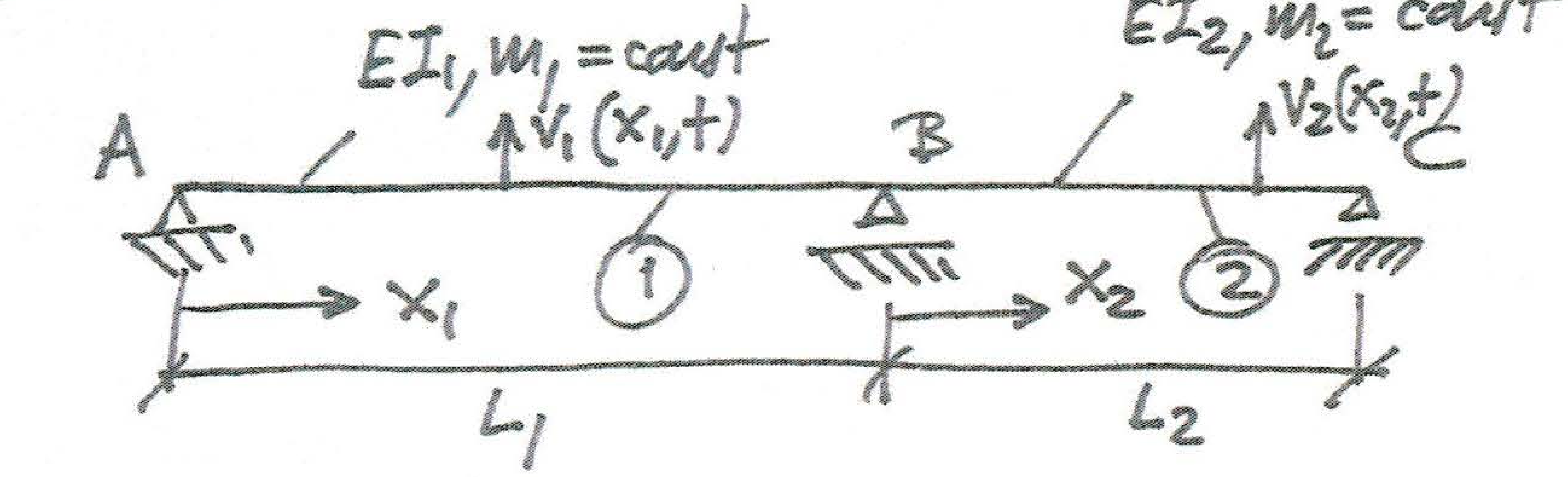
Example 5



We want to study the free vibrations of this statically indeterminate system.

Section 1

$$EI_1 \frac{\partial^4 v_1}{\partial x_1^4} + m_1 \frac{\partial^2 v_1}{\partial t^2} = 0, \quad 0 \le x_1 \le L_1, \; t \ge 0$$

$$v_1(0,t) = 0, \quad v_1(L_1,t) = 0$$

$$EI_1 \frac{\partial^2 v_1(0,t)}{\partial x_1^2} = 0$$

Section 2

$$EI_2 \frac{\partial^4 v_2}{\partial x_2^4} + m_2 \frac{\partial^2 v_2}{\partial t^2} = 0, \quad 0 \le x_2 \le L_2$$

$$v_2(0,t) = 0, \quad v_2(L_2,t) = 0, \quad EI_2 \frac{\partial^2 v_2(L_2,t)}{\partial x_2^2} = 0$$

$$\frac{\partial v_1(L_1,t)}{\partial x_1} = \frac{\partial v_2(0,t)}{\partial x_2}$$

$$EI_1 \frac{\partial^2 v_1(L_1,t)}{\partial x_1^2} = EI_2 \frac{\partial^2 v_2(0,t)}{\partial x_2^2}$$

Note that we assume identical time dependencies since we are seeking normal modes which are synchronous vibrations.

We proceed by separation of space and time $\Rightarrow$ $v_1(x_1,t) = \varphi_1(x_1)\, f(t)$

$v_2(x_2,t) = \varphi_2(x_2)\, f(t)$

Then it is easy to prove that $f(t)$ should be a harmonic function of $t \Rightarrow$

$\Rightarrow$ we reformulate two eigenvalue problems in space, one for each of the two segments of the beam.

