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Particle Physics

NOTES

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Contents

Chapter 1

Some Basic Concepts

1.1 Introduction

Particle physics studies the fundamental constituents and their interactions. **Standard model** tries to explain all the phenomena of particle physics. According to standard model, there are three distinct types of particles: **leptons** and **quarks** (spin $\frac{1}{2}$), **gauge bosons** (spin 1) and the **Higgs bosons** (spin 0). All those particles are assumed to be **elementary**. They are treated as point particles, without internal structure or excited states.

An example of a lepton is the **electron**, which is bound in atoms by the **electromagnetic interaction**. Another example is the **electron neutrino**, which is observed in the β-decays. The force responsible for the β-decays is called the **weak interaction**.

Another class of particles are called **hadrons**, including neutrons, protons, and pions. They are not elementary particles, but are made of quarks bound together by the **strong interaction**.

$$\text{Elementary Particles} \left\{ \begin{array}{ll} \text{Leptons and Quarks} & \text{Spin } \frac{1}{2} \\ \text{Gauge bosons} & \text{Spin } 1 \\ \text{Higgs bosons} & \text{Spin } 0 \end{array} \right.$$

$$\text{Forces of Nature} \left\{ \begin{array}{ll} \text{Electromagnetic Interaction} & \text{Protons} \\ \text{Weak Interaction} & W \text{ and } Z \text{ bosons} \\ \text{Strong Interaction} & \text{Gluons} \\ \text{Gravitational Interaction} & \end{array} \right.$$

The standard model specifies the origin of these forces. The electromagnetic interaction is transmitted by the exchange of spin-1 protons (force carriers). The weak interaction is associated with $\boldsymbol{W}$ and $\boldsymbol{Z}$ **bosons**. The equivalent particles for the strong interaction are called **gluons**.

1.2 Antiparticles

Every charged particle (elementary particles or hardons) has an associated particle of the same mass but opposite charge, called its **antiparticle**.

1.2.1 Relativistic Wave Equations

A particle moving with momentum $\boldsymbol{p}$ in free space is described by a de Brogile wavefunction

$$\Psi(\boldsymbol{r}, t) = Ne^{i(\boldsymbol{p}\cdot\boldsymbol{r}-Et)/\hbar},$$

where the frequency is $\nu = E/\hbar$ and the wavelength is $\lambda = \hbar/p$. Nonrelativistically, we have $E = p^2/2m$ and the wavefunction obeys the nonrelativistic Schrödinger equation

$$i\hbar \frac{\partial \Psi(\boldsymbol{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \partial \Psi(\boldsymbol{r}, t).$$

However, relativistically, we have

$$E^2 = p^2 c^2 + m^2 c^4,$$

where m is the rest mass and the wave equation is

$$\boxed{-\hbar^2 \frac{\partial^2 \Psi(\boldsymbol{r}, t)}{\partial t^2} = -\hbar^2 c^2 \nabla^2 \Psi(\boldsymbol{r}, t) + m^2 c^4 \Psi(\boldsymbol{r}, t).}$$

This is called the **Klein-Gordon equation**. One of its feature is the existence of solutions with negative energy. For every plane wave solution

$$\Psi(\boldsymbol{r}, t) = N \exp[i(\boldsymbol{p} \cdot \boldsymbol{r} - E_p t)/\hbar],$$

with momentum $\boldsymbol{p}$ and positive energy

$$E = E_p \equiv +(p^2 c^2 + m^2 c^4)^{1/2} \geq mc^2,$$

there is another solution

$$\boxed{\tilde{\Psi}(\boldsymbol{r}, t) \equiv \Psi^*(\boldsymbol{r}, t) \exp[i(-\boldsymbol{p} \cdot \boldsymbol{r} + E_p t)/\hbar],}$$

corresponding to momentum $-\boldsymbol{p}$ and negative energy

$$\boxed{E = -E_p = -(p^2 c^2 + m^2 c^4)^{1/2} \leq -mc^2.}$$

Klein-Gordon equation is not a sufficient foundation for relativistic quantum mechanics.

For spin-$\frac{1}{2}$ particles, Dirac proposed that

$$i\hbar \frac{\partial \Psi(\boldsymbol{r}, t)}{\partial t} = H(\boldsymbol{r}, \boldsymbol{p}) \Psi(\boldsymbol{r}, t),$$

where H is the Hamiltonian and $\boldsymbol{p} = -i\hbar\nabla$ is the momentum operator. And

$$H = i\hbar c \sum_{i=1}^{3} \alpha_i \frac{\partial}{\partial x_i} + \beta mc^2 = c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2,$$

where the coefficients β and α_i are determined by requiring that solutions of the Dirac equation are also solutions of the Klein-Gordon equation:

$$\begin{aligned}
&\alpha_i^2 = 1, \quad \beta^2 = 1, \\
&\alpha_i \beta + \beta \alpha_i = 0, \\
&\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \, (i \neq j).
\end{aligned}$$

The simplest assumption is that β and α_i are matrices. Thus we could interpret the **Dirac equation**

$$\boxed{i\hbar \frac{\partial \Psi}{\partial t} = H\Psi = -i\hbar c \sum_i \alpha_i \frac{\partial \Psi}{\partial x_i} + \beta mc^2 \Psi,}$$

as a 4-dimensional matrix equation where Ψ are 4-component wavefunctions called **spinors**. Plane wave solutions are

$$\Psi(\boldsymbol{r}, t) = u(\boldsymbol{p}) \exp[i(\boldsymbol{p} \cdot \boldsymbol{r} - Et)/\hbar],$$

where $u(\boldsymbol{p})$ is a 4-component spinor satisfying the eigenvalue equation

$$H_{\boldsymbol{p}} u(\boldsymbol{p}) = (c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2) u(\boldsymbol{p}) = E u(\boldsymbol{p}).$$

The equation has 4 solutions: two with positive energy $E = +E_p$ corresponding to the two possible spin states of a spin-$\frac{1}{2}$ particle, and two corresponding negative energy solutions with $E = -E_p$.

1.2.2 Hole Theory and the Positron

Dirac postulated that the negative energy states are almost always filled. For electrons, since they are fermions, they obey the Pauli exclusion principle. The Dirac picture of vacuum is a "sea" of negative energy states, each with two electrons (one spin up, one spin down), while the positive energy states are all unoccupied.

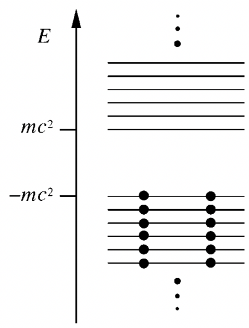

Figure 1.1: Dirac picture of the vacuum. The sea of negative energy states is occupied with two electrons in each level, one spin up, one spin down; one with momentum $\boldsymbol{p}$ and the other with momentum $-\boldsymbol{p}$.

For each state of momentum $\boldsymbol{p}$ there is a corresponding state with momentum $-\boldsymbol{p}$, thus the momentum of vacuum $\boldsymbol{p}_V = \sum \boldsymbol{p} = \boldsymbol{0}$. The same argument applies to spin. And since energies are measured relative to the vacuum, $E_V \equiv 0$ by definition. Also we could define the charge $Q_V \equiv 0$ (since the constant electrostatic potential produced by the sea is unobservable). Hence this state has all the measurable characteristics of the naive vacuum and the "sea" is unobservable.

$$\boxed{\boldsymbol{p}_V = 0, \quad \boldsymbol{S}_V = 0, \quad E_V = 0, \quad Q_V = 0.}$$

When an electron is added to the vacuum, it's confined to the positive energy region. And when an electron with negative energy $E = -E_p < 0$, momentum $-\boldsymbol{p}$, spin $-\boldsymbol{S}$ and charge $-e$ is removed from the vacuum, leaves a state with positive energy $E = E_p > 0$, momentum $\boldsymbol{p}$, spin $\boldsymbol{S}$ and charge $+e$. This state cannot be distinguished from a state formed by adding to the vacuum a particle with momentum $\boldsymbol{p}$, energy $E = E_p > 0$, spin $\boldsymbol{S}$ and charge $+e$. Dirac predicted the existence of a spin-$\frac{1}{2}$ particle e^+ with the same mass as the electron, but opposite charge, which is called the **positron** and the antiparticle of the electron.

The Dirac equation applies to any spin-$\frac{1}{2}$ particle, and hole theory predicts that all charged spin-$\frac{1}{2}$ particles have distinct antiparticles with opposite charge but the same mass.

1.3 Interactions and Feynman Diagrams

Feynman diagram is a convenient way of illustrating interactions involving elementary particles.

1.3.1 Basic Electromagnetic Process

The electromagnetic interactions of electrons and positrons could be understood in terms of 8 basic processes. The basic processes of an electron emits or absorbs a photon are

$$\text{(a) } e^- \to e^- + \gamma, \quad \text{(b) } \gamma + e^- \to e^-.$$

They correspond to transitions between positive energy states of the electron. Similarly, for

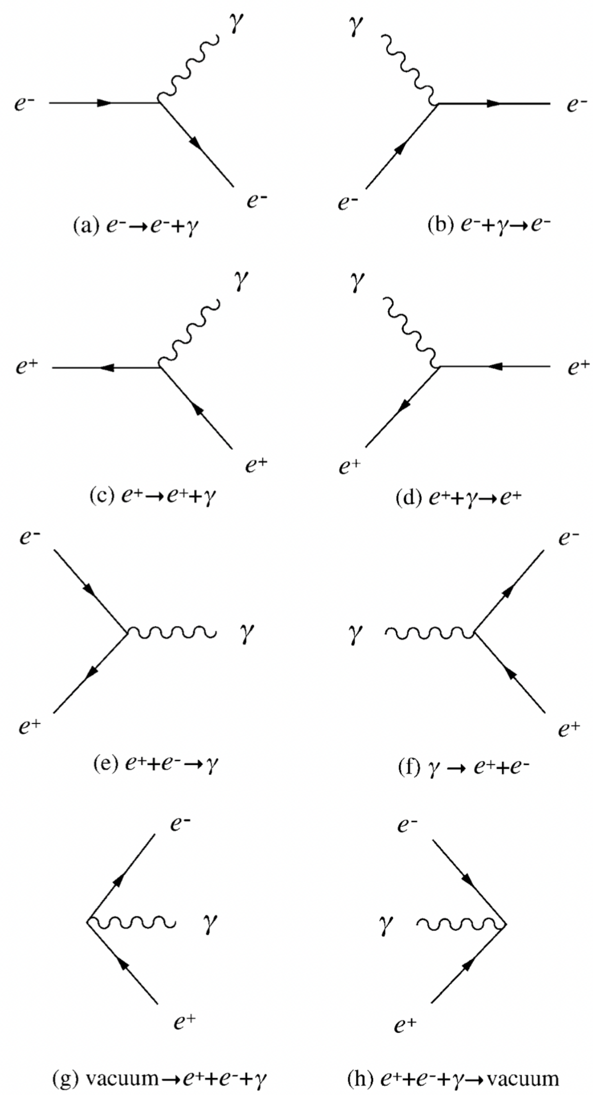

Figure 1.2: Feynman diagrams for the 8 basic processes whereby electrons and positrons interact with photons

positrons we have

$$\text{(a) } e^+ \to e^+ + \gamma, \quad \text{(b) } \gamma + e^+ \to e^+.$$

Also there are processes where an electron is excited from a negative energy state to a positive energy state, leaving a hole behind[1], or where a positive energy electron falls into a hole in the negative energy sea:

(e) $e + +e^- \to \gamma$, (f) $\gamma \to e^+ + e^-$,

(g) vacuum $\to e^+ + e^- + \gamma$, (h) $e^+ + e^- + \gamma \to$ vacuum.

This exhausts the possibilities in hole theory. Each process has an associated probability proportional to the strength of the electromagnetic fine structure constant

$$\alpha \equiv \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}.$$

1.3.2 Real Process

In the Feynman diagrams shown in Figure 5.2, time flows from the left to the right. The arrow directed towards the right indicates a particle while the arrow directed to the left indicates an antiparticle. Every vertex has a line corresponding to a photon being emitted or absorbed. One fermion line has the arrow pointing towards the vertex and the other away from the vertex, implying charge conservation at the vertex.

For virtual processes at the vertices		
	Charge	conserved
	Momentum and angular momentum	conserved
	Energy	NOT conserved

In a real process, two or more processes must be combined to make the energy conserved.

Momentum and angular momentum are also assumed to be conserved at the vertex. In free space, however, energy conservation is violated for all the basic processes. They are called **virtual** processes and cannot occur in isolation in free space. To make a real process, two or more virtual processes must be combined in such a way that energy conservation is only violated for a short period of time compatible with the energy-time uncertainty principle

$$\tau\Delta E \sim \hbar.$$

Particularly, the initial state ($t \to -\infty$) and final state ($t \to +\infty$) must have the same energy.

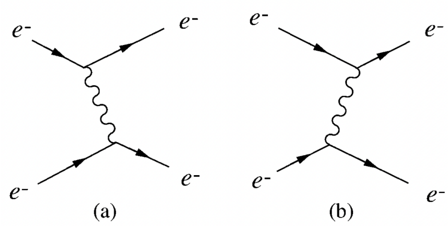

Figure 1.3: Single-photon exchange contributions to electron-electron scattering

Scattering can also occur via multiphoton exchange, as shown in Figure 5.4. The contributions of such diagrams are much smaller than the one-photon exchange contributions. Call the number of vertices in each diagram **order**. Each vertex represents a basic process whose probability of order α, any diagram of order n gives a contribution of order α^n. Thus the single-photon exchange is of order α^2 and two-photon exchange is of order α^4.

