Digression: Eingenvalue problems with nonsymmetric matrices

In many applications we encounted eigenvalue problems with nonsymmetric mass or stiffness matrices:

$$[K]\{u\} = \lambda [M]\{u\}, \quad [K] \neq [K]^T, [M] \neq [M]^T \quad , \quad u \in R^n \quad (*)$$

In such cases we cannot use the athogonality properties and the expansion theorem that hold for systems with symmetric system matrices. Instead we will develop a special tehnique that will allow us to decompose an arbitrary vector in R^n using the <u>bi-orthogonality property.</u>

Considering the nonsymmetric eigenvalue problem, let's assume that all eigenvalues $\lambda_i, i=1,\ldots,n$ are real and distinct $\Rightarrow$ It can be shown that the corresponding eigenvectors are linearly independent.

$$[K]\{u\} = \lambda [M]\{u\} \Rightarrow \underbrace{[M]^{-1}[K]}_{[\alpha]}\{u\} = \lambda \{u\} \Rightarrow [\alpha]\{u\} = \lambda\{u\}, \quad [\alpha] \neq [\alpha]^T$$

Suppose that the eigenvectors of this eigenvalue problem are $\{u_i\}$, $i=1,\ldots,n$ (Right eigenvectors)

Now consider the adjoint eigenvalue problem,

$$[\alpha]^T\{v\} = \lambda\{v\} \qquad \text{(Adjoint eigenvalue problem)}$$

which has the same set of eigenvalues but a different set of linearly independent eigenvectors $\{v_i\}, i=1,\ldots,n$ (left eigenvectors).

Let $[\alpha]\{u_i\} = \lambda_i\{u_i\} \Rightarrow \{u_i\}^T[\alpha]^T = \lambda_i\{u_i\}^T \Rightarrow$

$[\alpha]^T\{v_j\} = \lambda_j\{v_j\},\ \lambda_i \neq \lambda_j \Rightarrow$

$$\Rightarrow \{u_i\}^T[\alpha]^T\{v_j\} = \lambda_i\{u_i\}^T\{v_j\}$$

$$\Rightarrow \{u_i\}^T[\alpha]^T\{v_j\} = \lambda_j\{u_i\}^T\{v_j\}$$

$\Rightarrow$

$$\Rightarrow (\lambda_i-\lambda_j)\{u_i\}^T\{v_j\} = 0 \Rightarrow \boxed{\{u_i\}^T\{v_j\} = 0,\ i\neq j}$$

Biorthogonality condition

So, in order to solve the nonsymmetric eigenvalue problem, it is necessary to solve also the adjoint eigenvalue problem $\Rightarrow$

$\Rightarrow$ We obtain two sets of eigenvectors $\{u_i\}$ and $\{v_j\}$, $i,j=1,\dots,n$ that satisfy the biorthogonality conditions

$$\boxed{[u]^T[v] = [v]^T[u] = [I]}$$ Biorthonormality conditions

where we normalized the right- and left-eigenvectors so that it holds that $\{u_i\}^T\{v_i\}=1$, $i=1,\dots,n$. The vectors $\{u_i\}$ and $\{v_j\}$, $i=1,\dots,n$ are called conjugates of each other. Note that for a symmetric matrix, i.e., $[a]=[a]^T \Rightarrow \{u_i\}=\{v_j\}$.

Then, any vector in the n-dim space can be decomposed as a superpositions of either $\{u_i\}$ or $\{v_j\}$ provided that both of these sets of eigenvectors are computed $\Rightarrow$ Use the dual expansion theorem,

$$\forall \{x\} \in R^n \Rightarrow \{x\} = \sum_{r=1}^{n} \alpha_r \{u_r\}, \quad \alpha_r = \{v_r\}^T\{x\}, \quad r=1,\dots,n$$

or

$$\{x\} = \sum_{s=1}^{n} \beta_s \{v_s\}, \quad \beta_s = \{u_s\}^T\{x\}, \quad s=1,\dots,n$$

where α_r and β_s can be regarded as components of $\{x\}$ in the appropriate bases.