Section 1

$$\varphi_1^{IV}(x_1) - \frac{\omega^2}{c_1^4}\varphi_1(x_1) = 0$$

$$\varphi_1(0)=0, \quad \varphi_1(L_1)=0$$

$$\varphi_1''(0)=0$$

where prime denotes differentiation with respect to the argument of each function

$$\varphi_1'(L_1) = \varphi_2'(0)$$

$$EI_1\,\varphi_1''(L_1) = EI_2\,\varphi_2''(0)$$

Section 2

same frequency due to synchronicity

$$\varphi_2^{IV}(x_2) - \frac{\omega^2}{c_2^4}\varphi_2(x_2) = 0$$

$$\varphi_2(0)=0, \quad \varphi_2(L_2)=0, \quad \varphi_2''(L_2)=0$$

$$\varphi_1(x_1) = \cancel{A_1\cos\frac{\sqrt{\omega}}{c_1}x_1} + B_1\sin\frac{\sqrt{\omega}}{c_1}x_1 + \cancel{C_1\cosh\frac{\sqrt{\omega}}{c}x_1} + D_1\sinh\frac{\sqrt{\omega}}{c_1}x_1$$

$$\left.\begin{aligned}\varphi_1(0)=0 &\Rightarrow A_1 + C_1 = 0\\ \varphi_1''(0)=0 &\Rightarrow -A_1 + C_1 = 0\end{aligned}\right\} \Rightarrow A_1 = C_1 = 0$$

$$\varphi_1(L_1)=0 \Rightarrow \boxed{B_1\sin\frac{\sqrt{\omega}}{c_1}L_1 + D_1\sinh\frac{\sqrt{\omega}}{c_1}L_1 = 0}$$

$$\varphi_2(x_2) = \tilde{A}_2\cos\frac{\sqrt{\omega}}{c_2}x_2 + \tilde{B}_2\sin\frac{\sqrt{\omega}}{c_2}x_2 + \tilde{C}_2\cosh\frac{\sqrt{\omega}}{c_2}x_2 + \tilde{D}_2\sinh\frac{\sqrt{\omega}}{c_2}x_2 =$$

$$= \cancel{A_2\cos\frac{\sqrt{\omega}}{c_2}(L_2-x_2)} + B_2\sin\frac{\sqrt{\omega}}{c_2}(L_2-x_2) + \cancel{C_2\cosh\frac{\sqrt{\omega}}{c_2}(L_2-x_2)} + D_2\sinh\frac{\sqrt{\omega}}{c_2}(L_2-x_2)$$

$$\left.\begin{aligned}\varphi_2(L_2)=0 &\Rightarrow A_2 + C_2 = 0\\ \varphi_2''(L_2)=0 &\Rightarrow -A_2 + C_2 = 0\end{aligned}\right\} \Rightarrow A_2 = C_2 = 0$$

$$\varphi_2(0)=0 \Rightarrow \boxed{B_2\sin\frac{\sqrt{\omega}}{c_2}L_2 + D_2\sinh\frac{\sqrt{\omega}}{c_2}L_2 = 0}$$

$$\varphi_1'(L_1) = \varphi_2'(0) \Rightarrow \boxed{B_1 \frac{\sqrt{\omega}}{c_1} \cos \frac{\sqrt{\omega}}{c_1} L_1 + D_1 \frac{\sqrt{\omega}}{c_1} \cosh \frac{\sqrt{\omega}}{c_1} L_1 = -B_2 \frac{\sqrt{\omega}}{c_2} \cos \frac{\sqrt{\omega}}{c_2} L_2 - D_2 \frac{\sqrt{\omega}}{c_2} \cosh \frac{\sqrt{\omega}}{c_2} L_2}$$

$$EI_1 \varphi_1''(L_1) = EI_2 \varphi_2''(0) \Rightarrow \boxed{EI_1 \left(-B_1 \left(\frac{\sqrt{\omega}}{c_1}\right)^2 \sin \frac{\sqrt{\omega}}{c_1} L_1 + D_1 \left(\frac{\sqrt{\omega}}{c_1}\right)^2 \sinh \frac{\sqrt{\omega}}{c_1} L_1 \right) = EI_2 \left(-B_2 \left(\frac{\sqrt{\omega}}{c_2}\right)^2 \sin \frac{\sqrt{\omega}}{c_2} L_2 + D_2 \left(\frac{\sqrt{\omega}}{c_2}\right)^2 \sinh \frac{\sqrt{\omega}}{c_2} L_2 \right)}$$

