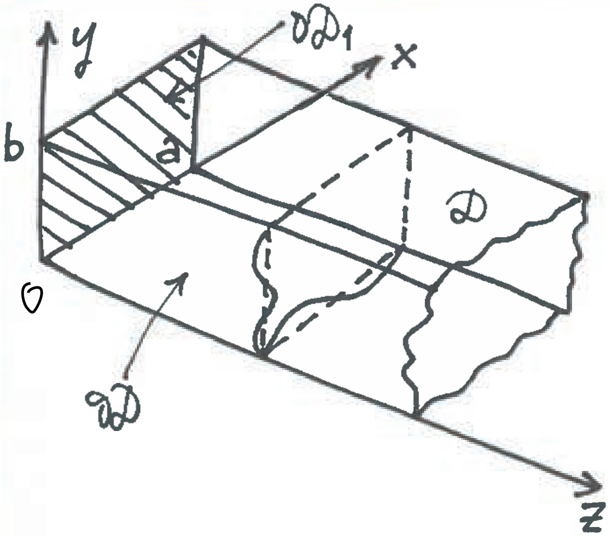


$$c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial^2 u}{\partial t^2} \quad \text{on } \mathcal{D}=\{0\le x\le a, 0\le y\le b, 0\le z<\infty\}$$

$$u(x,y,0,t)=f(x,y)e^{j\omega t} \quad \text{on } \partial\mathcal{D}_1$$

$$u(x,y,z,t)=0 \quad \text{on } \partial\mathcal{D}-\partial\mathcal{D}_1$$

$u(x,y,z,t)$ bounded and causal on $\mathcal{D}$

The response of the waveguide will be a linear superposition of travelling and standing/attenuating waves $\Rightarrow$

Traveling waves

$$u(x,y,z,t)=\sum_n\sum_m B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{j(\omega t-\lambda_{mn}z)}+$$
$$+\sum_q\sum_p D_{pq}\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}e^{-\lambda_{pq}z}e^{j\omega t} \quad \text{on } \mathcal{D} \quad \Rightarrow$$

Attenuating waves

But $u(x,y,0,t)=f(x,y)e^{j\omega t}\Rightarrow f(x,y)=\sum_i\sum_s K_{is}\sin\frac{i\pi x}{a}\sin\frac{s\pi y}{b}\Rightarrow$

Normalization coefficient

$$\Rightarrow K_{is}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{i\pi x}{a}\sin\frac{s\pi y}{b}\,dx\,dy$$

$$\Rightarrow B_{mn}=K_{mn}, \quad D_{pq}=K_{pq}$$

Note that as $z \to \infty$ only traveling modes propagate in the far field, while attenuating modes decay to zero $\Rightarrow$

$$\Rightarrow u(x,y,z,t) \to \sum_n \sum_m B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{j(\omega t - \lambda_{mn} z)} \Rightarrow$$

$\Rightarrow$ The phase velocity of each individual traveling mode is:

