1.1 Normal Modes of Vibration

$$[M]\{\ddot{q}\} + [K]\{q\} = \{0\} \longrightarrow \{q(t)\} = \{C\} f(t) \longrightarrow \boxed{([K] - \omega^2 [M])\{C\} = \{0\}} \quad (*)$$

Then, we derive an eigenvalue problem. This is the requirement for synchronous oscillations of the system. To compute ω and $\{C\}$ we consider the set of linear homogeneous equations $(*)$ and require that $\det([K] - \omega^2 [M]) = 0$ $(**)$ in order to get nontrivial solutions for $\{C\}$.

n eigenfrequencies squared
$0 \leq \omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_n^2$

Note that since $[K] = [K]^T$ and $[M] = [M]^T$ (as a result of the principle of reciprocity) and also are positive semi-definite, and positive-definite, respectivelly, that means that $\omega_i^2 \geq 0$, $i = 1, \ldots, n$. Moreover, all roots of $(**)$ are real. Note that it is possible that $\omega_i^2 = \omega_j^2$ for some $i, j \in [1, \ldots, n]$ (this is especially true for systems with symmetry).

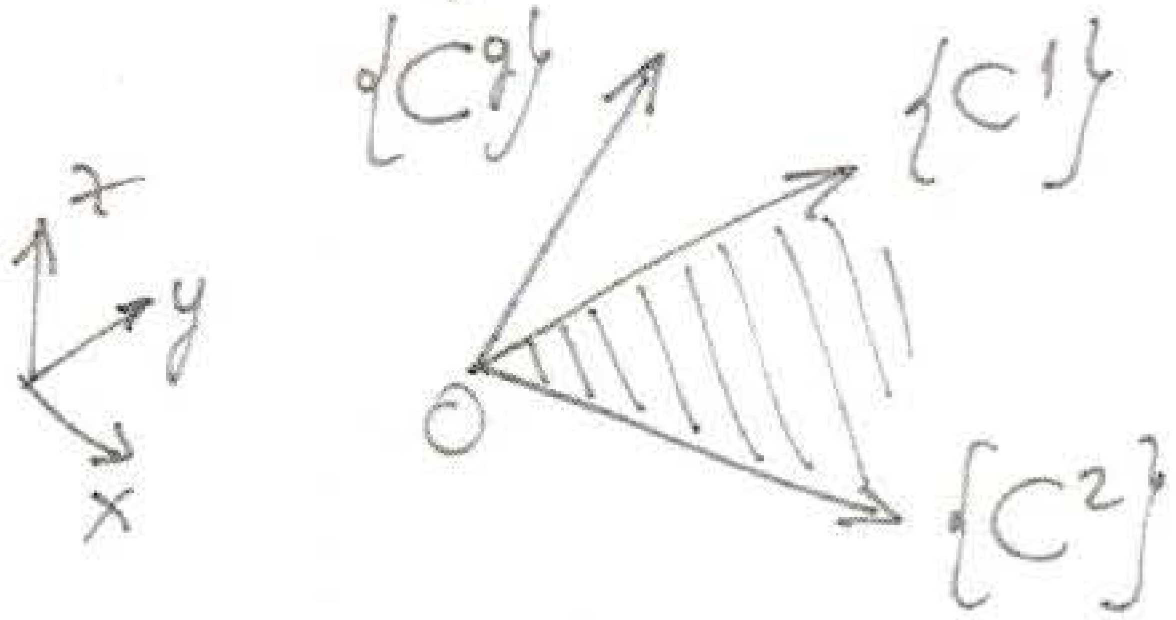
To each natural frequency squared ω_r^2 there corresponds a non-unique eigenvector computed by,

$$([K] - \omega_r^2 [M])\{C\} = \{0\} \Longrightarrow$$ Compute the eigenvector $\{C_r\}$ which is determined up to an arbitrary multiplicative constant

How to evaluate the undetermined multiplicative constant? Many ways:

- Set one of the elements of $\{C_r\}$ arbitrarily equal to 1
- Use mass-normalization, require that $\{C_r\}^T[M]\{C_r\} = 1$ ← Preferable for simplicity
- Require that the sum of the squares of all elements is equal to 1
- ...

There are at most n linearly independent eigenvectors $\{C_r\}$, $r=1,\dots,n$; although there are always n natural frequencies, the number of eigenvectors can be $\leq n$ since there are degenerate cases where we have multiplicities of eigenfrequencies leading to a fewer number of "**true** eigenvectors". In cases like these we can define "generalized eigenvectors" to complete the basis of eigenvectors.



Remarks

This holds for distinct or repeated eigenfrequencies

i) The set of eigenvectors $\{C_1\},\dots,\{C_n\}$ is complete in the sense that any n-dim vector representing a possible oscillation of the system can be expressed as a linear superposition of the eigenvectors $\Rightarrow$

$$\{q(t)\} = \alpha_1\{C_1\}\cos(\omega_1 t+\varphi_1) + \dots + \alpha_n\{C_n\}\cos(\omega_n t+\varphi_n)$$

Denote the 'participation' of each mode in the response

phase of each mode

The 'amplitudes' α_i, $i=1,\dots,n$ and the phases φ_i, $i=1,\dots,n$ are then determined by the initial conditions.

