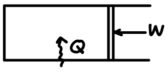
Energy in Thermal Physics.

Ideal Gas Law $PV=nRT$, $PV=Nk_BT$. (low density).

Equipartition Theorem $\frac{1}{2}k_BT$ per quadratic degree of freedom.

Heat & Work $\Delta U=Q+W$



Compression Work

Quasistatic: Slow and smooth enough to maintain equilibrium

For quasistatic $W=-P\Delta V$. Other: $W=-\int_{V_i}^{V_f} P\,dV$.

- Isothermal: T is same, $W=Nk_BT\ln\left(\frac{V_i}{V_f}\right)$. $\Delta U=0, Q=-W$.
- Adiabatic: $Q=0$. $VT^{f/2}=V_iT_i^{f/2}$

Heat Capacity

- $V=$ const, $C_V=\left(\frac{\partial U}{\partial T}\right)_V=\frac{f}{2}Nk_B$.
- $P=$ const, $C_P=\left(\frac{\partial U}{\partial T}\right)_P+P\left(\frac{\partial V}{\partial T}\right)_P=\left(\frac{f}{2}+1\right)Nk_BT$.

Enthalpy $H=U+PV$.

At constant P, $dH=dU+PdV=Q+W_{other}\Rightarrow C_P=\left(\frac{\partial H}{\partial T}\right)_P$.

The Second Law

Microstate: all info required to exactly specify state

Macrostate: Many-to-one function of microstate

Multiplicity: Number of microstates. $\Omega(n)$

Two-state system $\Omega(n)=\frac{N!}{n!(N-n)!}=\binom{N}{n}$.

Einstein Solid $E_j=j\hbar\omega$ for a SHO. N SHOs with total energy $E=q\hbar\omega$. $\Omega(N,q)=\frac{(q+N-1)!}{q!(N-1)!}$

Two Einstein Solids $E=E_A+E_B=\hbar\omega(q_A+q_B)=\hbar\omega q$.

Fundamental assumption: In a isolated system in thermal equilibrium, all accessible microstates are equally probable.

$\Omega(N_A,q_A,N_B,q_B)=\Omega(N_A,q_A)\times\Omega(N_B,q_B)$.

$P=\frac{\Omega(N_A,q_A,N_B,q_B)}{\Omega(N,q)}$.

Stirling's approximation $N!\approx\left(\frac{N}{e}\right)^N\sqrt{2\pi N}$

$\Rightarrow \ln N!\approx N\ln N-N$. $\ln\binom{N}{r}\approx N\ln N-r\ln r-(N-r)\ln(N-r)$

For Eistein solid: $\Omega=\left(\frac{eN}{q}\right)^q$ if $N\gg q$.

$q_A=\frac{q}{2}+x\Rightarrow P(x)=P(0)e^{-\frac{4Nx^2}{q^2}}$. Gaussian.

$\sigma(q_A)=\sigma(x)=\frac{q}{N}\sqrt{N}$

The Ideal Gas

$$\Omega(U,N,V)=\frac{1}{N!}\frac{2\pi^{DN/2}}{\left(\frac{DN}{2}-1\right)!}\frac{V^N\left(\sqrt{2mU}\right)^{DN}}{h^{DN}}\frac{dU}{2U}$$

$$=f(N)V^NU^{DN/2}\frac{dU}{2U}.$$

Entropy $S=k_B\ln\Omega$.

Second law: $\Delta S\geq 0$. Two parts: $S_{total}=S_A+S_B$

Entropy of an Ideal Gas

$k\ln\Omega\sim kN\left[\ln\left(\frac{V}{N}\sqrt{\frac{4\pi mU}{DNh^2}}^D\right)+\frac{5}{2}\right]$ Sackur-Tetrode.

$S(\lambda N,\lambda V,\lambda U)=\lambda S(N,V,U)\Rightarrow$ extensive

Reversible $\Delta S=0$ Irreversible $\Delta S>0$

Interactions and Implications

Temperature $\frac{1}{T}=\frac{\partial S}{\partial E}$

$\frac{1}{C_V}=\frac{\partial T}{\partial U}=\frac{\partial}{\partial U}\left(1/\frac{\partial S}{\partial U}\right)=-\frac{\partial^2S}{\partial U^2}\left(\frac{\partial S}{\partial U}\right)^{-2}=-T^2\frac{\partial^2S}{\partial U^2}$.