Hence,

$$\begin{bmatrix} \sin \frac{\sqrt{\omega}}{c_1} L_1 & \sinh \frac{\sqrt{\omega}}{c_1} L_1 & 0 & 0 \\ 0 & 0 & \sin \frac{\sqrt{\omega}}{c_2} L_2 & \sinh \frac{\sqrt{\omega}}{c_2} L_2 \\ \frac{\sqrt{\omega}}{c_1} \cos \frac{\sqrt{\omega}}{c_1} L_1 & \frac{\sqrt{\omega}}{c_1} \cosh \frac{\sqrt{\omega}}{c_1} L_1 & \frac{\sqrt{\omega}}{c_2} \cos \frac{\sqrt{\omega}}{c_2} L_2 & \frac{\sqrt{\omega}}{c_2} \cosh \frac{\sqrt{\omega}}{c_2} L_2 \\ -EI_1 \left(\frac{\sqrt{\omega}}{c_1}\right)^2 \sin \frac{\sqrt{\omega}}{c_1} L_1 & EI_1 \left(\frac{\sqrt{\omega}}{c_1}\right)^2 \sinh \frac{\sqrt{\omega}}{c_1} L_1 & EI_2 \left(\frac{\sqrt{\omega}}{c_2}\right)^2 \sin \frac{\sqrt{\omega}}{c_2} L_2 & -EI_2 \left(\frac{\sqrt{\omega}}{c_2}\right)^2 \sinh \frac{\sqrt{\omega}}{c_2} L_2 \end{bmatrix} \begin{Bmatrix} B_1 \\ D_1 \\ B_2 \\ D_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

We can proceed from this point to compute the eigenvalues and eigenvectors of this general problem, but for the sake of simplicity let's assume that $L_1 = L_2 = L$, $EI_1 = EI_2 = EI$, $m_1 = m_2 = m$

Then, let's define $\frac{\sqrt{\omega}}{c}L \equiv \alpha$ (non-dimensional frequency) $\Rightarrow$

$$\begin{bmatrix} \sin\alpha & \sinh\alpha & 0 & 0 \\ 0 & 0 & \sin\alpha & \sinh\alpha \\ \cos\alpha & \cosh\alpha & \cos\alpha & \cosh\alpha \\ -\sin\alpha & \sinh\alpha & \sin\alpha & -\sinh\alpha \end{bmatrix} \begin{Bmatrix} B_1 \\ D_1 \\ B_2 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

From the first and second equations $\Rightarrow$ $\frac{D_1}{B_1} = \frac{D_2}{B_2} = -\frac{\sin a}{\sinh a}$, $\sinh a \neq 0$ (which holds if $a \neq 0$)

From the third relation $\Rightarrow$ $B_1 \cos a - B_1 \frac{\sin a}{\sinh a} \overset{\times \cosh a}{+} B_2 \cos a - B_2 \frac{\sin a}{\sinh a} \cosh a = 0 \Rightarrow$

$$\Rightarrow B_1\left(\cos a - \frac{\sin a}{\sinh a}\cosh a\right) + B_2\left(\cos a - \frac{\sin a}{\sinh a}\cosh a\right) = 0$$

From the fourth equation $\Rightarrow$ $-B_1 \sin a - B_1 \frac{\sin a}{\cancel{\sinh a}} \cancel{\sinh a} + B_2 \sin a + B_2 \frac{\sin a}{\cancel{\sinh a}} \cancel{\sinh a} = 0 \Rightarrow$

$$\Rightarrow -B_1 \sin a + B_2 \sin a = 0 \Rightarrow \begin{cases} B_2 - B_1 = 0 \\ \text{or/and} \\ \sin a = 0 \end{cases}$$

