Assume that by this methodology we obtain a countable infinity of normal modes of vibration,

$$\{\omega_r^2, \varphi_r(x)\}, \quad r=1,2,\dots$$

We may mass-normalize each eigenfunction according to,

$$\boxed{\int_0^L m(x)\,\varphi_r^2(x)\,dx = 1, \quad r=1,2,\dots}$$ (Mass-normalization condition)

Then, the general solution to the beam equation is computed by linear superposition of the infinity of vibration modes,

$$u(x,t) = \sum_{i=1}^{\infty} f_i(t)\,\varphi_i(x)$$

Each of these functions contains two unknowns that are determined by the initial conditions

However, in order to compute the constants in the infinite modal superposition we need to study the orthogonality conditions of the eigenfunctions $\Rightarrow$ Start with simple boundary conditions, i.e., conditions involving either the zeroth, first or higher derivatives of $\varphi(x)$; examples are clamped BCs, $\varphi(0)=\varphi'(0)=0$ or free BCs $\varphi''(0)=\varphi'''(0)=0$.

Orthogonality of normal modes

Mass-normalized

Consider two modes $\{\omega_r, \varphi_r(x)\}$, $\{\omega_s, \varphi_s(x)\}$, $\omega_r \neq \omega_s \Rightarrow$ Then, these modes satisfy the eigenvalue problem $\Rightarrow$

$$[EI(x)\varphi_r'']'' - \omega_r^2 m(x)\varphi_r(x) = 0 \quad (*)$$

$$[EI(x)\varphi_s'']'' - \omega_s^2 m(x)\varphi_s(x) = 0 \quad (**)$$

$$\int_0^L (*)\varphi_s(x)\,dx = 0 \Rightarrow \int_0^L [EI(x)\varphi_r'']''\varphi_s\,dx - \omega_r^2\int_0^L m(x)\varphi_r\varphi_s\,dx = 0 \Rightarrow$$

$$\Rightarrow [EI(x)\varphi_r'']'\varphi_s\Big|_0^L - \int_0^L [EI(x)\varphi_r'']'\varphi_s'\,dx - \omega_r^2\int_0^L m(x)\varphi_r\varphi_s\,dx = 0 \Rightarrow$$

$$\Rightarrow \underbrace{[EI(x)\varphi_r'']'\varphi_s\Big|_0^L - [EI(x)\varphi_r''\varphi_s']\Big|_0^L}_{\text{Zero for simple boundary conditions}} + \int_0^L EI(x)\varphi_r''\varphi_s''\,dx - \omega_r^2\int_0^L m(x)\varphi_r\varphi_s\,dx = 0$$

$$\Rightarrow \int_0^L EI(x)\varphi_r''\varphi_s''\,dx - \omega_r^2\int_0^L m(x)\varphi_r\varphi_s\,dx = 0$$

$$\int_0^L (**)\varphi_r(x)\,dx = 0 \Rightarrow \int_0^L EI(x)\varphi_r''\varphi_s''\,dx - \omega_s^2\int_0^L m(x)\varphi_r\varphi_s\,dx = 0$$

$$\xrightarrow{-} (\omega_r^2-\omega_s^2)\int_0^L m(x)\varphi_r\varphi_s\,dx = 0 \Rightarrow \boxed{\int_0^L m(x)\varphi_r\varphi_s\,dx = 0}$$

Mass-orthogonality condition

Hence we can orthonormalized the modes so that,

$$\boxed{\int_0^L m(x)\,\varphi_r(x)\,\varphi_s(x)\,dx = \delta_{rs}\ , \quad r,s=1,2,\dots}$$ (Mass-orthonormality condition)

It immediately follows that,

Stiffness BCs can be nonsimple

$$\boxed{\int_0^L \left[EI(x)\,\varphi_r''(x)\right]''\varphi_s(x)\,dx = \omega_r^2\,\delta_{rs}, \quad r,s=1,2,\dots}$$ (Stiffness orthonormality condition)

or by performing integrations by parts $\Rightarrow$

Since we have simple BCs

$$\boxed{\int_0^L EI(x)\,\varphi_r''(x)\,\varphi_s''(x)\,dx = \omega_r^2\,\delta_{rs}\ , \quad r,s=1,2,\dots}$$ (Alternative stiffness orthonormality condition)

But note that this second condition is only valid for simple BCs!

Based on these conditions we may perform modal analysis and decouple the original continuous problem into a set of infinite uncoupled modal oscillators. To show this, we assume simple BCs and the governing equation of motion,

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right] + F(x,t) = m(x)\frac{\partial^2 v}{\partial t^2}\ , \quad \begin{matrix} 0 \le x \le L \\ t \ge 0 \end{matrix} \qquad (1)$$