$\Rightarrow C_V=-\frac{1}{T^2\partial^2S/\partial^2U}$

Measuring S via C: $\frac{\partial S}{\partial T}=\frac{\partial U}{\partial T}\frac{\partial S}{\partial U}=\frac{C_V}{T}\Rightarrow S_f-S_i=\int\frac{C_V}{T}dT$.

C_V For Einstein Solid $S=k_B[(N+q)\ln(N+q)-q\ln q-N\ln N]$.

$\frac{\partial S}{\partial q}=\frac{\partial S}{\partial U}\frac{\partial U}{\partial q}=\frac{\hbar\omega}{T}=k_B\ln\left(1+\frac{N}{q}\right)\Rightarrow q=\frac{N}{e^{\hbar\omega/k_BT}-1}$.

$C_V=\hbar\omega\frac{\partial q}{\partial T}=\frac{\hbar^2\omega^2}{k_BT^2}\frac{N}{(e^{\hbar\omega/k_BT}-1)^2}$.

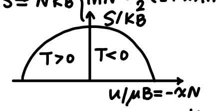
Paramagnetism N electrons: ↑↓⋯ $N=N_\uparrow+N_\downarrow$.

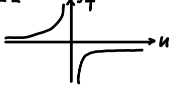
$\Omega(N,N_\uparrow)=\binom{N}{N_\uparrow}=\frac{N!}{N_\uparrow!N_\downarrow!}$. Magnetization $M=\mu(N_\uparrow-N_\downarrow)$.

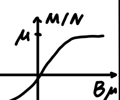
$U=-M\cdot B=\mu B(N-2N_\uparrow)$.

$x=\frac{M}{N\mu}$, $N_\uparrow=\frac{N}{2}(1+x)$, $N_\downarrow=\frac{N}{2}(1-x)$. $U=-BN\mu x$.

$S=Nk_B\left\{\ln N-\frac{1}{2}(1+x)\ln\left[\frac{N}{2}(1+x)\right]-\frac{1}{2}(1-x)\ln\left[\frac{N}{2}(1-x)\right]\right\}$.



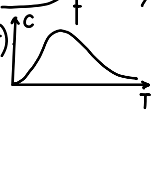




$\frac{1}{T}=\frac{\partial S}{\partial U}=\frac{\partial S}{\partial x}\frac{\partial x}{\partial U}=\frac{k}{2B\mu}\ln\left(\frac{1-U/NB\mu}{1+U/NB\mu}\right)$.

$U=-NB\mu\tanh\left(\frac{B\mu}{k_BT}\right)$ $M=N\mu\tanh\left(\frac{B\mu}{k_BT}\right)$

$C=Nk_B\frac{(\mu B/kT)^2}{\cosh^2(\mu B/kT)}$.



Pressure $P=T\frac{\partial S}{\partial V}$.

Thermodynamic Identity $dU=TdS-PdV$.

$\Rightarrow \left.\frac{\partial S}{\partial U}\right|_V=\frac{1}{T}$, $\left.\frac{\partial S}{\partial V}\right|_U=\frac{P}{T}$, $\left.\frac{\partial U}{\partial V}\right|_S=-P$.

Quasistatic: $TdS=Q$. If not, $dS\geq\frac{Q}{T}$.

$\xrightarrow[\text{Adiabatic}]{\text{Quasistatic}}$ $dS=0$. Isentropic

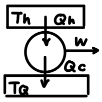
Chemical potential $\mu=-T\frac{\partial S}{\partial N}$.

$\Rightarrow dU=TdS-PdV+\mu dN$.

For ideal gas, $\mu=-k_BT\ln\left(\frac{V}{N}\left(\frac{2\pi mT}{h^2}\right)^{3/2}\right)$.

Heat Engines & Refrigerator

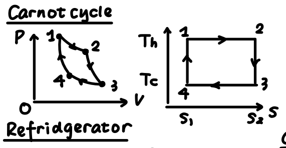
Heat Engine



$Q_h+W=Q_c$ $\frac{Q_c}{Q_h}\geq\frac{T_c}{T_h}$

$e=\frac{W}{Q_h}=1-\frac{Q_c}{Q_h}\leq 1-\frac{T_c}{T_h}$.

Carnot cycle

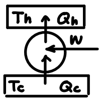


$1\to2$, $3\to4$: Isothermal

$2\to3$, $4\to1$: Adiabatic.

$\varepsilon=1-\frac{T_c}{T_h}$.

