Case of degeneracy

Suppose that $L[u]=0$ has a nontrivial solution $\Rightarrow$ $\lambda=0$ is an eigenvalue of the self-adjoint operator $L[\cdot]$; in this case we'll need to remove the degeneracy from our computation of Green's function $\Rightarrow$ To cancel the eigenfunction corresponding to the degenerate eigenvalue we solve the alternative problem,

$$L[u]=u_0(x)\,u_0(\xi)\,,$$ where $u_0(x)$ is the eigenfunction we wish to cancell.

Generalized Greens function

Then, we proceed to construct the Green's function $\bar{K}(x,\xi)$ as discussed previously considering the above augmented relation. However, it turns out that in this case the solution will have an arbitrary additive function $c(\xi)\,u_0(x)$, and to eliminate this we impose the following orthogonality condition for $\bar{K}(x,\xi)$:

boundaries of domain of solution

$$\int_{x_0}^{x_1} \bar{K}(x,\xi)\,u_0(x)\,dx=0 \Rightarrow \text{Compute } c(\xi)$$

Generalized Green's function

In similarity to the nondegenerate case, if $L[u]$ is a self-adjoint linear operator, then the generalized Green's function is symmetric $\Rightarrow$ $\bar{K}(x,\xi)=\bar{K}(\xi,x)$

Moreover, if $\lambda=0$ is a multiple eigenvalue (corresponding to multiple rigid body modes), then we construct $\bar{K}(x,\xi)$ by solving the augmented problem, $L[u]=u_0(\xi)u_0(x)+\ldots+u_n(\xi)u_n(x)$

where $u_0(x), \ldots, u_n(x)$ are the eigenfunctions corresponding to the degenerate eigenvalues $\lambda = 0$ of multiplicity $n+1$.

Equivalence of Integral and Differential Equations

Using Green's functions we can convert differential equations and boundary value problems to integral equations. Let's consider the nonhomogeneous ordinary differential equation,

$$\boxed{L[u] + \lambda \rho u = \psi(x)} \Rightarrow L[u] = \overbrace{\psi(x) - \lambda \rho u}^{-\varphi(x)} \quad + BCs \quad (1)$$

where $L[u]$ is a self-adjoint operator, $\psi(x)$ is piecewise continuous, and $\rho(x)$ is positive and continuous, and λ is a positive parameter. Considering homogeneous boundary conditions in the domain $[x_0, x_1]$, say $u(x_0) = u(x_1) = 0$, we can immediately write,

$$\boxed{u(x) = \lambda \int_{x_0}^{x_1} K(x,\xi)\rho(\xi)u(\xi)\,d\xi - \int_{x_0}^{x_1} K(x,\xi)\psi(\xi)\,d\xi} \;\cancel{+BCs}\; (2)$$

$K(x,\xi)$ is the Green's function of $L[u]$

Nonhomogeneous term of this integral equation

Considering λ as the eigenvalue of the resulting boundary value problem, there is an equivalence of the boundary value problem (1) and the integral equation (2).

Remark 1: Heuristic argument with a string (Generalized Green's functions)

If λ is an eigenvalue with a corresponding eigenfunction u, then the influence of an external force of the form $-\psi(x)\, e^{i\sqrt{\lambda}t}$ is that the response of the string becomes unstable since resonance occurs, unless the Fredholm alternative condition is satisfied, $\int_{x_0}^{x_1} \psi(x)\, u(x)\, dx = 0$.

Now suppose that a rigid body mode exists, so that there exists the eigenvalue $\lambda = 0 \Rightarrow$ Then the response of the string is unstable under the influence of any arbitrary force $-\psi(x) \Rightarrow$ The string response becomes unstable when a point force is applied at an arbitrary point $\Rightarrow$ To counterbalance this instability when a point force is applied, the string must be balanced by a fixed, time-independent opposing force, which may be chosen arbitrarily except that it may not be orthogonal to the eigenfunction $u_0(x)$ corresponding to $\lambda = 0$, since then it may not prevent the excitation of the mode corresponding to $\lambda = 0$ (and thus prevent the instability from happening) $\Rightarrow$

$\Rightarrow$ It is convenient to simply assume that this balancing force has the

the symmetric form $\psi(x) = -u_0(x)\,u_0(\xi) \Rightarrow$ Then, the Green's function $\bar{K}(x,\xi)$ of a point force acting at $x=\xi$ satisfies not only the boundary conditions but also the differential equation,

Generalized Green's function

$$L[\bar{K}] = u_0(x)\,u_0(\xi), \quad \text{for all } x \in [x_0, x_1] - \{x=\xi\} \quad (*)$$

and at $x=\xi$ the well-known discontinuity condition of the slope. Then, $\bar{K}$ is called the "generalized Green's function" and is the solution of $(*)$, composed of homogeneous + particular solutions. To determine the arbitrary constant of the homogeneous solution $c(\xi)\,u_0(x)$ we require that $\int_{x_0}^{x_1} \bar{K}(x,\xi)\,u_0(x)\,dx = 0 \Rightarrow$ Then, $\bar{K}(x,\xi) = \bar{K}(\xi,x)$.

Example: for the string with free ends we have $u_0(x) = \text{const}$ is an eigenfunction of $\lambda = 0 \Rightarrow$ for the opposing force that eliminates instability we take a force that is constant along the entire length of the string.

Note: Suppose that $L[u]=0$ has a non-trivial solution $u_0(x)$ satisfying the boundary conditions (i.e., that $\lambda=0$ is an eigenvalue). We can show that $L[u] = u_0(\xi)\,u_0(x)$ cannot have any such solution.

To show this, work as follows; suppose that the solution $u = u_0(x)$ exists $\Rightarrow$

$L[u_0] = u_0(\xi)\, u_0(x) \Rightarrow$ Multiply by $u_0(x)$ and $\int_{x_0}^{x_1} (\,)\, dx \Rightarrow$

$$\Rightarrow \int_{x_0}^{x_1} u_0(x) \underbrace{L[u_0]}_{0}\, dx = u_0(\xi) \int_{x_0}^{x_1} u_0^2(x)\, dx = 0 \Rightarrow \int_{x_0}^{x_1} u_0^2(x)\, dx = 0 \Rightarrow$$

$\Rightarrow$ Contradiction since $u_0(x)$ being an eigenfunction satisfies $\int_{x_0}^{x_1} u_0^2(x)\, dx \neq 0$

__Remark 2 (Generalized Green's functions)__

Let $w(x)$ be a function orthogonal to the eigenvector $u_0(x)$ of $\lambda = 0$, satisfies the boundary conditions, and has continuous first and piecewise continuous second derivatives. If $w(x)$ is the solution of the equation

Solution $w(x)$ determined only up to an additive function $c u_0(x)$
$\Rightarrow$ Constant c is determined by imposing $\int_{x_0}^{x_1} w u_0\, dx = 0$

$$L[w] = -\tilde{\varphi}(x), \text{ where } \tilde{\varphi}(x) \text{ is piecewise continuous}$$

then,
$$w(x) = \int_{x_0}^{x_1} \bar{K}(x, \xi)\, \tilde{\varphi}(\xi)\, d\xi.$$

Conversely, the latter relation implies the former if $\tilde{\varphi}(x)$ is orthogonal to $u_0(x)$.