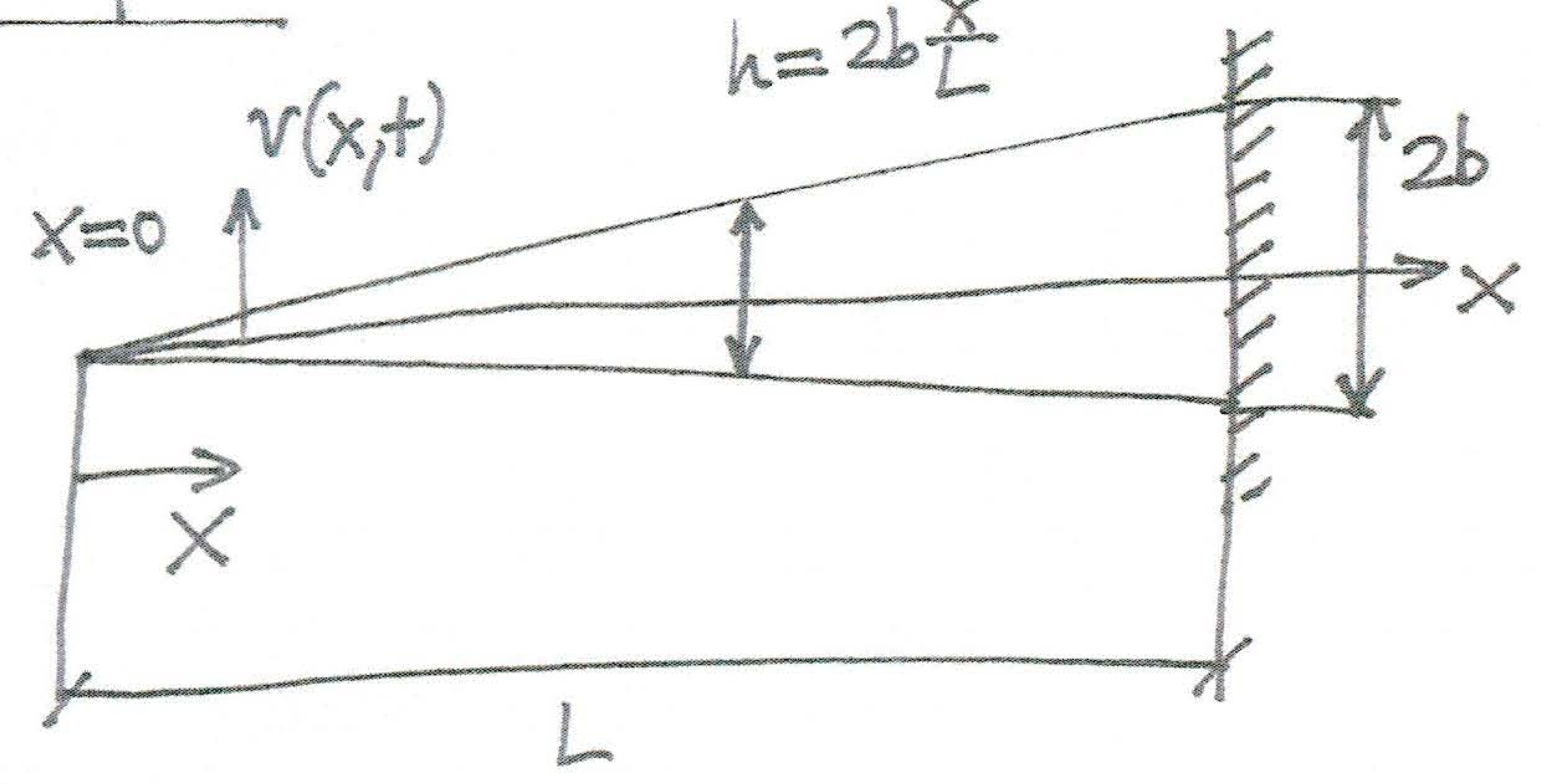
Example 6



Assuming that the thickness $t=1 \Rightarrow$

$\Rightarrow$ Area is $A(x) = h \cdot t = 2b\frac{X}{L}$

Assuming uniform density $\rho \Rightarrow$

$\Rightarrow m(x) = \rho A(x) = 2b\rho\frac{X}{L}$

Then assuming rectangular cross-section

$\Rightarrow I(x) = \frac{1 \cdot h^3}{12} = \frac{2}{3}b^3\left(\frac{X}{L}\right)^3 \Rightarrow$

