$$

# Waveforms in Numerical Relativity

GW150914-like numerical relativity simulation



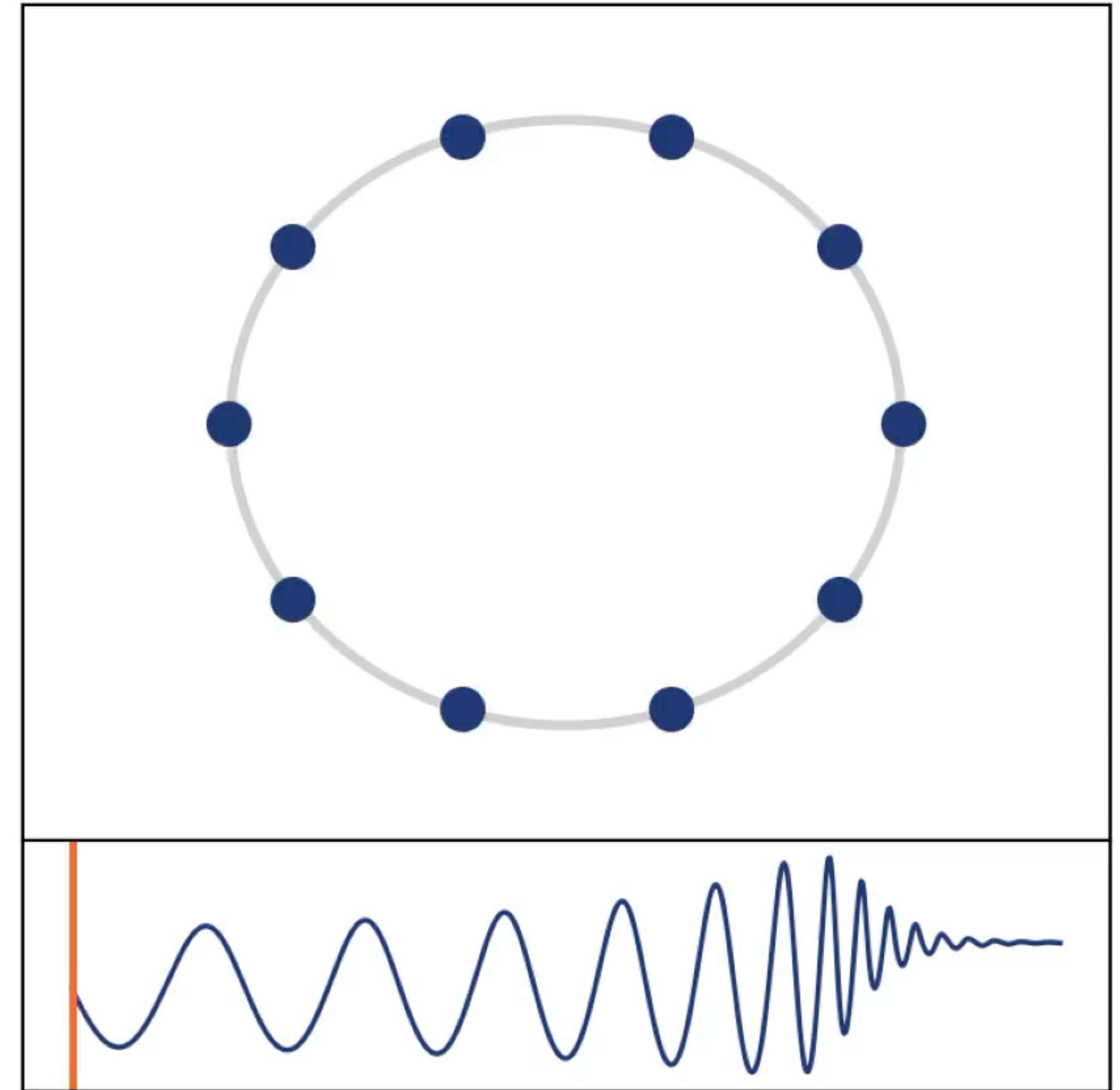
# Gravitational Wave Memory Effect

## ► What is it?

- ➡ Permanent, net displacement between two initially comoving observers
- ➡ Nonlinear\* prediction of general relativity
- ➡ (see, e.g., Mitman *et al.*, arXiv:2405.08868)

## ► Why do we care?

- ➡ Intimately connected to the symmetries of future null infinity
  - Think celestial holography or AdS/CFT
- ➡ Not yet observed!
- ➡ Intriguing and unique way to test Einstein's theory at low-frequencies



# Detecting Gravitational Memory

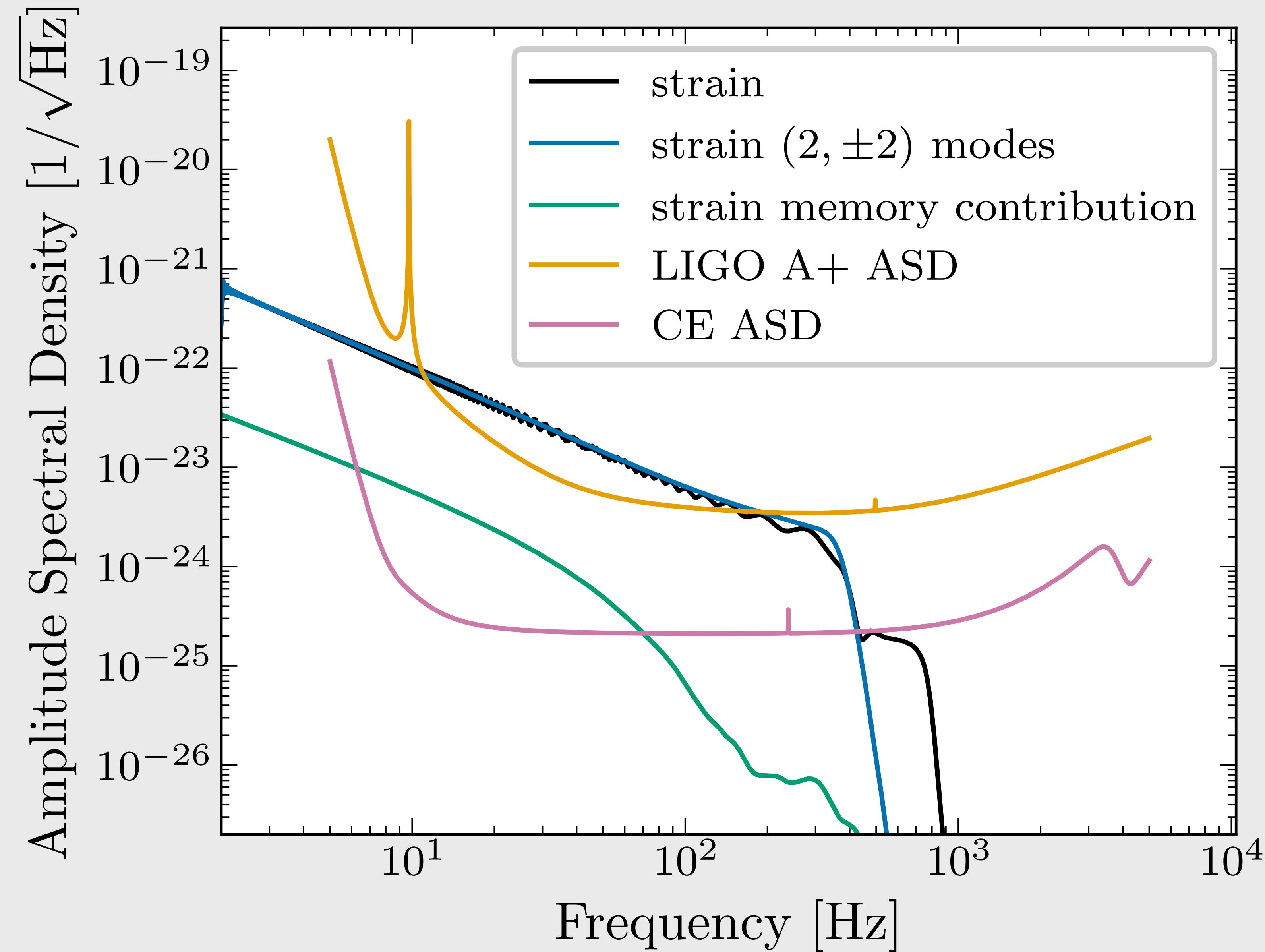
## ► Ground-based:

