Fowles & Cassiday's

Analytical Mechanics

NOTES

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Contents

Chapter 5

Noninertial Reference Systems

5.1 Accelerated Coordinates Systems and Inertial Forces

Now suppose $Oxyz$ are the primary coordinates axes and $O'x'y'z'$ are the moving axes. In the case of pure translation, the respective axes are parallel.

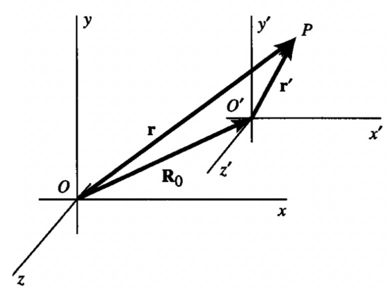

Figure 5.1: Two coordinates systems undergoing pure translation

We have

$$\boldsymbol{r} = \boldsymbol{R}_0 + \boldsymbol{r}'.$$

Taking derivatives we have

$$\boldsymbol{v} = \boldsymbol{V}_0 + \boldsymbol{v}', \quad \boldsymbol{a} = \boldsymbol{A}_0 + \boldsymbol{a}'.$$

If the moving system is accelerating (i.e. $\boldsymbol{A}_0 \neq 0$), Newton's second law becomes

$$\boldsymbol{F} = m\boldsymbol{A}_0 + m\boldsymbol{a}' \Rightarrow \boldsymbol{F} - m\boldsymbol{A}_0 = m\boldsymbol{a}'.$$

We may write

$$\boldsymbol{F}' = m\boldsymbol{a}',$$

where $\boldsymbol{F}' = \boldsymbol{F}' + (-m\boldsymbol{A}_0)$. That is, an acceleration $\boldsymbol{A}_0$ of the reference system can be taken into account by adding an **inertial term** $-m\boldsymbol{A}_0$ to the force $\boldsymbol{F}$ and equating the result to the product of mass and acceleration of the moving system. Inertial terms in the equations are called **inertial forces**, or **fictitious forces**.

5.2 Rotating Coordinate Systems

Suppose a primed coordinate system rotating with respect to an unprimed, fixed, inertial one. And the two systems have a common origin. The **angular velocity** of the rotating system is

$$\boldsymbol{\omega} = \omega \boldsymbol{n}.$$

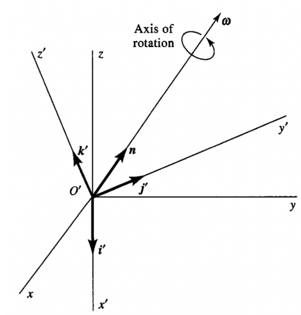

Figure 5.2: The angular velocity vector of a rotating coordinate system

Since the system have the same origin, we have

$$\boldsymbol{r} = \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z = \boldsymbol{r}' = \boldsymbol{i}'x' + \boldsymbol{j}'y' + \boldsymbol{k}'z'.$$

Note that the unit vectors in the rotating system are not constant, hence

$$\boldsymbol{i}\frac{dx}{dt} + \boldsymbol{j}\frac{dy}{dt} + \boldsymbol{k}\frac{dz}{dt} = \boldsymbol{i}'\frac{dx'}{dt} + \boldsymbol{j}'\frac{dy'}{dt} + \boldsymbol{k}'\frac{dz'}{dt} + x'\frac{d\boldsymbol{i}'}{dt} + y'\frac{d\boldsymbol{j}'}{dt} + z'\frac{d\boldsymbol{k}'}{dt}.$$

This can be written as

$$\boldsymbol{v} = \boldsymbol{v}' + x'\frac{d\boldsymbol{i}'}{dt} + y'\frac{d\boldsymbol{j}'}{dt} + z'\frac{d\boldsymbol{k}'}{dt}.$$

One may find that

$$\boxed{\frac{d\boldsymbol{i}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{i}', \quad \frac{d\boldsymbol{j}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{j}', \quad \frac{d\boldsymbol{k}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{k}'.}$$

