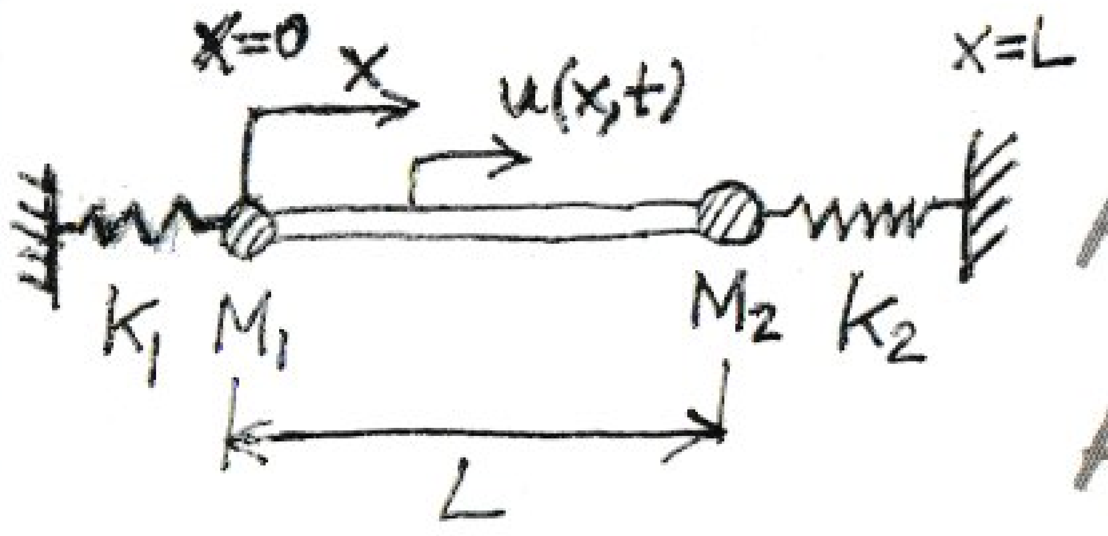
From last time...



$$\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] = B(x)\frac{\partial^2 u}{\partial t^2}, \quad 0<x\leq L, \quad t\geq 0$$

$$\left.\begin{aligned} &A(0)\frac{\partial u(0,t)}{\partial x} - k_1 u(0,t) - M_1\frac{\partial^2 u(0,t)}{\partial t^2} = 0 \\ &A(L)\frac{\partial u(L,t)}{\partial x} + k_2 u(L,t) + M_2\frac{\partial^2 u(L,t)}{\partial t^2} = 0 \end{aligned}\right\} (*)$$

We found that any two eigenfunctions of this problem satisfied the mass-orthonormality conditions

$$\boxed{\int_0^L B(x)\varphi_r(x)\varphi_s(x)\,dx + M_1\varphi_r(0)\varphi_s(0) + M_2\varphi_r(L)\varphi_s(L) = \delta_{rs},} \quad (4)$$

$$r,s = 1,2,\ldots$$

To derive the additional stiffness-orthogonality condition we consider

$$\frac{d}{dx}\left[A(x)\varphi_r'(x)\right] = -\omega_r^2 B(x)\varphi_r(x) \quad \text{for same } r \Rightarrow \text{Multiply by}$$

$\varphi_s(x)$, $s\neq r$ and integrate $\int_0^L dx \Rightarrow$

$$\Rightarrow \int_0^L \frac{d}{dx}\left[A(x)\varphi_r'(x)\right]\varphi_s(x)\,dx = -\omega_r^2\int_0^L B(x)\varphi_r(x)\varphi_s(x)\,dx \Rightarrow$$

$$\Rightarrow \boxed{\int_0^L \frac{d}{dx}\left[A(x)\varphi_r'(x)\right]\varphi_s(x)\,dx - \omega_r^2 M_1\varphi_r(0)\varphi_s(0) - \omega_r^2 M_2\varphi_r(L)\varphi_s(L) = -\omega_r^2\delta_{rs}} \quad (5a)$$

