David J. Griffith's

Introduction to Electrodynamics

NOTES

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Contents

Chapter 2

Electrostatics

2.1 The Electric Field

Suppose we have some **source charge** $q_1, q_2, q_3, \cdots$, and we wonder what force they exert on the **test charge** Q. To solve this problem, we have the **principle of superposition**, which states that the interaction between any two charges is completely unaffected by the presence of others. To begin with, we consider **electrostatics**, where are source charges are stationary.

Coulomb's law

Suppose we have a point charge q, which is at rest and at a distance r away from the test charge Q. The force on Q is given by **Coulomb's law**:

$$\boxed{\boldsymbol{F} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{\mathfrak{r}^2}\mathfrak{r}},$$

where ϵ_0 is the **permittivity of free space** and $\mathfrak{r}$[1] is the separation vector from $\boldsymbol{r}'$ (the location of q) to $\boldsymbol{r}$ (the location of Q):

$$\mathfrak{r} = \boldsymbol{r} - \boldsymbol{r}'.$$

The force is repulsive if q and Q have the same sign, and attractive if their signs are opposite.

The Electric Field

Suppose there are several point charges $q_1, q_2, \cdots, q_n$ at distance $\mathfrak{r}_1, \mathfrak{r}_2, \cdots, \mathfrak{r}_n$ from Q, then the total force on Q is

$$\begin{aligned}\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots &= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathfrak{r}_1^2}\mathfrak{r}_1 + \frac{q_2 Q}{\mathfrak{r}_2^2}\mathfrak{r}_2 + \cdots\right) \\ &= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathfrak{r}_1^2}\mathfrak{r}_1 + \frac{q_2}{\mathfrak{r}_2^2}\mathfrak{r}_2 + \cdots\right).\end{aligned}$$

[1] The original symbol in the textbook is difficult to find. So here I use mathfrak instead.

This can also be written as

$$\boxed{\boldsymbol{F} = Q\boldsymbol{E},}$$

where

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathfrak{r}_i}\boldsymbol{\mathfrak{r}}_i.$$

$\boldsymbol{E}$ is called the **electric field** of the source charge and is a function of position $\boldsymbol{r}$, since the separation vectors $\boldsymbol{\mathfrak{r}}_i$ depend on the location of the **field point** P.

Continuous Charge Distributions

If the charge is distributed continuously over some regions, we have

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0}\int\frac{q_i}{\mathfrak{r}}\boldsymbol{\mathfrak{r}}\,dq.$$

And

$$dq \to \lambda\,dl \sim \sigma\,da \sim \rho\,d\tau.$$

2.2 Divergence and Curl of Electrostatic Fields

Field Lines, Flux, and Gauss's Law

Suppose we have some point charges. The **flux** through a surface $\mathcal{S}$ is

$$\Phi_E \equiv \int_{\mathcal{S}}\boldsymbol{E}\cdot d\boldsymbol{a}.$$

This is a measure of the "number of field lines" passing through $\mathcal{S}$. The flux through any closed surface is a measure of the total charge inside. A charge outside the surface contributes nothing to the total flux since its field lines pass in one side and out the other. This is the essence of **Gauss's law**.

In the case of a point charge q at the origin, the flux of $\boldsymbol{E}$ through a spherical surface of radius r is

$$\oint\boldsymbol{E}\cdot d\boldsymbol{a} = \int\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\boldsymbol{r}\right)\cdot(r^2\sin\theta\,d\theta\,d\phi\,\boldsymbol{r}) = \frac{1}{\epsilon_0}q.$$

The flux through any surface enclosing the charge is q/ϵ_0.

