Fowles & Cassiday's

Analytical Mechanics

NOTES

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Contents

Chapter 5

Noninertial Reference Systems

5.1 Accelerated Coordinates Systems and Inertial Forces

Now suppose $Oxyz$ are the primary coordinates axes and $O'x'y'z'$ are the moving axes. In the case of pure translation, the respective axes are parallel.

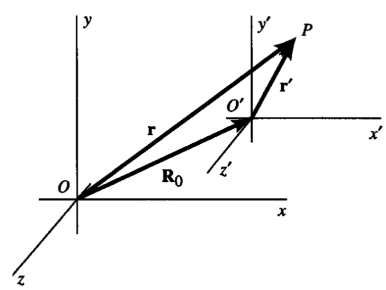

Figure 5.1: Two coordinates systems undergoing pure translation

We have

$$\boldsymbol{r} = \boldsymbol{R}_0 + \boldsymbol{r}'.$$

Taking derivatives we have

$$\boldsymbol{v} = \boldsymbol{V}_0 + \boldsymbol{v}', \quad \boldsymbol{a} = \boldsymbol{A}_0 + \boldsymbol{a}'.$$

If the moving system is accelerating (i.e. $\boldsymbol{A}_0 \neq 0$), Newton's second law becomes

$$\boldsymbol{F} = m\boldsymbol{A}_0 + m\boldsymbol{a}' \Rightarrow \boldsymbol{F} - m\boldsymbol{A}_0 = m\boldsymbol{a}'.$$

We may write

$$\boldsymbol{F}' = m\boldsymbol{a}',$$

where $\boldsymbol{F}' = \boldsymbol{F}' + (-m\boldsymbol{A}_0)$. That is, an acceleration $\boldsymbol{A}_0$ of the reference system can be taken into account by adding an **inertial term** $-m\boldsymbol{A}_0$ to the force $\boldsymbol{F}$ and equating the result to the product of mass and acceleration of the moving system. Inertial terms in the equations are called **inertial forces**, or **fictitious forces**.

5.2 Rotating Coordinate Systems

Suppose a primed coordinate system rotating with respect to an unprimed, fixed, inertial one. And the two systems have a common origin. The **angular velocity** of the rotating system is

$$\boldsymbol{\omega} = \omega \boldsymbol{n}.$$

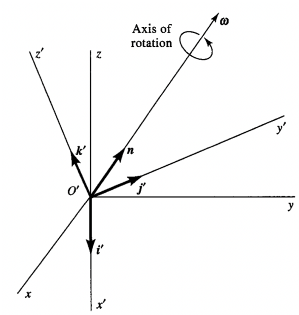

Figure 5.2: The angular velocity vector of a rotating coordinate system

Since the system have the same origin, we have

$$\boldsymbol{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z = \boldsymbol{r}' = \boldsymbol{i}'x' + \boldsymbol{j}'y' + \boldsymbol{k}'z'.$$

Note that the unit vectors in the rotating system are not constant, hence

$$\boldsymbol{i}\frac{dx}{dt} + \boldsymbol{j}\frac{dy}{dt} + \boldsymbol{k}\frac{dz}{dt} = \boldsymbol{i}'\frac{dx'}{dt} + \boldsymbol{j}'\frac{dy'}{dt} + \boldsymbol{k}'\frac{dz'}{dt} + x'\frac{d\boldsymbol{i}'}{dt} + y'\frac{d\boldsymbol{j}'}{dt} + z'\frac{d\boldsymbol{k}'}{dt}.$$

This can be written as

$$\boldsymbol{v} = \boldsymbol{v}' + x'\frac{d\boldsymbol{i}'}{dt} + y'\frac{d\boldsymbol{j}'}{dt} + z'\frac{d\boldsymbol{k}'}{dt}.$$

One may find that

$$\boxed{\frac{d\boldsymbol{i}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{i}', \quad \frac{d\boldsymbol{j}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{j}', \quad \frac{d\boldsymbol{k}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{k}'.}$$

Thus

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'.$$

More explicitly,

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{r}' = \left[\left(\frac{d}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\right]\boldsymbol{r}'.$$

This means that the operation of differentiating the position vector with respect to time in the fixed system is equivalent to the operation of taking the time derivative in the rotating system plus the operation $\boldsymbol{\omega}\times$. This apples to any vector $\boldsymbol{Q}$,

$$\boxed{\left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{Q}.}$$

If the vector is the velocity, we have

$$\boxed{\left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{v}.}$$

And $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}'$,

$$\begin{aligned}
\left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{fixed}} &= \left(\frac{d}{dt}\right)_{\text{rot}}(\boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}') + \boldsymbol{\omega}\times(\boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}') \\
&= \left(\frac{d\boldsymbol{v}'}{dt}\right)_{\text{rot}} + \left[\frac{d(\boldsymbol{\omega}\times\boldsymbol{r}')}{dt}\right]_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') \\
&= \left(\frac{d\boldsymbol{v}'}{dt}\right)_{\text{rot}} + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rot}}\times\boldsymbol{r}' + \boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}'}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}').
\end{aligned}$$

Consider $\boldsymbol{\omega}$, we have

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{\omega} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rot}} = \dot{\boldsymbol{\omega}}.$$

Since $\boldsymbol{v}' = (d\boldsymbol{r}'/dt)_{\text{rot}}, \boldsymbol{a}' = (d\boldsymbol{v}'/dt)_{\text{rot}}$, we have

$$\boldsymbol{a} = \boldsymbol{a}' + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}' + 2\boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$$

giving the acceleration in the fixed system in terms of position, velocity and acceleration in the rotating system.

In the general case, the primed system is undergoing both translation and rotation, then

$$\boxed{\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}' + \boldsymbol{V}_0,}$$

and

$$\boxed{\boldsymbol{a} = \boldsymbol{a}' + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}' + 2\boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') + \boldsymbol{A}_0.}$$

The term $2\boldsymbol{\omega}\times\boldsymbol{v}'$ is called **Coriolis acceleration** and the term $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$ is called the **centripetal acceleration**. The Coriolis acceleration appears whenever a particle moves in a rotating coordinate system and the centripetal acceleration is the result of particle being carried around a circular path in the rotating system. The centripetal acceleration is always directed toward the axis of rotation. The term $\dot{\boldsymbol{\omega}}\times\boldsymbol{r}'$ is called **transverse acceleration** since it is perpendicular to the position vector $\boldsymbol{r}'$.

5.3 Dynamics of a Particle in Rotating Coordinate System

The equation of motion of a particle in an inertial frame of reference is

$$\boldsymbol{F} = m\boldsymbol{a}.$$

Now we write the equation of motion in a noninertial frame of reference as

$$\boldsymbol{F} - m\boldsymbol{A}_0 - 2m\boldsymbol{\omega} \times \boldsymbol{v}' - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') = m\boldsymbol{a}'.$$

The equation represents the dynamical equation of motion in a noninertial frame of reference subjected to both real, physical forces and those inertial forces that appear as a result of the acceleration of the noninertial frame of reference. The **Coriolis force** is

$$\boxed{\boldsymbol{F}'_{Cor} = -2m\boldsymbol{\omega} \times \boldsymbol{v}'.}$$

The **transverse force** is

$$\boxed{\boldsymbol{F}'_{trans} = -m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}'.}$$

The **centrifugal force** is

$$\boxed{\boldsymbol{F}'_{centrif} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}').}$$

The remaining inertial force $-m\boldsymbol{A}_0$ appears whenever the (x', y', z') coordinate system is undergoing a translational acceleration.

A noninertial observer in the accelerated frame of reference writes the fundamental equation of motion as

$$\boldsymbol{F}' = m\boldsymbol{a}',$$

where

$$\boxed{\boldsymbol{F}' = \boldsymbol{F}_{physical} + \boldsymbol{F}'_{Cor} + \boldsymbol{F}'_{trans} + \boldsymbol{F}'_{centrif} - m\boldsymbol{A}_0.}$$

The Coriolis force is present only if a particle is moving in a rotating coordinate system. Its direction is perpendicular to the velocity vector of the particle in the moving system. In the northern hemisphere, the air would move toward the right. In the southern hemisphere the reverse is true.

The transverse force is present only if there is an angular acceleration of the rotating coordinate system. It is perpendicular to the radius vector $\boldsymbol{r}'$ in the rotating coordinate system.

The centrifugal force arises from the rotation about an axis. It is directed toward away from the axis of rotation and is perpendicular to the axis. In general if the angle between $\boldsymbol{\omega}$ and $\boldsymbol{r}'$ is θ, the magnitude of the centripetal force is $mr'\omega^2 \sin\theta$.

