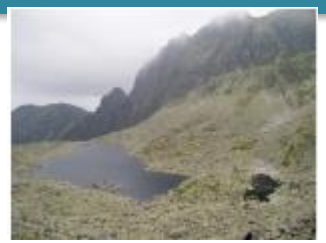
中国科学院大学
University of Chinese Academy of Sciences

Mining Massive datasets
Finding Similar Items:
Locality Sensitive Hashing

Outline

Scene Completion Problem

Scene Completion Problem





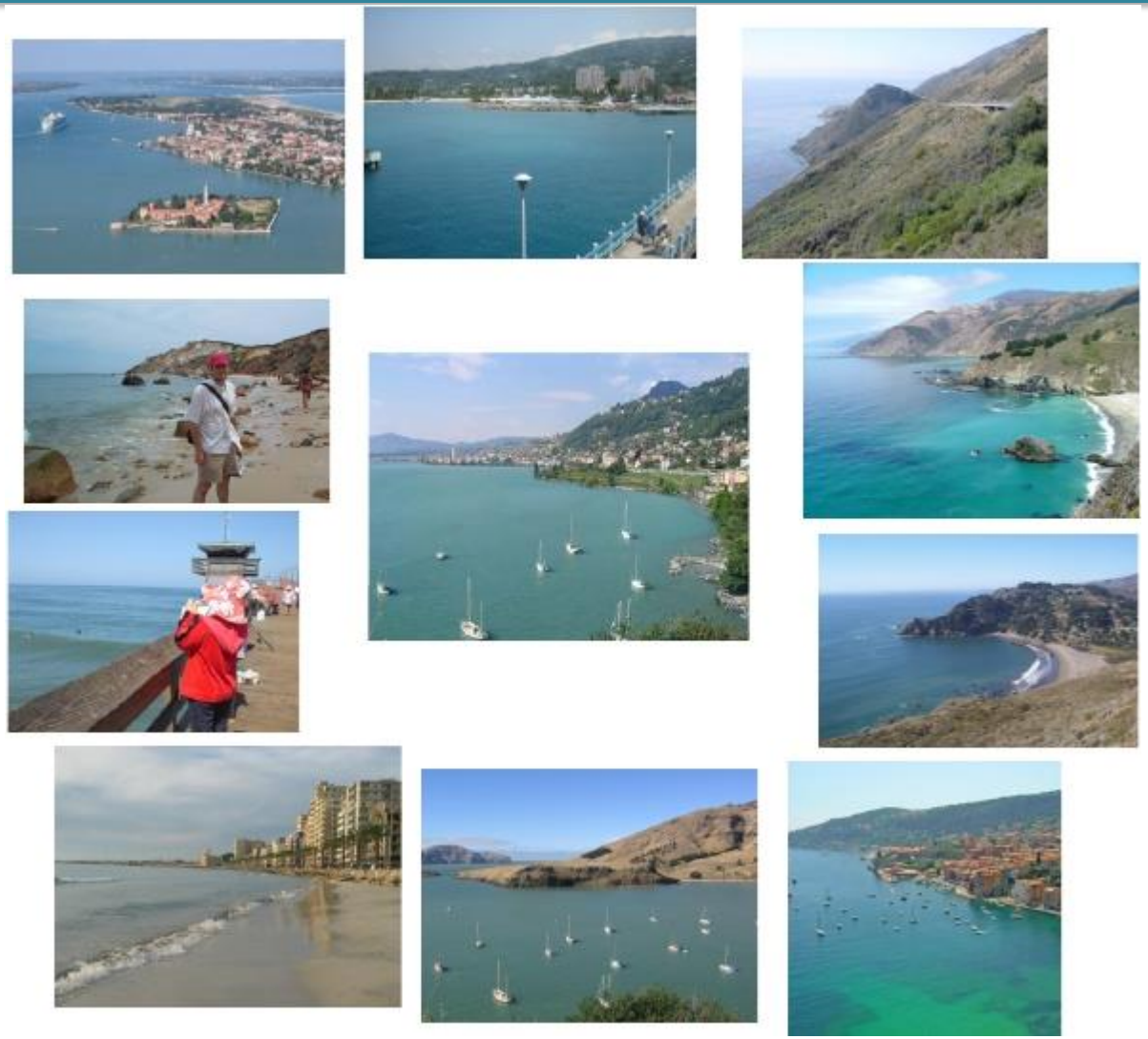






10 nearestneighbors from a collection of 20,000 images

Scene Completion Problem



10 nearestneighbors from a collection of 2 million images

Motivation

- Finding similar documents/webpages/images
  - (Approximate) mirror sites.
    Application: Don't want to show both when Google.
  - Plagiarism, large quotations
    Application: I am sure you know
  - Similar topic/articles from various places
    Application: Cluster articles by \same story".
  - Google image

- Social network analysis
  - Finding NetFlix users with similar tastes in movies.
  - e.g., for personalized recommendation systems.

A Common Metaphor

- Many problems can be expressed as finding "similar" sets:
  - **Find near-neighbors in high-dim space**
- **Example**
  - **Pages with similar words**
    - For duplicate detection, classification by topic
  - **Customers who purchased similar products**
    - Products with similar customer sets
  - **Images with similar features**
    - Users who visited similar websites

How can we compare similarity?

Convert the data (homework, webpages, images) into an object in an abstract space that we know how to measure distance, and how to do it efficiently.

What abstract space?

For example, the Euclidean space over R^d

L^1 space, L^∞ space, …

Problem for today's lecture

- Given: High dimensional data points x_1, x_2
  - For example: Image is a long vector of pixel colors
  - $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow [1\ 2\ 1\ 0\ 2\ 1\ 0\ 1\ 0]$
- And some distance function
  - Which quantifies the "distance" between x_1 and x_2
- Goal: Find all pairs of data points (x_i, x_j) that are within some distance threshold $d(x_i, x_j) \leq s$
- Note: Naïve solution would take $O(N^2)$ where N is the number of data points
- MAGIC: This can be done in O(N)!! How?

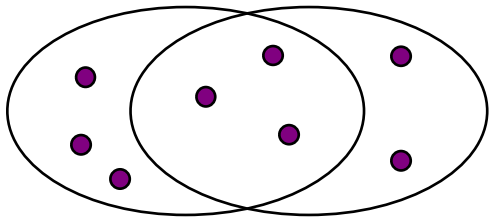
Outline

Distance Measures

- Goal: Find near-neighbors in high-dim. space
  - We formally define "near neighbors" as points that are a "small distance" apart
- For each application, we first need to define what "distance" means
- Today: Jaccard distance/similarity
  - The Jaccard similarity of two sets is the size of their intersection divided by the size of their union:

$$\mathbf{sim(C1, C2) = |C1 \cap C2| / |C1 \cup C2|}$$

  - Jaccard distance: $d(C1, C2) = 1 - |C1 \cap C2| / |C1 \cup C2|$



3 in intersection
8 in union
Jaccard similarity= 3/8
Jaccard distance = 5/8

Jaccard similarity with clusters

- Consider two sets A = {0,1,2,5,6}, B = {0,2,3,5,7,9}. What is the Jaccard similarity of A and B?

- With clusters: We may have some items which basically represent the same thing. We group objects to clusters. E.g.,

$$C1 = \{0,1,2\};\ C2 = \{3,4\};\ C3 = \{5,6\};\ C4 = \{7,8,9\}$$

For instance: C1 may represent action movies, C2 may represent comedies, C3 may represent documentaries, C4 may represent horror movies.

Now we can represent $A_{clu} = \{C1, C3\}$, $B_{clu} = \{C1, C2, C3, C4\}$

$$JS_{clu}(A; B) = JS(Aclu; Bclu) = |\{C1, C2\}|/|\{C1, C2, C3, C4\}| = 0.5$$

Finding Similar Documents

- Goal: Given a large number (N in the millions or billions) of documents, find "near duplicate" pairs
- Applications:
  - Mirror websites, or approximate mirrors
    - Don't want to show both in search results
  - Similar news articles at many news sites
    - Cluster articles by "same story"
- Problems:
  - Many small pieces of one document can appear out of order in another
  - Too many documents to compare all pairs
  - Documents are so large or so many that they cannot fit in main memory

3 Essential Steps for Similar Docs

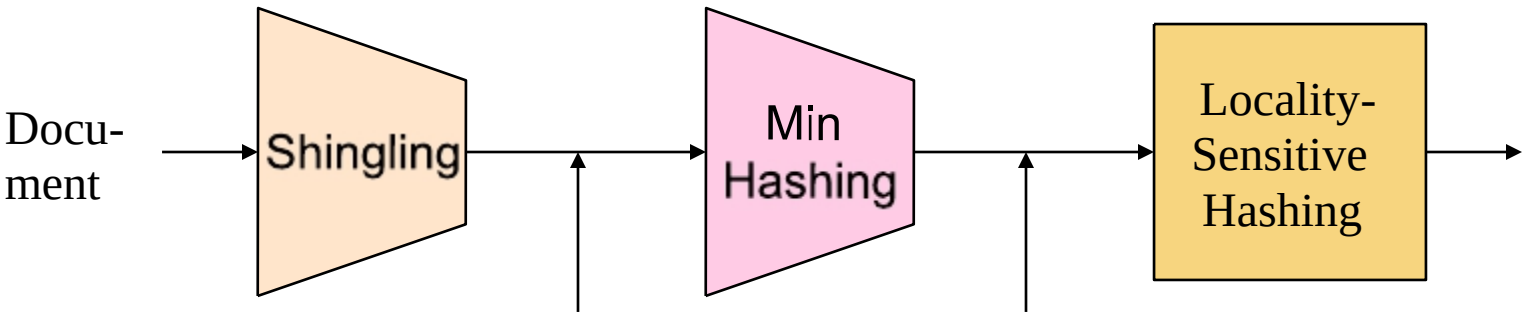
1. Shingling: Convert documents to sets

2. Min-Hashing: Convert large sets to short signatures, while preserving similarity

3. Locality-Sensitive Hashing: Focus on pairs of signatures likely to be from similar documents

   - **Candidate pairs!**

The Big Picture



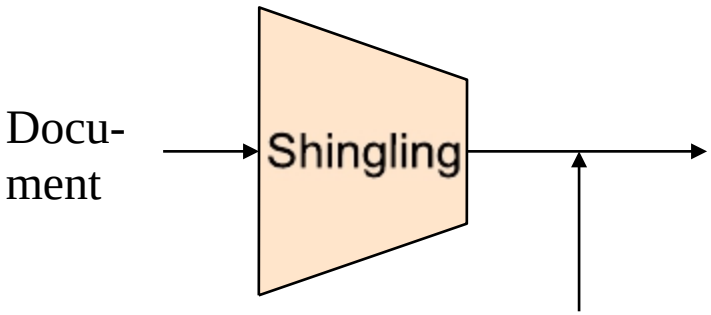
The set of strings of length ***k*** that appear in the document

Signatures: short integer vectors that represent the sets, and reflect their similarity

Candidate pairs: those pairs of signatures that we need to test for similarity

Outline

Shingling



Step 1: Shingling:

Convert documents to sets

Documents as High-Dim Data

- Step 1: Shingling: Convert documents to sets

- **Simple approaches:**
  - Document = set of words appearing in document
  - Document = set of "important" words
  - Don't work well for this application. Why?

- Need to account for ordering of words!
- A different way: Shingles!

