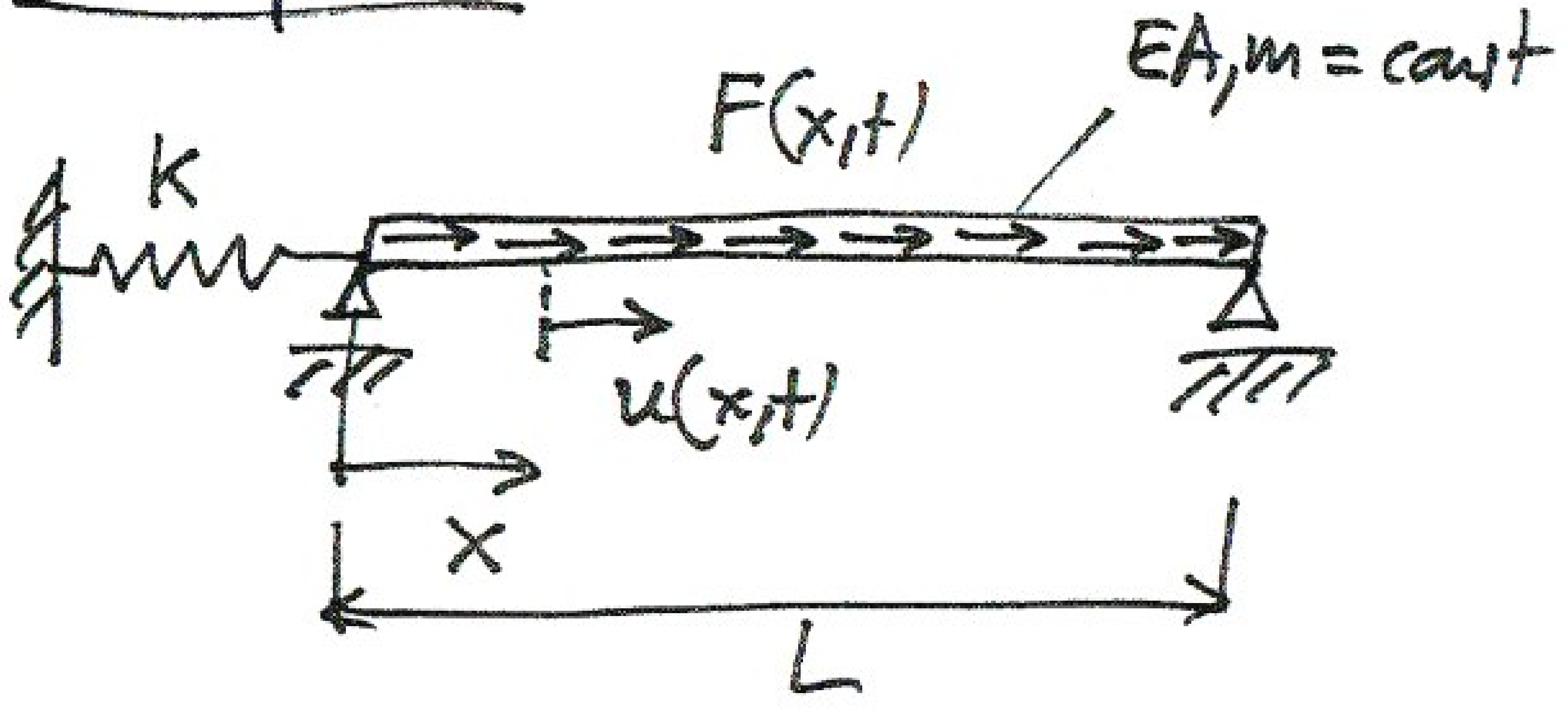
Example 3



$$EA\frac{\partial^2 u}{\partial x^2}+F(x,t)=m\frac{\partial^2 u}{\partial t^2},\quad 0\le x\le L,\ t\ge 0 \quad (1)$$

$$\left.\begin{aligned} &EA\frac{\partial u(0,t)}{\partial x}-ku(0,t)=0\\ &\frac{\partial u(L,t)}{\partial x}=0\end{aligned}\right\}\ \text{BCs} \quad (1a)$$

$$u(x,0)=g(x),\ \frac{\partial u}{\partial t}(x,0)=h(x) \quad (1b)$$

first we compute the modes of this system $\Rightarrow$

$$\varphi''(x)+\left(\frac{\omega}{c}\right)^2\varphi(x)=0 \quad (2)$$

$$\left.\begin{aligned} &EA\varphi'(0)-k\varphi(0)=0\\ &\varphi'(L)=0\end{aligned}\right\} \quad (2a)$$