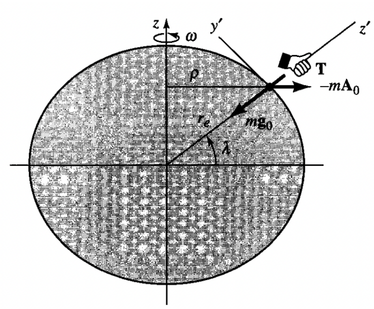

Figure 5.3: The plumb line

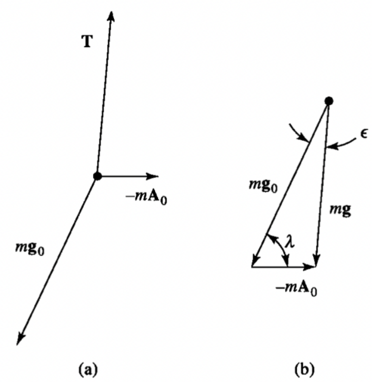

Figure 5.4: Force acting on the plumb bob

5.4 Effects of Earth's Rotation

Static Effects: The Plumb Line

Consider the frame of reference attached to the surface of the Earth, whose origin is at the position of the bob. It undergoes translation and rotation. The translation of the frame takes place along a circle with radius $\rho = r_e \cos\lambda$. The rate of rotation is $\boldsymbol{\omega}$, which is the same as the Earth. Note that $\boldsymbol{a}' = 0, \dot{\boldsymbol{\omega}} = 0, \boldsymbol{r}' = 0, \boldsymbol{v}' = 0$. Thus we have

$$\boldsymbol{F} - m\boldsymbol{A}_0 = 0.$$

Then we have

$$(\boldsymbol{T} + m\boldsymbol{g}_0) - m\boldsymbol{A}_0 = 0,$$

where $\boldsymbol{T}$ is the tension, $m\boldsymbol{g}_0$ is the real gravitational force. When we hang a plumb bob, we think that $\boldsymbol{T}$ balances out the local force of gravity $m\boldsymbol{g}$. Then

$$m\boldsymbol{g}_0 - m\boldsymbol{g} - m\boldsymbol{A}_0 = 0 \quad \Rightarrow \quad \boldsymbol{g}_0 - \boldsymbol{g} - \boldsymbol{A}_0 = 0.$$

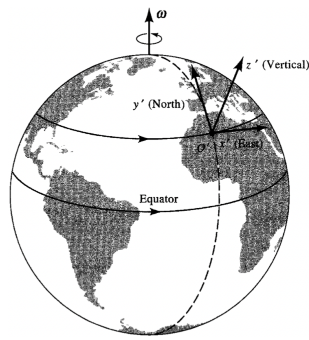

Figure 5.5: Coordinates $O'x'y'z'$ for the projectile motion: z'-axis is vertial in the direction of the plumb line

The local acceleration $\boldsymbol{g}$ due to gravity contains a term $\boldsymbol{A}_0$ due to the rotation of the earth. Also we have

$$\frac{\sin\epsilon}{m\omega^2 r_e\cos\lambda} = \frac{\sin\lambda}{mg},$$

since ϵ is small, hence

$$\sin\epsilon \approx \epsilon = \frac{\omega^2 r_e}{2g}\sin 2\lambda.$$

The maximum deviation of the direction of the plumb from the center of the Earth occurs at $\lambda = 45°$:

$$\epsilon_{\max} = \frac{\omega^2 r_e}{2g} \approx 1.7\times 10^{-3} \text{ radian} \approx 0.1°.$$

Dynamics Effects: Motion of a Projectile

The equation of motion for a projectile near the Earth's surface is

$$m\ddot{\boldsymbol{r}}' = \boldsymbol{F} + m\boldsymbol{g}_0 - m\boldsymbol{A}_0 - 2m\boldsymbol{\omega}\times\dot{\boldsymbol{r}}' - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}'),$$

where $\boldsymbol{F}$ is any applied force other than gravity. This can also be written as

$$m\ddot{\boldsymbol{r}}' = \boldsymbol{F} + m\boldsymbol{g} - 2m\boldsymbol{\omega}\times\dot{\boldsymbol{r}}' - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}').$$

Consider the motion of a projectile. Ignoring the air resistance, we have $\boldsymbol{F} = 0$. Also we may ignore $m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$ since it's very small. Then

$$m\ddot{\boldsymbol{r}}' = m\boldsymbol{g} - 2m\boldsymbol{\omega}\times\dot{\boldsymbol{r}}'.$$

We have $\boldsymbol{g} = -g\boldsymbol{k}'$. And

$$\omega_{x'} = 0, \quad \omega_{y'} = \omega\cos\lambda, \quad \omega_{z'} = \omega\sin\lambda.$$

The solutions can be found at Page 205 in the textbook.

Chapter 6

Gravitation and Central Forces

Chapter 7

Dynamics of Systems of Particles

7.1 Center of Mass and Linear Momentum of a System

A system is called **rigid body** if the relative positions of all the particles in it are fixed. Suppose the system consists of n particles of masses $m_1, m_2, \cdots, m_n$ whose position vectors are respectively $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_n$. Define the **center of mass** of the system as the point whose position vector $\boldsymbol{r}_{cm}$ is given by

$$\boldsymbol{r}_{cm} = \frac{\sum_i m_i \boldsymbol{r}_i}{m}.$$

Define the **linear momentum $\boldsymbol{p}$** of the system as

$$\boldsymbol{p} = \sum_i \boldsymbol{p}_i = \sum_i m_i \boldsymbol{v}_i.$$

Since $\dot{\boldsymbol{r}}_{cm} = \boldsymbol{v}_{cm}$, we have

$$\boldsymbol{p} = m\boldsymbol{v}_{cm}.$$

Now suppose there are external forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_n$ acting on the respective particles. And we denote the internal forces by $\boldsymbol{F}_{ij}$, meaning the force exerted on particle i by j. Note that $\boldsymbol{F}_{ii} = 0$. The equation of motion of the particle i is

$$\boldsymbol{F}_i + \sum_{j=1}^{n} \boldsymbol{F}_{ij} = m_i \ddot{\boldsymbol{r}}_i = \dot{\boldsymbol{p}}_i.$$

For n particles, we have

$$\sum_{i=1}^{n} \boldsymbol{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{F}_{ij} = \sum_{i=1}^{n} \dot{\boldsymbol{p}}_i.$$

For every $\boldsymbol{F}_{ij}$ there's a $\boldsymbol{F}_{ji}$ such that $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, from Newton's third law. Thus the double sum vanishes. And hence we can write:

$$\boxed{\sum_i \boldsymbol{F}_i = \dot{\boldsymbol{p}} = m\boldsymbol{a}_{cm}.}$$

When there's no external forces acting on a system, the linear momentum of the system remains constant:

$$\sum_i \boldsymbol{p}_i = \boldsymbol{p} = m\boldsymbol{v}_{cm} = \text{constant}.$$

This is the **principle of conservation of linear momentum**.

7.2 Angular Momentum and Kinetic Energy of a System

The **angular momentum $\boldsymbol{L}$** of a system of particles is defined as

$$\boldsymbol{L} = \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i).$$

Taking derivative we have

$$\frac{d\boldsymbol{L}}{dt} = \sum_{i=1}^{n} (\boldsymbol{v}_i \times m_i \boldsymbol{v}_i) + \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{a}_i).$$

The first term vanishes and hence

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \sum_{i=1}^{n} \left[\boldsymbol{r}_i \times \left(\boldsymbol{F}_i + \sum_{j=1}^{n} \boldsymbol{F}_{ij} \right) \right] \\ &= \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}. \end{aligned}$$

Note that

$$(\boldsymbol{r}_i \times \boldsymbol{F}_{ij}) + (\boldsymbol{r}_j \times \boldsymbol{F}_{ji}) = (\boldsymbol{r}_i \times \boldsymbol{F}_{ij}) - (\boldsymbol{r}_j \times \boldsymbol{F}_{ij}) = -\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ij}.$$

Thus the double sum vanishes. Denote the total external torque (or moment of force)

$$\boldsymbol{N} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i,$$

then we have

$$\boxed{\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}}.$$

If a system is isolated, then $\boldsymbol{N} = 0$, and hence

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = \text{constant vector}.$$

This is the **principle of conservation of angular momentum**.

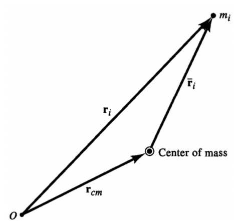

Figure 7.1: Vector $\bar{\boldsymbol{r}}_i$

Sometimes we may write

$$\boldsymbol{r}_i = \boldsymbol{r}_{cm} + \bar{\boldsymbol{r}}_i \quad \Rightarrow \quad \boldsymbol{v}_i = \boldsymbol{v}_{cm} + \bar{\boldsymbol{v}}_i.$$

And thus

$$\begin{aligned}
\boldsymbol{L} &= \sum_i (\boldsymbol{r}_{cm} + \bar{\boldsymbol{r}}_i) \times m_i(\boldsymbol{v}_{cm} + \bar{\boldsymbol{v}}_i) \\
&= \sum_i (\boldsymbol{r}_{cm} \times m_i \boldsymbol{v}_{cm}) + \sum_i (\boldsymbol{r}_{cm} \times m_i \bar{\boldsymbol{v}}_{cm}) + \sum_i (\bar{\boldsymbol{r}}_{cm} \times m_i \boldsymbol{v}_{cm}) + \sum_i (\bar{\boldsymbol{r}}_{cm} \times m_i \bar{\boldsymbol{v}}_{cm}) \\
&= \boldsymbol{r}_{cm} \times \left(\sum_i m_i \right) \boldsymbol{v}_{cm} + \boldsymbol{r}_{cm} \times m_i \bar{\boldsymbol{v}}_i + \left(\sum_i m_i \bar{\boldsymbol{r}}_i \right) \times \boldsymbol{v}_{cm} + \sum_i (\bar{\boldsymbol{r}}_{cm} \times m_i \bar{\boldsymbol{v}}_{cm}).
\end{aligned}$$

We have

$$\sum_i m_i \bar{r}_i = \sum_i m_i (\boldsymbol{r}_i - \boldsymbol{r}_{cm}) = \sum_i m_i \boldsymbol{r}_i - m \boldsymbol{r}_{cm} = 0,$$

and

$$\sum_i m_i \bar{v}_i = \sum_i m_i (\boldsymbol{v}_i - \boldsymbol{v}_{cm}) = \sum_i m_i \boldsymbol{v}_i - m \boldsymbol{v}_{cm} = 0.$$

Thus

$$\boxed{\boldsymbol{L} = \boldsymbol{r}_{cm} \times m \boldsymbol{v}_{cm} + \sum_i \bar{\boldsymbol{r}}_i \times m_i \bar{\boldsymbol{v}}_i.}$$

This expresses the angular momentum of a system in terms of an orbital part (motion of the center of mass) and a spin part (motion about the center of mass).

