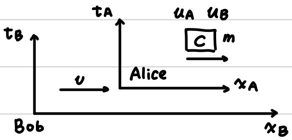
PHYS 2022 Modern Physics
Part 1. Special Relativity



1 Principles of Special Relativity

1.1 Galileo's Principle of Relativity (R)

Laws of nature take the same form in all inertial frames

$$t_B = t_A.\ x_B = x_A + v t_A$$

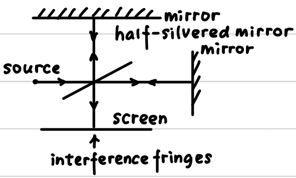
In Alice's frame. $$F = m a_A.$$

In Bob's frame. $$F = m a_A = m\ddot{x}_A = m\ddot{x}_B = m a_B.$$

$$u_A = \dot{x}_A.\ u_B = \dot{x}_B = u_A + v.$$

(R) is respected by Newtonian mechanics.

1.2 The speed of light



From Maxwell equations, the velocity addition rule is inconsistent with the observer-independent speed of light. The Michelson-Morley interferometer experiment proved that.

1.3 Einstein's Relativity

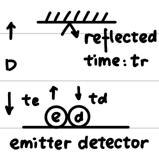
The postulates of special relativity

(R) Laws of nature take the same form in all inertial frames

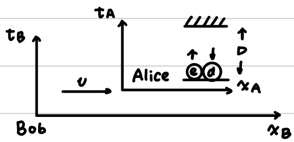
(C) The vaccum speed of light is c in all inertia frames

The postulate of observer-independent events

(E) Event: happen to a particular small object at a particular moment (localized at a spacetime point). The occurrence of an event is observer-independent.



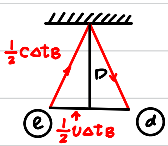
2 Time Dilation



Construction of a light clock

$$\Delta t \equiv t_d - t_e = \frac{2D}{c}.$$

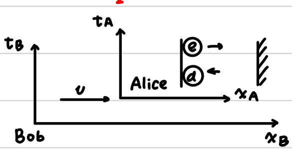
In Alice's view, $$\Delta t_A = \frac{2D}{c}.$$



In Bob's view, $$\left(\tfrac{1}{2}c\Delta t_B\right)^2 = \left(\tfrac{1}{2}v\Delta t_B\right)^2 + D^2$$

$$\Rightarrow \boxed{\Delta t_B = \frac{2D}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t_A}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma \Delta t_A,\ \gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} > 1}$$

4 Length Contraction

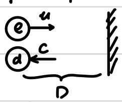


Construct a light ruler $D = \frac{1}{2}c\Delta t.$

In Alice's view. $D_A = \frac{1}{2}c\Delta t_A.$

Velocity addition

1. Define a speed meter



$\Delta t = D(\frac{1}{u}+\frac{1}{c})$.

2. Load it in a moving car

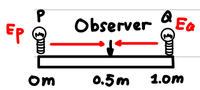
For Alice: $\Delta t_A = D_A(\frac{1}{u_A}+\frac{1}{c})$

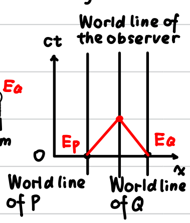
For Bob: $\Delta t_B = \Delta t_{B1}+\Delta t_{B2}$

$\Delta t_{B1} = \frac{D_B}{u_B - v}$, $\Delta t_{B2} = \frac{D_B}{c+v}$

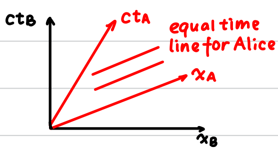
$$\Rightarrow u_B = \frac{u_A + v}{1+\frac{u_A v}{c^2}}.$$

The meaning of simultaneity wrt an observer

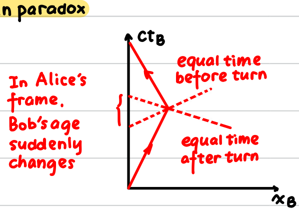




Equal time slices



Twin paradox



In Bob's view, $c\Delta t_{B_1} = D_B + v\Delta t_{B1}$

$c\Delta t_{B2} = D_B - v\Delta t_{B2}$

Thus $\Delta t_B = \Delta t_{B1}+\Delta t_{B2} = \frac{D_B}{c-v}+\frac{D_B}{c+v} = \frac{2}{c}\frac{D_B}{1-\frac{v^2}{c^2}} = \frac{2}{c}\frac{D_B}{\gamma^2}$.