Now suppose we have many charges. And by the principal of superposition:

$$\boldsymbol{E} = \sum_{i=1}^{n}\boldsymbol{E}_i.$$

And thus the flux through a surfacing enclosing all of them is

$$\oint\boldsymbol{E}\cdot d\boldsymbol{a} = \sum_{i=1}^{n}\left(\oint\boldsymbol{E}_i\cdot d\boldsymbol{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\epsilon_0}q_i\right).$$

Then for any closed surface, we have

$$\boxed{\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0} Q_{encl}.}$$

Applying the divergence theorem,

$$\oint_{\mathcal{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = \int_{\mathcal{V}} (\nabla \cdot \boldsymbol{E}) d\tau.$$

Writing Q_{enc} in terms of the charged density ρ, we have

$$Q_{enc} = \int_{\mathcal{V}} \rho \, d\tau.$$

Then

$$\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{E}) d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0} \right) d\tau.$$

Hence

$$\boxed{\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} \rho.}$$

This is the **Gauss's law in differential form**[2].

The Curl of $\boldsymbol{E}$

Suppose we have a point charge at the origin:

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \boldsymbol{r}.$$

The line integral of this field from point $\boldsymbol{a}$ to $\boldsymbol{b}$ is

$$\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l}.$$

And $d\boldsymbol{l} = dr\, \boldsymbol{r} + r\, d\theta\, \boldsymbol{\theta} + r\, \sin\theta\, d\phi\, \boldsymbol{\phi}$, thus

$$\boldsymbol{E} \cdot d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dr.$$

Therefore,

$$\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0} \int_a^b \frac{q}{r^2} dr = -\frac{1}{4\pi\epsilon_0} \left. \frac{q}{r} \right|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right).$$

The integral around a closed path is 0:

$$\boxed{\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0.}$$

Applying the Stokes' theorem,

$$\boxed{\nabla \times \boldsymbol{E} = \boldsymbol{0}.}$$

[2] Also this shows the divergence of $\boldsymbol{E}$.

2.3 Electric Potential

Define

$$\boxed{V(\boldsymbol{r}) \equiv -\int_{\mathcal{O}}^{\boldsymbol{r}} \boldsymbol{E} \cdot d\boldsymbol{l}.}$$

$V(\boldsymbol{r})$ is called the **electric potential**. The **potential difference** between two points $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_{\mathcal{O}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l} + \int_{\mathcal{O}}^{\boldsymbol{a}} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l}.$$

The fundamental theorem for gradients is

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} (\nabla V) \cdot d\boldsymbol{l},$$

thus

$$\boxed{\boldsymbol{E} = -\nabla V.}$$

Comments on Potential

1. A surface over which potential is constant is called an **equipotential**.
2. Advantage of the potential formulation: if we know V, we can easily get $\boldsymbol{E}$ since $\boldsymbol{E} = -\nabla V$.
3. The choice of reference point $\mathcal{O}$ is arbitrary. Changing the reference point:

   $$V'(\boldsymbol{r}) = -\int_{\mathcal{O}'}^{\boldsymbol{r}} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{\mathcal{O}'}^{\mathcal{O}} \boldsymbol{E} \cdot d\boldsymbol{l} - \int_{\mathcal{O}}^{\boldsymbol{r}} \boldsymbol{E} \cdot d\boldsymbol{l} = K + V(\boldsymbol{r}).$$

   However, the potential difference between two points won't be affected:

   $$V'(\boldsymbol{b}) - V'(\boldsymbol{a}) = V(\boldsymbol{b}) - V(\boldsymbol{a}).$$

   And

   $$\nabla V' = \nabla V.$$

4. Potential obeys the superposition principle.
5. Unit of potential: volt.

Poisson's Equation and Laplace's Equation

We have $\nabla \cdot \boldsymbol{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V$, then we have

$$\boxed{\nabla^2 V = \frac{\rho}{\epsilon_0}.}$$

This is the **Poisson's equation**. In region where there is no charge, $\rho = 0$, Poisson's equation reduces to **Laplace's equation**:

$$\nabla^2 V = 0.$$

And

$$\nabla \times \boldsymbol{E} = \nabla \times (-\nabla V) = \boldsymbol{0}.$$

There's no condition on V since curl of gradient is always zero.