[1] One may consider this as removing an electron from the sea (which is equivalent to adding a positron), and then adding an electron with positive energy.

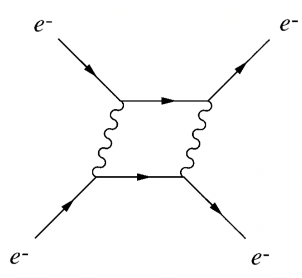

Figure 1.4: Two-photon exchange contributions to electron-electron scattering

1.3.3 Electron-Positron Pair Production and Annihilation

[Example 1] Consider the reaction

$$\boxed{e^+ + e^- \to p\gamma}.$$

To conserve energy at least two photons must be produced, thus $p \geq 2$. To produce p photons we must combine p vertices. For $p = 2$, we could have the top vertex occurring first or after, thus there are $2! = 2$ "time orderings" diagrams. For $p = 3$, there are $3! = 6$ diagrams. In practice we usually draw only one time ordering, leaving the others implied. Generally,

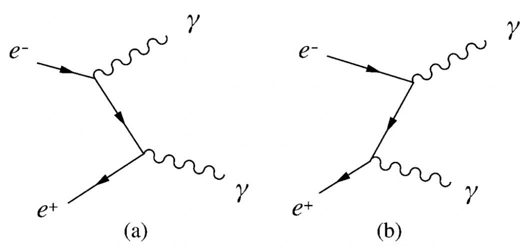

Figure 1.5: Lowest-order contributions to $e^+ + e^- \to \gamma + \gamma$. There are $2! = 2$ ways to draw the diagrams.

the process $e^+ + e^- \to p\gamma$ has an associated probability of order α^p. Thus we expect that many-photon annihilation is very rare compared to few-photon annihilation.

[Example 2] Consider the pair production reaction $\gamma \to e^+ + e^-$. This basic process cannot conserve both energy and momentum simultaneously, but is allowed in the presence of a nucleus,

$$\boxed{\gamma + (Z, A) \to e^+ + e^- + (Z, A),}$$

where Z is the charge number and A is the mass number of the nucleus. Since one of the vertices involves a charge Ze, the corresponding factor $\alpha \to Z^2\alpha$ and the expected rate is of order $Z^2\alpha^2$.

1.3.4 Other Process

Feynman diagrams can also be used to describe the fundamental weak and strong interactions, as shown in Figure 5.7.

1.4 Particle Exchange

The forces of elementary particle physics are associated with the exchange of particles.

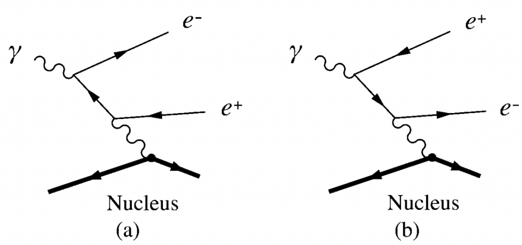

Figure 1.6: The pair production process in lowest order. The two diagrams represent distinct contributions and are not relate.

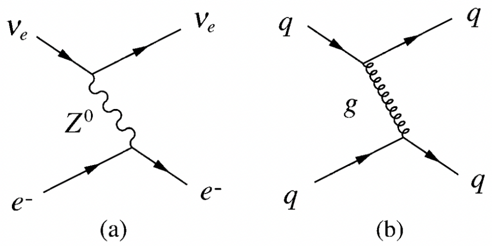

Figure 1.7: (a) Contribution of Z^0 exchange to the elastic weak scattering reaction $e^- + \nu_e \to e^- + \nu_e$; (b) The gluon exchange contribution to the strong interaction $q + q \to q + q$.

1.4.1 Range of Forces

Consider the elastic scattering of two particles A and B with masses M_A, M_B, via the exchange of a third particle X of mass M_X, with equal coupling strengths g to particles A and B. In the rest frame of the incident particle A, the lower vertex represents the virtual

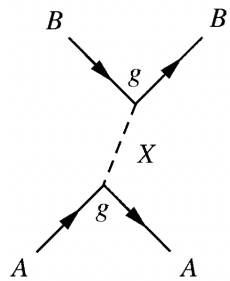

Figure 1.8: Contribution to the reaction $A + B \to A + B$ from the exchange of a particle X.

process

$$A(M_A c^2, \mathbf{0}) \to A(E_A, \boldsymbol{p}) + X(E_X, -\boldsymbol{p}),$$

where

$$E_A = (p^2c^2 + M_A^2c^4)^{1/2}, \quad E_X = (p^2c^2 + M_X^2c^4)^{1/2}, \quad p \equiv |\boldsymbol{p}|.$$

The energy difference between the initial state and final state is

$$\Delta E = E_X + E_A - M_A c^2.$$

As $p \to 0$, $\Delta E \to M_X c^2$. Thus we have $\Delta E \geq M_X c^2$ for all p. By the uncertainty principle, such an energy violation is only allowed for a time $\tau \approx \hbar/\Delta E$, thus the maximum distance over which X can propagate before being absorbed by B is

$$\boxed{r \approx R \equiv \frac{\hbar}{M_X c}.}$$

R is called the **range** of the interaction.

The electromagnetic interaction has an infinite range since the exchanged particle is a massless photon. Also, the strong force has an infinite range since the exchanged particles are massless gluons. For weak interaction, the exchange particles are W and Z bosons (which are not massless), corresponding to ranges of order

$$R_{W,Z} \equiv \frac{\hbar}{M_W c} \approx 2 \times 10^{-3}\text{fm}.$$

1.4.2 The Yukawa Potential

If $M_A \to \infty$, B can be considered as being scattered by a static potential of which A is the source. We could consider X as a spin-0 boson, and thus it obeys the Klein-Gordon equation. The static solution satisfies

$$\nabla^2 \phi(\boldsymbol{r}) = \frac{M_X^2 c^4}{\hbar^2} \phi(\boldsymbol{r}),$$

where $\phi(\boldsymbol{r})$ is considered as a static potential. For $M_X^2 \neq 0$, the solution is

$$\boxed{V(r) = -\frac{g^2}{4\pi} \frac{e^{-r/R}}{r},}$$

where R is the range and g is the coupling constants. $V(r)$ is called a **Yukawa potential**. Also we could introduce a dimensionless strength parameter

$$\alpha_X = \frac{g^2}{4\pi\hbar c,}$$

that characterizes the strength of the interaction at short distances.

1.4.3 The Zero-Range Approximation

In lowest-order perturbation theory, the probability amplitude for a particle with initial momentum $\boldsymbol{q}_i$ to be scattered to a final state with momentum $\boldsymbol{q}_f$ by a potential $V(\boldsymbol{r})$ is proportional to

$$\mathcal{M}(\boldsymbol{q}) = \int d^2\boldsymbol{r} V(\boldsymbol{r}) \exp(i\boldsymbol{q} \cdot \boldsymbol{r}/\hbar),$$

where $\boldsymbol{q} \equiv \boldsymbol{q}_i - \boldsymbol{q}_f$. For the Yukawa potential, we have

$$\mathcal{M}(\boldsymbol{q}) = \frac{-g^2\hbar^2}{|\boldsymbol{q}|^2 + M_X^2 c^2}.$$

And

$$\boxed{\mathcal{M}(q^2) = \frac{g^2\hbar^2}{q^2 - M_X^2 c^2},}$$

where

$$q^2 \equiv (E_f - E_i)^2 - (\boldsymbol{q}_f - \boldsymbol{q}_i)^2 c^2.$$

In the zero-range approximation, $\mathcal{M}(q^2)$ reduces to a constant. When $q^2 \ll M_X^2 c^2$, we have

$$\boxed{\mathcal{M}(q^2) = -G,}$$

where the constant G is given by

$$\boxed{\frac{G}{(\hbar c)^3} = \frac{1}{\hbar c}\left(\frac{g}{M_X c^2}\right)^2 = \frac{4\pi\alpha_X}{(M_X c^2)^2}.}$$

thus in the zero-range approximation, the resulting point interaction between A and B is characterized by a single dimensioned coupling constant G.

1.5 Units and Dimensions

In elementary particle physics, **natural units** are adopted, such that

$$\boxed{\hbar = 1, \quad c = 1.}$$

The unit of energy is taken to be the electronvolt (eV). In natural units (nu), all quantities have the dimensions of a power of energy. In particular, we have

$$M = \frac{E}{c^2}, \quad L = \frac{\hbar c}{E}, \quad T = \frac{\hbar}{E}.$$

Hence a quantity with meter-kilogram-second (mks) dimensions $M^p L^q T^r$ has the nu dimensions E^{p-q-r}.

Chapter 2

Leptons and the Weak Interaction

2.1 Lepton Multiplets and Lepton Numbers

Leptons are spin-$\frac{1}{2}$ fermions without strong interactions. There are 6 leptons which occur in pairs called **generations**, written as doublets:

$$\boxed{\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}, \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}, \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}.}$$

The 3 charged leptons are called the electron, the **muon** and the **tauon**. All of them have charge $Q = -e$. Associated with them are 3 neutral leptons, or **neutrinos**, called the **electron neutrino**, **mu-neutrino** and **tau-neutrino** respectively. All of them have very small masses, and the 6 distinct types of leptons are also referred to as having different "flavours". In addition, there are 6 corresponding antileptons:

$$\boxed{\begin{pmatrix}e^+ \\ \bar{\nu}_e\end{pmatrix}, \begin{pmatrix}\mu^+ \\ \bar{\nu}_\mu\end{pmatrix}, \begin{pmatrix}\tau^+ \\ \bar{\nu}_\tau\end{pmatrix}.}$$

The charged leptons interact via both electromagnetic and weak forces, and the neutral leptons interact only via weak interactions.

Note that each generation of leptons has an associated quantum number, called the **lepton numbers**. The **electron number** is defined as

$$\boxed{L_e \equiv N(e^-) - N(e^+) + N(\nu_e) - N(\bar{\nu}_e).}$$

For single particle states, we have $L_e = 1$ for e^- and ν_e, $L_e = -1$ for e^+ and $\bar{\nu}_e$, and $L_e = 0$ for all other particles. Similarly we have the **muon number** and **tauon number**, defined by

$$\boxed{L_\mu \equiv N(\mu^-) - N(\mu^+) + N(\nu_\mu) - N(\bar{\nu}_\mu),}$$

and

$$\boxed{L_\tau \equiv N(\tau^-) - N(\tau^+) + N(\nu_\tau) - N(\bar{\nu}_\tau).}$$

We can conclude that

In the standard model, lepton numbers are individually conserved in all known interactions.

2.1.1 Electron Neutrinos

The existence of the electron neutrino was postulated by Pauli in 1930 in order to understand the observed β-decays:

$$(Z, A) \to (Z+1, A) + e^- + \bar{\nu}_e,$$
$$(Z', A') \to (Z'-1, A) + e^+ + \nu_e.$$

These reactions are actually decays of bound neutrons and protons via the basic processes:

$$\boxed{n \to p + e^- + \bar{\nu}_e,}$$

and

$$\boxed{p \to n + e^+ + \nu_e.}$$

The neutrinos were not initially observed in experiments, but inferred from energy and angular momentum conservation. In the second reaction, if the antineutrino were not presented, the reaction would be a two-body decay, and the energy of the emitted electron would have the unique value

$$E_e = \Delta M = M(Z, A) - M(Z+1, A),$$

where the nuclear recoil energy is neglected. However, if the antineutrino is present, the electron energy will not be unique and lie in the range

$$m_e \leq E_e \leq \Delta M - m_{\bar{\nu}_e}.$$

In experiments, the observed spectrum spans the whole range with $m_{\bar{\nu}_e} \approx 0$. Careful study of the spectrum near the endpoint $E_e = \Delta M - m_{\bar{\nu}_e}$ allows an upper limit to be set on neutrino mass. The best results come from the tritium β-decay

$$^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \bar{\nu}_e,$$

which gives us

$$\boxed{m_{\bar{\nu}_e} < 2\mathrm{eV}/c^2 \approx 4 \times 10^{-6} m_e.}$$

Neutrinos and antineutrinos can be detected by the observation of the **inverse $\boldsymbol{\beta}$-decay**:

$$\boxed{\nu_e + n \to e^- + p,}$$

and

$$\boxed{\bar{\nu}_e + p \to e^+ + n.}$$

2.1.2 Further Generations

Muon is a very penetrating particle of mass $105.7\mathrm{MeV}/c^2$ and have the Dirac magnetic moment

$$\boldsymbol{\mu} = \frac{e}{m_e}\boldsymbol{S},$$

and generally their electromagnetic properties are identical with those of electrons, provided the mass difference is taken into account.

The tauon is much heavier with mass $1777\mathrm{MeV}/c^2$, and compatible with a point-like spin-$\frac{1}{2}$ particle whose electromagnetic interactions are identical with those of the electron and muon.