$$V_{mn}^{(phase)} = \frac{\omega}{\lambda_{mn}}$$

Frequency

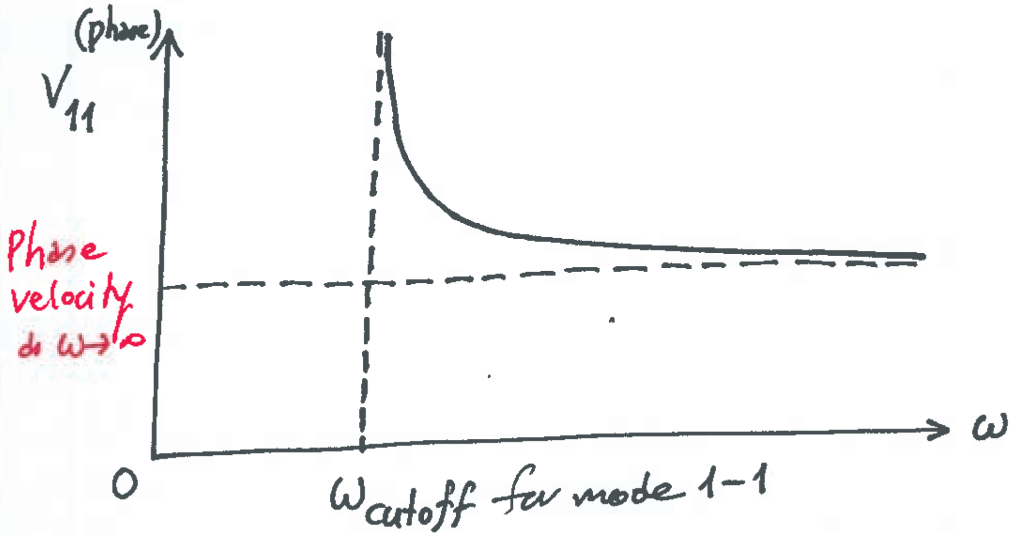
Wavenumber

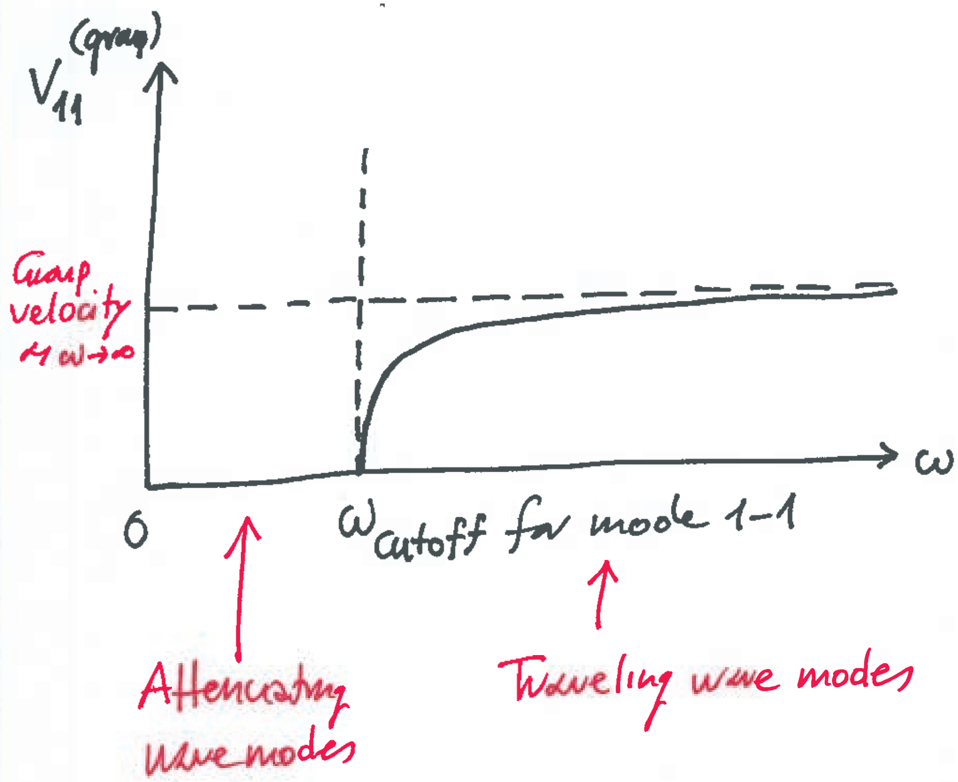
Hence,

$$V_{mn}^{(phase)} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \Rightarrow V_{mn}^{(phase)} \to \infty \text{ as } \omega \to \omega_{cutoff}$$

where $\omega_{cutoff} = \omega_{MN}$ such that $\omega_{MN}^2 = \left(\frac{\omega}{c}\right)^2$. The phase velocity indicates the speed of wave propagation of each individual traveling mode component of a propagating wavepacket (collection of many wave modes). The group velocity indicates how a group of waves propagates as an entity (wavepacket) $\Rightarrow$ $V_{mn}^{(group)} = \frac{d\omega}{d\lambda_{mn}} = \frac{1}{\left(\frac{d\lambda_{mn}}{d\omega}\right)} \Rightarrow$

$$\Rightarrow V_{mn}^{(group)} = \frac{c^2}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \Rightarrow V_{mn}^{(group)} \to 0 \text{ as } \omega \to \omega_{cutoff}$$