ii) Orthogonality property of eigenvectors: Let's consider two eigenvectors corresponding to two distinct natural frequencies ω_r and $\omega_s \Rightarrow$ From (*),

$$\left.\begin{array}{l}\omega_r^2[M]\{C_r\}=[K]\{C_r\}\\ \omega_s^2[M]\{C_s\}=[K]\{C_s\}\end{array}\right\}\Rightarrow \begin{array}{l}\omega_r^2\{C_s\}^T[M]\{C_r\}=\{C_s\}^T[K]\{C_r\}\\ \omega_s^2\{C_r\}^T[M]\{C_s\}=\{C_r\}^T[K]\{C_s\}\xRightarrow{T}\\ \Rightarrow\omega_s^2\{C_s\}^T[M]^T\{C_r\}=\{C_s\}^T[K]^T\{C_r\}\Rightarrow\\ \Rightarrow\omega_s^2\{C_s\}^T[M]\{C_r\}=\{C_s\}^T[K]\{C_r\}\end{array}\Bigg|\; - \;\Rightarrow$$

$$\Rightarrow \underbrace{(\omega_r^2-\omega_s^2)}_{\neq 0}\{C_s\}^T[M]\{C_r\}=0\Rightarrow$$

$$\Rightarrow \boxed{\{C_s\}^T[M]\{C_r\}=0},\ r\neq s \text{ (Mass-orthogonality condition)}$$

But then, it also holds that $\boxed{\{C_s\}^T[K]\{C_r\}=0}$, $r\neq s$ (Stiffness-orthogonality condition)

~~But~~ recall that one of the ways to normalize the eigenvectors was mass normalization, i.e., $\{C_r\}^T[M]\{C_r\}=1$, $r=1,\dots,n \Rightarrow$ If we define the matrix of eigenvectors

The modal matrix

$$[C]=[\{C_1\},\{C_2\},\dots,\{C_n\}]\Rightarrow \left.\begin{array}{l}[C]^T[M][C]=[I] \leftarrow \text{Identity matrix}\\ [C]^T[K][C]=[\omega_r^2]\end{array}\right\} \cancel{(***)}$$

(orthonormalization wrt mass matrix)

For the special matrix of eigenvectors that satisfies the mass orthonormality conditions (***), we will reserve the lable <u>Mass-orthonormalized Modal Matrix</u> $[C] \equiv [\Phi]$, $[\Phi]^T[M][\Phi] = [I]$, $[\Phi]^T[K][\Phi] = [\omega_r^2]$

Then, the original equations of motion assume a very nice form!

$$\left.\begin{array}{l}[M]\{\ddot{q}\} + [K]\{q\} = \{0\} \\ \{q\} = [\Phi]\{\eta\}\end{array}\right\} \Rightarrow [M][\Phi]\{\ddot{\eta}\} + [K][\Phi]\{\eta\} = \{0\} \Rightarrow$$

$$\Rightarrow \underbrace{[\Phi]^T[M][\Phi]}_{[I]}\{\ddot{\eta}\} + \underbrace{[\Phi]^T[K][\Phi]}_{[\omega_r^2]}\{\eta\} = \{0\} \Rightarrow$$

change from generalized coordinates to modal coordinates

$$\Rightarrow [I]\{\ddot{\eta}\} + [\omega_r^2]\{\eta\} = \{0\} \Rightarrow$$

$$\Rightarrow \ddot{\eta}_r + \omega_r^2 \eta_r = 0, \; r = 1, \ldots, n$$

In terms of modal coordinates we get a set of uncoupled linear oscillators!

<u>Initial conditions:</u>

$$\{q(0)\} = [\Phi]\{\eta(0)\} \Rightarrow$$

$$\Rightarrow \{\eta(0)\} = [\Phi]^{-1}\{q(0)\}$$

$$\{\dot{\eta}(0)\} = [\Phi]^{-1}\{\dot{q}(0)\}$$

$$\eta_r(t) = \alpha_r \cos(\omega_r t + \varphi_r) \Rightarrow$$

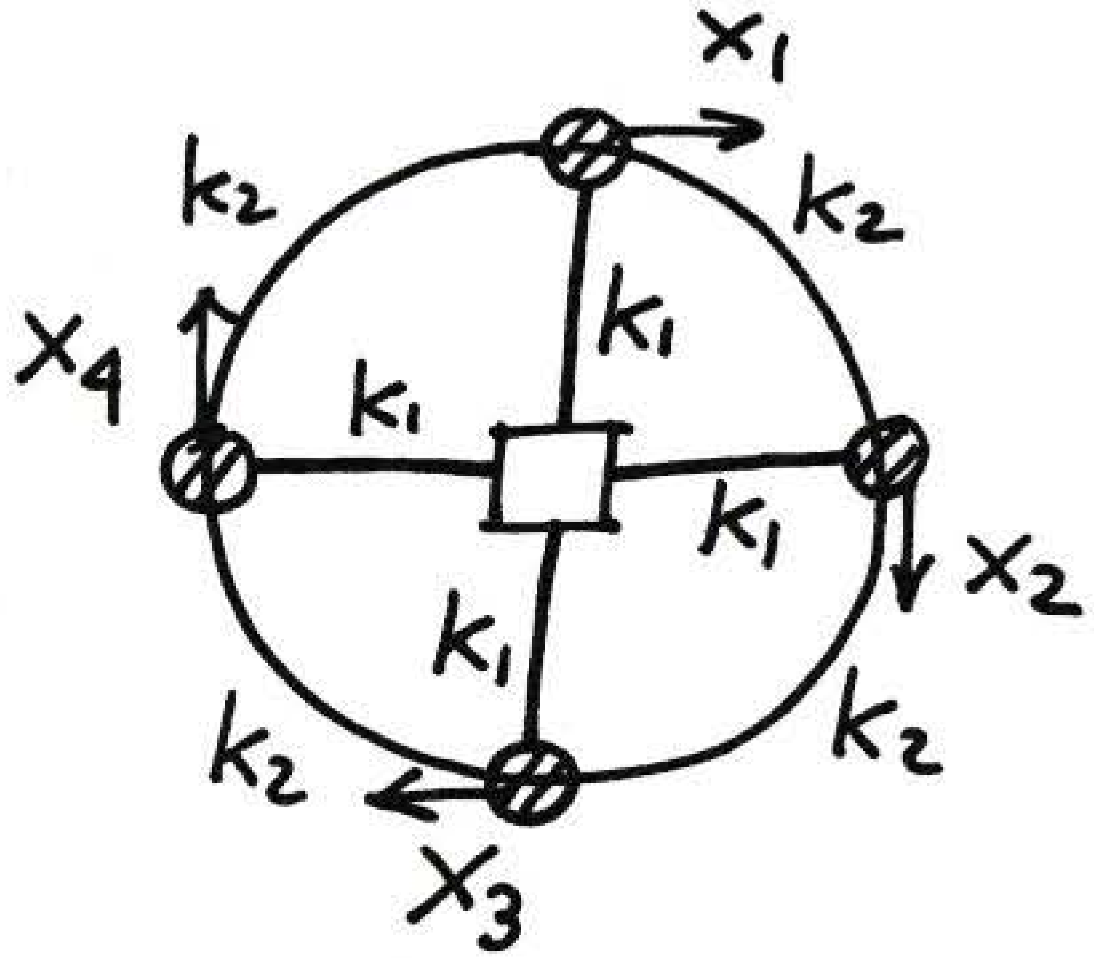
$$\Rightarrow \{q\} = [\Phi]\begin{Bmatrix}\alpha_1 \cos(\omega_1 t + \varphi_1) \\ \vdots \\ \alpha_n \cos(\omega_n t + \varphi_n)\end{Bmatrix} \Rightarrow$$