Since $\Delta t_B = \gamma\Delta t_A$, $\boxed{D_B = \frac{c}{2}\frac{\Delta t_B}{\gamma^2} = \frac{c}{2}\frac{\Delta t_A}{\gamma} = \frac{D_A}{\gamma} = \frac{D_A}{\sqrt{1-\frac{v^2}{c^2}}}}$

5 Meaning of the "Same Time" (Simultaneity)

5.1 Simultaneity Depends on Which Observer

Spacetime diagram: 1. An event is a point

2. Light travels 45° line

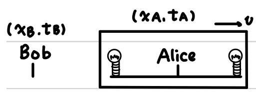
3. An object (or observer) is a line (called world line) with |slope| > 45° everywhere $\left(\frac{\Delta(ct)}{\Delta x} = \frac{c}{v} > 1 = \tan 45°\right)$.

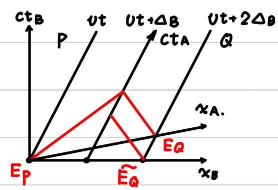
4. An intertial observer is a straight line

5. A static object is a line parallel to the ct axis

6. Events at the same time are parallel to the x-axis.

Simultaneity is a relative concept





Bob's spacetime diagram

5.2 Causality and Types of Separations

Time order associated with cause and effect

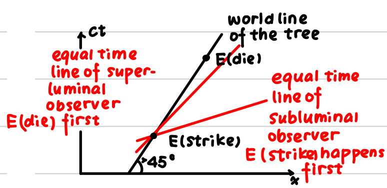
Event E(strike): Lightning strikes on the tree

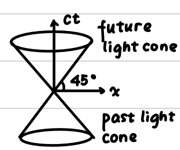
Event E(die): The tree dies

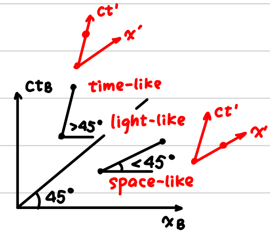
Causality

Special relativity without superluminal motion is consistent with causality

Causal structure of spacetime: past and future light cones







No perfect rigid body in relatively

Space-like, null and time like intervals

Space-like: exists observer wrt whom 2 space-like separated happen at the same time. The events are pure space separated wrt this observer. The time order of the events can be flipped for different observers.

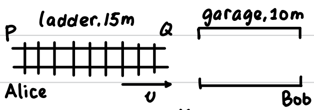
$ds^2 > 0$

Time-like: exisits observer wrt whom two time-like events happen at the same position. The events thus separated only in time wrt this observer. The time order of the events is absolute and has to be agreed on for all observers. Causally-related events are separated by time-like intervals

$ds^2 < 0$

Light-like (null): Light travels with light-like lines.

6 Example: The Ladder Paradox

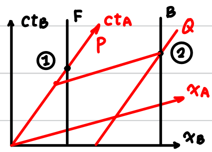


$v = 0.87c$, i.e. $\gamma = 2$

Alice: garage moves, length is 10m/2 = 5m. ladder doesn't fit in the garage

Bob: ladder moves, length is 15m/2 = 7.5m. ladder fits in the garage.

Two events: 1. P enters front door F 2. Q exists back door B



For Alice: ② earlier than ①, doesn't fit

For Bob: ① earlier than ②, fit

Both are right

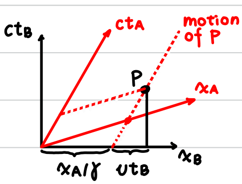
7 The Lorentz Transformation

Space Transformation of a boost

Suppose Alice holds a ruler with length x_A. Wrt Bob, the length is $\frac{x_A}{\gamma}$

$t_B = 0$, $x_B = \frac{x_A}{\gamma}$.

Then $x_B = \frac{x_A}{\gamma} + vt_B \Rightarrow x_A = \gamma(x_B - vt_B)$.



Following (R), we get $x_B = \gamma(x_A + vt_A)$.