Thus

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'.$$

More explicitly,

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{r}' = \left[\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\right]\boldsymbol{r}'.$$

This means that the operation of differentiating the position vector with respect to time in the fixed system is equivalent to the operation of taking the time derivative in the rotating system plus the operation $\boldsymbol{\omega}\times$. This apples to any vector $\boldsymbol{Q}$,

$$\boxed{\left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{Q}.}$$

If the vector is the velocity, we have

$$\boxed{\left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{v}.}$$

And $\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}'$,

$$\begin{aligned}
\left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{fixed}} &= \left(\frac{d}{dt}\right)_{\text{rot}}(\boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}') + \boldsymbol{\omega}\times(\boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}') \\
&= \left(\frac{d\boldsymbol{v}'}{dt}\right)_{\text{rot}} + \left[\frac{d(\boldsymbol{\omega}\times\boldsymbol{r}')}{dt}\right]_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') \\
&= \left(\frac{d\boldsymbol{v}'}{dt}\right)_{\text{rot}} + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rot}}\times\boldsymbol{r}' + \boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}'}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}').
\end{aligned}$$

Consider $\boldsymbol{\omega}$, we have

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{\omega} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rot}} = \dot{\boldsymbol{\omega}}.$$

Since $\boldsymbol{v}' = (d\boldsymbol{r}'/dt)_{\text{rot}}, \boldsymbol{a}' = (d\boldsymbol{v}'/dt)_{\text{rot}}$, we have

$$\boldsymbol{a} = \boldsymbol{a}' + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}' + 2\boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$$

giving the acceleration in the fixed system in terms of position, velocity and acceleration in the rotating system.

In the general case, the primed system is undergoing both translation and rotation, then

$$\boxed{\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega}\times\boldsymbol{r}' + \boldsymbol{V}_0,}$$

and

$$\boxed{\boldsymbol{a} = \boldsymbol{a}' + \dot{\boldsymbol{\omega}}\times\boldsymbol{r}' + 2\boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}') + \boldsymbol{A}_0.}$$

The term $2\boldsymbol{\omega}\times\boldsymbol{v}'$ is called **Coriolis acceleration** and the term $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$ is called the **centripetal acceleration**. The Coriolis acceleration appears whenever a particle moves in a rotating coordinate system and the centripetal acceleration is the result of particle being carried around a circular path in the rotating system. The centripetal acceleration is always directed toward the axis of rotation. The term $\dot{\boldsymbol{\omega}}\times\boldsymbol{r}'$ is called **transverse acceleration** since it is perpendicular to the position vector $\boldsymbol{r}'$.

5.3 Dynamics of a Particle in Rotating Coordinate System

The equation of motion of a particle in an inertial frame of reference is

$$\boldsymbol{F} = m\boldsymbol{a}.$$

Now we write the equation of motion in a noninertial frame of reference as

$$\boldsymbol{F} - m\boldsymbol{A}_0 - 2m\boldsymbol{\omega} \times \boldsymbol{v}' - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') = m\boldsymbol{a}'.$$

The equation represents the dynamical equation of motion in a noninertial frame of reference subjected to both real, physical forces and those inertial forces that appear as a result of the acceleration of the noninertial frame of reference. The **Coriolis force** is

$$\boxed{\boldsymbol{F}'_{Cor} = -2m\boldsymbol{\omega} \times \boldsymbol{v}'.}$$

The **transverse force** is

$$\boxed{\boldsymbol{F}'_{trans} = -m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}'.}$$

The **centrifugal force** is

$$\boxed{\boldsymbol{F}'_{centrif} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}').}$$

The remaining inertial force $-m\boldsymbol{A}_0$ appears whenever the (x', y', z') coordinate system is undergoing a translational acceleration.