Define: Shingles

- A k-shingle (or k-gram) for a document is a sequence of k tokens that appears in the doc
  - Tokens can be characters, words or something else, depending on the application
  - Assume tokens = characters for examples (ignoring back-space)

- Example: **k=2**; document **D1**= abcab
- Set of 2-shingles: **S(D1)** = {ab, bc, ca}
  - Option: Shingles as a bag (multiset), count ab twice: **S'(D1)** = {ab, bc, ca, ab}

Compressing Shingles

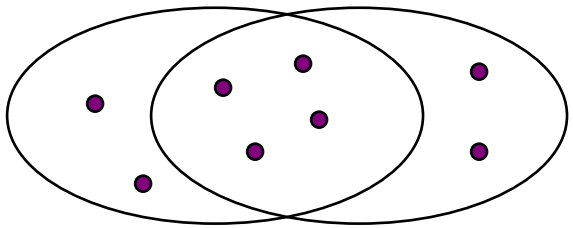
- To compress long shingles, we can hash them to (say) 4 bytes
- Represent a document by the set of hash values of its k-shingles
  - Idea: Two documents could (rarely) appear to have shingles in common, when in fact only the hash-values were shared
- Example: k=2; document D1= abcab
- Set of 2-shingles: S(D1) = {ab, bc, ca}
- Hash the singles: h(D1) = {1, 5, 7}

Similarity Metric for Shingles

- Document D1 is a set of its k-shingles C1=S(D1)
- Equivalently, each document is a 0/1 vector in the space of k-shingles
  - Each unique shingle is a dimension
  - Vectors are very sparse
- **A natural similarity measure is the**

  Jaccard similarity:

$$sim(D1, D2) = |C1 \cap C2| / |C1 \cup C2|$$



Working Assumption

- **Documents that have lots of shingles in common have similar text, even if the text appears in different order**

- Careful: You must pick **k** large enough, or most documents will have most shingles
  - **k** = 5 is OK for short documents (e.g. e-mails and short essays)
  - **k** = 10 is better for long documents (e.g. novels and papers)

Motivation for Minhash/LSH

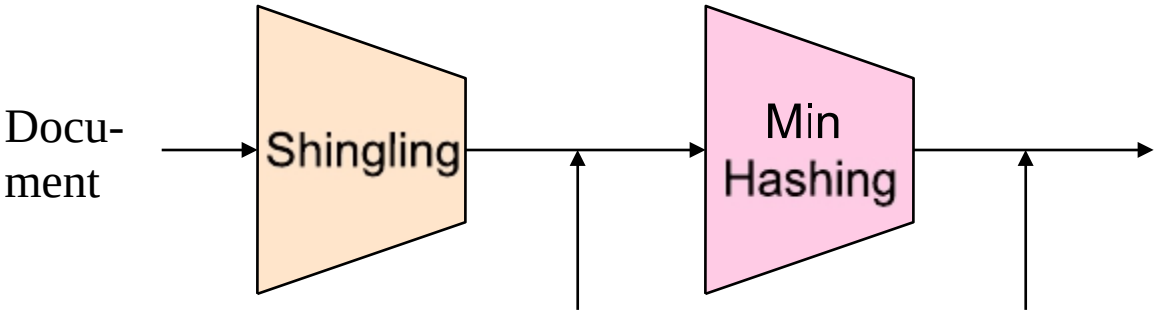
- Suppose we need to find near-duplicate documents among N = 1 million documents

- Naïvely, we would have to compute pairwise Jaccard similarities for every pair of docs
  - N(N-1)/2 ≈$5*10^{11}$ comparisons
  - At 10^5 secs/day and 10^6 comparisons/sec it would take 5 days

- For N = 10 million, it takes more than a year…

Outline

Min Hashing



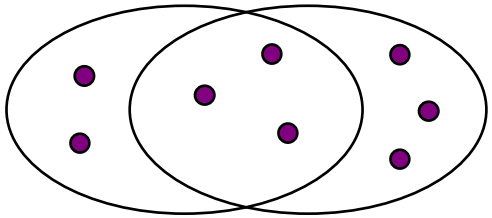
The set of strings of length ***k*** that appear in the doc-ument

Signatures: short integer vectors that represent the sets, and reflect their similarity

Step 2: Min-Hashing: Convert large sets to short signatures, while preserving similarity

Encoding Sets as Bit Vectors

- Many similarity problems can be formalized as **finding subsets that have significant intersection**
- Encode sets using 0/1 (bit, boolean) vectors
  - One dimension per element in the universal set
- Interpret set intersection as bitwise AND, and set union as bitwise OR



- **Example**: C1 = 10111; C2 = 10011
  - Size of intersection = 3; size of union = 4,
  - Jaccard similarity (not distance) = 3/4
  - Distance: d(C1,C2) = 1 – (Jaccard similarity) = 1/4

From Sets to Boolean Matrices

- Rows = elements (shingles)
- Columns = sets (documents)
  - 1 in row e and column s if and only if e is a member of s
  - Column similarity is the Jaccard similarity of the corresponding sets (rows with value 1)
  - Typical matrix is sparse!
- Each document is a column:
  - Example: sim(C1 ,C2) = ?
    - Size of intersection = 3; size of union = 6,
    - Jaccard similarity (not distance) = 3/6
    - d(C1,C2) = 1 – (Jaccard similarity) = 3/6

Shingles \ Documents				
	1	1	1	0
	1	1	0	1
	0	1	0	1
	0	0	0	1
	1	0	0	1
	1	1	1	0
	1	0	1	0

Outline: Finding Similar Columns

- So far:
  - Documents → Sets of shingles
  - Represent sets as boolean vectors in a matrix
- Next goal: Find similar columns while computing small signatures
  - Similarity of columns == similarity of signatures

Outline: Finding Similar Columns

- Next Goal: Find similar columns, Small signatures
- Naïve approach:
  - 1) Signatures of columns: small summaries of columns
  - 2) Examine pairs of signatures to find similar columns
    - Essential: Similarities of signatures and columns are related
  - 3) Optional: Check that columns with similar signatures are really similar
- Warnings:
  - Comparing all pairs may take too much time: Job for LSH
    - These methods can produce false negatives, and even false positives
    - Optional check can cancel false positives

Hashing Columns (Signatures)

- Key idea: "hash" each column C to a small signature $h(C)$, such that:
  - (1) $h(C)$ is small enough that the signature fits in RAM
  - (2) $sim(C1, C2)$ is the same as the "similarity" of signatures $h(C1)$ and $h(C2)$

- Goal: Find a hash function $h(\cdot)$ such that:
  - If $sim(C1,C2)$ is high, then with high prob. $h(C1) = h(C2)$
  - If $sim(C1,C2)$ is low, then with high prob. $h(C1) \neq h(C2)$

- Hash docs into buckets. Expect that "most" pairs of near duplicate docs hash into the same bucket!

Min-Hashing

- Goal: Find a hash function $h(\cdot)$ such that:
  - if $sim(C1,C2)$ is high, then with high prob. $h(C1) = h(C2)$
  - if $sim(C1,C2)$ is low, then with high prob. $h(C1) \neq h(C2)$

- Clearly, the hash function depends on the similarity metric:
  - Not all similarity metrics have a suitable hash function

- **There is a suitable hash function for the Jaccard similarity:** It is called Min-Hashing

Min-Hashing

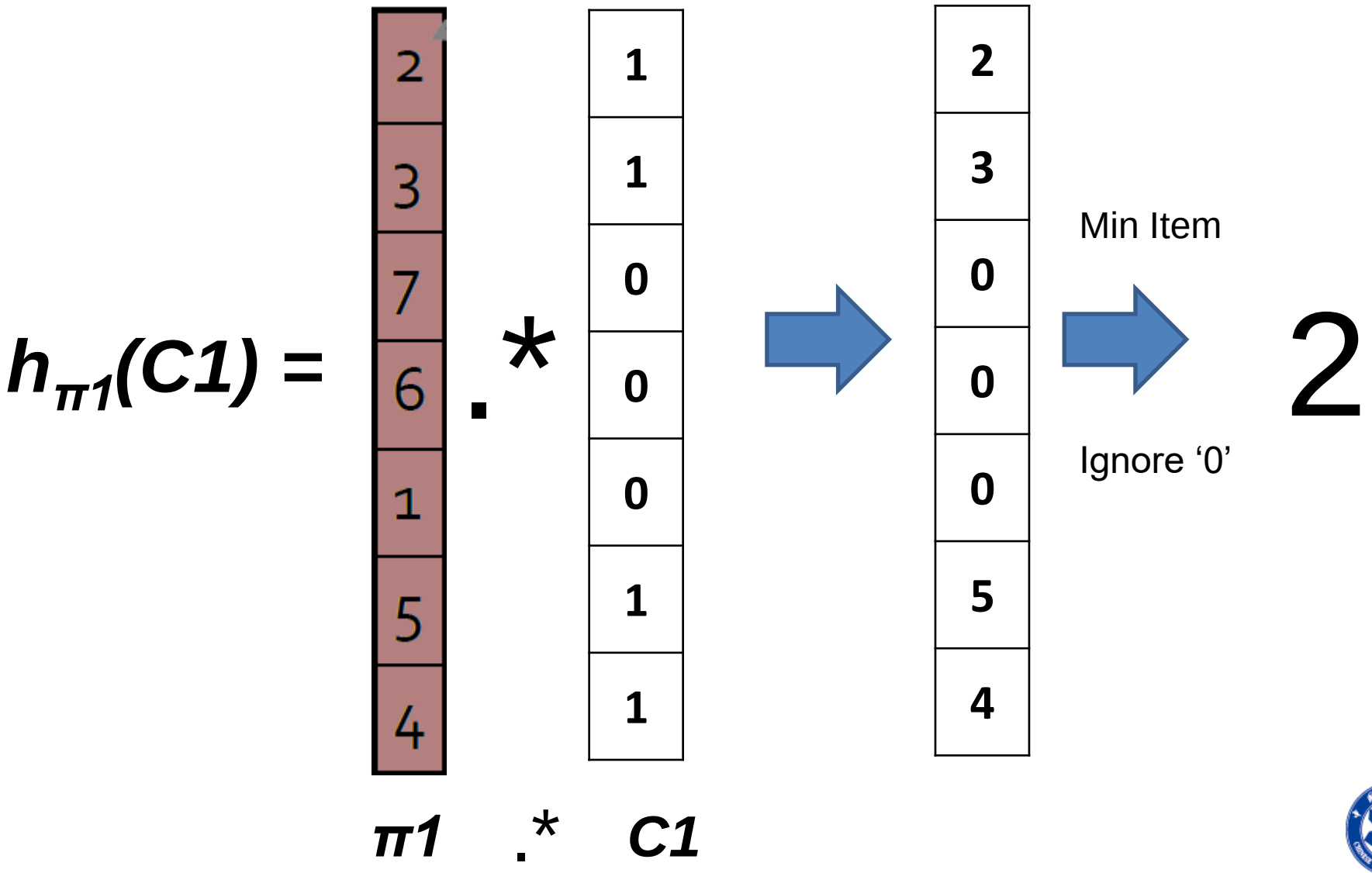
- Imagine the rows of the boolean matrix permuted under random permutation π
- Define a "hash" function $h_\pi(C)$ = the index of the first (in the permuted order π) row in which column C has value 1:

$$h_\pi(C) = \min \pi(C)$$

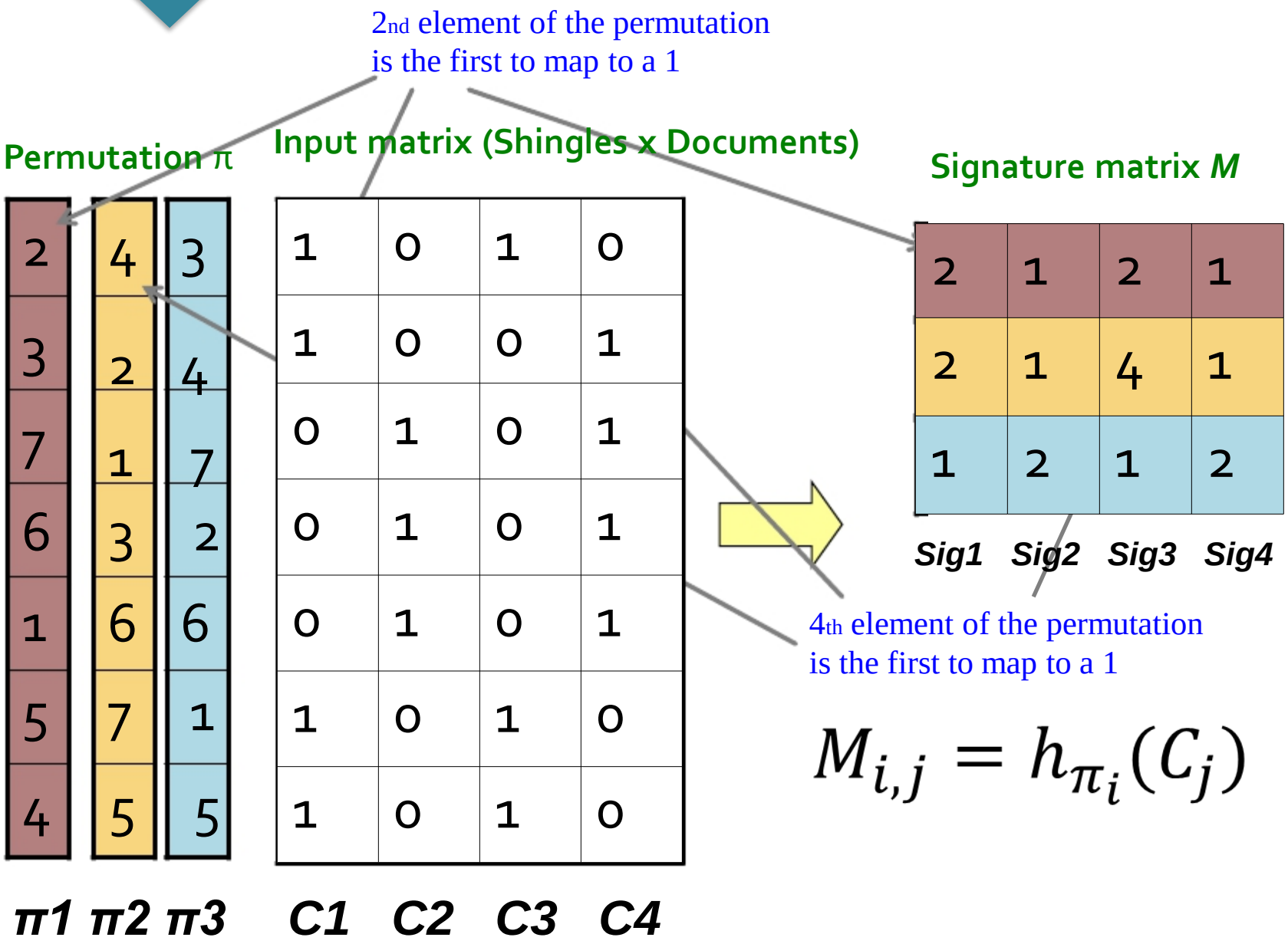
- Use several (e.g., 100) independent hash functions (that is, permutations) to create a signature of a column