Time transformation of a boost

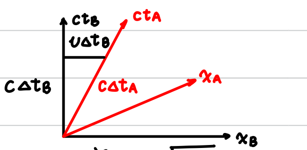
$x_B = \gamma[\gamma(x_B - vt_B) + vt_A] \Rightarrow t_A = \gamma(t_B - \frac{v}{c^2}x_B)$, $t_B = \gamma(t_A + \frac{v}{c^2}x_A)$

Summary: the Lorentz Transformation

$$\boxed{\begin{cases} ct_A = \gamma(ct_B - \beta x_B) \\ x_A = \gamma(x_B - \beta ct_B) \\ y_A = y_B \\ z_A = z_B \end{cases} \quad \begin{cases} ct_B = \gamma(ct_A + \beta x_A) \\ x_B = \gamma(x_A + \beta ct_A) \\ y_B = y_A \\ z_B = z_A. \end{cases} \quad (\beta \equiv \frac{v}{c}).}$$

$$\begin{bmatrix} ct_A \\ x_A \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} ct_B \\ x_B \end{bmatrix}$$

Rotation $\begin{bmatrix} x_A \\ y_A \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_B \\ y_B \end{bmatrix}$

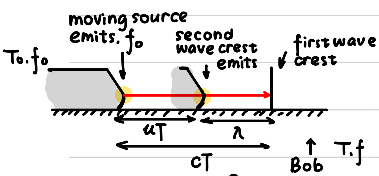


$$\Delta t_B = \gamma \Delta t_A \Rightarrow \sqrt{1-\frac{v^2}{c^2}}\,\Delta t_B = \Delta t_A$$

$$\Rightarrow (c\Delta t_B)^2 - (v\Delta t_B)^2 = (c\Delta t_A)^2$$

$$\Rightarrow (ic\Delta t_B)^2 + (v\Delta t_B)^2 = (ic\Delta t_A)^2$$

Doppler Effect for Electromagnetic Waves



$\lambda = (c-u)T$, $f = \frac{c}{(c-u)T}$.

$$T = \frac{T_0}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{cT_0}{\sqrt{c^2-u^2}}.$$

$$\frac{1}{T} = \frac{\sqrt{c^2-u^2}}{c}\cdot\frac{1}{T_0} = \frac{\sqrt{c^2-u^2}}{c} f_0$$

$$f = \frac{c}{c-u}\,\frac{\sqrt{c^2-u^2}}{c} f_0 = \sqrt{\frac{c+u}{c-u}}\, f_0.$$

$$\vec{v}_A = (v_{Ax}, v_{Ay}, v_{Az}) = \left(\frac{dx_A}{dt_A}, \frac{dy_A}{dt_A}, \frac{dz_A}{dt_A}\right).$$

$$\vec{v}_B = (v_{Bx}, v_{By}, v_{Bz}) = \left(\frac{dx_B}{dt_B}, \frac{dy_B}{dt_B}, \frac{dz_B}{dt_B}\right).$$

$$v_{Bx} = \frac{dx_B}{dt_B} = \frac{\gamma\, d(x_A + v t_A)}{\gamma\, d\left(t_A + \frac{x_A v}{c^2}\right)} = \frac{dx_A + v\,dt_A}{dt_A + \frac{v}{c^2}dx_A} = \frac{v_{Ax}+v}{1+\frac{v_{Ax}v}{c^2}}$$

$$v_{By} = \frac{dy_B}{dt_B} = \frac{dy_A}{\gamma\, d\left(t_A + \frac{v}{c^2}x_A\right)} = \frac{v_{Ay}}{\gamma\left(1+\frac{v_{Ax}v}{c^2}\right)}$$

$$v_{Bz} = \frac{v_{Az}}{\gamma\left(1+\frac{v_{Ax}v}{c^2}\right)}.$$

8 The Geometry of Spacetime

rapidity

Rewrite the Lorentz transformation using $\phi = \operatorname{arctanh}\beta$

$$\beta = \tanh\phi = \frac{v}{c} = \frac{\sinh\phi}{\cosh\phi} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{\cosh^2\phi} \Rightarrow \cosh\phi = \gamma.$$

$\Rightarrow \sinh\phi = \beta\gamma$. Then

$$\begin{cases} ct_B = ct_A\cosh\phi + x_A\sinh\phi \\ x_B = ct_A\sinh\phi + x_A\cosh\phi \end{cases}$$

