W.K. Tung's

Group Theory in Physics

NOTES

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CHAPTER 1 PRE-REQUISITE

1.1 Notations and Symbols

(a) Summation Convention:

$$\sum_i A_i B^i = A_i B^i,$$

but $A_i B_i$ indicates no summation.

(b) In $n-$dimensional Euclidean space,

$$\mathbf{x} = \hat{\mathbf{e}}_i x^i,$$

where $\hat{\mathbf{e}}_i$ are orthonormal basis vectors.

(c) Certain linear spaces have non-trivial invariant metric tensors g_{ij}. The *contravariant components* of a vector are labelled by an upper index, and *convariant components* of the same vector are labelled by a lower index

$$x_i = g_{ij} x^j,$$

such that $x_i y^i$ is an invariant. In Euclidean spaces, $g_{ij} = \delta_{ij}$ and $x_i = x^i$.

(d) Multiplication of a vector $|x\rangle$ by a number α:

$$|\alpha x\rangle = \alpha |x\rangle = |x\rangle \alpha.$$

(e) We have

$$|x\rangle = |e_i\rangle x^i, \quad \langle x| = x_i \langle e^i|.$$

The scalar product can be written as

$$\langle x|y\rangle = x_i^\dagger y^i.$$

And

$$x_i{}^\dagger = x^{i*},$$

where * indicates the complex conjugation.

(f) Elements of a matrix are labelled by a row index followed by a column index. The transpose of a matrix implies the interchange of the row and column indices,

$$A^T_{ij} = A_{ji}.$$

The notation for matrix multiplication is

$$(ABC)^i{}_j = A^i{}_k B^k{}_m C^m{}_j.$$

1.2 Summary of Linear Vector Spaces

(1) Linear Vector Space

Definition (Linear Vector Space): A *linear vector space* V is a set $\{|x\rangle, |y\rangle, ..., \text{etc.}\}$, on which two operations $+$ (addition) and $\cdot$ (multiplication by a number) are defined, such that the following basic axioms hold:
(i) If $|x\rangle, |y\rangle \in V$, then $|x\rangle + |y\rangle \equiv |z\rangle \in V$;
(ii) If $|x\rangle \in V$ and α is a (real or complex) number, then $|\alpha x\rangle \equiv \alpha|x\rangle \equiv |x\rangle\alpha \in V$;

(iii) There exits a *null vector* $|0\rangle$, such that $|x\rangle + |0\rangle = |x\rangle$ for all $|x\rangle \in V$;
(vi) For every $|x\rangle \in V$, there exists a $|-x\rangle \in V$ such that $|x\rangle + |-x\rangle = |0\rangle$;
(v) The operation $+$ is *commutative* and *associative*;
(vi) $1 \cdot |x\rangle = |x\rangle$;
(vii) Multiplication by a number is associative, $\alpha \cdot |\beta x\rangle = (\alpha\beta)|x\rangle \equiv |x\rangle(\alpha\beta)$;
(viii) The two operations satisfy the *distributive* properties: $(\alpha+\beta)|x\rangle = |x\rangle\alpha + |x\rangle\beta$ and $\alpha(|x\rangle + |y\rangle) = |x\rangle\alpha + |y\rangle\alpha$.

Definition (Linear Independence): A set of vectors $\{\mathbf{x}_i \in V, i = 1, ..., m\}$ are *linearly independent* if $|x_i\rangle\alpha^i = 0$ necessarily implies that $\alpha^i = 0$ for all i; conversely, the vectors $\mathbf{x}_i$ are *linearly dependent* if there exists a set of numbers $\{\alpha^i\}$, not all zero, such that $|x_i\rangle\alpha^i = 0$.

Definition (Basis): a set of vectors $\{\mathbf{e}_i, i =\hat{} 1, ..., n\}$ forms a *basis* of V if (i) They are linearly independent; (ii) Every $\mathbf{x} \in V$ can be written as a linear combination of $\{\hat{\mathbf{e}}_i\}$, i.e. $|x\rangle = |e_i\rangle x^i$, where x^i are ordinary numbers.

The numbers $\{x^i, i = 1, ..., n\}$ are the *components* of $\mathbf{x}$ with respect to the basis $\{\hat{\mathbf{e}}_i\}$. Vector spaces which have a basis with a finite number of elements are said to be *finite dimensional.*

Theorem: All bases of a finite dimensional vector space V have the same number of elements.