● LIGO:  
 $\mathcal{O}(2,000)$  events\*  
arXiv:1911.12496  
arXiv:2105.02879  
arXiv:2210.16266  
arXiv:2404.11919

● CE/ET:  
 $\mathcal{O}(1)$  event per year  
arXiv:2210.16266

## ► LISA:

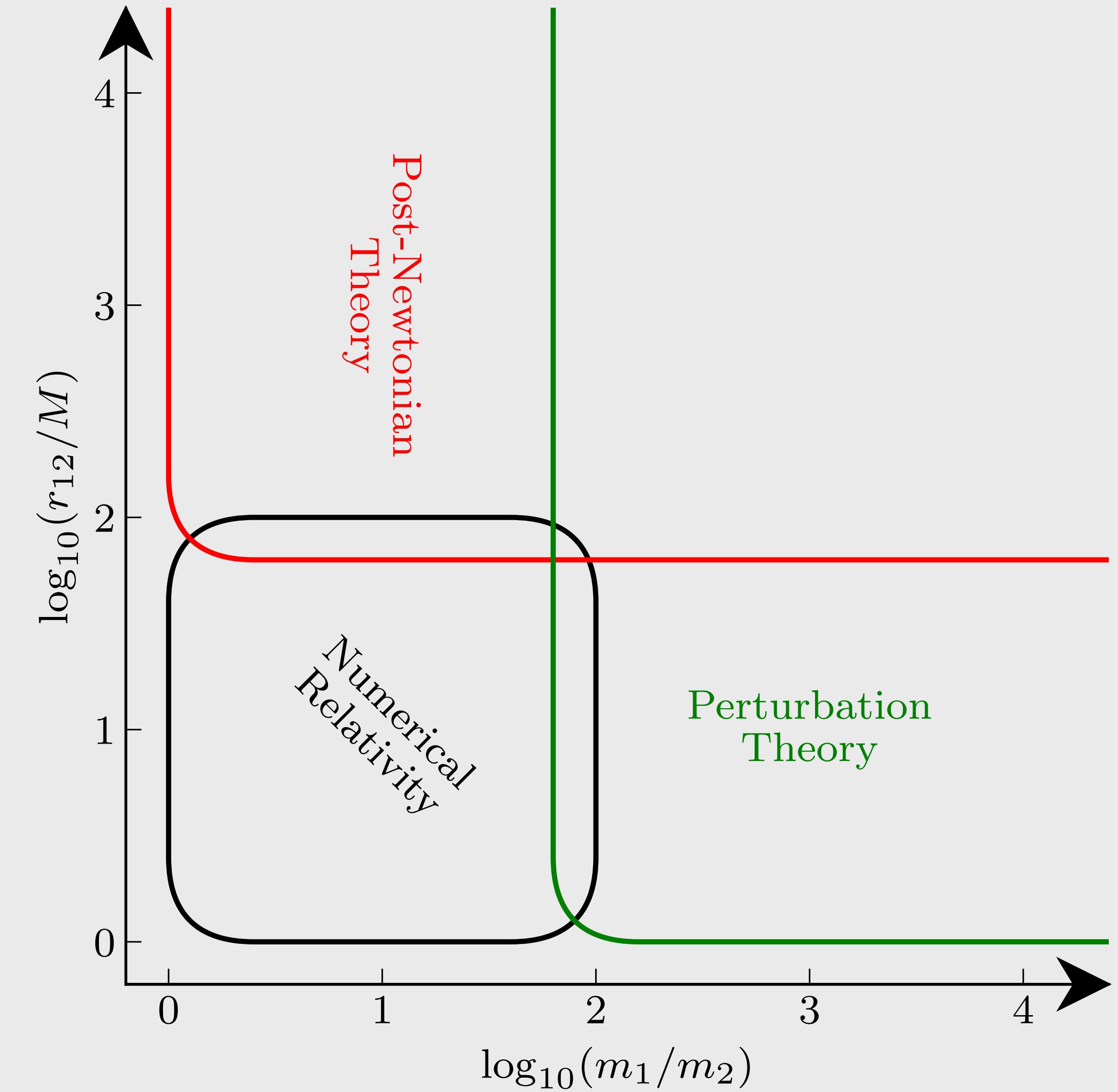
●  $\mathcal{O}(10)$  event per 4 years!  
arXiv:1906.11936