Refridgerator



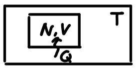
$Q_h=W+Q_c$. $\frac{Q_h}{Q_c}\geq\frac{T_h}{T_c}$

$COP=\frac{Q_c}{Q_h-Q_c}\leq\frac{T_c}{T_h-T_c}$

Heat Pump $COP=\frac{Q_h}{W}=\frac{Q_h}{Q_h-Q_c}\leq\frac{T_h}{T_h-T_c}$.

Free energies and Chemical Thermodynamics.

Free energy $F=U-TS$.



Quasistatic: System in equilibrium with itself, has single T,P,μ ⇑

Reversible: System is also in equilibrium with environment.

Always have $\Delta F=\Delta U-T\Delta S=Q+W-T\Delta S$ (T const).

$\Rightarrow \Delta F=W$ (T const, reversible $T=T_{env}$).

$\Rightarrow \Delta F=\Delta U-T\Delta S\leq\Delta U-Q=W$ (T const).

Gibbs Free Energy $G=U-TS+PV$.

$\Rightarrow \Delta G=\Delta U-T\Delta S+P\Delta V$ (constant P,T).

$=Q+W_{mech}+W_{other}-T\Delta S+P\Delta V$.

$\xrightarrow{\text{Reversible}} T\Delta S=Q, -P\Delta V=W_{mech}\Rightarrow\Delta G=W_{other}$

Generally $\Delta G\leq W_{other}$ (P,T const).

$G=H-TS\Rightarrow\Delta G=\Delta H-T\Delta S$

Thermodynamic Identities

$$dU=TdS-PdV+\mu dN$$
$$dH=TdS+VdP+\mu dN$$
$$dF=-SdT-PdV+\mu dN$$
$$dG=-SdT+VdP+\mu dN$$

$\left.\frac{\partial F}{\partial T}\right|_{V,N}=-S$. $\left.\frac{\partial F}{\partial V}\right|_{T,N}=-P$. $\left.\frac{\partial F}{\partial N}\right|_{T,V}=\mu$.

$\left.\frac{\partial G}{\partial T}\right|_{P,N}=-S$, $\left.\frac{\partial G}{\partial P}\right|_{T,N}=V$, $\left.\frac{\partial G}{\partial N}\right|_{T,P}=\mu$.

Thermodynamic Forces System exchange U at fixed V,N. $dS_{uni}=dS+dS_{env}=dS-\frac{dU}{T}\geq0$

$\Rightarrow TdS-dU=-dF\geq0$ ($T=T_{env}$) $\Rightarrow dF\leq0$. F behaves like a "potential".

Fixed P,T,N: $dS_{total}=dS-\frac{1}{T}dU-\frac{P}{T}dV=-\frac{1}{T}dG\geq0$.

- Const U,V,N, S tends to ↗
- Const T,V,N, F tends to ↘
- Const T,P,N, G tends to ↘

Extensive: U,S,V,N, PV. F. $S(\lambda U,\lambda V,\lambda N)=\lambda S(U,V,N)$.

Intensive: T, P, μ, $\frac{N}{V}=n$.

Int + Int = Int, Ext + Ext = Ext, Int × Int = Int,

Int × Ext = Ext

$\Rightarrow G=U-TS+PV=\mu N\Longrightarrow G=\sum\mu_iN_i$.

Latent Heat $1\to2$, $S_1\neq S_2$. Tc is const

$\Delta Q=T_c(S_2-S_1)$, $L=\frac{\Delta Q}{m}=T_c\left(\frac{S_2}{m}-\frac{S_1}{m}\right)$.

Clausius-Clapeyron Relation $\frac{dP}{dT}=\frac{L}{T\cdot\Delta V}$.

Boltzmann Statistics

Canonical Ensemble $P(s)=\frac{1}{Z}e^{-E(s)/kT}$.

Partition Function $Z=\sum_s e^{-E(s)/kT}$.

Average Values:

$\langle E\rangle=\sum E(s)P(s)=\frac{1}{Z}\sum_s E(s)e^{-\beta E(s)}$

Or $\langle E\rangle=-\frac{1}{Z}\frac{\partial Z}{\partial\beta}=-\frac{\partial\ln Z}{\partial\beta}$.

$C_V=\frac{d\langle E\rangle}{dT}=k_B\beta^2\partial_\beta^2\ln Z$.