Definition: Two vector spaces V and V' are said to be *isomorphic* to each other if there exits a one-to-one mapping $\mathbf{x} \in V \to \mathbf{x}' \in V'$, such that $(|x\rangle + |y\rangle)' = |x'\rangle + |y'\rangle\alpha$ for all $\mathbf{x}, \mathbf{y}, \alpha$.

Theorem: Every $n-$dimensional linear vector space V_n is isomorphic to the space of n ordered complex numbers C_n; hence all $n-$dimensional linear vector space are isomorphic to each other.

Definition (Subspace): A subset V_n of a linear space V with the property that it forms a linear vector space with the same definitions of the two operations $+$ and $\cdot$ is called a *subspace* of V.

Theorem: Given V_n and a subspace V_m, one can always choose a basis $\{\hat{\mathbf{e}}_i, i = 1, ..., n\}$ in such a way that the first m basis vectors lie in V_m.

Definition (Direct Sum): Let V_1 and V_2 be subspaces of V, and we say that V is the *direct sum* of V_1 and V_2, denoted by $\mathrm{V} = \mathrm{V}_1 \oplus \mathrm{V}_2$, provided (i) $\mathrm{V}_1 \cap \mathrm{V}_2 = 0$ (disjoint); and (ii) every vector $\mathbf{x} \in V$ can be written as $|x\rangle = |x_1\rangle + |x_2\rangle$, where $\mathbf{x}_1 \in V_1$ and $\mathbf{x}_2 \in V_2$.

(2) Linear Transformation (Operators) on Vector Spaces

Definition (Linear Transformation): A *linear transformation* (operator) A is a mapping of the elements of one vector space V onto those of another V', such that (i) $|x\rangle \in V \xrightarrow{A} |Ax\rangle \in V'$; (ii) if $|y\rangle = |x_1\rangle\alpha_1 + |x_2\rangle\alpha_2 \in V$, then $|Ay\rangle = |Ax_1\rangle\alpha_1 + |Ax_2\rangle\alpha_2 \in V'$.

Two important linear operators: (i) the *null* operator $A = 0$: $|x\rangle \to 0|x\rangle 0|0\rangle$, for all $\mathbf{x} \in V$; (ii) the *identity* operator $A = E$: $|x\rangle \to E|x\rangle = |x\rangle$, for all $\mathbf{x} \in V$.

Definition (Linear Functional): *Linear functionals* are scalar-valued linear functions over vector space.

Definition: The *multiplication of linear transformation* is defined by $(AB)|x\rangle = A|Bx\rangle$ for all $\mathbf{x} \in V$. The *addition of linear transformations* and *multiplication by numbers* are defined as $(\alpha A + \beta B)|x\rangle = A|x\rangle\alpha + B|x\rangle\beta$. The above operations form the *algebra of linear transformations* on V.

Remarks on linear transformations:
(i)$A0 = 0A = 0$, and $AE = EA = A$;
(ii) With respect to addition and multiplication by numbers, the set of operators forms a vector space.

(iii) The algebra of operators satisfies $A(B+C) = AB+AC$ (distributive) and $A+(B+C) = (A+B)+C$, $(AB)C = A(BC)$ (associative). However, the multiplication is not necessarily commutative: $AB \neq BA$.

Definition: An operator A is said to be *idempotent* if $A^2 = A$, and $A \neq E$. Such operators are also called *projection operators*.

Definition: An operator is said to be *nilpotent* if $A \neq$, and there is a positive integer m such that $A^m = 0$.

Definition (Inverse): An operator is said to be *invertible* if (i) $|Ax\rangle = |Ay\rangle$ implies $|x\rangle = |y\rangle$ for all $\mathbf{x}, \mathbf{y} \in V$; (ii) for every $y \in V$ there exists a $\mathbf{x} \in V$ such that $|Ax\rangle = |y\rangle$. It is called the *inverse* of A, denoted by A^{-1}.