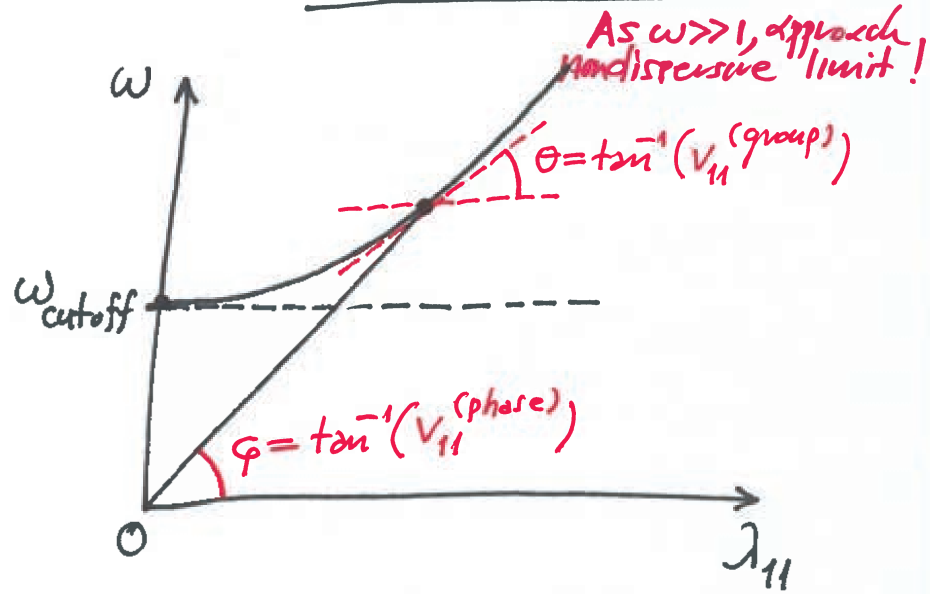




Plots for $c=a=b=1 \Rightarrow$

$\Rightarrow$ We focus on wave mode 1-1

Dispersion relation



The relation between ω and the wave number λ_{11} is the <u>dispersion relation</u> for that traveling mode $\Rightarrow$

$\Rightarrow$ Note that this differs for each mode.

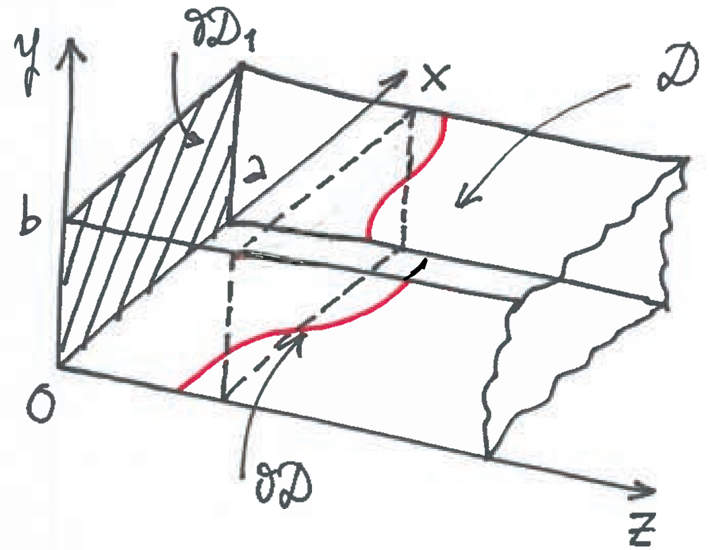
Remarks

1) For a waveguide if the dispersion relation is $\omega = ck$ (wavenumber), then the wave guide is <u>non-dispersive</u>, all traveling wave modes travel with the same phase velocity irrespective of frequency, and the phase velocity equals the group velocity. If the dispersion relation $\omega = \omega(k)$ is any function differing than the proportional one, then the waveguide is <u>dispersive</u>, and the phase velocity is different than the group velocity. Hence, the 3D waveguide considered here is dispersive.

2) In a waveguide energy in the far field is transmitted with the group velocity $\Rightarrow$ As we approach the cut-off frequency the group velocity tends to zero and no energy transmission in the far field can occur. The cut-off frequency represents the boundary between the regime of attenuating waves and the regime of traveling waves $\Rightarrow$
$\Rightarrow$ At the cut-off frequency the system response is a <u>standing wave</u>.