The electron is the lightest charged particle and is necessarily stable. However, the muon and the tauon are unstable with lifetimes 2.2×10^{-6}s and 2.9×10^{-13}s respectively. Both decay by weak interactions, the difference in their lifetimes is a consequence of the mass difference. For muon, we have the purely leptonic decay

$$\boxed{\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu, \quad \mu^- \to e^- + \bar{\nu}_e + \nu_\mu,}$$

conserving both charge and lepton number. For the tauon, many decay modes are observed, and most of them involves hadrons in the final state. And the purely leptonic modes

$$\boxed{\tau^+ \to e^+ + \nu_e + \bar{\nu}_\tau, \quad \tau^- \to e^- + \bar{\nu}_e + \nu_\tau,}$$

are also observed. The relative importance of any decay mode is measured by its **branching ratio** B, which is defined as the fraction of all decays leading to that particular final state. For the leptonic decays of tauon, the measured branching ratios are 0.178 and 0.174 respectively.

Well-defined muon-neutrino beams can be created in the laboratory and used to study reactions like the elastic scattering from electrons

$$\boxed{\nu_\mu + e^- \to \nu_\mu + e^-,}$$

inverse muon decay

$$\boxed{\nu_\mu + e^- \to \mu^- + \nu_e,}$$

and reaction on nucleons, such as

$$\boxed{\nu_\mu + p \to \nu_\mu + p,}$$

and

$$\boxed{\bar{\nu}_\mu + p \to \mu^+ + n.}$$

Well-defined tau-neutrino beams are not available in the laboratory and it was not until 2000 that tau-neutrinos were directly detected. The masses of ν_μ and ν_τ can be inferred from the μ^- and τ^- energy spectra in the leptonic decays using energy conservation. The present limits are

$$\boxed{m_{\nu_\mu} < 0.19\text{MeV}/c^2, \quad m_{\nu_\tau} < 18.2\text{MeV}/c^2.}$$

2.2 Leptonic Weak Interactions

2.2.1 W$^\pm$ and Z^0 Exchange

Weak interactions involving only leptons are described by exchange processes in which a $W^\pm$ or Z^0 is emitted by one lepton and absorbed by another. When drawing the Feynman diagrams, we must remember that these absorption and emission processes conserve the lepton numbers and electric charge. A simple example is elastic $\nu_\mu e^-$ scattering and two

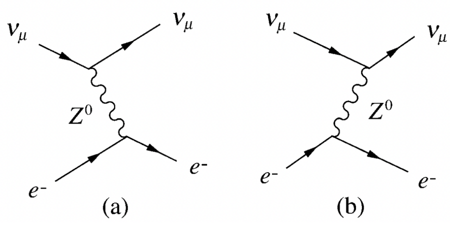

Figure 2.1: Two time-ordered diagrams for elastic $\nu_\mu e^-$ scattering.

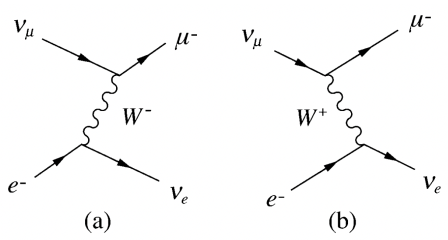

Figure 2.2: Dominant Feynman diagrams for inverse muon decay.

possible Feynman diagrams are drawn in Figure 2.1. It's conventional to draw one such diagram, leaving the other implied.

The diagrams for inverse muon decay are shown in Figure 2.2. In Figure 2.2(a), the initial process corresponds to the lower vertex where an electron converts into an electron-neutrino by emitting a W^- boson, conserving both the electric charge and lepton numbers. The W^- is then absorbed at the upper vertex, conversing both charge and lepton numbers again. Thus Figure 2.2(a) corresponds to W^- exchange. Similarly, (b) corresponds to W^+ exchange. It's conventional to draw just one such diagram, leaving the other implied. In

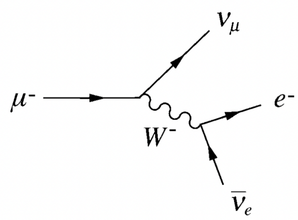

Figure 2.3: Dominant Feynman diagrams for muon decay.

Figure 2.3, for the muon decay, the first vertex corresponds to a muon emitting a W^- boson, and then the W^- boson is converted into a lepton pair.

These processes can also occur via higher-order diagrams, which containing more vertices. The contribution of such diagram is expected to be much smaller.

At low energies, the de Brogile wavelengths of all the particles involved are large compared with the range $R_W \approx 2 \times 10^{-3}$fm of the W-exchange interaction. Thus it could be approximated by a zero-range point interaction. The strength of the interaction is characterized by the Fermi coupling constant

$$G_F = 1.166 \times 10^{-5}\text{GeV}^{-2}.$$

Taking the spin complications into account, the coupling constant relation for spinless particles becomes

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g_M^2}{M_W^2} = \frac{4\pi\alpha_W}{M_W^2},}$$

where g_W is the coupling constant associated with the W boson-lepton vertices and

$$\alpha \equiv \frac{g_W^2}{4\pi}$$

is the dimensionless strength parameter analogous with the fine structure constant of electromagnetism α. Substituting the measured value, we have

$$\alpha_W = 4.2 \times 10^{-3} = 0.58\alpha.$$

Thus the weak and electromagnetic interactions are of comparable intrinsic strength.

2.2.2 Lepton Decays and Universality

All known experiments are consistent with the assumption that the interactions of the electron and its neutrino are identical with those of the muon and its associated neutrino, and the tauon and its neutrino, provided the mass differences taken into account. This fundamental assumption is called the **universality** of lepton interactions.

Consider the leptonic decays of the muon and tauon at rest. We will work on lowest order only, and use the zero-range approximation. We also assume that we can neglect the masses of the leptons in the final state relative to the mass of the decaying μ or τ. For the muon decay, the decay rate is in the form of

$$\Gamma(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu) = KG_F^2 m_\mu^5,$$

since G_F has a dimension $[E]^{-2}$ and the rate should have a dimension of $[E]$. K is a dimensionless constant. If we assume the $\mu - \tau$ universality, we have

$$\Gamma(\tau^- \to e^- + \bar{\nu}_e + \nu_\tau) = KG_F^2 m_\tau^5,$$

while the $e - \mu$ universality gives

$$\Gamma(\tau^- \to e^- + \bar{\nu}_e + \nu_\tau) = \Gamma(\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau).$$

This explains why the branching ratios for the two leptonic decay modes of the tauon are approximately equal.

The lifetimes τ_l are related to decay rates by

$$\boxed{\tau_l = \frac{1}{\Gamma_{tot}} = \frac{B(l^- \to e^-\bar{\nu}_e\nu_l)}{\Gamma(l^- \to e^-\bar{\nu}_e\nu_l)},}$$

where Γ is the total decay rate and

$$B(l^- \to e^-\bar{\nu}_e\nu_l) \equiv \frac{\Gamma(l^- \to e^-\bar{\nu}_e\nu_l)}{\Gamma_{tot}}$$

is the branching ratio. Experimentally we have $B = 1$ and 0.178 ± 0.004 for μ and τ respectively. Thus

$$\frac{\tau_\tau}{\tau_\mu} = \frac{B(\tau^- \to e^- + \bar{\nu}_e + \nu_\tau)}{B(\mu^- \to e^- + \bar{\nu}_e + \nu_\mu)} \left(\frac{m_\mu}{m_\tau}\right)^5 = (1.328 \pm 0.004) \times 10^{-7},$$

which is consistent with the ratio of the experimental results $(1.3227 \pm 0.0005) \times 10^{-7}$. These are the evidence of the universality of the lepton interactions.

2.3 Neutrino Masses and Neutrino Mixing

2.3.1 Neutrino Mixing

Neutrino mixing is the assumption that the neutrino states ν_e, ν_μ and ν_τ that couple to electrons, muons and tauons respectively, do not have definite masses. Instead they are linear combinations of three other states ν_1, ν_2 and ν_3 that do have definite masses m_1, m_2 and m_3.

Now we consider the mixing between two of the flavour states, say ν_α and ν_β. In order to preserve the orthonormality of the states, we write

$$\begin{aligned}\nu_\alpha &= \nu_i \cos\theta_{ij} + \nu_j \sin\theta_{ij},\\ \nu_\beta &= -\nu_i \sin\theta_{ij} + \nu_j \cos\theta_{ij},\end{aligned}$$

where ν_i and ν_j are the two mass eigenstates involved and θ_{ij} is a **mixing angle** that is determined from the experiment.

Consider the case where $\theta_{ij} \neq 0$. For example, a ν_α neutrino is produced with momentum $\boldsymbol{p}$ at $t = 0$, the ν_i and ν_j components will have slightly different energies E_i and E_j due to their slightly different masses. As time evolves, the original beam of ν_α particles develops a ν_β component whose intensity oscillates as it travels through space, and the intensity of the ν_α neutrino beam is correspondingly reduced. These are called the **neutrino oscillations.**

Consider a ν_α produced with momentum $\boldsymbol{p}$ at time $t = 0$, and the initial state is

$$|\nu_\alpha, \boldsymbol{p}\rangle = |\nu_i, \boldsymbol{p}\rangle \cos\theta_{ij} + |\nu_j, \boldsymbol{p}\rangle \sin\theta_{ij}.$$

After time t the state will become

$$a_i(t)|\nu_i, \boldsymbol{p}\rangle + a_j(t)|\nu_j, \boldsymbol{p}\rangle \sin\theta_{ij},$$

where

$$a_i(t) = e^{-iE_i t}, \quad a_j(t) = e^{-iE_j t}.$$

Now the state is not a pure ν_α state, but can be written as a linear combination

$$A(t)|\nu_\alpha, \boldsymbol{p}\rangle + B(t)|\nu_\beta, \boldsymbol{p}\rangle,$$

of ν_α and ν_β states. The latter is given by

$$|\nu_\beta, \boldsymbol{p}\rangle = -|\nu_i, \boldsymbol{p}\rangle \sin\theta_{ij} + |\nu_j, \boldsymbol{p}\rangle \cos\theta_{ij}.$$

And hence we have

$$\begin{aligned} A(t) &= a_i(t)\cos^2\theta_{ij} + a_j(t)\sin^2\theta_{ij}, \\ B(t) &= \sin\theta_{ij}\cos\theta_{ij}[a_j(t) - a_i(t)]. \end{aligned}$$

The probability of finding a ν_β state is

$$\boxed{P(\nu_\alpha \to \nu_\beta) = |B(t)|^2 = \sin^2(2\theta_{ij})\sin^2\left[\frac{1}{2}(E_j - E_i)t\right]},$$

and oscillates with time. The oscillations vanish if the mixing angle is 0 or the corresponding mass eigenstates have equal masses, and hence equal energies. Also, such oscillations are not possible if both the ν_i and ν_j have zero masses.

Practically the time t is determined by measuring the distance L that the neutrinos travel from their source. Note that the momenta of the neutrinos are much greater than their possible masses. To a very good approximation we have $t = L$ and

$$E_j - E_i = \sqrt{m_j^2 + p^2} - \sqrt{m_i^2 + p^2} \approx \frac{m_j^2 - m_i^2}{2p},$$

thus

$$\boxed{P(\nu_\alpha \to \nu_\beta) = \sin^2(2\theta_{ij})\sin^2[L/L_0],}$$

where L_0 is the **oscillation length**

$$\boxed{L_0 = \frac{4E}{(m_j^2 - m_i^2)},}$$

and $E = p$. Also we have

$$\boxed{P(\nu_\alpha \to \nu_\alpha) = 1 - P(\nu_\alpha \to \nu_\beta).}$$

In the previous situation we assume that the neutrinos are propagating in a vacuum, since the mean free paths for neutrinos to interact with matter is enormous. If the neutrinos travel long distances through dense matter, the propagation can be significantly modified by their interactions with the dense material they encounter. This is called the **MSW effect**. It arises from the fact that electron neutrino interact with electrons in a different way from muon and tauon neutrinos, and hence have different forward scattering amplitudes. When this is taken into account, $\sin(2\theta_{ij})$ and $\Delta m_{ij}^2 \equiv m_j^2 - m_i^2$ would be modified as

$$\Delta m_{ij}^2 \to C\Delta m_{ij}^2, \quad \sin(2\theta_{ij}) \to \sin(2\theta_{ij})/C,$$

where

$$C = \sqrt{[\cos(2\theta_{ij}) - A]^2 + \sin^2(2\theta_{ij})},$$

and

$$A = \pm\frac{2\sqrt{2}G_F N_e E}{\Delta m_{ij}^2}.$$

N_e is the electron density in matter, G_F is the Fermi weak interaction coupling and the positive/negative sign refers to neutrinos/antineutrinos.

2.3.2 Neutrino Oscillations

The experiments that established the existence of neutrino oscillations fall into four main categories: (a) atmospheric neutrino experiments, (b) neutrino beam experiments, (c) solar neutrino experiments and (d) reactor experiments. The first two study muon neutrino oscillations and the second two study electron neutrino oscillations. Details of the experiments could be found in the textbook.

2.4 Lepton Number Revisited

Lepton number violation in weak interactions is completely negligible within the standard model, but could occur if one abandons the standard model.