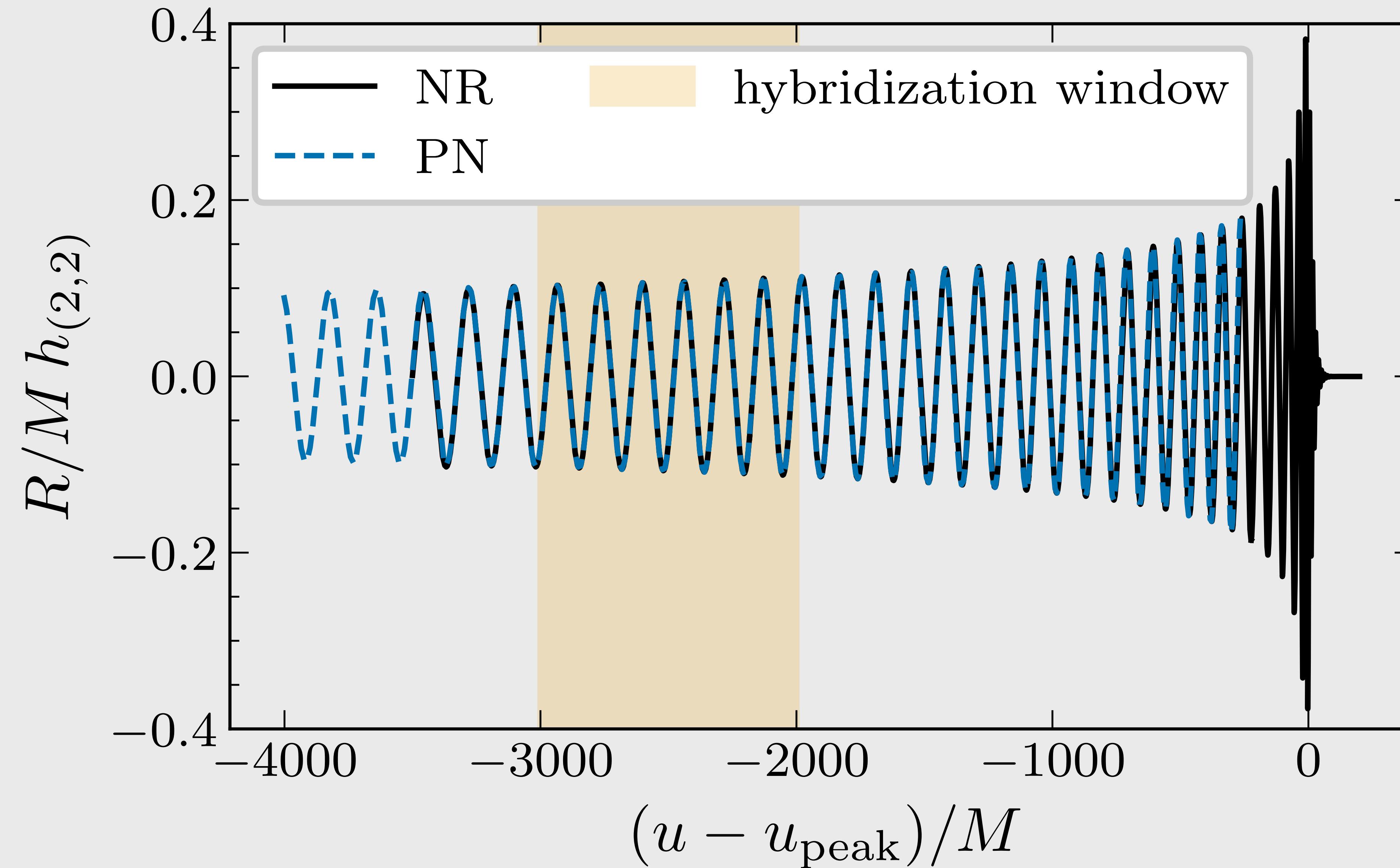
# Modeling Binary Black Hole Waveforms

## ► What's needed for waveform modeling?

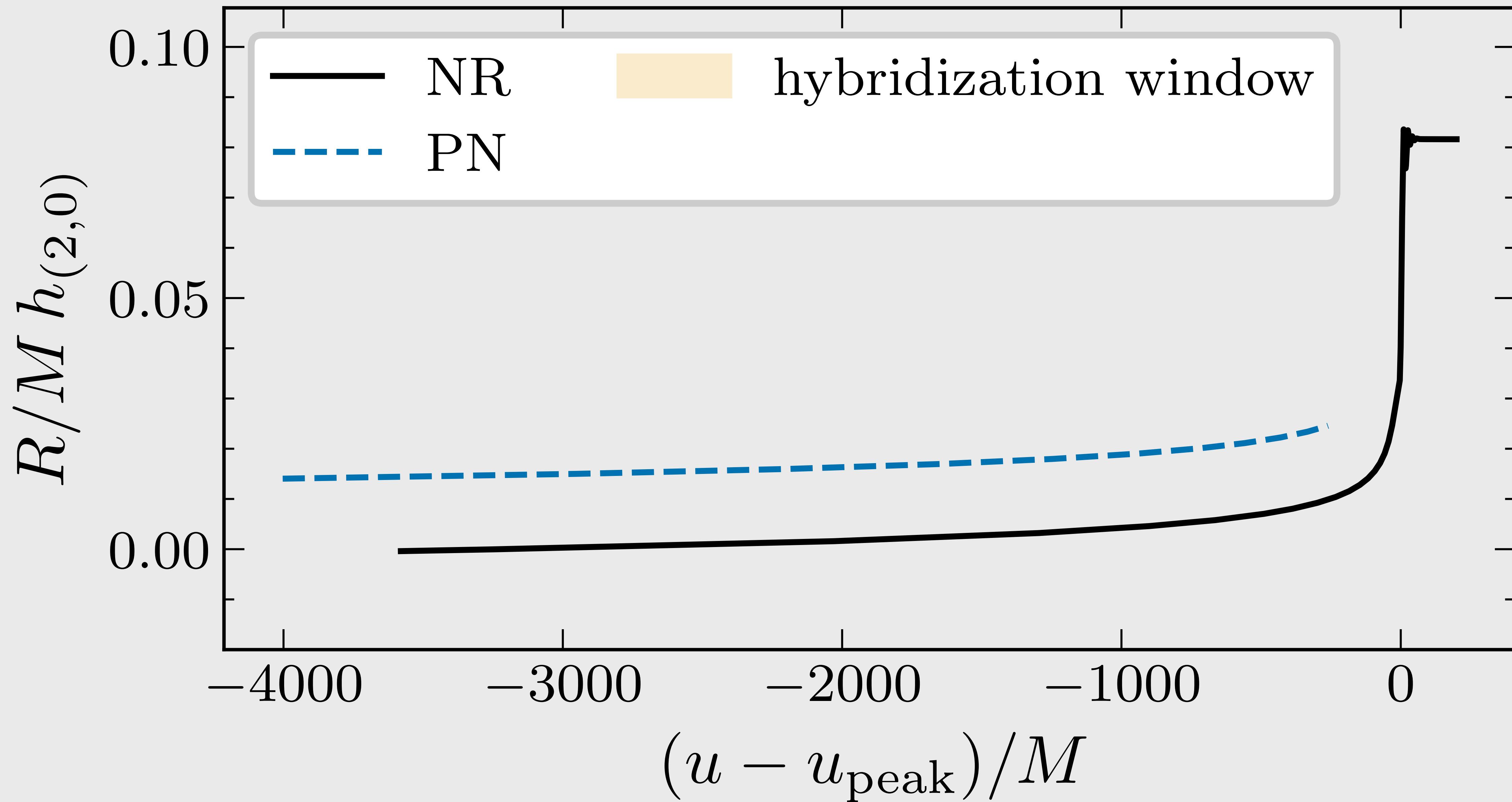
- Need infinitely long waveforms for accurate frequency domain spectra
- NR waveforms are finite in length
- Can be extended by “hybridizing” with analytic models, like post-Newtonian (PN) waveforms
- Crucial for building robust waveform models for GW detectors



# Comparing NR Waveforms to post-Newtonian Waveforms



# Comparing NR Waveforms to post-Newtonian Waveforms



## The Hard Way:

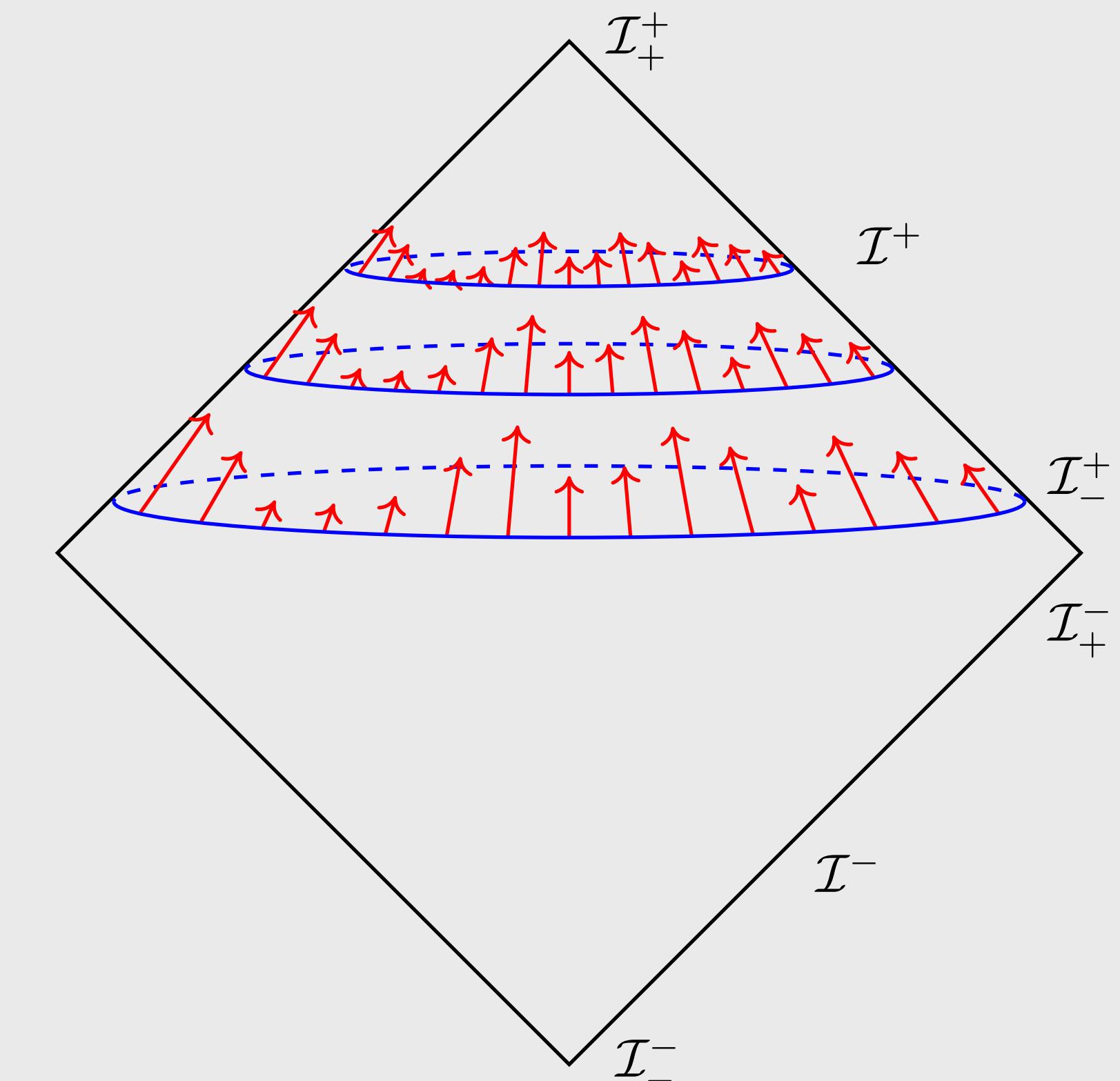
- ▶ Find an equivalence class of vector fields  $\vec{\xi}$  satisfying Killing's equation  $\mathcal{L}_{\vec{\xi}} g_{ab} = 0$  (approximately, i.e., w.r.t. fall-off conditions) as one approaches future null infinity  $\mathcal{J}^+$
- ▶ Yields
$$\vec{\xi} = \left[ \alpha(\theta^A) + \frac{1}{2} u D_A Y^A(\theta^B) \right] \partial_u + Y^A(\theta_B) \partial_A$$