3) In linear waveguides the dispersion relation does not depend on the wave amplitude (provided that the system is time-invariant).

Waveguide with traction-free boundary conditions



$$c^2 \nabla^2 u = u_{tt} \text{ on } D = \{0 \le x \le a, 0 \le y \le b, 0 \le z < \infty\}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial D - \partial D_1$$

$$u(x,y,0,t) = f(x,y)\, e^{j\omega t} \text{ on } \partial D_1$$

At the steady state,

?

$$u(x,y,z,t) = e^{j\omega t} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\, Z(z) \Rightarrow$$

$\Rightarrow Z(z)$ is governed by,

$$\frac{d^2 Z}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right] Z = 0 \Rightarrow$$

0 due to causality

$$\Rightarrow Z(z) = \cancel{A_{mn}\, e^{j\lambda_{mn} z}} + B_{mn}\, e^{-j\lambda_{mn} z}, \quad \lambda_{mn} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \Rightarrow$$

$$n, m = 0, 1, 2 \dots$$

Note that now the indices start from zero

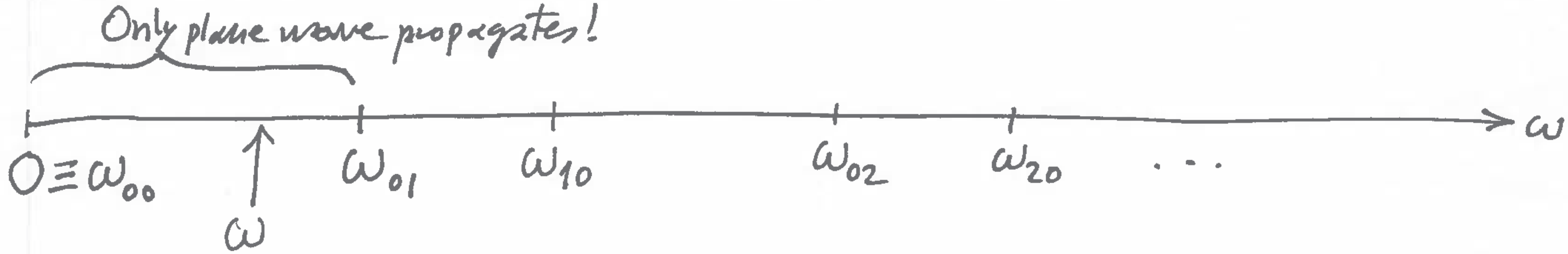
$$\Rightarrow u(x,y,z,t) = \sum_{m=0,1,\dots}^{\infty} \sum_{n=0,1,\dots}^{\infty} B_{mn}\, e^{j(\omega t - \lambda_{mn} z)} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \Rightarrow$$

$\Rightarrow$ If $\omega \le c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$, then the (m,n) wavemode is attenuating $\Rightarrow$

$\Rightarrow \omega = c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$ is the cut-off frequency of the wavemode (m,n).

Important observation: If $m=n=0 \Rightarrow \omega_{cutoff}^{(0,0)}=0 \Rightarrow$ The $(0,0)$ mode, i.e., the plane wave mode propagates for all frequencies $\omega \geq 0$ without dispersion $\Rightarrow u^{(0,0)}(x,y,z,t) = B_{00}\,e^{j(\omega t - cz)} \Rightarrow \lambda_{00} = \frac{\omega}{c} \Rightarrow$

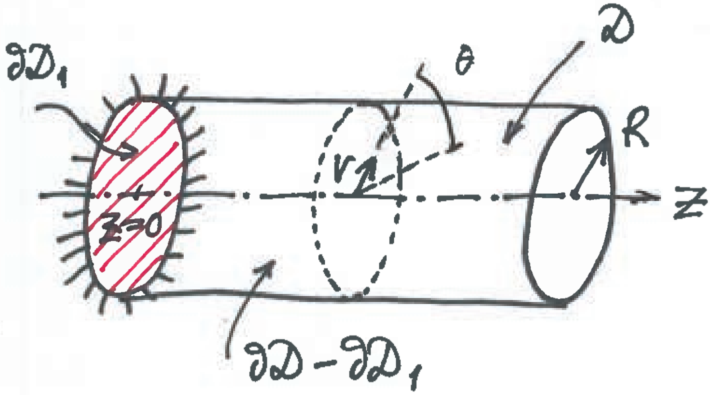
$\Rightarrow \omega = c\lambda_{00}$. Let $\omega_{mn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$, $m,n = 0,1,2,\ldots$



If follows that in the frequency range $0 \equiv \omega_{00} < \omega < \omega_{01}$ only the plane mode can propagate in the far field of this 3D waveguide!