Min-Hashing Example (one element)

$$h_\pi(C) = \min\pi(C)$$



Min-Hashing Example



$$M_{i,j} = h_{\pi_i}(C_j)$$

The Min-Hash Property

- Choose a random permutation π
- Claim: $Pr[h\pi(C1) = h\pi(C2)] = sim(C1, C2)$
- Why?
  - Let X be a doc (set of shingles), $y \in X$ is a shingle
  - Then: $Pr[\pi(y) = min(\pi(X))] = 1/|X|$
    - It is equally likely that any $y \in X$ is mapped to the min element
- Let y be s.t. $\pi(y) = min(\pi(C1 \cup C2))$
- Then either: $\pi(y) = min(\pi(C1))$ if $y \in C1$, or
  $\pi(y) = min(\pi(C2))$ if $y \in C2$

One of the two cols had to have 1 at position *y*

- So the prob. that **both** are true is the prob. $y \in C1 \cap C2$
- **$Pr[min(\pi(C1))=min(\pi(C2))]$**

$= |C1 \cap C2| / |C1 \cup C2| =$ *sim(C1, C2)*

Similarity for Signatures

- We know: $Pr[h_\pi(C1) = h_\pi(C2)] = sim(C1, C2)$
- Now generalize to multiple hash functions
- The similarity of two signatures is the fraction of the hash functions in which they agree
- Note: Because of the Min-Hash property, the similarity of columns is the same as the expected similarity of their signatures

Hence, sim(C1, C2) = sim(Sig1,Sig2) when the number of permutation function is large enough(e.g. 100)

Min-Hashing Example

Permutation □

2	4	3
3	2	4
7	1	7
6	3	2
1	6	6
5	7	1
4	5	5

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

Signature matrix *M*

2	1	2	1
2	1	4	1
1	2	1	2

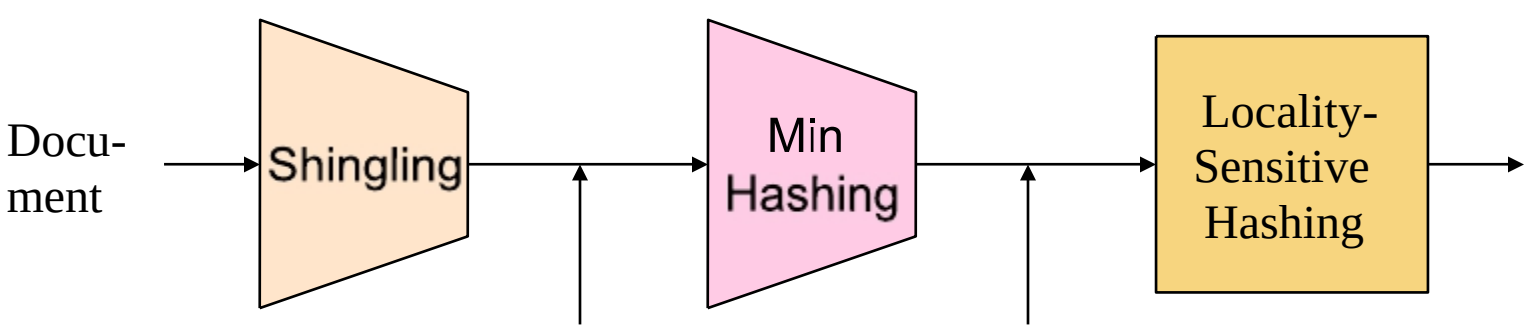
Similarities:

	1-3	2-4	1-2	3-4
Col/Col	0.75	0.75	0	0
Sig/Sig	0.67	1.00	0	0

We achieved our goal! **We "compressed" long bit vectors into short signatures**

Outline

Locality Sensitive Hashing



***Candidate pairs*:** those pairs of signatures that we need to test for similarity

The set of strings of length ***k*** that appear in the doc-ument

Signatures: short integer vectors that represent the sets, and reflect their similarity

Step 3: Locality-Sensitive Hashing:

Focus on pairs of signatures likely to be from similar documents

LSH: First Cut

- Goal: Find documents with Jaccard similarity at least s (for some similarity threshold, e.g., s=0.8)
- LSH – General idea: Use a function ***f(x,y)*** that tells whether x and y is a candidate pair: a pair of elements whose similarity must be evaluated
- For Min-Hash matrices:
  - Hash columns of signature matrix M to many buckets
  - Each pair of documents that hashes into the same bucket is a candidate pair

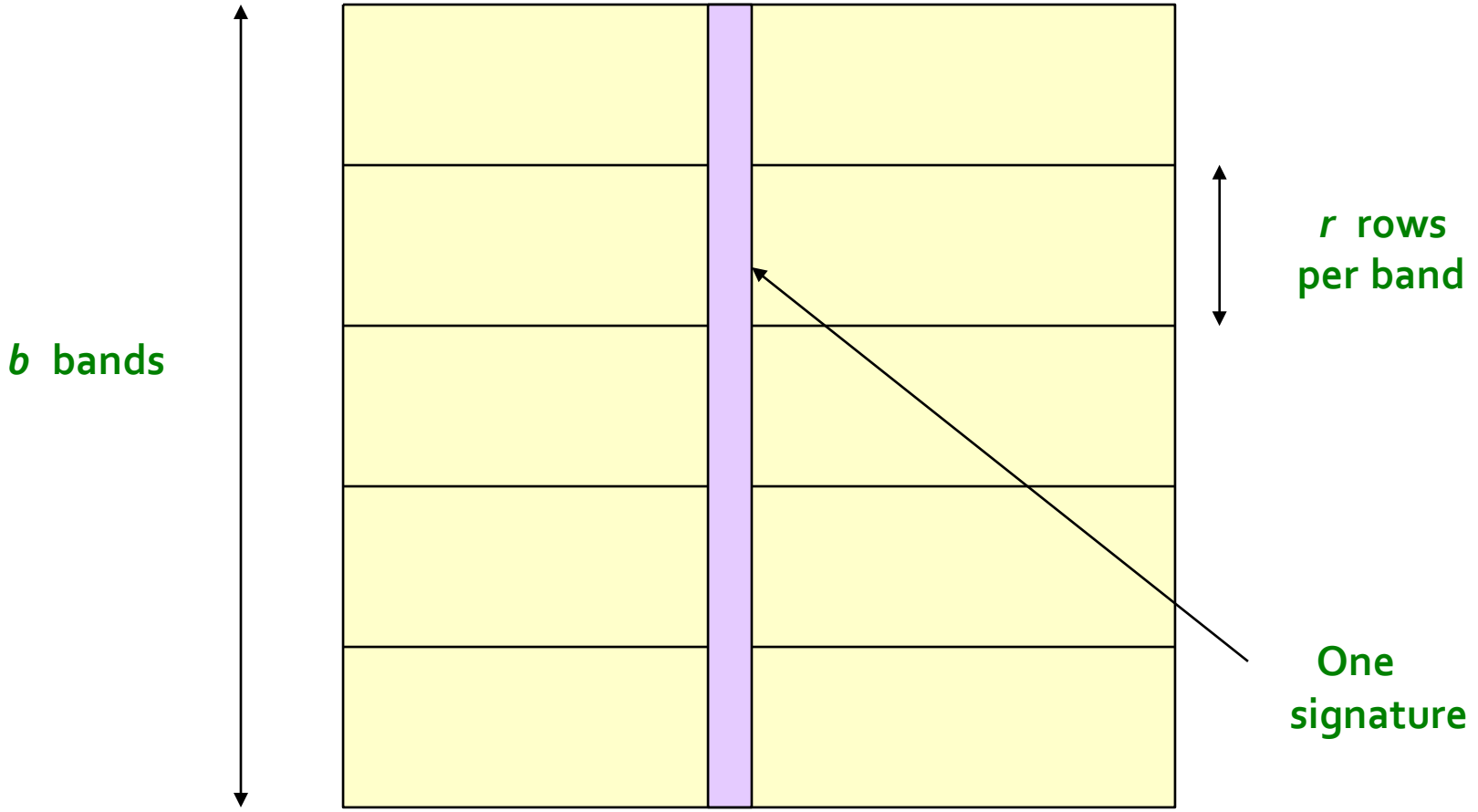
Candidates from Min-Hash

- Pick a similarity threshold s ($0 < s < 1$)
- Columns x and y of M are a candidate pair if their signatures agree on at least fraction s of their rows:
- $M(i, x) = M(i, y)$ for at least frac. s values of i
  - We expect documents x and y to have the same (Jaccard) similarity as their signatures

LSH for Min-Hash

- Big idea: Hash columns of signature matrix M several times instead of computing all couples of columns
- Arrange that (only) similar columns are likely to hash to the same bucket, with high probability
- Candidate pairs are those that hash to the same bucket

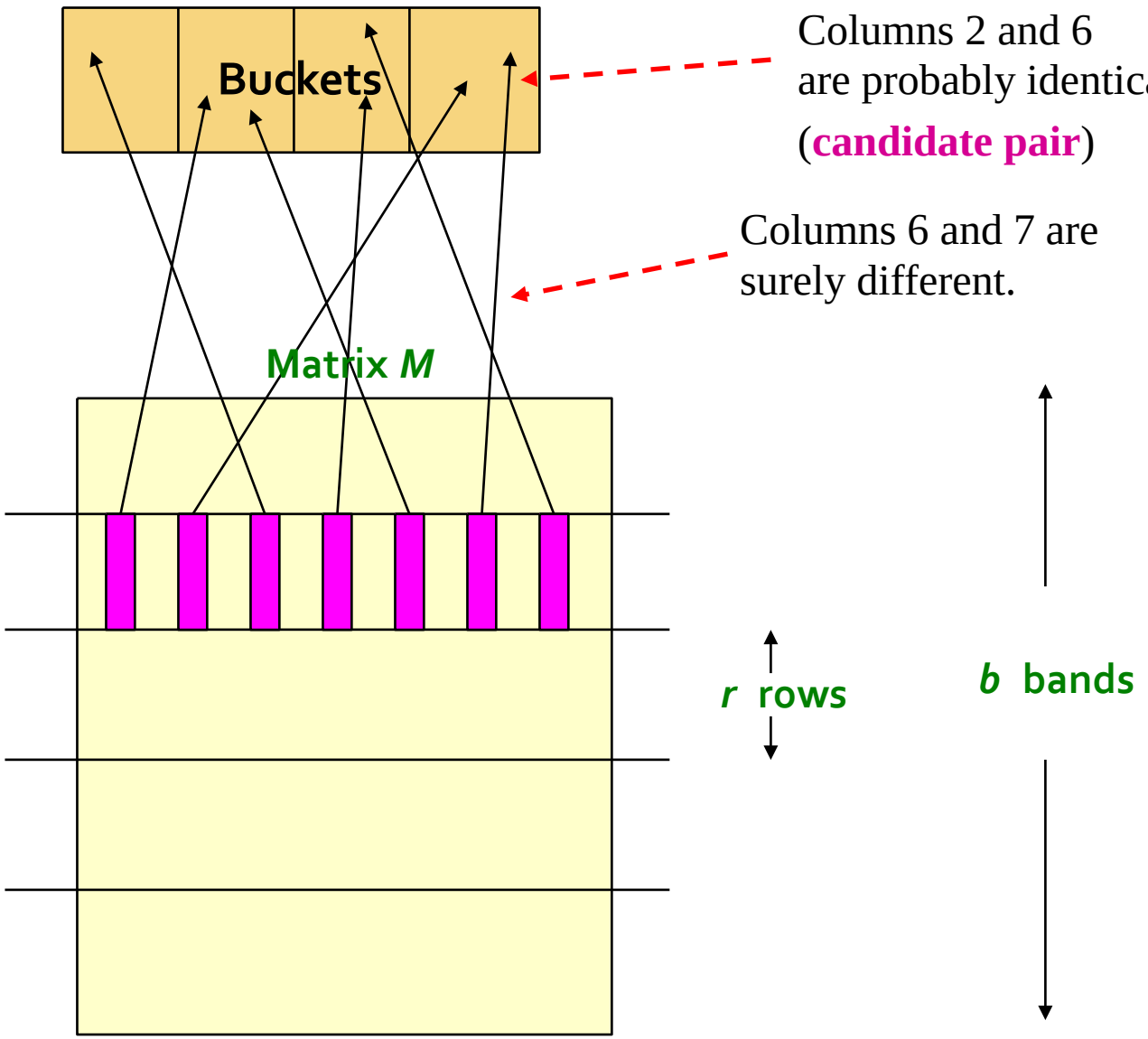
Partition M into b Bands



Partition M into Bands

- Divide matrix M into b bands of r rows
- For each band, hash its portion of each column to a hash table with k buckets
  - Make k as large as possible (at least k >= number of documents)
- Candidate column pairs are those that hash to the same bucket for ≥ 1 band
- Tune b and r to catch most similar pairs, but few non-similar pairs