Chapter 3

Quarks and Hadrons

3.1 Quarks

There are six quarks now known to exist. These six distinct types (or **flavours**) occur in pairs (or generations):

$$\boxed{\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix}.}$$

Each generation consists of a quark with charge $+\frac{2}{3}e$ (u, c, t) together with a quark of charge $-\frac{1}{3}e$ (d, s, b). They are called the **down** (d), **up** (u), **strange** (s), **charmed** (b) and **top** (t) quarks. The corresponding antiquarks are denoted by

$$\boxed{\begin{pmatrix} \bar{d} \\ \bar{u} \end{pmatrix}, \quad \begin{pmatrix} \bar{s} \\ \bar{c} \end{pmatrix}, \quad \begin{pmatrix} \bar{b} \\ \bar{t} \end{pmatrix},}$$

with charges $+\frac{1}{3}e$ $(\bar{d}, \bar{s}, \bar{b})$ and $-\frac{2}{3}e$ $(\bar{u}, \bar{c}, \bar{t})$. The masses of the quarks are inferred indirectly from the observed masses of their hadron bound states, together with models of quark binding. And there's no convincing evidence for the existence of isolated free quarks, or any other fractionally charged particles.

Name	Symbol	Mass	Q	B	S	C	$\tilde{B}$	T
Down	d	$m_d \approx 0.3$	-1/3	1/3	0	0	0	0
Up	u	$m_u \approx m_d$	2/3	1/3	0	0	0	0
Strange	s	$m_d \approx 0.5$	-1/3	1/3	-1	0	0	0
Charmed	c	$m_d \approx 1.5$	2/3	1/3	0	1	0	0
Bottom	b	$m_d \approx 4.5$	-1/3	1/3	0	0	-1	0
Top	t	$m_d \approx 174$	2/3	1/3	0	0	0	1

Table 3.1: Quarks. B baryon number, S strangeness, C charm, $\tilde{B}$ bottom, T top

3.2 General Properties of Hadrons

There are only three types of quark bound states are allowed. They are (1) the **baryons**, which have half-integer spin and are assumed to be bound states of three quarks $(3q)$; (2) the **antibaryons**, which are the antiparticles of baryons and assumed to be bound states of three antiquarks $(3\bar{q})$; (3) the **mesons**, which have integer spin and are assumed to be bound states of a quark and an antiquark $(q\bar{q})$.

$$\text{Quark Bound States} \begin{cases} \text{Baryon} & 3q \\ \text{Antibaryon} & 3\bar{q} \\ \text{Mesons} & q\bar{q} \end{cases}$$

Particle		Mass (MeV/c^2)	Q	S	C	$\tilde{B}$
p	uud	938	1	0	0	0
n	udd	940	0	0	0	0
Λ	uds	1116	0	-1	0	0
Λ_c	udc	2285	1	0	1	0
Λ_d	udd	5624	0	0	0	-1

Table 3.2: Some examples of baryons.

Particle		Mass (MeV/c^2)	Q	S	C	$\tilde{B}$
π^+	$u\bar{d}$	140	1	0	0	0
K^-	$s\bar{u}$	494	-1	-1	0	0
D^-	$d\bar{c}$	1869	-1	0	-1	0
D_s^+	$c\bar{s}$	1969	1	1	1	0
B^-	$b\bar{u}$	5279	-1	0	0	-1
Υ	$b\bar{b}$	9460	0	0	0	0

Table 3.3: Some examples of mesons.

Several quantum numbers are associated with the state and the quark content. The **strangeness** S is defined by

$$S \equiv -N_s \equiv -[N(s) - N(\bar{s})],$$

where $N(s)$ and $N(\bar{s})$ are the number of s quarks and $\bar{s}$ antiquarks present in the state. Clearly we have $S = -1$ for an s quark, $S = 1$ for an $\bar{s}$ antiquark and $S = 0$ for all other quarks and antiquarks. The **charm** (C), **bottom** ($\tilde{B}$) and **top** (T) quantum numbers are similarly defined by

$$\begin{aligned} C &\equiv N_c \equiv N(c) - N(\bar{c}), \\ \tilde{B} &\equiv -N_b \equiv -[N(b) - N(\bar{b})], \\ T &\equiv N_t \equiv N(t) - N(\bar{t}). \end{aligned}$$

The top quantum number $T = 0$ for all known hadrons. The **baryon number** B is defined by

$$B \equiv= \frac{1}{3}[N(q) - N(\bar{q})],$$

where $N(q)$ and $N(\bar{q})$ are the total numbers of quarks and antiquarks present. We have $B = 1$ for baryons, $B = -1$ for antibaryons and $B = 0$ for mesons. The remaining quark numbers

$$N_u = N(u) - N(\bar{u}), \quad N_d = N(d) - N(\bar{d}),$$

do not have special name since given $S, C, \tilde{B}, T$ and Q, their values can be inferred. We have

$$\begin{aligned}
B &= \frac{1}{3}(N_u + N_d + N_s + N_c + N_b + N_t) \\
&= \frac{1}{3}(N_u + N_d - S + C - \tilde{B} + T), \\
Q &= \frac{2}{3}(N_u + N_c + N_t) - \frac{1}{3}(N_d + N_s + N_b), \\
&= \frac{2}{3}(N_u + C + T) - \frac{1}{3}(N_d - S - \tilde{B}).
\end{aligned}$$

$$\begin{array}{c} \text{Quark Numbers} \\ \text{(Internal Quantum Numbers)} \end{array} \left\{ \begin{array}{ll} Q & \text{Charge} \\ S & \text{Strangeness} \\ C & \text{Charm} \\ \tilde{B} & \text{Bottom} \\ T & \text{Top} \\ B & \text{Baryon Number} \end{array} \right.$$

All these quantum numbers are called **internal quantum numbers** since they are not associated with motion or the special properties of wave functions. In strong and electromagnetic interactions quarks and antiquarks are only created and destroyed in particle-antiparticle pairs. For example, the quark description of the strong interaction process

$$p + p \to p + n + \pi^+,$$

is

$$(uud) + (uud) \to (uud) + (udd) + (u\bar{d}).$$

The final state contains the same number of quarks of each flavour as the initial state, plus an additional $(d\bar{d})$, so that the quark numbers N_u and N_d are separately conserved.

The characteristic of strong and electromagnetic interactions is that all the quark numbers are separately conserved.

For the neutron β-decay

$$n \to p + e^- + \bar{\nu}_e,$$

the quark interpretation is

$$(udd) \to (uud) + e^- + \bar{\nu}_e,$$

In this reaction, a d is replaced by a u. The decay is represented by the **quark diagram**

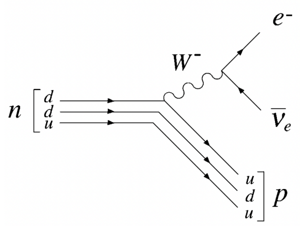

Figure 3.1: Quark diagram.

shown in Figure 7.1. Each hadron is represented by lines corresponding to its constituent quarks and antiquarks, with quarks and antiquarks labelled by arrows pointing to the right and left respectively.

The characteristic of weak interactions is that quark flavours can change and only the baryon number and the total electric charge are conserved.

The quark numbers is very important to understand the long lifetimes of some hadrons. Hadrons have typical effective radii of order 1 fm, with a time scale r/c of order 10^{-23}s. The majority of hadrons are very unstable and would decay to lighter hadrons by the strong interaction with lifetimes of this order. Since each hadron is characterized by the quark numbers, in some cases there are no lighter hadron states with the same values of these quantum numbers to which they can decay. These hadrons are called **stable particles** (or **long-lived particles**), having a lifetime of order 10^{-23}s. Except for proton they are not absolutely stable, and can decay by the electromagnetic or weak interaction.

Interaction	Lifetime (s)
Strong	$10^{-22} \sim 10^{-24}$
Electromagnetic	$10^{-16} \sim 10^{-21}$
Weak	$10^{-7} \sim 10^{-13}$

Table 3.4: Typical lifetimes of hadrons decaying by the three interactions.

3.3 Pions and Nucleons

The lightest known mesons are the pions (or pi-mesons) $\pi^{\pm}$ (140[1]), π^0 (135). These particles are produced copiously in may hadronic reactions that conserve both charge and baryon number. For example, in proton-proton collisions

$$\begin{aligned} p + p &\to p + n + \pi^+, \\ &\to p + p + \pi^0, \\ &\to p + p + \pi^- + \pi^+. \end{aligned}$$

The charge pions decay predominantly by the reactions

$$\boxed{\pi^+ \to \mu^+ + \nu_\mu, \quad \pi^- \to \mu^- + \bar{\nu}_\mu,}$$

with lifetimes of 2.6×10^{-8}s (weak interaction). And the neutral pions decay by the electromagnetic interaction

$$\boxed{\pi^0 \to \gamma + \gamma,}$$

with a lifetime of 0.8×10^-16s.

Pions and nucleons are the lightest known mesons and baryons respectively, and are bound states of the lightest quarks (u, d) and their antiquarks $(\bar{u}, \bar{d})$. We have

$$\boxed{p = uud, \quad n = udd,}$$

and

$$\boxed{\pi^+ = ud, \quad \pi^0 = u\bar{u}, d\bar{d}, \quad \pi^- = d\bar{u}.}$$

Yukawa proposed that nuclear forces are due to the exchange of spin-0 mesons. In the Yukawa theory, the nucleons and pions are treated as point particles. Neutral pion exchange gives rise to normal direct forces, while $\pi^{\pm}$ exchange gives rise to exchange forces whereby the neutron and proton are interchanged. The resulting potential is of the Yukawa type, which could reproduce the longest range part of the nuclear force $r \geq 2$ fm accurately.

[1] MeV/c^2.

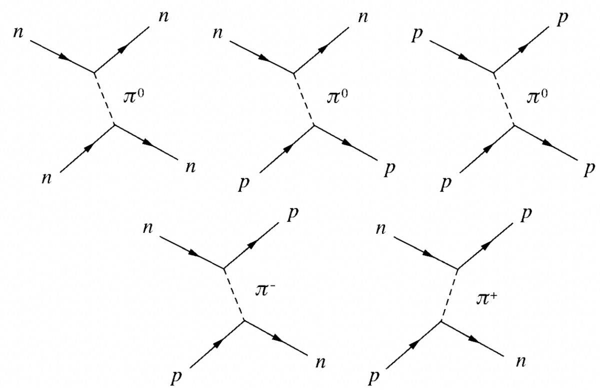

Figure 3.2: The Yukawa model for nuclear forces: direct forces (upper 3) and exchange forces (lower 2).

3.4 Strange Particles, Charm and Bottom

After the discovery of the pion, it was discovered that other hadrons were produced in strong interaction, but decayed by weak interaction. These hadrons are called **strange particles**. One example for the strange particle is K^+, with mass 494 MeV/c^2 and lifetime 1.2×10^{-8}s. For the K^+ meson (kaon) decay

$$\boxed{K^+ \to \mu^+ + \nu_\mu,} \quad B = 0.63,$$

where B is the branching ratio. Also

$$\boxed{K^+ \to \pi^+ + \pi^0,} \quad B = 0.21.$$

Another example of a strange particle is the Λ (lambda) baryon, with mass 1116 MeV/c^2. It decays mainly into pions and nucleons:

$$\boxed{\Lambda \to \pi^- + p,} \quad B = 0.64,$$

and

$$\boxed{\Lambda \to \pi^0 + n,} \quad B = 0.36,$$

with a lifetime of 2.6×10^{-10}s. It is clear from the long lifetimes of the K^+ and Λ that they both decay via the weak interaction.

Now a **strange particle** is defined as a particle with a nonzero value of the strangeness quantum number. Most of them, like most hadrons with $S = 0$, decay by the strong interaction. However, it has been shown that if a particle is the lightest state with a given non-zero set of B, Q and S values, it can only decay by weak interactions, and will be relatively long-lived.

For example, the lightest strange baryon Λ has $B = 1, Q = 0, S = -1$. And the lightest strange mesons are the kaons. In addition to the K^+, we also have the negative and neutral kaons, which also decay by the weak interaction:

$$\boxed{K^+(494) = u\bar{s}, \quad K^0(498) = d\bar{s},}$$

and

$$\boxed{K^-(494) = s\bar{u}, \quad \bar{K}^0(498) = s\bar{d},}$$

where K^+ and K^0 have $S = +1$ and K^- and $\bar{K}^0$ have $S = -1$.

The production of strange particles in strong interactions is an example of **associated production**, in which more than one strange particle is produced, giving strangeness conservation overall.

For example, we have

$$\boxed{\pi^- + p \to K^0 + \Lambda,}$$

with $S = 0, 0, 1$ and -1 respectively for the particles.

After that, J/ψ is discovered, which is one of the lightest of a family of particles that are bound states of a charmed quark and its antiparticle:

$$\boxed{J/\psi(3097) = c\bar{c}. \quad (C = 0)}$$

Such $c\bar{c}$ states are collectively called **charmonium**. Also since $C = 0$, these states are said to contain "hidden charm". Particles with "naked charm" $C \neq 0$ were also discovered.

Charm is conserved in strong and electromagnetic interactions, and the lightest charmed particles decay by weak interaction.

