Mathematical Methods for Physics and Engineering

First-Order Differential Equations

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The *order* of an ODE is the order of the highest derivative it contains. The *degree* of an ODE is the power to which the highest-order derivative is raised, after the equation has been rationalised to contain only integer powers of derivatives. The *general solution* to an ODE is the most general function $y(x)$ that satisfies the equation; it will contain *constants of integration* which may be determined by the application of some suitable *boundary conditions*.

1 First-Degree First-Order Equations

First-degree first-order ODEs can be written as

$$\frac{dy}{dx} = F(x,y), \quad A(x,y)dx + B(x,y)dy = 0,$$

where $F(x,y) = -A(x,y)/B(x,y)$.

1.1 Separable-Variable Equations

A separable-variable equation is one which may be written as

$$\frac{dy}{dx} = f(x)g(y).$$

Rearranging this equation we obatin

$$\int \frac{dy}{g(y)} = \int f(x)dx.$$

Solve

$$\frac{dy}{dx} = x + xy.$$

The RHS could be factorised as $x(1+y)$, then

$$\int \frac{dy}{1+y} = \int x dx.$$

Integrating both sides, we have

$$\ln(1+y) = \frac{x^2}{2} + c,$$

and hence

$$1 + y = \exp\left(\frac{x^2}{2} + c\right) = A\exp\left(\frac{x^2}{2}\right),$$

where c and A are arbitrary constants. □

1.2 Exact Equations

An *exact* first-degree first-order ODE is one of the form

$$A(x,y)dx + B(x,y)dy = 0, \quad \frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}.$$

In this case $A(x,y)dx + B(x,y)dy$ is an exact differential $dU(x,y)$, say

$$Adx + Bdy = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy.$$

From this we can obtain

$$A(x,y) = \frac{\partial U}{\partial x}, \quad B(x,y) = \frac{\partial U}{\partial y}.$$

Since $\partial^2 U/\partial x \partial y = \partial^2 U/\partial y \partial x$, we require

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}.$$

Thus if this holds we can write $dU(x,y) = 0$, which has the solution $U(x,y) = c$, where c is a constant and

$$U(x,y) = \int A(x,y)dx + F(y).$$

The function $F(y)$ can be found by differentiating with respect to y and comparing with $B(x,y)$.

Solve

$$x\frac{dy}{dx} + 3x + y = 0.$$

Rearranging this equation we have

$$(3x+y)dx + xdy = 0,$$

i.e. $A = 3x + y$ and $B = x$. Since $\partial A/\partial y = \partial B/\partial x = 1$, the equation is exact and hence

$$U(x,y) = \int (3x+y)dx + F(y) = c_1 \quad \Rightarrow \quad U(x,y) = \frac{3x^2}{2} + xy + F(y) = c_1.$$

Since $\partial U/\partial y = x + dF/dy$ and $B(x,y) = x$, thus $dF/dy = 0$, which gives that $F(y) = c_2$. Then the solution is

$$\frac{3x^2}{2} + x = c. \square$$

1.3 Inexact Equations: Integrating Factors

An inexact equation can be written in the form

$$A(x,y)dx + B(x,y)dy = 0, \quad \frac{\partial A}{\partial x} \neq \frac{\partial B}{\partial y}.$$

However, the differential $Adx + Bdy$ can be made exact by multiplying an *integrating factor* $\mu(x,y)$, which obeys

$$\frac{\partial(\mu A)}{\partial x} = \frac{\partial(\mu B)}{\partial y}.$$

There are no general method to find $\mu(x,y)$. However, if an integrating factor is only a function of x or y then we can solve the above equation to find it. For example, say $\mu = \mu(x)$. Then the above equation reads

$$\mu\frac{\partial A}{\partial y} = \mu\frac{\partial B}{\partial x} + B\frac{d\mu}{dx}.$$

Rearranging this expression we have

$$\frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)dx = f(x)dx,$$

where we require $f(x)$ also be a function of x only. Then the integrating factor is given by

$$\mu(x) = \exp\left\{\int f(x)dx\right\}, \quad f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right).$$