Hashing Bands



Simplifying Assumption

- There are enough buckets that columns are unlikely to hash to the same bucket unless they are identical in a particular band
- Hereafter, we assume that "***same bucket***" means "***identical in that band***"
- Assumption needed only to simplify analysis, not for correctness of algorithm

C1,C2 are 80% Similar

- Find pairs of $\geq s = 0.8$ similarity, set b =20 and r =5
- Assume: sim(C1,C2) =0.8
  - Since Sim(C1,C2) $\geq$ s, we want C1,C2 to be candidate pair: We want them to hash to at least 1 common bucket (at least one band is identical)
- Probability C1, C2 identical in one particular band: $(0.8)^5 = 0.328$
- Probability C1, C2 are not similar in all of the 20 bands: $(1-0.328)^{20} = 0.00035$
  - i.e., about 1/3000th of the 80%-similar column pairs are false negatives (we miss them)
  - We would find 99.965% pairs of truly similar documents

C1,C2 are 30% Similar

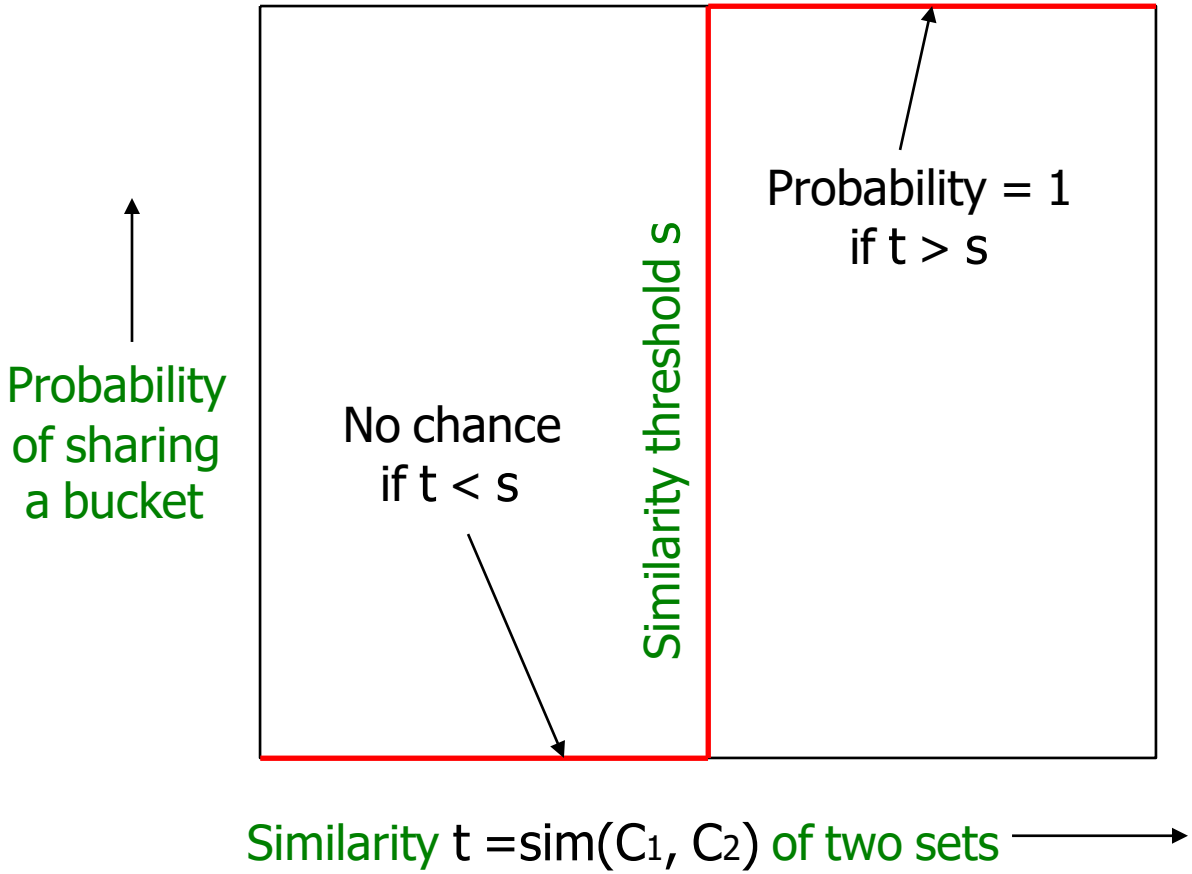
- Find pairs of $\geq s = 0.8$ similarity, set b =20 and r =5
- Assume: sim(C1,C2) =0.3
  - Since Sim(C1,C2) < s, we want C1,C2 to be No candidate pair (all bands should be different)
- Probability C1, C2 identical in one particular band: $(0.3)^5 = 0.00243$
- Probability C1, C2 identical in at least 1 of 20 bands: $1-(1-0.00243)^{20} = 0.00474$
  - In other words, approximately 0.474% pairs of docs with similarity 0.3 end up becoming candidate pairs
  - They are false positives since we will have to examine them (they are candidate pairs) but then it will turn out their similarity is below threshold s

b Bands, r rows/band

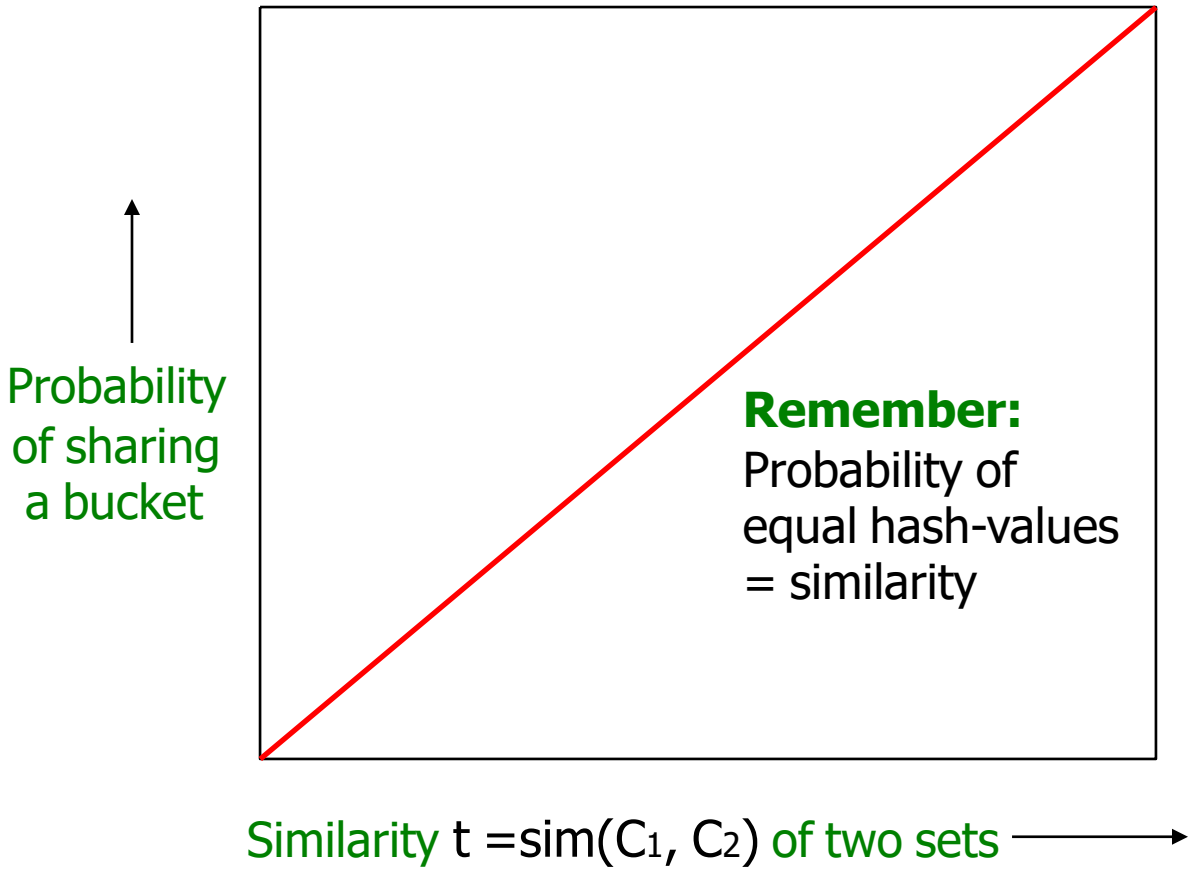
- Columns C_1 and C_2 have similarity t
- Pick any band (r rows)
  - Prob. that all rows in band equal = t^r
  - Prob. that some row in band unequal = $1 - t^r$
- Prob. that no band identical = $(1 - t^r)^b$
- Prob. that at least 1 band identical =