For example, the lightest charmed mesons are the D mesons:

$$\boxed{D^+(1870) = c\bar{d}, \quad D^0(1865) = c\bar{u}, \quad (C = +1)}$$

and

$$\boxed{D^-(1870) = d\bar{c}, \quad \bar{D}(1865) = u\bar{c}. \quad (c = -1)}$$

Also we have the D_s mesons:

$$\boxed{D_s^+(1968) = c\bar{s}, \quad (C = +1, S = +1)}$$

and

$$\boxed{D_s^-(1968) = s\bar{c}. \quad (C = -1, S = -1)}$$

The lightest charmed baryon is

$$\boxed{\Lambda_c^+(2286) = udc. \quad (C = +1)}$$

These particles all have lifetimes of order 10^{-13}s, which is in the expected range for weak decays. Charmed particles can be produced in strong and electromagnetic interactions by associated production reactions.

The bottom quark b with its associated quantum number $\tilde{B}$ came from the discovering of one of the lightest **bottomonium** states

$$\boxed{\Upsilon(9460) = b\bar{b}, \quad (\tilde{B} = 0)}$$

which is a hidden bottom state called the **upsilon**. Also mesons with "naked bottom" $\tilde{B} \neq 0$ were discovered, and the lightest example are the B-mesons

$$\boxed{B^+(5279) = u\bar{b}, \quad B^0(5280) = d\bar{b}, \quad (\tilde{B} = +1)}$$

and

$$\boxed{B^-(5279) = b\bar{u}, \quad \bar{B}^0(5280) = b\bar{d}, \quad (\tilde{B} = -1)}$$

with lifetimes of order 10^{-12}s, consistent with their decay via the weak interactions. Also we have baryons with $\tilde{B} \neq 0$:

$$\boxed{\Lambda_b^0(5620) = udb,}$$

and the doublet

$$\boxed{\Xi_b^0(5793) = usb, \quad \Xi_b^-(5795) = dsb,}$$

which decay via the weak interaction, with lifetimes of order 10^{-12}s.

3.5 Short-Lived Hadrons

3.6 Allowed Quantum Numbers and Exotics

Chapter 4

Experimental Methods

Chapter 5

Space-Time Symmetries

In this chapter we concentrate on symmetries and conservation laws in strong and electromagnetic interactions. Some of the conservation laws discussed in this chapter are universal laws of nature, but others, like parity, are only convered in the approximation that weak interactions are neglected.

5.1 Translational Invariance

Translational invariance means that all positions in space are physically indistinguishable. When a close system of particles (no external forces) is moved from one position to another, its physical properties are unaltered.

In quantum mechanics, this is expressed as an invariance of the Hamiltonian. If the system is displaced a distance $\boldsymbol{a}$, the position vector $\boldsymbol{r}_i$ of a particle i becomes

$$\boldsymbol{r}_i \to \boldsymbol{r}'_i = \boldsymbol{r}_i + \boldsymbol{a}.$$

Here we consider infinitesimal displacements $\boldsymbol{a} = \delta\boldsymbol{r}$. And

$$H(\boldsymbol{r}'_1, \boldsymbol{r}'_2, \cdots) = H(\boldsymbol{r}_1 + \delta\boldsymbol{r}, \boldsymbol{r}_2 + \delta\boldsymbol{r}, \cdots).$$

For a single particle of mass m, the Hamiltonian

$$H = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right).$$

And easily we have $H(\boldsymbol{r}') = H(\boldsymbol{r} + \delta\boldsymbol{r}) = H(\boldsymbol{r})$. Generally we have

$$\boxed{H(\boldsymbol{r}'_1, \boldsymbol{r}'_2, \cdots) = H(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots),}$$

for a closed system and the Hamiltonian is said to be invariant under the translation.

Now we specialize to s system containing a single particle. Introduce a translation operator $\hat{D}$

$$\boxed{\hat{D}\psi(\boldsymbol{r}) \equiv \psi(\boldsymbol{r} + \delta\boldsymbol{r}),}$$

where $\psi(\boldsymbol{r})$ is an arbitrary wavefunction. Expand the RHS to first order in $\delta\boldsymbol{r}$:

$$\psi(\boldsymbol{r} + \delta\boldsymbol{r}) = \psi(\boldsymbol{r}) + \delta\boldsymbol{r} \cdot \nabla\psi(\boldsymbol{r}).$$

Thus we have

$$\boxed{\hat{D} = 1 + i\delta\boldsymbol{r} \cdot \hat{\boldsymbol{p}}.}$$

Applying $\hat{D}$ to a wavefunction

$$\psi'(\boldsymbol{r}) = H(\boldsymbol{r})\psi(\boldsymbol{r})$$

gives

$$\hat{D}\psi'(\boldsymbol{r}) = \hat{D}H(\boldsymbol{r})\psi(\boldsymbol{r}).$$

Also we have

$$\begin{aligned}\hat{D}\psi'(\boldsymbol{r}) &= \psi'(\boldsymbol{r}+\delta\boldsymbol{r}) = H(\boldsymbol{r}+\delta\boldsymbol{r})\psi(\boldsymbol{r}+\delta\boldsymbol{r}) \\ &= H(\boldsymbol{r})\psi(\boldsymbol{r}+\delta\boldsymbol{r}) = H(\boldsymbol{r})\hat{D}\psi(\boldsymbol{r}).\end{aligned}$$

Thus we have

$$\left[\hat{D}H(\boldsymbol{r}) - H(\boldsymbol{r})\hat{D}\right]\psi(\boldsymbol{r}) = 0,$$

which is equivalent as

$$\boxed{[\hat{D}, H] = 0.}$$

This commutation relation leads to the conservation law for linear momentum

$$\boxed{[\hat{\boldsymbol{p}}, H] = 0}$$

for a single-particle state whose Hamiltonian is invariant under the translation. Generally for an N-particle state, we have the conservation of the total linear momentum

$$\boldsymbol{p} = \sum_{i=1}^{N} \boldsymbol{p}_i.$$

This works for other conservation laws in translational symmetry.

5.2 Rotational Invariance

> Rotational invariance means that all directions in space are physically indistinguishable. Particularly, when a closed system of particles is rotated as a whose about its center-of-mass to a new orientation in space, its physical properties are unchanged.

The Hamiltonian $H(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots)$ will be replaced by $H(\boldsymbol{r}'_1, \boldsymbol{r}'_2, \cdots)$. If the system is rotationlly invariant, we have

$$\boxed{H(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots) = H(\boldsymbol{r}'_1, \boldsymbol{r}'_2, \cdots).}$$

The invariance property holds for any closed system. For a particle moving in a central potential $V(r)$, the Hamiltonian

$$H = -\frac{1}{2m}\nabla^2 + V(r), \quad r = \sqrt{x^2+y^2+z^2}.$$

5.2.1 Angular Momentum Conservation

Now we derive the law of angular momentum conservation for a single spin-0 particle moving in a central potential. Consider an infinitesimal rotation through an angle $\delta\theta$ about the z-axis:

$$x' = x - y\delta\theta, \quad y' = x\delta\theta + y, \quad z' = z.$$

Introduce a rotation operator $\hat{R}_z$

$$\boxed{\hat{R}_z(\delta\theta)\psi(\boldsymbol{r}) \equiv \psi(\boldsymbol{r}') = \psi(x - y\delta\theta, x\delta\theta + y, z).}$$

Expand the RHS to the first order in $\delta\theta$ gives

$$\psi(\boldsymbol{r}') = \psi(\boldsymbol{r}) - \delta\theta\left(y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right) = (1 + i\delta\theta\,\hat{L}_z)\psi(\boldsymbol{r}).$$

Hence we have

$$\boxed{\hat{R}_z(\delta\theta) = 1 + i\delta\theta\,\hat{L}_z.}$$

For a rotation about an arbitrary axis specified by the unit vector $\boldsymbol{n}$, we have

$$\boxed{\hat{R}_{\boldsymbol{n}}(\delta\theta)\psi(\boldsymbol{r}) \equiv \psi(\boldsymbol{r}'),}$$

and

$$\boxed{\hat{R}_{\boldsymbol{n}} = 1 + i\delta\theta\hat{\boldsymbol{L}}\cdot\boldsymbol{n}}.$$

Similarly, apply $\hat{R}_{\boldsymbol{n}}$ to the wavefunction $\psi'(\boldsymbol{r}) = H(\boldsymbol{r})\psi(\boldsymbol{r})$ we have

$$\hat{R}_{\boldsymbol{n}}\psi'(\boldsymbol{r}) = \hat{R}_{\boldsymbol{n}}H(\boldsymbol{r})\psi(\boldsymbol{r}),$$

also

$$\begin{aligned}\hat{R}_{\boldsymbol{n}}\psi'(\boldsymbol{r}) &= \psi'(\boldsymbol{r}') = H(\boldsymbol{r}')\psi(\boldsymbol{r}')\\ &= H(\boldsymbol{r})\psi(\boldsymbol{r}') = H(\boldsymbol{r})\hat{R}_{\boldsymbol{n}}\psi(\boldsymbol{r}).\end{aligned}$$

Hence

$$\boxed{\left[\hat{R}_{\boldsymbol{n}}, H\right] = 0.}$$

Finally by the commutation relation we have the conservation laws for orbital angular momentum

$$\boxed{\left[\hat{\boldsymbol{L}}, H\right] = 0,}$$

for a spinless particle whose Hamiltonian is invariant under rotation.

If the particle has spin, the total angular momentum

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S},$$

and the wavefunction Ψ is written as

$$\Psi = \psi(\boldsymbol{r})\chi.$$

For spin-$\frac{1}{2}$ particles, we use $\chi = \alpha$ to denote a spin up state and $\chi = \beta$ to denote a spin down particle. We claim that under a rotation through a small angle $\delta\theta$ about a direction $\boldsymbol{n}$, the rotation operator generalizes to

$$\boxed{\hat{R}_{\boldsymbol{n}}(\delta\theta) = 1 + i\delta\theta\hat{\boldsymbol{J}}\cdot\boldsymbol{n},}$$

so that

$$\Psi \to \Psi' = (1 + i\delta\theta\hat{\boldsymbol{J}}\cdot\boldsymbol{n})\Psi.$$

On the RHS, $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{S}}$ act only on the space and spin wavefunctions respectively. Similarly we have

$$\boxed{\left[\hat{\boldsymbol{J}}, H\right] = 0.}$$

Rotational invariance does not lead to the conservation of $\boldsymbol{L}$ and $\boldsymbol{S}$ separately. And generally the orbital and spin angular momentum are not conserved.

5.2.2 Classification of Particles

One of the quantum numbers that characterize the particle is its spin, which is defined as the particle's angular momentum in its own rest frame. For a composite particle, the frame corresponds to the center-of-mass frame of its constituents.

In the rest frame of the particle, the total angular momentum $\boldsymbol{J}$ of the constituents is conserved, but the total orbital angular momentum $\boldsymbol{L}$ and the total spin angular momentum $\boldsymbol{S}$ are not separately conserved. However, it's often a good approximation to assume that $\boldsymbol{L}^2$ and $\boldsymbol{S}^2$ are conserved. In this approximation, L and S are also good quantum numbers, thus the particle is characterized by $S_P = J, L, S$, and a fourth J_z depends on the orientation of its spin. This lead to the **spectroscopic notation**, where the states are denoted by

$$\boxed{{}^{2S+1}L_J.}$$

In this notation we write $S, P, D, F, \cdots$ for $L = 0, 1, 2, 3, \cdots$.

5.2.3 Angular Momentum in the Quark Model

In the simple quark model we assume that it's a good approximation to treat L and S as good quantum numbers, and that the lightest states for any combinations of $q\bar{q}$ and $3q$ have zero orbital angular momentum.

Mesons are $q\bar{q}$ bound states. The rest frame of the meson corresponding to the centre-of-mass frame of the $q\bar{q}$ system. In this frame we have a single orbital angular momentum $\boldsymbol{L}$, but two constituent spins, thus

$$\boldsymbol{S} = \boldsymbol{S}_q + \boldsymbol{S}_{\bar{q}}.$$

Both q and $\bar{q}$ have spin $\frac{1}{2}$, thus $S = 0$ or $S = 1$.

5.3 Parity

5.3.1 Leptons and Antileptons

5.3.2 Quarks and Hadrons

5.4 Charge Conjugation

5.5 Positronium

5.6 Time Reversal

Chapter 6

The Quark Model

Chapter 7

QCD, Jets and Gluons

Chapter 8

Quarks and Partons

Chapter 9

Weak Interactions: Quarks and Leptons

The weak interaction is associated with elementary spin-1 bosons that act as "force carriers" between quarks and/or leptons. These bosons are massive particles and the interactions are hence of very short range. There are 3 such bosons: the charged bosons W^+ and W^- and the neutral Z^0. There masses

$$M_W = 80.38\text{GeV}/c^2, \quad M_Z = 91.19\text{GeV}/c^2,$$

which give ranges

$$R_W \approx R_Z \approx 2 \times 10^{-3}\text{fm}.$$

for the weak interactions. The distance are very small, and at low energies the weak interaction could be treated as a zero-range interaction. For high-energy weak interactions, the zero-range approximation is no longer appropriate.

> **Neutral current** reactions are those which involve the emission, absorption, or exchange of Z^0 bosons. And **charged current** reactions arise from the charged $W^\pm$ boson exchange.

In this chapter we will discuss about the charged-current reactions.