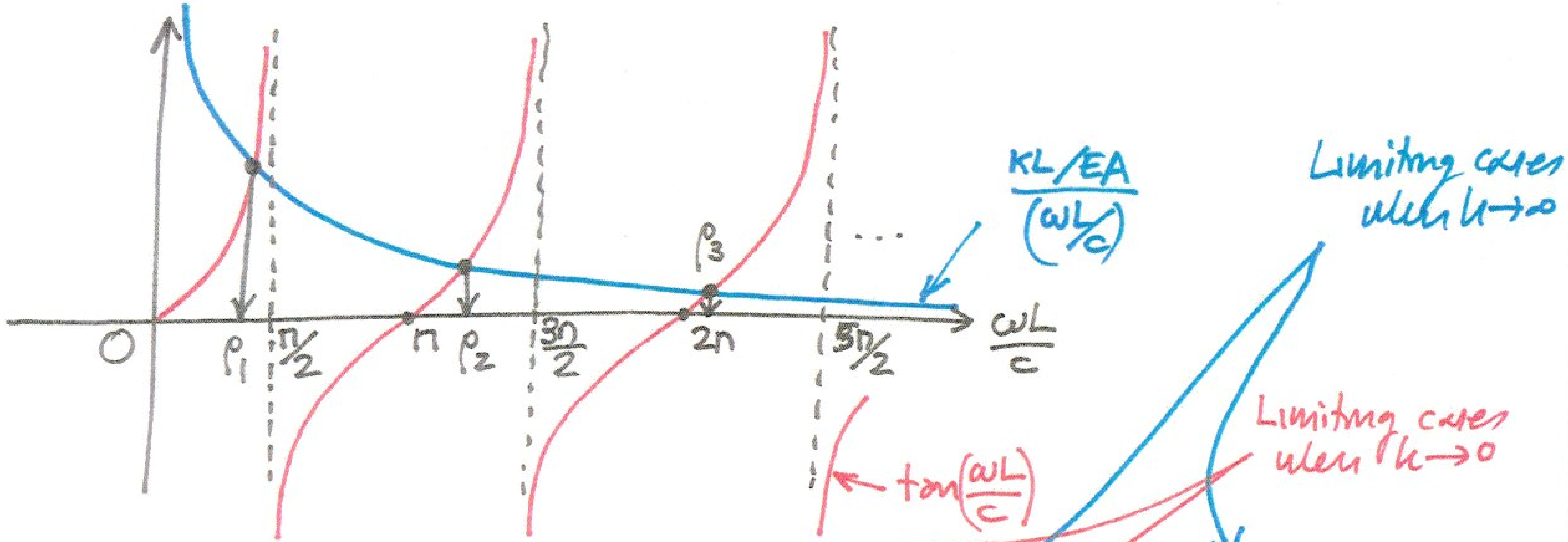
Solving for $\varphi(x)\Rightarrow \varphi(x)=C_1\cos\frac{\omega x}{c}+C_2\sin\frac{\omega x}{c}\Rightarrow$ Applying (2a) $\Rightarrow$

$$\Rightarrow \left.\begin{aligned} &EA\,C_2\frac{\omega}{c}-kC_1=0\\ &-C_1\cancel{\frac{\omega}{c}}\sin\frac{\omega L}{c}+C_2\cancel{\frac{\omega}{c}}\cos\frac{\omega L}{c}=0\end{aligned}\right\}\Rightarrow \begin{bmatrix} -k & \frac{EA\omega}{c}\\ -\sin\frac{\omega L}{c} & \cos\frac{\omega L}{c}\end{bmatrix}\begin{Bmatrix} C_1\\ C_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}\Rightarrow$$

$\Rightarrow$ For nontrivial solutions we require that $\det[\quad]=0\Rightarrow$

Frequency equation

$$\Rightarrow -k\cos\frac{\omega L}{c}+EA\frac{\omega}{c}\sin\frac{\omega L}{c}=0\Rightarrow \tan\left(\frac{\omega L}{c}\right)=\frac{k/EA}{\omega/c}=\frac{\frac{k}{EA}}{\left(\frac{\omega L}{c}\right)\frac{1}{L}}\Rightarrow \boxed{\tan\left(\frac{\omega L}{c}\right)=\frac{\frac{kL}{EA}}{\left(\frac{\omega L}{c}\right)}}$$