$$1 - (1 - t^r)^b$$

Analysis of LSH – What We Want

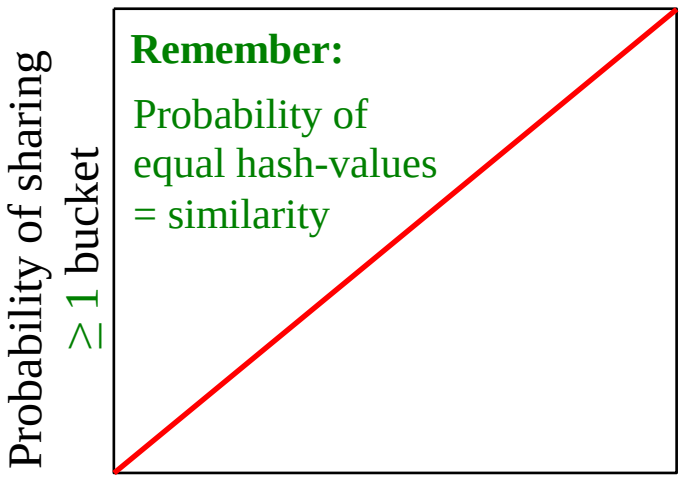


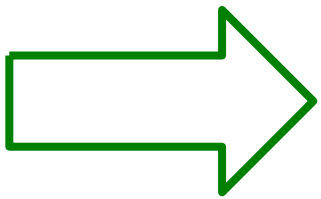
What 1 band of 1 Row Gives You

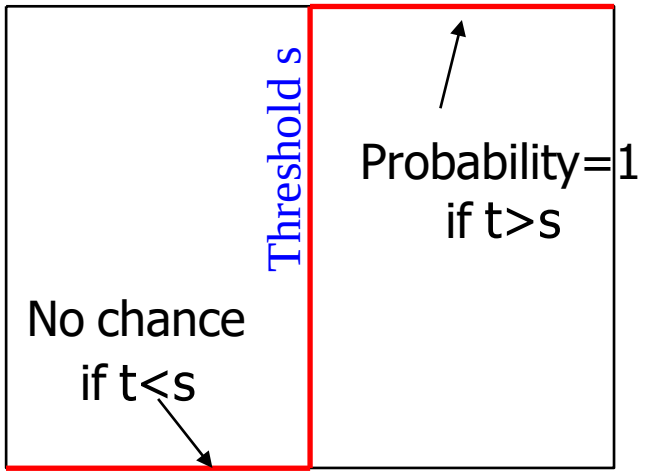


Recap: The S-Curve

- **The S-curve is where the "magic" happens**







This is what 1 hash-code gives you

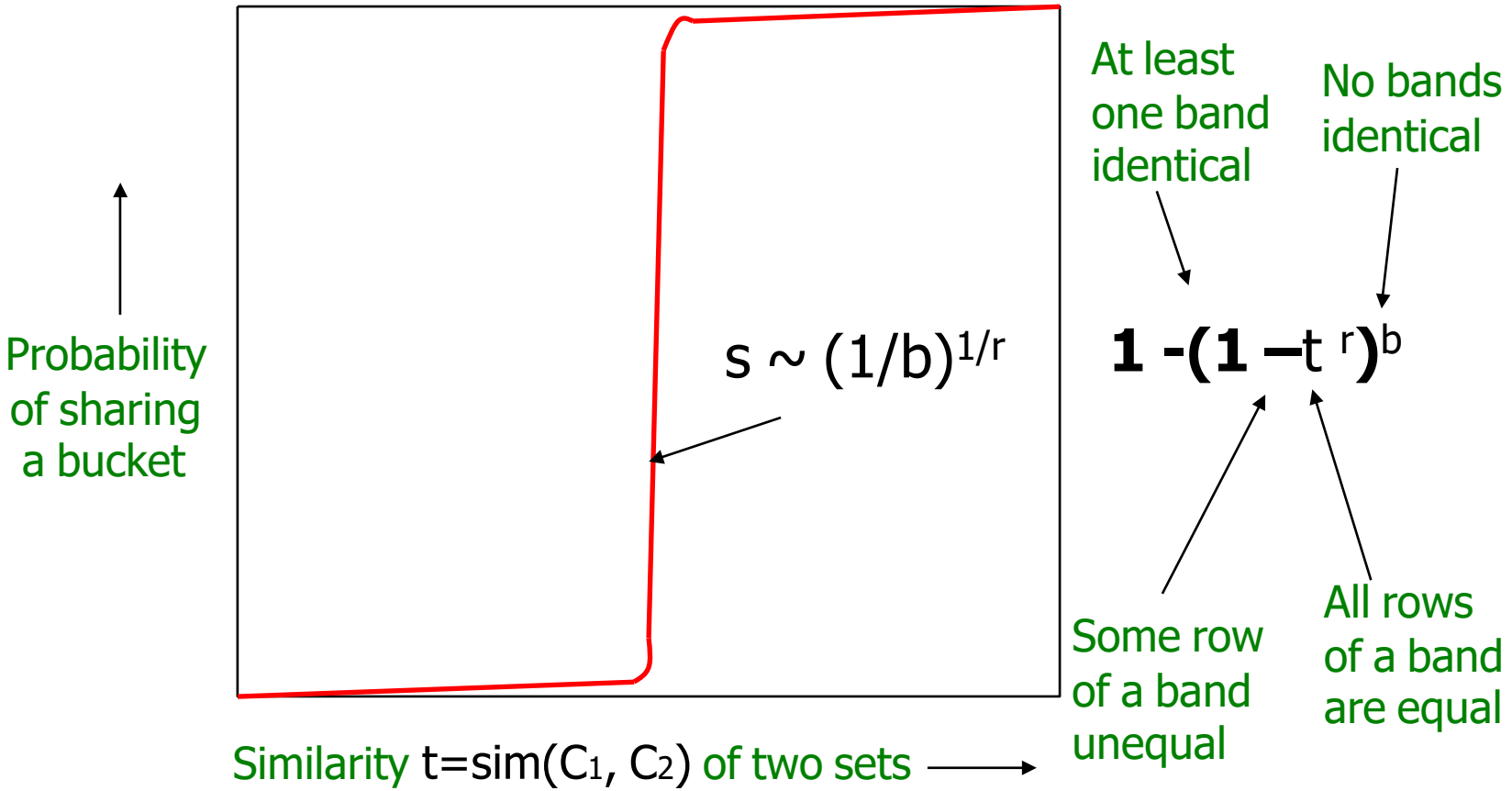
$\Pr[h(C_1) = h(C_2)] = sim(D_1, D_2)$

This is what we want!

How to get a step-function?

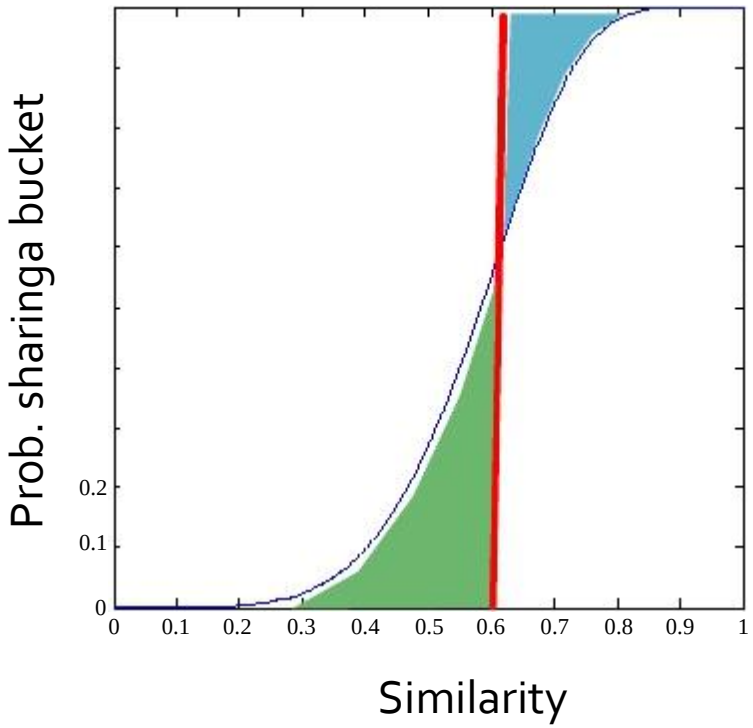
By choosing *r* and *b*!

What b Bands of r Rows Gives You



Picking r and b: The S-curve

- Picking r and b to get the best S-curve
  - 50 hash-functions (r=5, b=10)



Blue area: False Negative rate
Green area: False Positiverate

Optional Check

- Because of false positive/negative existing, we need the optional check work to solve the false positive
- Check in main memory that candidate pairs really do have similar signatures
- Optional: In another pass through data, check that the remaining candidate pairs really represent similar documents

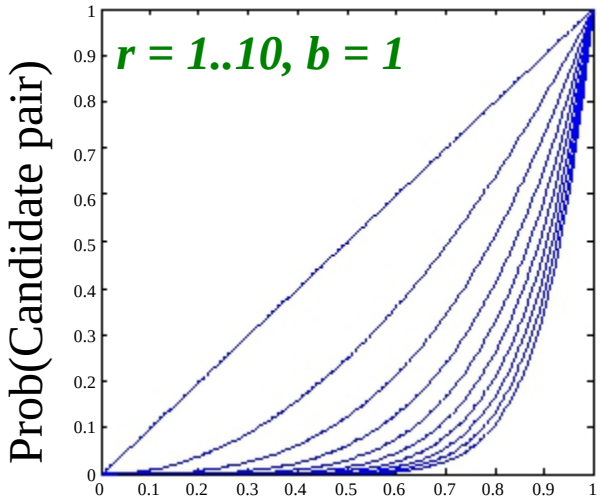
Summary: 3Steps

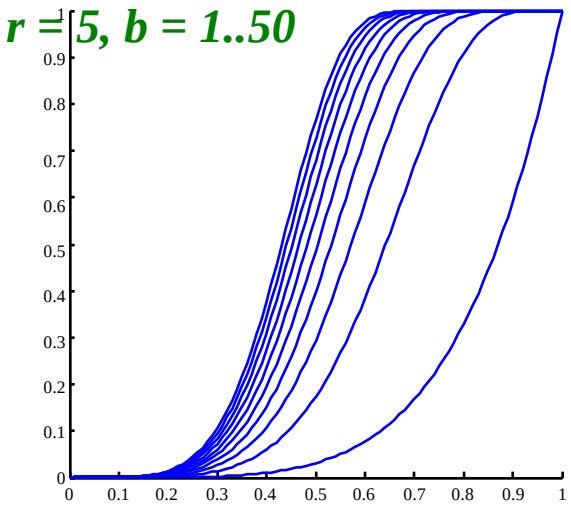
- Shingling: Convert documents to sets
  - We used hashing to assign each shingle an ID
- Min-Hashing: Convert large sets to short signatures, while preserving similarity
  - We used similarity preserving hashing to generate signatures with property $Pr[h_\pi(C1) = h_\pi(C2)] = sim(C1, C2)$
  - We used hashing to get around generating random permutations
- Locality-Sensitive Hashing: Focus on pairs of signatures likely to be from similar documents
  - We used hashing to find candidate pairs of similarity $\geq s$

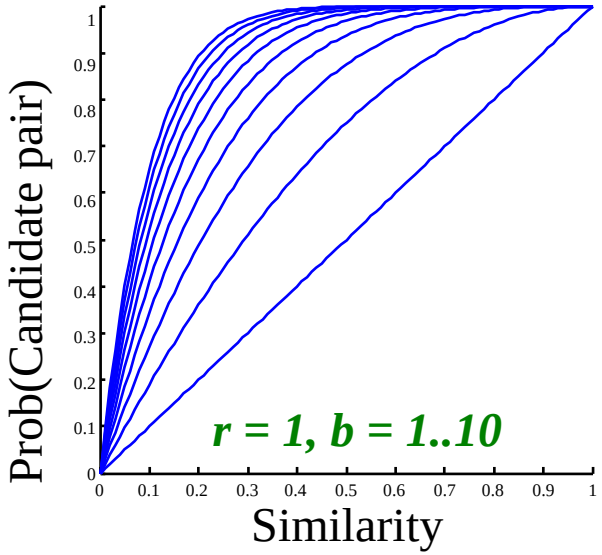
S-curves as a func. of *b* and *r*

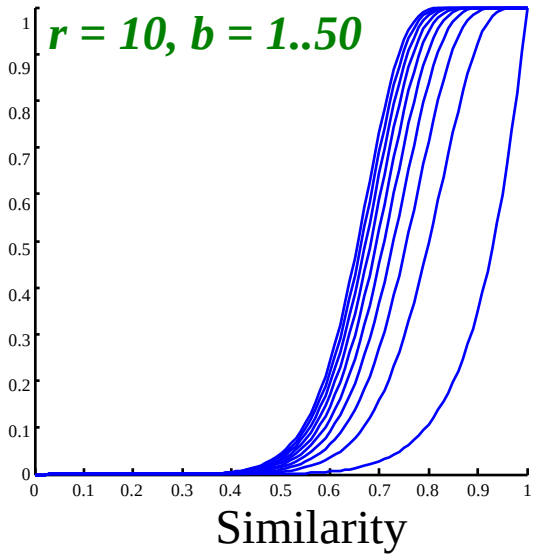
Given a fixed threshold ***s***.

We want choose ***r*** and ***b*** such that the ***P(Candidate pair)*** has a "step" right around ***s***.









P(C1,C2 is a candidate pair) = 1 - (1 - t^r)^b

Outline

Families of Hash Functions

- For Min-Hashing signatures, we got a Min-Hash function for each permutation of rows
- A "hash function" is any function that takes two elements and says whether they are "equal"
  - Example: h(x)=h(y) means "h says x and y are equal"
- A family of hash functions is any set of hash functions from which we can pick one at random efficiently
  - Example: The set of Min-Hash functions generated from permutations of rows

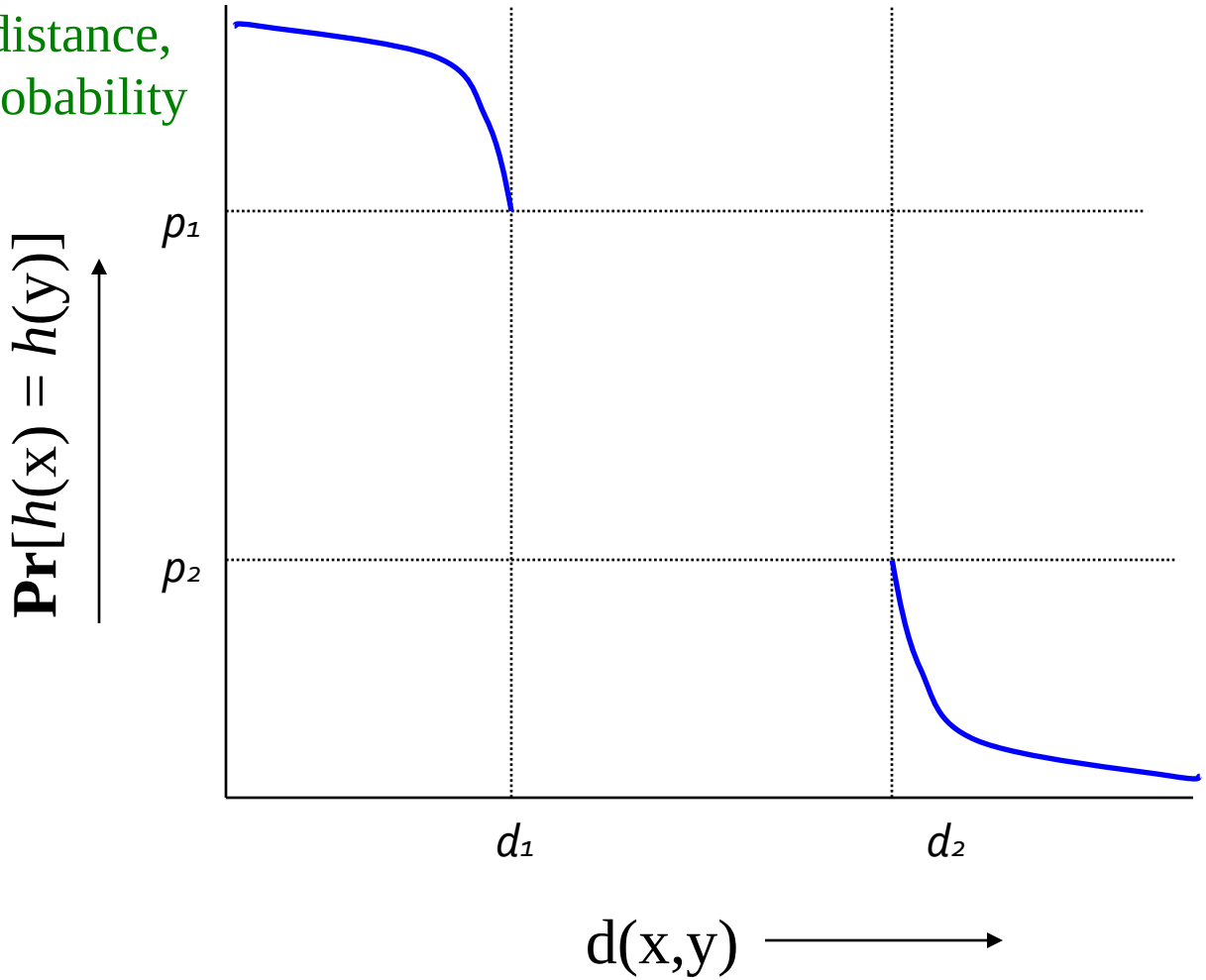
Locality –Sensitive(LS) Families

- Suppose we have a space S of points with a distance measure $d(x,y)$

- A family H of hash functions is said to be *(d1, d2, p1, p2)-sensitive* if for any x and y in S:
  - 1. If $d(x, y) < d1$, then the probability over all $h \in H$, that $h(x) = h(y)$ is at least $p1$
  - 2. If $d(x, y) > d2$, then the probability over all $h \in H$, that $h(x) = h(y)$ is at most $p2$