9.1 Charged Current Reactions

The simplest charged current reactions are purely leptonic processes, like the muon decay:

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu.$$

Also we have the hadronic processes, like the lambda decay

$$\Lambda \to p + \pi^-.$$

And there are **semileptonic** reactions, such as neutron decay

$$n \to p + e^- + \bar{\nu}_e,$$

which involve both hadrons and leptons.

9.1.1 W$^\pm$-Lepton Interactions

From Chapter 1 we know that all electromagnetic interactions of electrons and positrons can be built from 8 basic interactions where a photon is emitted or absorbed. These processes could be summarised by the vertex of Figure 9.1(a), from which they can be obtained by

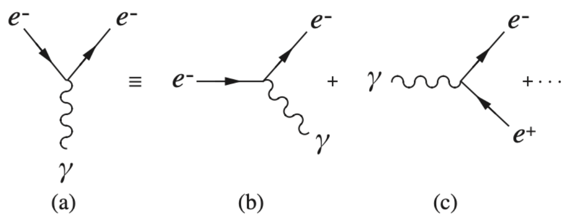

Figure 9.1: The basic vertex for electron-photon interactions.

associated different particle lines with the initial and final states. Leptonic weak interaction processes like the muon decay and inverse muon decay

$$\nu_\mu + e^- \to \mu^- + \nu_e,$$

could also be built from a limited number of basic vertices in the same way. In the case of charged current reactions, each lepton type $\ell = e, \mu, \tau$, there are 16 such basic reactions corresponding to the two vertices of Figure 9.2. The 8 processes corresponding to Figure

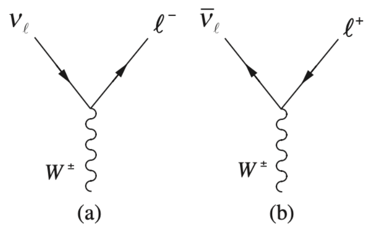

Figure 9.2: The two basic vertices for $W^\pm$-lepton interactions.

9.2(a) are shown in Figure 9.3. The 8 processes corresponding to Figure 9.2(b) can be obtained from those of Figure 9.3 by replacing all the particles by their antiparticles, as shown in Figure 9.4.

An important property of weak interaction is that it conserves the lepton numbers L_e, L_μ, L_τ. This is guaranteed by the fact that in Figure 9.2 there is one arrow pointing into the vertex and one pointing out of it, and the lepton label ℓ is the same on both lines. In contrast, vertices like those in Figure 9.5(a), which give rises to process that violate the lepton number, are excluded from the scheme.

Although the processes in Figure 9.3 conserve the lepton numbers, this does not mean that they can occur as isolated reactions in free space. As we saw before, the virtual processes violate energy conservation if momentum conservation is assumed, and they can only contribute to physical processes. However, Figure 9.3(e) and (f) are compatible with both energy and momentum conservation provided that

$$M_W > M_\ell + M_{\nu_\ell},$$

which is obviously the case. Particularly, Figure 9.3(f) and its antiparticle reaction, are the dominant mechanisms for the leptonic decays, which were used to detect the $W^\pm$ bosons in the experiments.

Now we determine the strength of the interaction. We could obtain an order-of-magnitude estimate for α_W by applying the method of dimensions to the rate for the leptonic decays in Figure 9.3(a) and (b). This decay rate has a natural unit $[E]$, and the measured value is

$$\Gamma(W \to e\nu) \simeq 0.223 \pm 0.007\text{GeV}.$$

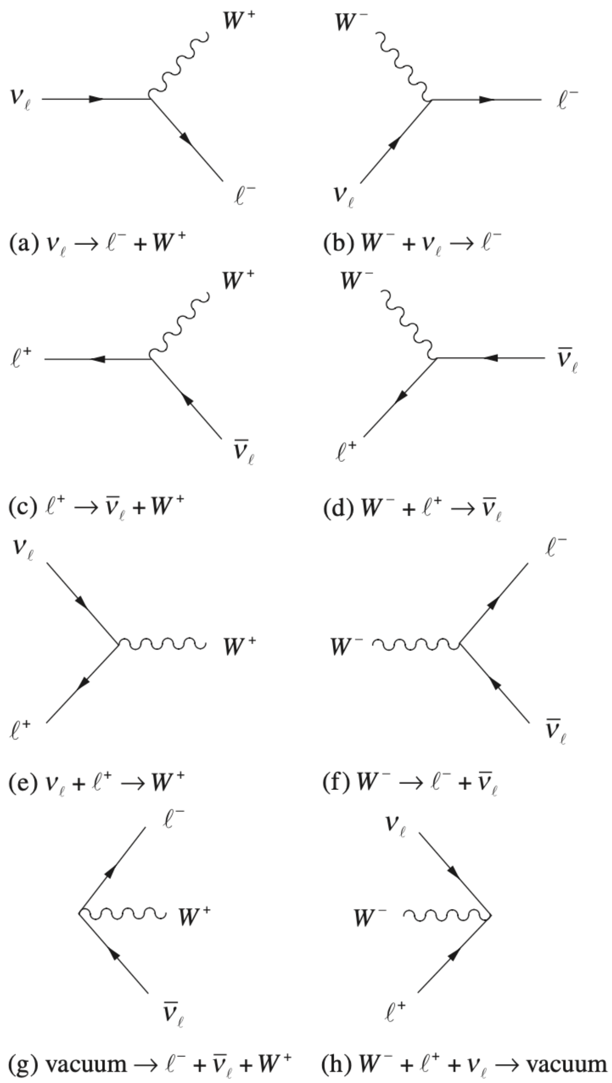

Figure 9.3: The 8 basic vertices derived from the vertex of Figure 9.2(a).

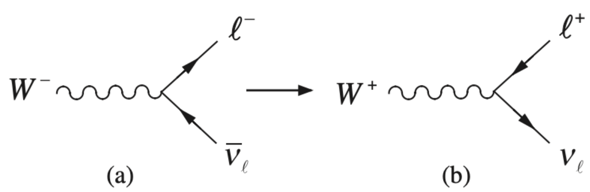

Figure 9.4: A pair of weak interaction processes that are related by replacing all particles by their antiparticles.

A detailed calculation gives

$$\Gamma(W \to e\nu) = 2\alpha_W M_W/3,$$

and hence

$$\boxed{\alpha_W = 0.0042 \pm 0.0002.}$$

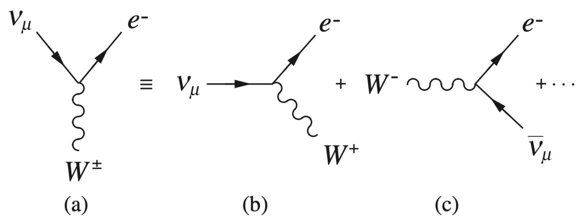

Figure 9.5: Example of a vertex that violates lepton number conservation.

9.1.2 Lepton-Quark Symmetry and Mixing

The weak interactions of hadrons are understood in terms of basic processes in which $W^\pm$ bosons are emitted or absorbed by their constituent quarks. These can give rise to semileptonic processes like the neutron decay (Figure 9.6), and purely hadronic decays such as the Λ decay (Figure 9.7).

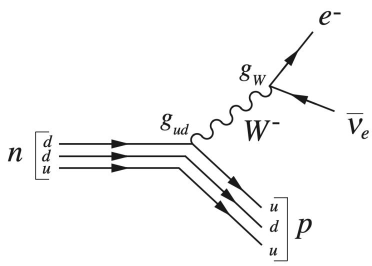

Figure 9.6: Dominant diagram for neutron decay.

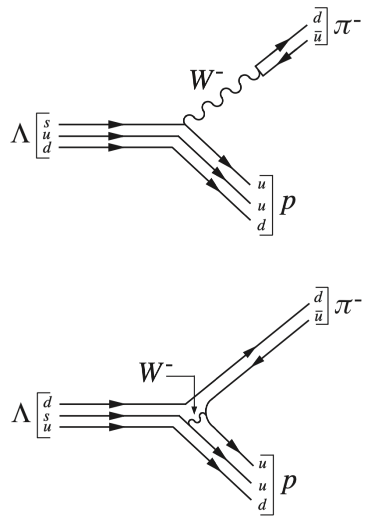

Figure 9.7: Dominant diagrams for Λ decay.

The weak interactions of quarks are understood in terms of two ideas: **lepton-quark symmetry** and **quark mixing**.

Lepton-quark symmetry asserts that the two generations of quarks

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}$$

and the two generations of leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}$$

have identical interactions.

This means that one obtains the basic $W^{\pm}$-quark vertices by making the replacements

$$\boxed{\nu_e \to u, \quad e^- \to d, \quad \nu_\mu \to c, \quad \mu^- \to s}$$

in the basic $W^{\pm}$-lepton vertices in Figure 9.2, leaving the coupling constant g_W unchanged. By this way we could obtain the vertices of Figure 9.8, where the coupling constant

$$\boxed{g_{ud} = g_{cs} = g_W.}$$

This works for many reactions, like the pion decay

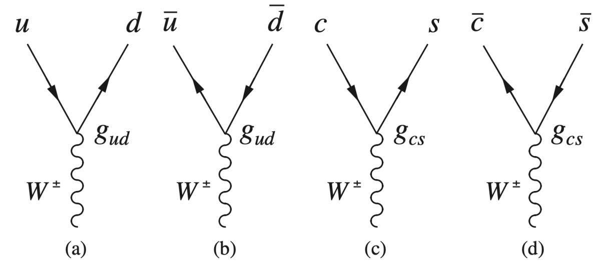

Figure 9.8: The $W^{\pm}$-quark vertices obtained from lepton-quark symmetry when quark mixing is ignored.

$$\pi^- \to \mu^- + \bar{\nu}_\mu,$$

which corresponds to

$$d\bar{u} \to \mu^- + \bar{\nu}_\mu.$$

However, many other decays that are experimentally observed are forbidden in this scheme,

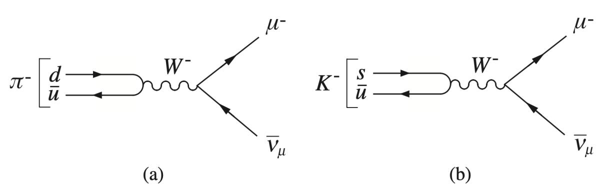

Figure 9.9: Feynman diagrams for the semileptonic decays: (a) pion decay, and (b) kaon decay.

like the kaon decay

$$K^- \to \mu^- + \bar{\nu}_\mu,$$

which corresponds to

$$s\bar{u} \to \mu^- + \bar{\nu}_\mu.$$

Note that the usW vertex is not included in the vertices of Figure 9.8. This could be incorporated by introducing an hypothesis called quark mixing, proposed by Cabibbo.

According to quark mixing, the d and s quarks participate in the weak interactions via the linear combinations

$$d' = d\cos\theta_C + s\sin\theta_C,$$

and

$$s' = -d\sin\theta_C + s\cos\theta_C,$$

where θ_C is called the **Cabibbo angle**.

This means that lepton-quark symmetry is assumed to apply to the doublets

$$\boxed{\begin{pmatrix} u \\ d' \end{pmatrix}, \quad \begin{pmatrix} c \\ s' \end{pmatrix}.}$$

This is shown in Figure 9.10. The coupling of the previously allowed udW vertex is suppressed by a factor $\cos\theta_C$, while the previously forbidden usW vertex is now allowed with a coupling $g_W \sin\theta_C$. This applies to the other vertices in Figure 9.8. Thus in addition to

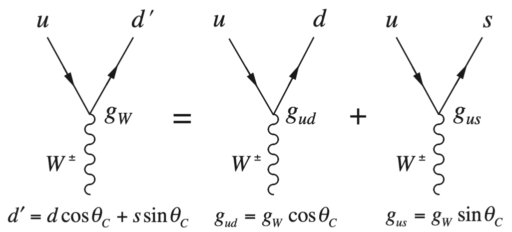

Figure 9.10: The $ud'W$ vertex and its interpretation in terms of udW and usW vertices.

the 4 vertices of Figure 9.8 with couplings

$$\boxed{g_{ud} = g_{cs} = g_W \cos\theta_C,}$$

we have the vertices of Figure 9.11 with coupling

$$\boxed{g_{us} = -g_{cd} = g_W \sin\theta_C.}$$

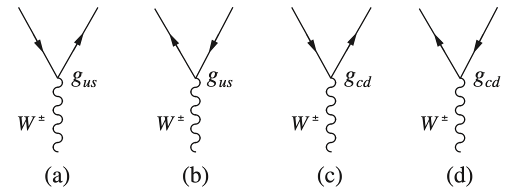

Figure 9.11: The additional vertices arising from lepton-quark symmetry when quark mixing is taken into account.

The set of $W^{\pm}$-quark couplings successfully accounts for the charged current interactions of hadrons. The Cabibbo angle is determined by deducing the values of the coupling g_{ud} and g_{us} from the measured rates of hadron decays

$$\frac{\Gamma(K^- \to \mu^- + \bar{\nu}_\mu)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} \propto \frac{g_{us}^2}{g_{ud}^2} = \tan^2 \theta_C.$$

We have

$$\theta_C = 13.02 \pm 0.004°.$$

Note that decays that involving the first kind of couplings are called **Cabibbo-suppressed** since their rates are reduced by a factor of order

$$\frac{g_{us}^2}{g_{ud}^2} = \frac{g_{cd}^2}{g_{cs}^2} = \tan^2 \theta_C \approx \frac{1}{20}$$

compared with similar **Cabibbo-allowed** decays which involve the second kind of couplings. The Cabibbo-allowed decays

$$c \to s + \ell^+ + \nu_\ell,$$

and

$$c \to s + u + \bar{d}$$

of a charmed quark to lighter quarks and leptons are shown in Figure 9.12. They necessarily produce an s quark in the final state[1]. The result is that the decay of a charmed hadron will always lead to a strange hadron in the final state.

