Herbert Goldstein's

Classical Mechanics

NOTES

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Contents

Chapter 1

Survey of Elementary Principle

1.1 Mechanics of a Particle

Suppose $\mathbf{r}$ is the radius vector of a particle from some given origin and $\boldsymbol{v}$ is the velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

The *linear momentum* $\mathbf{p}$ of the particle is

$$\mathbf{p} = m\mathbf{v}.$$

The mechanics of the particle is contained in *Newton's second law of motion*:

$$\boxed{\mathbf{F} = \frac{d\mathbf{p}}{dt} \equiv \dot{\mathbf{p}},}$$

or

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}).$$

In most instants, the mass is constant and then

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

where $\mathbf{a}$ is the acceleration of the particle defined by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}.$$

Conservation Theorem for the Linear Momentum of a Particle: If the total force $\mathbf{F}$ is zero, then $\dot{\mathbf{p}} = 0$ and the linear momentum $\mathbf{p}$ is conserved.

The angular momentum of the particle about point O, denoted by $\mathbf{L}$, is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

The *torque* (moment of force) about O is defined as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}.$$

We may find that

$$\boxed{\mathbf{N} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{L}}{dt} \equiv \dot{\mathbf{L}}.}$$

Conservation Theorem for the Angular Momentum of a Particle: If the total torque $\mathbf{N}$ is zero then $\dot{\mathbf{L}} = 0$ and the angular momentum $\mathbf{L}$ is conserved.

The work done by the external force $\mathbf{F}$ upon the particle in going from point 1 to point 2 is defined as

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}.$$

For constant mass, the equation reduces to

$$\int \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = \frac{m}{2} \int \frac{d}{dt}(v^2) dt,$$

and hence

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2).$$

The scalar quantity $mv^2/2$ is called the kinetic energy of the particle and is denoted by T, so the work done is equal to the change in the kinetic energy:

$$W_{12} = T_2 - T_1.$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force is said to be *conservative*. An alternative description of a conservative system is

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0.$$

Also, for a conservative force, W_{12} is independent of the physical path taken by the particle, and $\mathbf{F}$ is the gradient of a scalar function of position:

$$\mathbf{F} = -\nabla V(\mathbf{r}),$$

where V is called the *potential*. For a differential path length we have

$$\mathbf{F} \cdot d\mathbf{s} = -dV,$$

or

$$F_s = -\frac{\partial V}{\partial s}.$$

It's easy to find that *the zero level* of V is arbitrary.

For a conservative system, the work done by the forces is

$$W_{12} = V_1 - V_2.$$

Combining the kinetic energy we have

$$T_1 + V_1 = T_2 + V_2.$$

This symbols that

> **Energy Conservation Theorem for a Particle:** If the forces acting on a particle are conservative, then the total energy of the particle $T + V$ is conserved.

1.2 Mechanics of a System of Particles

The *external forces* are the forces that acts on the particles due to sources outside the system, and the *internal forces* are the ones acting on some particles i due to all other particles in the system. Thus the equation of motion for the ith particle is written as

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i,$$

where $\mathbf{F}_i^{(e)}$ stands for an external force and $\mathbf{F}_{ji}$ is the internal force on the ith particle due to the jth particle.

Sum up all the particles, we have

$$\frac{d^2}{dt^2}\sum_i m_i \mathbf{r}_i = \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i\neq j}} \mathbf{F}_{ji}.$$

The first term on RHS is the total external force $\mathbf{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\mathbf{F}_{ij} + \mathbf{F}_{ij}$ is zero. For LHS, we define $\mathbf{R}$ as the average of the radii vectors of the particles, weighted in portion to their mass:

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}.$$

$\mathbf{R}$ is known as the *center of mass* of the system. Thus

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}.$$

The total linear momentum of the system is

$$\mathbf{P} = \sum m_i \frac{d\mathbf{r}_i}{dt} = M\frac{d\mathbf{R}}{dt}.$$

Thus the equation of motion for the center of mass can be restated as

> **Conservation Theorem for the Linear Momentum of a System of Particles:** If the total external force is zero, the total linear momentum is conserved.

The total angular momentum of the system is formed by $\mathbf{r}_i \times \mathbf{p}_i$ and summing over i. Then

$$\dot{\mathbf{L}} = \sum_i \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i) = \sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i\neq j}} \mathbf{r}_i \times \mathbf{F}_{ji}.$$

The last term on the RHS can be written as

$$\mathbf{r}_{ij} \times \mathbf{F}_{ji}.$$

The cross-product terms vanish due to the action and reaction. Thus

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{(e)}.$$

This gives that

> **Conservation Theorem for the Total Angular Momentum:** $\mathbf{L}$ is constant in time if the applied (external) torque is zero.