A *(d1, d2, p1, p2)-sensitive* function

Small distance, high probability



Large distance, low probability of hashing to the same value

LS Family of Min-Hash

- For any hash function $h \in H$:

$$\Pr[h(x) = h(y)] = 1 - d(x, y)$$

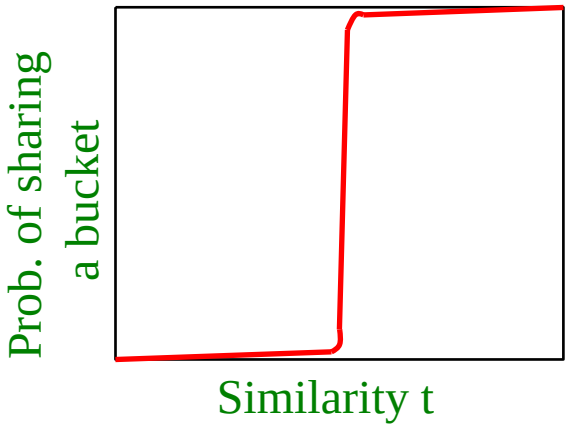
  - Simply restates theorem about Min-Hashing in terms of distances rather than similarities

- For Jaccard similarity, Min-Hashing gives a (d1,d2,(1-d1),(1-d2))-sensitive family for any d1<d2

- Theory leaves unknown what happens to pairs that are at distance between d1 and d2
  - Consequence: No guarantees about fraction of false positives in that range

Amplifying a LS-Family

- Can we reproduce the "S-curve" effect we saw before for any LS family?



- The "bands" technique we learned for signature matrices carries over to this more general setting
  - So we can do LSH with any (d1, d2, p1, p2)-sensitive family
- Two constructions:
  - AND construction like "rows in a band"
  - OR construction like "many bands"

Outline

AND of Hash Functions

- Given family H, construct family H' consisting of r functions from H
- For $h = [h_1,\ldots,h_r]$ in H', we say $h(x) = h(y)$ if and only if $h_i(x) = h_i(y)$ for all i, $1 \le i \le r$
  - Note this corresponds to creating a band of size r
- Theorem: If H is (d_1, d_2, p_1, p_2)-sensitive, then H' is $(d_1, d_2, (p_1)^r, (p_2)^r)$-sensitive
- Proof: Use the fact that h_i's are independent

Subtlety Regarding Independence

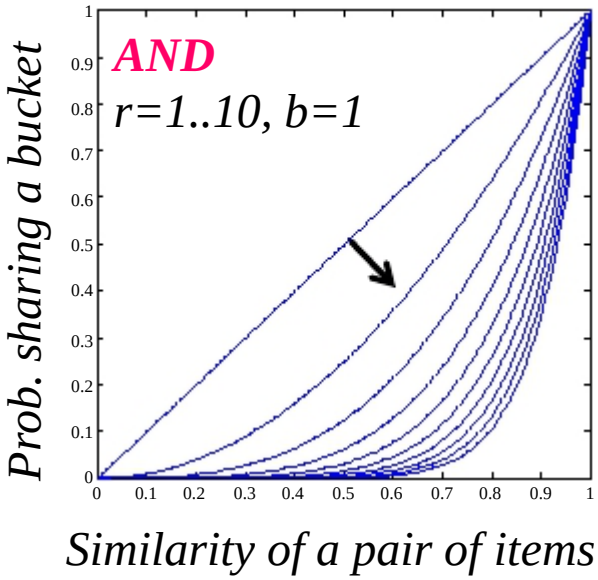
- Independence of hash functions (HFs) really means that the prob. of two HFs saying "yes" is the product of each saying "yes"
  - But two hash functions could be highly correlated
    - For example, in Min-Hash if their permutations agree in the first one million entries
  - However, the probabilities in definition of a LSH-family are over all possible members of H, H'

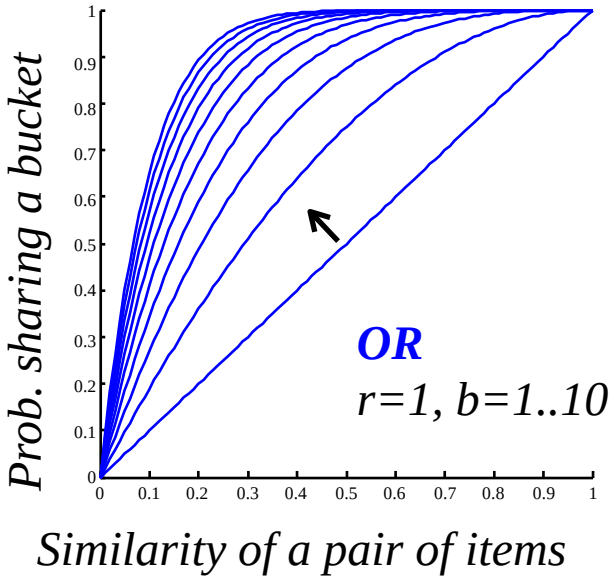
OR of Hash Functions

- Given family H, construct family H' consisting of b functions from H
- For $h = [h_1,\ldots,h_b]$ in H', we say $h(x) = h(y)$ if and only if $h_i(x) = h_i(y)$ for at least 1 i
- Theorem: If H is (d_1, d_2, p_1, p_2)-sensitive, then H' is $(d_1, d_2, 1-(1-p_1)^b, 1-(1-p_2)^b)$-sensitive
- Proof: Use the fact that h_i's are independent

Effect of AND and OR Constructions

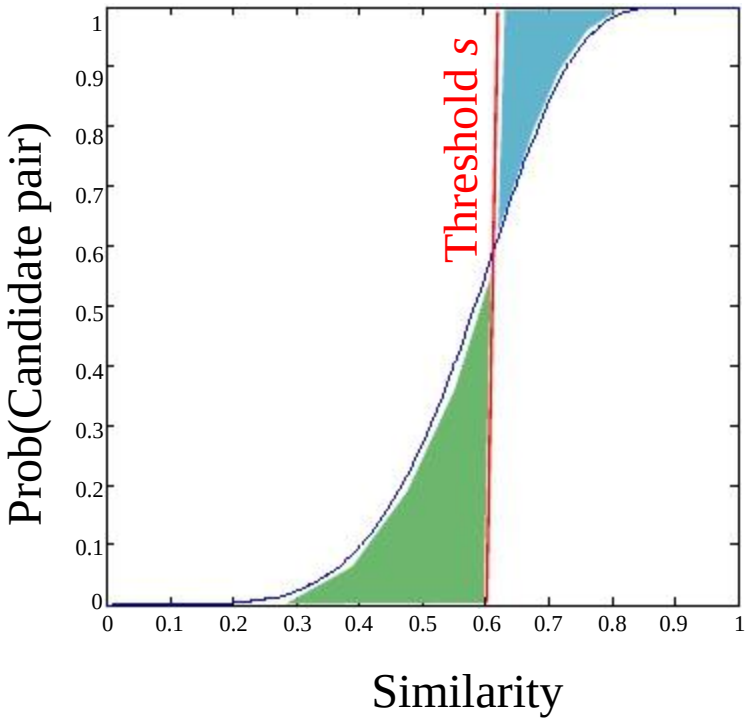
- AND makes all probs. shrink, but by choosing r correctly, we can make the lower prob. approach 0 while the higher does not
- OR makes all probs. grow, but by choosing b correctly, we can make the upper prob. approach 1 while the lower does not

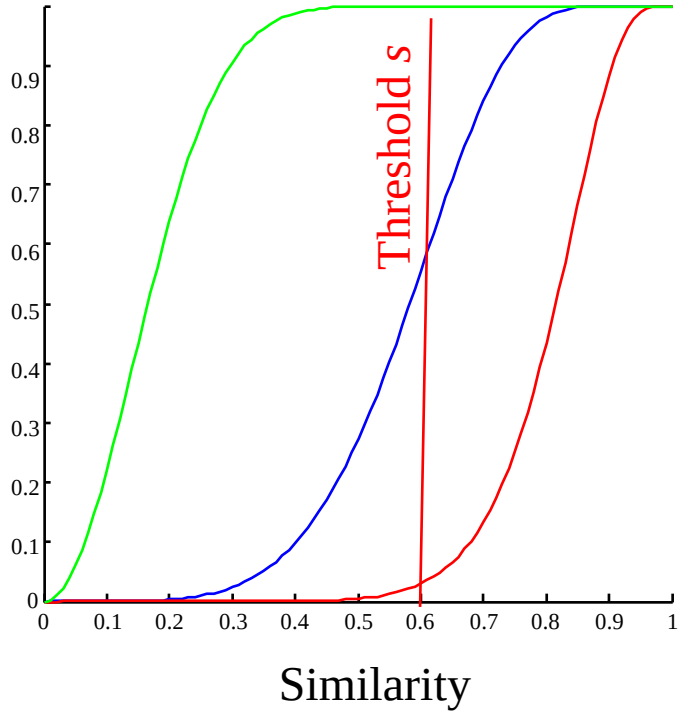




Picking r and b: the S-curve

- Picking r and b to get desired performance
  - 1. 50 hash-functions (r = 5, b = 10)
  - 2. 50 hash-functions (r * b = 50)





r=2, b=25
r=5, b=10
r=10, b=5

Composing Constructions

- Exactly what we did with Min-Hashing
  - If bands match in all r values hash to same bucket
  - Cols that are hashed into ≥ 1 common bucket -> Candidate
- Take points x and y s.t. Pr[h(x) = h(y)] = p
  - H will make (x,y) a candidate pair with prob. P
- Construction makes (x,y) a candidate pair with probability $1-(1-p^r)^b$
  - The S-Curve!
  - Example: Take H and construct H′ by the AND construction with $r = 4$. Then, from H′, construct H″ by the OR construction with $b = 4$

AND-OR Composing Constructions

- Each of the two probabilities p is transformed into $1-(1-p^r)^b$.
  - The "S-curve" studied before.

- Example: Take **H** and construct **H'** by the AND construction with $r = 4$. Then, from **H'**, construct **H''** by the OR construction with $b = 4$.

Table for Function $1-(1-p^4)^4$

p	$1-(1-p^4)^4$
.2	.0064
.3	.0320
.4	.0985
.5	.2275
.6	.4260
.7	.6666
.8	.8785
.9	.9860

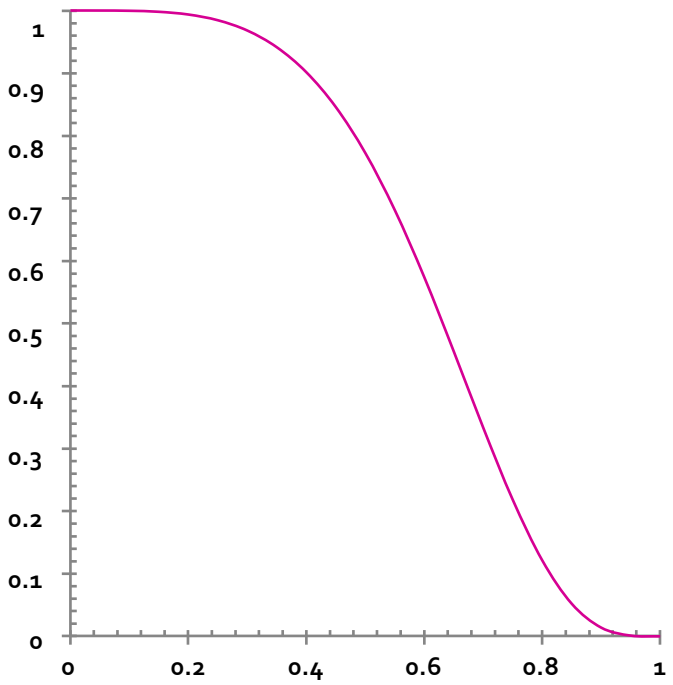
Example: Transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.8785,.0064)-sensitive family.

OR-AND Composing Constructions

- Each of the two probabilities p is transformed into $(1-(1-p)^b)^r$.
  - The same S-curve, mirrored horizontally and vertically.
- Example: Take H and construct H′ by the OR construction with b = 4. Then, from H′, construct H″ by the AND construction with r =4.