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh(i\theta) = \cos\theta$$

$$\sinh(i\theta) = i\sin\theta.$$

Define $\phi = i\theta$, $ct = iw$ — rotation of the w-x plane.

$$\begin{cases} iw_B = iw_A\cos\theta + x_A(i\sin\theta) \\ x_B = iw_A(i\sin\theta) + x_A\cos\theta \end{cases} \Rightarrow \begin{cases} w_B = w_A\cos\theta + x_A\sin\theta \\ x_B = -w_A\sin\theta + x_A\cos\theta. \end{cases}$$

In Euclidean space $ds^2 = dw^2 + dx^2 + dy^2 + dz^2 = -d(ct)^2 + dx^2 + dy^2 + dz^2$.

ds^2 is an invariant quantity under Lorentz transformation

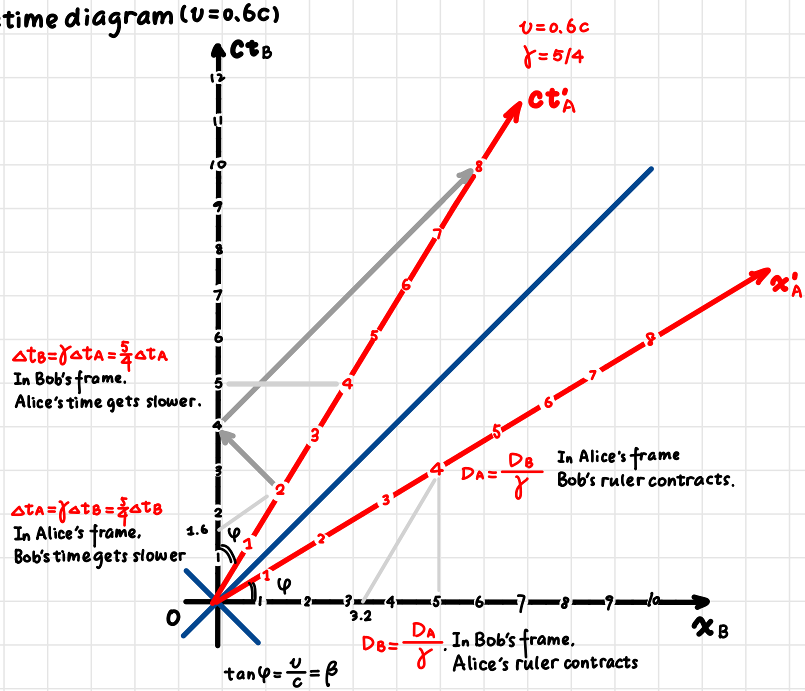
$x^\mu = (ct, x, y, z)$ a 4-vector in Minkowski space.

$ds^2 < 0 \Leftrightarrow$ time-like, $ds^2 > 0 \Leftrightarrow$ space-like, $ds^2 = 0 \Leftrightarrow$ light-like, null

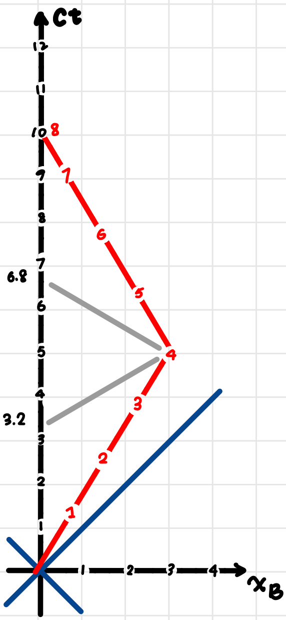
for an observer $dx^2 + dy^2 + dz^2 = 0$

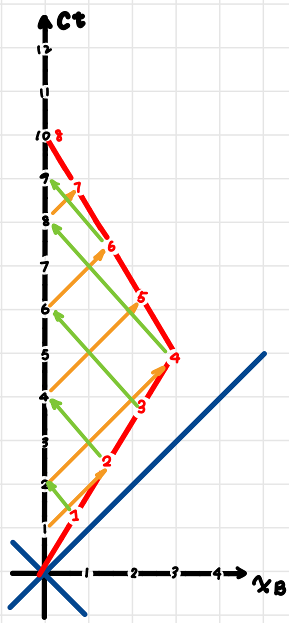
for an observer $d(ct)^2 = 0$

The Spacetime diagram ($v=0.6c$)



The twin paradox ($v=0.6c$)





$1eV = 1.602\times10^{-19}J.$

9 Relativistic Momentum and Energy

Relativistic Momentum

Define proper time

$$d\tau = \sqrt{-\frac{ds^2}{c^2}} = \sqrt{dt^2 - \frac{dx^2}{c^2}} = \frac{dt}{\gamma}.$$

Proper time is the time measured by the moving object itself.