Example where nonsymmetric system matrices can arise: The <u>collocation method</u> for solving continuous vibration problems assumes an approximate solution where it can be represented as superposition of some set of functions v_j and associated coefficients a_j. According to our selection of the basis functions v_j we have:

- The boundary method: The functions satisfy the governing dif. equations but not the boundary conditions.
- The interior method: The functions satisfy the boundary conditions but not the governing dif. equations.
- The mixed method: The functions satisfy neither the BCs nor the dif. equations.

Then the coefficients a_r are computed by requiring that the approximation satisfies the dif. equation and/or the BCs at a certain set of points (sensors) or locations s_r, $r = 1, \dots, n$. For example, taking the interior method, let's solve the eigenvalue problem

$$L[w] = \lambda M[w] \quad , \quad w = w(s) \Rightarrow \text{Approximation: } W(s) \cong W_n(s) = \sum_{j=1}^{n} a_j v_j(s)$$

Substituting into the governing dif. eq. the error that we obtain is,

$$\mathcal{E} = L[W_n] - {}^{n}\lambda\, M[W_n] \Rightarrow$$ Require that $\mathcal{E}=0$ at the set of sensors or locations chosen $\Rightarrow$

Approximation to the eigenvector

Approximation of the eigenvalue

But $w_n(s) = \sum_{j=1}^{n} a_j v_j(s)$

$$\Rightarrow L[W_n(s_r)] - {}^{n}\lambda\, M[W_n(s_r)] = 0, \quad r = 1, \dots, n$$

Note that $\mathcal{E} \neq 0$ $\forall s$ in the domain of the problem, since otherwise we would have solved exactly the original problem! Now use the fact that $L[\cdot]$ and $M[\cdot]$ are linear integrodifferential operators $\Rightarrow$

$$\Rightarrow \sum_{j=1}^{n} a_j \underbrace{L[v_j(s_r)]}_{k_{rj}} - {}^{n}\lambda \sum_{j=1}^{n} a_j \underbrace{M[v_j(s_r)]}_{m_{rj}} = 0, \quad r = 1, \dots, n \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{n} a_j k_{rj} - {}^{n}\lambda \sum_{j=1}^{n} a_j m_{rj} = 0 \Rightarrow \sum_{j=1}^{n} (k_{rj} - {}^{n}\lambda\, m_{rj}) a_j = 0, \quad r = 1, \dots, n \Rightarrow$$

$$\Rightarrow [K]\{a\} = {}^{n}\lambda [M]\{a\}$$ Eingenvalue problem, where in general $[K] \neq [K]^T$, $[M] \neq [M]^T$

Example of non-symmetric eigenvalue problems

Consider $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Let $\{x\} = \{x_1 \; x_2\}^T$

$\Rightarrow \{\dot{x}\} + [-A]\{x\} = \{0\}$

where $[A] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

Let $\{x\} = \{u\}e^{\lambda t}$,

where $\{u\} = \{u_1 \; u_2\}^T$ $\Rightarrow$

$[-A]$

$$\Rightarrow \lambda\{u\}e^{\lambda t} + [-A]\{u\}e^{\lambda t} = \{0\} \Rightarrow \lambda\{u\} = [A]\{u\}$$

where $[A] \neq [A]^T$

Non-symmetric eigenvalue problem

Right eigenvectors

First solve the eigenvalue problem

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 1 & -1-\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Computation of the eigenvalues:

$$-(1-\lambda)(1+\lambda)-2=0 \Rightarrow \lambda_{1,2}=\pm\sqrt{3}$$

For $\lambda=\lambda_1=-\sqrt{3}$ the eigenvector is computed as,

$$(1-\lambda_1)u_1+2u_2=0 \Rightarrow u_1=-\frac{2}{1+\sqrt{3}}u_2 \Rightarrow \{u_1\}=\begin{Bmatrix}-\frac{2}{1+\sqrt{3}}\\ 1\end{Bmatrix}c,$$

$c\in R$
(multiplicative const)

For $\lambda=\lambda_2=\sqrt{3}$ the eigenvector is computed as,

$$(1-\lambda_2)u_1+2u_2=0 \Rightarrow u_1=-\frac{2}{1-\sqrt{3}}u_2 \Rightarrow \{u_2\}=\begin{Bmatrix}-\frac{2}{1-\sqrt{3}}\\ 1\end{Bmatrix}d,$$