$\Rightarrow EI(x) = \frac{2}{3}Eb^3\left(\frac{X}{L}\right)^3$

The equation of motion is given by,

$$\frac{\partial^2}{\partial X^2}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right] + m(x)\frac{\partial^2 v}{\partial t^2} = 0$$

$$\left\{EI(x)\frac{\partial^2 v}{\partial X^2}\right\}_{x=0+} = 0, \quad \left\{\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v}{\partial X^2}\right]\right\}_{x=0+} = 0$$

$$v(L,t) = 0, \quad \frac{\partial v}{\partial X}(L,t) = 0$$

$\Rightarrow$ We can formulate the eigenvalue problem

$$\left(EI(x)\varphi''(x)\right)'' - m(x)\omega^2\varphi(x) = 0$$

$$\varphi(L) = \varphi'(L) = 0$$

$$EI(0)\varphi''(0) = 0$$

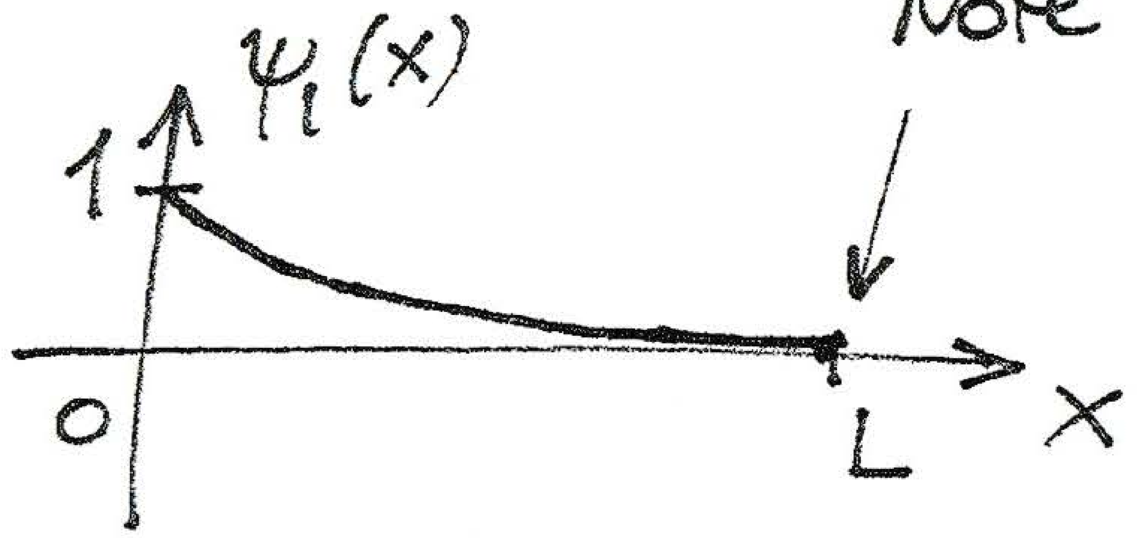
$$\left(EI(0)\varphi''(0)\right)' = 0$$

Geometric BCs

Natural BCs

To solve the problem using the Rayleigh–Ritz approximate discretization method we need to pick admissible functions satisfying the geometric BCs only (in this case we have 'simple' BCs) $\Rightarrow$

$$\Rightarrow \quad \phi(x) = a_1 \psi_1(x) + a_2 \psi_2(x) + a_3 \psi_3(x) + \dots$$