Similarly, if $\mu = \mu(y)$, then

$$\mu(y) = \exp\left\{\int g(y)dy\right\}, \quad g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right).$$

Solve

$$\frac{dy}{dx} = -\frac{2}{y} - \frac{3y}{2x}.$$

Rearranging the equation we have

$$(4x + 3y^2)dx + 2xy\,dy = 0,$$

i.e. $A = 4x + 3y^2$ and $B = 2xy$. Now

$$\frac{\partial A}{\partial y} = 6y, \quad \frac{\partial B}{\partial x} = 2y,$$

so the ODE is not exact. However, we have

$$\frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = \frac{2}{x}.$$

Thus the integrating factor is given by

$$\mu(x) = \left\{2\int \frac{dx}{x}\right\} = \exp(2\ln x) = x^2.$$

Multiplying the original equation by $\mu(x) = x^2$ we have

$$(4x^3 + 3x^2y^2)dx + 2x^3ydy = 0.$$

This gives the solution

$$x^4 + y^2x^3 = c,$$

where c is a constant. □

1.4 Linear Equations

Linear first-order ODEs are a special case of inexact ODEs and can be written as

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Such equations can be made exact by multiplying through an appropriate integrating factor. In this case, the integrating factor is always a function of x alone, say

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \frac{d}{dx}[\mu(x)y] = \mu(x)Q(x).$$

We integrate this equation and then $\mu(x)y = \int \mu(x)Q(x)dx$. The required integrating factor $\mu(x)$ is determined by

$$\frac{d}{dx}(\mu y) = \mu\frac{dy}{dx} + \frac{d\mu}{y} = \mu\frac{dy}{dx} + \mu Py,$$

which gives that

$$\frac{d\mu}{dx} = \mu(x)P(x) \quad \Rightarrow \quad \mu(x) = \exp\left\{\int P(x)dx\right\}.$$

Solve

$$\frac{dy}{dx} + 2xy = 4x.$$

The integrating factor is given by

$$\mu(x) = \exp\left\{\int 2x dx\right\} = \exp x^2.$$

Multiplying through the ODE by $\mu(x) = \exp x^2$ we have

$$y \exp x^2 = 4 \int x \exp x^2 \, dx = 2 \exp x^2 + c.$$

Then the solution is given by $y = 2 + c \exp(-x^2)$. $\square$

1.5 Homogeneous Equations

Homogeneous equation are ODEs that can be written in the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right),$$

where $A(x,y)$ and $B(x,y)$ are homogeneous functions of the same degree. A function $f(x,y)$ is homogeneous of degree n if for any λ it obeys

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

For functions of the form of A and B, we require the sum of the powers in each term of A and B to be the same. The RHS of a homogeneous ODE can be written as a function of y/x. Then the equation can be solved by making the substitution $y = vx$, so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = F(v).$$

This is a separable equation and can be integrated to give

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}.$$

Solve

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right).$$

Substituting $y = vx$ we obtain

$$v + x\frac{dv}{dx} = v + \tan v.$$

Then

$$\int \cot v dv = \int \frac{dx}{x} = \ln x + c_1.$$

But

$$\int \cot v dv = \int \frac{\cos v}{\sin v} dv = \ln(\sin v) + c_2.$$

So the solution is

$$y = x \sin^{-1} Ax,$$

where A is a constant. □

1.6 Isobaric Equations

An isobaric ODE is a generalisation of the homogeneous ODE and of the form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)},$$

where the equation is dimensionally consistent if the substitution $y = vx^m$ makes it separable.