By combining these two relations we end up having a (2x2) eigenvalue problem!

$$\begin{bmatrix} \cos a - \dfrac{\sin a}{\tanh a} & \cos a - \dfrac{\sin a}{\tanh a} \\ -\sin a & \sin a \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$[A]$

$\Rightarrow$ for nontrivial solutions we request that $\det[A] = 0 \Rightarrow$

$$\Rightarrow \boxed{\sin a \left(\cos a - \frac{\sin a}{\tanh a} \right) = 0} \text{ (Frequency equation)}$$

$\sin a = 0 \Rightarrow a_k = k\pi,\ k = 1, 2, \ldots \Rightarrow$

$\Rightarrow \dfrac{\sqrt{\omega_k}}{c} L = a_k \Rightarrow \omega_k = \left(\dfrac{c a_k}{L} \right)^2,\ k = 1, 2, \ldots$

Then $D_1 = D_2 = 0$

Then from the first of the above equations $\Rightarrow$

$\cos a\, B_1 + \cos a\, B_2 = 0 \Rightarrow B_1 = -B_2 \Rightarrow$

$$\Rightarrow \varphi_1(x_1) = B_1 \sin \frac{\sqrt{\omega}}{c} x_1 \Rightarrow \left. \begin{aligned} \varphi_{1k}(x_1) &= B_k \sin \frac{\sqrt{\omega_k}}{c} x_1 \\ \varphi_{2k}(x_2) &= -B_k \sin \frac{\sqrt{\omega_k}}{c} (L_2 - x_2) \end{aligned} \right\}$$

$k = 1, 2, \ldots$

We may mass-normalize $\displaystyle\int_0^{L_1} m \varphi_{1k}^2(x_1)\, dx_1 = 1 \Rightarrow$

$\Rightarrow$ Compute $B_k,\ k = 1, 2, \ldots$

$\cos a - \dfrac{\sin a}{\tanh a} = 0 \Rightarrow$

$\Rightarrow \tan a = \tanh a \Rightarrow$

$\Rightarrow \tan \dfrac{\sqrt{\omega}}{c} L = \tanh \dfrac{\sqrt{\omega}}{c} L \Rightarrow$