where

$$Y^A = D^A \chi + \epsilon^{AB} D_B \kappa = \text{boost} + \text{rotation}$$

## The Easy Way:

- ▶ Consider a collection of observers on the celestial two sphere



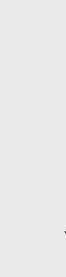
# Action of a BMS Transformation

When acted on by a BMS transformation  $\left( \alpha(\zeta, \bar{\zeta}), (a, b, c, d) \right) \dots$

$$\rightarrow (u, \zeta) \rightarrow (u', \zeta') = \left( k(u - \alpha), \frac{a\zeta + b}{c\zeta + d} \right)$$

$$\sigma(u, \zeta, \bar{\zeta}) \rightarrow \sigma'(u', \zeta', \bar{\zeta}) = \sigma(u', \zeta, \bar{\zeta}) - \partial^2 \alpha(\zeta, \bar{\zeta})$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\alpha(\zeta, \bar{\zeta}) \frac{\partial}{\partial u} \right)^n \sigma(u, \zeta, \bar{\zeta}) - \partial^2 \alpha(\zeta, \bar{\zeta})$$

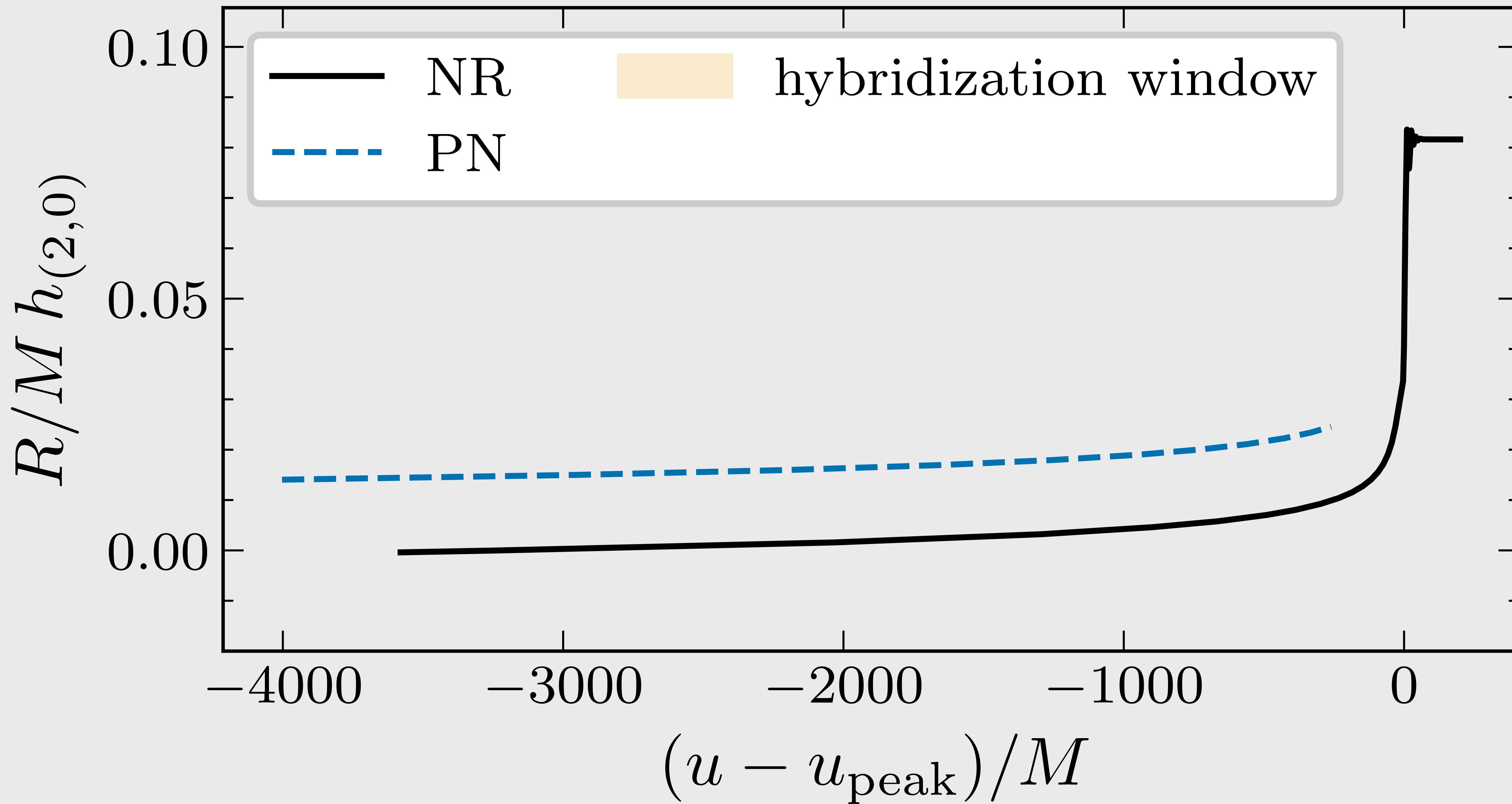


Mixing between  
the modes



Constant change  
in the modes

# Can a BMS Transformation Explain NR/PN Differences?



## Wald-Zoupas/Dray Charges:

$$Q_{\vec{\xi}} = \frac{1}{4\pi} \int d^2\Omega \left[ \alpha m + \frac{1}{2} Y^A \hat{N}_A \right]$$

where

$$m \equiv - \operatorname{Re} [\psi_2 + \sigma \dot{\bar{\sigma}}]$$

$$\hat{N} \equiv - \left( \psi_1 + \sigma \eth \bar{\sigma} + \frac{1}{2} \eth (\sigma \bar{\sigma}) + u \eth m \right)$$

## Balance Laws:

For  $u_0$  a non-radiative section of  $\mathcal{J}^+$ ,

$$\operatorname{Re} [\eth^2 \sigma] = \operatorname{Re} [m + \mathcal{E} + (\eth^2 \sigma - m) |^{u_0}]$$