Note $\psi_i(L)=0$
$\psi_i'(L)=0$
$i=1,2,\dots$

$$\psi_1(x) = \left(1-\frac{x}{L}\right)^2$$

$$\psi_2(x) = \frac{x}{L}\left(1-\frac{x}{L}\right)^2$$

$$\psi_3(x) = \left(\frac{x}{L}\right)^2\left(1-\frac{x}{L}\right)^2$$

$\dots$

Using R–R procedure we can assemble the discretized matrices:

$$k_{ij} = \int_0^L EI(x)\,\psi_i''(x)\,\psi_j''(x)\,dx \Rightarrow k_{11} = \frac{8}{3}\frac{Eb^3}{L^3},\; k_{12}=k_{21}=\frac{16}{15}\frac{Eb^3}{L^3},\dots$$

$$k_{22} = \frac{16}{15}\frac{Eb^3}{L^3},\dots$$

$$m_{ij} = \int_0^L m(x)\,\psi_i(x)\,\psi_j(x)\,dx \Rightarrow m_{11} = \frac{4}{15}\rho bL,\; m_{12}=m_{21}=\frac{8}{105}\rho bL,\dots$$

$$m_{22} = \frac{1}{35}\rho bL,\dots$$

Then assemble the discretized eigenvalue problem

$$\left([K]-\Omega^2[M]\right)\underline{a}=\underline{0}$$

<u>1st approximation</u> $\Rightarrow \varphi(x)=a_1\psi_1(x) \Rightarrow \Omega_1^2=\frac{k_{11}}{m_{11}}=10\frac{Eb^2}{\rho L^4} \Rightarrow \Omega_1 = 3.162\frac{b}{L^2}\sqrt{\frac{E}{\rho}}$

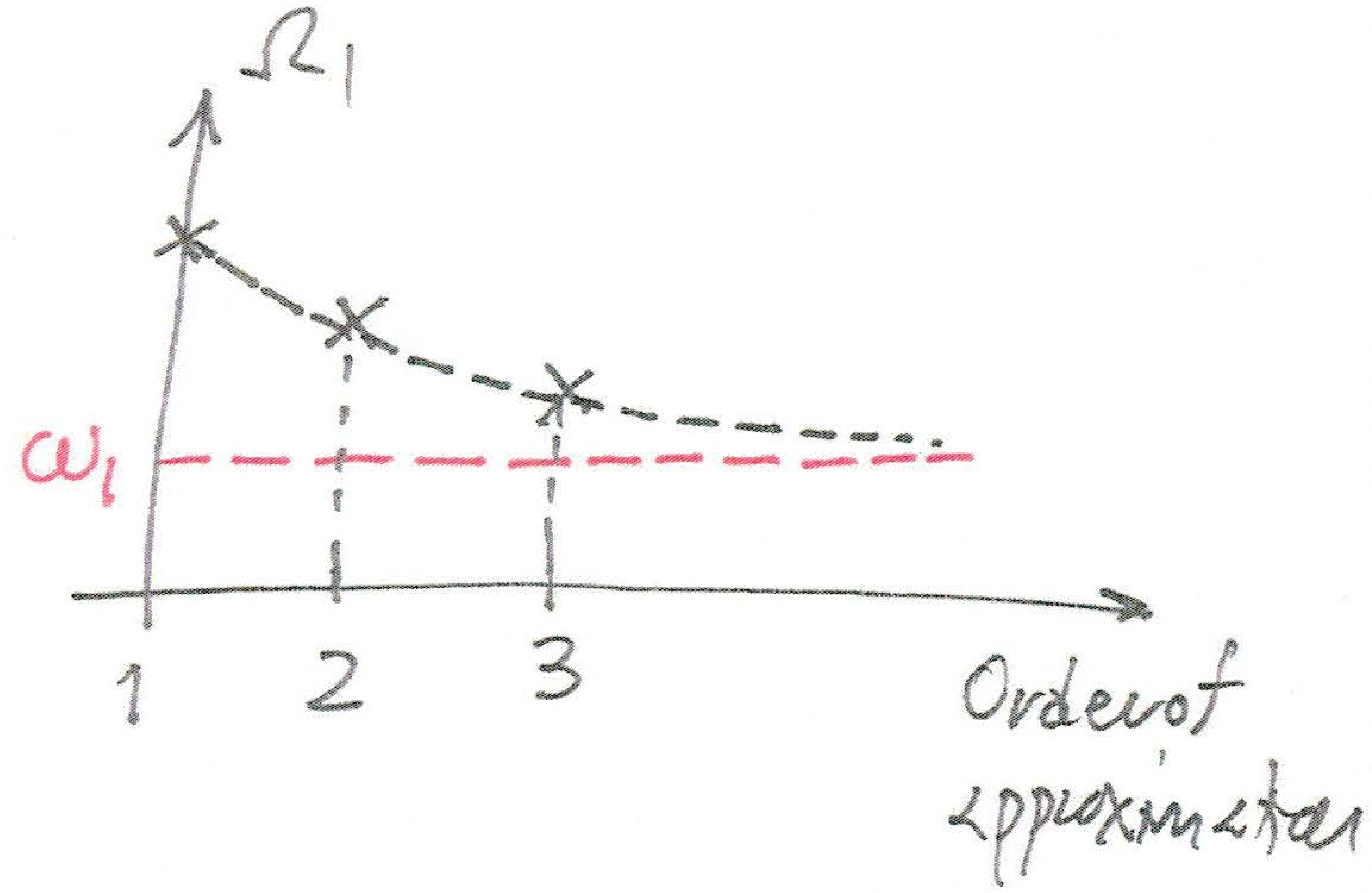
<u>2end approximation</u> $\Rightarrow \varphi(x)=a_1\psi_1(x)+a_2\psi_2(x) \Rightarrow$