(3) Matrix Representation of Linear Operators

Given two vector spaces V_n, V_m and bases $\{\hat{\mathbf{e}}_i, i = 1, ..., n\}, \{\hat{\mathbf{e}}'_i, i = 1, ..., m\}$, every linear operator A from V_n to V_m can be represented by a $m \times n$ matrix $A^j{}_i$.
(i) $\hat{\mathbf{e}}_i \in V_n$, then each of the n vectors $A\hat{\mathbf{e}}_i \in V_m$ can be expressed as a linear combination of $\{\hat{\mathbf{e}}'_i, i = 1, ..., m\}$:

$$A|e_i\rangle = |e'_j\rangle A^j{}_i.$$

(ii) For $|x\rangle \in V_n$ and $|y\rangle \in V_m$,

$$|y\rangle = A|x\rangle = A|e_i\rangle x^i = |e'_j\rangle A^j_i x^i = |e'_j\rangle y^j,$$

which implies $y^j = A^j{}_i x^i$.

Theorem: A change of basis on a vector space causes the matrix representation of the linear operators to undergo a *similarity transformation*. The transformation matrix is the same for all operators A.

Definition (Trace): The *trace* of an operator A, denoted by Tr A, and the *determinant* of A, denoted by det A, are defined to be the corresponding quantities for its matrix representation with respect to any given basis on the vector space. The results are independent of the choice of basis.

Theorem: A linear operator on V_n is invertible if and only if det $A \neq 0$.

(4) Dual Spaces, Adjoint Operators

The set of all linear functionals f on a vector space V forms a vector space $\tilde{V}$. A linear functional f assigns a (complex) number to each $\mathbf{x} \in V$, denoted by $\langle f|x\rangle$:

$$\mathbf{x} \in V \xrightarrow{f} \langle f|x\rangle \in C_1.$$

Then we have $\langle f|\alpha_1 x_1 + \alpha_2 x_2\rangle = \langle f|x_1\rangle\alpha_1 + \langle f|x_2\rangle\alpha_2$. In fact $\langle f|x\rangle = \langle f|e_i\rangle x^i$.

Definition: The *addition* of linear functionals f_1, f_2 and the *multiplication* of linear functionals by number α_1, α_2 are defined by

$$\langle \alpha_1 f_1 + \alpha_2 f_2|x\rangle = \alpha_1^*\langle f_1|x\rangle + \alpha_2^*\langle f_2|x\rangle.$$

Given any basis $\{\hat{\mathbf{e}}_i, i = 1, ..., n\}$ of V, one can define a set of n linear functionals $\{\tilde{\mathbf{e}}_j \in \tilde{V}, j = 1, ..., n\}$ by

$$\langle \tilde{\mathbf{e}}^j|\hat{\mathbf{e}}_i\rangle = \delta^i{}_j.$$

Theorem: (i) The linear functionals $\tilde{\mathbf{e}}_j$ are linearly independent, and (ii) every linear functional on V can be written as a linear combination of $\{\tilde{\mathbf{e}}_j\}$, which forms a basis of $\tilde{V}$.

Then (i) $\tilde{V}$ has dimension n, and it is isomorphic to V. $\{\tilde{\mathbf{e}}^j\}$ is called the *dual basis* to $\{\hat{\mathbf{e}}_i\}$.

Let f be a linear functional $f \in \tilde{V}$ and $\mathbf{x} \in V$. The mapping $\mathbf{x} \to \langle f|Ax\rangle$ defines another linear functional on $\tilde{V}$, called f'. The mapping $f \to f'$ is a linear transformation on $\tilde{V}$, denoted by $A^\dagger$.

Definition: For every linear operator A on V, the *adjoint operator* $A^\dagger$ on $\tilde{V}$ is defined by

$$\langle A^\dagger f|x\rangle = \langle f|Ax\rangle.$$

We have

$$\begin{aligned}(A+B)^\dagger &= A^\dagger + B^\dagger,\\ (AB)^\dagger &= B^\dagger A^\dagger,\\ (\alpha A)^\dagger &= A^\dagger \alpha^*,\\ (A^\dagger)^\dagger &= A.\end{aligned}$$

(5) Inner (Scalar) Product and Inner Product Space

Definition (Scalar Product): Let V be a vector space. An *inner* (or *scalar*) *product* on V is defined to be a scalar-valued function of order pair of vectors, denoted by (x, y), such that (i) $(x, y) = (y, x)^*$; (ii) $(x, \alpha_1 y_1 + \alpha_2 y_2) = \alpha_1(x, y_1) + \alpha_2(x, y_2)$; (iii) $(x, x) \geq 0$; and (vi) $(x, x) = 0$.

