4.2. Sturm – Liouville Problems

A Sturm–Liouville (S–L) eigenvalue problem is of the general form:

$$\boxed{(pv')' - qv + \lambda\rho v = 0\ , \quad 0 \le x \le \pi} \quad + \text{BCs} \qquad p, \rho > 0,\ 0 \le x \le \pi \qquad (13)$$

where $v = v(x)$ and q is a continuous function. By the previous discussion this is a self-adjoint differential equation. This equation can be brought into normal form by setting $z = v\sqrt{\rho} \Rightarrow$

$$\Rightarrow \boxed{\frac{d}{dx}(p^* z') - (q^* - \lambda) z = 0} \quad p^* = \frac{q}{\rho},\ q^* = -\frac{1}{\sqrt{\rho}}\frac{d}{dx}\left(p\frac{d}{dx}\frac{1}{\sqrt{\rho}}\right) + \frac{q}{\rho}$$

If $q = 0 \Rightarrow$ Equation (13) can be transformed to

$$q = 0 \rightarrow \boxed{\frac{d^2 v}{d\xi^2} + \lambda\sigma v = 0},\quad \sigma = \rho p,\ \xi = \int \frac{dx}{p(x)} \qquad (14)$$

Also, if we set $u = \sqrt[4]{p\rho}\,v$, $t = \int_0^x \sqrt{\frac{\rho}{p}}\,dx$, $l = \int_0^\pi \sqrt{\frac{\rho}{p}}\,dx$, then (13) can be written as,

$$q \neq 0 \rightarrow \boxed{u'' - ru + \lambda u = 0},\quad u = u(t),\ 0 \le t \le l,\ r = \frac{f''}{f} + \frac{q}{\rho},\ f = \sqrt[4]{p\rho},\ r \text{ continuous} \qquad (15)$$

Considering the S-L problem (13),

Also homogeneous

$$(p v')' - q v + \lambda \rho v = 0 \ , \quad 0 \leq x \leq \pi \ + \text{Simple BCs} \quad (13)$$

Simple BCs
$v(0) = v(\pi) = 0$
$v'(0) = v'(\pi) = 0$
$h_0 v(0) = v'(0)$,
$-h_1 v(\pi) = v'(\pi)$
$h_0, h_1 > 0$

to the eigenfunctions v and the positive eigenvalues λ correspond eigenvibrations of the system at frequency $\nu = \sqrt{\lambda}$ represented by functions:

$$v(x)\left(a_\nu \cos\nu t + b_\nu \sin\nu t\right) \qquad (16)$$

Moreover, the eigenfunctions of the S-L problem furnish systems of orthogonal functions; this general property follows from the very structure of the S-L differential equation. In fact, if $\lambda_n \neq \lambda_m$ are two distinct eigenvalues and v_n, v_m the corresponding eigenfunctions, it holds:

$$(\lambda_n - \lambda_m)\int_0^\pi \rho v_m v_n \, dx + \underbrace{\int_0^\pi \frac{d}{dx}\left(p\left[v_n' v_m - v_n v_m'\right]\right) dx}_{0 \text{ for simple BCs}} = 0 \Rightarrow \qquad (17)$$

$$p\left[v_n' v_m - v_n v_m'\right]_0^\pi = 0$$

$\Rightarrow \sqrt{\rho}\, v_n$ is orthogonal to $\sqrt{\rho}\, v_m$, $m \neq n \Rightarrow \int_0^\pi \rho v_n v_m \, dx = 0$, $m \neq n$

Also may orthonormalize $\rightarrow \int_0^\pi \rho v_n^2 = 1$, $n = 1, 2 \ldots$