$d\in R$

So, the right eigenvector matrix is,

$$[U]=[\{u_1\}\ \{u_2\}]=\begin{bmatrix}-\frac{2c}{1+\sqrt{3}} & -\frac{2d}{1-\sqrt{3}}\\ 1\cdot c & 1\cdot d\end{bmatrix}$$

Right Eigenvector for $\lambda=\lambda_1=-\sqrt{3}$

Right Eigenvector for $\lambda=\lambda_2=\sqrt{3}$

Left eigenvectors

Second, solve the eigenvalue problem

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^T \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \lambda \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This adjoint eigenvalue problem has the same eigenvalues,

$\lambda = \pm\sqrt{3} \Rightarrow \lambda_1 = -\sqrt{3}$ and $\lambda_2 = \sqrt{3}$.

For $\lambda = \lambda_1 = -\sqrt{3}$, the left eigenvector is computed as,

$$(1-\lambda_1)\, v_1 + v_2 = 0 \Rightarrow v_1 = -\frac{1}{1+\sqrt{3}}\, v_2 \Rightarrow \{v_1\} = \begin{Bmatrix} -\frac{1}{1+\sqrt{3}} \\ 1 \end{Bmatrix} f, \quad f \in R$$

For $\lambda = \lambda_2 = \sqrt{3}$, the left eigenvector is computed as,

$$(1-\lambda_2)\, v_1 + v_2 = 0 \Rightarrow v_1 = -\frac{1}{1-\sqrt{3}}\, v_2 \Rightarrow \{v_2\} = \begin{Bmatrix} -\frac{1}{1-\sqrt{3}} \\ 1 \end{Bmatrix} g, \quad g \in R$$

So, the left eigenvector matrix is,

$$[V] = [\{v_1\}\ \{v_2\}] = \begin{bmatrix} -\frac{1\cdot f}{1+\sqrt{3}} & -\frac{1\cdot g}{1-\sqrt{3}} \\ 1\cdot f & 1\cdot g \end{bmatrix}$$

Hence, $[V]^T[U] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $[V]^T$ is orthogonal to $[U]$; similarly, $[U]^T$ is orthogonal to $[V]$ (bi-orthogonality)

Compute multiplicative constants c, d, f, g: $\{U_i\}^T\{V_i\} = 1$, $i = 1,2$

So, to solve the original problem we express the solution as,

$$\boxed{\{X\} = C_1\{U_1\}e^{\lambda_1 t} + C_2\{U_2\}e^{\lambda_2 t}}$$

where C_1, C_2 are computed by initial conditions,

or

$$\{X\} = D_1\{V_1\}e^{\lambda_1 t} + D_2\{V_2\}e^{\lambda_2 t}, \quad D_1, D_2 \in R$$

$$\left.\begin{matrix} X_1(0) = X_{10} \\ X_2(0) = X_{20} \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \{X(0)\} = \begin{Bmatrix} X_{10} \\ X_{20} \end{Bmatrix}$$

Hence, $\{X(0)\} = C_1\{U_1\} + C_2\{U_2\} \Rightarrow$

$$\Rightarrow \{V_1\}^T\{X(0)\} = C_1\{V_1\}^T\{U_1\} + C_2\underbrace{\cancel{\{V_1\}^T\{U_2\}}}_{0} \Rightarrow$$

$$\Rightarrow \quad C_1 = \frac{\{V_1\}^T\{X(0)\}}{\underbrace{\{V_1\}^T\{U_1\}}_{1}}; \text{ similarly, } \quad C_2 = \frac{\{V_2\}^T\{X(0)\}}{\underbrace{\{V_2\}^T\{U_2\}}_{1}}$$

Remarks:

1) Note that we can also express the solution as,

$$\boxed{\{x\} = D_1\{v_1\}e^{\lambda_1 t} + D_2\{v_2\}e^{\lambda_2 t}}$$

where D_1 and D_2 are computed in terms of the bi-orthogonality condition using the initial conditions.