Kinetic Energy of a System

The total kinetic energy T of a system of particles is given by

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\boldsymbol{v}_i \cdot \boldsymbol{v}_i).$$

Express the velocities related to the mass center

$$\begin{aligned} T &= \sum_i \frac{1}{2} m_i (\boldsymbol{v}_{cm} + \bar{\boldsymbol{v}}_i) \cdot (\boldsymbol{v}_{cm} + \bar{\boldsymbol{v}}_i) \\ &= \sum_i \frac{1}{2} m_i v_{cm}^2 + \sum_i m_i (\boldsymbol{v}_{cm} \cdot \bar{\boldsymbol{v}}_i) + \sum_i \frac{1}{2} m_i \bar{v}_i^2 \\ &= \frac{1}{2} v_{cm}^2 \sum_i + \boldsymbol{v}_{cm} \cdot \sum_i m_i \bar{\boldsymbol{v}}_i + \sum_i \frac{1}{2} m_i \bar{v}_i^2. \end{aligned}$$

The second term vanishes, and hence

$$\boxed{T = \frac{1}{2} m v_{cm}^2 + \sum_i \frac{1}{2} m_i \bar{v}_i^2.}$$

The first term is the kinetic energy of translation of the whole system, and the second is the kinetic energy of motion relative to the center of mass.

7.3 Motion of Two Interacting Bodies: The Reduced Mass

Consider a system consisting two particles, interacting with each other by a central force. Assume the system is isolated, thus the center of mass moves with constant velocity. Take the center of mass as the origin. Then

$$m_1 \bar{\boldsymbol{r}}_1 + m_2 \bar{\boldsymbol{r}}_2 = 0.$$

If $\boldsymbol{R}$ is the position vector of particle 1 relative to 2, then we have

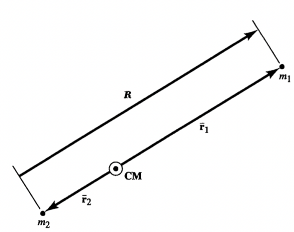

Figure 7.2: The relative vector $\boldsymbol{R}$ for the two-body problem

$$\boldsymbol{R} = \bar{\boldsymbol{r}}_1 - \bar{\boldsymbol{r}}_2 = \bar{\boldsymbol{r}}_1 \left(1 + \frac{m_1}{m_2}\right).$$

The equation of motion relative to the center of mass is

$$m_1 \frac{d^2 \bar{\boldsymbol{r}}_1}{dt^2} = \boldsymbol{F}_1 = f(R) \frac{\boldsymbol{R}}{R},$$

where $|f(R)|$ is the magnitude of the mutual force between the two particles. Then

$$\mu \frac{d^2 \boldsymbol{R}}{dt^2} = f(R) \frac{\boldsymbol{R}}{R},$$

where

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2}.}$$

μ is called the **reduced mass**. The equation is the same as the ordinary equation of motion of a single particle of mass μ moving in a central field of force $f(R)$.

7.5 Collisions

When two body undergoes a collision, the force is an internal force. And thus the total linear momentum is unchanged:

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}'_1 + \boldsymbol{p}'_2.$$

And for energy, we have

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{{p'}_1^2}{2m_1} + \frac{{p'}_2^2}{2m_2} + Q,$$

where Q is introduced to indicate the net loss or gain in kinetic energy. In an **elastic collision**, $Q = 0$. If $Q > 0$, there is energy loss, and it's called an **exoergic** collision. If $Q < 0$, there is energy gain, and it's called an **endoergic** collision.

Direct Collisions

Suppose we have a head-on collision of two bodies, where the motions takes place on a single straight line.

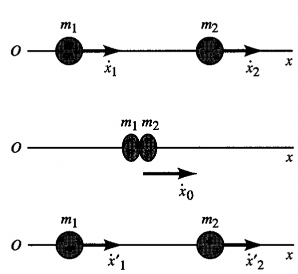

Figure 7.3: Head-on collision of two particles

For momentum, we have

$$m_1 \dot{x}_1 + m_2 \dot{x}_2 = m_1 \dot{x}'_1 + m_2 \dot{x}'_2.$$

Introducing a parameter ϵ called the **coefficient of restitution**, which is defined as the ratio of the speed o separation v' to the speed of approach v. Thus

$$\epsilon = \frac{|\dot{x}_2' - \dot{x}_1'|}{|\dot{x}_2 - \dot{x}_2|} = \frac{v'}{v}.$$

In an elastic collision, $\epsilon = 1$. In the case of a **totally inelastic** collision, we have $\epsilon = 0$. Calculate the final velocities from the definition of ϵ:

$$\dot{x}_1' = \frac{(m_1 - \epsilon m_2)\dot{x}_1 + (m_2 + \epsilon m_2)\dot{x}_2}{m_1 + m_2},$$
$$\dot{x}_2' = \frac{(m_1 + \epsilon m_1)\dot{x}_1 + (m_2 - \epsilon m_1)\dot{x}_2}{m_1 + m_2}.$$

In the general case of a direct nonelastic collision, we can verify that

$$Q = \frac{1}{2}\mu v^2 (1 - \epsilon^2),$$

where μ is the reduced mass, and v is the relative speed before impact.

Impulse in Collision

Forces of extremely short duration in time (in collisions) are called **impulsive forces**. The equation of motion is $d(m\boldsymbol{v})/dt = \boldsymbol{F}$. Then we have

$$\Delta(m\boldsymbol{v}) = \int_{t_1}^{t_2} \boldsymbol{F} dt.$$

The time integral of the force is the impulse, denoted by $\boldsymbol{P}$, and thus

$$\Delta(m\boldsymbol{v}) = \boldsymbol{P}.$$

7.6 Oblique Collisions and Scattering: Comparison of Laboratory and Center of Mass Coordinates

Now suppose that in the collisions, the motion is not confined to a single straight line. Ann incident particle of mass m_1 with initial velocity $\boldsymbol{v}_1$ that strikes a target particle of mass m_2 that is initially at rest. Then we have

$$\boldsymbol{p}_1 = \boldsymbol{p}_1' + \boldsymbol{p}_2',$$

and

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{{p'}_1^2}{2m_1} + \frac{{p'}_2^2}{2m_2} + Q.$$

Consider the case where $m_1 = m_2 = m$, then

$$p_1^2 = p_1'^2 + p_2'^2 + 2mQ.$$

Also we have

$$p_1^2 = (\boldsymbol{p}_1' + \boldsymbol{p}_2') \cdot (\boldsymbol{p}_1' + \boldsymbol{p}_2') = p_1'^2 + p_2'^2 + 2\boldsymbol{p}_1' \cdot \boldsymbol{p}_2'.$$

Hence

$$\boldsymbol{p}_1' \cdot \boldsymbol{p}_2' = mQ.$$

For an elastic collision, $Q = 0$, and $\boldsymbol{p}_1' \cdot \boldsymbol{p}_2' = 0$.

Center of Mass Coordinates

In collisions we have two coordinate systems: the center of mass system and the laboratory system.

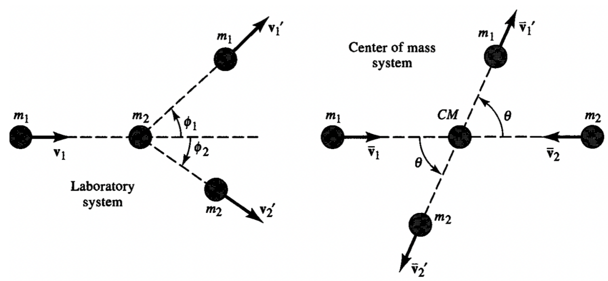

Figure 7.4: Comparison of laboratory and center of mass system

In the center of mass system we have

$$\bar{\boldsymbol{p}}_1 + \bar{\boldsymbol{p}}_2 = 0,$$
$$\bar{\boldsymbol{p}}_1' + \bar{\boldsymbol{p}}_2' = 0,$$

and the energy equation is

$$\frac{\bar{p}_1^2}{2m_1} + \frac{\bar{p}_2^2}{2m_1} = \frac{\bar{p}_1'^2}{2m_1} + \frac{\bar{p}_2'^2}{2m_2} + Q.$$

Using the reduced mass,

$$\frac{\bar{p}_1^2}{2\mu} = \frac{\bar{p}_1'^2}{2\mu} + Q.$$

The velocity of the center of mass is

$$\boldsymbol{v}_{cm} = \frac{m_1 \boldsymbol{v}_1}{m_1 + m_2}.$$

Hence,

$$\bar{\boldsymbol{v}}_1 = \boldsymbol{v}_1 - \boldsymbol{v}_{cm} = \frac{m_2 \boldsymbol{v}_1}{m_1 + m_2}.$$

7.7 Motion of a Body with Variable Mass: Rocket Motion

Chapter 8

Mechanics of Rigid Bodies: Planar Motion

8.1 Center of Mass of a Rigid Body

The center of mass of a system of particles is

$$x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i}, \quad y_{cm} = \frac{\sum_i y_i m_i}{\sum_i m_i}, \quad z_{cm} = \frac{\sum_i z_i m_i}{\sum_i m_i}.$$

For a rigid extended body, replace the summation by an integration:

$$x_{cm} = \frac{\int_v \rho x \, dv}{\int_v \rho \, dv}, \quad y_{cm} = \frac{\int_v \rho y \, dv}{\int_v \rho \, dv}, \quad z_{cm} = \frac{\int_v \rho z \, dv}{\int_v \rho \, dv},$$

where ρ is the density and dv is the element of volume. If a rigid body is in the form of a thin shell, we have

$$x_{cm} = \frac{\int_s \rho x \, ds}{\int_s \rho \, ds}, \quad y_{cm} = \frac{\int_s \rho y \, ds}{\int_s \rho \, ds}, \quad z_{cm} = \frac{\int_s \rho z \, ds}{\int_s \rho \, ds},$$

where ρ is the mass per unit area. And if the body is in the form of a thin wire,

$$x_{cm} = \frac{\int_l \rho x \, dl}{\int_l \rho \, dl}, \quad y_{cm} = \frac{\int_l \rho y \, dl}{\int_l \rho \, dl}, \quad z_{cm} = \frac{\int_l \rho z \, dl}{\int_l \rho \, dl},$$

where ρ is the mass per unit length.