Remark: Note that no such plane wave mode can exist in the waveguide with fixed edges (i.e., boundary conditions $u(x,y,z,t)=0$ on $\partial D - \partial D_1$).

Dispersion in a cylinder



$u(r,\theta,z,t)$ finite and causal

Consider the cylindrical waveguide with fixed boundary conditions,

$$c^2\nabla^2 u = u_{tt} \text{ on } D=\{0<r\le R,\ 0\le\theta<2\pi,\ z\ge 0\}$$

$$u(R,\theta,z,t)=0 \text{ on } \partial D-\partial D_1$$

$$u(r,\theta,0,t)=f(r,\theta)e^{j\omega t} \text{ on } \partial D_1$$

where we used cylindrical coordinates.

We seek a steady state solution in the ~~form~~ $u_{nl}(r,\theta,z,t)=e^{j\omega t}\Phi_{nl}(r,\theta)Z(z)\Rightarrow$

$$\Rightarrow \Phi_{nl}(r,\theta)=\frac{\sqrt{2}}{\sqrt{\pi\gamma}\,R\,J_{n+1}\left(\frac{\omega_{nl}R}{c}\right)}(\cos n\theta+\sin n\theta)\,J_n\left(\frac{\omega_{nl}r}{c}\right)\Rightarrow$$

Where ω_{nl} is the l-th zero of $J_n\left(\frac{\omega_{nl}R}{c}\right)$

$$\Rightarrow c^2\left(\frac{\partial^2 u_{nl}}{\partial r^2}+\frac{1}{r}\frac{\partial u_{nl}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_{nl}}{\partial\theta^2}+\frac{\partial^2 u_{nl}}{\partial z^2}\right)=-\omega^2 e^{j\omega t}Z(z)\Phi_{nl}(r,\theta)\Rightarrow$$

$$\left.\begin{aligned}&\Rightarrow c^2\left(\frac{\partial^2\Phi_{nl}}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_{nl}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi_{nl}}{\partial\theta^2}\right)Z(z)+c^2\Phi_{nl}\frac{d^2Z}{dz^2}=-\omega^2 Z(z)\Phi_{nl}\\ &\text{but } \frac{\partial^2\Phi_{nl}}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi_{nl}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi_{nl}}{\partial\theta^2}=-\left(\frac{\omega_{nl}}{c}\right)^2\Phi_{nl}\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow -\left(\frac{\omega_{nl}}{\cancel{c}}\right)^2\cancel{c^2}\cancel{\Phi_{nl}}\,Z(z)+\omega^2 Z(z)\cancel{\Phi_{nl}}+c^2\cancel{\Phi_{nl}}\frac{d^2Z}{dz^2}=0\Rightarrow$$

$$\Rightarrow \frac{d^2Z}{dz^2}(z) + \left(\frac{\omega^2-\omega_{n\ell}^2}{c^2}\right)Z(z)=0 \Rightarrow Z_{n\ell}(z) = B_{n\ell}e^{-j\lambda_{n\ell}z} + \cancel{A_{n\ell}e^{j\lambda_{n\ell}z}}$$

0 due to causality

where $\lambda_{n\ell}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{\omega_{n\ell}}{c}\right)^2}$ $\Big\} \Rightarrow$

$$\Rightarrow Z_{n\ell}(z) = B_{n\ell}e^{-\lambda_{n\ell}z} \Rightarrow \boxed{u_{n\ell} = B_{n\ell}\,\Phi_{n\ell}(r,\theta)\,e^{j(\omega t-\lambda_{n\ell}z)}},\quad z\geq 0$$

Satisfies the BC at $r=R$

Consider row the waveguide with free BCs.