The eigenvalues λ of the S-L problem (13) for given (simple) BCs, ordered with respect to magnitude, form a countable infinity of values, or a denumerable sequence,

$$\lambda_1, \lambda_2, \lambda_3, \ldots \quad \leftarrow \text{Discrete spectrum of eigenvalues}$$

and the corresponding system of eigenfunctions is a complete orthogonal system. Every continuous function $f(x)$ which has piecewise continuous first and second derivatives and satisfies the BCs of the eigenvalue problem (13) can be expanded in an absolutely and uniformly convergent series,

$$f(x) = \sum_{n=1}^{\infty} c_n v_n(x), \quad c_n = \int_0^{\pi} \rho f v_n \, dx, \quad 0 \le x \le \pi \qquad (18)$$

in terms of the eigenfunctions. This expansion makes it possible to fit the solution $u(x,t) = \sum_{n=1}^{\infty} v_n(x) \left(a_n \cos\nu_n t + b_n \sin\nu_n t \right)$ to a prescribed initial state.

Simple eigenvalues of the S-L problem for simple (non-periodic) homogeneous BCs

All eigenvalues of S-L problem are simple (except those of periodic BCs) ⇒ No two linearly independent eigenfunctions v and v^* can correspond to the same eigenvalue λ. Proof by contradiction:

Suppose that that there were two such eigenfunctions, v and v^*, of $(pv')' - qv + \lambda\rho v = 0$ corresponding to the same eigenvalue λ ⇒

⇒ Then, <u>every</u> solution of the S-L problem could be expressed as superposition of these two linearly independent solutions, $cv + c^*v^*$, <u>in superposition</u> of these two linearly independent solutions would hold for any choice of the BCs ⇒ <u>The solution $cv + c^*v^*$ would hold for any choice of the BCs</u> ⇒ Contradiction, since each of v and v^* (and hence, their superposition) must satisfy the prescribed homogeneous simple BCs of the S-L problem, as they are eigenfunctions of the problem

$$(pv')' - qv + \lambda\rho v = 0 \ + \ \text{Simple homogeneous BCs.}$$

All eigenvalues of S–L problems (except those with periodic BCs, where $\lambda = n^2$, $n = 1, 2, \ldots$ is a double eigenvalue of $y'' + \lambda y = 0$ with two eigenfunctions $\sin nx$ and $\cos nx$) are <u>simple</u>, i.e., no two linearly independent eigenfunctions v and v^* can correspond to the same eigenvalue.

All eigenvalues are positive

In problems in mechanics, separation of variables often leads to eigenvalue problems of the S-L type,

$$(pv')' - qv + \lambda \rho v = 0 \;, \quad v = v(x), \quad x \in \mathcal{D}, \text{ BCs on } \partial\mathcal{D}$$

for which singularities occur at the end points of the fundamental domain $\mathcal{D}$ $\Rightarrow$ for example $p \to 0$ at $\partial\mathcal{D}$. The nature of these problems is such that certain conditions are imposed at the end points; e.g., solution continuous, or bounded, or becomes infinite but of an order less than that prescribed $\Rightarrow$ one obtains <u>singular S–L BVPs</u>! We show this by some examples. Note that in each case the eigenfunctions form a complete orthogonal basis in $\mathcal{D}$ (orthogonal functions).

All eigenvalues of the S-L problem with simple BCs are positive ($q \geq 0$)

This follows from the self-adjointness of the S-L problem $\Rightarrow$

$\Rightarrow$ Let $q \geq 0$, $h_0 \geq 0$, $h_1 \geq 0 \Rightarrow$

$\Rightarrow (pv')' - qv + \lambda\rho v = 0 \Rightarrow$ Multiply by v and integrate $\int_0^\pi (\;)\,dx \Rightarrow$

$$\Rightarrow \int_0^\pi (pv')'v\,dx - \int_0^\pi qv^2\,dx + \lambda\int_0^\pi \rho v^2\,dx = 0 \Rightarrow$$

$$\Rightarrow \lambda\int_0^\pi \rho v^2\,dx = -\int_0^\pi (pv')'v\,dx + \int_0^\pi qv^2\,dx =$$

$$= \underbrace{-pv'v\Big|_0^\pi}_{0 \text{ for simple BCs}} + \int_0^\pi (pv'^2 + qv^2)\,dx > 0 \quad \text{for } q \geq 0,\ h_0, h_1 \geq 0$$