Symmetry Considerations

If the body has a plane of symmetry, then the center of mass lies on that plane. And if the body has a line of the symmetry, the center of mass lies on that line.

Type	z_{cm}
Hemispherical shell	$\frac{1}{2}a$
Solid hemisphere	$\frac{3}{8}a$
Semicircle	$\frac{2a}{\pi}$
Semicircular lamina	$\frac{4a}{3\pi}$

Table 8.1: Some common results of the center of mass

8.2 Rotation of a Rigid Body about a Fixed Axis: Moment of Inertia

Choose the z axis as the rotation axis. The path of a particle m_i at the point (x_i, y_i, z_i) is a circle of radius $r_i = (x_i^2 + y_i^2)^{1/2}$. Then the speed of the particle is

$$v_i = r_i\omega = (x_i^2 + y_i^2)^{1/2}\omega.$$

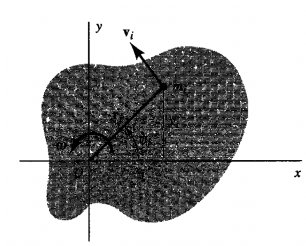

Figure 8.1: Cross-section of a rigid body rotating about the z-axis

The component of the velocity is

$$\begin{aligned}
\dot{x}_i &= -v_i \sin\phi_i = -\omega_i y, \\
\dot{y}_i &= v_i \cos\phi_i = \omega_i x, \\
\dot{z}_i &= 0.
\end{aligned}$$

This can also be written as

$$\boxed{\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i,}$$

where $\boldsymbol{\omega} = \boldsymbol{k}\omega$. The kinetic energy of rotation of the body is

$$T_{rot} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I_z \omega^2,$$

where

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2)$$

is called the **moment of inertia** about the z-axis. The angular momentum of single particle is $\boldsymbol{r}_i \times m_i \boldsymbol{v}_i$, the z-component is

$$m_i(x_i \dot{y}_i - y_i \dot{x}_i) = m_i(x_i^2 + y_i^2)\omega = m_i r_i^2 \omega.$$

Then we have

$$L_z = \sum_i m_i r_i^2 \omega = I_z \omega.$$

And

$$N_z = \frac{dL_z}{dt} = \frac{d(I_z \omega)}{dt},$$

where N_z is the total moment of all the applied forces about the axis of rotation. If the body is rigid, then I_z is constant, and hence

$$N_z = I_z \frac{d\omega}{dt}.$$

To conclude,

Translation along x-axis	**Rotation about the z-axis**
Linear momentum $p_x = mv_x$	Angular momentum $L_z = I_z \omega$
Force $F_x = m\dot{v}_x$	Torque $N_z = I_z \dot{\omega}$
Kinetic energy $T = \frac{1}{2}mv^2$	Kinetic energy $T_{rot} = \frac{1}{2}I_z \omega^2$

8.3 Calculation of the Moment of Inertia

For any axis,

$$I = \int r^2 \, dm.$$

Type	$\boldsymbol{I_z}$
A thin rod of length a, axis at one end	$\frac{1}{3}ma^2$
A thin rod of length a, axis at the center	$\frac{1}{12}ma^2$
A thin circular hoop with radius a, axis at the center	ma^2
Circular disc or cylinder with radius a, axis at center	$\frac{1}{2}ma^2$
Sphere with radius a, axis across the center	$\frac{2}{5}ma^2$
Spherical shell with radius a, axis across the center	$\frac{2}{3}ma^2$

Table 8.2: Some common results of the center of mass

Perpendicular-Axis Theorem for a Plane Lamina

Consider a rigid body in the form of a plane lamina of any shape. Place it in the xy-plane. Then the moment of inertia about the z-axis is given by

$$I_z = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2.$$

This can also be written as

$$\boxed{I_z = I_x + I_y.}$$

This is the **perpendicular-axis theorem**. The moment of inertia of any plane lamina about an axis normal to the plane of the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the given axis and lying in the plane of the lamina.

Parallel-Axis Theorem for Any Rigid Body

The moment of inertia about the z-axis is

$$I_z = \sum_i m_i(x_i^2 + y_i^2).$$

Write x_i and y_i in terms of the coordinates of the center of mass and the coordinates relative to the center of mass:

$$x_i = x_{cm} + \bar{x}_i, \quad y_i = y_{cm} + \bar{y}_i.$$

Thus we have

$$I_z = \sum_i m_i(\bar{x}_i^2 + \bar{y}_i^2) + \sum_i m_i(x_{cm}^2 + y_{cm}^2) + 2x_{cm}\sum_i m_i\bar{x}_i + 2y_{cm}\sum_i m_i\bar{y}_i.$$

The first sum is the moment of inertia about an axis parallel to the z-axis and passing through the center of mass, denoted by I_{cm}. Let $l^2 = x_{cm}^2 + y_{cm}^2$. And from the definition of center of mass,

$$\sum_i m_i\bar{x}_i = \sum_i m_i\bar{y}_i = 0.$$

Thus the final result is

$$\boxed{I = I_{cm} + ml^2.}$$

This is the **parallel-axis theorem**. The moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the center of mass plus the product of the mass of the body and the square of the distance between the two axes.

8.5 The Angular Momentum of a Rigid Body in Laminar Motion

Laminar motion takes place when all the particles making up a rigid body move parallel to some fixed plane. The rigid body undergoes both translational and rotational acceleration. We have

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N},$$

or

$$\frac{d}{dt}\sum_i \boldsymbol{r}_i \times m_i\boldsymbol{v}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i.$$

Consider a system of particles rotating about an axis whose direction is fixed

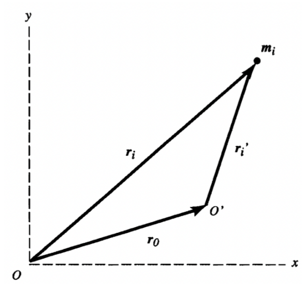

Figure 8.2: A particle in a rigid body in a laminar motion

in space (it might be accelerating). Let O represents the origin of an inertial frame of reference and O' the origin of the axis. The total torque about O' is

$$\boldsymbol{N}' = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i.$$

And we have

$$\boldsymbol{r}_i = \boldsymbol{r}_0 + \boldsymbol{r}_i' \quad \Rightarrow \quad \boldsymbol{v}_i = \boldsymbol{v}_0 + \boldsymbol{v}_i'.$$

In the inertial frame of reference we have

$$\boldsymbol{F}_i = \frac{d}{dt}(m\boldsymbol{v}_i).$$

Then

$$\begin{aligned}
\boldsymbol{N}' = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i &= \sum_i \boldsymbol{r}_i' \times \frac{d}{dt}(\boldsymbol{v}_0 + \boldsymbol{v}_i') \\
&= -\dot{\boldsymbol{v}}_0 \times \sum_i m_i\boldsymbol{r}_i' + \sum_i \boldsymbol{r}_i' \times \frac{d}{dt}m_i\boldsymbol{v}_i' \\
&= -\dot{\boldsymbol{v}}_0 \times \sum_i m_i\boldsymbol{r}_i' + \frac{d}{dt}\sum_i \boldsymbol{r}_i' \times m_i\boldsymbol{v}_i'.
\end{aligned}$$

The last term is the rate of change of the angular momentum $\boldsymbol{L}'$ about the O' axis. Thus

$$\boldsymbol{N}' = -\ddot{\boldsymbol{r}}_0 \times \sum_i m_i \boldsymbol{r}'_i + \frac{d}{dt}\boldsymbol{L}'.$$

The first term vanishes when

1. The acceleration $\ddot{\boldsymbol{r}}_0$ of the axis of rotation O' vanishes.

2. O' is the center of mass of the system of particles.

3. The O' axis passes through the point of contact between the cylinder and the plane. The vector $\sum_i m_i \boldsymbol{r}'_i$ passes through the center of the mass. If $\ddot{\boldsymbol{r}}_0$ also passes through the center of mass, then the cross product vanishes.