$$\Rightarrow \left(\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{bmatrix} - \Omega^2\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}\right)\begin{Bmatrix} a_1 \\ a_2\end{Bmatrix}=\begin{Bmatrix} 0 \\ 0\end{Bmatrix} \Rightarrow$$

$$\Rightarrow \Omega_1 = 3.0707\frac{b}{L^2}\sqrt{\frac{E}{\rho}}$$

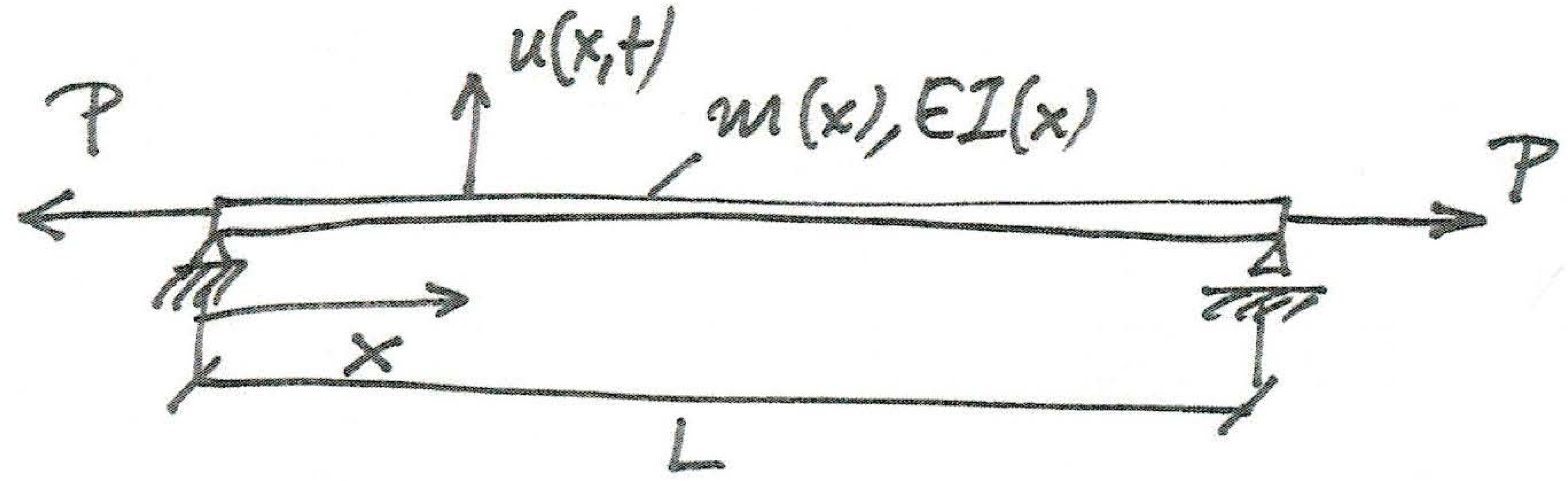
$$\Omega_2 = 9.9886\frac{b}{L^2}\sqrt{\frac{E}{\rho}}$$

...