$$\Rightarrow \{q(t)\} = \sum_{i=1}^{n} \alpha_r \{\Phi_r\} \cos(\omega_r t + \varphi_r)$$, α_r, and φ_r determined by initial conditions

iii) Suppose $\exists$ p repeated eigenfrequencies of an n-DOF linear system, where $2 \leq p < n \Rightarrow$ Since $[K]$ and $[M]$ are symmetric square matrices we can always define a complete, orthogonal basis of eigenvectors. However, there are $n-p$ uniquely defined eigenvectors (corresponding to the $n-p$ distinct eigenfrequencies), and p non-unique eigenvectors that span an p-dimensional invariant modal subspace; any orthogonal base of that p-dim invariant subspace can be "generalized" eigenvectors of the system.

Example

All masses are identical and normalized to unity



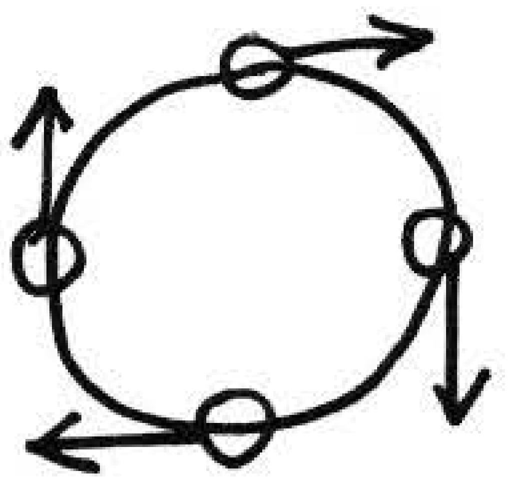
The equations of motion are given by:

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{Bmatrix} + \begin{bmatrix} k_1+2k_2 & -k_2 & 0 & -k_2 \\ -k_2 & k_1+2k_2 & -k_2 & 0 \\ 0 & -k_2 & k_1+2k_2 & -k_2 \\ -k_2 & 0 & -k_2 & k_1+2k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The symmetry of this system yields degeneracy of its vibration modes, with two repeated eigenfrequencies.

Mode 1

(Unique eigenvector)



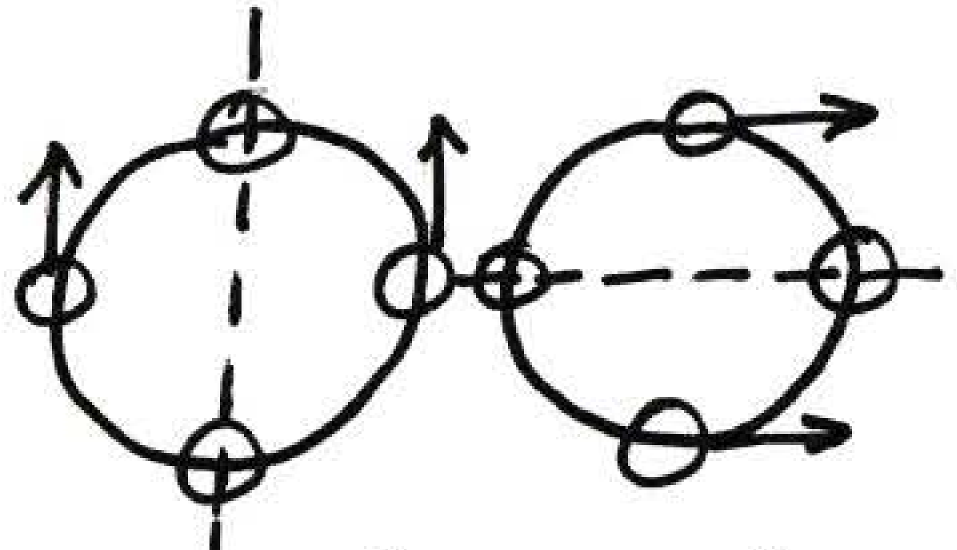
$$\{C_1\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Nat. frequency ω_1

Represents a 1-dim invariant line in the configuration space (x_1, x_2, x_3, x_4), or a 2-dim invariant subspace in the 8-dim phase space of the system $(x_1, \dot{x}_1, x_2, \dot{x}_2, x_3, \dot{x}_3, x_4, \dot{x}_4)$ $\Rightarrow$ On this mode system behaves like a SDOF system.