Summing torques about an axis passing through the center of mass, we have

$$\boldsymbol{N}_{cm} = \frac{d}{dt}\boldsymbol{L}_{cm} = I_{cm}\dot{\boldsymbol{\omega}}$$

Chapter 9

Motion of Rigid Bodies in Three Dimensions

9.1 Rotation of a Rigid Body about an Arbitrary Axis: Moments and Products of Inertia – Angular Momentum and Kinetic Energy

Consider a rigid body rotating about an arbitrary axis. And the axis passes through a fixed point O, taken as the origin of the coordinate system. By the

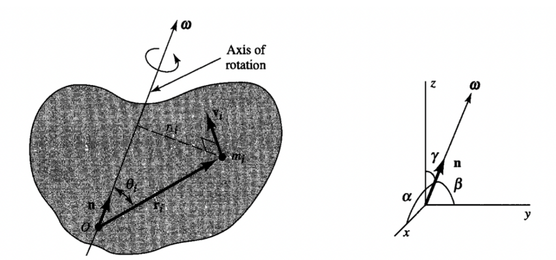

Figure 9.1: Angular velocity vector $\boldsymbol{\omega} = \omega \boldsymbol{n}$

definition of moment of inertia, we have

$$I = \sum_i m_i r_{\perp i}^2,$$

where $r_{\perp i}$ is the perpendicular distance from the particle of m_i to the axis of rotation. If the direction of the axis of rotation is defined by the unit vector $\boldsymbol{n}$, then

$$r_{\perp i} = |r_i \sin\theta_i| = |\boldsymbol{r}_i \times \boldsymbol{n}|.$$

Let

$$\boldsymbol{r}_i = \boldsymbol{i}x_i + \boldsymbol{j}y_i + \boldsymbol{k}z_i,$$

and

$$\boldsymbol{n} = \boldsymbol{i}\cos\alpha + \boldsymbol{j}\cos\beta + \boldsymbol{k}\cos\gamma.$$

Then by calculating the cross product we have

$$\begin{aligned} r_{\perp i}^2 &= (y_i^2 + z_i^2)\cos^2\alpha + (z_i^2 + x_i^2)\cos^2\alpha + (x_i^2 + y_i^2)\cos^2\alpha \\ &\quad - 2y_i z_i \cos\beta\cos\gamma - 2z_i x_i \cos\gamma\cos\alpha - 2x_i y_i \cos\alpha\cos\beta. \end{aligned}$$

And thus the moment of inertia about the axis of rotation is

$$\begin{aligned} I &= \sum_i m_i(y_i^2 + z_i^2)\cos^2\alpha + \sum_i m_i(z_i^2 + x_i^2)\cos^2\alpha + \sum_i m_i(x_i^2 + y_i^2)\cos^2\alpha \\ &\quad - 2\sum_i m_i y_i z_i \cos\beta\cos\gamma - 2\sum_i m_i z_i x_i \cos\gamma\cos\alpha - 2\sum_i m_i x_i y_i \cos\alpha\cos\beta. \end{aligned}$$

Note that the sum involving the squares of coordinates are moments of inertia of the body about the three coordinate axes:

$$\begin{aligned} \sum_i m_i(y_i^2 + z_i^2) &= I_{xx}, \quad \text{moment of inertia about the } x\text{-axis} \\ \sum_i m_i(z_i^2 + x_i^2) &= I_{yy}, \quad \text{moment of inertia about the } y\text{-axis} \\ \sum_i m_i(x_i^2 + y_i^2) &= I_{zz}. \quad \text{moment of inertia about the } z\text{-axis} \end{aligned}$$

The sums involving the product of coordinates are called **products of inertia**:

$$\begin{aligned} -\sum_i m_i x_i y_i &= I_{xy} = I_{yx}, \quad xy \text{ product of inertia} \\ -\sum_i m_i y_i z_i &= I_{yz} = I_{zy}, \quad yz \text{ product of inertia} \\ -\sum_i m_i z_i x_i &= I_{zx} = I_{xz}. \quad zx \text{ product of inertia} \end{aligned}$$

In an extended rigid body, we may replace the summations by integrations:

$$\begin{aligned} I_{zz} &= \int (x^2 + y^2)dm, \cdots \\ I_{xy} &= -\int xy\, dm, \cdots \end{aligned}$$

Now we can write the moment of inertia in the matrix form. Define the **moment of inertia tensor $\boldsymbol{I}$**:

$$\boldsymbol{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix},$$

where we have $I_{ij} = I_{ji}$ and the matrix is symmetric. And since

$$\boldsymbol{n} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix},$$

the moment of inertia could be written as

$$\boxed{I = \boldsymbol{n}^T \boldsymbol{I} \boldsymbol{n} = \boldsymbol{I} = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix}.}$$

Angular Momentum Vector

A **tyad product** is defined as

$$\boldsymbol{ab} \equiv \boldsymbol{a}\boldsymbol{b}^T = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \begin{pmatrix} b_x & b_y & b_z \end{pmatrix} = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix}.$$

This means that $\boldsymbol{ab}$ acts like a 3×3 matrix. And we have

$$(\boldsymbol{ab}) \cdot \boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}).$$

This dot product[1] yields a vector. In three dimensions, the component of a tensor is given by

$$T_{ij} = \boldsymbol{i} \cdot \boldsymbol{T} \cdot \boldsymbol{j}.$$

The unit tensor is

$$\mathbf{1} = \boldsymbol{ii} + \boldsymbol{jj} + \boldsymbol{kk}.$$

Consider a rigid body rotating about an arbitrary axis. The angular momentum of a system of particles is given by

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times m_i \boldsymbol{v}_i.$$

The rotational velocity of the particle of the rigid body is given by

$$\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i.$$

Thus we have

$$\boldsymbol{L} = \sum_i m_i \boldsymbol{r}_i \times \boldsymbol{v}_i = \sum_i m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i).$$

The triple product can be reduced to

$$\boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) = r_i^2 \boldsymbol{\omega} - \boldsymbol{r}_i(\boldsymbol{r}_i \cdot \boldsymbol{\omega}).$$

[1] It seems the author mixed the multiplication of matrices and dot product of vectors together. My understanding is that if $\cdot$ is between two matrices, then it's multiplication of matrices, and if $\cdot$ is between two vectors, then it's dot product of vectors.

Then the angular momentum can be written as

$$\begin{aligned}\boldsymbol{L} &= \sum_i m_i r_i^2 \boldsymbol{\omega} - \sum_i m_i \boldsymbol{r}_i (\boldsymbol{r}_i \cdot \boldsymbol{\omega}) \\ &= \left(\sum_i m_i r_i^2\right) \boldsymbol{\omega} - \left(\sum_i m_i \boldsymbol{r}_i \boldsymbol{r}_i\right) \cdot \boldsymbol{\omega} \\ &= \left[\left(\sum_i m_i r_i^2 \mathbf{1}\right) - \left(\sum_i m_i \boldsymbol{r}_i \boldsymbol{r}_i\right)\right] \cdot \boldsymbol{\omega},\end{aligned}$$

since $\mathbf{1} \cdot \boldsymbol{\omega} = \boldsymbol{\omega}$. Note that

$$\boxed{\boldsymbol{I} = \sum_i m_i r_i^2 \mathbf{1} - \sum_i m_i \boldsymbol{r}_i \boldsymbol{r}_i,}$$

we have

$$\boxed{\boldsymbol{L} = \boldsymbol{I} \cdot \boldsymbol{\omega}.}$$

Note that the direction of angular momentum vector is not necessarily aligned along the axis of rotation, and $\boldsymbol{L}$ and $\boldsymbol{\omega}$ are not necessarily parallel.

Rotational Kinetic Energy of a Rigid Body

The velocity of a particle is given by $\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i$. The rotational kinetic energy is given by

$$T_{rot} = \sum_i \frac{1}{2} m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i = \sum_i \frac{1}{2} (\boldsymbol{\omega} \times \boldsymbol{r}_i) \cdot m_i \boldsymbol{v}_i.$$

Since $(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$,

$$T_{rot} = \frac{1}{2} \sum_i \boldsymbol{\omega} \cdot (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i) = \frac{1}{2} \boldsymbol{\omega} \cdot \sum_i (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i).$$

By definition, $\sum_i (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i) = \boldsymbol{L}$, we have

$$\boxed{T_{rot} = \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{L}.}$$

This is the **rotational kinetic energy** of a rigid body. Also we may express it in the tensor form,

$$\boxed{T_{rot} = \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega}^T \boldsymbol{I} \boldsymbol{\omega}.}$$

If the origin is chosen to be the center of mass [2], the total kinetic energy of the rigid body is given by

$$T = T_{rot} + T_{trans} = \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{L} + \frac{1}{2} \boldsymbol{v}_{cm} \cdot \boldsymbol{p}.$$

[2]See **here** for a much clearer explanation. The author is not very rigorous here. It holds only when we are in a center of mass frame.

9.2 Parallel Axis Theorem

[3]Recall that we have

$$\boldsymbol{I} = \sum_i m_i r_i^2 \boldsymbol{1} - \sum_i m_i \boldsymbol{r}_i \boldsymbol{r}_i.$$

Decompose $\boldsymbol{r}_i$ with respect to the center of mass:

$$\boldsymbol{r}_i = \boldsymbol{r}_{cm} + \bar{\boldsymbol{r}}_i.$$

Then

$$\begin{aligned}
\boldsymbol{I} &= \sum_i m_i (\boldsymbol{r}_{cm} + \bar{\boldsymbol{r}}_i)^2 \boldsymbol{1} - \sum_i m_i (\boldsymbol{r}_{cm} + \bar{\boldsymbol{r}}_i)(\boldsymbol{r}_{cm} + \bar{\boldsymbol{r}}_i)^T \\
&= \sum_i m_i r_{cm}^2 \boldsymbol{1} - \sum_i m_i \boldsymbol{r}_{cm} \boldsymbol{r}_{cm} + \sum_i m_i \bar{r}_i^2 \boldsymbol{1} - \sum_i m_i \bar{\boldsymbol{r}}_i \bar{\boldsymbol{r}}_i \\
&= \left(M r_{cm}^2 \boldsymbol{1} - M \boldsymbol{r}_{cm} \boldsymbol{r}_{cm}\right) + \left(\sum_i m_i \bar{r}_i^2 \boldsymbol{1} - \sum_i m_i \bar{\boldsymbol{r}}_i \bar{\boldsymbol{r}}_i\right).
\end{aligned}$$

This is equivalent as

$$\boxed{\boldsymbol{I} = \boldsymbol{I}_{cm} + M\left(r_{cm}^2 \boldsymbol{1} - \boldsymbol{r}_{cm} \boldsymbol{r}_{cm}\right).}$$

That is, the moment of inertia tensor about a point is the sum of that evaluated at the center of mass plus that of a mass sitting at the center of mass.