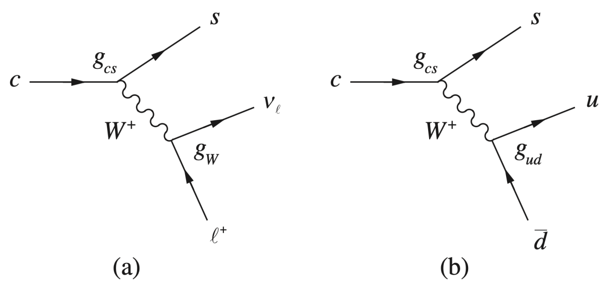

Figure 9.12: Cabibbo-allowed decays of a charmed quark

9.1.3 W Boson Decays

9.2 The Third Generation

[1]Note that all these quarks must be bound into hadrons.

Chapter 10

Weak Interactions: Electroweak Unification

Appendix A

Relativistic Kinematics

A.1 The Lorentz Transformation for Energy and Momentum

Suppose in an inertial frame S a particle has coordinate $(\boldsymbol{r}, t) \equiv (x, y, z, t)$. Then its coordinates $(\boldsymbol{r}', t')$ in a second inertial frame S' moving with uniform velocity relative to S are given by **Lorentz transformation**. Assume the two frames coincide at $t = 0$ and frame S' is moving with uniform speed $v = |\boldsymbol{v}|$ in $+x$-direction relative to S, we have

$$\boxed{x' = \gamma(v)(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(v)\left(t - \frac{vx}{c^2}\right),}$$

where $\gamma(v) \equiv (1 - \beta^2)^{-1/2}$ and $\beta = v/c$. Now consider a particle i of rest mass m_i with uniform velocity $\boldsymbol{u}_i$ in S, then its energy E_i and momentum $\boldsymbol{p}_i$ are

$$\boxed{E_i = m_i c^2 \gamma(u_i), \quad \boldsymbol{p}_i = m_i \gamma(u_i) \boldsymbol{u}_i,}$$

where $u_i = |\boldsymbol{u}_i|$. For simplicity, take $\boldsymbol{u}_i$ in the x-direction, i.e. the same direction as $\boldsymbol{v}$. The velocity of the particle in S' is

$$u_i' = \frac{u_i - v}{1 - u_i v/c^2}.$$

Then

$$\gamma(u_i') \equiv (1 - u_i'^2/c^2)^{-1/2} = \gamma(u_i)\gamma(v)(1 - u_i v/c^2).$$

And the energy and momentum in S' are

$$\begin{aligned} E_i' &= m_i c^2 \gamma(u_i') = \gamma(v)(E_i - v p_i), \\ p_i' &= m_i u_i' \gamma(u_i') = \gamma(v)(p_i - v E_i/c^2). \end{aligned}$$

Generalize to an arbitrary direction for the particle's velocity and momentum, we have

$$\begin{aligned} (p_i')_x &= \gamma(v)[(p_i)_x - v E_i/c^2], \\ (p_i')_y &= (p_i)_y, \\ (p_i')_z &= (p_i) z, \\ E_i' &= \gamma(v)[E_i - v(p_i)_x]. \end{aligned}$$

Since those equations are linear in both energy and momentum, then

$$\begin{aligned} p_x' &= \gamma(v)(p_x - vE/c^2), \\ p_y' &= p_y, \\ p_z' &= p_z, \\ E' &= \gamma(v)(E - v p_x), \end{aligned}$$

for the total energy and momentum

$$E = E_1 + E_2 + \cdots + E_N, \quad \boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \cdots + \boldsymbol{p}_N$$

of N particles that are sufficiently far apart for their mutual interaction energies to be neglected.

A.2 The Invariant Mass

The **invariant mass** W of a set of N particles is defined by

$$\boxed{W^2c^4 \equiv E^2 - \boldsymbol{p}^2c^2,}$$

where E and $\boldsymbol{p}$ are the total energy and momentum. W has the same value in any reference frame,

$$(E')^2 - (\boldsymbol{p}')^2c^2 = E^2 - \boldsymbol{p}^2c^2.$$

In the center-of-mass (CM) frame, we have

$$\boldsymbol{p} = \boldsymbol{0},$$

thus

$$Wc^2 = E_{CM}$$

is the total energy in the CM frame. For a single particle i, the invariant mass W_i is identical with the rest mass m_i, i.e. $W_i = m_i$.

A.2.1 Beam Energies and Thresholds

Consider a beam of particles b with momentum $\boldsymbol{p}_L$ incident upon target particles t, which are at rest in the **laboratory**, i.e. $\boldsymbol{p}_t = \boldsymbol{0}$. Then the energies of particles in the laboratory frame are

$$E_L = (m_b^2c^4 + \boldsymbol{p}_L^2c^2)^{1/2}, \quad E_t = m_tc^2.$$

The invariant mass W

$$W^2c^4 = (E_L + m_tc^2)^2 - \boldsymbol{p}_L^2c^2 = m_b^2c^4 + m_t^2c^4 + 2m_tc^2E_L.$$

Then we have

$$\boxed{E_{CM} = (m_b^2c^4 + m_t^2c^4 + 2m_tc^2E_L)^{1/2}.}$$

A reaction can only occur if

$$\boxed{E_{CM} \geq \sum m_ic^2,}$$

where i's are the final state particles. This corresponds to all the final state particles are at rest in the center-of-mass frame. Then in the laboratory frame, the threshold energy $E_{\min}$ could be solved from the above inequality. And we have $E_L \geq E_{\min}$.

A.2.2 Masses of Unstable Particles

Consider a particle A decays to N particles in the final state:

$$A \to 1 + 2 + \cdots + N,$$

then the invariant mass of the final state particles is given by

$$W^2c^4 = \left(\sum_i E_i\right)^2 - \left(\sum_i \boldsymbol{p}_ic\right)^2 = E_A^2 - (\boldsymbol{p}_Ac)^2 = m_A^2c^4.$$

The mass of the decaying particle is equal to the invariant mass of its decay products. This could be used to determine the decaying particle if it's short-lived.

A.3 Transformation of the Scattering Angle

Now we consider the transformation of scattering angles between the laboratory frame (measured in fixed-target experiments) and the center-of-mass frame (used in theoretical discussions). Consider a reaction of the form

$$b(E_L, \boldsymbol{p}_L) + t(m_t c^2, \boldsymbol{0}) \to P(E, \boldsymbol{q}) + \cdots ,$$

where $E = (m_P^2 c^4 + \boldsymbol{q}^2 c^2)^{1/2}$. Without loss of generality, we could choose axes such that

$$\boldsymbol{p}_L = (p_L, 0, 0), \quad \boldsymbol{q} = (q\cos\theta_L, q\sin\theta_L, 0),$$

which means that the beam is in the x-direction while $\boldsymbol{q}$ is in the xy-plane. The angle θ_L is called the **production angle** or **scattering angle**. Now we want to find it's value in the center-of-mass frame.

For any Lorentz transformation along the beam direction, the beam and energy momenta become

$$p_b' = \gamma(v)(p_L - vE_L/c^2), \quad p_t' = -m_t v\gamma(v).$$

And in the center-of-mass frame, we have

$$p_b' + p_t' = 0.$$

Thus we could solve the velocity v of the Lorentz transformation

$$\boxed{v = \frac{p_L c^2}{E_L + m_t c^2}.}$$

Then we want to work out $\boldsymbol{q}'$ of the produced particle in the center-of-mass frame. We have

$$q_x' = \gamma(v)(q\cos\theta_L - vE/c^2), \quad q_y' = q\sin\theta_L, \quad q_z' = 0.$$

Writing

$$\boldsymbol{q}' = (q'\cos\theta_C, q'\sin\theta_C, 0),$$

where $q' = |\boldsymbol{q}'|$ and θ_C is the angle between $\boldsymbol{q}'$ and the x-axis. Thus

$$\boxed{\tan\theta_C = \frac{q_y'}{q_x'} = \frac{1}{\gamma(v)}\frac{q\sin\theta_L}{q\cos\theta_L - vE/c^2}.}$$

The dominance of small laboratory scattering angles θ_L at high energies follows the inverse of the above equation:

$$\boxed{\tan\theta_L = \frac{1}{\gamma(v)}\frac{q'\sin\theta_C}{q'\cos\theta_C + vE'/c^2},}$$

where $E' = (m_P^2 c^4 + q'^2 c^2)^{1/2}$ is the energy of the produced particle in the center-of-mass frame. At high energies, $E_L \approx p_L c \gg m_b c^2, m_t c^2$. Thus $v \approx c(1 - m_t c/p_L) \approx c$ and $\gamma(v) = (1 - v^2/c^2)^{-1/2} \approx (p_L/2m_t c)^{1/2}$. Also we have

$$E' = m_P c^2 \gamma(u), \quad q' = m_P u\gamma(u),$$

where u is the magnitude of the particle's velocity in the center-of-mass frame, we have

$$\boxed{\tan\theta_L \approx \sqrt{\frac{2m_t c}{p_L}}\frac{u\sin\theta_C}{u\cos\theta_C + c}}$$

Appendix B

Amplitudes and Cross-Sections

B.1 Rates and Cross-Sections

In a scattering experiment, a beam of ideally monoenergetic particles is directed on to a target and the rates of production of various particles are measured. This could be considered as the sums of contributions resulting from the interactions of individual beam particles with individual target particles provided the following conditions:

(a) The target particles are separated by distances that are much greater than the de Brogile wavelength of the incident particles, so that interference effects between waves scattered from different target particles can be neglected/

(b) The target is small enough (or of sufficiently low density), for multiple scattering to be neglected, so that any particles produced in the interaction will almost certainly leave the target without further interactions.

(c) The collision energy is sufficiently high that the binding energies of the particles in the target can be neglected.

(d) The beam density is low enough that mutual interactions of its particles can be neglected.

Since these conditions are usually satisfied in particle physics experiments, we can focus on interactions between individual beams and target particles.

For each initial state, a number of final states are possible. Whatever reaction we choose, the rate is proportional to (a) the number N of particles in the target illuminated by the bream; (b) the rate per unit area at which beam particles cross a small surface placed in the beam at rest with respect to the target and perpendicular to the beam direction. The latter rate is called the **flux**,

$$J = n_b v_i,$$

where n_b is the density of particles in the beam and v_i is their velocity in the rest frame of the target. The rate W_r at which a specific reaction r occurs in a particular experiment can be written as

$$\boxed{W_r = JN\sigma_r,}$$

where σ_r is called the **cross-section** for reaction r. And

$$\boxed{L \equiv JN,}$$

is called the **luminosity**. The cross-section is characteristic of the particular reaction r and has the dimension of an area. Note that the cross-section has the same value in the laboratory and center-of-mass frames.

B.2 The Total Cross-Section

σ_r is called the **partial cross-section** for a particular reaction r. The **total cross-section** σ is defined by

$$\boxed{\sigma \equiv \sum_r \sigma_r.}$$

The sum extends over all reactions r that satisfy the conservation laws. The total reaction rate is

$$W \equiv \sum_r W_r = JN\sigma = L\sigma.$$

Consider a beam of cross-sectional area A traversing a stationary target of length d. The

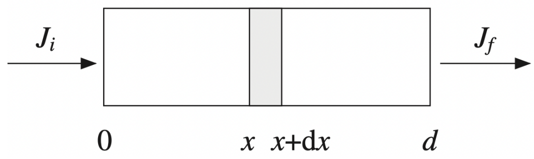

Figure B.1: Beam of particles with initial flux J_i being transmitted through a target of length d with reduced flux $J_f < J_i$.

reduction in the beam intensity on crossing the segment dx is equal to the reaction rate dW within the segment,

$$AdJ(x) = -dW.$$

We also have

$$dW = J(x)\sigma n_t A dx,$$

where n_t is the density of particles in the target. Thus

$$dJ(x) = -n_t \sigma J(x) dx,$$

and if $J(0) = J_i$, we can solve that

$$\boxed{J(x) = J_i \exp(-x/l_c), \quad 0 \leq x \leq d,}$$

where the **collision length**

$$l_c \equiv \frac{1}{n_t \sigma},$$

is the mean free path travelled by a beam particle in an infinitely long target ($d = \infty$) before it interacts. Beyond the target, the transmitted flux is

$$\boxed{J_f \equiv J(x \geq d) = J_i \exp(-d/l_c).}$$

The results shows that: (1) multiple scattering within the target can only be neglected if $d \ll l_c$; (2) the measured depletion of a beam flux on passing through a long target gives a relatively simple determination of the total cross-section.

B.3 Differential Cross-Sections

Now we consider the angular distributions of particles produced in a scattering reaction. For reactions with two-body final states, only the direction of one particle need to be specified, since the direction of the other follows the conservation of energy and momentum. In a convenient reference frame (we let z-axis coincides with the beam direction), the direction of the chosen particle is specified by a polar angle θ and an azimuthal angle ϕ. θ is called the **scattering angle** in an elastic reaction or the **production angle** in an inelastic reaction.