Equipartition Theorem $E=\frac{1}{2}k_BT$. A state $x=(x_1,x_2,\cdots)$

$E(x)=\sum\frac{1}{2}k_ix_i^2\Rightarrow$

$Z=\int\prod_{i=1}^N dx_i e^{-\beta\sum\frac{k_i}{2}x_i^2}=\prod_{i=1}^N\int dx_ie^{-\beta k_ix_i^2/2}=\prod_{i=1}^N\sqrt{2\pi/\beta k_i}$

$\Rightarrow E=-\partial_\beta\ln Z=\frac{N}{2}k_BT$

Maxwell Speed Distribution $E=\frac{mv^2}{2}$. $P(\vec v)=\frac{1}{N}e^{-\beta m\vec v^2/2}$.

$N=\left(\frac{2\pi}{\beta m}\right)^{3/2}$. $\langle v\rangle=\int_0^\infty P(v)dv=\sqrt{\frac{8k_BT}{m}}$.

$P(v)=4\pi v^2P(\vec v)$

Free Energy $F=-kT\ln Z=-\frac{1}{\beta}\ln Z$. $Z=e^{-F/kT}=e^{-\beta F}$.

$\Rightarrow S=-\frac{\partial F}{\partial T}$, $P=-\frac{\partial F}{\partial V}$, $\mu=\frac{\partial F}{\partial N}$.

Composite System

Noninteracting ($E(x_1,x_2)=E(x_1)+E(x_2)$) and distinguishable system of N particles:

$$Z=\prod^N Z_i$$

For N indistinguishable particles, $Z=\frac{1}{N!}Z_1^N$.

Ideal Gas $\sum_x\to\frac{1}{h^3}\int d^3x\int d^3p$.

$Z_1=\frac{1}{h^3}\int d^3x\int d^3p\ e^{-\beta\frac{p^2}{2m}}=V\sqrt{\frac{2\pi mk_BT}{h^2}}^3$

$\ell_Q=\sqrt{\frac{h^2}{2\pi mkT}}\Rightarrow Z_1=\frac{V}{\ell_Q^3}$. $\Rightarrow Z_N=\frac{1}{N!}\left(\frac{V}{\ell_Q^3}\right)^N$.

$\Rightarrow \ln Z_N=N\left[\ln\left(\frac{V}{N}\frac{1}{\ell_Q^3}\right)+1\right]$

$\langle E\rangle=-\partial_\beta\ln Z=\frac{3}{2}k_BT$.

$F=-kT\ln Z=-k_BTN\left[\ln\left(\frac{V}{N}\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2}\right)+1\right]$.

$P=-\frac{\partial F}{\partial V}=\frac{Nk_BT}{V}$

$S=-kN\left[\ln\left(\frac{V}{N\ell_Q^3}\right)+\frac{5}{2}\right]$. $\mu=-k_BT\left(\frac{V}{N\ell_Q^3}\right)$.

In reality, $Z_1=\frac{V}{\ell_Q^3}Z_{int}$, $Z_{int}=\sum_{x_{int}}e^{-\beta E_{int}(x_{int})}$

$\Rightarrow Z_N=\frac{1}{N!}\left(\frac{N}{\ell_Q}\right)^3Z_{int}^N$, $F=F_{cm}+NF_{int}$

$\mu=-kT\ln\left(\frac{VZ_{int}}{N\ell_Q^3}\right)$.

$\xrightarrow{PV=NKT}\mu(T,P)=-kT\ln P-kT\ln\left(\frac{kTZ_{int}}{\ell_Q^3}\right)$.

$\xrightarrow{\mu^0=\mu(T,P^0)}\mu(T,P)=\mu^0(T)+kT\ln\left(\frac{P}{P^0}\right)$.

Quantum Statistics.

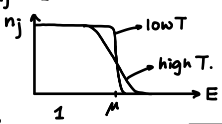
Grand Canonical Ensemble

$P(x)=\frac{1}{Z}e^{-\beta[E(x)-\mu N(x)]}$, $Z=\sum_x e^{-\beta[E(x)-\mu N(x)]}$

$\langle N\rangle=k_BT\frac{d}{d\mu}\ln Z$.

$Z=\prod_j Z_j$, $Z_j=\sum_{n_j}e^{-\beta n_j(E_j-\mu)}$

Fermion $Z_j^F=1+e^{-\beta(E_j-\mu)}\Rightarrow\langle n_j\rangle=\frac{1}{e^{\beta(E_j-\mu)}+1}$



Bosons: $Z_j^B=\frac{1}{1-e^{-\beta(E_j-\mu)}}$, $\langle n_j\rangle=\frac{1}{e^{\beta(E_j-\mu)}-1}$

Degenerate Fermi Gas

At $T=0$, $n_j=\begin{cases}1, & E_j<\mu \\ 0, & E_j>\mu\end{cases}$ filled → Fermi sea. empty

$T=0$, $H=p^2/2m$, in a L^3 box. $\vec p=\hbar\vec k$. $\vec k=\frac{2\pi}{L}(n_x,n_y,n_z)$.