9.3 Principal Axes of a Rigid Body

There exists a coordinate system where the products of inertia all vanish. The axes of this coordinate system are **principal axes** for the body at the point O. Then we have $I_{xy} = I_{xz} = I_{yz} = 0$. Denote $I_{xx} = I_1, I_{yy} = I_2, I_{zz} = I_3$ (called **principal moments** of the rigid body at O) and $\omega_x = \omega_1, \omega_y = \omega_2, \omega_z = \omega_3$. Also denote the three bases as $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. Then the moment of inertia tensor is

$$\boldsymbol{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

The problem of finding the principal axes of a rigid body is equivalent of diagonalizing a 3×3 matrix. Since $\boldsymbol{I}$ is always a symmetric matrix, it can always be diagonalized. That is, **a set of principal axes exists for any rigid body at any point in space**. In the principal axes, the moment of inertia, angular momentum and rotational kinetic energy about an axis are in a simple form:

$$\begin{aligned}
I &= \boldsymbol{n}^T \boldsymbol{I} \boldsymbol{n} = I_1 \cos^2\alpha + I_2 \cos^2\beta + I_3 \cos^2\gamma, \\
\boldsymbol{L} &= \boldsymbol{I}\boldsymbol{\omega} = I_1 \omega_1 \boldsymbol{e}_1 + I_2 \omega_2 \boldsymbol{e}_2 + I_3 \omega_3 \boldsymbol{e}_3, \\
T_{rot} &= \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{I} \boldsymbol{\omega} = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2).
\end{aligned}$$

[3]This section is not included in the textbook. But it's covered PHYS3032.

Dynamic Balancing

Suppose a body is rotating about one of its principal axes, then we have $\omega = \omega_1$, $\omega_2 = \omega_3 = 0$. And

$$\boldsymbol{L} = \boldsymbol{e}_1 I_1 \omega_1,$$

or

$$\boldsymbol{L} = I_1 \boldsymbol{\omega}.$$

In this case the angular momentum vector has the same direction as the angular velocity vector. The angular momentum vector is either in the same direction as the axis of rotation, or is not, depending on whether the axis of rotation is, or is not, a principal axis.

Determining Principal Axes by Diagonalizing the Moment of Inertia Matrix

Suppose we have found the coordinate system where the moment of inertia tensor is represented by a diagonal matrix, whose diagonal elements are the principal moments of inertia. Let $\boldsymbol{e}_i$ be the unit vector of such coordinate system. Then

$$\boldsymbol{I}\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i,$$

where λ_i is the principal moments of inertia about the respective principal axes[4]. This is equivalent as

$$\boxed{\det(\boldsymbol{I} - \lambda \mathbf{1}) = 0.}$$

The three roots $\lambda_1, \lambda_2, \lambda_3$ are the three principal moments of inertia. To find $\boldsymbol{e}_i$, we calculate the basis of $\mathrm{Nul}(\boldsymbol{I} - \lambda_i \mathbf{1})$ and normalize it[5].

9.4 Euler's Equations of Motion of a Rigid Body

Now we consider the actual 3-dimensional rotation of rigid body under action of external forces.

$$\boldsymbol{N} = \frac{d\boldsymbol{L}}{dt},$$

where $\boldsymbol{N}$ is the net applied torque and $\boldsymbol{L}$ is the angular momentum. Generally, we must employ a coordinate system that is fixed in the body and rotates with it, i.e. the angular velocity of the body and the angular velocity of the coordinate system are one and the same (Of course, the coordinate system is NOT an inertial one).

Note that

$$\left(\frac{d\boldsymbol{L}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{L}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{L}.$$

[4] Note that $\boldsymbol{I}$ here is the original moment of inertia tensor. And the equation are computed in the original coordinate system.

[5] The method used by the author is confused. Simply apply the linear algebra knowledge here.

Thus the equation of motion in the rotating system[6] is

$$\boldsymbol{N} = \left(\frac{d\boldsymbol{L}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{L}.$$

We take the coordinates axes to be the principal axes, then

$$\boldsymbol{L} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}, \quad \boldsymbol{\omega} \times \boldsymbol{L} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix}.$$

Then we have

$$\boxed{\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) \\ I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) \\ I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) \end{pmatrix}.}$$

This is the **Euler's equations** for the motion of a rigid body in components along the principal axes of the body.

9.5 Free Rotation of a Rigid Body: Geometric Description of the Motion

Consider the case of a rigid body that is free to rotate in any direction about a certain point O. Let O be the center of mass. Since $N = 0$, then $\boldsymbol{L}$ is conserved. In the rotating frame[7], we have

$$\boldsymbol{L} \cdot \boldsymbol{L} = \text{constant}.$$

Thus

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2 = \text{constant}.$$

Also, since there is zero torque, the total rotational kinetic energy must remain constant:

$$\boldsymbol{\omega} \cdot \boldsymbol{L} = 2T_{rot} = \text{constant}.$$

Equivalently,

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = L^2 = \text{constant}.$$

[6] This is actually an very important point. In the equation, $\boldsymbol{\omega}$ is the vector referred to the fixed frame. And $\boldsymbol{L}$ should be referred to the rotating frame. In this case we express $\boldsymbol{\omega}$ in the coordinates of the rotating frame, since $\boldsymbol{L} = \boldsymbol{I\omega}$ and we need the $\boldsymbol{\omega}$ in the rotating frame to express $\boldsymbol{L}$ in the rotating frame. This actually doesn't matter, since the direction and norm of $\boldsymbol{\omega}$ in the fixed frame doesn't change at all. Also, N_1, N_2, N_3 are components in the rotating frame.

[7] In the fixed frame, the torque is zero, then $\boldsymbol{L}$ is conserved in the fixed frame, i.e. its direction and magnitude remains unchanged. In the rotating frame the direction of $\boldsymbol{L}$ might change, but its norm remains constant.

9.6 Free Rotation of a Rigid Body with an Axis of Symmetry: Analytical Treatment

Consider a body possesses an axis of symmetry, so that two of the three principal moments of inertia are equal. Choose 3-axis as the axis of symmetry, denote

$$\begin{aligned} I_s = I_3 &\quad \textit{moment of inertia about the symmetry axis} \\ I = I_1 = I_2 &\quad \textit{moment about the axes normal to the symmetry axis} \end{aligned}$$

In the case of zero torque, Euler's equations become

$$\begin{aligned} I\dot{\omega}_1 + \omega_2\omega_3(I_s - I) &= 0, \\ I\dot{\omega}_2 + \omega_3\omega_1(I - I_s) &= 0, \\ I_s\dot{\omega}_3 &= 0. \end{aligned}$$

For the last equation,

$$\omega_3 = \text{constant}.$$

Define

$$\Omega = \omega_3 \frac{I_s - I}{I}.$$

Then

$$\begin{aligned} \dot{\omega}_1 + \Omega\omega_2 &= 0, \\ \dot{\omega}_2 - \Omega\omega_1 &= 0. \end{aligned}$$

Differentiate the first equation,

$$\ddot{\omega}_1 + \Omega\dot{\omega}_2 = 0,$$

hence

$$\ddot{\omega}_1 + \Omega^2\omega_1 = 0.$$

A solution is

$$\begin{aligned} \omega_1 &= \omega_0 \cos \Omega t, \\ \omega_2 &= \omega_0 \sin \Omega t, \end{aligned}$$

where ω_0 is a constant. Note that ω_1 and ω_3 vary harmonically in time with angular frequency Ω, and their phases differ by $\pi/2$. The projection of $\boldsymbol{\omega}$ on the 12 plane describes a circle of radius ω_0 at the angular frequency Ω.

In the free rotation of a rigid body with an axis symmetry, the angular velocity vector describes a conical motion (precesses) about the symmetry axis. An observer in the rotating frame would see $\boldsymbol{\omega}$ trace out a cone around the symmetry axis of the body (called **body cone**). Let α denote the angle between the symmetry axis and $\boldsymbol{\omega}$, then

$$\Omega = \left(\frac{I_s}{I} - 1\right) \omega \cos \alpha.$$

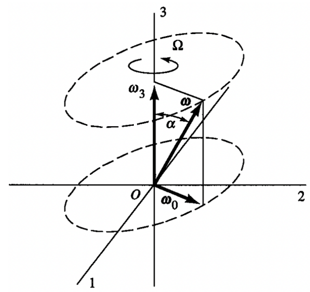

Figure 9.2: Angular velocity vector and its components for the free precession of a body with an axis of symmetry

9.7 Description of the Rotation of a Rigid Body Relative to a Fixed Coordinate System: The Eulerian Angles

The coordinate $O123$ is defined by the three principal axes fixed to the rigid body. The coordinate system fixed in space is labeled $Oxyz$. A rotating system

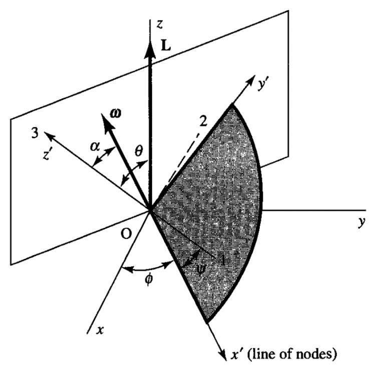

Figure 9.3: Relation of the Eulerian angles to the fixed and rotating coordinate axes

$O'x'y'z'$ is defined as follows: The z'-axis coincides with the 3-axis of the body. The x'-axis is defined by the intersection of the body's 1-2 plane with the fixed xy plane, called the **line of nodes**. The angle between x- and x'- axes is denoted by ϕ. The angle between the z- and $z'-$ is denoted by θ. The rotation of the body about its 3-axis is represented by the angle ψ between the 1-axis and the x'-axis. The three angles ϕ, θ and ψ completely define the orientation of the body in space and are called the **Eulerian angles**.