A noninertial observer in the accelerated frame of reference writes the fundamental equation of motion as

$$\boldsymbol{F}' = m\boldsymbol{a}',$$

where

$$\boxed{\boldsymbol{F}' = \boldsymbol{F}_{physical} + \boldsymbol{F}'_{Cor} + \boldsymbol{F}'_{trans} + \boldsymbol{F}'_{centrif} - m\boldsymbol{A}_0.}$$

The Coriolis force is present only if a particle is moving in a rotating coordinate system. Its direction is perpendicular to the velocity vector of the particle in the moving system. In the northern hemisphere, the air would move toward the right. In the southern hemisphere the reverse is true.

The transverse force is present only if there is an angular acceleration of the rotating coordinate system. It is perpendicular to the radius vector $\boldsymbol{r}'$ in the rotating coordinate system.

The centrifugal force arises from the rotation about an axis. It is directed toward away from the axis of rotation and is perpendicular to the axis. In general if the angle between $\boldsymbol{\omega}$ and $\boldsymbol{r}'$ is θ, the magnitude of the centripetal force is $mr'\omega^2 \sin\theta$.

Chapter 6

Gravitation and Central Forces

Chapter 7

Dynamics of Systems of Particles

7.1 Center of Mass and Linear Momentum of a System

A system is called **rigid body** if the relative positions of all the particles in it are fixed. Suppose the system consists of n particles of masses $m_1, m_2, \cdots, m_n$ whose position vectors are respectively $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_n$. Define the **center of mass** of the system as the point whose position vector $\boldsymbol{r}_{cm}$ is given by

$$\boldsymbol{r}_{cm} = \frac{\sum_i m_i \boldsymbol{r}_i}{m}.$$

Define the **linear momentum** $\boldsymbol{p}$ of the system as

$$\boldsymbol{p} = \sum_i \boldsymbol{p}_i = \sum_i m_i \boldsymbol{v}_i.$$

Since $\dot{\boldsymbol{r}}_{cm} = \boldsymbol{v}_{cm}$, we have

$$\boldsymbol{p} = m\boldsymbol{v}_{cm}.$$

Now suppose there are external forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \cdots, \boldsymbol{F}_n$ acting on the respective particles. And we denote the internal forces by $\boldsymbol{F}_{ij}$, meaning the force exerted on particle i by j. Note that $\boldsymbol{F}_{ii} = 0$. The equation of motion of the particle i is

$$\boldsymbol{F}_i + \sum_{j=1}^{n} \boldsymbol{F}_{ij} = m_i \ddot{\boldsymbol{r}}_i = \dot{\boldsymbol{p}}_i.$$

For n particles, we have

$$\sum_{i=1}^{n} \boldsymbol{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{F}_{ij} = \sum_{i=1}^{n} \dot{\boldsymbol{p}}_i.$$

For every $\boldsymbol{F}_{ij}$ there's a $\boldsymbol{F}_{ji}$ such that $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$, from Newton's third law. Thus the double sum vanishes. And hence we can write:

$$\boxed{\sum_i \boldsymbol{F}_i = \dot{\boldsymbol{p}} = m\boldsymbol{a}_{cm}.}$$

When there's no external forces acting on a system, the linear momentum of the system remains constant:

$$\sum_i \boldsymbol{p}_i = \boldsymbol{p} = m\boldsymbol{v}_{cm} = \text{constant}.$$

This is the **principle of conservation of linear momentum**.

7.2 Angular Momentum and Kinetic Energy of a System

The **angular momentum $\boldsymbol{L}$** of a system of particles is defined as

$$\boldsymbol{L} = \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{v}_i).$$

Taking derivative we have

$$\frac{d\boldsymbol{L}}{dt} = \sum_{i=1}^{n} (\boldsymbol{v}_i \times m_i \boldsymbol{v}_i) + \sum_{i=1}^{n} (\boldsymbol{r}_i \times m_i \boldsymbol{a}_i).$$

The first term vanishes and hence

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \sum_{i=1}^{n} \left[\boldsymbol{r}_i \times \left(\boldsymbol{F}_i + \sum_{j=1}^{n} \boldsymbol{F}_{ij} \right) \right] \\ &= \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}. \end{aligned}$$