A vector space endowed with an inner product is called an *inner product space*.

Definition: (i) The *length* (or *norm*) of a vector $\mathbf{x}$ in an inner product space V_n is defined by

$$|x| = (x, x)^{1/2},$$

(ii) The cosine of the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\cos\theta = \frac{(x, y)}{|x||y|}.$$

Theorem: (i) For fixed $\mathbf{x}$, the scalar product (x, y) is a linear functional on V, called f_x. Then $f_x(y) \equiv \langle f_x|y\rangle = (x, y)$;
(ii) Given any linear functional f on an inner product space V, there is always a vector $\mathbf{x}_f$, such that $\langle f|y\rangle = (x_f, y)$ for all $\mathbf{y} \in V$;
(iii) The one-to-one correspondence between $\mathbf{x} \in V$ and $f \in \tilde{V}$ established by (i) and (ii) is an isomorphism.

Then we can identify the two spaces and write

$$\langle f_x|y\rangle = (x_f, y) \equiv \langle x|y\rangle.$$

Definition: (i) Two vectors $\mathbf{x}, \mathbf{y} \in V$ are said to be *orthogonal* if $\langle x|y\rangle = 0$; (ii) A set of vectors $\{\mathbf{x}_i, i = 1, 2, ...\}$ are *orthonormal* if $\langle x^j|x_i\rangle = \delta^j{}_i$ for all i, j.

Theorem: Any set of n orthonormal vectors $\{\hat{\mathbf{u}}_i\}$ in $n-$dimensional vector space V_n forms an *orthonormal basis*, which has the following properties: (i) $|x\rangle = |u_i\rangle x^i, x^i = \langle u^i|x\rangle$; (ii) $\langle x|y\rangle = x_i^\dagger y^i = \langle x|u_i\rangle\langle u^i|y\rangle$; (iii) $|x|^2 = x_i^\dagger x^i = \sum_{i=1}^n |x^i|^2$.

Theorem: Let $E_i = |e_i\rangle\langle e^i|$ be the mapping $|x\rangle \to E_i|x\rangle = |e_i\rangle\langle e^i|x\rangle = |e_i\rangle x^i$. Then
(i) E_i, $i = 1, 2, ..., n$ are linear operators on V;
(ii) E_i are *projection operators*;
(iii) $\sum_{i=1}^n E_i = E$ is the identity operator (*completeness relation*).

Suppose $\{\hat{\mathbf{e}}_i\}$ and $\{\hat{\mathbf{u}}_i\}$ are two orthonormal bases on V_n.

Theorem: If $|u_k\rangle = |e_i\rangle S^i{}_k$ and $S^{\dagger k}{}_j = (S^j{}_k)^*$, then
(i) $S^\dagger l_i S^i{}_k = \delta^l{}_k$;

(ii) $S^i{}_k S_j^{\dagger k} = \delta^i{}_j$;
(iii) $|e_i\rangle = |u_k\rangle (S^\dagger)^k{}_i$, hence $S^\dagger = S^{-1}$, S is a unitary matrix.

Given an arbitrary basis in an inner product space, an orthonormal basis can be constructed by the *Gram-Schmidt process.*

(6) Linear Transformations (Operators) on Inner Product Spaces

Lemma: A linear operator A on an inner product space V is equal to zero if and only if $\langle x|Ax\rangle = 0$ for all $\mathbf{x} \in V$.

Definition: Given the operator A on V, its *adjoint* $A^\dagger$ on V is defined by

$$\langle x|A^\dagger y\rangle \equiv \langle Ax|y\rangle.$$

We have

$$\langle x|Ay\rangle = \langle Ay|x\rangle^* = \langle y|A^\dagger x\rangle^* = \langle A^\dagger x|y\rangle.$$

Since $\langle x|A|y\rangle \equiv \langle x|Ay\rangle = \langle A^\dagger x|y\rangle$, we can interpret that $A|y\rangle = |Ay\rangle$ and $\langle x|A = \langle A^\dagger x|$. Similarly, $A^\dagger|y\rangle = |A^\dagger y\rangle$ and $\langle x|A^\dagger = \langle Ax|$.