Bessel functions

$$(x v'(x))' - \frac{n^2}{x} v(x) + \lambda x v(x) = 0, \quad 0 \le x \le 1 \equiv \mathcal{D}, \; n = 0,1,2,\ldots$$

The singularity is $p(0) = 0 \Rightarrow$ the BC at $x=0$ is that the solution is bounded $\Rightarrow$ The eigenvalues $\lambda = \lambda_{nm}$ are roots of the equation $J_n(\sqrt{\lambda}) = 0$ and the corresponding eigenfunctions are $\sqrt{x}\, J_n(\sqrt{\lambda}\, x)$.

Bessel function of the 1st kind, order $n = 0,1,2,\ldots$

Legendre functions of arbitrary order

$$\left[(1-x^2) v'(x)\right]' + \lambda x = 0, \quad -1 \le x \le 1 \equiv \mathcal{D} \Rightarrow p(\pm 1) = 0$$

The BCs are that $v(x)$ remains bounded at $x = \pm 1 \Rightarrow$ The eigenvalues $\lambda = \lambda_n = n(n+1)$ correspond the eigenfunctions $P_n(x)$, Legendre polynomials. The Legendre polynomials are the only solutions of this eigenvalue problem and form an orthogonal and complete basis on $\mathcal{D}$.

Jacobi and Tchebycheff polynomials

$$\left[(1-x)^{m-q+1} x^{q} v'(x)\right]' + \lambda (1-x)^{m-q} x^{q-1} v(x) = 0, \quad 0 \leq x \leq 1 \equiv \mathcal{D}$$

Then the singularities are $p(0) = p(1) = 0 \Rightarrow$ The BCs ensure that the solution will be finite at $x=0$ and $x=1 \Rightarrow$ The eigenvalues are $\lambda = n(m+n)$ and the eigenfunctions are Jacobi's polynomials.

$$\left(\sqrt{1-x^2}\, v'(x)\right)' + \frac{\lambda}{\sqrt{1-x^2}} v(x) = 0, \quad -1 \leq x \leq 1 \equiv \mathcal{D} \Rightarrow p(\pm 1) = 0$$

The BCs are such that the solution will be bounded at $x = \pm 1 \Rightarrow$
$\Rightarrow$ The eigenvalues are $\lambda_n = n^2$ and the eigenfunctions are Tchebycheff polynomials $T_n(x)$.

Recall eigenvalue problem for wave equation,

$$\frac{d}{dx}\Big(\underbrace{A(x)}_{p(x)} \frac{du}{dx}\Big) + \underbrace{B(x)}_{\rho(x)}\, \underbrace{\omega^2}_{\lambda} u = 0 \Rightarrow \text{with } q=0 \text{ we write this as,}$$

$$(pu')' - qu + \lambda \rho u = 0$$

Consider now the asymptotic behavior of the solutions of the S-L eigenvalue problem,

$$\boxed{(pV')' - qV + \lambda\rho V = 0} \xrightarrow[\text{transformations}]{\text{Suitable}} u'' - ru + \lambda u = 0 \Rightarrow$$

$$\Rightarrow u'' + \mu(x)u = 0$$

Assume that as $x \to \infty$, $\mu(x)$ approaches a positive limit, which, without loss of generality can be taken as $1 \Rightarrow$ Let $\lim_{x\to\infty} \mu(x) = 1$

Hence, we set $\mu(x) = 1 + \rho(x) \Rightarrow \boxed{u'' + u + \rho u = 0}$ $(*)$

Now, replace the assumption $\lim_{x\to\infty} \rho(x) = 0$ by the more strict assumptions that,

$$|\rho| < \frac{\alpha}{x}, \quad |\rho'| < \frac{\alpha}{x^2}, \quad \alpha > 0 \text{ as } x \to \infty$$