Modes 2, 3

(Non-unique pairs of eigenvectors)



Nat. frequencies $\omega_2 = \omega_3$

$$\{C_2\} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix}, \{C_3\} = \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix}$$

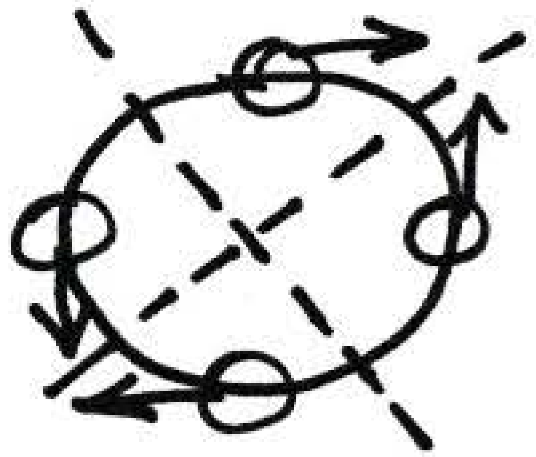
But $\{C_{23}\} = \alpha_1 \{C_2\} + \beta_1 \{C_3\}$ is also an eigenvector, $\alpha_1, \beta_1 \in R$

These generalized eigenvectors span a 4-dimensional invariant modal subspace in the 8-dim phase space of the system.

Mode 4

(Unique eigenvector)

$$\{C_4\} = \begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix}$$

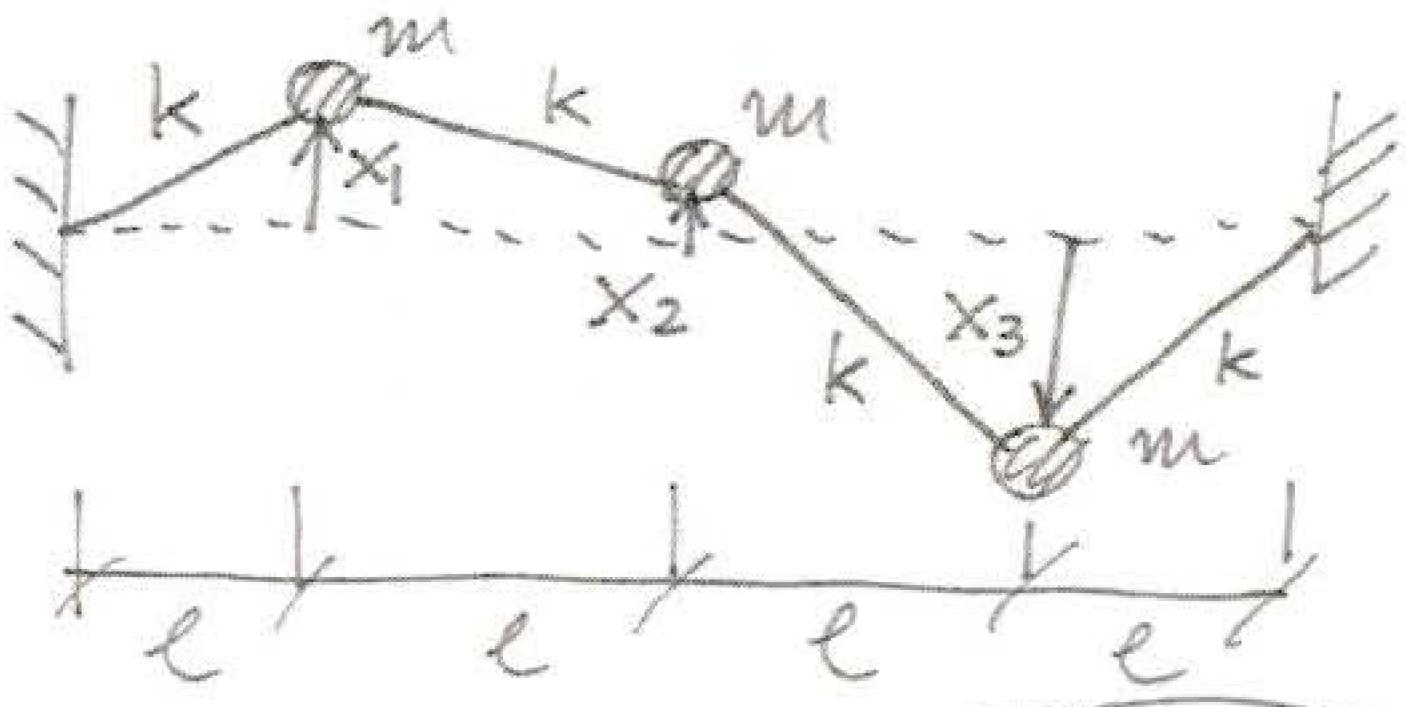


Nat. frequency ω_4

Spans a 2-dim invariant manifold in the 8-dim phase space $\Rightarrow$ When the system oscillates in this mode it behaves as a SDOF linear oscillator.