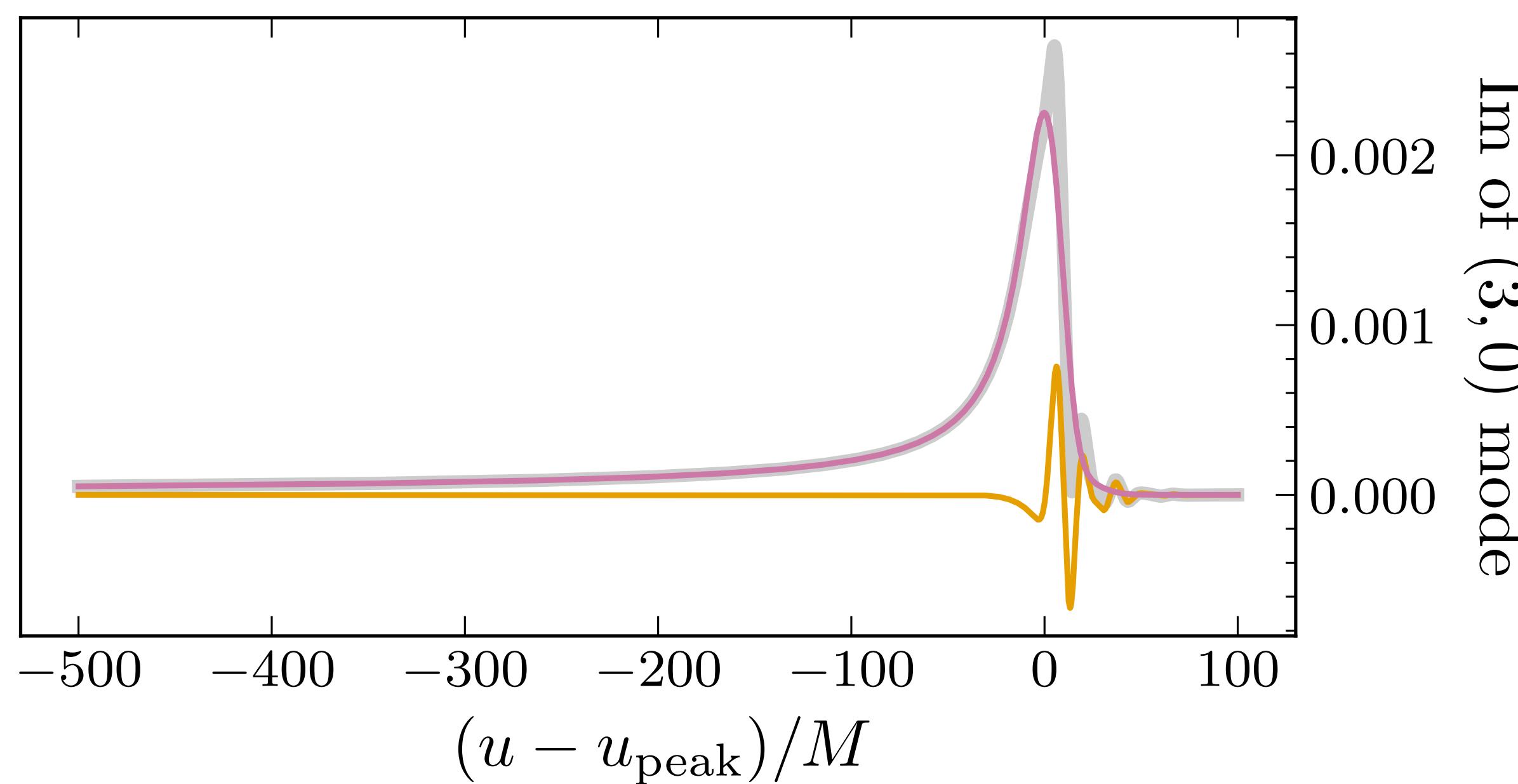
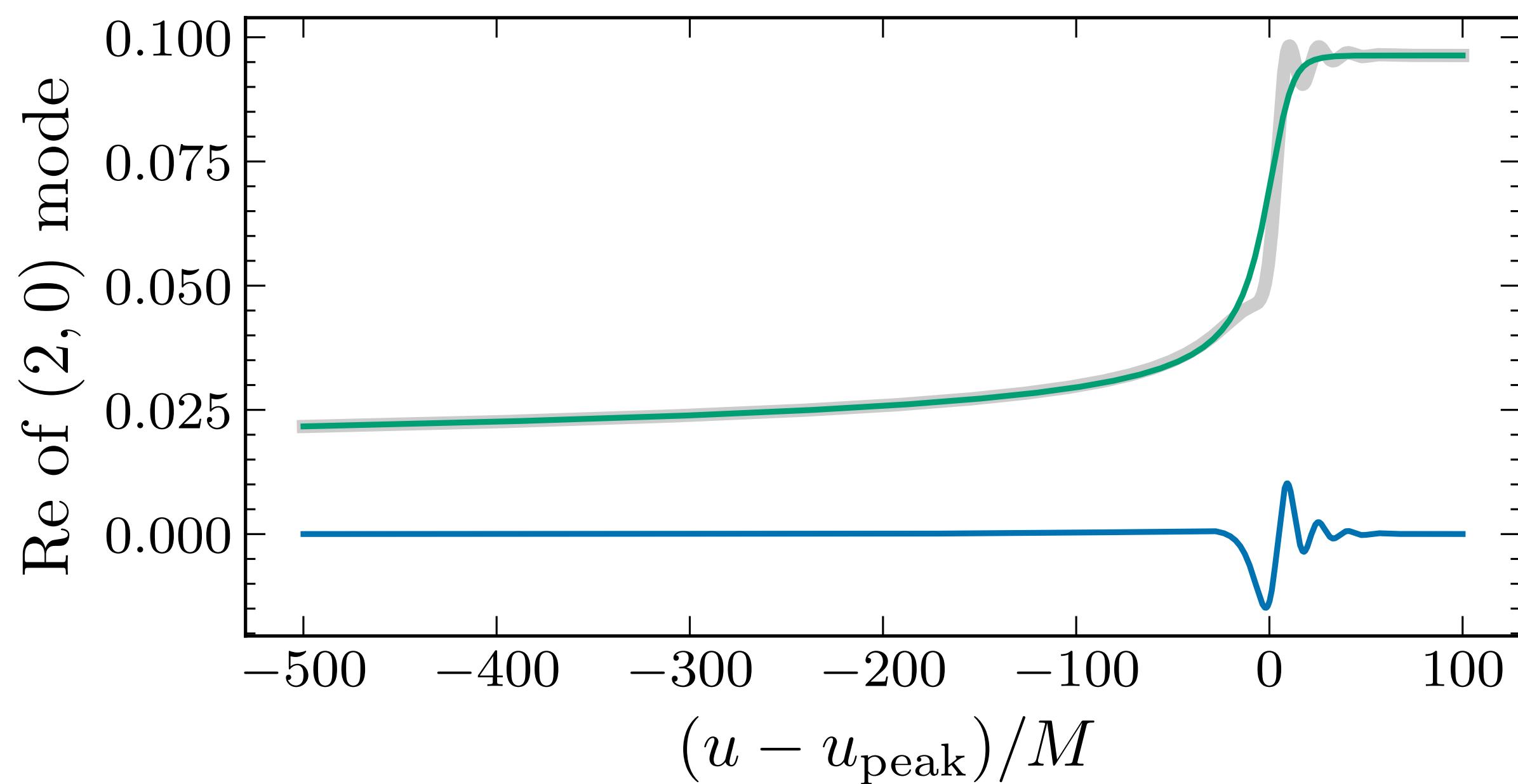
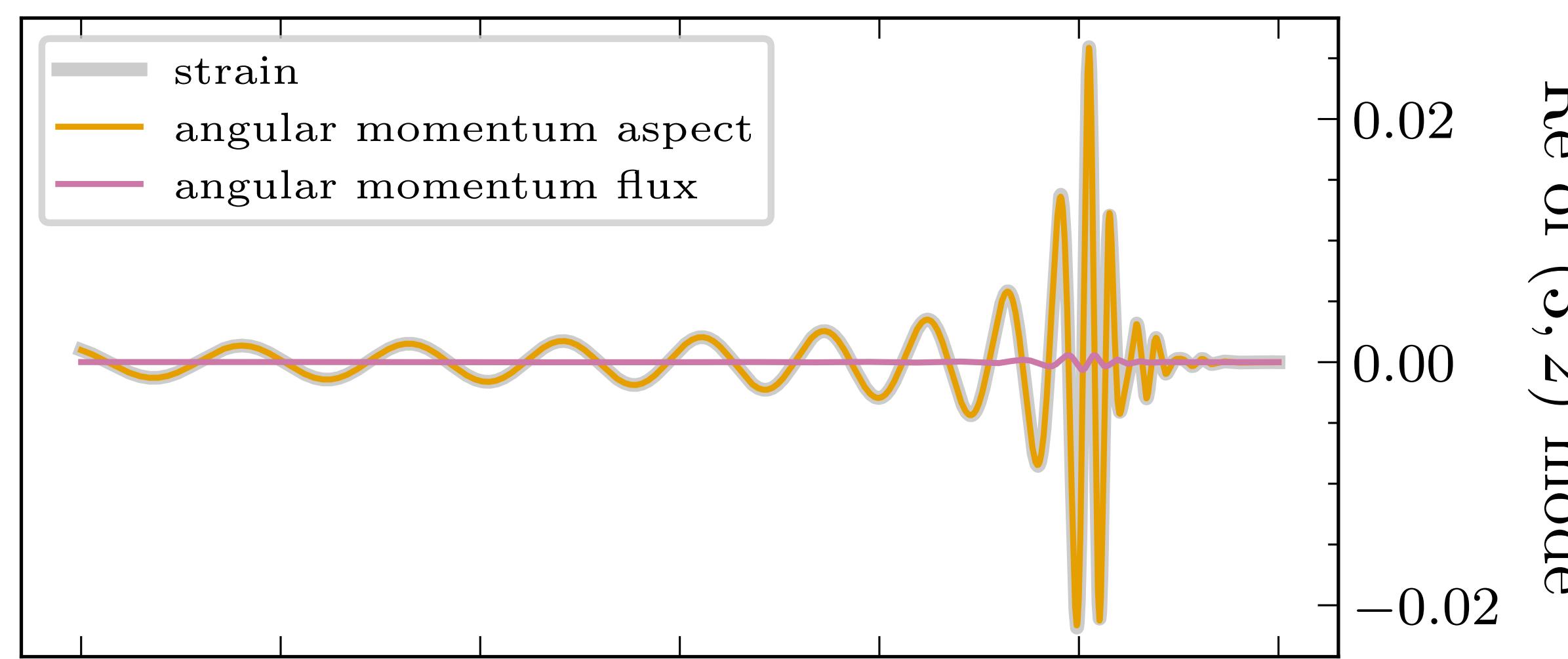
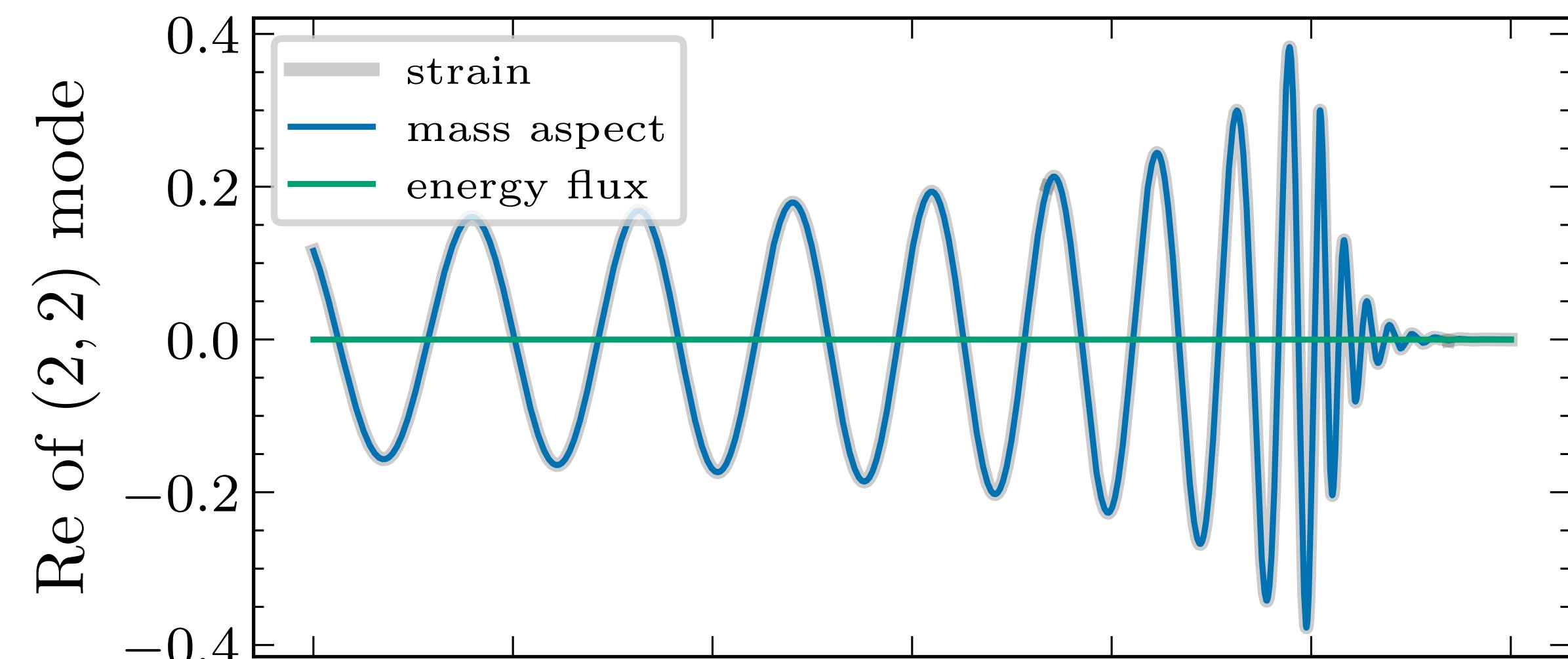
$$\operatorname{Im} [\eth^2 \sigma] = - \frac{d}{du} (\eth \bar{\eth})^{-1} \operatorname{Im} [\eth (\hat{N} + \mathcal{J})]$$