Momentum $\vec{p} = m\frac{d\vec{x}}{d\tau} = \gamma m\vec{v}.$

Force $\vec{F} \equiv \frac{d\vec{p}}{dt} = \frac{d(\gamma m\vec{v})}{dt} = \gamma m\dot{\vec{v}} + \gamma^3 m\vec{v}\frac{\vec{v}\cdot\dot{\vec{v}}}{c^2}$

As $v\to c$, $F\to\infty \Rightarrow c$ is the speed limit

$$\int_0^v v\,d(\gamma m v) = \gamma m v^2 - \int_0^v \gamma m v\,dv$$

$$\int_0^v \gamma v\,dv = \int_0^v \frac{v}{\sqrt{1-\frac{v^2}{c^2}}}dv \xlongequal[du=-\frac{2}{c^2}v\,dv]{u=1-\frac{v^2}{c^2}}$$

$$\int_0^u \frac{-\frac{c^2}{2}du}{\sqrt{u}} = -c^2\sqrt{u}\Big|_0^u = -c^2\sqrt{1-\frac{v^2}{c^2}}\Big|_0^v$$

$$= -c^2\sqrt{1-\frac{v^2}{c^2}} + c^2 = \frac{v^2-c^2}{\sqrt{1-\frac{v^2}{c^2}}} + c^2 = c^2 - \gamma c^2 + \gamma c^2$$

$$\int_0^v v\,d(\gamma m v) = \gamma m v^2 - mc^2 + \gamma mc^2 - \gamma m v^2$$

$$= (\gamma-1)mc^2$$

Relativistic Energy

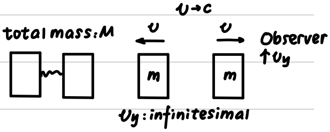
$$K = \int F dx = \int \frac{d(\gamma m v)}{dt}dx = \int v\,d(\gamma m v) = (\gamma-1)mc^2$$

Status: independent from process

Taylor expansion at $v=0$: $\gamma = (1-\frac{v^2}{c^2})^{-\frac{1}{2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots$

In Newtonian mechanics, $K = \frac{1}{2}mv^2$

$v\to c$



Before: $P_y = Mv_y$ $\Rightarrow M = 2\gamma m$

After: $P_y = 2\gamma m v_y$

Mass m is a constant of boost (const wrt all observers)

Consider $M\to 2m$ split process Consider $\gamma\to\infty$, $m\to 0$, $\gamma mc^2\to$ finite quantity

Kinetic Energy $K \simeq 2\gamma mc^2$ (Final state) $\xleftrightarrow{\text{Energy Conservation}}$ $E = 2\gamma mc^2 = Mc^2$. (Initial state, not moving)

The rest energy $E_{rest} = mc^2$ (m: rest mass, invariant mass)

Total energy (relativistic energy) $E = \gamma mc^2$ (γm: relativistic mass)

The 4-dimensional vector

4-D vector $(cdt, d\vec{x})$, inner product $ds^2 = -c^2dt^2 + d\vec{x}^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$.

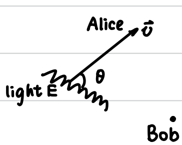
4-D vector of momentum $\quad \mathbf{p} = \gamma m\frac{d\vec{x}}{dt} \Rightarrow \mathbf{p} = (\gamma mc, \gamma m\vec{v}) = (\frac{E}{c}, \vec{p})$

$$\mathbf{p}\cdot\mathbf{p} = -(\frac{E}{c})^2 + \vec{p}^2 = -\gamma^2m^2c^2 + \gamma^2m^2v^2 = -\gamma^2m^2c^2(1-\frac{v^2}{c^2}) = -m^2c^4 \Rightarrow |\vec{p}|^2 = \frac{E^2}{c^2} - m^2c^4$$

$\Rightarrow \boxed{E^2 = |\vec{p}|^2c^2 + m^2c^4}$. A particle may have energy and momentum even it has no rest mass. In such a case, $E = pc$, the particle always travel at c in vacuum.