Fermi level $\frac{\hbar^2k_F^2}{2m}=\mu\Rightarrow k_F=\frac{\sqrt{2m\mu}}{\hbar}$.

$N=\underset{\text{spin}}{2}\sum_k\langle n_k\rangle=2\sum_{|k|<k_F}1\to2\times\left(\frac{L}{2\pi}\right)^3\int_{k<k_F}d^3k$.

$=2\frac{V}{(2\pi)^3}\frac{4}{3}\pi k_F^3$, $k_F=\left(\frac{N}{V}3\pi^2\right)^{1/3}$.

$\mu(T=0,N,V)=\frac{\hbar^2}{2m}\left(\frac{N}{V}\cdot3\pi^2\right)^{2/3}=E_F$. Fermi energy

$\langle E\rangle=\sum_j E_j\langle n_j\rangle=2\sum_{k<k_F}\frac{\hbar^2k^2}{2m}1$.

$=2\frac{\hbar^2}{2m}\frac{V}{(2\pi)^3}\int_{k<k_F}k^2d^3k$.

$=2\frac{\hbar^2}{2m}\frac{V}{(2\pi)^3}4\pi\frac{k_F^5}{5}=\frac{3}{5}NE_F$.

At $T=0$, $F=E$. The Fermi pressure

$P(T=0)=-\frac{\partial F}{\partial V}=\frac{2}{3}\frac{E}{V}=\frac{2}{5}\frac{N}{V}E_F\propto\frac{\hbar^2}{2m}\left(\frac{N}{V}\right)^{1+2/3}$.

$B=-V\frac{\partial P}{\partial V}=\frac{10}{9}\frac{E}{V}$.

Small $T>0$: $E_F=k_BT_F$ Fermi temperature. $T\ll T_F$:

$E=E(T=0)+\frac{\pi^2}{6}g(E_F)(k_BT)^2$

$C_V=\frac{\pi^2}{3}g(E_F)k_B^2T=k_B\frac{\pi^2}{2}N\frac{T}{T_F}$.

Density of states:

$N=\int g(E)n(E)d(E)$, $E=\int g(E)n(E)EdE$.

$E_k=\frac{\hbar^2k^2}{2m}\Rightarrow N=2\sum_k1=2\left(\frac{L}{2\pi}\right)^3\int d^3k=2\left(\frac{L}{2\pi}\right)^34\pi\int k^2dk$.

$\int k^2dk=\frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int\sqrt{E}dE$.

$\Rightarrow g(E)=\frac{L^3}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{E}=\frac{3}{2}\frac{N}{E_F}$.

Blackbody Radiation

Light in L^3 box. $i=\{\vec k$, polarization$\}$. $\vec k=\frac{2\pi\vec n}{L}$.

$H^i=\hbar\omega^i\left(n^i+\frac{1}{2}\right)$, $\omega^i=kc$.

$H=\sum_i H^i=\sum_i\hbar\omega_i\left(n^i+\frac{1}{2}\right)$.

Total energy

$\langle E\rangle=\sum_i\hbar\omega^i\langle n^i\rangle+\frac{1}{2}\sum_i\hbar\omega_i$ ← zero-point energy. neglect

$\langle E\rangle=2\hbar c\sum_k\frac{k}{e^{\beta\hbar kc}-1}$.

$L\to\infty$, $\Delta k\to0$. $\Rightarrow\sum_k\to\left(\frac{1}{\Delta k}\right)^3\int d^3k$.