where

$$\mathcal{E} \equiv \int_{u_0}^u |\dot{\sigma}|^2 du$$

$$\mathcal{J} \equiv \frac{1}{2} \int_{u_0}^u (3\dot{\sigma} \eth \bar{\sigma} - 3\sigma \eth \dot{\bar{\sigma}} + \bar{\sigma} \eth \dot{\sigma} - \dot{\bar{\sigma}} \eth \sigma) du$$

# Noether's Theorem and BMS Charges



## Poincaré Frame:

- ▶ translations
- ▶ rotations
- ▶ boosts

## Poincaré Frame:

- ▶ translations
  - ▶ rotations
  - ▶ boosts
- 

## Poincaré Charges:

$$P^a(u) = \frac{1}{4\pi} \int_{S^2} n^a m d\Omega;$$

$$J^a(u) = \frac{1}{4\pi} \int_{S^2} \text{Im} \left[ (\bar{\delta}n^a) \hat{N} \right] d\Omega;$$

$$K^a(u) = \frac{1}{4\pi} \int_{S^2} \text{Re} \left[ (\bar{\delta}n^a) \hat{N} \right] d\Omega$$

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## Other “Charges”:

Center-of-Mass Charge:

$$G^a(u) = (K^a + u P^a) / P^t$$

Angular Velocity:

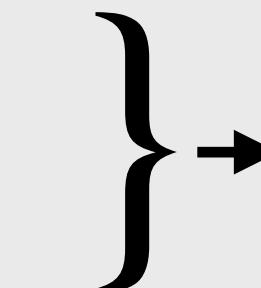
$$\vec{\omega} = - \langle \overrightarrow{L} \overrightarrow{L} \rangle^{-1} \cdot \langle \overrightarrow{L} \partial_t \rangle$$

$$\langle \overrightarrow{L} \overrightarrow{L} \rangle^{ab} \equiv \sum_{\ell, m, m'} \bar{f}^{\ell, m'} \langle \ell, m' | L^{(a} L^{b)} | \ell, m \rangle f^{\ell, m}$$

$$\langle \overrightarrow{L} \partial_t \rangle^a \equiv \sum_{\ell, m, m'} \text{Im} \left[ \bar{f}^{\ell, m'} \langle \ell, m' | L^a | \ell, m \rangle \partial_t f^{\ell, m} \right]$$