Solve

$$\frac{dy}{dx} = -\frac{1}{2xy} + \left(y^2 + \frac{2}{x}\right).$$

Rearranging we have

$$\left(y^2 + \frac{2}{x}\right) dx + 2xy\, dy = 0.$$

Giving y and dy the weight m, and x and dx the weight 1, the sum of the powers in each term on the LHS are $2m+1$, 0 and $2m+1$ respectively. They are equal if $2m+1=0$. Then we substitute $y = vx^{-1/2}$, $dy = x^{-1/2}dv - \frac{1}{2}vx^{-3/2}dx$, and then

$$vdv + \frac{dx}{x} = 0.$$

The result is

$$\frac{1}{2}v^2 + \ln x = c.$$

Replacing v by $y\sqrt{x}$ we have $\frac{1}{2}y^2x + \ln x = c$. □

1.7 Bernoulli's Equation

Bernoulli's equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 0 \text{ or } 1.$$

The equation can be made linear by substituting $v = y^{1-n}$ and then

$$\frac{dy}{dx} = \left(\frac{y^n}{1-n}\right)\frac{dv}{dx}.$$

Hence

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x),$$

which is a linear equation.

> *Solve*
> $$\frac{dy}{dx} + \frac{y}{x} = 2x^3y^4.$$

Let $v = y^{1-4} = y^{-3}$, then

$$\frac{dy}{dx} = -\frac{y^4}{3}\frac{dv}{dx}.$$

Substituting this into the ODE, we have

$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3.$$

The integrating factor is

$$\exp\left\{-3\int\frac{dx}{x}\right\} = \exp(-3\ln x) = \frac{1}{x^3}.$$

This yields the solution

$$\frac{v}{x^3} = -6x + c.$$

Since $v = y^{-3}$, we have $y^{-3} = -6x^4 + cx^3$. □

1.8 Miscellaneous Equations

Firstly we consider

$$\frac{dy}{dx} = F(ax + by + c),$$

where a, b and c are constants. This equation can be solved by the substitution $v = ax + by + c$, then

$$\frac{dv}{dx} = a + b\frac{dy}{dx} = a + bF(v).$$

> *Solve*
> $$\frac{dy}{dx} = (x + y + 1)^2.$$

Substituting $v = x + y + 1$, then

$$\frac{dv}{dx} = 1 + v^2,$$

which gives that

$$\int\frac{dv}{v^2+1} = \int dx \quad \Rightarrow \quad \tan^{-1} v + x + c.$$

Thus the solution is $\tan^{-1}(x+y+1) = x + c$. □

Secondly we consider

$$\frac{dy}{dx} = \frac{ax+by+c}{ex+fy+g},$$

where a, b, c, e, f, and g are constants. This can be solved by letting $x = X + \alpha$ and $y = Y + \beta$ where α and β are constants found from

$$a\alpha + b\beta + c = 0, \quad e\alpha + f\beta + g = 0.$$

Then

$$\frac{dY}{dX} = \frac{aX+bY}{eX+fY},$$

which is homogeneous.

Solve

$$\frac{dy}{dx} = \frac{2x-5y+3}{2x+4y-6}.$$

Let $x = X + \alpha$ and $y = Y + \beta$, where α and β obeys

$$2\alpha - 5\beta + 3 = 0, \quad 2\alpha + 4\beta = 0.$$

This gives that $\alpha = \beta = 1$. Hence

$$\frac{dY}{dX} = \frac{2X-5Y}{2X+4Y}.$$

Substituting $Y = vX$, we have

$$\frac{dv}{dX} = \frac{2-7v-4v^2}{X(2+4v)}.$$

This is separable, and

$$\int \frac{2+4v}{2-7v-4v^2} dv = -\frac{4}{3}\int \frac{dv}{4v-1} - \frac{2}{3}\int \frac{dv}{v+2} = \int \frac{dX}{X},$$

which gives that

$$\ln X + \frac{1}{3}\ln(4v-1) + \frac{2}{3}\ln(v+2) = c_1,$$

or

$$X^3(4v-1)(v+2)^2 = \exp 3c_1.$$

Thus the solution of the original ODE is $(4y-x-3)(y+2x-3)^2 = c_2$. □

2 Higher-Degree First-Order Equations

Higher-degree first-order equations can be written as $F(x, y, dy/dx) = 0$. The most general standard form is

$$p^n + a_{n-1}(x,y)p^{n-1} + \cdots a_1(x,y)p + a_0(x,y) = 0.$$

where $p = dy/dx$.