Example

Small angles



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_1 - x_2)^2 + \frac{1}{2} k (x_2 - x_3)^2 + \frac{1}{2} k x_3^2$$

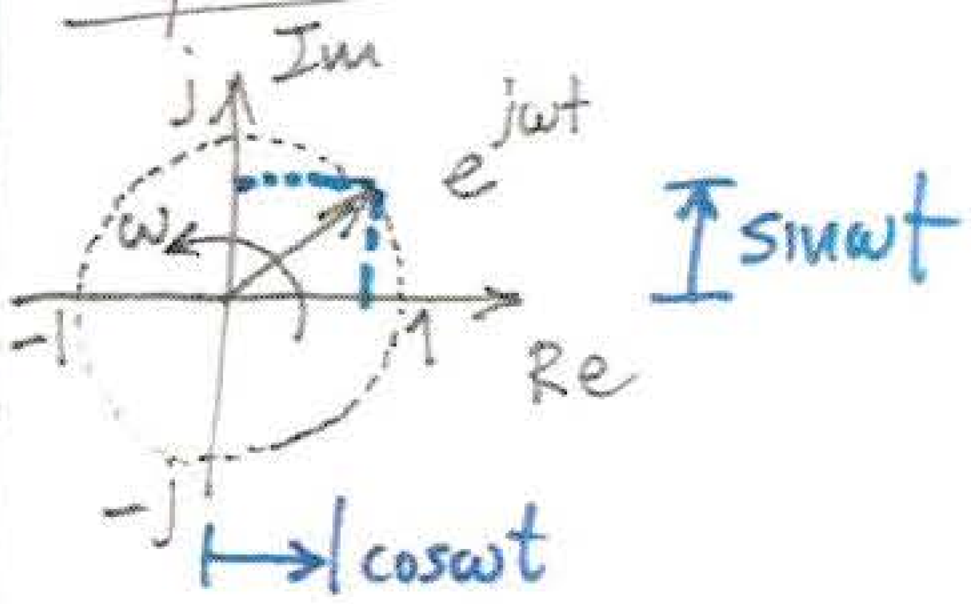
$k = T/e$, T = Internal constant tension in each elastic string.

Then, $T = \frac{1}{2} \{\dot{x}\}^T \underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{[M]} \{\dot{x}\}$, $V = \frac{1}{2} \{x\}^T \underbrace{\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}}_{[K]} \{x\}$, $\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$

So, the equations of motion can be written as $[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$

To compute the normal modes of this system require that $\{x\} = \{C\} e^{j\omega t}$, $j = (-1)^{1/2}$

Digression



$$e^{j\omega t} = \cos\omega t + j \sin\omega t$$

Now the usual notation for oscillating systems is real:

$m\ddot{x} + kx = F\cos\omega t \Rightarrow x(t) = A\cos(\omega t + \varphi)$ (Real steady state solution), $\varphi = 0$ or π

$m\ddot{x} + kx = F e^{j\omega t} \Rightarrow x(t) = X^* e^{j\omega t} = |X^*| e^{j\varphi} e^{j\omega t} = |X^*| e^{j(\omega t + \varphi)}$

Substituting into the equations of motion,

$$\{x\}=\{C\}e^{j\omega t} \Rightarrow \{\ddot{x}\}=-\omega^2\{C\}e^{j\omega t} \Rightarrow -\omega^2[M]\{C\}e^{j\omega t}+[K]\{C\}e^{j\omega t}=\{0\}$$

$$\Rightarrow \underbrace{\left([K]-\omega^2[M]\right)}_{[A]}\{C\}\cancel{e^{j\omega t}}=\{0\} \Rightarrow \text{Eigenvalue problem } \boxed{\left([K]-\omega^2[M]\right)\{C\}=\{0\}}$$

for this example,

$$\begin{bmatrix} 2k-m\omega^2 & -k & 0 \\ -k & 2k-m\omega^2 & -k \\ 0 & -k & 2k-m\omega^2 \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

for nontrivial solutions we require that $\det\left[A\right]=0 \Rightarrow$

$$\Rightarrow J_1\omega^6+J_2\omega^4+J_3\omega^2+J_4=0 \Rightarrow \omega_1^2<\omega_2^2<\omega_3^2$$

Once ω_r^2 is computed, $r=1,2,3$ we substitute back into the eigenvalue problem,

$$(2k-m\omega_r^2)C_1-kC_2=0 \Rightarrow \frac{C_1}{C_2}=\frac{k}{2k-m\omega_r^2}$$

$$-kC_1+(2k-m\omega_r^2)C_2-kC_3=0$$

$$-kC_2+(2k-m\omega_r^2)C_3=0 \Rightarrow \frac{C_2}{C_3}=\frac{2k-m\omega_r^2}{k}$$