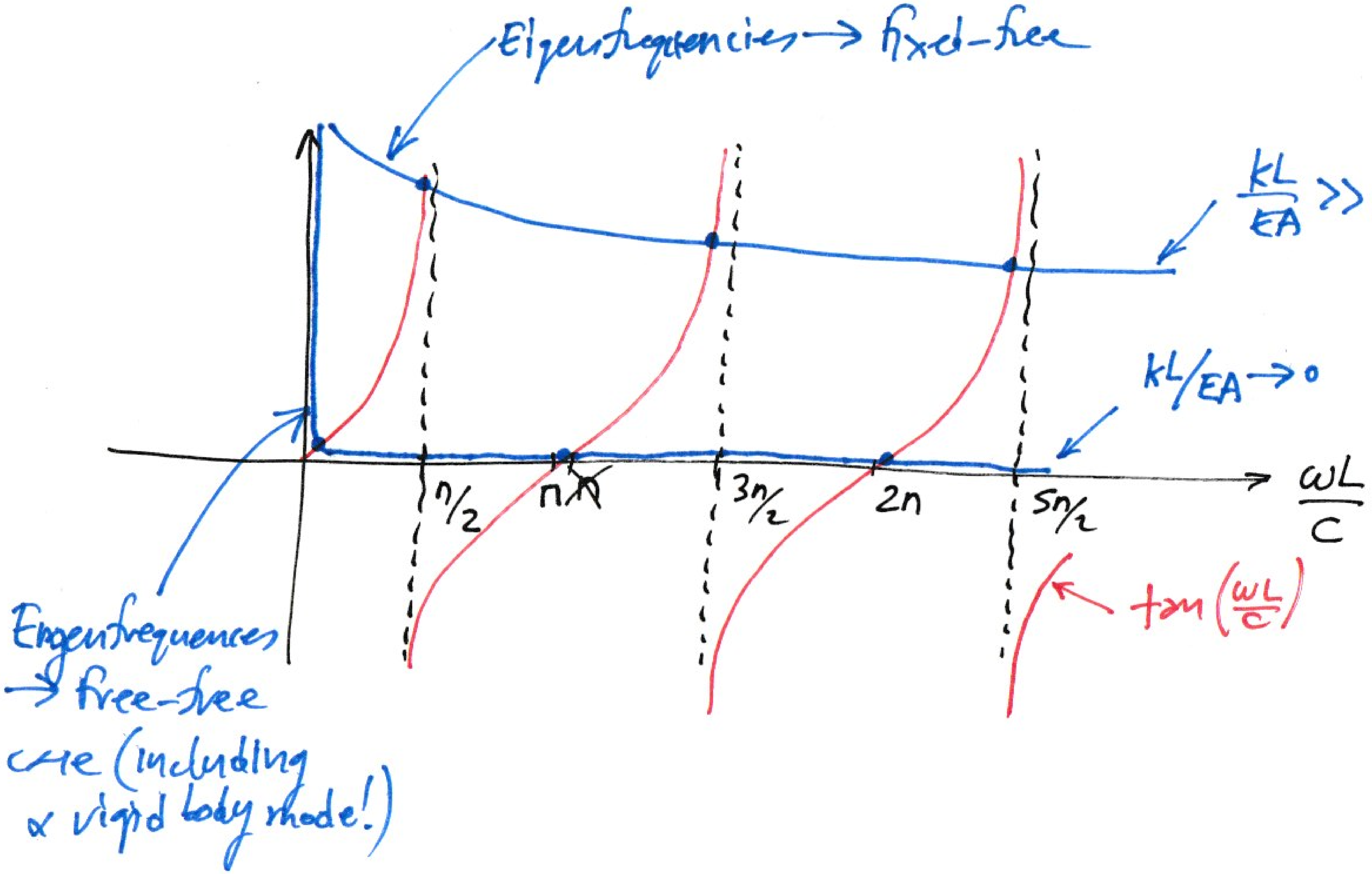


So we get a countable infinity of roots $0<\rho_1<\frac{\pi}{2}$, $\pi<\rho_2<\frac{3\pi}{2}, \ldots \Rightarrow$

$\Rightarrow$ Then we obtain the corresponding natural frequencies as $\frac{\omega_i L}{c}=\rho_i \Rightarrow$

$\Rightarrow \omega_i = \frac{c\rho_i}{L},\ i=1,2,\ldots \Rightarrow$ Then we can compute the corresponding eigenfunctions by noting that $c_2=c_1 \tan\frac{\omega_i L}{c} \Rightarrow \varphi_i(x)= c_i\left(\cos\frac{\omega_i x}{c}+\tan\frac{\omega_i L}{c}\sin\frac{\omega_i x}{c}\right) \Rightarrow$

$\Rightarrow \varphi_i(x)= c_i\,\frac{\cos\left(\rho_i \frac{L-x}{L}\right)}{\cos\rho_i}$ Eigenfunctions for $i=1,2,\ldots$ Of course, we may orthonormalize the eigenfunctions $\varphi_i(x)$ using the formulas for 'non-simple' BCs derived in earlier lecture.