Doppler Effect*

momentum of light $\quad E = h\nu \xlongequal{c=\lambda\nu} h\frac{c}{\lambda} = \frac{c}{p}$



Alice: $k_A = (mc, \vec{0})$, $p_A = (\frac{E_A}{c}, \vec{p}_A)$ $\quad k_A\cdot p_A = -mE_A$

Alice's vector wrt Bob

Bob: $k_B = (\gamma mc, \gamma m\vec{v})$, $p_B = (\frac{E_B}{c}, \vec{p}_B)$, $k_B\cdot p_B = -\gamma m(E_B - \vec{v}\cdot\vec{p}_B)$

$$k_A\cdot p_A = k_B\cdot p_B \Rightarrow E_A = \gamma(E_B - \vec{v}\cdot\vec{p}_B) = \gamma(E_B - vp_B\cos\theta)$$

$$\Rightarrow E_A = \gamma E_B(1 - \frac{v}{c}\cos\theta)$$

$$\xRightarrow{E=\hbar\omega} \omega_A = \gamma\omega_B(1-\frac{v}{c}\cos\theta)$$

PHYS 2022 Modern Physics
PART 2. General Relativity

1 The Equivalence Principle

Two quantities of mass

Inertial mass: $m_I = \frac{F}{a}$, is a measure of inertia - laziness in changing its velocity

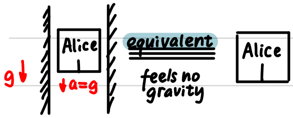
Gravitational mass: $m_G = \frac{F}{g}$ for gravity - how strongly gravity attracts the matter.

Alice would feel inertial force downward

Einstein's equivalence principle

Assume the complete physical equivalence of a (uniform) gravitational field and a corresponding acceleration of the reference frame.

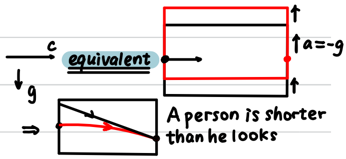
This answered: $m_I = m_G$, because inertia (a) equals to gravity.



Uniform gravity can be cancelled by constant acceleration

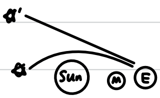
g is equivalent to $a=-g$, and $a=-g$ could be cancelled by $a=g$.

e.g. One feels no gravity in free falling lifts.



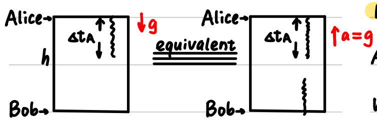
Light bends in uniform gravity

Light bends the same way as in an accelerated environment.



2 Time with Uniform Gravity

Higher is faster



$t=0$, the pulse starts to pass through A. A, B has velocity $v=a$. s is the length of the light pulse. s should be a constant.

Alice sends two light pulses with interval Δt_A. Assume $c\Delta t_A \ll h$, $gh \ll c^2$.

Wrt Alice, $v=0$ when she sends the signal. $s = c\Delta t_A$

Wrt Bob, $v = \frac{gh}{c}$ when he receives the signal. $s = (c + \frac{gh}{c})\Delta t_B$

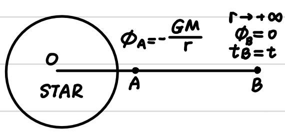
$\Rightarrow \Delta t_A = (1 + \frac{gh}{c^2})\Delta t_B$

Bob finds Alice faster. $t=0$ Alice and Bob

Time dilation with a gravitational potential

Gravitational potential ϕ, $dt_A = (1 + \frac{\phi_A - \phi_B}{c^2}) dt_B$.