$\langle E\rangle = \left(\frac{L}{2\pi}\right)^3 2\int_{\vec k} \frac{\hbar k c}{e^{\beta\hbar k c}-1} d^3k$

$= \left(\frac{L}{2\pi}\right)^3 2\int_0^\infty d^3k\, 4\pi k^2 \frac{\hbar kc}{e^{\beta\hbar kc}-1}$

$\underline{\underline{x=\beta\hbar kc}}\ k_BT\left(\frac{L}{2\pi}\right)^3 2\left(\frac{k_BT}{\hbar c}\right)^3 4\pi\int_0^\infty \frac{x^3}{e^x-1}$

$= k_BT\frac{\pi^2}{15}L^3\left(\frac{k_BT}{\hbar c}\right)^3.$

$C_V = \frac{dE}{dT} = k_B\frac{4\pi^2}{15}V\left(\frac{k_BT}{\hbar c}\right)^3.$

$P=-\frac{\partial F}{\partial V},\ F=-\frac{1}{\beta}\ln Z.$

$Z=\prod_i Z_i = \prod \frac{e^{-\beta\hbar kc/2}}{1-e^{\beta\hbar kc}}$

$F=\underbrace{\sum_k \hbar kc}_{E_0} + 2k_BT\frac{L^3}{(2\pi)^3}\int d^3k \ln(1-e^{-\beta\hbar ck}).$

$P=-\frac{\partial E_0}{\partial V}+2k_BT\int\frac{d^3k}{(2\pi)^3}\ln(1-e^{-\beta\hbar ck})$

$=P_0+\frac{1}{3}\frac{E}{V}\sim P_0+k_BT\frac{\pi^2}{45}\left(\frac{k_BT}{\hbar c}\right)^3\sim T^4.$

Planck distribution

$\frac{\langle E\rangle}{V}=\frac{\hbar}{\pi^2c^3}\int_0^\infty d\omega\frac{\omega^3}{e^{\beta\hbar\omega}-1}=\int \underset{\text{spectrum density}}{\underline{u(\omega)}}\,d\omega$

$J=\frac{c}{4}\frac{\langle E\rangle}{V}=\frac{c}{4}k_BT\frac{\pi^2}{15}\left(\frac{k_BT}{\hbar c}\right)^3=\sigma_B T^4.$

Debye Theory of solids Dispersion relation $\omega(\vec k)$

Only $\vec k$ in the first BZ ($\frac{2\pi}{a}\times\frac{2\pi}{a}\times\frac{2\pi}{a}$) are counted.

$H=\sum_{x,y,z}\sum_{\vec k\in BZ}\hbar\omega(\vec k)\left(n_\alpha+\frac12\right).$

$\langle n_\alpha\rangle=\frac{1}{e^{\beta\hbar\omega(\vec k)}-1}$

$\langle H\rangle=3\sum_{\vec k\in BZ}\hbar\omega(\vec k)\left(\frac{1}{e^{\beta\hbar\omega(\vec k)}-1}+\frac12\right)$

$=E_0+3\sum_{\vec k\in BZ}\frac{\hbar\omega(\vec k)}{e^{\beta\hbar\omega(\vec k)}-1}.$

Debye's approximation:

1. $\omega(k)=v_s k$
2. $\int_{BZ}d^3k=\left(\frac{2\pi}{a}\right)^3=(2\pi)^3\frac{N_{atom}}{V}=\frac43\pi k_D^3\Rightarrow k_D^3=\frac{6\pi^2N_{atom}}{V}.$

$L\to\infty.\ \sum_{\vec k\in BZ}\to\left(\frac{L}{2\pi}\right)^3\int_{k<k_D}d^3k.$

$\langle H\rangle=E_0+3\frac{L^3}{(2\pi)^3}4\pi\int_0^{k_D}\frac{\hbar vk^3}{e^{\beta\hbar vk}-1}dk.$

$x=\beta\hbar vk,\ x_D=\beta\hbar vk_D$

$\Rightarrow\langle H\rangle=E_0+\frac{3Vk_BT}{2\pi}\left(\frac{k_BT}{\hbar v}\right)^3\int_0^{x_D}\frac{x^3}{e^x-1}dx$

$x_D=\frac{\hbar vk_D}{k_BT}=\frac{T_D}{T}$ Debye Temperature.

Case I: $T\to0,\ x_D\to\infty.$

$\int_0^{x_D}dx\frac{x^3}{e^x-1}=\int_0^\infty dx\frac{x^3}{e^x-1}=\frac{\pi^4}{15}.$

$\langle H\rangle=E_0+3Vk_BT\frac{\pi^2}{30}\left(\frac{k_BT}{\hbar v}\right)^3\sim T^4.$

$C_V=3Vk_B\frac{2\pi^2}{15}\left(\frac{k_BT}{\hbar v}\right)^3\sim T^3$

Case II: $T\to\infty,\ x_D\to0.$

$\int_0^{x_D}dx\frac{x^3}{e^x-1}=\frac{x_D^3}{3}.$

$\langle H\rangle=E_0+3Nk_BT.$

$C_V=\frac{dE}{dT}=3Nk_B.$

Bose-Einstein Condensation $E_k=\frac{\hbar^2k^2}{2m}.$

At $T=0$, all bosons pile up into min(E_k). $E_0=0$.