Let $\mathbf{R}$ be the radius vector from O to the center of mass and let $\mathbf{r}_i'$ be the radius vector from the center of mass to the ith particle. Then we have

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R},$$

and then

$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{v},$$

where

$$\mathbf{v} = \frac{d\mathbf{R}}{dt}$$

is the velocity of the center of mass relative to O and

$$\mathbf{v}_i' = \frac{d\mathbf{r}_i'}{dt}$$

is the velocity of the ith particle relative to the center of mass of the system. The total angular momentum takes the form

$$\mathbf{L} = \sum_i \mathbf{R} \times m_i \mathbf{v} + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' + \left(\sum_i m_i \mathbf{r}_i'\right) \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} \sum_i m_i \mathbf{r}_i'.$$

The last two terms vanish, since they both contain the factor $\sum m_i \mathbf{r}_i'$. Thus the total angular momentum about O is

$$\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i'.$$

This says that the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass.

Now consider the energy equation. We calculate the work done by all forces in moving the system from an initial configuration 1 to a final configuration 2:

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s}_i = \sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i + \sum_{\substack{i,j \\ i \neq j}} \int_1^2 \mathbf{F}_{ji} \cdot d\mathbf{s}_i.$$

The equations of motion can be used to reduce the integrals to

$$\sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{s} = \sum_i \int_1^2 m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i dt = \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right).$$

The work done can be written as

$$W_{12} = T_2 - T_1,$$

where T is the total kinetic energy of the system

$$T = \frac{1}{2} \sum_i m_i v_i^2.$$

This could also be written as

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i (\mathbf{v} + \mathbf{v}_i') \cdot (\mathbf{v} + \mathbf{v}_i') \\ &= \frac{1}{2} \sum_i m_i v^2 + \frac{1}{2} \sum_i m_i {v'}_i^2 + \mathbf{v} \cdot \frac{d}{dt} \left(\sum_i m_i \mathbf{r}_i' \right). \end{aligned}$$

The last term vanishes, leaving

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_i m_i {v'}_i^2.$$

The kinetic energy consists of two parts: the kinetic energy obtained if all the mass were concentrated at the center of mass, plus the kinetic energy of motion about the center of mass.

Now we consider the RHS of W_{12}. In the special case that the external forces are derivable in terms of the gradient of a potential, then

$$\sum_i \int_1^2 \mathbf{F}_i^{(e)} \cdot d\mathbf{s}_i = -\sum_i \int_1^2 \nabla_i V_i \cdot d\mathbf{s}_i = -\sum_i V_i \Big|_1^2.$$

If the internal forces are also conservative, then the mutual forces between the ith and jth particles, $\mathbf{F_{ij}}$ and $\mathbf{F_{ji}}$, can be obtained from a potential function V_{ij}:

$$V_{ij} = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|).$$

The two forces are equal and opposite:

$$\mathbf{F}_{ji} = -\nabla_i V_{ij} = +\nabla_j V_{ij} = -\mathbf{F}_{ij},$$

and lie along the line joining the two particles:

$$V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = (\mathbf{r}_i - \mathbf{r}_j) f,$$

where f is some scalar function.

When the forces are all conservative, the second term of W_{12} can be written as a sum over pairs of particles, the terms for each pair being of the form

$$-\int_1^2 (\nabla_i V_{ij} \cdot d\mathbf{s}_i + \nabla_j V_{ij} \cdot d\mathbf{s}_j).$$

If the difference vector $\mathbf{r}_i - \mathbf{r}_j$ is denoted by $\mathbf{r}_{ij}$, and ∇_{ij} stands for the gradient with respect to $\mathbf{r}_{ij}$, then

$$\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij},$$

and

$$d\mathbf{s}_i - d\mathbf{s}_j = d\mathbf{r}_i - d\mathbf{r}_j = d\mathbf{r}_{ij}.$$

Then the term for the ij pair is

$$-\int \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij}.$$

The total work done by the internal forces reduces to

$$-\frac{1}{2} \sum_{i,j\; i\neq j} \int_1^2 \nabla_{ij} V_{ij} \cdot d\mathbf{r}_{ij} = -\frac{1}{2} \sum_{\substack{i,j \\ i\neq j}} V_{ij} \Big|_1^2.$$

The factor $1/2$ appears because we count each particles twice.