The metric for a spherical star



For Alice, $dt_A = (1 + \frac{\phi_A}{c^2}) dt_B \Rightarrow dt_A = (1 - \frac{GM}{rc^2}) dt_B$

Alice is static at a fixed position wrt the star, $d\vec{x} = 0$.

$$ds^2 = -c^2 dt_A^2 = -c^2 \left(1 - \frac{GM}{rc^2}\right)^2 dt_B^2 = -c^2\left(1 - \frac{2GM}{rc^2}\right) dt_B^2 = -\left(1 + \frac{2\phi_A}{c^2}\right) c^2 dt_B^2$$

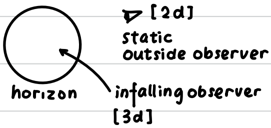
The Schwarzschild metrix

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

The Schwarzschild radius $r_s \equiv \frac{2GM}{c^2}$

Observation at $r=r_s$:

$\begin{cases} \infty \text{ time dilation} \\ \infty \text{ gravitational potential} \end{cases} \Rightarrow$ horizon



A Classical Black Hole:

mass, angular momentum, charge

c.f. a fundamental particle

3 Black Holes

What happens near $r=r_s$

For any finite interval Δt_A according to Alice

$$\Delta t_B = \Delta t_A \times \left(1-\frac{r_s}{r}\right)^{-\frac{1}{2}} \to \infty$$

Bob finds Alice frozen on the $r=r_s$ surface. Light emitted at r_s need ∞ time to reach Bob. Bob cannot see anything happen at $r<r_s$.

Event horizon and black hole

$r=r_s$ is a surface to limit the events that Bob can see. Thus $r=r_s$ is known as an **event horizon**. The dense object hides inside this event horizon and thus is invisible. No light from this object can reach outside. The object is known as a **black hole**

"Inside" the horizon: the future is doomed at a singularity

At $r<r_s$, the metric could be written as

$$ds^2 = -\underbrace{\left(\frac{r_s}{r}-1\right)}_{>0}^{-1} dr^2 + \underbrace{\left(\frac{r_s}{r}-1\right)}_{>0} c^2 dt^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Now dr is like time and dt is like space. For Alice, the $-r$ direction is now the time direction.

Everything inside the black hole falls towards the future at $r=0$. It's known as **singularity**.

4 Gravitational Waves

Special relativity and Newtonian gravity are inconsistent

$F=G_N\frac{Mm}{r^2}$. If Alice waves her hand, the Bob from a light year away could measure a slightly different gravitational attraction from Alice immediately.

But from special relativity, no information can travel faster than light.

Learn from E&M

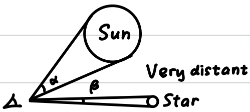
The source electric charge accelerates $\Rightarrow$ E&M wave is emitted.

The gravitational waves are the fluctuations of spacetime. They travel at the speed of light and are transverse waves.

PHYS 2022 Modern Physics
PART 3. Cosmology

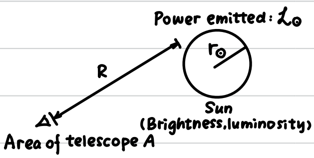
0 Introduction

Olber's Paradox



Two possible reasons why the sun is bright:

1. The sun spreads a larger solid angle.
2. The energy emitted by the sun is greater than the energy emitted by the star per solid angle

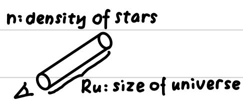


(Power received per solid angle) $= \mathcal{L}_\odot \dfrac{A}{4\pi R^2} \Big/ \dfrac{\pi r^2}{4\pi R^2} = A\dfrac{\mathcal{L}_\odot}{\pi r_\odot^2}$

Surface brightness of the sun $\Sigma_\odot \equiv \dfrac{\mathcal{L}_\odot}{\pi r_\odot^2} = 5\times10^{-3}$. The sun is as bright as other stars per solid angle. It's brighter because it has a large solid angle.

Surface brightness of the night sky: 5×10^{-17}.

Olber's paradox If the universe is infinite, then there would be a star at any solid angle $\Rightarrow$ The night sky is as bright as the sun.

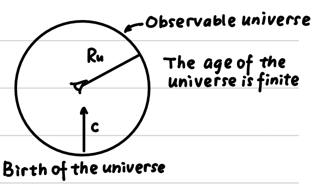


Another approach: the luminosity decays as $1/r^2$, the number of stars at distance r should scale as r^2. Sum up and all the stars should be as bright as the sun.

Mpc $\approx 3\times10^{22}$ m
$\approx 3\times10^{6}$ lyr

The Observable Universe

To solve the paradox, assume the universe is finite.