Suppose $\{\hat{\mathbf{e}}_i\}$ is an orthonormal basis and $A|e_i\rangle = |e_j\rangle A^j{}_i$, then

$$A^j{}_i = \langle e^j|A|e_i\rangle, \quad (A^\dagger)^k{}_l = \langle e^k|A^\dagger|e_l\rangle = \langle e^l|Ae_k\rangle^* = (A^l{}_k)^*.$$

The matrix corresponding to the adjoint operator $A^\dagger$ is the *hermitian conjugate* of the matrix corresponding to A.

Definition: If $A = A^\dagger$ on V, A is said to be *hermitian* or self adjoint.

Theorem: A linear transformation A on an inner product space V is hermitian if and only if $\langle x|A|x\rangle$ is real for all $\mathbf{x} \in V$.

Theorem: If A and B are hermitian operators, then (i) $A + B$ is hermitian; (ii) αA is hermitian if and only if α is real; (iii) AB is hermitian if and only if $AB = BA$.

Definition: An operator U on inner product space is said to be *unitary* if $UU^\dagger = U^\dagger U = E$.

Theorem: If U is a unitary operator on V, then (i) $\langle Ux|Uy\rangle = \langle x|y\rangle$; (ii) $|Ux| = |x|$.

Thus lengths of vectors and angles between vectors are variant when they undergo unitary transformations.

CHAPTER 2 BASIC GROUP THEORY

2.1 Basic Definitions and Simple Examples

Definition (A Group): A set $\{G : a, b, c...\}$ is said to form a *group* if there is an operation $\cdot$, called *group multiplication*, which associates any given (ordered) pair of elements $a, b \in G$ with a well-defined *product* $a \cdot b$ which is also an element of G, such that the following conditions are satisfied:
(i) The operation $\cdot$ is *associative*, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in G$;
(ii) Among the elements of G, there is an element e, called the *identity*, which has the property that $a \cdot e = a$ for all $a \in G$;
(iii) For each $a \in G$, there is an element $a^{-1} \in G$, called the *inverse* of a, which has the property that $a \cdot a^{-1} = e$.

e	a	b
a	b	e
b	e	a

Table 2.1: Group Multiplication Table of C_3

The rules of multiplication can be summarized in a *group multiplication table*. The *cyclic groups* C_n have the general structure $\{e, a, a^2, ..., a^{-1}; a^n = e\}$, where n is an positive integer.

Definition (Abelian Group): An *Abelian group* G is one for which the group multiplication is commutative, $ab = ba$ for all $a, b \in G$.

Definition (Order): The *order* of a group is the number of elements of the group if it is finite.

The cyclic groups C_n are all abelian groups. The simplest non-cyclic group is of order 4, called *four group* or the *dihedral group*, denoted by D_2.

e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

Table 2.2: Group Multiplication Table of D_2

2.2 Further Examples, Subgroups

The smallest *non-abelian group* is of order 6. A geometric configuration: (i) the identity transformation; (ii) reflections about the axes (1,1'), (2,2'), (3,3'); (iii) rotations around the center by angles $2\pi/3$ and $4\pi/3$. They form the *dihedral group* D_3. Denote the reflections by (12), (23), (31), and the rotations by (123), (321).

Definition (Subgroup): A subset H of a group G which forms a group under the same multiplication law as G is said to form a *subgroup* of G.

In some cases, the group elements carry labels which are continuous parameters. These are *continuous groups*. Refer to the groups of rotations in 2- and 3-dimensional space as R_2 and R_3; and the combined groups of rotations and translations in the same space as E_2 and E_3.

Any set of invertible $n \times n$ matrices forms a *matrix group*. For example:
(i) the *general linear group* $GL(n)$ consisting of all invertible $n \times n$ matrices;
(ii) the *unitary group* $U(n)$ consisting of all unitary matrices: $UU^\dagger = 1$;

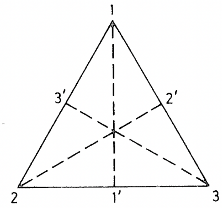

Figure 2.1: A configuration with D_3 symmetry

(iii) the *special unitary group* $\mathrm{SU}(n)$ consisting of unitary matrices with unit determinant;
(iv) the *orthogonal group* $\mathrm{O}(n)$ consisting of real orthogonal matrices: $OO^T = 1$.