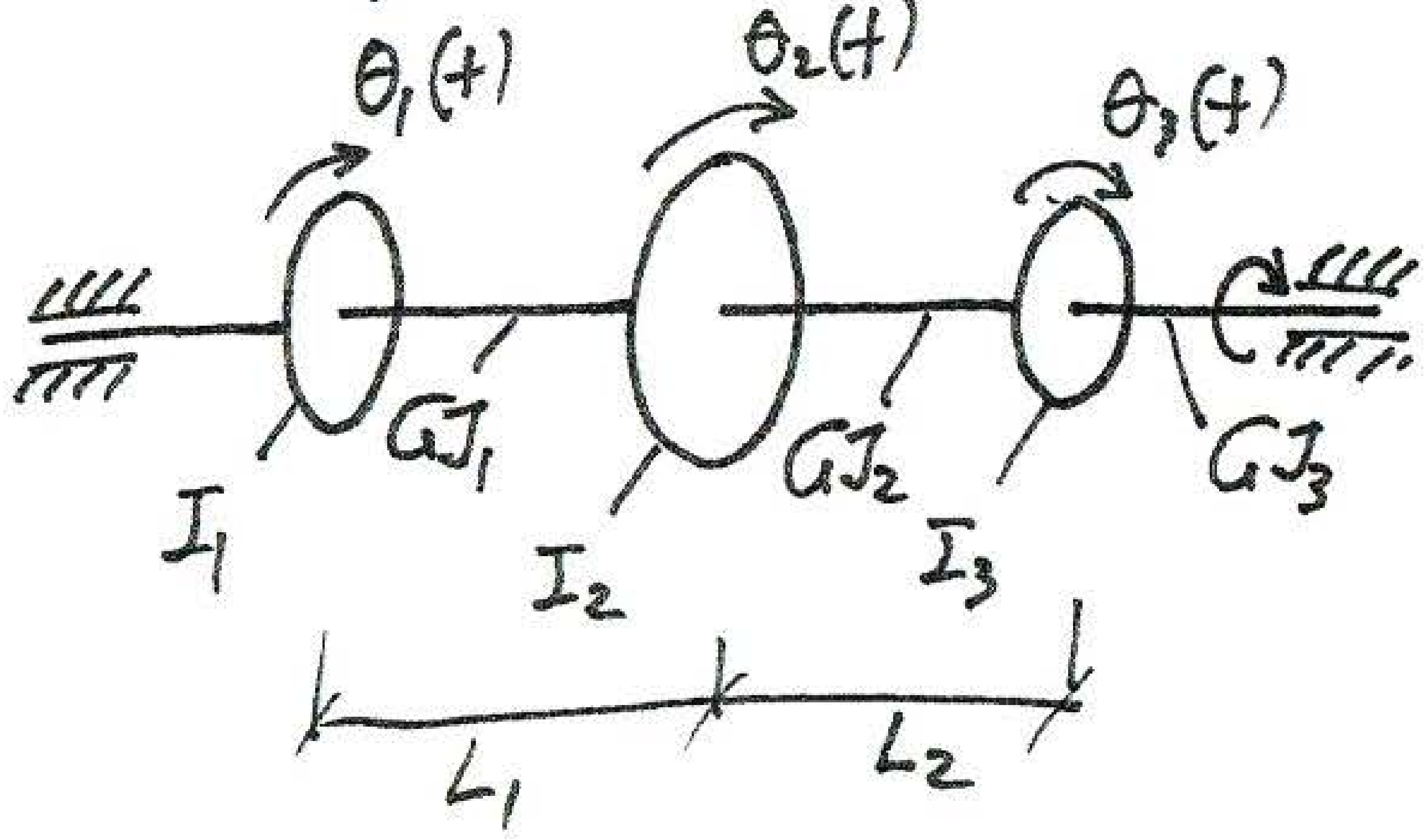
2) We can also not bi-orthonormalize the right- and left-eigenvectors so that,

$$[v]^T[u] \neq [I], \text{ but diagonal matrix}$$

That, however, would not change the form of the solutions.

Digression: Systems with Rigid-Body modes

There are cases where the stiffness matrix $[K]$ is <u>positive-semidefinite</u>, i.e., is <u>singular</u>. We will demonstrate a technique for decomposing the rigid body modes from the elastic modes (i.e., the modes corresponding to nonzero nat. frequencies) by an example.