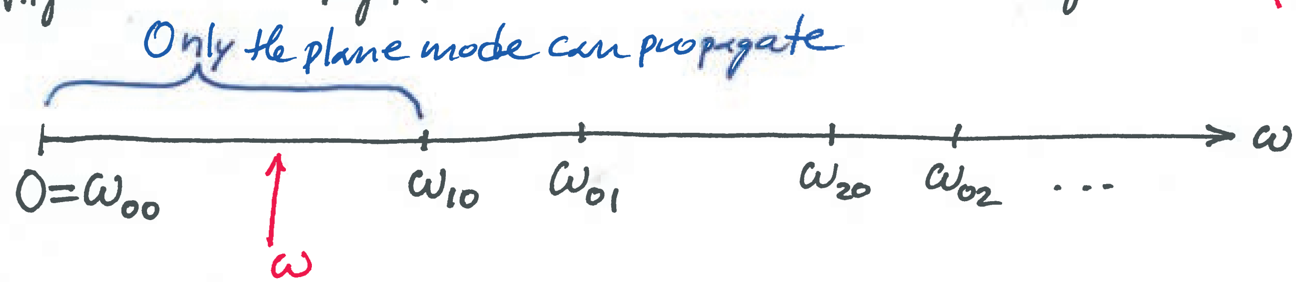
So, for propagation of waves it must be satisfied that $\lambda_{n\ell}$ is real $\Rightarrow$

$\Rightarrow \omega > \omega_{n\ell}$. Note that if $\omega < \min(\omega_{01},\omega_{10}) \Rightarrow \omega_{cutoff} = \omega_{00} = 0 \Rightarrow$

$\Rightarrow$ Then for $0 \leq \omega < \min(\omega_{01},\omega_{10})$ only the plane wave mode $(0,0)$ can propagate in the cylindrical waveguide. But $\Phi_{00}(r,\theta) = \frac{\sqrt{2}}{\sqrt{\pi}\gamma R}J_0\left(\frac{\omega_{00}r}{c}\right) =$

$$= \frac{\sqrt{2}}{\sqrt{\pi}\gamma R}J_0(0) = \frac{\sqrt{2}}{\sqrt{\pi}\gamma R} \Rightarrow u_{00}(r,\theta,z,t) = B_{00}\frac{\sqrt{2}}{\sqrt{\pi}\gamma R}e^{j(\omega t - \frac{\omega}{c}z)}.$$

Plane wave mode



Hence, if we want to transmit waves in the far field with zero (or minimal) dispersion, excite the (0,0) mode

Note that the plane mode is non-dispersive as the wavenumber λ_{00} is proportional to the frequency $\Rightarrow \lambda_{00} = \omega/c$. It's is unique in that sense.

The phase velocity of the (n,l) mode is,

$$V_{nl}^{(phase)} = \frac{\omega}{\lambda_{nl}} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{nl}}{c}\right)^2}}$$

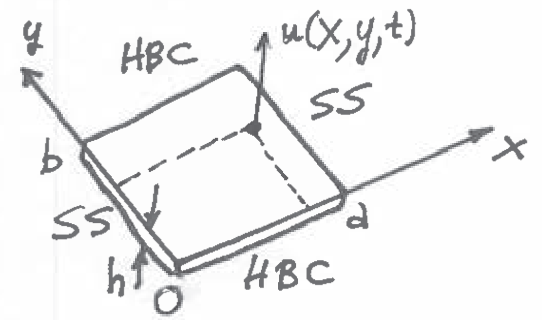
So, $V_{nl}^{(phase)} \to \infty$ as $\omega \to \omega_{nl}^{cutoff} = \omega_{nl}$.

The group velocity of the (n,l) mode is,

$$V_{nl}^{(group)} = \frac{d\omega}{d\lambda_{nl}} = \frac{1}{\left(\frac{d\lambda_{nl}}{d\omega}\right)} = \frac{c^2}{\omega}\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega_{nl}}{c}\right)^2}$$

Hence, $V_{nl}^{(group)} \to 0$ as $\omega \to \omega_{nl}$.

Vibrating rectangular plates



In order to apply separation of variables to this problem we must restrict attention to the combinations of boundary conditions that have two opposite edges simply supported and the other two with homogeneous boundary conditions. Note that in this case we must prescribe two BCs at each edge. This is because, in contrast to the membrane, the plate can support bending moments (2D extension of beam).