2.3 The Rearrangement Lemma and the Symmetric (Permutation) Group

Rearrangement Lemma: If $p, b, c \in G$ and $pb = pc$, then $b = c$.

Consider a finite group of order n: $\{g_1, g_2, ..., g_n\}$. Then $\{hg_1, hg_2, ..., hg_n\} = \{g_{h_1}, g_{h_2}, ..., g_{h_n}\}$, where $(h_1, h_2, ..., h_n)$ is a permutation of the numbers $(1, 2, ..., n)$. This is a relation between a group element $h \in G$ and a permutation. An arbitrary permutation of n objects is denoted by

$$p = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ p_1 & p_2 & p_3 & \cdots & p_n \end{pmatrix},$$

where each entry in the first row is to be replaced by the corresponding one in the second row. The set of $n!$ permutations of n objects form a group S_n called the *permutation group* or the *symmetric group*. The identity element corresponds to no permutation:

$$p = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix},$$

and the inverse to p is

$$p^{-1} = \begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}.$$

Consider

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 2 & 6 \end{pmatrix}.$$

The *cycle notation* (134)(25)(6) uniquely specifies the permutation.

Definition (Isomorphism): Two groups G and G' are said to be *isomorphic* if there exists a one-to-one correspondence between their elements which preserves the law of group multiplication, i.e. if $g_i \in G \leftrightarrow g'_i \in G'$ and $g_1 g_2 = g_3$ in G, then $g'_1 g'_2 = g'_3$ in G' and vice versa. Denote by $G \simeq G'$.

Theorem (Cayley): Every group G of order n is isomorphic to a subgroup of S_n.

Theorem: If the order n of a group is a prime number, it must be isomorphic to C_n.

2.4 Classes and Invariant Subgroups

Definition (Conjugate Elements): An element $b \in G$ is said to be *conjugate* to $a \in G$ if there exists another group element $p \in G$ such that $b = pap^{-1}$, denoted by $b \sim a$.

Conjugation is an *equivalence relation*: (i) reflexive: each element is conjugate to itself $a \sim a$; (ii) symmetric: if $a \sim b$, then $b \sim a$; (iii) transitive if $a \sim b$ and $b \sim c$, then $a \sim c$.

Definition (Conjugate Class): Elements of a group which are conjugate to each other are said to form a (conjugate) class.

If H is a subgroup of G and $a \in G$, then $H' = \{aha^{-1}; h \in H\}$ also forms a subgroup of G. H' is said to be a *conjugate subgroup* to H. They have the same number of elements.

Definition (Invariant Subgroup): An *invariant subgroup* H of G is one which is identical to all its conjugate subgroups.

A subgroup H is invariant if and only if it contains elements of G in complete classes. All subgroups of an abelian group are invariant subgroups. Every group G has at least two trivial subgroups: $\{e\}$ and G itself.

Definition (Simple and Semi-simple Groups): A group is *simple* if it does not contain any non-trivial invariant subgroup. A group is *semi-simple* if it does not contain any abelian invariant subgroup.

2.5 Cosets and Factor (Quotient) Groups

Definition (Cosets): Let $H = \{h_1, h_2, ...\}$ be a subgroup of G and let p be an element of G (one which is not in H), then the set of elements $pH = \{ph_1, ph_2, ...\}$ is called a *left coset* and $Hp = \{h_1p, h_2p, ...\}$ is called a *right coset* of H.

Lemma: Two left cosets of a subgroup H either coincide completely, or else have no elements in common at all.

Given a subgroup H of order n_H, the distinct left cosets of H partition the elements of the full group of G into disjoint sets of n_H each.

Theorem (Lagrange): The order of a finite group must be an integer multiple of the order of any of its subgroups.

Theorem: If H is an invariant subgroup of G, the set of cosets endowed with the law of multiplication $pH \cdot qH = (pq)H$ form a group, called the *factor* (or *quotient*) *group* of G. The factor group is denoted by G/H, it is of order n_G/n_H.