Table for Function $(1-(1-p^4))^4$

p	$1-(1-p^4)^4$
.1	.0140
.2	.1215
.3	.3334
.4	.5740
.5	.7725
.6	.9015
.7	.9680
.8	.9936



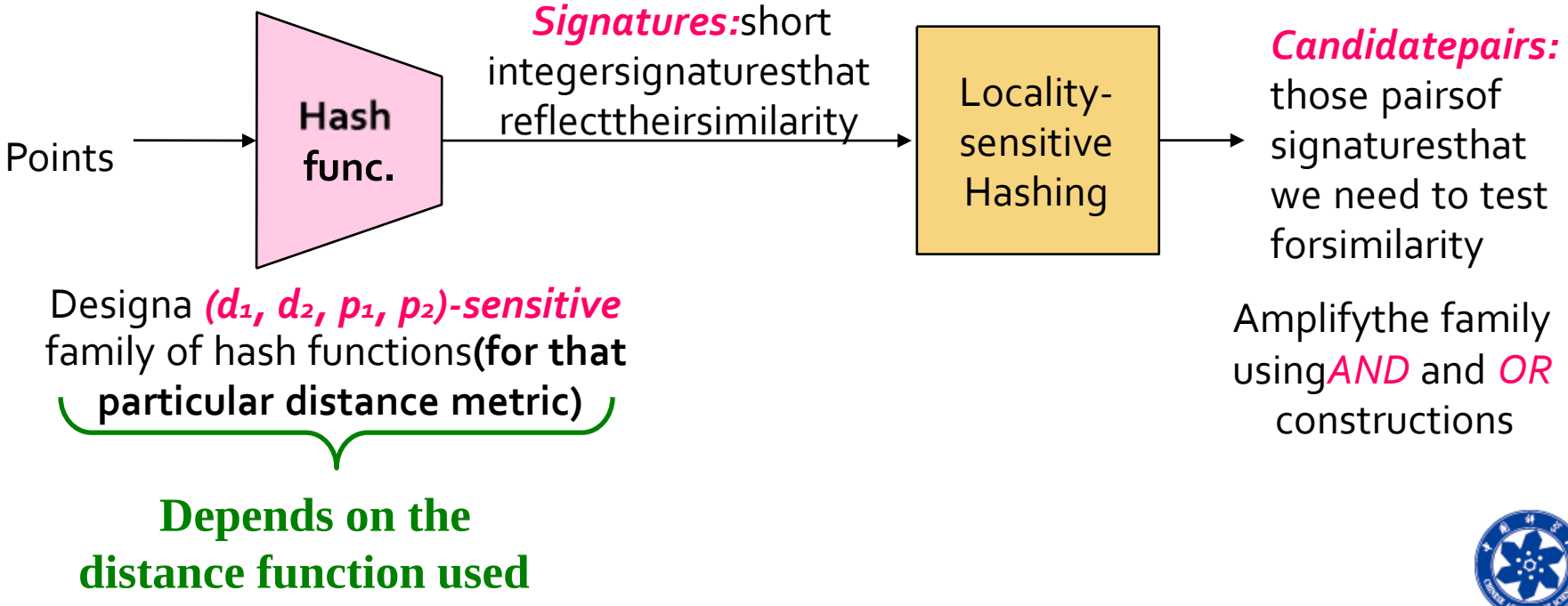
The example transforms a (.2,.8,.8,.2)-sensitive family into a (.2,.8,.9936,.1215)-sensitive family

Cascading Constructions

- Transforms a (.2, .8, .8, .2)-sensitive family into a (.2, .8, . 9936, . 1215)-sensitive family
  - Note this family uses 256 (=4*4*4*4) of the original hash functions
  - Note by using 256 hash functions, the (.2, .8, .8, .2)-sensitive family was transformed into a (.2, .8, .9999996, .0008715)-sensitive family
- The closer to 0 and 1 we get, the more hash functions must be used!

LSH for other Distance Metrics

- **Problem:** More generally, we found similar columns in large sparse matrices with high Jaccard similarity
- **Can we use LSH for other distance measures?**
  - E.g., cosine distance: Random hyperplanes
  - E.g., euclidean distance: Project on lines



Outline

Distance Measures

- Generalized LSH is based on some kind of "distance" between points.
  - Similar points are "close."
  - Jaccard similarity is not a distance; 1 minus Jaccard similarity is.

- Two major classes of distance measure:

  1. Euclidean

  2. Non-Euclidean

Euclidean VS. Non- Euclidean

- A *Euclidean space* has some number of real valued dimensions and "dense" points.
  - There is a notion of "average" of two points.
  - A Euclidean distance is based on the locations of points in such a space.
- Any other space is Non-Euclidean.
  - Distance measures for non-Euclidean spaces are based on properties of points, but not their "location" in a space.

Axioms of a Distance Measure

- *d* is a *distance measure* if it is a function from pairs of points to real numbers such that:

1. $d(x,y) \geq 0$.
2. $d(x,y) = 0$ iff $x = y$.
3. $d(x,y) = d(y,x)$.
4. $d(x,y) \leq d(x,z) + d(z,y)$ (*triangle inequality*).

Some Euclidean Distances

- L_2 *norm*: d(x,y) = square root of the sum of the squares of the differences between *x* and *y* in each dimension.
  - The most common notion of "distance."
- L_1 *norm*: sum of the differences in each dimension.
  - *Manhattan distance* = distance if you had to travel along coordinates only.

Some Euclidean Distances

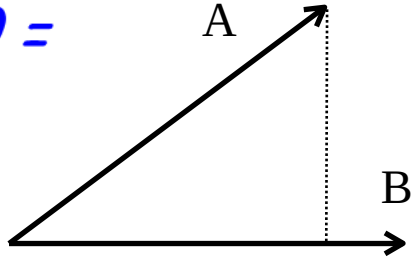
- L_∞ *norm*: d(x,y) = the maximum of the differences between *x* and *y* in any dimension.
- Note: the maximum is the limit as *r* goes to ∞ of the L_r norm: what you get by taking the r^{th} power of the differences, summing and taking the r^{th} root.

Axioms of a Distance Measure

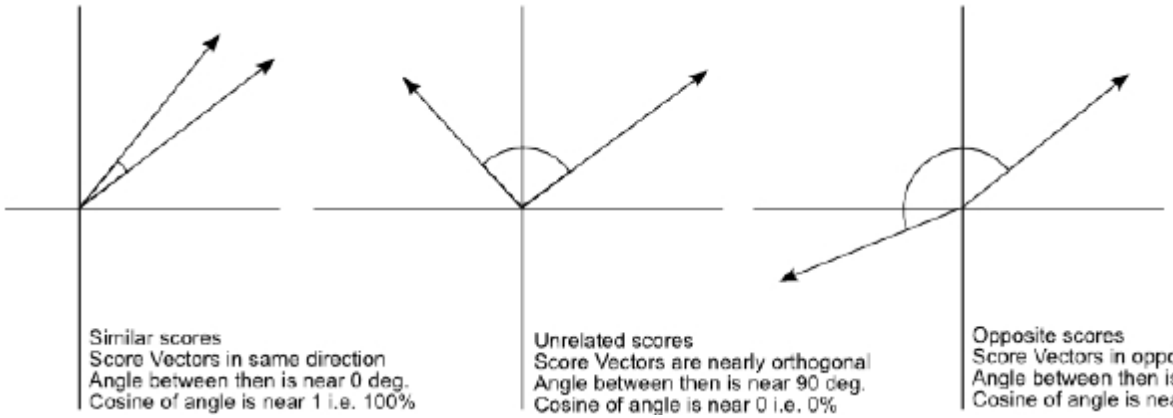
- *Jaccard distance* for sets = 1 – Jaccard similarity.
- *Cosine distance* for vectors = angle between the vectors.
- *Edit distance* for strings = number of inserts and deletes to change one string into another.
- *Hamming Distance* for bit vectors = the number of positions in which they differ.

Cosine Distance

- Cosine distance = angle between vectors from the origin to the points in question $d(A, B) = \theta = \arccos(A \cdot B / \|A\| \cdot \|B\|)$
  - Has range $0 \ldots \pi$ (equivalently 0…180°)
  - Can divide θ by $\pi\pi$ to have distance in range 0…1



- Cosine similarity = 1-d(A,B)
  - But often defined as cosine sim: $\cos(\theta) = \frac{A \cdot B}{\|A\| \|B\|}$



- Has range -1…1 for general vectors
- Range 0..1 for non-negative vectors (angles up to 90°)

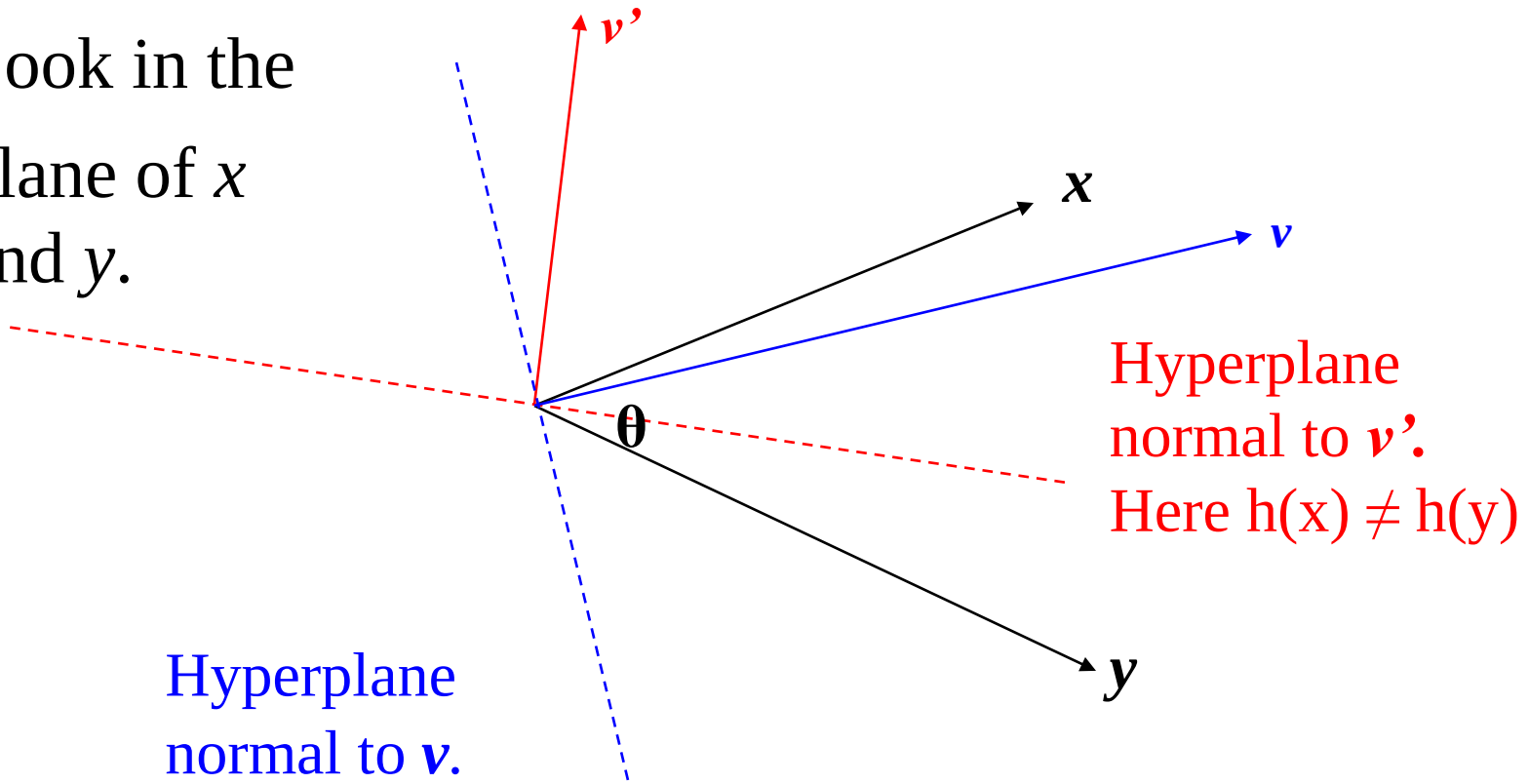
LSH for Cosine Distance

- For cosine distance, there is a technique called Random Hyperplanes
  - Technique similar to Min-Hashing
- Random Hyperplanes method is a $(d_1, d_2, (1-d_1/180), (1-d_2/180))$-sensitive family for any d_1 and d_2
- Pick a random vector v, which determines a hash function h_v with two buckets
- $h_v(x) = +1$ if $v \cdot x \geq 0$; $= -1$ if $v \cdot x < 0$
- LS-family H = set of all functions derived from any vector
- Claim: For points x and y,

$$Pr[h_v(x) = h_v(y)] = 1 - d(x,y)/180$$

Proof of Claim

Look in the plane of x and y.