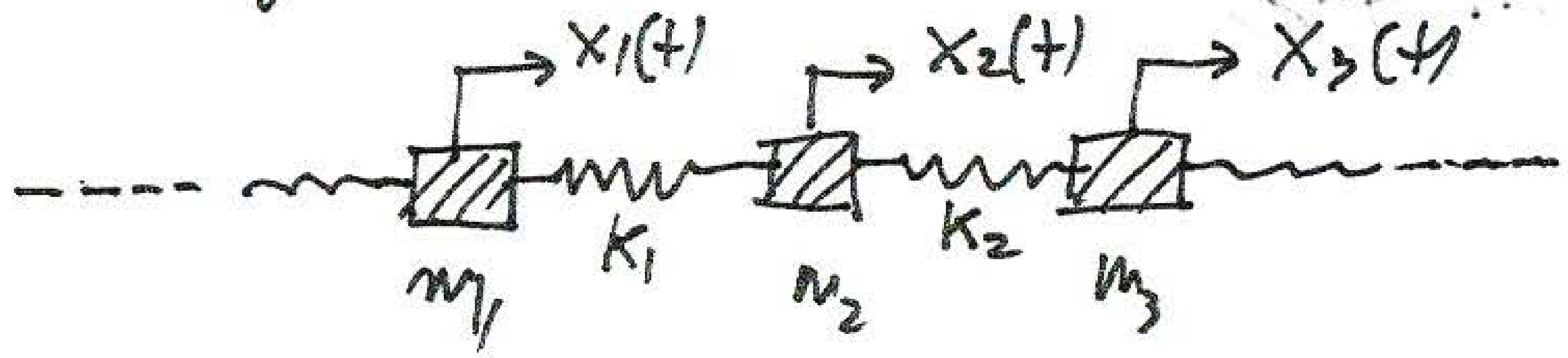
The kinetic energy of this system is

$$T = \frac{1}{2}\left\{I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2\right\} =$$

$$= \frac{1}{2}\{\dot{\theta}\}^T[I]\{\theta\},$$

$$\{\theta\} = \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix},\quad [I] = \begin{bmatrix} I_1 & & 0\\ & I_2 & \\ 0 & & I_3\end{bmatrix}$$

Equivalent translational system



The potential energy is

$$V = \frac{1}{2}\left[k_1(\theta_1-\theta_2)^2 + k_2(\theta_2-\theta_3)^2\right] =$$

$$= \frac{1}{2}\{\theta\}^T[K]\{\theta\},$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0\\ -k_1 & k_1+k_2 & -k_2\\ 0 & -k_2 & k_2\end{bmatrix},\quad k_i = \frac{GJ_i}{L_i},\ i=1,2$$

Assuming synchronous torsional oscillations we require that $\theta_i(t) = \Theta_i\, f(t)$, $i=1,2,3$, $f(t)$ is harmonic function $\Rightarrow$ Derive the eigenvalue problem,

$$\omega^2 [I]\{\Theta\} = [k]\{\Theta\}$$

I could denote the rigid body mode as $\{\Theta_0\} = \begin{Bmatrix} A_0 \\ A_0 \\ A_0 \end{Bmatrix} = A_0 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

We can easily show that one of the eigenvalues is $\omega = 0$ corresponding to the eigenvector $\{\Theta_0\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$. This fact is consistent with the singularity of the stiffness matrix and the corresponding mode is the rigid body mode of this system $\Rightarrow$ The other flexible modes of the system should be orthogonal to the rigid body mode $\Rightarrow$

$$\{\Theta_0\}^T [I] \{\Theta\} = 0 \Rightarrow A_0 (I_1 \Theta_1 + I_2 \Theta_2 + I_3 \Theta_3) = 0 \Rightarrow$$

$\{\Theta_0\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} A_0$

Denotes a flexible mode corresponding to nat. freq. > 0

$$\Rightarrow \underbrace{I_1 \Theta_1 + I_2 \Theta_2 + I_3 \Theta_3}_{\text{for the flexible modes}} \underset{\forall t \geq 0}{=} 0 \Rightarrow I_1 \dot{\theta}_1 + I_2 \dot{\theta}_2 + I_3 \dot{\theta}_3 = 0$$

Satisfied by the flexible modes

<u>This implies that for all flexible modes the angular momentum of the system is zero.</u>

The general motion of the system would be a combination of the rigid body and the flexible modes $\Rightarrow$ How could we separate the flexible modes?