Examine then the case of three simply supported edges and one traction-free edge ⇒ Also assume a uniform plate ⇒

$$\nabla^4 u(x,y,t) + \frac{\gamma}{D} u_{tt}(x,y,t) = 0 \quad \text{on} \quad \mathcal{D} = \{0 \le x \le a,\ 0 \le y \le b\} \qquad (*)$$

$$\left.\begin{aligned}
&u(x,y,t)=0 \quad \text{on} \quad \overset{0<x<a}{\cancel{x=0,\ x=a}},\ y=0 \\
&u_{xx}(x,y,t)=0 \quad \text{on} \quad x=0,\ x=a \\
&u_{yy}(x,y,t)=0 \quad \text{on} \quad y=0 \\
&u_{yy}(x,y,t)=0 \quad \text{on} \quad y=b \\
&u_{yyy}(x,y,t)=0 \quad \text{on} \quad y=b
\end{aligned}\right\}$$

where $\nabla^4 = (\nabla^2)^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$

$D = \frac{Eh^3}{12(1-\nu^2)}$, γ = mass per unit area

D: Flexural rigidity per unit width
h: Thickness of the plate
ν: Poisson's ratio
E: Modulus of elasticity

Then separation of variables is possible and the free solution is expressed as,

$u(x,y,t)=e^{j\omega t}\sin\frac{n\pi x}{a}\,Y(y) \Rightarrow$ Substituting into (*) we get,

$$\left.\begin{aligned}&\frac{d^4Y}{dy^4}-\left(\frac{n\pi}{a}\right)^2\frac{d^2Y}{dy^2}+\left[\left(\frac{n\pi}{a}\right)^4-\frac{\gamma}{D}\omega^2\right]Y=0\\ &\text{Let } \left(\frac{n\pi}{a}\right)^4-\frac{\gamma}{D}\omega^2\equiv\mu^4>0 \leftarrow \text{Note assumption}\end{aligned}\right\}\Rightarrow \begin{aligned}&\text{Seek solution of the form}\\ &Y(y)=Ce^{\lambda y}\Rightarrow\end{aligned}$$

$$\Rightarrow \lambda^4-\left(\frac{n\pi}{a}\right)^2\lambda^2+\mu^4=0 \Rightarrow \lambda^2=\frac{1}{2}\left(\frac{n\pi}{a}\right)^2\pm\sqrt{\frac{1}{4}\left(\frac{n\pi}{a}\right)^4-\left(\frac{n\pi}{a}\right)^4+\frac{\gamma}{D}\omega^2}\Rightarrow$$

$$\Rightarrow \lambda^2=\frac{1}{2}\left(\frac{n\pi}{a}\right)^2\pm\sqrt{\frac{\gamma}{D}\omega^2-\frac{3}{4}\left(\frac{n\pi}{a}\right)^4}\Rightarrow\begin{cases}\lambda_{1,3}=\pm\left[\frac{1}{2}\left(\frac{n\pi}{a}\right)^2+\sqrt{\frac{\gamma}{D}\omega^2-\frac{3}{4}\left(\frac{n\pi}{a}\right)^4}\right]^{1/2}\\ \lambda_{2,4}=\pm\left[\frac{1}{2}\left(\frac{n\pi}{a}\right)^2-\sqrt{\frac{\gamma}{D}\omega^2-\frac{3}{4}\left(\frac{n\pi}{a}\right)^4}\right]^{1/2}\end{cases}$$

Hence, the most general solution of the 4th order dif. equation is,

$$\left.Y(y)=C_1\cos\lambda_1 y+C_2\sin\lambda_1 y+C_3\cosh\lambda_2 y+C_4\sinh\lambda_2 y\right\}\Rightarrow\left.\begin{aligned}&C_1+C_3=0\\ &\lambda_1^2C_1+\lambda_2^2C_3=0\end{aligned}\right\}\Rightarrow$$