The Potential of a Localized Charge Distribution

Recall that

$$\boldsymbol{E} \cdot d\boldsymbol{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr.$$

Setting the reference point at infinity, the potential of a point charge q at the origin is

$$V(r) = -\int_{\mathcal{O}}^{\boldsymbol{r}} \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}dr' = \frac{1}{4\pi\epsilon_0}\left.\frac{q}{r'}\right|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}.$$

Generally, the potential of a point charge is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathfrak{r}},$$

where $\mathfrak{r}$ is the distance from q to $\boldsymbol{r}$. By the superposition principle, the potential of a collection of charges is

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathfrak{r}}.$$

For a continuous distribution

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{1}{\mathfrak{r}_i}dq.$$

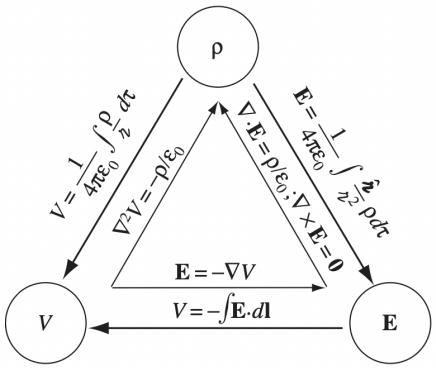
For a volume charge,

$$\boxed{V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}')}{\mathfrak{r}}d\tau'.}$$

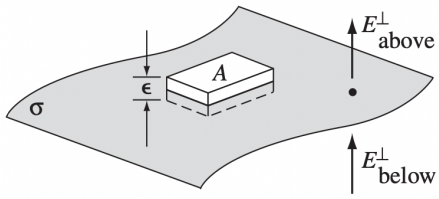
Note that the above formulae are derived based on $\mathcal{O} = \infty$.

Boundary Conditions

A summary on the relations between $\rho, \boldsymbol{E}$ and V is shown below.



Note that the electric field always undergoes a discontinuity when we cross a surface charge σ. Consider a Gaussian pillbox with A being the area of the



pillbox lid. By Gauss's law we have

$$\oint_S \boldsymbol{E}\cdot d\boldsymbol{a} = \frac{1}{\epsilon_0}Q_{encl} = \frac{1}{\epsilon_0}\sigma A.$$

In the limit as the thickness $\epsilon \to 0$, we have

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma.$$

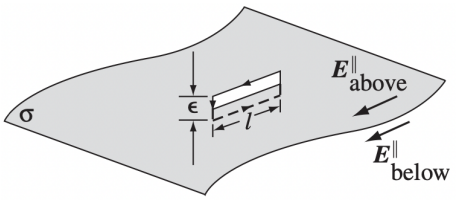
Thus the **normal** component of $\boldsymbol{E}$ is discontinuous by an amount σ/ϵ_0 at any boundary.

However, the **tangential** component of $\boldsymbol{E}$ is always continuous. Consider the rectangular loop, as $\epsilon \to 0$, since $\oint \boldsymbol{E}\cdot d\boldsymbol{l} = 0$, we have

$$\boldsymbol{E}^{||}_{\text{above}} = \boldsymbol{E}^{||}_{\text{below}}.$$

The boundary conditions on $\boldsymbol{E}$ can be then written as

$$\boxed{\boldsymbol{E}_{\text{above}} - \boldsymbol{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}},}$$



where $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to the surface, pointing from below to above.

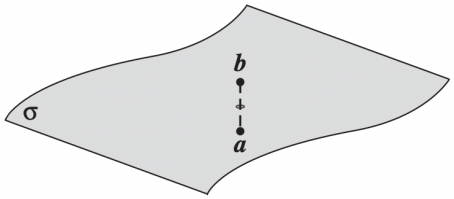
The potential is continuous across any boundary since

$$V_{\text{above}} - V_{\text{below}} = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l}.$$

As the path length shrinks to 0, thus

$$V_{\text{above}} = V_{\text{below}}.$$

The gradient of V inherits the discontinuity of E since $\boldsymbol{E} = -\nabla V$,



$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{\sigma}{\epsilon_0}\hat{\boldsymbol{n}}.$$

Or

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\epsilon_0},$$

where

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\boldsymbol{n}}$$

denotes the **normal derivative** of V, i.e. the rate of change in the direction perpendicular to the surface.