Special Topics

1) Beam under axial force



It can be shown that in this case the classical Euler-Bernoulli beam equation is modified as follows:

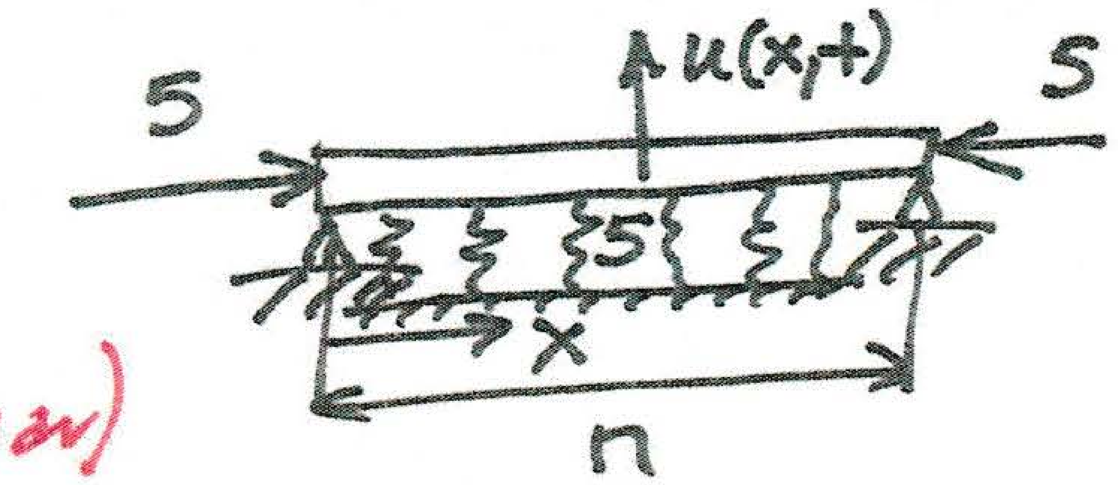
$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 u}{\partial x^2}\right) - P\frac{\partial^2 u}{\partial x^2} + m(x)\frac{\partial^2 u}{\partial t^2} = 0$$

Accounts for the axial force (in extension)

We can show that the extra term arises when we perform balance of moments of a differential beam element. It can be proven that we can still have separation of variables, so this system admits real normal modes $\Rightarrow u(x,t) = \varphi(x)\, f(t) \Rightarrow$

$\Rightarrow$ We can derive an eigenvalue problem for $\varphi(x)$.

Note that typically finite beams with homogeneous boundary conditions possess a discrete spectrum of eigenfrequencies which are distinct. However, the application of an axial load can lead to repetitive eigenfrequencies! That is, it is possible for certain values of the parameters to have $\omega_i = \omega_j$, $i \neq j$.



Example

Compressive axial load

$$\frac{\partial^4 u}{\partial x^4} + 5\frac{\partial^2 u}{\partial x^2} + 5u(x,t) + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 \le x \le \pi$$

Elastic foundation (distributed and linear)

$$\left.\begin{aligned} &u(0,t) = u(\pi,t) = 0 \\ &\frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(\pi,t) = 0 \end{aligned}\right\} \text{ Simply supported BCs at } x = 0, \pi.$$

Harmonic function $\ddot{f} + \omega^2 f = 0$

We perform separation of variables $\Rightarrow u(x,t) = \varphi(x)\, f(t) \Rightarrow$

$$\Rightarrow \frac{d^4\varphi(x)}{dx^4} + 5\frac{d^2\varphi(x)}{dx^2} + 5\varphi(x) - \omega^2\varphi(x) = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} &\frac{d^4\varphi(x)}{dx^4} + 5\frac{d^2\varphi(x)}{dx^2} + (5-\omega^2)\varphi(x) = 0 \\ &\varphi(0) = \varphi(\pi) = \varphi''(0) = \varphi''(\pi) = 0 \\ &0 \le x \le \pi \end{aligned}\right\} \Rightarrow$$