$\Rightarrow$ We get the nat. freq. $\omega_k,\ k = 1, 2, \ldots$

$\Rightarrow$ Identical equation to clamped-simply supported beam of length L

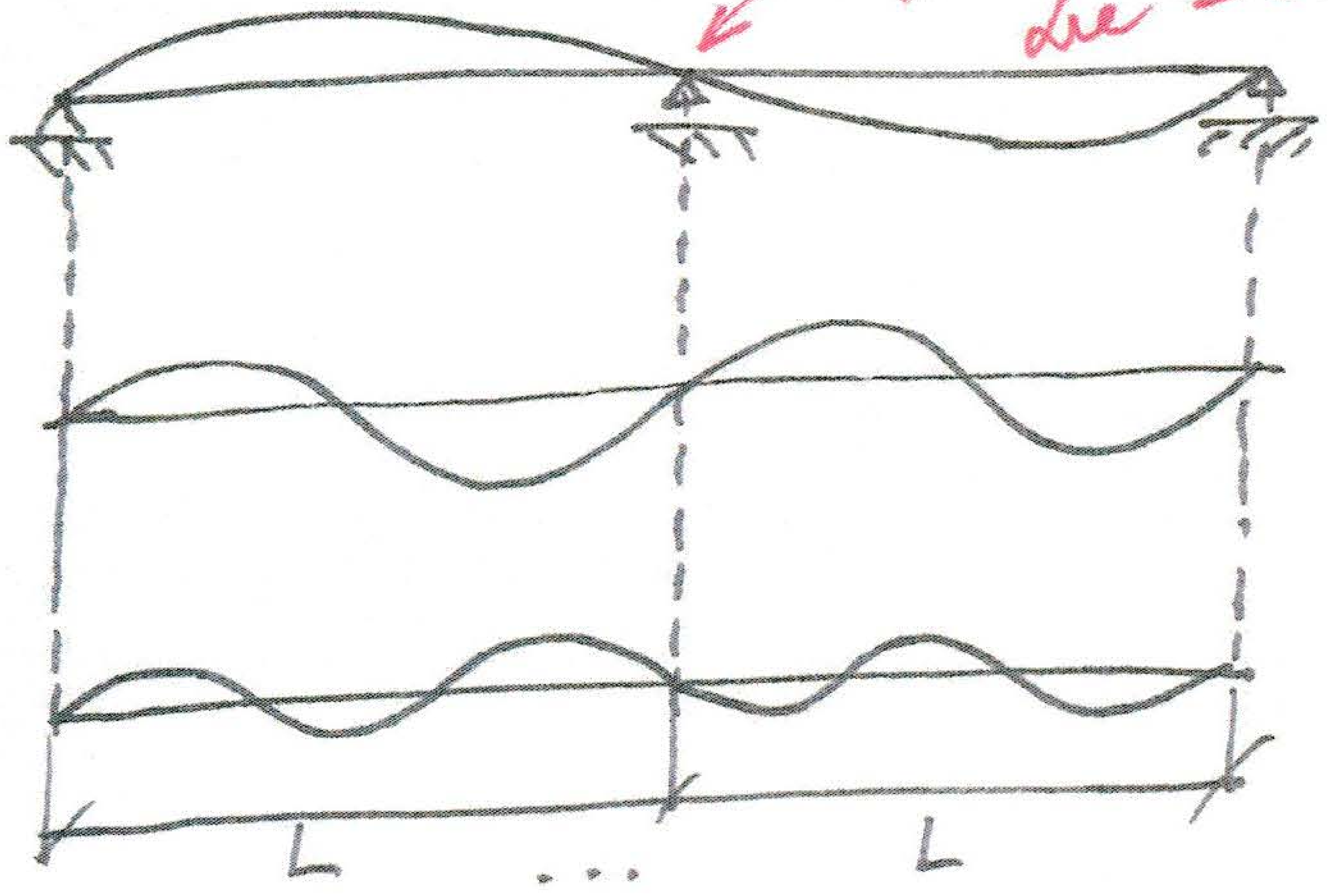
From the second of the above equations, $-B_1 \sin a + B_2 \sin a = 0$

$\Rightarrow B_1 = B_2$

Then, $D_1 = -B_1 \dfrac{\sin \frac{\sqrt{\omega_k}}{c} L}{\sinh \frac{\sqrt{\omega_k}}{c} L} = D_2$

$\sin a = 0$

Note that equal slopes are $\neq 0$, but equal moments are $= 0$!



$k=1$

$k=2$

$k=3$

Then,

$$\varphi_{1k}(x_1) = B_k\left[\sin\frac{\sqrt{a_k}}{c}x_1 - \frac{\sin\frac{\sqrt{a_k}}{c}L}{\sinh\frac{\sqrt{a_k}}{c}L}\times \sinh\frac{\sqrt{a_k}}{c}x_1\right],$$

$$0 \le x_1 \le L_1$$

$$\varphi_{2k}(x_2) = B_k\left[\sin\frac{\sqrt{a_k}}{c}(L-x_2) - \frac{\sin\frac{\sqrt{a_k}}{c}L \;\sinh\frac{\sqrt{a_k}}{c}(L-x_2)}{\sinh\frac{\sqrt{a_k}}{c}(L-x_2)}\right]$$