To be conclude, if the external and internal forces are both conservative, then the *total potential energy* V of the system is

$$V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i,j \\ i\neq j}} V_{ij},$$

and the total energy $T + V$ is conserved.

1.3 Constraints

All problems in mechanics can be reduced to solve the set of differential equations:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ji}.$$

If the condition of constraint can be expressed as equations connecting the coordinates of the particles having form

$$f(\mathbf{r}_1, \mathbf{r}_2, ..., t) = 0,$$

then the constraints are said to be *holonomic*. Constraints not expressible in this fashion are called nonholonomic.

Constraints are further classified according to whether the equations of constraint contain the time as an explicit variable (rheonomous) or are not explicitly dependent on time (scleronomous).

Constraints introduce two types of difficulties in the solution of mechanical problems. First, the coordinates r_i are no longer all independent. Second, the forces of constraint are not priori.

In the case of holonomic constraints, the first difficulty is solved by the introduction of *generalized coordinates*. In Cartesian coordinates, a system of N particles, free from constraints, has $3N$ independent coordinates or *degrees of freedom*. If there exist holonomic constraints, expressed in k equations, then we may use these equations to eliminate k of the $3N$ coordinates, and we are left with $3N-k$ independent coordinates, and the system is said to have $3N-k$ degrees of freedom:

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{r}_1(q_1, q_2, \cdots, q_{3N-k}, t), \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N(q_1, q_2, \cdots, q_{3N-k}, t),\end{aligned}$$

where $q_1, q_2, \cdots, q_{3N-k}$ are new independent variables.

1.4 D'Alembert's Principle and Lagrange's Equations

A *virtual displacement* (or infinitesimal displacement) of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta\mathbf{r}_i$, consistent with the forces and constraints imposed on the system at the given instant t.

Suppose the system is in equilibrium, i.e., the total force on each particle vanishes, $\mathbf{F}_i = 0$. Then $\mathbf{F}_i \cdot \delta\mathbf{r}_i$ vanishes as well. Thus

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0.$$

Decompose $\mathbf{F}_i$ into the applied force $\mathbf{F}_i^{(a)}$ and the force of constraint $\mathbf{f}_i$:

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i,$$

and then

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0.$$

We now restrict the systems for which the net virtual work of the forces of constraint is zero. This holds true for rigid bodies. Therefore we have

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0.$$

This is called the *principle of virtual work*.

The equation of motion writes

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \quad \Rightarrow \quad \mathbf{F}_i - \dot{\mathbf{p}}_i = 0,$$

which states that the particles in the system will be in equilibrium under a force equal to the actual force plus a "reserved effective force" $-\dot{\mathbf{p}}_i$. Then

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0.$$

And similarly,

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

Again we restrict the systems for which the virtual work of the forces of constraint vanishes and therefore

$$\boxed{\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0,}$$

which is the *D'Alembert's principle.* The translation from $\mathbf{r}_i$ to q_j is given by

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2, \cdots, q_n, t).$$

And $\mathbf{v}_i$ is expressed by

$$\mathbf{v}_i \equiv \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}.$$

Similarly, the arbitrary virtual displacement $\delta \mathbf{r}_i$ can be expressed by

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

In terms of the generalized coordinates, the virtual work of the $\mathbf{F}_i$ becomes

$$\begin{aligned}
\mathbf{F}_i \cdot \delta \mathbf{r}_i &= \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\
&= \sum_j Q_j \delta q_j,
\end{aligned}$$

where Q_j are called the components of the *generalized force*, defined as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Then we consider

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{i,j} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

Consider the relation

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right].$$

For the last term,

$$\frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}$$

1.5 Velocity-Dependent Potentials and the Dissipation Function

1.6 Simple Applications of the Lagrangian Formulation

Chapter 2

Variational Principle and Lagrange's Equations

2.1 Hamilton's Principle

Hamilton's Principle: The motion of the system from time t_1 to t_2 is such that the line integral (called the *action* or *action integral*)

$$I = \int_{t_1}^{t_2} L dt,$$

where $L = T - V$ has a stationary value for the actual path of the motion.

2.2 Some Techniques of Calculus of Variations

2.3 Derivation of Lagrange's Equations from Hamilton's Principle

2.4 Extending Hamilton's Principle to Systems with Constraints

2.5 Advantages of a Variational Principle Formulation

2.6 Conservation Theorems and Symmetry Properties

2.7 Energy Function and the Conservation of Energy

Chapter 3

The Central Force Problems