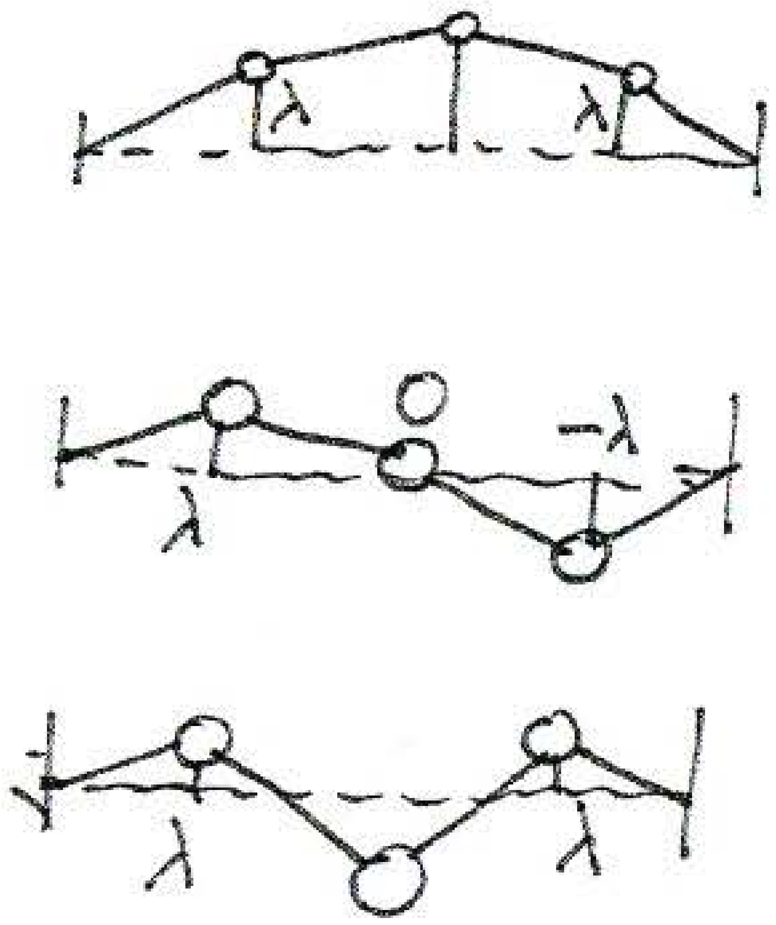
Let $C_1=\lambda$

$$\Rightarrow C_2=\frac{(2k-m\omega_r^2)\lambda}{k}$$

$$C_3=\frac{(2k-m\omega_r^2)\lambda}{2k-m\omega_r^2}=\lambda$$

Then, $\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix}=\begin{Bmatrix} \lambda \\ \frac{(2k-m\omega_r^2)\lambda}{k} \\ \lambda \end{Bmatrix}$; λ is arbitrary multiplicative constant

Eigenvector 1



Corresponds to ω_1^2

$\wedge$

Corresponds to ω_2^2

$\wedge$

Corresponds to ω_3^2

iv) Geometric interpretation of normal modes.

Consider again the eigenvalue problem $[K]\{C\} = \omega^2 [M]\{C\} \Rightarrow$

$$\Rightarrow \underbrace{[M]^{-1}[K]}_{[\alpha]}\{C\} = \omega^2 \underbrace{[M]^{-1}[M]}_{I=[I]}\{C\} = \omega^2\{C\} \Rightarrow [\alpha]\{C\} = \omega^2\{C\} \Rightarrow$$

Let $\omega^2 = \lambda$

$$\Rightarrow \{C\}^T[\alpha]\{C\} = \lambda\{C\}^T\{C\}$$

Let's normalize $\{C\}$ so that $\{C\}^T\{C\} = 1$ $\Rightarrow$ $\boxed{\{C\}^T[\alpha]\{C\} = \lambda}$ $\Rightarrow$

Assume that we have a two-degree-of-freedom system, $\{C\} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$,

$[\alpha] = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$, where $\alpha_{12} = \alpha_{21}$ due to the symmetries of $[M]$ and $[K]$

$$\frac{1}{\lambda}\left[\alpha_{11} c_1^2 + \underbrace{(\alpha_{12}+\alpha_{21})}_{2\alpha_{12}} c_1 c_2 + \alpha_{22} c_2^2\right] = 1 \Rightarrow \qquad \leftarrow \{c\} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$\Rightarrow \boxed{\frac{1}{\lambda}\left[\alpha_{11} c_1^2 + 2\alpha_{12} c_1 c_2 + \alpha_{22} c_2^2\right] = 1}$$