Under this assumption it can be shown that <u>every solution of the S-L problem $(*)$ is bounded as $x \to \infty$.</u> This is to be expected since as $x \to \infty$ the system $(*)$ approaches the limiting system

$$u'' + u = 0 \quad \text{as } \rho(x) \to 0, \; x \to \infty$$

<u>Proof</u>

$u'' + u + \rho u = 0 \Rightarrow$ Multiply by u' and $\int_a^x (\ ) dx \Rightarrow$

$$\Rightarrow u''u' + uu' + \rho uu' = 0 \Rightarrow \int_a^x (u''u' + uu' + \rho uu')\, dx = 0 \Rightarrow$$

$$\Rightarrow u'^2\Big|_a^x + u^2\Big|_a^x = -2\int_a^x \rho uu'\, dx = -\rho u^2\Big|_a^x + \int_a^x \rho' u^2 dx \Rightarrow$$

$$\Rightarrow u'^2(x) - u'^2(a) + u^2(x) - u^2(a) = -\rho\left(u^2(x) - u^2(a)\right) + \int_a^x \rho' u^2 dx \Rightarrow$$

$$\Rightarrow u^2(x) \leq u'^2(x) + u^2(x) \leq \underbrace{u'^2(a) + u^2(a)}_{C(a) > 0} + |\rho(x)|\, u^2(x) + \int_a^x |\rho'(x)| u^2 dx$$

Let $M = M(x) = \sup\limits_{a \leq \xi \leq x} \{u(\xi)\} \Rightarrow$ Taking into account the previous inequality we have that $M^2 \leq C(a) + \frac{M^2 \alpha}{\xi} + M^2 \alpha\left(\frac{1}{a} - \frac{1}{\xi}\right) \Rightarrow$

$\Rightarrow M^2\left(1 - \frac{\alpha}{a}\right) \leq C(a) \Rightarrow$ If we take $a \geq 2\alpha \Rightarrow M^2 \leq 2C(a) > 0$, independent of x ✓

Consider again $u'' + u + \rho(x) = 0$, and assume that $\rho(x) \to 0$ as $x \to \infty$, at an order higher than the first $\Rightarrow \rho(x) = O\left(\frac{1}{x^2}\right)$ ← Bessel functions $\Rightarrow$ Then, there is a closer agreement between the solutions of this equation and those of $u'' + u = 0$

$\Rightarrow$ Solutions are not only bounded, but they also approach trigonometric functions asymptotically $\Rightarrow$ $u = \alpha_\infty \sin(x + \delta_\infty) + O\left(\frac{1}{x}\right)$, where α_∞ and

constants

δ_∞ are the solutions of,

$$\frac{\alpha'}{\alpha} = -\frac{\delta'}{\tan(x+\delta)} = -\rho \sin(x+\delta)\cos(x+\delta)$$

Solving these two equations determines the functions $\alpha(x)$ and $\delta(x)$

So, as $x \to \infty$ it holds that $\delta_\infty = \lim_{x\to\infty} \delta(x)$, $\alpha_\infty = \lim_{x\to\infty} \alpha(x) \Rightarrow$

$\Rightarrow \alpha(x) = \alpha_\infty + O\left(\frac{1}{x}\right)$, $\delta(x) = \delta_\infty + O\left(\frac{1}{x}\right)$. Hence, the shapes of the eigenfunctions approach well-defined trigonometric functions as $x \to \infty \Rightarrow$

$\Rightarrow$ The effect of $\rho(x)$ is confined only to small values of x in the neighborhood of $x = 0$.

Hint for proof (Courant + Hilbert): Require that as $x \to \infty$,

$u \sim \alpha \sin(x+\delta)$ and $u' \sim a\cos(x+\delta)$

Example

$$u'' + \left(1 - \frac{m^2 - 1/4}{x^2}\right)u = 0 \Rightarrow u'' + u + p(x)u = 0, \quad p(x) = -\frac{m^2 - 1/4}{x^2} = O\left(\frac{1}{x^2}\right) \Rightarrow$$