## Poincaré Frame:

- ▶ translations
- ▶ boosts



**center-of-mass charge  
“rest frame”**

- ▶ rotations

**rotation charge  
“aligned with  $\hat{z}$ -axis”**

## Poincaré Charges:

$$P^a(u) = \frac{1}{4\pi} \int_{S^2} n^a m d\Omega;$$

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# Fixing the Poincaré Frame

## Poincaré Frame:

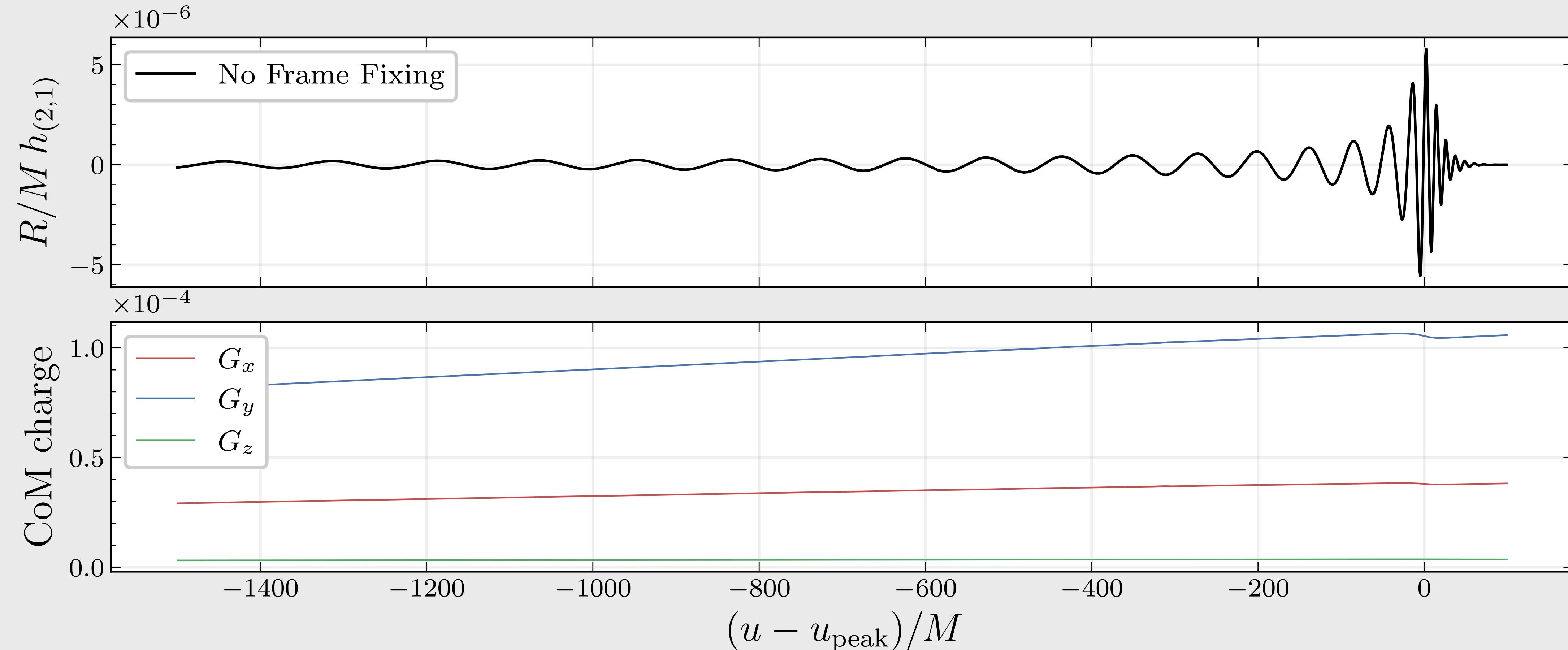
- ▶ translations
- ▶ boosts

}

center-of-mass charge  
“rest frame”

- ▶ rotations

rotation charge  
“aligned with  $\hat{z}$ -axis”