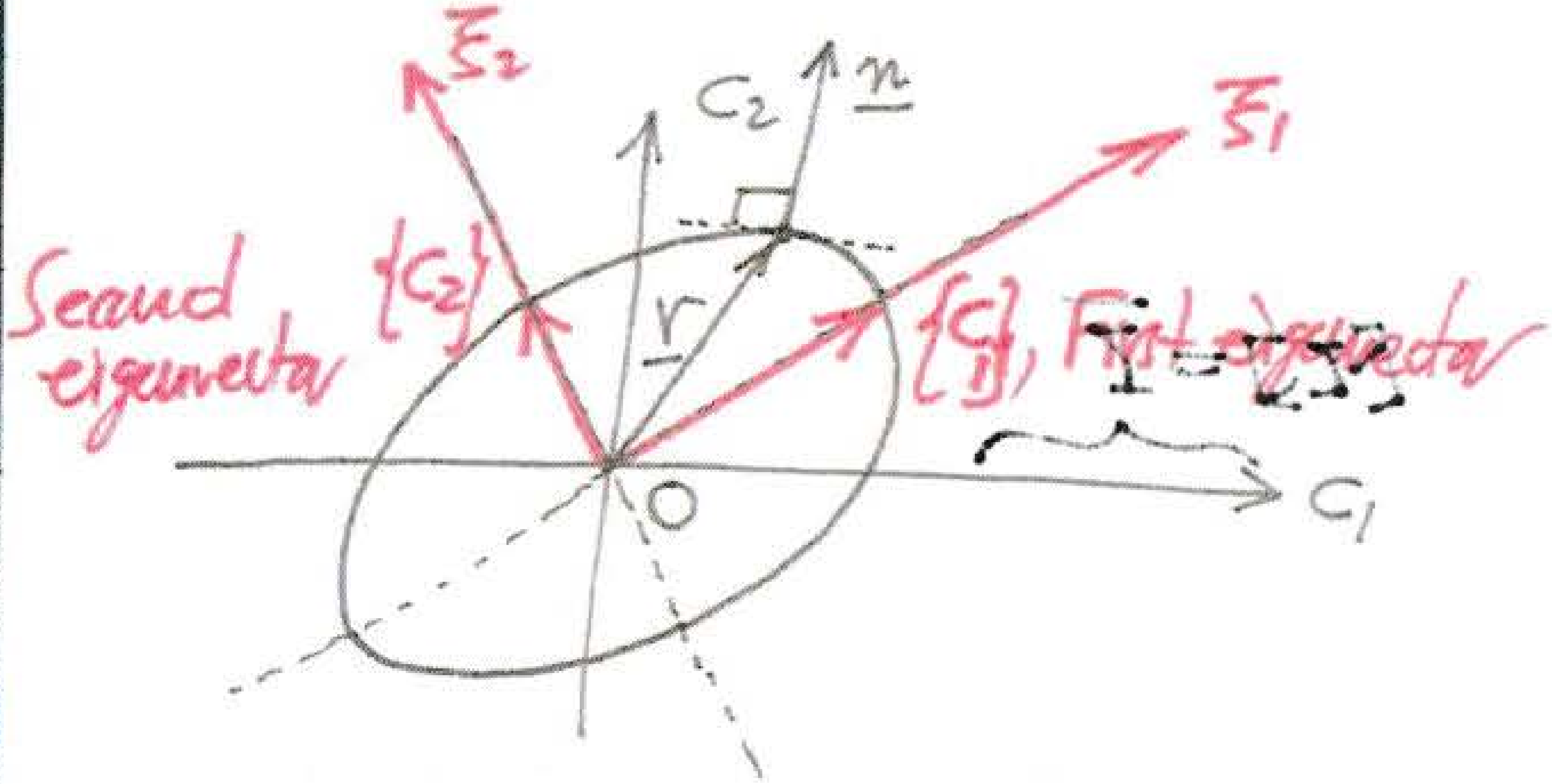
This is a conic section, which for vibrations of mechanical system is an <u>ellipse</u>. Note that the normal to the ellipse at any point is $\underline{n} \sim [\alpha]\{c\} \Rightarrow$

$\Rightarrow$ <u>The eigenvalue problem can be interpreted as the problem of finding λ such that the direction $\underset{\sim}{n} \sim \underset{\sim}{r}$,</u>

which the problem of finding the principal axes of the ellipse! In other words, it will be satisfied that

$$\overbrace{[\alpha]\{c\}}^{n} \sim \overbrace{\lambda\{c\}}^{r} \Rightarrow$$

$$\Rightarrow \{c\}^T[\alpha]\{c\} = \lambda \underbrace{\{c\}^T\{c\}}_{1} = \lambda$$



Eigenvalue problem is:

$$[\alpha]\{c\} = \lambda\{c\}$$

Once these principal components are determined we may introduce new modal coordinates (ξ_1, ξ_2) in terms of which the ellipse is written in simplest form,
$\frac{1}{\lambda} b_{11} \xi_1^2 + \frac{1}{\lambda} b_{22} \xi_2^2 = 1$,

this is identical to setting $\{C\} = \underset{\text{rotation}}{[R]}\{\xi\} \Rightarrow [\alpha][R]\{\xi\} = \lambda[R]\{\xi\}$

$$\Rightarrow \frac{1}{\lambda}\{\xi\}^T \underbrace{[R]^T[\alpha][R]}_{[b] = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}}\{\xi\} = 1 \Rightarrow \left.\begin{array}{l} \frac{1}{\lambda}\{\xi\}^T[b]\{\xi\} = 1 \\ \text{But } \{\xi\}^T\{\xi\} = 1 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_1 = b_{11}, \{\xi_1\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \lambda_2 = b_{22}, \{\xi_2\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \end{array}$$

These represent the two eigenvalues and the two eigenvectors of the system

v) Forced response by modal analysis

It is easy to extend modal analysis to the forced case. Suppose that the original equations of motion are:

$$\left.\begin{array}{l}[M]\{\ddot{q}\}+[K]\{q\}=\{F(t)\}\\ \text{Let } \{q\}=[\Phi]\{y\}\end{array}\right\}\Rightarrow \underbrace{[M][\Phi]\{\ddot{y}\}}_{I}+\underbrace{[K][\Phi]\{y\}}_{[\omega_r^2]}=\{F(t)\}\Rightarrow$$

Orthonormalized modal matrix

$$\Rightarrow [\Phi]^T[M][\Phi]\{\ddot{y}\}+[\Phi]^T[K][\Phi]\{y\}=$$

$$=[\Phi]^T\{F(t)\}\Rightarrow$$

$$\Rightarrow \{\ddot{y}\}+[\omega_r^2]\{y\}=\underbrace{[\Phi]^T\{F(t)\}}_{\{Q(t)\}}\Rightarrow$$

$$\Rightarrow \ddot{y}_r+\omega_r^2 y_r=Q_r,\ r=1,\dots,n$$

Then we get an uncoupled set of forced modal oscillators, with each representing the forced response of an individual mode of vibration. So, each modal oscillator is solved independently of the others with initial conditions:

$$\{y(0)\}=[\Phi]^{-1}\{q(0)\},\ \{\dot{y}(0)\}=[\Phi]^{-1}\{\dot{q}(0)\}$$

Remark: If $[M]=[I]$, then $[\Phi]^T[\Phi]=[I]\Rightarrow$

$\Rightarrow [\Phi]^T=[\Phi]^{-1}$

Then, $[\Phi]$ is an orthogonal matrix.