But, due to conservation (actually nullification) of the angular momentum of the system in a flexible mode, it holds that,

$$\theta_3(t) = -\frac{I_1}{I_3}\theta_1(t) - \frac{I_2}{I_3}\theta_2(t)$$ $\Rightarrow$ We reduce the number of independent coordinates by one in a flexible mode.

Then,

$$\underbrace{\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}_f}_{\{\theta\}_f} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -\frac{I_1}{I_3} & -\frac{I_2}{I_3} & 0\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\\ -\frac{I_1}{I_3} & -\frac{I_2}{I_3}\end{bmatrix}}_{[C]}\underbrace{\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}}_{\{\theta\}} \Rightarrow$$

$\Rightarrow$ $\{\theta\}_f = [C]\{\theta\}$ Applies only for a flexible mode! where $[C]$ plays the role of a constraint matrix.

Then I can express the kinetic and potential energy of the system for motion on a flexible mode $\Rightarrow$ Then I can reduce the dimensionality of the problem by extracting the rigid body mode out of the problem and working only with flexible modes $\Rightarrow$

$$T = \frac{1}{2}\{\dot{\theta}\}_f^T[I]\{\theta\}_f = \frac{1}{2}\{\dot{\theta}\}^T\underbrace{[C]^T[I][C]}_{[I']}\{\theta\} = \frac{1}{2}\{\dot{\theta}\}^T[I']\{\theta\}$$

Reduced mass matrix

Do the same for elastic matrix,

$$V=\frac{1}{2}\{\theta\}_f^T[K]\{\theta\}_f=\frac{1}{2}\{\theta\}^T\underbrace{[C]^T[K][C]}_{[K']}\{\theta\}=\frac{1}{2}\{\theta\}^T[K']\{\theta\}$$

Reduced stiffness matrix.

Explicitly, $[I']=\frac{1}{I_3}\begin{bmatrix} I_1(I_1+I_3) & I_1I_2 \\ I_1I_2 & I_2(I_2+I_3)\end{bmatrix}$

$$[K']=\frac{1}{I_3^2}\begin{bmatrix} k_1I_3^2+k_2I_1^2 & -k_1I_3^2+k_2I_1(I_2+I_3) \\ -k_1I_3^2+k_2I_1(I_2+I_3) & (k_1+k_2)I_3^2+k_2I_2(2I_2+I_3)\end{bmatrix}$$

which are (2×2) <u>positive definite</u> matrices $\Rightarrow$ The reduced eigenvalue problem for the flexible modes is,

$$\omega^2[I']\{\Theta\}=[K]\{\Theta\}\Rightarrow \text{Compute the flexible modes.}$$

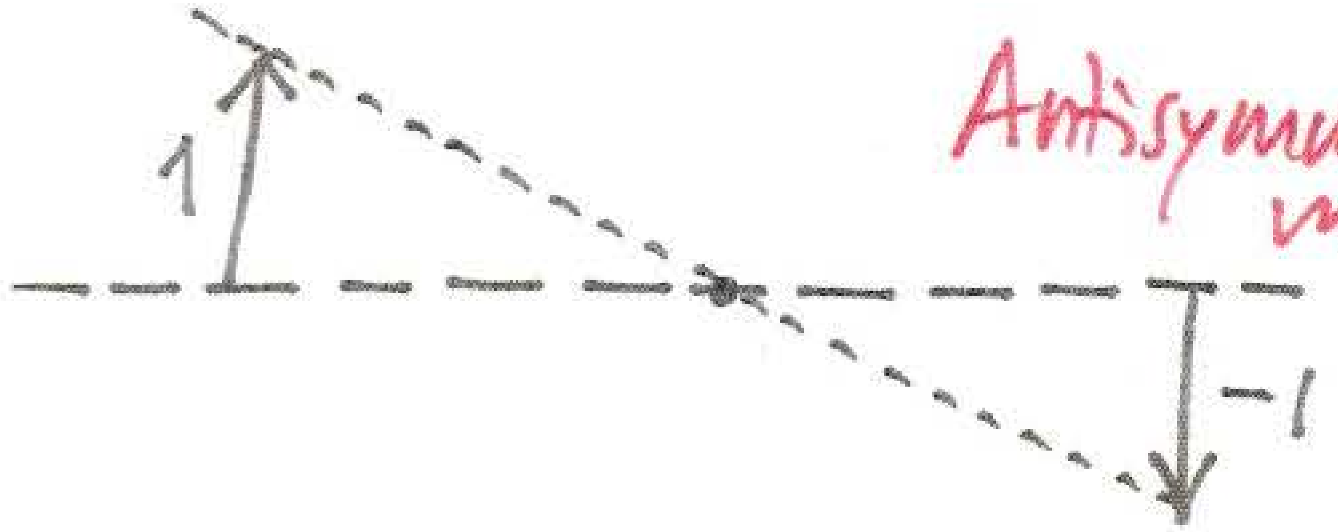
for example, if $k_1 = k_2 = k$, $I_1 = I_2 = I_3 = I \Rightarrow$