Note that

$$(\boldsymbol{r}_i \times \boldsymbol{F}_{ij}) + (\boldsymbol{r}_j \times \boldsymbol{F}_{ji}) = (\boldsymbol{r}_i \times \boldsymbol{F}_{ij}) - (\boldsymbol{r}_j \times \boldsymbol{F}_{ij}) = -\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ij}.$$

Thus the double sum vanishes. Denote the total external torque (or moment of force)

$$\boldsymbol{N} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_i,$$

then we have

$$\boxed{\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}}.$$

If a system is isolated, then $\boldsymbol{N} = 0$, and hence

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = \text{constant vector}.$$

This is the **principle of conservation of angular momentum**.

Chapter 8

Mechanics of Rigid Bodies: Planar Motion

8.1 Center of Mass of a Rigid Body

The center of mass of a system of particles is

$$x_{cm} = \frac{\sum_i x_i m_i}{\sum_i m_i}, \quad y_{cm} = \frac{\sum_i y_i m_i}{\sum_i m_i}, \quad z_{cm} = \frac{\sum_i z_i m_i}{\sum_i m_i}.$$

For a rigid extended body, replace the summation by an integration:

$$x_{cm} = \frac{\int_v \rho x \, dv}{\int_v \rho \, dv}, \quad y_{cm} = \frac{\int_v \rho y \, dv}{\int_v \rho \, dv}, \quad z_{cm} = \frac{\int_v \rho z \, dv}{\int_v \rho \, dv},$$

where ρ is the density and dv is the element of volume. If a rigid body is in the form of a thin shell, we have

$$x_{cm} = \frac{\int_s \rho x \, ds}{\int_s \rho \, ds}, \quad y_{cm} = \frac{\int_s \rho y \, ds}{\int_s \rho \, ds}, \quad z_{cm} = \frac{\int_s \rho z \, ds}{\int_s \rho \, ds},$$

where ρ is the mass per unit area. And if the body is in the form of a thin wire,

$$x_{cm} = \frac{\int_l \rho x \, dl}{\int_l \rho \, dl}, \quad y_{cm} = \frac{\int_l \rho y \, dl}{\int_l \rho \, dl}, \quad z_{cm} = \frac{\int_l \rho z \, dl}{\int_l \rho \, dl},$$

where ρ is the mass per unit length.

Symmetry Considerations

If the body has a plane of symmetry, then the center of mass lies on that plane. And if the body has a line of the symmetry, the center of mass lies on that line.

Type	z_{cm}
Hemispherical shell	$\frac{1}{2}a$
Solid hemisphere	$\frac{3}{8}a$
Semicircle	$\frac{2a}{\pi}$
Semicircular lamina	$\frac{4a}{3\pi}$

Table 8.1: Some common results of the center of mass

8.2 Rotation of a Rigid Body about a Fixed Axis: Moment of Inertia

Choose the z axis as the rotation axis. The path of a particle m_i at the point (x_i, y_i, z_i) is a circle of radius $r_i = (x_i^2 + y_i^2)^{1/2}$. Then the speed of the particle is

$$v_i = r_i\omega = (x_i^2 + y_i^2)^{1/2}\omega.$$

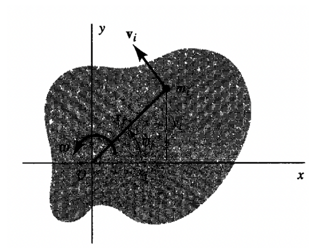

Figure 8.1: Cross-section of a rigid body rotating about the z-axis

The component of the velocity is

$$\begin{aligned}
\dot{x}_i &= -v_i \sin\phi_i = -\omega_i y, \\
\dot{y}_i &= v_i \cos\phi_i = \omega_i x, \\
\dot{z}_i &= 0.
\end{aligned}$$