Role of Damping

Necessary + sufficient condition for proportional damping $\longrightarrow$ Theorem by T.K. Caughey

Sufficient condition $[D] = \alpha[M] + \beta[K]$
$\alpha, \beta \in \mathbb{R}$

$$[M]\{\ddot{q}\} + [K]\{q\} + [D]\{\dot{q}\} = \{F(t)\}$$

Can we use modal analysis to solve this problem?

Can we use the same modal transformation $\{q\} = [\Phi]\{y\}$ to decouple the equation of motion?

As for the undamped problem

$$\underbrace{[\Phi]^T[M][\Phi]}_{[I]}\{\ddot{y}\} + \underbrace{[\Phi]^T[K][\Phi]}_{[\omega_r^2]}\{y\} + \underbrace{[\Phi]^T[D][\Phi]}_{?}\{\dot{y}\} = [\Phi]^T\{F(t)\}$$

Either $[\Phi]^T[D][\Phi] = [\theta]$
Proportional damping

Yes! we can use the same modal matrix to uncoupled the damped equations!

or

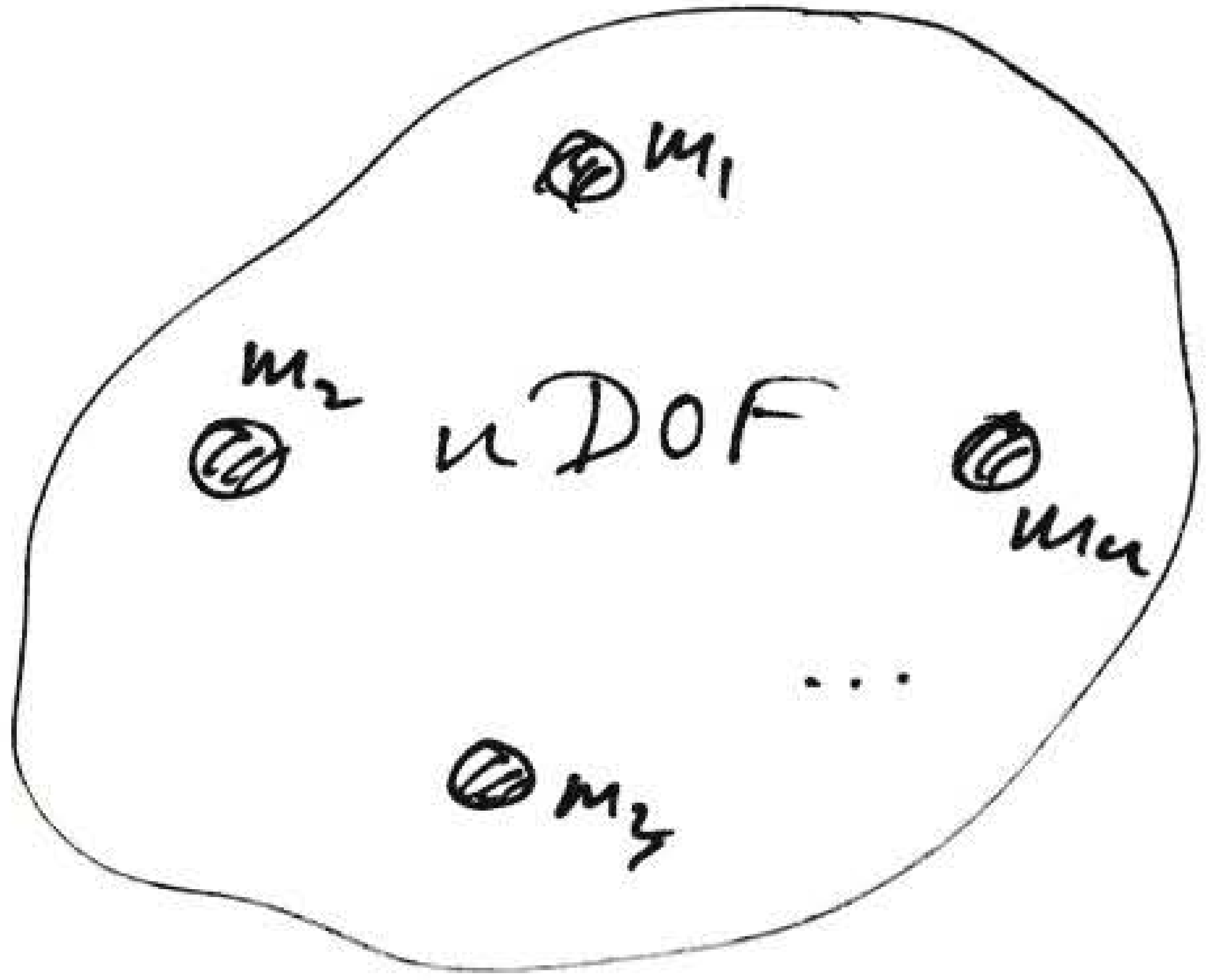
$[\Phi]^T[D][\Phi] \neq [\theta]$
Non-proportional damping

No, modal analysis does not work here.

Undamped system: $t=0 \to$ ICs $\to$

$$\left.\begin{array}{l} \text{Mode } 1 \to E_1 \\ \text{Mode } 2 \to E_2 \\ \vdots \\ \text{Mode } n \to E_n \end{array}\right\} \text{Energy cannot be exchanged between modes!}$$

Correct!



Damped system: (light)
$t=0 \to$ ICs $\to$ Mode i gets initial energy E_i, this energy cannot be exchanged between modes!

Almost correct: Correct if damping distribution is proportional

Not correct if damping is not proportional and we have closely spaced modes!

Closely spaced modes with non-proportional damping do exchange energy between them!