2.6 Homomorphisms

Definition (homomorphism): A *homomorphism* from a group G to another group G' is a mapping (not necessarily one-to-one) which preserves group multiplication. If $g_i \in G \to g'_i \in G$ and $g_1g_2 = g_3$, then $g'_1g'_2 = g'_3$.

Theorem: Let f be a homomorphism from G to G'. Denote by K the set of all elements of G which are mapped to the identity element of G', i.e. $K = \{a \in G; a \xrightarrow{f} e' \in G'\}$. Then K forms an invariant subgroup of G. And the factor group G/K is isomorphic to G'.

2.7 Direct Products

Definition (Direct Product Group): Let H_1 and H_2 be subgroups of a group G with the following properties: (i) every element of H_1 commutes with any element of H_2, i.e. $h_1h_2 = h_2h_1$ for all $h_1 \in H$ and $h_2 \in H_2$; (ii) every element g of G can be written uniquely as $g = h_1h_2$ where $h_1 \in H_1$ and $h_2 \in H_2$. In this case, G is said to be the *direct product* of H_1 and H_2, i.e. $G = H_1 \otimes H_2$.

If $G = H_1 \otimes H_2$, then both H_1 and H_2 are invariant subgroups of G.

CHAPTER 3 GROUP REPRESENTATIONS

3.1 Representations

A set of invertible linear transformations, closed with respect to operator multiplication, satisfies the group axioms. Such a set forms a *group of linear transformations* or *group of operators.*

Definition (Representations of a Group): If there is a homomorphism from a group G to a group of operators $U(G)$ on a linear vector space V, we say that $U(G)$ forms a *representation* of the group G. The *dimension of the representation* is the dimension of the vector space V. A representation is said to be *faithful* if the homomorphism is also an isomorphism. A *degenerate representation* is one which is not faithful.

The representation is a mapping

$$g \in G \xrightarrow{U} U(g),$$

where $U(g)$ is an operator on V, such that

$$U(g_1)U(g_2) = U(g_1 g_2),$$

that is, the representation operators satisfy the same rules of multiplications as the original group elements.

The group of $n \times n$ matrices $D(G) = \{D(g), g \in G\}$ forms a *matrix representation* of G.

Theorem: (i) If the group is a non-trivial invariant subgroup H, then any representation of the factor group $K = G/H$ is also a representation of G. This representation must be degenerate; (ii) Conversely, if $U(G)$ is a degenerate representation of G, then G has at least one invariant subgroup such that $U(G)$ defines a faithful representation of the factor group G/H.

All representations of simple groups are trivial.

3.2 Irreducible, Inequivalent Representations

Suppose $U(G)$ is a representation of the group G on the vector space V, and S is any non-singular (i.e. invertible) operator on V. Then $U'(G) = SU(G)S^{-1}$ also form a representation of G on V, which is of the same dimension as the original one. The two sets of operators $U(G)$ and $U'(G)$ are said to be related by the similarity transformation S.

Definition (Equivalence of Representation): Two representations of a group G related by a similarity transformation are said to be *equivalent.*

Definition (Characters of a Representation): The *character* $\chi(g)$ of $g \in G$ in a representation $U(g)$ is defined to be $\chi(g) = \mathrm{Tr}\, U(g)$. If $D(G)$ is a matrix realization of $U(G)$, then

$$\chi(g) = \sum_i D(g)^i{}_i.$$

The group character is a function of the class-label only.

Let $U(G)$ be a representation of the group G on a vector space V_n. If for some choice of basis on V_n, the matrix representation of $U(G)$ appear in the form:

$$D(g) = \begin{pmatrix} D_1(g) & O \\ O & D_2(g) \end{pmatrix} \quad \text{for all } g \in G,$$

where $D_1(g)$ and $D_2(g)$ are square matrices. Then $D(G)$ is the direct sum of $D_1(G)$ and $D_2(G)$.

Definition (Invariant Subspace): Let $U(G)$ be a representation of G on the vector space V, and V_1 be a subspace of V with the property that $U(g)|x\rangle \in \mathrm{V}_1$ for all $\mathbf{x} \in \mathrm{V}_1$ and $g \in G$. V_1 is said to be an *invariant subspace* of V with respect to $U(G)$. An invariant subspace is *minimal* or *proper* if it does not contain any non-trivial invariant subspace with respect to $U(G)$.