This can also be written as

$$\boxed{\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i,}$$

where $\boldsymbol{\omega} = \boldsymbol{k}\omega$. The kinetic energy of rotation of the body is

$$T_{rot} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I_z \omega^2,$$

where

$$I_z = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2)$$

is called the **moment of inertia** about the z-axis. The angular momentum of single particle is $\boldsymbol{r}_i \times m_i\boldsymbol{v}_i$, the z-component is

$$m_i(x_i\dot{y}_i - y_i\dot{x}_i) = m_i(x_i^2 + y_i^2)\omega = m_i r_i^2\omega.$$

Then we have

$$L_z = \sum_i m_i r_i^2\omega = I_z\omega.$$

And

$$N_z = \frac{dL_z}{dt} = \frac{d(I_z\omega)}{dt},$$

where N_z is the total moment of all the applied forces about the axis of rotation. If the body is rigid, then I_z is constant, and hence

$$N_z = I_z\frac{d\omega}{dt}.$$

To conclude,

Translation along x-axis	**Rotation about the z-axis**
Linear momentum $p_x = mv_x$	Angular momentum $L_z = I_z\omega$
Force $F_x = m\dot{v}_x$	Torque $N_z = I_z\dot{\omega}$
Kinetic energy $T = \frac{1}{2}mv^2$	Kinetic energy $T_{rot} = \frac{1}{2}I_z\omega^2$

8.3 Calculation of the Moment of Inertia

For any axis,

$$I = \int r^2\,dm.$$

Type	$\boldsymbol{I_z}$
A thin rod of length a, axis at one end	$\frac{1}{3}ma^2$
A thin rod of length a, axis at the center	$\frac{1}{12}ma^2$
A thin circular hoop with radius a, axis at the center	ma^2
Circular disc or cylinder with radius a, axis at center	$\frac{1}{2}ma^2$
Sphere with radius a, axis across the center	$\frac{2}{5}ma^2$
Spherical shell with radius a, axis across the center	$\frac{2}{3}ma^2$

Table 8.2: Some common results of the center of mass

Perpendicular-Axis Theorem for a Plane Lamina

Consider a rigid body in the form of a plane lamina of any shape. Place it in the xy-plane. Then the moment of inertia about the z-axis is given by

$$I_z = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2.$$

This can also be written as

$$\boxed{I_z = I_x + I_y.}$$

This is the **perpendicular-axis theorem**. The moment of inertia of any plane lamina about an axis normal to the plane of the lamina is equal to the sum of the moments of inertia about any two mutually perpendicular axes passing through the given axis and lying in the plane of the lamina.

Parallel-Axis Theorem for Any Rigid Body

The moment of inertia about the z-axis is

$$I_z = \sum_i m_i(x_i^2 + y_i^2).$$

Write x_i and y_i in terms of the coordinates of the center of mass and the coordinates relative to the center of mass:

$$x_i = x_{cm} + \bar{x}_i, \quad y_i = y_{cm} + \bar{y}_i.$$

Thus we have

$$I_z = \sum_i m_i(\bar{x}_i^2 + \bar{y}_i^2) + \sum_i m_i(x_{cm}^2 + y_{cm}^2) + 2x_{cm}\sum_i m_i\bar{x}_i + 2y_{cm}\sum_i m_i\bar{y}_i.$$

The first sum is the moment of inertia about an axis parallel to the z-axis and passing through the center of mass, denoted by I_{cm}. Let $l^2 = x_{cm}^2 + y_{cm}^2$. And from the definition of center of mass,

$$\sum_i m_i\bar{x}_i = \sum_i m_i\bar{y}_i = 0.$$

Thus the final result is

$$\boxed{I = I_{cm} + ml^2.}$$

This is the **parallel-axis theorem**. The moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axis passing through the center of mass plus the product of the mass of the body and the square of the distance between the two axes.