The angular distribution of the chosen particle produced in a two-body reaction r is described by the **differential cross-section**

$$\frac{d\sigma_r(\theta,\phi)}{d\Omega}$$

defined by

$$\boxed{dW_r \equiv JN\frac{d\sigma_r(\theta,\phi)}{d\Omega}d\Omega,}$$

where dW_r is the measured rate for the produced particle to be emitted into an element of solid angle $d\Omega = d\cos\theta d\phi$ in the direction (θ,ϕ). The cross-section can be obtained by

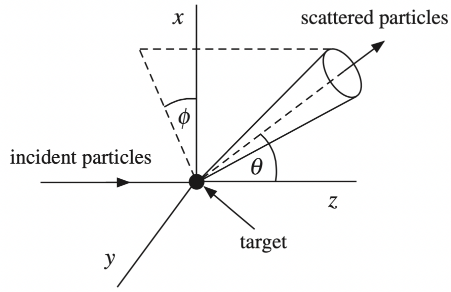

Figure B.2: Geometry of the differential cross-section

integrating the differential cross-section over all solid angles

$$\boxed{\sigma_r = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \frac{d\sigma_r(\theta,\phi)}{d\Omega}.}$$

In practice, most experiments are done using unpolarized beams and targets, the cross-sections in such experiments are called **unpolarized cross-sections**. There is complete cylindrical symmetry about the beam direction. Thus the unpolarized cross-sections are independent of ϕ. Define the unpolarized differential cross-section

$$\boxed{\frac{d\sigma_r}{d\cos\theta} \equiv \int_0^{2\pi} d\phi \frac{d\sigma_r}{d\Omega} = 2\pi\frac{d\sigma_r}{d\Omega},}$$

with

$$\boxed{\sigma_r = \int_{-1}^{1} d\cos\theta \frac{d\sigma_r}{d\cos\theta}.}$$

B.4 The Scattering Amplitude

Consider a single beam particle interacting with a single target particle and confine the system in a large cube defined by $-L/2 \le x, y, z \le L/2$ so that $V = L^2$. The incident flux is then

$$J = n_b v_i = v_i/V.$$

And since $N = 1$, we have

$$dW_r = \frac{v_i}{V}\frac{d\sigma_r(\theta,\phi)}{d\Omega}d\Omega.$$

Impose the periodic boundary conditions

$$\begin{aligned}\psi(-L/2, y, z) &= \psi(L/2, y, z),\\ \psi(x, -L/2, z) &= \psi(x, L/2, z),\\ \psi(x, y, -L/2) &= \psi(x, y, L/2),\end{aligned}$$

and take the limit $V = L^3 \to \infty$ at the end of calculation. The results are independent of V and also of the detailed form of the boundary conditions chosen.

Consider a simple case: the scattering of a nonrelativistic particle by a potential $V(\boldsymbol{r})$, using lowest-order perturbation theory. Suppose there are transitions from an initial state

$$\psi_i = \frac{1}{\sqrt{V}} \exp(i\boldsymbol{q}_i \cdot \boldsymbol{r})$$

to final states

$$\psi_f = \frac{1}{\sqrt{V}} \exp(i\boldsymbol{q}_f \cdot \boldsymbol{r}),$$

where the final momentum $\boldsymbol{q}_f$ lies within a small solid angle $d\Omega$ located in the direction (θ, ϕ). The transition rate for any process is given in first-order perturbation theory by the Born approximation

$$dW_r = 2\pi \left| \int d^3\boldsymbol{r} \psi_f^* V(r) \psi_i \right|^2 \rho(E_f),$$

where $\rho(E_f)$ is the **density of states factor**. Substituting ψ_i and ψ_f we have

$$dW_r = \frac{2\pi}{V} \rho(E_f) |\mathcal{M}_{if}|^2,$$

where the **scattering amplitude** $\mathcal{M}_{if}$ is

$$\boxed{\mathcal{M}_{if} = \int d^3\boldsymbol{r} V(\boldsymbol{r}) \exp[i(\boldsymbol{q}_i - \boldsymbol{q}_f) \cdot \boldsymbol{r}].}$$

$\rho(E)$ is defined by setting $\rho(E)dE$ equal to the number of possible quantum states of final state particles that have a total energy between E and $E + dE$. To find $\rho(E)$, first evaluate $\rho(q)$, where $\rho(q)$ is the number of possible final states with $q = |\boldsymbol{q}|$ lying between q and $q + dq$, and then

$$\rho(q) \frac{dq}{dE} dE = \rho(E) dE.$$

The possible values of the momentum $\boldsymbol{q}$ are determined by the boundary conditions:

$$q_x = \frac{2\pi}{L} n_x, \quad q_y = \frac{2\pi}{L} n_y, \quad q_z = \frac{2\pi}{L} n_z,$$

where n_x, n_y and n_z are integers. Then the number of final states with momenta lying in the momentum space volume

$$d^3\boldsymbol{q} = q^2 dq\, d\Omega$$

is given by

$$\rho(q) dq = \left(\frac{L}{2\pi}\right)^3 d^3\boldsymbol{q} = \frac{V}{8\pi^3} q^2 dq\, d\Omega.$$

The derivative

$$\frac{dq}{dE} = \frac{1}{v},$$

and thus

$$\boxed{\rho(E_f) = \frac{V}{8\pi^3} \frac{q_f^2}{v_f} d\Omega.}$$

For elastic scattering, $v_i = v_f = v$ and $q_i = q_f = mv$, we have

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2}|\mathcal{M}_{if}|^2.$$

The result is for nonrelativistic elastic scattering in lowest-order perturbation theory.

For reaction of the form

$$a(\boldsymbol{q}_a, m_a) + b(\boldsymbol{q}_b, m_b) \to c(\boldsymbol{q}_c, m_c) + d(\boldsymbol{q}_d, m_d),$$

where m specifies the spin states of the particles and $\boldsymbol{q}$ are the momentum in the center-of-mass frame. We can show that

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\frac{q_f^2}{v_i v_f}|\mathcal{M}_{if}|^2.}$$

The result applies to the scattering of polarized particles, and the amplitude depends on the particular spin states involved. In an unpolarized scattering experiment,

$$\boxed{\frac{d\sigma}{d\cos\theta} = \frac{(2S_c+1)(2S_d+1)}{2\pi}\frac{q_f^2}{v_i v_f}\langle|\mathcal{M}_{if}|^2\rangle.}$$

B.5 The Breit-Wigner Formula

An unstable particle could be characterized by its lifetime at rest τ, or its **natural decay width** $\Gamma = \hbar/\tau$. In natural units, the decay width is equal to the decay rate

$$\boxed{\Gamma = \frac{1}{\tau}.}$$

Generally the particle may decay to many different channels f, and the total decay rate is the sum of partial decay rates for various channels

$$\Gamma = \sum_f \Gamma_f.$$

The **branching ratio**, the fraction of decays leading to a particular channel f is given by

$$B_f \equiv \frac{\Gamma_f}{\Gamma}.$$

The lifetimes and branching ratios for "long-lived" particles with $\tau \geq 10^{-16}s$ can be measured directly. For some particles having lifetimes much shorter, the decay widths are derived from the observation of peaks in cross-sections and mass distribution.

B.5.1 Decay Distributions

Denote the wavefunction of the unstable state by ψ_0 and the wavefunctions of possible final states by $\psi_n (n \geq 1)$. We could always choose the wavefunction so that they are orthonormal, i.e. $\langle\psi_m|\psi_n\rangle = \delta_{mn}$. The wavefunction ψ_0 describes the system at $t = 0$, then we have

$$\Psi(\boldsymbol{r}, t) = \sum_{n=0}^{\infty} a_n(t)e^{-iE_n t}\psi_n(\boldsymbol{r}), \quad a_0(0) = 1, \quad a_n(0) = 0 (n \geq 1).$$

And

$$E_n \equiv \langle\psi_n|H|\psi_n\rangle.$$

The time dependence of the coefficients $a_n(t)$ is determined by the Schrödinger equation. Multiplying the equation by ψ_n^* (using the orthonormality property), we have

$$\boxed{i\frac{da_n}{dt} = \sum_{m\neq n} \langle\psi_n|H|\psi_m\rangle e^{-i(E_m-E_n)t} a_m.}$$

We assume H_{nm} to be small when $n \neq m$. Since a_m is of order H_{0m} for $m \neq 0$, we have

$$i\frac{da_n}{dt} = \langle\psi_n|H|\psi_0\rangle e^{-i(E_0-E_n)t} a_0.$$

Specialize to the rest frame of the decaying particle so that $E_0 = M$, where M is the rest mass. Assume $a_0(t) = e^{-\Gamma t/2}$, which is consistent with the decay law $|a_0(t)|^2 = e^{-\Gamma t} = e^{-t/\tau}$. And then we have

$$ia_n = -i\langle\psi_n|H|\psi_0\rangle \left\{\frac{\exp\{-i[(M-E_n) - i\Gamma/2]t\} - 1}{(E_n - M) + i\Gamma/2}\right\}.$$

For $t \gg 1/\Gamma$, the exponential term tends to be zero, and the probability of finding a final state n is given by

$$\boxed{P_n = |a_n(\infty)|^2 = \frac{\langle\psi_n|H|\psi_0\rangle^2}{(E_n - M)^2 + \Gamma^2/4} = \frac{2\pi}{\Gamma}\langle\psi_n|H|\psi_0\rangle^2 P(E_n - M),}$$

where the distribution function

$$P(E-M) = \frac{\Gamma/2\pi}{(E-M)^2 + \Gamma^2/4}, \quad \int_{-\infty}^{\infty} P(E-M)dE = 1.$$

P_n is the probability for decay to a specific quantum state n. Generally, a particle may decay to several channels f and we must sum over all the quantum states n contained in that channel. Ignoring the spin, the probability $P_f(E)dE$ for decay to a particular channel f with energy in the interval E to $E + dE$ is

$$P_f(E)dE = \frac{2\pi}{\Gamma}\langle\psi_f|H|\psi_0\rangle^2 P(E_n - M)\rho_f(E)dE,$$

where $\rho_f(E)$ is the density of states. If Γ is small, $\rho_f(E)$ peaks sharply at $E = M$. And thus by approximation, we have

$$P_f(E)dE = \frac{2\pi}{\Gamma}\langle\psi_f|H|\psi_0\rangle^2 P(E_n - M)\rho_f(M)dE.$$

By integration we have

$$\Gamma_f = 2\pi|\langle\psi_f|H|\psi_0\rangle|^2\rho_f(M),$$

and hence

$$\boxed{P_f(E) = \frac{1}{2\pi}\frac{\Gamma_f}{(E-M)^2 + \Gamma^2/4}.}$$

This is the **Breit-Wigner formula for the decay distribution of a spin-0 particle**. The formula holds in the rest frame of the decaying particle. Since it's identical to the center-of-mass frame of the decay products, the invariant mass $W = E$, and

$$\boxed{P_f(W) = \frac{1}{2\pi}\frac{\Gamma_f}{(W-M)^2 + \Gamma^2/4}.}$$

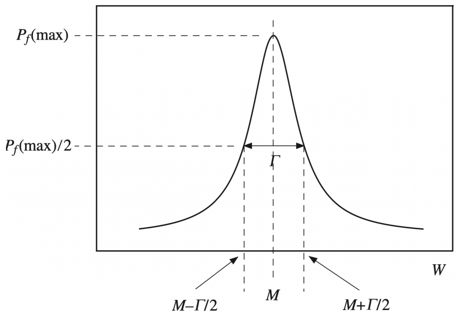

Figure B.3: The Breit-Wigner formula

B.5.2 Resonant Cross-Sections

Now consider an isolated unstable particle (or resonance) is formed in a scattering process and subsequently decays into a final channel f. We need to solve the Schrödinger equation with boundary conditions

$$a_1(0) = 1, \quad a_n(0) = 0 (n \neq 1),$$

where $n = 1$ refers to the initial state and $n = 0$ to the resonant state. It can be solved that

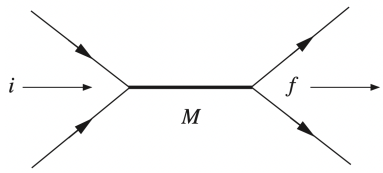

Figure B.4: Formation and decay of an unstable particle of mass M in a two-body scattering process

the rate

$$W_f = \frac{|\langle\psi_0|H|\psi_1\rangle|^2\Gamma_f}{(E - E_0)^2 + \Gamma^2/4}.$$

And the cross-section

$$\boxed{\sigma_{if} = \frac{\pi}{q_i^2}\frac{\Gamma_i\Gamma_f}{(E - E_0)^2 + \Gamma^2/4}.}$$

If the initial particles have spins S_1 and S_2 and the unstable particle has spin j, we have

$$\boxed{\sigma_{if} = \frac{\pi}{q_i^2}\frac{2j + 1}{(2S_1 + 1)(2S_2 + 1)}\frac{\Gamma_i\Gamma_f}{(E - E_0)^2 + \Gamma^2/4}.}$$

Appendix C

The Isospin Formalism

Appendix D

Gauge Theories