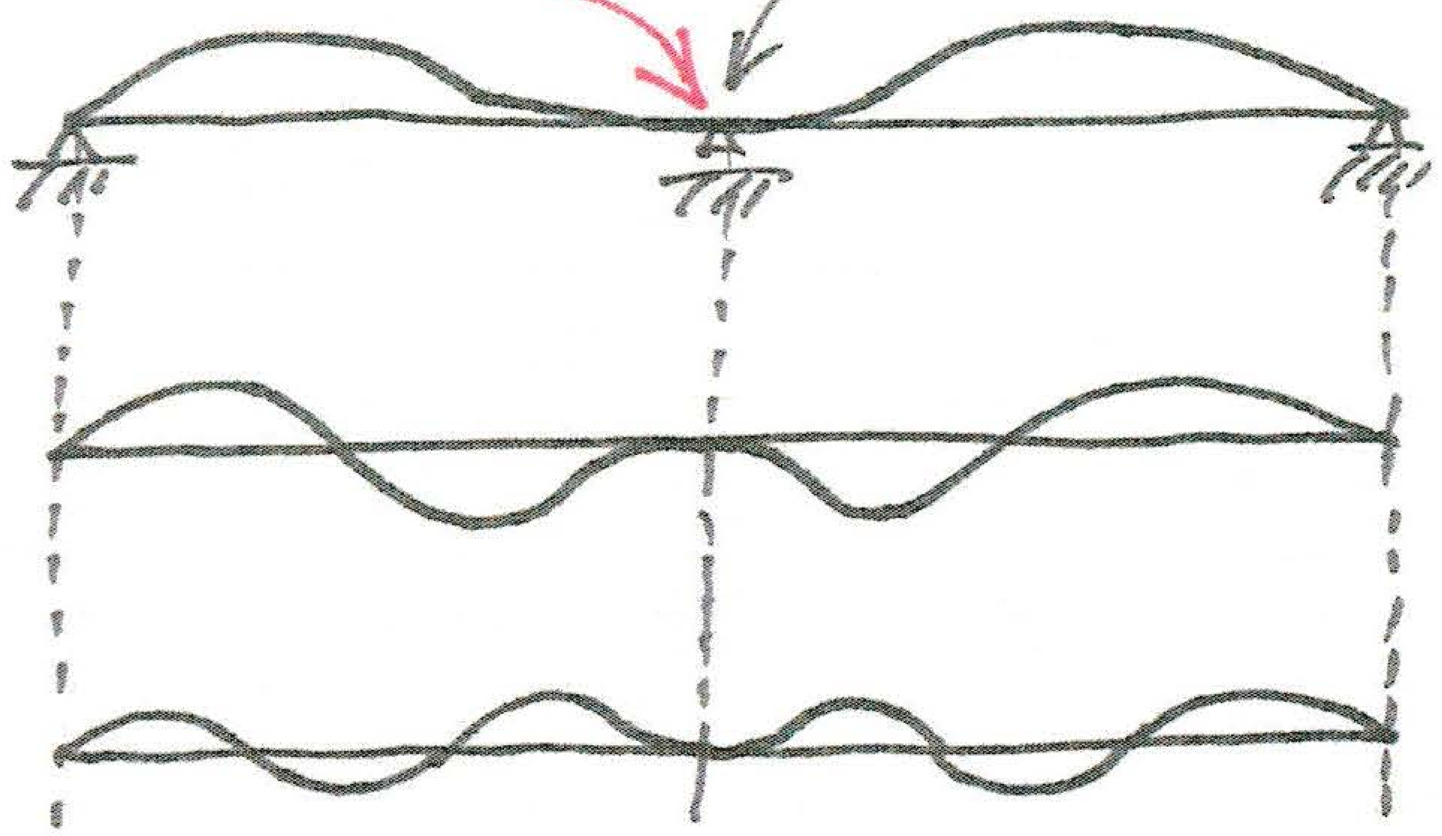
$$0 \le x_2 \le L_2$$

Again may mass-normalize,

$$\int_0^{L_1} m\,\varphi_{1k}^2(x_1)\,dx_1 = 1 \Rightarrow B_k,\ k=1,2,\ldots$$

Note effective clamped BC!

Note that equal slopes are $=0$ but equal moments $\neq 0$!



$k=1$

$k=2$

$k=3$

Remark

For the general case the mass-orthonormalization should be carried out throughout the length of the beam,

$$\int_0^{L_1} m_1\varphi_{1k}^2(x_1)\,dx_1 + \int_0^{L_2} m_2\varphi_{2k}^2(x_2)\,dx_2 = 1 \Rightarrow$$

$\Rightarrow$ We evaluate the single unknown coefficient in the expressions of the mode shapes.