Suppose at $t = 0$, $O123$ coincides with $Oxyz$. Some times dt later, the rigid body rotates through some infinitesimal angle $\boldsymbol{\omega} dt = d\beta$. We have

$$d\boldsymbol{\beta} = \boldsymbol{\omega} dt = d\boldsymbol{\phi} + d\boldsymbol{\theta} + d\boldsymbol{\psi}.$$

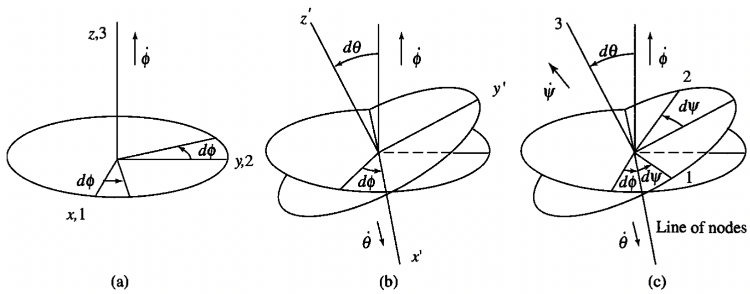

Figure 9.4: Generation of any arbitrary infinitesimal rotation of a rigid body as the vector sum of infinitesimal rotations through the free Eulerian angles, (a) $d\boldsymbol{\phi}$, (b) $d\boldsymbol{\theta}$, and (c) $d\boldsymbol{\psi}$

Thus

$$\boxed{\boldsymbol{\omega} = \dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\psi}}.}$$

Note that $\boldsymbol{\omega}$ consists of a rotation about the 3-axis with angular velocity $\dot{\boldsymbol{\psi}}$

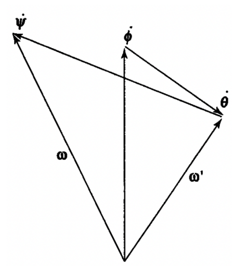

Figure 9.5: Relation between $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$

superimposed on the rotation of the $Ox'y'z'$ system whose angular velocity we call $\boldsymbol{\omega}'$. That is

$$\boldsymbol{\omega}' = \dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\theta}}.$$

Find the components of $\boldsymbol{\omega}'$ in the $Ox'y'z'$ system:

$$\begin{array}{lll} \dot{\phi}_{x'} = 0 & \dot{\theta}_{x'} = \dot{\theta} & \dot{\psi}_{x'} = 0 \\ \dot{\phi}_{y'} = \dot{\phi}\sin\theta & \dot{\theta}_{y'} = 0 & \dot{\psi}_{y'} = 0 \\ \dot{\phi}_{z'} = \dot{\phi}\cos\theta & \dot{\theta}_{z'} = 0 & \dot{\psi}_{z'} = \dot{\psi} \end{array}$$

Thus

$$\begin{aligned} \omega'_{x'} &= \dot{\theta}, \\ \omega'_{y'} &= \dot{\phi}\sin\theta, \\ \omega'_{z'} &= \dot{\phi}\cos\theta. \end{aligned}$$

Since $\boldsymbol{\omega}$ differs from $\boldsymbol{\omega}'$ only $\dot{\psi}$, we have

$$\begin{aligned}
\omega_{x'} &= \dot{\theta}, \\
\omega_{y'} &= \dot{\phi} \sin\theta, \\
\omega_{z'} &= \dot{\phi} \cos\theta + \dot{\psi}.
\end{aligned}$$

Now we express $\dot{\boldsymbol{\phi}}, \dot{\boldsymbol{\theta}}, \dot{\boldsymbol{\psi}}$ in the $O123$ system,

$$\begin{array}{lll}
\dot{\phi}_1 = \dot{\phi}_{y'} \sin\psi = \dot{\phi} \sin\theta \sin\psi & \dot{\theta}_1 = \dot{\theta}_{x'} \cos\psi = \dot{\theta} \cos\psi & \dot{\psi}_1 = 0 \\
\dot{\phi}_2 = \dot{\phi}_{y'} \cos\psi = \dot{\phi} \sin\theta \cos\psi & \dot{\theta}_2 = -\dot{\theta}_{x'} \sin\psi = \dot{\theta} \sin\psi & \dot{\psi}_2 = 0 \\
\dot{\phi}_3 = \dot{\phi} \cos\theta & \dot{\theta}_3 = 0 & \dot{\psi}_3 = \dot{\psi}
\end{array}$$

Thus the components of $\boldsymbol{\omega}$ in the $O123$ system is

$$\begin{aligned}
\omega_1 &= \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi, \\
\omega_2 &= \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi, \\
\omega_3 &= \dot{\phi} \cos\theta + \dot{\psi}.
\end{aligned}$$

9.8 Motion of a Top

Chapter 10

Lagrangian Mechanics

10.1 Hamilton's Variational Principle: An Example

Hamilton's variational principle states that the integral

$$J = \int_{t_1}^{t_2} L\,dt$$

taken along a path of the possible motion of a physical system is an extremum when evaluated along the path of motion that is the one actually taken. $L = T - V$ is the **Lagrangian**. The can be expressed as

$$\boxed{\delta J = \delta \int_{t_1}^{t_2} L\,dt = 0,}$$

where δ is an operation that represents a variation of any particular system parameter by an infinitesimal amount away from that value taken by the parameter then the integral is an maximum.[1]

Now suppose we have a particle dropped from rest in a uniform gravitational field. Then we have

$$\delta J = \delta \int_{t_1}^{t_2} \left[\frac{m\dot{y}^2}{2} - mgy\right] dt = \int_{t_1}^{t_2} (m\dot{y}\,\delta\dot{y} - mg\,\delta y)dt.$$

Note that[2]

$$\boxed{\delta\dot{y} = \frac{d}{dt}\delta y.}$$

Then

$$\int_{t_1}^{t_2} m\dot{y}\,\delta\dot{y}\,dt = \int_{t_1}^{t_2} m\dot{y}\frac{d}{dt}\delta y\,dt = m\dot{y}\;\delta y|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{y}\delta y\,dt.$$

[1] In this chapter I would put many examples in the notes since it's a very important part in analytical mechanics.

[2] The mathematics here is not very rigorous. A more rigorous derivation could be found in *Classical Mechanics* by Goldstein. The advantage of this textbook is that it gives many examples, which is very friendly to new comers.

The first term vanishes since $\delta y(t_1) = \delta y(t_2) = 0$. And hence

$$\delta J = \delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (-m\ddot{y} - mg)\delta y\,dt = 0.$$

Thus

$$-m\ddot{y} - mg = 0.$$

This is just Newton's second law of motion.

Example. Assume a particle travels along a sinusoidal path from a point $x = 0$ to $x = x_1$ during a time interval Δt in a force-field free region of space. Use Hamilton's principle to show that the amplitude of the assumed sinusoidal path is zero, implying that the path is really a straight line between two points.

Solution. We have $x = v_x t$, and the motion is constrained to be completed during $\Delta t = x/v_1$. We may vary the sinusoidal paths

$$x = v_x t, \quad y = \pm\eta \sin(\pi v_x t/x_1).$$

Then the Lagrangian is

$$L = T - V = \frac{1}{2}m\left[v_x^2 + \left(\frac{\eta\pi v_x}{x_1}\right)^2 \cos^2\frac{\pi v_x t}{x_1}\right] - V,$$

where the potential energy V is a constant. Thus

$$J = \int_0^{x_1/v_x} Ldt = \frac{mv_x x_1}{2} + \frac{mv_x\eta^2\pi^2}{4x_1} - V\frac{x_1}{v_x}.$$

Varying the path by varying η,

$$\delta J = \left(\frac{\pi^2 m v_x}{2x_1}\right)\eta\delta\eta = 0.$$

Thus we requires $\eta = 0$.

10.2 Generalized Coordinates

Consider the motion of a pendulum, which is constrained to move in the xy-plane along an arc of radius r. Then

$$z = 0, \quad r^2 - (x^2 + y^2) = 0.$$

Only one scalar coordinate is needed to specify the position of the pendulum. This means that the pendulum has **one degree of freedom**.

Generalized coordinates are any collection of independent coordinates q_i (not connected by any equations of constraint) that just suffice to specify uniquely the configuration of a system of particles. The required number of generalized coordinates is equal to the system's number of degrees of freedom.

In general, if N particles are free to move in 3-dimensional space but their $3N$ coordinates are connected by m conditions of constraint, then there exist $n = 3N - m$ independent generalized coordinates sufficient to describe uniquely the position of the N particles and n independent degrees of freedom available for the motion, provided the constraints are of the type

$$\boxed{f_j(x_i, y_i, z_i, t) = 0, \quad i = 1, 2, \cdots, N, \quad j = 1, 2, \cdots, m}$$

Such constraints are called **holonomic**. These equations are equalities, integrable in form, and may or may not be explicitly time-independent.

Constraints that cannot be expressed as equations of equality or that are nonintegrable in form are called **nonholonomic**, and the equations representing such constraints cannot be used to eliminate from consideration any dependent coordinates describing the configuration of the system.