8.4 The Physical Pendulum

8.5 The Angular Momentum of a Rigid Body in Laminar Motion

Laminar motion takes place when all the particles making up a rigid body move parallel to some fixed plane. The rigid body undergoes both translational and rotational acceleration. We have

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N},$$

or

$$\frac{d}{dt}\sum_i \boldsymbol{r}_i \times m_i\boldsymbol{v}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i.$$

Consider a system of particles rotating about an axis whose direction is fixed

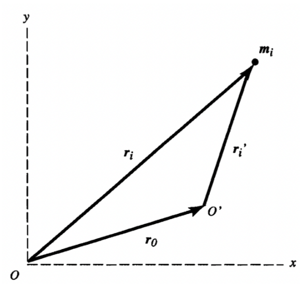

Figure 8.2: A particle in a rigid body in a laminar motion

in space (it might be accelerating). Let O represents the origin of an inertial frame of reference and O' the origin of the axis. The total torque about O' is

$$\boldsymbol{N}' = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i.$$

And we have

$$\boldsymbol{r}_i = \boldsymbol{r}_0 + \boldsymbol{r}_i' \quad \Rightarrow \quad \boldsymbol{v}_i = \boldsymbol{v}_0 + \boldsymbol{v}_i'.$$

In the inertial frame of reference we have

$$\boldsymbol{F}_i = \frac{d}{dt}(m\boldsymbol{v}_i).$$

Then

$$\begin{aligned}
\boldsymbol{N}' = \sum_i \boldsymbol{r}_i' \times \boldsymbol{F}_i &= \sum_i \boldsymbol{r}_i' \times \frac{d}{dt}(v_i = \boldsymbol{v}_0 + \boldsymbol{v}_i') \\
&= -\dot{\boldsymbol{v}}_0 \times \sum_i m_i\boldsymbol{r}_i' + \sum_i \boldsymbol{r}_i' \times \frac{d}{dt} m_i\boldsymbol{v}_i' \\
&= -\dot{\boldsymbol{v}}_0 \times \sum_i m_i\boldsymbol{r}_i' + \frac{d}{dt}\sum_i \boldsymbol{r}_i' \times m_i\boldsymbol{v}_i'.
\end{aligned}$$

The last term is the rate of change of the angular momentum $\boldsymbol{L}'$ about the O' axis. Thus

$$\boldsymbol{N}' = -\ddot{\boldsymbol{r}}_0 \times \sum_i m_i \boldsymbol{r}_i' + \frac{d}{dt}\boldsymbol{L}'.$$

The first term vanishes when

1. The acceleration $\ddot{\boldsymbol{r}}_0$ of the axis of rotation O' vanishes.

2. O' is the center of mass of the system of particles.

3. The O' axis passes through the point of contact between the cylinder and the plane. The vector $\sum_i m_i \boldsymbol{r}_i'$ passes through the center of the mass. If $\ddot{\boldsymbol{r}}_0$ also passes through the center of mass, then the cross product vanishes.

Summing torques about an axis passing through the center of mass, we have

$$\boldsymbol{N}_{cm} = \frac{d}{dt}\boldsymbol{L}_{cm} = I_{cm}\dot{\boldsymbol{\omega}}$$

Chapter 9

Motion of Rigid Bodies in Three Dimensions

9.1 Rotation of a Rigid Body about an Arbitrary Axis: Moments and Products of Inertia – Angular Momentum and Kinetic Energy

Consider a rigid body rotating about an arbitrary axis. And the axis passes through a fixed point O, taken as the origin of the coordinate system. By the

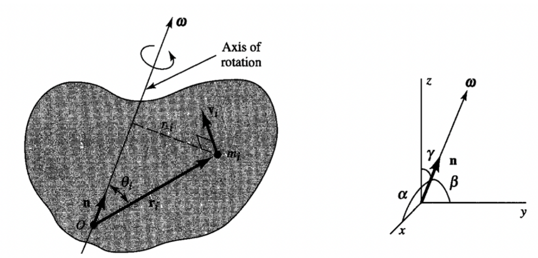

Figure 9.1: Angular velocity vector $\boldsymbol{\omega} = \omega \boldsymbol{n}$

definition of moment of inertia, we have

$$I = \sum_i m_i r_{\perp i}^2,$$

where $r_{\perp i}$ is the perpendicular distance from the particle of m_i to the axis of rotation.

Angular Momentum Vector