# Fixing the Poincaré Frame

## Poincaré Frame:

- ▶ translations
- ▶ boosts

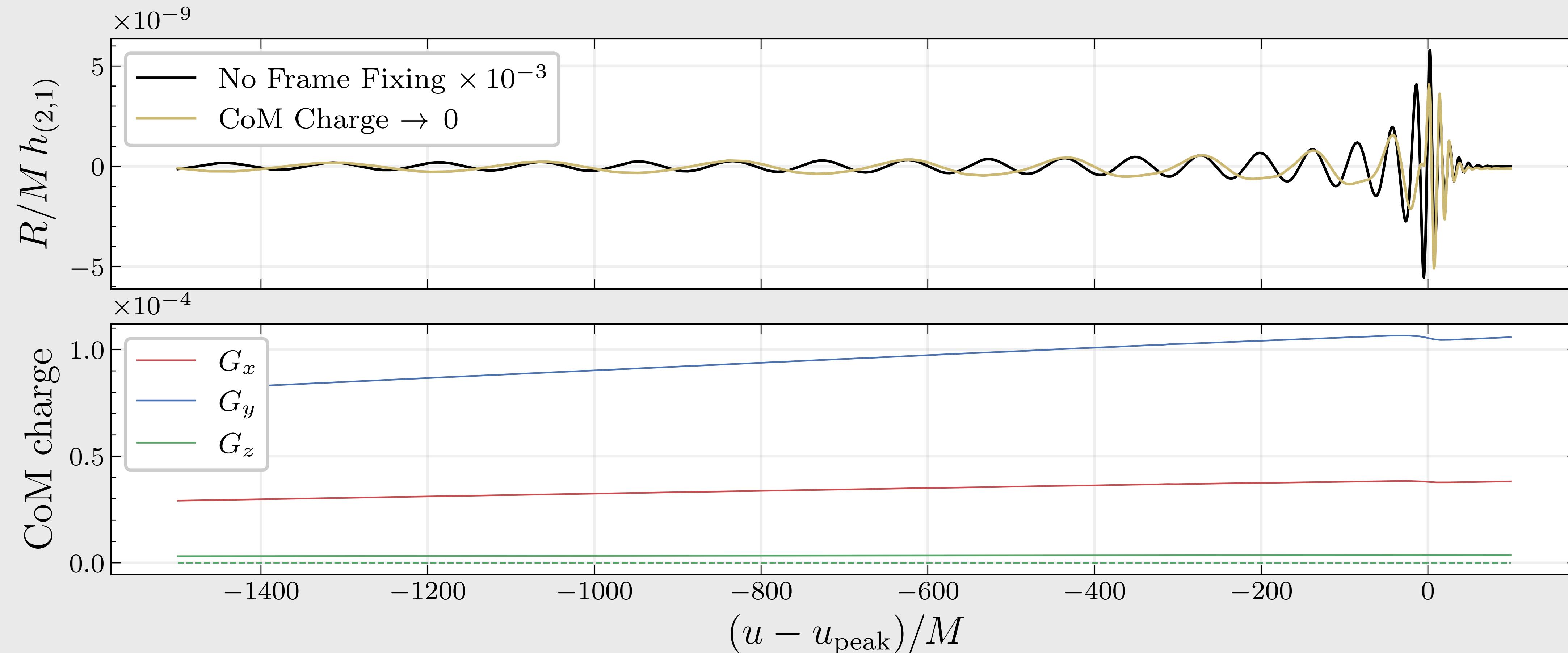
} →

**center-of-mass charge  
“rest frame”**

- ▶ rotations

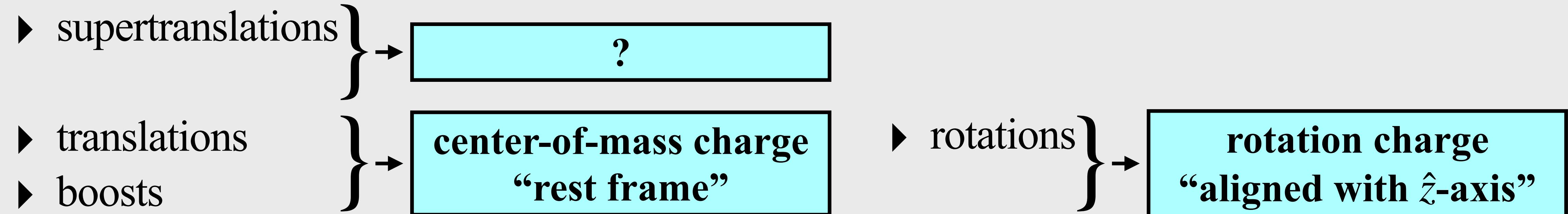
} →

**rotation charge  
“aligned with  $\hat{z}$ -axis”**



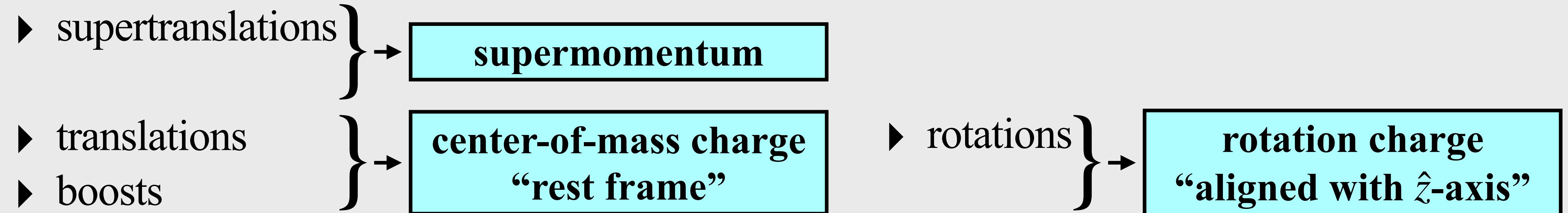
# Fixing the BMS Frame

## BMS Frame:



# Fixing the BMS Frame

## BMS Frame:



# Fixing the BMS Frame

- ▶ Use the supermomentum from Dray/Streubel:

$$\Psi_{p,q} = \Psi_2 + \sigma \dot{\bar{\sigma}} + p (\eth^2 \bar{\sigma}) - q (\bar{\eth}^2 \sigma)$$

- $p = q \Rightarrow$  no supermomentum flux in Minkowski space;
- $p + q = 1 \Rightarrow$  supermomentum is real
- For  $p = q = 1/2$ , we have the Geroch supermomentum:  $\Psi_G = \Psi_2 + \sigma \dot{\bar{\sigma}} + \frac{1}{2} (\eth^2 \bar{\sigma} - \bar{\eth}^2 \sigma)$   
(which is just the Bondi mass aspect), but this is supertranslation-covariant in non-radiative regimes of  $\mathcal{I}$

- ▶ Moreschi supermomentum ( $p = 1, q = 0$ ):

$$\Psi_M = \Psi_2 + \sigma \dot{\bar{\sigma}} + \eth^2 \bar{\sigma} = \int_{-\infty}^u |\dot{\sigma}|^2 du - M_{ADM}$$

- Measures the “supertranslation flux” in Minkowski space, i.e., the “instantaneous null memory”
- Enables “superrest frames” to be computed iteratively:  $\eth^2 \bar{\eth}^2 \alpha = \Psi_M (u = \alpha, \zeta, \bar{\zeta}) + k_{rest} (\alpha, \zeta, \bar{\zeta})^3 M_B(\alpha)$

# Fixing the BMS Frame to Match PN Waveforms

**PN waveforms are in a “canonical” BMS frame**

$$m(u \rightarrow -\infty, \zeta, \bar{\zeta}) = m_0 = \text{constant}$$

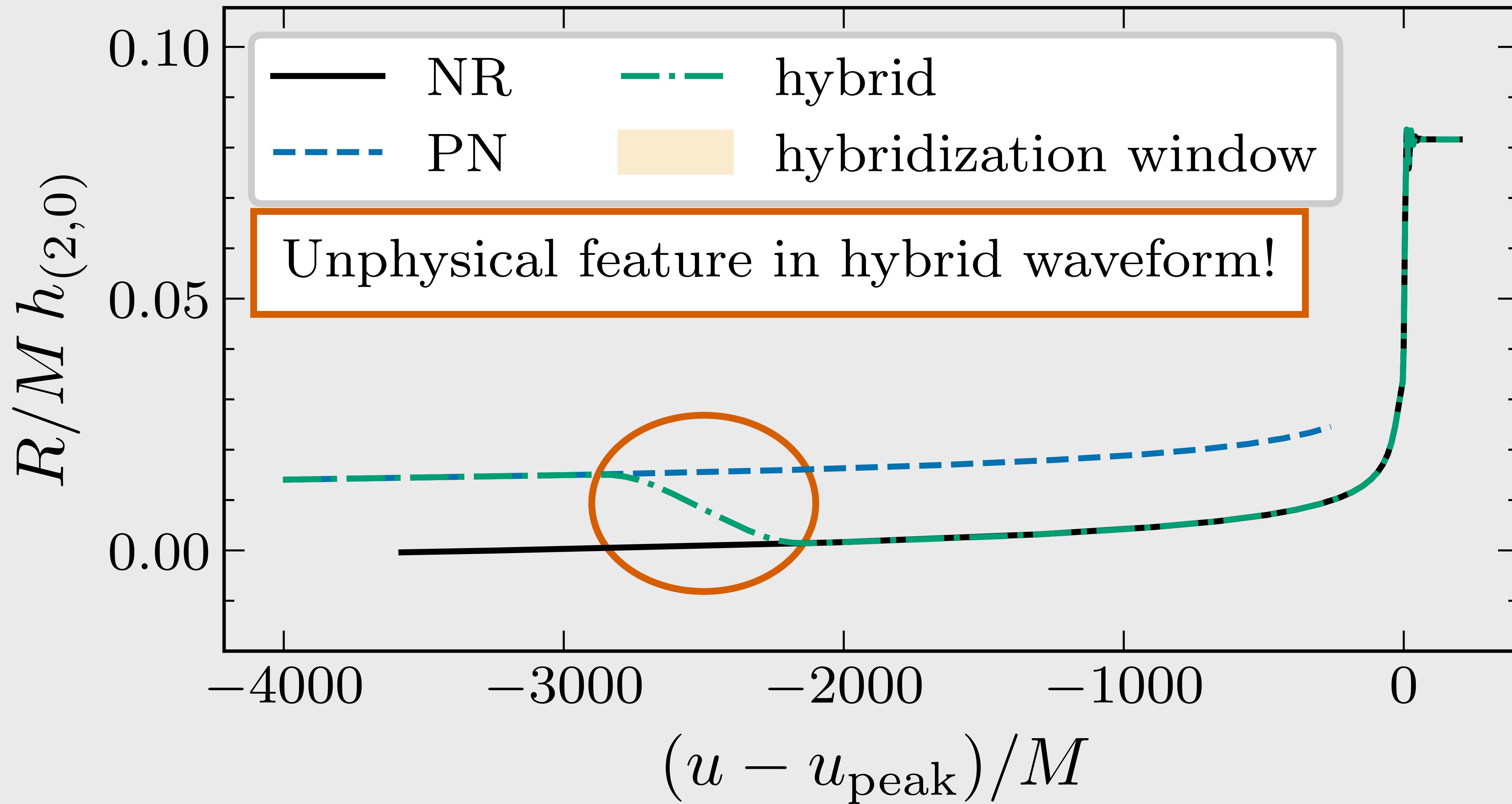
$$\sigma(u \rightarrow -\infty, \zeta, \bar{\zeta}) = 0$$

$$\hat{N}(u \rightarrow -\infty, \zeta, \bar{\zeta}) = \text{magnetic parity, } \ell = 1 \text{ (only rotational)}$$

- ▶ Map to this frame by mapping NR Poincaré charges to PN Poincaré charges and...
- ▶ by mapping the NR supermomentum to the PN supermomentum
- ▶ Need to know the PN Moreschi supermomentum
  - ➡ Can compute this “easily” because

$$\Psi_M(u, \zeta, \bar{\zeta}) = \int_{-\infty}^u |\dot{\sigma}|^2 du - M_{ADM}$$

# Comparing NR Waveforms to post-Newtonian Waveforms



# Thank you!