But $Y(0)=Y''(0)=0$

$\Rightarrow C_1=C_3=0$ for $\lambda_1\neq\lambda_2$

$$\text{Also, }\left.\begin{aligned}&Y''(b)=0\Rightarrow\lambda_1^2C_2\sinh\lambda_1 b+\lambda_2^2C_4\sinh\lambda_2 b=0\\ &Y'''(b)=0\Rightarrow\lambda_1^3C_2\cosh\lambda_1 b+\lambda_2^3C_4\cosh\lambda_2 b=0\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \begin{bmatrix} \lambda_1^2 \sinh\lambda_1 b & \lambda_2^2 \sinh\lambda_2 b \\ \lambda_1^3 \cosh\lambda_1 b & \lambda_2^3 \cosh\lambda_3 b \end{bmatrix} \begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\Rightarrow$ For nontrivial solutions require that the determinant of coefficients is zero $\Rightarrow$ Obtain the following frequency equation:

$$\boxed{\lambda_2 \tanh\lambda_1 b = \lambda_1 \tan\lambda_2 b} \quad (**)$$

Solving this equation compute the natural frequencies ω_{ij}
$i \rightarrow$ refers to $\sin\frac{i\pi x}{a}$
$j \rightarrow$ refers to the j-th root of $(**)$

Also, obtain, $C_4 = -\left(\frac{\lambda_1}{\lambda_2}\right)^2 \frac{\sinh\lambda_1 b}{\sinh\lambda_2 b} C_2$

Hence, the corresponding (i,j)-th eigenfunction is,

Where i enters implicitly in the computation of λ_{1j} and λ_{2j}

$$Y_{ij}(y) = C_2\left[\sinh\lambda_{1j} y - \left(\frac{\lambda_{1j}}{\lambda_{2j}}\right)^2 \frac{\sinh\lambda_{1j} b}{\sinh\lambda_{2j} b} \sinh\lambda_{2j} y\right]$$

Then, the response of the plate is expressed as,

$$u(x,y,t) = \sum_i \sum_j A_{ij}(t) \sin\frac{i\pi x}{a}\left[\sinh\lambda_{1j} y - \left(\frac{\lambda_{1j}}{\lambda_{2j}}\right)^2 \frac{\sinh\lambda_{1j} b}{\sinh\lambda_{2j} b} \sinh\lambda_{2j} y\right]$$

Using the orthogonality properties of the eigenfunctions we can obtain the infinite series of modal oscillators

$$\ddot{A}_{ij}(t) + \omega_{ij}^2 A_{ij}(t) = 0$$

Remark

As mentioned previously, separation of variables in the equation of motion of the plate requires that two opposite edges have simply-supported BCs. Then, we obtain the following fourth-order differential equation (say in the y-variable):

$$\frac{d^4Y}{dy^4} - a_n^2 \frac{d^2Y}{dy^2} - (\beta^4 - a_n^4)\,Y = 0$$

Appropriate BCs:

(1) SS–SS–SS–SS
(2) SS–C–SS–C
(3) SS–C–SS–SS
(4) SS–F–SS–SS
(5) SS–F–SS–F
(6) SS–F–SS–C

Examined in previous derivation

There are three possible cases:

I. $\beta^4 > a_n^4 \Rightarrow Y(y) = C_1 \sin J_2 y + C_2 \cos J_2 y + C_3 \sinh J_1 y + C_4 \cancel{\sinh}\ \cosh J_1 y$

$$J_1 = (\beta^2 + a_n^2)^{1/2};\quad J_2 = (\beta^2 - a_n^2)^{1/2}$$

Can be used to solve the six combinations of BCs.

II. $\beta^4 = a_n^4 \Rightarrow Y(y) = C_1 \sinh J_1 y + C_2 y \sinh J_1 y + C_3 \cosh J_1 y + C_4 y \cosh J_1 y$

III. $\beta^4 < a_n^4 \Rightarrow Y(y) = C_1 \sinh J_2' y + C_2 \cosh J_2' y + C_3 \sinh J_1 y + C_4 \cosh J_1 y$

$$J_1 = (\beta^2 + a_n^2)^{1/2};\quad J_2'^2 = -J_2^2 = a_n^2 - \beta^2 \Rightarrow J_2' = \sqrt{a_n^2 - \beta^2}$$

The BCs (1–3) cannot be solved by case III, however, the last three combinations of BCs (4–6) correspond to this case.