Eigenfrequencies → fixed-free
$\frac{kL}{EA} \gg 1$
$kL/EA \to 0$
$\pi/2$
π
$3\pi/2$
2π
$5\pi/2$
$\frac{\omega L}{c}$
$\tan\left(\frac{\omega L}{c}\right)$
Eigenfrequences → free-free case (including a rigid body mode!)

$\varphi_1(x)$

$\varphi_2(x)$

$\varphi_3(x)$

$\vdots$

To perform orthonormalization require that for each eigenfunction $\varphi_i(x)$ it holds that,

$$\int_0^L m\,\varphi_i^2(x)\,dx = 1\,,\quad i=1,2,\ldots \Rightarrow$$

In simple form since the mass is uniform

$$\Rightarrow \int_0^L m\,C_i^2\,\frac{\cos^2\left(\rho_i\frac{L-x}{L}\right)}{\cos^2\rho_i}\,dx = 1 \Rightarrow$$

$$\Rightarrow C_i = 2\cos\rho_i\sqrt{\frac{\rho_i}{Lm(2\rho_i+\sin 2\rho_i)}}$$

$i=1,2,\ldots$

Then, the orthonormalized eigenfunctions are,

$$\boxed{\varphi_i(x) = 2\sqrt{\frac{\rho_i}{Lm(2\rho_i+\sin 2\rho_i)}}\,\cos\rho_i\frac{L-x}{L}\,,\quad \omega_i=\rho_i\frac{c}{L}\,,\quad i=1,2,\ldots}$$

Note that the stiffness-orthogonality condition won't be simple due to the local stiffness at $x=0$.

Remark: What happens when $k \to 0$? Suppose that $\frac{kL}{EA} \ll 1 \Rightarrow$

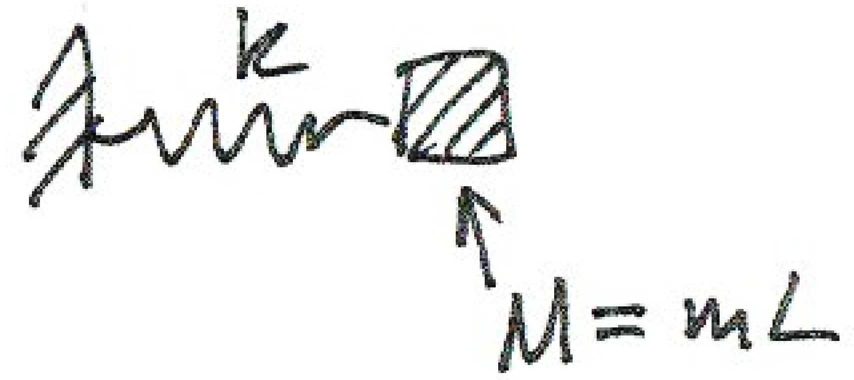
$$\Rightarrow \tan\left(\frac{\omega L}{c}\right) = \frac{kL/EA}{\left(\frac{\omega L}{c}\right)} \Rightarrow \tan\frac{\omega L}{c} \ll 1 \Rightarrow \tan\frac{\omega L}{c} \sim \frac{\omega L}{c} \Rightarrow$$

For the first branch of tan

$$\Rightarrow \frac{\omega_1 L}{c} \sim \frac{kL/EA}{\frac{\omega_1 L}{c}} \Rightarrow \left(\frac{\omega_1 L}{c}\right)^2 \sim \frac{kL}{EA} \Rightarrow \omega_1^2 \sim \frac{c^2}{L^2}\frac{kL}{EA} = \frac{\cancel{EA}}{mL}\frac{k}{\cancel{EA}} = \frac{k}{mL} \Rightarrow$$

$$\Rightarrow \omega_1^2 \sim \frac{k}{mL}$$

Then we get the approximation that the rod moves as a rigid body



So as $k \to 0 \Rightarrow \omega_1^2 \to 0 \Rightarrow \varphi_1(x) = \text{const}$

Hence the first mode becomes the rigid-body mode.

But what happens to higher modes? As $k \to 0 \Rightarrow \tan\frac{\omega L}{c} \to 0 \Rightarrow$

For other branches of tan

$$\Rightarrow \frac{\omega_i L}{c} \to \pi, 2\pi, 3\pi, \ldots$$ (the natural frequencies of free-free rod)

What happens when $k \to \infty$? In that case, $\rho_i \to \frac{(2(i-1)+1)\pi}{2} = \frac{(2i-1)\pi}{2}, \; i=1,2,3,\ldots$

Natural frequencies of fixed-free rod!