$N_0=\frac{1}{e^{-\beta\mu}-1}.\ N_0\approx N\gg1\Rightarrow z=e^{\beta\mu}-1$

$e^{-\beta\mu}=1-e^{\beta\mu}\Rightarrow N_0\approx-k_BT/\mu\approx N\Rightarrow\mu\to k_BT/N.$

As $T\to\infty$, recover classical

$N=\frac{V}{\lambda^3}e^{\beta\mu}\Rightarrow\mu=k_BT\ln\left(\frac{N}{V}\lambda^3\right).$

$N=\sum_k N_k=\int_0^\infty g(E)\frac{1}{e^{\beta(E-\mu)}-1}dE.$

$g(E)=\frac{L^3}{2\sqrt\pi}\left(\frac{2\pi m}{h^2}\right)^{3/2}\sqrt E.$

$\Rightarrow N=\frac{V}{2\sqrt\pi}\left(\frac{2\pi m}{h^2}\right)^{3/2}\int_0^\infty\frac{\sqrt E\, dE}{e^{\beta(E-\mu)}-1}$

$\underline{\underline{x=\beta E}}\ N=\frac{V}{2\sqrt\pi}\left(\frac{2\pi mk_BT}{h^2}\right)^{3/2}\int_0^\infty\frac{\sqrt x\, dx}{e^xe^{\beta\mu}-1}.$

$\lambda_T=\frac{h}{\sqrt{2\pi mk_BT}}.\Rightarrow\frac{N}{V}=\frac{1}{\lambda_T^3}\frac{1}{2\sqrt\pi}\int_0^\infty\frac{\sqrt x\, dx}{e^xe^{\beta\mu}-1}.$

$\mu=0\Rightarrow N=2.612\frac{V}{\lambda_T^3}.$

Define T_c: the temperature satisfies this

$N=2.612\left(\frac{2\pi mk_BT_c}{h^2}\right)^{3/2}V,$

$T_c=\frac{0.562}{2\pi}\left(\frac{N}{V}\right)^{3/2}\frac{h^2}{2m}.$

$T<T_c$ more N is allowed.

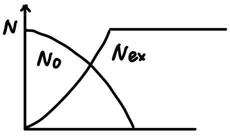
$N=N_0+N_{ex}=\frac{1}{e^{-\beta\mu}-1}+\int_{-\infty}^{+\infty}\frac{g(E)dE}{e^{\beta(E-\mu)}-1}.$

$T<T_c,\ \mu\sim-k_BT/N_0\to0.$

$N=N_0+\underbrace{2.612\frac{V}{\lambda_T^3}}_{N\left(\frac{T}{T_c}\right)^{3/2}}.$

$N_0=N\left[1-\left(\frac{T}{T_c}\right)^{3/2}\right]$

$N_{ex}=N\left(\frac{T}{T_c}\right)^{3/2}$



Ferromagnet Neiboring dipoles align parallel to each other.

Curie Temperature Net magnetization → 0

Ising model N: total # of dipole. $s_i=\pm1$. Energy due to the interaction of a pair of neighboring dipoles: $-\varepsilon$ parallel, $+\varepsilon$ antiparallel $\Rightarrow -\varepsilon s_is_j$

$U=-\varepsilon\sum_{\substack{\text{neighbor}\\ i,j}}s_is_j.$

$Z=\sum_{\{s_i\}}e^{-\beta U}.$

1 Dimension $U=-\varepsilon(s_1s_2+s_2s_3+\cdots+s_{N-1}s_N).$

$Z=\sum_{s_1}\sum_{s_2}\cdots\sum_{s_N}e^{\beta\varepsilon s_1s_2}e^{\beta\varepsilon s_2s_3}\cdots e^{\beta\varepsilon s_{N-1}s_N}$

$\sum_{s_N}e^{\beta\varepsilon s_{N-1}s_N}=e^{\beta\varepsilon}+e^{-\beta\varepsilon}=2\cosh\beta\varepsilon.$

$Z=2^N(\cosh\beta\varepsilon)^{N-1}\approx(2\cosh\beta\varepsilon)^N$

$\langle U\rangle=-\frac{\partial}{\partial\beta}\ln Z=-N\varepsilon\tanh\beta\varepsilon.$

Mean Field Approximation

Consider a single dipole i.