Hyperplane normal to ***v'***. Here h(x) ≠ h(y)

Hyperplane normal to ***v***. Here h(x) = h(y)

Signatures for Cosine Distance

Random vector

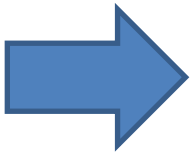
+1	-1	0
-1	-1	0
0	0	0
+1	-1	-1
+1	0	+1
-1	0	+1
-1	0	-1

v1 v2 v3

Input matrix (Shingles x Documents)

1	0	1	0
1	0	0	1
0	1	0	1
0	1	0	1
0	1	0	1
1	0	1	0
1	0	1	0

C1 C2 C3 C4



Signature matrix *M*

-1	+1	-1	+1
-1	-1	-1	-1
0	0	0	0

Sig1 Sig2 Sig3 Sig4

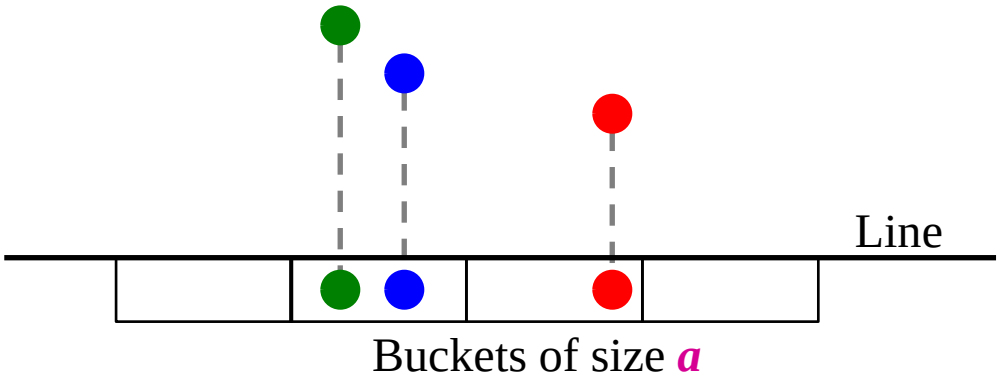
$$M_{i,j} = h_{v_i}(C_j)$$

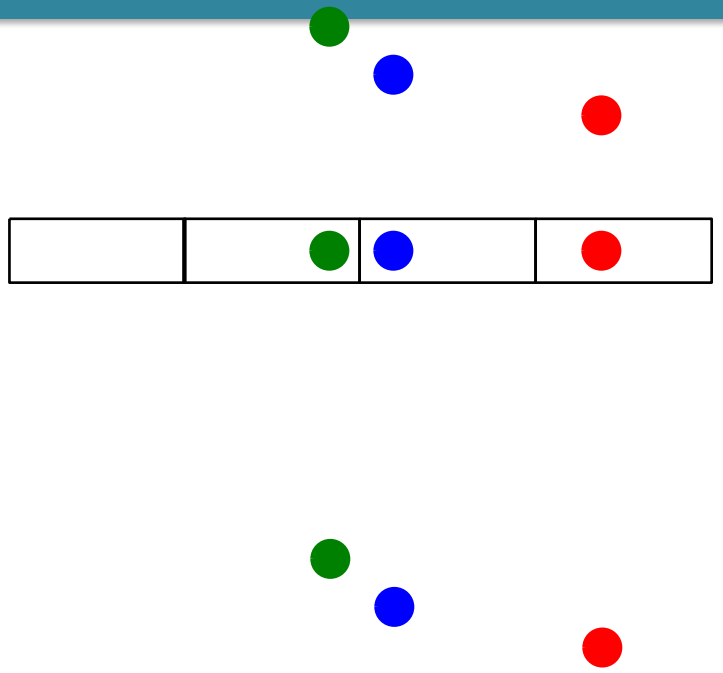
- LSH of cosine distance likes we used the Min-Hash signatures for Jaccard distance
- Amplify using **AND/OR** constructions(*b* bands and *r* rows)

LSH for Euclidean Distance

- Simple idea: Hash functions correspond to lines
- Partition the line into buckets of size a
- Hash each point to the bucket containing its projection onto the line
- **Nearby points are always close;**
  distant points are rarely in same bucket

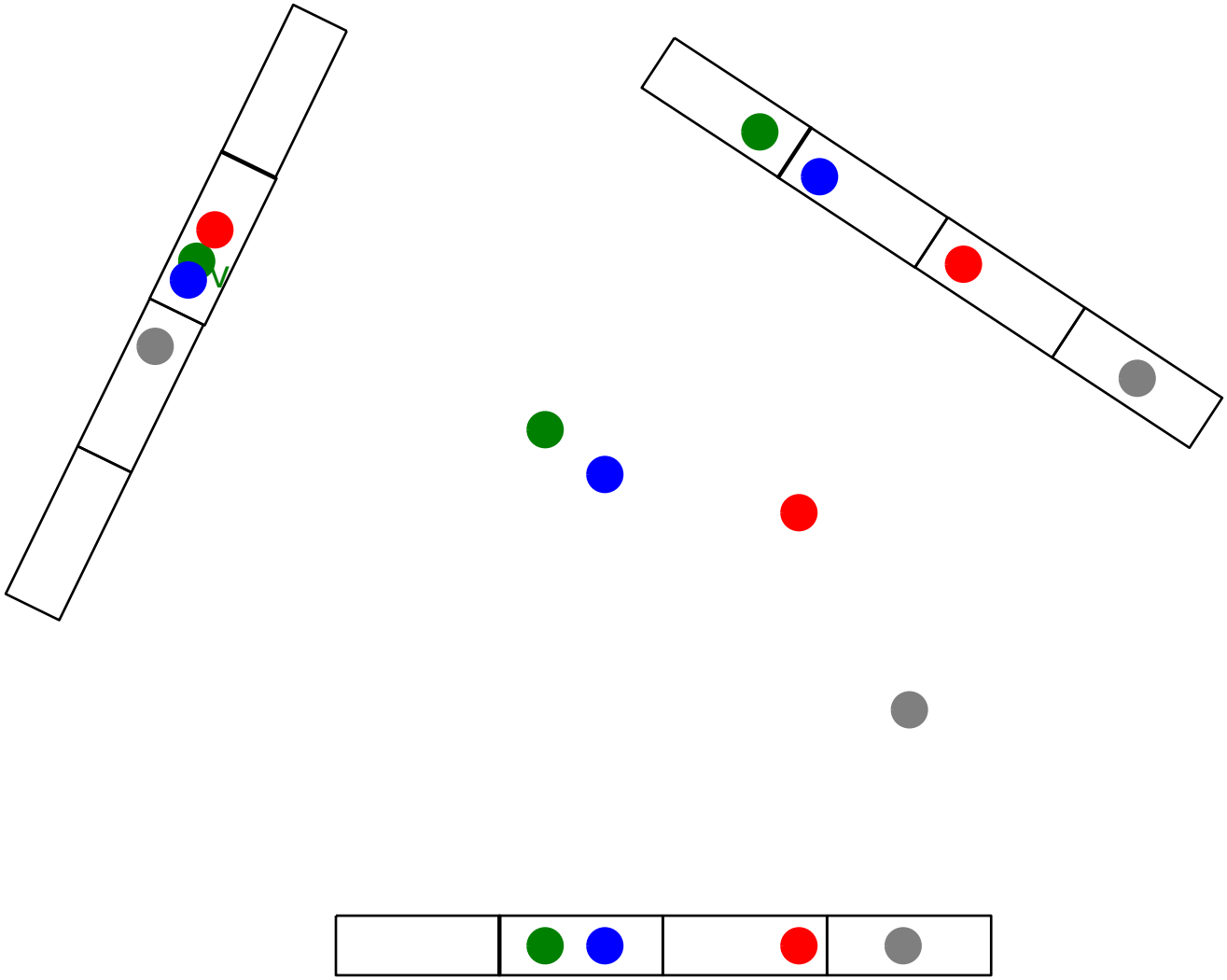
Projection of Points



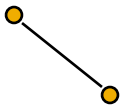


- **"Lucky" case:**
  - Points that are close hash in the same bucket
  - Distant points end up in different buckets
- **Two "unlucky" cases:**
  - **Top:** unlucky quantization
  - **Bottom:** unlucky projection

Multiple Projections



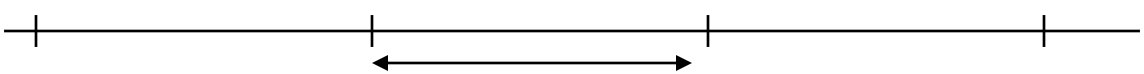
Projection of Points



Points at distance ***d***

If ***d*** << ***a***, then

the chance the points are in the same bucket is at least **1 – *d*/*a***.

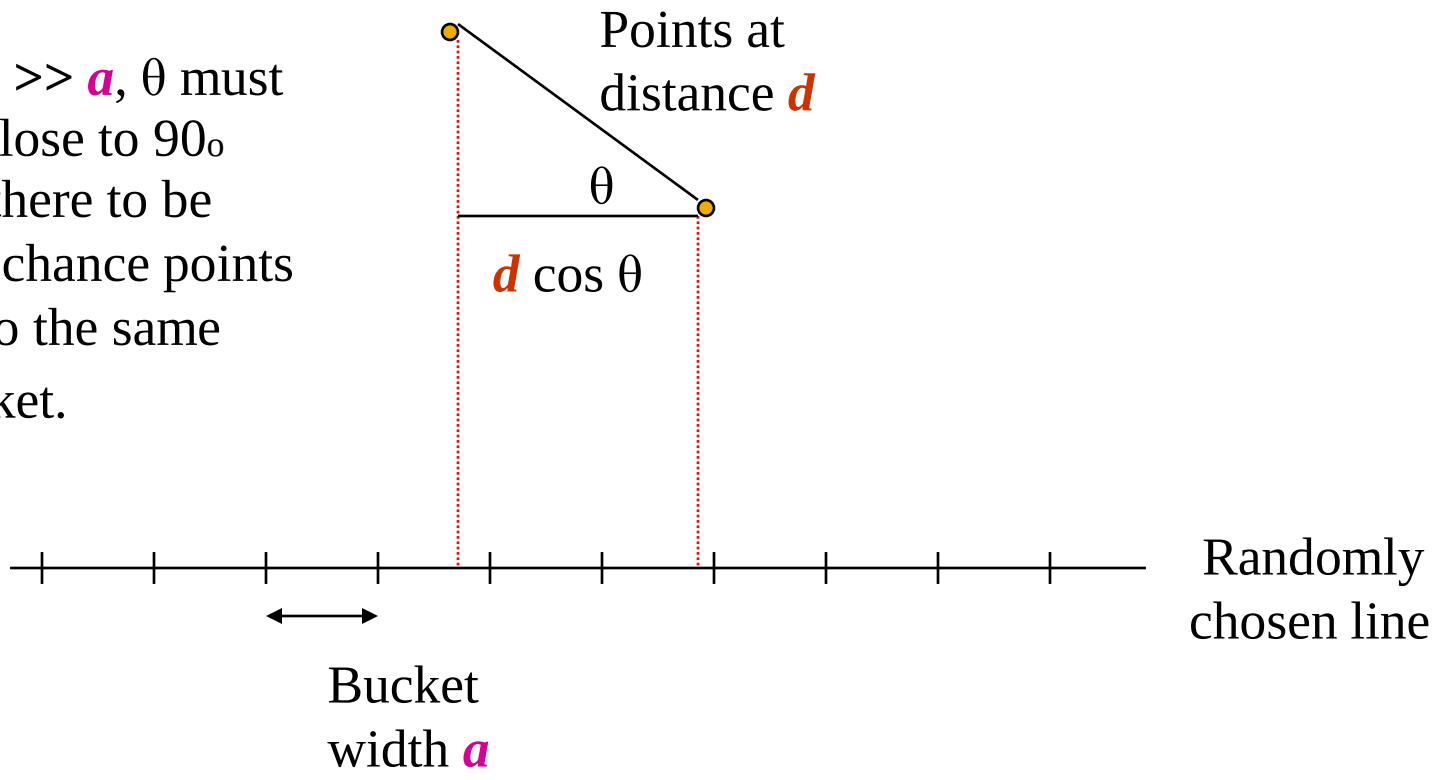


Randomly chosen line

Bucket width ***a***

Projection of Points

If ***d*** >> ***a***, θ must be close to 90o for there to be any chance points go to the same bucket.



An LS-Famlily for Euclidean Distance

- If points are distance $d < a/2$, prob. they are in same bucket $\geq 1 - d/a = \frac{1}{2}$
- If points are distance $d > 2a$ apart, then they can be in the same bucket only if $d \cos \theta \leq a$
  - $\cos \theta \leq \frac{1}{2}$
  - $60 < \theta < 90$, i.e., at most 1/3 probability
- Yields a $(a/2, 2a, 1/2, 1/3)$-sensitive family of hash functions for any a
- Amplify using AND-OR cascades

Fixup: Euclidean Distance

- Projection method yields a *(a/2, 2a, 1/2, 1/3)-sensitive* family of hash functions
- For previous distance measures, we could start with an *(d_1, d_2, p_1, p_2)-sensitive* family for any $d_1 < d_2$, and drive p_1 and p_2 to 1 and 0 by AND/OR constructions
- Note: Here, we seem to need $d_1 \leq 4\, d_2$
- In the calculation on the previous slide we only considered cases $d < a/2$ and $d > 2a$

Fixup-(2)

- But as long as $d_1 < d_2$, the probability of points at distance d_1 falling in the same bucket is greater than the probability of points at distance d_2 doing so

- Thus, the hash family formed by projecting onto lines is an (d_1, d_2, p_1, p_2)-sensitive family for some $p_1 > p_2$
  - Then, amplify by AND/OR constructions

Tell me and I forget.
Show me and I remember.
Involve me and I understand.

Thank you! Q&A