2.1 Equations Soluble for p

Sometimes the LHS of the above equation can be factorised into the form

$$(p - F_1)(p - F_2) \cdots (p - F_n) = 0.$$

Then we are left with solving the n first-degree equations $p = F_i(x, y)$. Writing the solution as $G_i(x, y) = 0$, the general solution is given by

$$G_1(x, y)\, G_2(x, y) \cdots G_n(x, y) = 0.$$

Solve

$$(x^3 + x^2 + x + 1)p^2 - (3x^2 + 2x + 1)yp + 2xy^2 = 0.$$

The equation can be factorised as

$$[(x + 1)p - y][(x^2 + 1)p - 2xy] = 0.$$

Thus

$$(x + 1)\frac{dy}{dx} - y = 0, \quad (x^2 + 1)\frac{dy}{dx} - 2xy = 0.$$

Then the general solution is

$$[y - c(x + 1)][y - c(x^2 + 1)] = 0. \square$$

2.2 Equations Soluble for x

Equations that can be solved for x is written in the form

$$x = F(y, p),$$

can be reduced to first-degree first-order equations in p by differentiating both sides with respect to y, so that

$$\frac{dx}{dy} = \frac{1}{p} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial p}\frac{dp}{dy}.$$

The results are in an equation of the form $G(y, p) = 0$, which can be used to eliminate p and give the general solution.

Solve

$$6y^2p^2 + 3xp - y = 0.$$

Rearranging we have

$$3x = \frac{y}{p} - 6y^2p.$$

Differentiating both sides with respect to y we have

$$3\frac{dx}{dy} = \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 6y^2\frac{dp}{dy} - 12yp,$$

which factorises to give

$$(1 + 6yp^2)\left(2p + y\frac{dp}{dy}\right) = 0.$$

For the second factor $py^2 = c$. Substituting p we have

$$y^3 = 3cx + 6c^2.$$

For the first factor we have $6p^2y = -1$, then

$$8y^3 + 3x^2 = 0,$$

which is a singular solution. □

2.3 Equations Soluble for y

Equations that can be solved for y can be written in the form

$$y = F(x, p),$$

which can be reduced to first-degree first-order equations in p by differentiating both sides with respect to x:

$$\frac{dy}{dx} = p = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{dp}{dx}.$$

This results in an equation of the form $G(x, p) = 0$, which can be used to eliminate p to give the general solution.

Solve

$$xp^2 + 2xp - y = 0.$$

This equation can be written as $y = xp^2 + 2xp$. Differentiating both sides with respect to x, we have

$$\frac{dy}{dx} = p = 2xp\frac{dp}{dx} + p^2 + 2x\frac{dp}{dx} + 2p.$$

This could be factorised as

$$(p + 1)\left(p + 2x\frac{dp}{dx}\right) = 0.$$

Consider the second term as 0. Then the solution is

$$xp^2 = c.$$

Thus the general solution is $(y - c)^2 = 4cx$. If we set the other factor equal to zero, then $p = -1$. This gives that

$$x + y = 0,$$

which is a singular solution. □

2.4 Clairaut's Equation

Consider the Clairaut's equation, which has the form

$$y = px + F(p),$$

which is a special case of equations soluble for y. Differentiating both sides with respect to x we obtain

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{dF}{dp}\frac{dp}{dx} \quad \Rightarrow \quad \frac{dp}{dx}\left(\frac{dF}{dp} + x\right) = 0.$$

Consider the first term we have

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad y = c_1x + c_2.$$

Since $p = dy/dx = c_1$, then $c_1x + c_2 = c_1x + F(c_1)$. Thus the general solution is

$$y = c_1x + F(c_1).$$

For the second factor, we have

$$\frac{dF}{dp} + x = 0,$$

which has the form $G(x, p) = 0$. This relation can be used to give a singular solution.

Solve

$$y = px + p^2.$$

The general solution is $y = cx + c^2$. And we also have $2p + x = 0 \Rightarrow p = -x/2$. Substituting this into the original equation we have $x^2 + 4y = 0$. □