10.3 Calculating Kinetic and Potential Energies in Terms of Generalized Coordinates: An Example

The Lagrangian $L = T - V$ must be expressed as a function of the generalized coordinates and time derivatives (generalized velocities) appropriate for a given physical situation.

Consider a pendulum of mass m attached to a support of mass M that is free to move in a single dimension along a frictionless, horizontal surface. Each mass

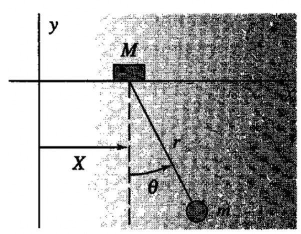

Figure 10.1: A simple pendulum attached to a movable support

needs three Cartesian coordinates, but there are four holonomic constraints

$$\begin{aligned} Z = 0, &\quad Y = 0, \\ z = 0, &\quad [(x - X)^2 + y^2] - r^2 = 0. \end{aligned}$$

Then there are two degrees of freedom for the motion and two generalized coordinates necessary to describe the configuration of the system. Choose X, and θ, the angular displacement of the pendulum away from the vertical. Then

$$x = X + r\sin\theta, \quad y = -r\cos\theta.$$

And

$$\dot{x} = \dot{X} + r\dot{\theta}\cos\theta, \quad \dot{y} = r\dot{\theta}\sin\theta.$$

The kinetic and potential energy of the system can be expressed as

$$\begin{aligned} T &= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \\ &= \frac{M}{2}\dot{X}^2 + \frac{m}{2}[\dot{X}^2 + (r\dot{\theta}^2 + 2\dot{x}r\dot{\theta}\cos\theta)]. \\ V &= mgy = -mgr\cos\theta. \end{aligned}$$

Another way of obtaining T and V is writing the coordinates in vector form and calculate.[3]

When problems involve only holonomic constraints, there always exist transformation equations that relate the Cartesian coordinates of an ensemble of particles to their generalized coordinates, and the required generalized velocities can be obtained by differentiation.

10.4 Lagrange's Equations of Motion for Conservative Systems

Now we derive the Lagrange's equations from Hamilton's variational principle. Note that

$$\delta J = \delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right) dt = 0.$$

And

$$\delta\dot{q}_i = \frac{d}{dt}\delta q_i.$$

Also,

$$\int_{t_1}^{t_2} \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)dt = \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q_i\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\,dt.$$

The first term on RHS vanishes since the variation $\delta q_i = 0$ at the endpoints t_1 and t_2. Thus

$$\delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i\,dt = 0.$$

[3] Generally speaking, the first method can hardly get wrong.

Hence

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad (i = 1, 2, \cdots, n)}$$

These are the **Lagrangian equations of motion** for a conservative system subject to holonomic constraints.

10.5 Some Applications of Lagrange's Equations

Example. The Harmonic Oscillator
Let x be the displacement coordinate. The Lagrangian is

$$L(x, \dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

Then

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial x} = -kx,$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + kx = 0.$$

This is the equation of motion of the undamped harmonic oscillator.

Example. Euler's Equations for

10.6 Generalized Momenta: Ignorable Coordinates

In the case of a system described by the generalized coordinates $q_1, q_2, \cdots, q_k, \cdots, q_n$, the quantities p_k defined by

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

are called the **generalized momenta conjugate to the generalized coordinate** q_k. Lagrange's equations for a conservative system can be written as

$$\boxed{\dot{p}_k = \frac{\partial L}{\partial q_k}.}$$

If the Lagrangian doesn't explicitly contain the coordinate q_k, then

$$\dot{p}_k = \frac{\partial L}{\partial q_k} = 0, \quad p_k = \text{constant}.$$

10.7 Forces of Constraint: Lagrange Multipliers

10.8 D'Alembert's Principle: Generalized Forces

The fundamental condition of equilibrium is that all the forces acting on all the bodies that make a a physical system vanish when the system is in equilibrium:

$$\sum_{i=1}^{N} \boldsymbol{F}_i = 0.$$

The method of finding the conditions necessary for equilibrium is: imagine that a system of bodies in a given configuration undergo small displacements away from their assumed positions, calculating the resultant work done on the system and then demanding that it sum to zero. The displacement are called **virtual displacement**. The key point is that, if the work vanishes for a system subjected to such virtual displacements, then the system is in equilibrium. This is the **principle of virtual work**:

$$\delta W = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0.$$

Problems in **dynamics** deal with bodies are not in equilibrium. They need to be solved using Newton's second law:

$$\boldsymbol{F}_i = \dot{\boldsymbol{p}}_i.$$

Thus

$$\boxed{\sum_{i=1}^{N} (\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i) \cdot \delta \boldsymbol{r}_i = 0.}$$

This is called **D'Alembert's principle**, which is equivalent to Newton's second law of motion.

Now let's derive Lagrange's equation of motion from D'Alembert's principle. Consider a particle expressed in 3-dimensional Cartesian coordinates

$$\sum_{i=1}^{3} (F_i - \dot{p}_i) \delta x_i = 0,$$

where F_i are the sum of all the force components acting on the body along the ith direction. Assume that the motion of the particle is also describable in terms of its generalized coordinates q_j. (Assume they are not connected by equations of constraint.) The first term in the sum is the virtual work done on the system

$$\delta W = \sum_i F_i \delta x_i = \sum_j \left[\sum_i \left(F_i \frac{\partial x_i}{\partial q_j} \right) \right] \delta q_j = \sum_j Q_j \, \delta q_j,$$

where

$$\boxed{Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j}.}$$

10.9 The Hamilton's Function: Hamilton's Equations

Chapter 11

Dynamics of Oscillating Systems

11.1 Potential Energy and Equilibrium: Stability

In a conservative system, its restoring force is derived from a potential energy function

$$V(x) = \frac{1}{2}kx^2,$$
$$F(x) = -\frac{dV(x)}{dx} = -kx.$$

The equilibrium position is $x = 0$, where the restoring force vanishes. If the oscillator was initially placed at $x = 0$, it would remain there at rest.

Consider a pendulum of length r constrained to swing in a vertical plane. Its position is described by the single generalized coordinate θ. The potential energy and the force is given by

$$V = mgr(1 - \cos\theta),$$
$$N_\theta = -\frac{\partial V}{\partial \theta} = -mg(r\sin\theta = -mgx.$$

The pendulum is in its equilibrium position when the restoring torque is equal to 0. In the two cases, equilibrium corresponds to the configuration at which the derivative of the potential energy function vanishes.

Suppose a system with n degrees of freedom whose generalized coordinates $q_1, q_2, \cdots, q_n$ completely specify its configuration. Assume the system is conservative:

$$V = V(q_1, q_2, \cdots, q_n).$$

All forces and torques acting on the system vanish when

$$\boxed{\frac{\partial V}{\partial q_k} = 0, \quad (k = 1, 2, \cdots, n)}$$

If the system is given a sufficiently small displacement, and it always tends to return to equilibrium, then the equilibrium is **stable**; otherwise it's **unstable**.

An intuitive thought is that the potential energy must be a **minimum** in all cases for stable equilibrium.

Extended Criteria for Stable Equilibrium

Consider a system with one degree of freedom. Expand $V(q)$ as a Taylor series about $q = 0$,

$$V(q) = V_0 + qV_0' + \frac{q^2}{2!}V_0'' + \cdots + \frac{q^n}{n!}V_0^{(n)} + \cdots$$

If $q = 0$ is a position of equilibrium, then $V_0' = 0$. Also we may let $V_0 = 0$ without loss of generality. Then

$$V(q) = \frac{q^2}{2}V_0'' + \cdots$$

If V_0'' is not zero, then for a small displacement q from the equilibrium the force is approximately linear in the displacement:

$$F(q) = -\frac{dV}{dq} = -qV_0''.$$

This is a restorative (or stabilizing) force if V_0'' is positive[1], and antirestroing (unstable equilibrium) if V_0'' is negative. If $V_0'' = 0$, we must examine the first nonvanishing term in the expansion.

- If the term is of even order in n, then the equilibrium is stable (or unstable), depending on whether the derivative $V_0^{(n)}$ is positive or negative.
- If the first nonvanishing term in the expansion is of odd order in n, then the equilibrium is always unstable.
- If all derivatives vanish, the potential energy is a constant, and the equilibrium is neutral.

For the case of system with n degrees of freedom, effect a linear transformation so that $q_1 = q_2 = \cdots = q_n = 0$ is the configuration of the equilibrium. The potential energy can be expanded in the form

$$V(q_1, q_2, \cdots, q_n) = \frac{1}{2}(K_{11}q_1^2 + 2K_{12}q_1q_2 + K_{22}q_2^2 + \cdots),$$

where

$$K_{11} = \left(\frac{\partial^2 V}{\partial q_1^2}\right)_{q_1=q_2=\cdots=q_n=0}$$

$$K_{12} = \left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)_{q_1=q_2=\cdots=q_n=0}$$

[1]Like a spring, the force is of opposite direction of the displacement and will always drag it to the equilibrium point, hence the equilibrium is stable.

and so on. We have set $V(0, 0, \cdots, 0) = 0$, and the linear terms in the expansion are absent because the expansion is about an equilibrium configuration. This is known as a **quadratic form**. If the quadratic form is positive definite, i.e.

$$\boxed{K_{11} > 0, \quad \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} > 0, \quad \text{and so on}}$$

then the equilibrium configuration $q_1 = q_2 = \cdots = q_n = 0$ are stable.

Consider

11.2 Oscillation of a System with One Degree of Freedom about a Position of Stable Equilibrium

11.3 Coupled Harmonic Oscillator: Normal Coodinates