Seek a solution of the form $\varphi(x) = Ce^{sx} \Rightarrow$

$$\Rightarrow s^4 + 5s^2 + (5-\omega^2) = 0 \Big\} \Rightarrow$$

Set $s^2 = \lambda$

$$\Rightarrow \lambda^2 + 5\lambda + \underbrace{5-\omega^2}_{-\alpha^4} = 0 \Rightarrow \lambda_{1,2} = \frac{-5 \pm \sqrt{25+4a^4}}{2} \Rightarrow \begin{cases} \lambda_1 = \dfrac{-5+\sqrt{25+4a^4}}{2} \equiv \delta^2 > 0 \\ \lambda_2 = \dfrac{-5-\sqrt{25+4a^4}}{2} \equiv -\varepsilon^2 < 0 \end{cases}$$

Then, $s_{1,2} = \pm\delta$, $s_{3,4} = \pm j\varepsilon$, $j = (-1)^{1/2}$

Then, the general solution for $\varphi(x)$ can be expressed as:

$$\varphi(x) = \cancel{D_1 \cos\varepsilon x} + D_2 \sin\varepsilon x + \cancel{D_3 \cosh\delta x} + D_4 \sinh\delta x$$

$$\varphi'(x) = -D_1 \varepsilon \sin\varepsilon x + D_2 \varepsilon \cos\varepsilon x + \delta D_3 \sinh\delta x + \delta D_4 \cosh\delta x$$

$$\varphi''(x) = \cancel{-D_1 \varepsilon^2 \cos\varepsilon x} - D_2 \varepsilon^2 \sin\varepsilon x + \cancel{\delta^2 D_3 \cosh\delta x} + \delta^2 D_4 \sinh\delta x$$

$$\left.\begin{array}{l} \varphi(0) = 0 \Rightarrow D_1 + D_3 = 0 \\ \varphi''(0) = 0 \Rightarrow -D_1\varepsilon^2 + D_3\delta^2 = 0 \end{array}\right\} \Rightarrow D_1 = D_3 = 0$$

$$\Downarrow$$

$$D_1 = -D_3 \Rightarrow D_3\varepsilon^2 + D_3\delta^2 = 0 \Rightarrow D_3 = 0$$

$$\varphi(\pi) = 0 \Rightarrow D_2 \sin\varepsilon\pi + D_4 \sinh\delta\pi = 0 \Rightarrow D_2 \varepsilon^2 \sin\varepsilon\pi + D_4 \varepsilon^2 \sinh\delta\pi = 0 \;\Big|\; \overset{+}{\Rightarrow} D_4 = 0$$

$$\varphi''(\pi) = 0 \Rightarrow -D_2 \varepsilon^2 \sin\varepsilon\pi + D_4 \delta^2 \sinh\delta\pi = 0$$

Hence, $D_2 \sin\varepsilon\pi = 0 \Rightarrow$ for nontrivial solutions $\varepsilon_k \pi = k\pi,\ k = 1, 2, \dots \Rightarrow$

$$\Rightarrow \varepsilon_k = k,\ k = 1, 2, \dots \Rightarrow$$

$$\left.\begin{array}{l} \Rightarrow \left(\dfrac{+5 + \sqrt{25 + 4\alpha_k^4}}{2}\right)^{1/2} = k \\ \alpha_k^4 = \omega_k^2 - 5 \end{array}\right\} \text{Compute the eigen-frequencies}$$

After manipulations we should be getting that $\omega_k^2 = 5 + k^2(k^2-5)$, $k=1,2,\ldots$
Then, we see that $\omega_k^2 > 0 \ \forall k=1,2,\ldots \Rightarrow$ All modes are stable modes, and the corresponding eigenfunctions are, $\varphi_k = D_2 \sin kx$, $k=1,2,\ldots \Rightarrow$
$\Rightarrow$ After mass normalization, $\int_0^\pi \varphi_k^2 dx = 1 \Rightarrow D_2 = \sqrt{\frac{2}{\pi}}$

Hence, all modes are stable (the compressive load is not strong enough to cause buckling), and the eigenfunctions are identical to those of the simply supported Euler-Bernoulli beam (that is, the extra terms $5\frac{\partial^2 u}{\partial x^2}$, $5u(x,t)$ do not affect the eigenfunctions! They do affect, however, the natural frequencies).