(Probability encounter a star) $= nR_u\pi r_\odot^2 \approx \dfrac{10^{-17}}{10^{-3}} = 10^{-14}$

$\Rightarrow R_u = \dfrac{10^{-14}}{n\pi r_\odot^2} = \dfrac{10^{-4}}{(10^{9}\text{Mpc}^{-3})(10^{-27}\text{Mpc}^2)} = 10^{4}\text{Mpc} \approx 10^{10}\text{ lyr}$

Radius of observable universe: $R_u = 10^{10}$ lyr.

(Speed of light) × (Age of universe) $= R_u \Rightarrow$ Age of universe: 10^{10} years.

1 The dynamics of the universe

The cosmological principle

Our universe is approximately homogeneous and isotopic on larger scales.
(100 Mpc ~ 10^8 lyr)

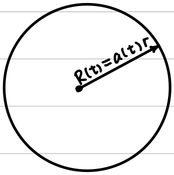
Distant galaxies are leaving us

Hubble measured the spectrum of light from other galaxies. One can find out the velocity of galaxies by Doppler-like effects, through measuring the observed spectrum.

From the Copernicus principle to the expanding universe

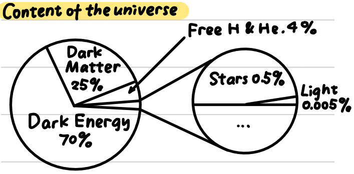
The universe does not have a center. Wherever you are on the universe, you find the neighboring points leaving you $\Rightarrow$ The universe is expanding.

Copernicus Principle:
Humans are not privileged observers of the universe, that observations from the Earth are representative of observations from the average position in the universe.



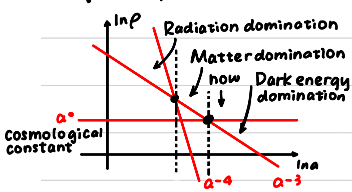
Physical distance : R(t)
real distance. actual physical measurements.
Comoving distance : r

Content of the universe



$$p = w\rho \Rightarrow \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

- Pressureless dust: $p=0$ (non-relativistic matter) $\rho \propto a^{-3}$
- Radiation: $p=\frac{\rho}{3}$. (relativistic matter) $\rho \propto a^{-4}$
- Dark energy: $p=-\rho$ (ρ is const) $\rho \propto a^{0}$



Imagine a universe filled with pressureless dust. The dust particles expand together with the universe expansion. without additional motion by themselves.

The comoving coordinate is related to the physical coordinate by $R(t)=a(t)r$

$a(t)$ is known as the scale factor, quantifying the time dependence of the universe

Define the Hubble parameter, the expansion rate of the universe

$$H(t) \equiv \frac{\dot{a}}{a}.$$

The universe tells matter how to get diluted

Consider a component in the universe with energy density ρ and pressure p.

First law of thermodynamics $dE = TdS - pdV$

$dS=0$ as heat will not be conducted in a homogenous universe.

$$\frac{dE}{dt} = p\frac{dV}{dt} \Rightarrow \frac{d(a^3\rho)}{dt} = p\frac{da^3}{dt} \Rightarrow \dot{\rho} + 3H(\rho+p) = 0$$

This is called the continuity equation in cosmology.

Matter tells the universe how to expand

Consider a dust particle with mass m, comoving distance r. The energy conservation equation is

$$\frac{m}{2}\left(\frac{dR}{dt}\right)^2 - \frac{GMm}{R} = E = \text{constant}.$$

E is determined by the initial condition of the universe. Observation tell that $E \simeq 0$.

M is the mass inside the sphere $Mc^2 = \frac{4\pi R^3}{3}\rho$.

Then we obtain the Friedmann Equation telling the universe how to expand

$$H^2 = \frac{8\pi G\rho}{3c^2}$$

2 The early universe

time	temperature	
20ps	100GeV	A spontaneous symmetry breaking generated mass for known massive fundamental particles.
20ms	150MeV	Free quarks are binded into protons and neutrons. Their binding energy is the main source of mass of atomic matter
3min	0.1MeV	Light elements, especially helium, are created from p.n.
50.000 years	1eV	Dominated by non-relativistic radiation
400.000 years	0.3eV	Universe becomes transparent, light can travel freely
		Structures grow in the universe.
14 billion years (Present)		Start to be dominated by dark energy.