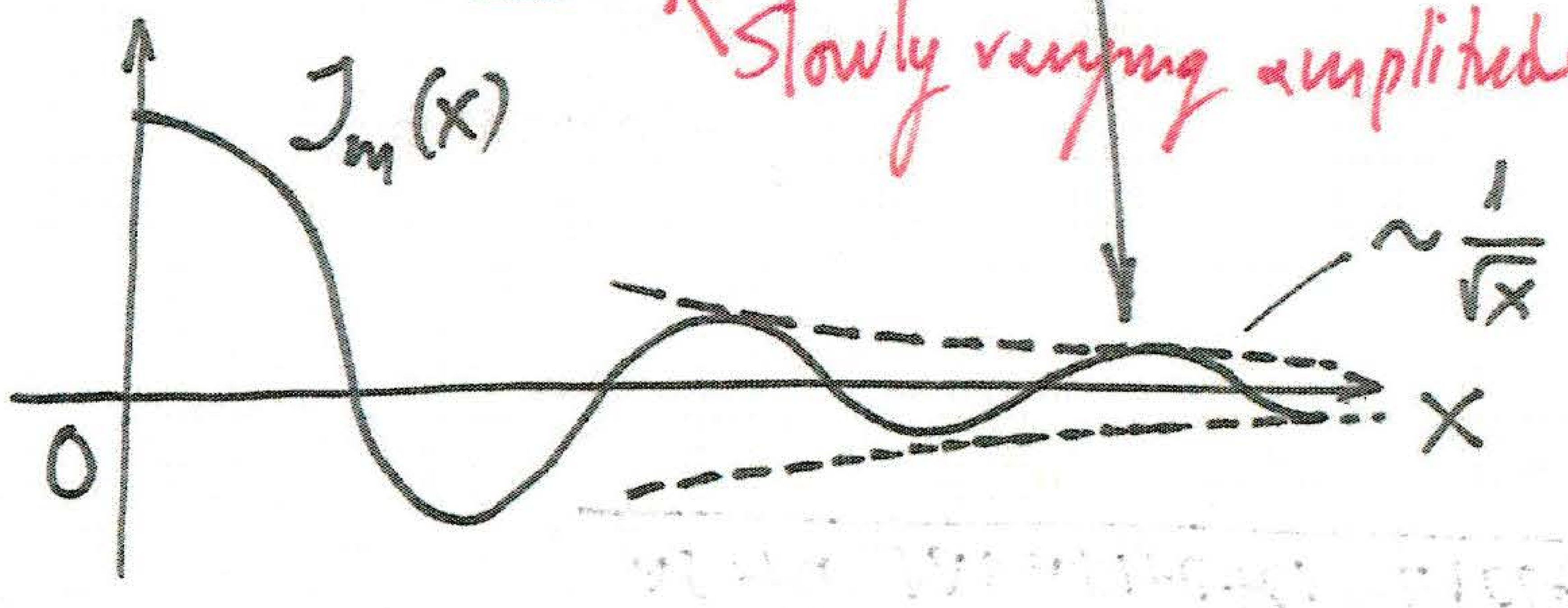
$\Rightarrow$ We can estimate asymptotically how the solutions of this system will look like in the limit as $x \to \infty \Rightarrow$

$$\Rightarrow u_\infty \sim \alpha_\infty \cos(x + \delta_\infty) + O\left(\frac{1}{x}\right), \quad \alpha_\infty = \sqrt{\frac{2}{\pi}}, \; \delta_\infty = -\frac{m\pi}{2} - \frac{\pi}{4}$$

We note that the solutions of this system are connected to the solutions $y_m(x)$ of the Bessel equation through the relation, $u = y_m\sqrt{x} \Rightarrow$

$\Rightarrow$ We can prove that the asymptotic behavior of the Bessel function $J_m(x)$ is

$$J_m(x) \sim \underbrace{\frac{1}{\sqrt{x}}\sqrt{\frac{2}{\pi}}}_{\text{Slowly varying amplitude}} \underbrace{\cos\left(x + \left[-\frac{m\pi}{2} - \frac{\pi}{4}\right]\right)}_{\text{Fast varying oscillation}} + O\left(\frac{1}{x^{3/2}}\right)$$



Note the two scales (slow and fast) in the solution.