2.4 Work and Energy in Electrostatics

The Work to Move a Charge

Suppose we have a stationary configuration of source charges, and we want to move a test charge Q from point $\boldsymbol{a}$ to $\boldsymbol{b}$. The force we must exert is $-Q\boldsymbol{E}$. Then the work done is

$$W = \int_a^b \boldsymbol{F} \cdot d\boldsymbol{l} = -Q \int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = Q[V(\boldsymbol{b}) - V(\boldsymbol{a})].$$

Note that the work done is independent of time since the electrostatic force is conservative. And

$$\boxed{V(\boldsymbol{b}) - V(\boldsymbol{a}) = \frac{W}{Q}.}$$

The potential difference between $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to the work per unit charge required to carry a particle from $\boldsymbol{a}$ to $\boldsymbol{b}$. If we bring Q from ∞ to point $\boldsymbol{r}$,

$$W = Q[V(\boldsymbol{r}) - V(\infty)].$$

If we set the reference point at infinity, i.e. $V(\infty) = 0$,

$$W = QV(\boldsymbol{r}).$$

The Energy of a Point Charge Distribution

Now we want to find how much work it takes to assemble an entire collection of point charge. Firstly, we bring in q_1, this takes no work since there's no field yet. Then we bring in q_2, the work is

$$W_2 = q_2 V_1(\boldsymbol{r}_2) = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathfrak{r}_{12}} \right),$$

where $\mathfrak{r}_{12}$ is the distance between q_1 and q_2. Then we bring in q_3, the work is

$$W_3 = q_3[V_1(\boldsymbol{r}_3) + V_2(\boldsymbol{r}_3)] = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathfrak{r}_{13}} + \frac{q_2}{\mathfrak{r}_{23}} \right).$$

Then the total work done is

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{\mathfrak{r}_{ij}}.$$

Count each pair twice,

$$W = \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{\mathfrak{r}_{ij}}.$$

Pull out the factor q_i:

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i \left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathfrak{r}_{ij}} \right).$$

The term in parentheses is the potential at $\boldsymbol{r}_i$ due to all the other charges. Thus

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r}_i).}$$

The Energy of a Continuous Charge Distribution

For a volume charge density ρ, the work done is

$$W = \frac{1}{2}\int \rho V \, d\tau.$$

By Gauss's law, we have

$$\rho = \epsilon_0 \nabla \cdot \boldsymbol{E},$$

thus

$$W = \frac{\epsilon_0}{2}\int (\nabla \cdot \boldsymbol{E}) V \, d\tau.$$

Using integrating by parts[3],

$$W = \frac{\epsilon_0}{2}\left[-\int \boldsymbol{E}\cdot(\nabla V) d\tau + \oint V\boldsymbol{E}\cdot d\boldsymbol{a} \right].$$

Since $\nabla V = -\boldsymbol{E}$, we have

$$W = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 d\tau + \oint_{\mathcal{S}} V\boldsymbol{E}\cdot d\boldsymbol{a} \right).$$

$\mathcal{V}$ represents the volume encloses all the charge. If we integrate over all space, then the surface integral goes to zero[4]. Thus

$$\boxed{W = \frac{\epsilon_0}{2}\int E^2 d\tau.} \quad \text{(all space)}$$

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[3] We have

$$\nabla\cdot(f\boldsymbol{A}) = f(\nabla\cdot\boldsymbol{A}) + \boldsymbol{A}\cdot(\nabla f),$$

thus

$$\int \nabla\cdot(f\boldsymbol{A}) d\tau = \int (\nabla\cdot\boldsymbol{A}) d\tau + \int \boldsymbol{A}\cdot(\nabla f) d\tau = \oint f\boldsymbol{A}\cdot d\boldsymbol{a},$$

or

$$\int_{\mathcal{V}} (\nabla\cdot\boldsymbol{A}) d\tau = -\int_{\mathcal{V}} \boldsymbol{A}\cdot(\nabla f) d\tau + \oint_{\mathcal{S}} f\boldsymbol{A}\cdot d\boldsymbol{a}.$$

The key is to treat ∇ as d in the single-value differentiation.

[4] The author claimed that the surface integral act like $1/r$, so it vanishes as $r\to\infty$.