But we want to formulate the second orthogonality condition only in terms of stiffness terms $\Rightarrow$ Recall the boundary conditions satisfied by the eigenfunctions $\Rightarrow$

$$\left.\begin{aligned}\omega_r^2 M_1 \varphi_r(0) &= K_1 \varphi_r(0) - A(0)\varphi_r'(0) \Rightarrow (\cdot) \text{ by } \varphi_s(0)\\ \omega_r^2 M_2 \varphi_r(L) &= K_2 \varphi_r(L) + A(L)\varphi_r'(L) \Rightarrow (\cdot) \text{ by } \varphi_s(L)\end{aligned}\right\} \Rightarrow$$

$\Rightarrow$ Then, substituting into (5a) we derive the alternative expression for the orthogonality condition:

$$\boxed{\begin{aligned}\int_0^L \frac{d}{dx}\left[A(x)\frac{d\varphi_r}{dx}\right]\varphi_s(x)\,dx - K_1\varphi_r(0)\varphi_s(0) + A(0)\varphi_r'(0)\varphi_s(0) - \\ - K_2\varphi_r(L)\varphi_s(L) - A(L)\varphi_r'(L)\varphi_s(L) = -\omega_r^2\delta_{rs}\end{aligned}} \quad (5b)$$

But we can simplify even more by performing integration by parts of the first integral $\Rightarrow$

Here I used (5a)

$$\Rightarrow \underbrace{A(x)\varphi_r'(x)\varphi_s(x)\Big|_0^L}_{A(L)\varphi_r'(L)\varphi_s(L) - A(0)\varphi_r'(0)\varphi_s(0)} - \int_0^L A(x)\varphi_r'(x)\varphi_s'(x)\,dx \boxed{-\omega_r^2 M_1\varphi_r(0)\varphi_s(0)} - \omega_r^2 M_2\varphi_r(L)\varphi_s(L) + \omega_r^2\delta_{rs} = 0$$

$-K_1\varphi_r(0)\varphi_s(0)$

$-K_2\varphi_r(L)\varphi_s(L)$

It follows that the stiffness-orthogonality condition can be expressed in simplest form as:

$$\boxed{\int_0^L A(x)\,\varphi_r'(x)\varphi_s'(x)\,dx + K_1\varphi_r(0)\varphi_s(0) + K_2\varphi_r(L)\varphi_s(L) = \omega_r^2\,\delta_{rs} \qquad r,s=1,2,\ldots}$$

Note that this relation holds for the mass-orthonormalized eigenfunctions (6)

Modal Analysis

Now consider the forced generalized wave equation with non-simple BCs,

$$\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] + F(x,t) = B(x)\frac{\partial^2 u}{\partial t^2}, \quad 0\le x\le L,\ t\ge 0$$

Initial conditions $u(x,0) = g(x)$, $\frac{\partial u}{\partial t}(x,0) = h(x)$

Boundary conditions (*)

first we solve the eigenvalue problem and mass-orthonormalize the eigenfunctions, so that $\varphi_r(x)$ satisfy (4), (6), $r=1,2,\ldots$

Then, use modal superposition,

$$u(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\,\varphi_i(x)$$

Substituting into the governing pde and the boundary conditions,

$$\sum_{i=1}^{\infty} \ddot{\eta}_i(t) \int_0^L B(x)\,\varphi_i(x)\,\varphi_j(x)\,dx = \sum_{i=1}^{\infty} \eta_i(t) \int_0^L \frac{d}{dx}\left[A(x)\frac{d\varphi_i}{dx}\right]\varphi_j(x)\,dx + \underbrace{+\int_0^L F(x,t)\,\varphi_j(x)\,dx}_{N_j(t)} \; ; \quad j \text{ arbitrary but fixed}$$