Definition (Irreducible Representations): A representation $U(G)$ on V is *irreducible* if there is no non-trivial invariant subspace in V with respect to $U(G)$. Otherwise the representation is *reducible*. In the latter case, if the orthogonal complement of the invariant subspace is also invariant with respect to $U(G)$, then the representation is said to be *fully reducible* or *decomposable*.

If $U(G)$ is a representation of the group G on V and V^μ is an invariant subspace of V with respect to G, then by restricting the action of $U(G)$ to V^μ, we obtain a lower-dimension representation $U^\mu(G)$. If the subspace V^μ cannot be further reduced, $U^\mu(G)$ is an irreducible representation, and we say that V^μ is a *proper* or *irreducible invariant subspace* with respect to G.

3.3 Unitary Representations

Definition (Unitary Representation): If the group representation space is an inner product space, and if the operators $U(g)$ are unitary for all $g \in G$, then the representation $U(g)$ is said to be a *unitary representation.*

Theorem: If a unitary representation is reducible, then it is also decomposable (fully reducible).

Theorem: Every representation $D(G)$ of a finite group on an inner product space is equivalent to a unitary representation.

All reducible representations of finite groups are fully reducible. Let V_1 and V_2 be complementary invariant subspaces with respect to $U(G)$, and $U_1(G)$ and $U_2(G)$ denote the operators which coincide with $U(G)$ on these subspaces. Then $V = V_1 \oplus V_2$ and $U(g) = U_1(g) \oplus U_2(g)$.

Direct Sum Representation: Given the above situation, the representation $U(G)$ is said to be the *direct sum representation* of $U_1(G)$ (on V_1) and $U_2(G)$ (on V_2).

If either V_1 or V_2 is reducible with respect to G, then it can be further decomposed. The process can be repeated until $U(G)$ is fully reduce.

$$U(G) = \underbrace{U^1(G) \oplus \cdots \oplus U^1(G)}_{a_1} \oplus \underbrace{U^2(G) \oplus \cdots \oplus U^2(G)}_{a_2} \oplus \cdots = \sum_{\mu \oplus} a_\mu U^\mu(G),$$

where $U^\mu(G)$ denotes *inequivalent irreducible representations.* With proper choice of bases, $U(G)$ will appear in block-diagonal form with $U^\mu(G)$ appearing as diagonal blocks.

3.4 Schur's Lemmas

Schur's Lemma 1: Let $U(G)$ be an irreducible representation of a group G on the vector space V, and A be an arbitrary operator on V. If A commutes with all the operators $\{U(g), g \in G\}$, i.e. $AU(g) = U(g)A$, then A must be a multiple of the identity operator, $A = \lambda E$.

Theorem: Irreducible representations of any abelian group must be of dimension one.

Schur's Lemma 2: Let $U(G)$ and $U'(G)$ be two irreducible representations of G on the vector spaces V and V' respectively. Let A be a linear transformation from V' to V which satisfies $AU'(g) = U(g)A$ for all $g \in G$. It follows then either (i) $A = 0$ or (ii) V and V' are isomorphic and $U(G)$ is equivalent to $U'(G)$.

3.5 Orthonormality and Completeness Relations of Irreducible Representation Matrices

Consider the notations:
(1) n_G: order of the group G;
(2) μ, ν: labels of inequivalent, irreducible representations of G;
(3) n_μ: the dimension of the $\mu-$representation;
(4) $D^\mu(g)$: the matrix corresponding to $g \in G$ in the $\mu-$representation with respect to an orthonormal basis;
(5) χ_i^μ: character of class ζ_i elements in the $\mu-$representation;
(6) n_i: number of elements in class ζ_i;
(7) n_e: number of classes in the group G;

Theorem (Orthonormality of Irreducible Representation Matrices): The following *orthonormality condition* holds:

$$\frac{n_\mu}{n_G} \sum_g D_\mu^\dagger(g)^k{}_i D^\nu(g)^j{}_l = \delta_\mu^\nu \delta_i^j \delta_l^k,$$

where $D_\mu^\dagger(g)^k{}_i = [D^\mu(g)^i{}_k]^*$.