Comments on Electrostatic Energy

1. Now we have two equations about the work done. Equation 1 is

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\boldsymbol{r}_i).$$

   Equation 2 is

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau.$$

   Equation 1 does not take into account the work necessary to make the point charges in the first place. Equation 2 is more complete since it tells the total energy stored in a charge configuration, but 1 is more appropriate when we deal with point charges.

2. The energy is stored in the field with a density $\epsilon_0 E^2/2$.

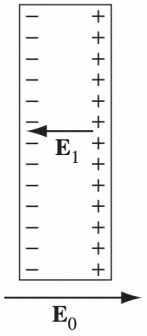
3. The energy of a compound system does not obey a superposition principle:

$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int (\boldsymbol{E}_1 + \boldsymbol{E}_2)^2 d\tau \\ &= \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2 + 2\boldsymbol{E}_1 \cdot \boldsymbol{E}_2) d\tau \\ &= W_1 + W_2 + \epsilon_0 \int \boldsymbol{E}_1 \cdot \boldsymbol{E}_2) d\tau. \end{aligned}$$

2.5 Conductors

In an **insulator**, each electron is on a short leash attached to a particular atom. And in a **conductor**, one or more electrons per atom are free to roam. The basic electrostatic properties of ideal conductors are

i. $\boldsymbol{E} = \boldsymbol{0}$ *inside a conductor.* When a conductor is put into an external



field $\boldsymbol{E}_0$, the field drives free positive charges to the right and the negative

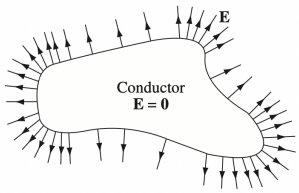
charges to the left. The **induced charges** produce a field $\boldsymbol{E}_1$, which is in the opposite direction to $\boldsymbol{E}_0$. Charge will continue to flow until $\boldsymbol{E}_1$ completely cancels out $\boldsymbol{E}_0$.

ii. $\rho = 0$ *inside a conductor.* By Gauss's law $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$, since $\boldsymbol{E} = 0$, then $\rho = 0$.

iii *Any net charge resides on the surface.*

iv *A conductor is an equipotential.* If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two points within (or on the surface of) a given conductor, $V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{E} \cdot d\boldsymbol{l} = 0$, and hence $V(\boldsymbol{a}) = V(\boldsymbol{b})$.

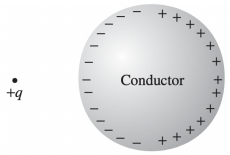
v. $\boldsymbol{E}$ *is perpendicular to the surface, just outside a conductor.* Otherwise the



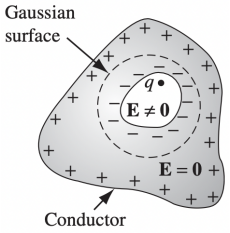
charge would flow around the surface until all the tangential components are killed off.

Induced Charges

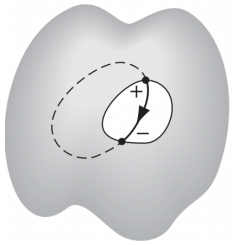
If we hold a charge $+q$ near an uncharged conductor, they will attract each other. q will pulls negative charges over to the near side and repel positive



charges to the far side. The negative induced charge is closer to q, there is a net force of attraction.



If there are some hollow cavity in the conductor, and we put some charge inside the cavity, the field in the cavity will not be zero. However, no external fields penetrate the conductor, since they are canceled at the outer surface by the induced charge there. The total charge induced on the cavity wall is equal and opposite to the charge inside, by Gauss's law. If the conductor as a whole is electrically neutral, there must be a charge $+q$ on its outer surface.



If a cavity is surrounded by conducting material is itself empty of charge, then the field within the cavity is zero. If the field line goes as shown, then $\oint \boldsymbol{E} \cdot d\boldsymbol{l}$ is positive, which is impossible. Thus $\boldsymbol{E} = \boldsymbol{0}$ inside an empty cavity, and there is no charge on the surface of the cavity.

Surface Charge and the Force on a Conductor

Capacitors

Chapter 3

Potential

3.1 Laplace's Equation