$$\left.\begin{array}{l}[I'] = \frac{1}{I}\begin{bmatrix} 2I^2 & I^2 \\ I^2 & 2I^2 \end{bmatrix} = I\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \\ [k'] = k\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}\end{array}\right\} \Rightarrow$$

Flexible mode 1

$$\omega_1 = \sqrt{\frac{k}{I}}, \quad \{\Theta_1\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \{\Theta_1\}_f = [C]\{\Theta_1\} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$



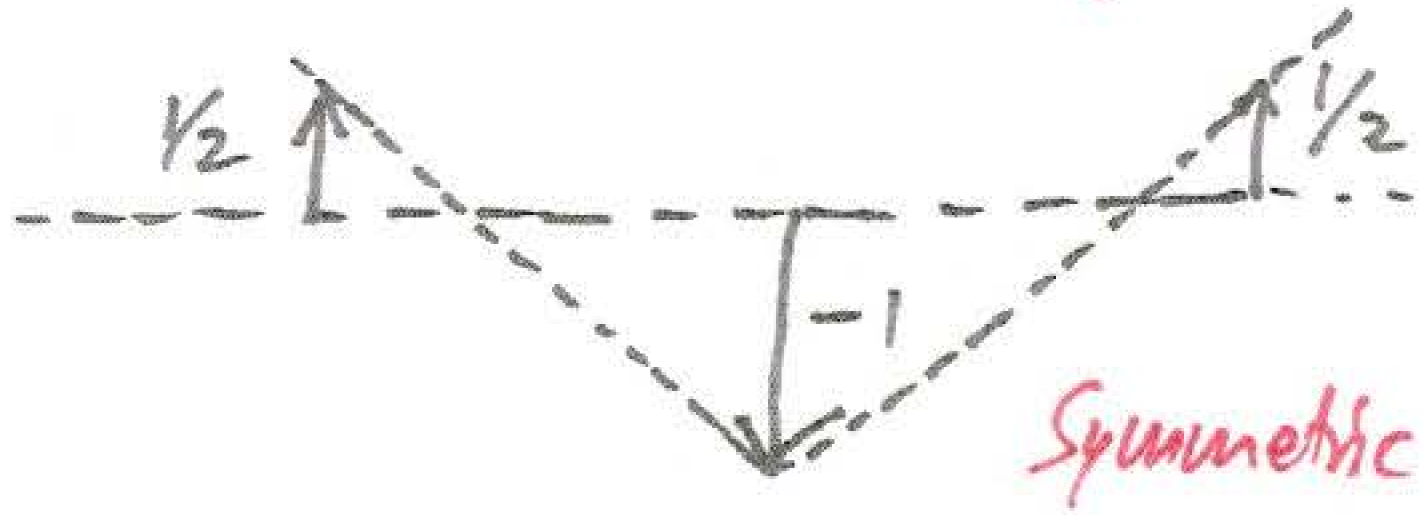
Antisymmetric mode

Flexible mode 2

$$\omega_2 = \sqrt{\frac{2k}{I}}, \quad \{\Theta_2\} = \begin{Bmatrix} 1/2 \\ -1 \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \{\Theta_2\}_f = [C]\{\Theta_2\} = \begin{Bmatrix} 1/2 \\ -1 \\ 1/2 \end{Bmatrix}$$

Symmetric mode



Rigid body mode

$$\omega_0 = 0, \quad \{\Theta\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

Symmetric mode