Example

$$\left.\begin{aligned} &\frac{\partial^4 u}{\partial x^4} + 6\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 \le x \le \pi \\ &u(0,t) = u(\pi,t) = 0 \\ &\frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(\pi,t)}{\partial x^2} = 0 \end{aligned}\right\}$$

Since the eigenfunctions of this system are identical to the eigenfunctions of the simple Euler-Bernoulli simply supported beam $\Rightarrow$
$$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} A_k(t) \sqrt{\frac{2}{\pi}} \sin kx$$

Substituting the expansion into the governing equation of motion $\Rightarrow$

$$\sum_{k=1}^{\infty}\left\{A_k(t)\sqrt{\frac{2}{\pi}}k^4\sin kx - 6A_k(t)\sqrt{\frac{2}{\pi}}k^2\sin kx + \ddot{A}_k(t)\sqrt{\frac{2}{\pi}}\sin kx\right\}=0 \Rightarrow$$

$\Rightarrow \int_0^{\pi}(x)\sqrt{\frac{2}{\pi}}\sin px\,dx \Rightarrow$ Using the orthonormality properties of the modes – normalized eigenfunctions $\Rightarrow$ $\ddot{A}_p(t) + A_p(t)\underbrace{(p^4 - 6p^2)}_{\bar{\omega}_p^2} = 0 \Rightarrow$

$$\Rightarrow \ddot{A}_p(t) + \bar{\omega}_p^2 A_p(t) = 0\ ,\quad p=1,2,\ldots,\quad \bar{\omega}_p^2 = p^4 - 6p^2$$

Since we derive a set of uncoupled modal oscillators, the system possesses real normal modes, with eigenfunctions identical to the underlying Euler-Bernoulli system.

For stability it must hold that $\bar{\omega}_p^2 \geq 0 \Rightarrow p^4 - 6p^2 \geq 0 \Rightarrow p^2(p^2-6)\geq 0 \Rightarrow$

$\Rightarrow p^2 \geq 6 \Rightarrow p \geq \sqrt{6} \Rightarrow$ for $p=1$ and $p=2$ we have unstable modes! Whereas all higher modes are stable.

$\underline{p=1} \Rightarrow \bar{\omega}_1^2 = -5 \Rightarrow \ddot{A}_1 - 5A_1 = 0 \Rightarrow A_1(t) = k_1\, e^{-\sqrt{5}t} + k_2\, e^{\sqrt{5}t} =$ ← causes instability as $t \to \infty$

$= \bar{k}_1 \sinh\sqrt{5}t + \bar{k}_2 \cosh\sqrt{5}t$

$\underline{p=2} \Rightarrow \bar{\omega}_2^2 = -8 \Rightarrow \ddot{A}_2 - 8A_2 = 0 \Rightarrow$ Mode unstable.

$\underline{p=3} \Rightarrow \bar{\omega}_3^2 = 27 \Rightarrow \ddot{A}_3 + 27A_3 = 0 \Rightarrow A_3(t) = K_1 \sin\sqrt{27}t + K_2 \cos\sqrt{27}t$

...

Hence, this elastodynamic analysis tells us that only the first two modes are becoming unstable due to buckling, whereas all higher modes are stable. So, depending on the initial conditions the dynamics can be stable or unstable:

Excite first mode ... Excite second mode

for unstable dynamics: $u(x,0) = 2\sin x + 3\cos 2x + 4\sin 2x + 6\sin 3x$

$\frac{\partial u}{\partial t}(x,0) = 0$

The unstable modes are not excited ⇒ Bounded response

for stable dynamics: $u(x,0) = 10\sin 5x + 11\sin 7x$

$\frac{\partial u}{\partial t}(x,0) = 0$