# of nearest neighbors:

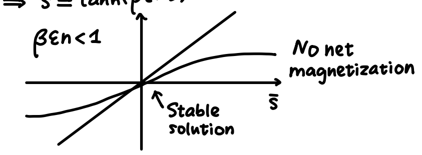
$n=\begin{cases}2 & 1D\\ 4 & 2D\\ 6 & 3D \text{ (simple cubic)}\\ 8 & 3D \text{ (body centered)}\\ 12 & 3D \text{ (face-centered)}.\end{cases}$

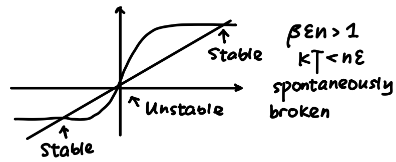
$E_\uparrow=-\varepsilon\sum_{nei}s=-\varepsilon n\langle s\rangle$

$E_\downarrow=+\varepsilon n\langle s\rangle.\Rightarrow Z_i=e^{\beta\varepsilon n\langle s\rangle}+e^{-\beta\varepsilon n\langle s\rangle}$
$=2\cosh(\beta\varepsilon n\langle s\rangle).$

$\bar s_i=\frac{1}{Z_i}\left[1e^{\beta\varepsilon n\langle s\rangle}+(-1)e^{-\beta\varepsilon n\langle s\rangle}\right]=\tanh(\beta\varepsilon n\langle s\rangle).$

$\bar s_i=\bar s\Rightarrow\bar s=\tanh(\beta\varepsilon n\bar s)$





Fermi gas in D dimension

$N_s=\underset{\text{spin}}{(D-1)}\left(\frac{L}{2\pi}\right)^D\times V_D$ → max k.

$V_D=\int r^{D-1}dr\,d\Omega_D=\frac{\pi^{D/2}k^D}{\Gamma(1+D/2)}.$

$d\Omega_D=d\theta_1\sin\theta_2d\theta_2\sin^2\theta_3d\theta_3\cdots\sin^{D-2}\theta_{D-1}d\theta_{D-1}$

$\Rightarrow$ Since $E=\frac{\hbar^2k^2}{2m}.$

$\underset{\text{number of states}}{N_s}=(D-1)\left(\frac{L}{2\pi}\right)^D\frac{\pi^{D/2}}{\Gamma(1+D/2)}\left(\frac{2m}{\hbar^2}\right)^{D/2}E^{D/2}.$

$g(E)=\frac{dN_s}{dE}=\frac{D(D-1)}{2}\left(\frac{L}{2\pi}\right)^D\frac{\pi^{D/2}}{\Gamma(1+D/2)}\left(\frac{2m}{\hbar^2}\right)^{D/2}E^{(D-2)/2}$

(*) $\underset{\#\text{ of particles}}{N}=\int_0^{E_F}g(E)dE=(D-1)\left(\frac{L}{2\pi}\right)^D\frac{\pi^{D/2}}{\Gamma(1+D/2)}\left(\frac{2m}{\hbar^2}\right)^{D/2}E_F^{D/2}.$

$\Rightarrow E_F=\frac{\hbar^2}{2m}4\pi\left[\frac{\Gamma(1+D/2)}{D-1}\frac{N}{V}\right]^{2/D}$

Total energy at $T=0$: $n(E)=1$

$E_{total}=\int_0^{E_F}Eg(E)dE$

$=\frac{D(D-1)V}{(D+2)(2\pi)^D}\left(\frac{2m}{\hbar^2}\right)^{D/2}\frac{\pi^{D/2}}{\Gamma(1+D/2)}E_F^{(D+2)/2}$

$=\frac{D}{D+2}NE_F$

(*) $N=(D-1)\frac{V}{(4\pi)^{D/2}}\frac{k_F^D}{\Gamma(1+D/2)}$

$\Rightarrow k_F=\left[\frac{(4\pi)^{D/2}}{D-1}\Gamma\left(1+\frac{D}{2}\right)\rho\right]^{1/D}$

$D=1.\ \Gamma(\frac32)=\frac{\sqrt\pi}{2}$

$D=2.\ \Gamma(2)=1$

$D=3.\ \Gamma(\frac52)=\frac{3\sqrt\pi}{4}$

$D=4.\ \Gamma(3)=2$

$D=5.\ \Gamma(\frac72)=\frac{15\sqrt\pi}{8}$