Doing the same for the BCs,

$$\sum_{i=1}^{\infty} \ddot{\eta}_i(t)\, M_1\, \varphi_i(0) = \sum_{i=1}^{\infty} \eta_i(t)\left[-K_1\varphi_i(0) + A(0)\varphi_i'(0)\right] \Rightarrow (\cdot)\,\varphi_j(0)$$

$$\sum_{i=1}^{\infty} \ddot{\eta}_i(t)\, M_2\, \varphi_i(L) = \sum_{i=1}^{\infty} \eta_i(t)\left[-K_2\varphi_i(L) - A(L)\varphi_i'(L)\right] \Rightarrow (\cdot)\,\varphi_j(L)$$

$\Rightarrow$

$\Rightarrow$ Then add the resulting expressions $\Rightarrow$

$$\Rightarrow \sum_{i=1}^{\infty} \ddot{\eta}_i(t)\Big[\underbrace{\int_0^L B(x)\varphi_i(x)\varphi_j(x)\,dx + M_1\varphi_i(0)\varphi_j(0) + M_2\varphi_i(L)\varphi_j(L)}_{\delta_{ij}}\Big] =$$

$$= \sum_{i=1}^{\infty} \eta_i(t)\Big[\underbrace{\int_0^L \frac{d}{dx}\left[A(x)\varphi_i'(x)\right]\varphi_j(x)\,dx - K_1\varphi_i(0)\varphi_j(0) + A(0)\varphi_i'(0)\varphi_j(0) - K_2\varphi_i(L)\varphi_j(L) - A(L)\varphi_i'(L)\varphi_j(L)}_{-\omega_j^2\delta_{ij}}\Big] + N_j(t) \Rightarrow$$

$$\Rightarrow \boxed{\ddot{\eta}_j(t) + \omega_j^2 \eta_j(t) = N_j(t), \; j = 1, 2, \dots} \quad \text{Modal oscillators in simple form!} \tag{7}$$

Finally we need to compute the initial conditions for these modal oscillators.

$$u(x,0) = g(x) \Rightarrow \sum_{i=1}^{\infty} \eta_i(0)\varphi_i(x) = g(x), \quad 0 \le x \le L \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{\infty} \eta_i(0) \int_0^L B(x)\varphi_i(x)\varphi_j(x)\,dx = \int_0^L B(x) g(x) \varphi_j(x)\,dx$$

But also, it holds that $\sum_{i=1}^{\infty} \eta_i(0)\varphi_i(0) = g(0) \Rightarrow$

$$\Rightarrow \sum_{i=1}^{\infty} \eta_i(0) M_1 \varphi_i(0)\varphi_j(0) = g(0) M_1 \varphi_j(0)$$

Also, $\sum_{i=1}^{\infty} \eta_i(0)\varphi_i(L) = g(L) \Rightarrow \sum_{i=1}^{\infty} \eta_i(0)\varphi_i(L) M_2 \varphi_j(L) = g(L) M_2 \varphi_j(L)$

$$\xRightarrow{+} \sum_{i=1}^{\infty} \eta_i(0) \Big[\underbrace{\int_0^L B(x)\varphi_i(x)\varphi_j(x)\,dx + M_1 \varphi_i(0)\varphi_j(0) + M_2 \varphi_i(L)\varphi_j(L)}_{\delta_{ij}} \Big] =$$

$$= \int_0^L B(x) g(x)\varphi_j(x)\,dx + M_1 g(0)\varphi_j(0) + M_2 g(L)\varphi_j(L) \Rightarrow$$

$$\Rightarrow \eta_j(0) = \int_0^L B(x) g(x) \varphi_j(x)\, dx + M_1 g(0) \varphi_j(0) + M_2 g(L) \varphi_j(L) \quad , j = 1, 2, \ldots \quad (8)$$

Similarly we find the initial velocities for the modal oscillators

$$\dot{\eta}_j(0) = \int_0^L B(x) h(x) \varphi_j(x)\, dx + M_1 h(0) \varphi_j(0) + M_2 h(L) \varphi_j(L), \; j = 1, 2, \ldots \quad (9)$$

Then, the solution of the forced problem is given by,

$$u(x,t) = \sum_{i=1}^{\infty} \eta_i(t) \varphi_i(x)$$

where the modal amplitudes $\eta_i(t)$ are solved by the modal solutions of (7) subject to the initial conditions (8) and (9).