Problem 1

Compute the Green's function for the eigenvalue problem of the simply supported linear Euler-Bernoulli beam:

$$u''''(x) - \lambda u(x) = f(x)$$
$$u(0) = u(1) = u''(0) = u''(1) = 0$$

Use two alternative definitions for the linear operator, and construct two different (but equivalent) Green's function formulations. In each case verify that the Green's function is symmetric with respect to the arguments. Then, use the derived Green's functions to convert this boundary value problem to an integral equation. (Hint: in the first case define $L[u] = u''''(x)$, and in the second case use $L[u] = u''''(x) - \lambda u(x)$)

Solution

Case #1 - $L[u] = u''''(x)$

For the first case we want to solve for the following problem:

$$u''''(x) = 0 \tag{1}$$

In this case the solution takes on the general form:

$$\begin{aligned} u(x) &= c_1 + c_2 x + c_3 x^2 + c_4 x^3 && x < \xi \\ \tilde{u}(x) &= d_1 + d_2(1-x) + d_3(1-x)^2 + d_4(1-x)^3 && x > \xi \end{aligned} \tag{2}$$

Taking the derivatives of each expression we get:

$$\begin{aligned} u'(x) &= c_2 + 2c_3 x + 3c_4 x^2 && x < \xi \\ \tilde{u}'(x) &= -d_2 - 2d_3(1-x) - 3d_4(1-x)^2 && x > \xi \\ \\ u''(x) &= 2c_3 + 6c_4 x && x < \xi \\ \tilde{u}''(x) &= 2d_3 + 6d_4(1-x) && x > \xi \\ \\ u'''(x) &= 6c_4 && x < \xi \\ \tilde{u}'''(x) &= -6d_4 && x > \xi \end{aligned} \tag{3}$$

Imposing the four boundary conditions we get:

$$\begin{aligned} u(0) &= c_1 = 0 \\ \tilde{u}(1) &= d_1 = 0 \\ u''(0) &= c_3 = 0 \\ \tilde{u}''(1) &= d_3 = 0 \end{aligned} \tag{4}$$

At ξ we need to impose continuity in the Green's function ξ, along with its first and second derivatives and we impose a unit jump in the third derivative at ξ:

$$\begin{aligned} & u(\xi) = \tilde{u}(\xi) \\ & \Rightarrow c_2\xi + c_4\xi^3 = d_2(1-\xi) + d_4(1-\xi)^3 \\ \\ & u'(\xi) = \tilde{u}'(\xi) \\ & \Rightarrow c_2 + 3c_4\xi^2 = -d_2 - 3d_4(1-\xi)^2 \\ \\ & u''(\xi) = \tilde{u}''(\xi) \\ & \Rightarrow 6c_4\xi = 6d_4(1-\xi) \\ \\ & u'''(\xi^+) - u'''(\xi^-) = -1 \\ & \Rightarrow -6d_4 - 6c_4 = -1 \end{aligned} \tag{5}$$

From the third and fourth equation we have:

$$\begin{aligned} c_4 &= \frac{(1-\xi)}{6} \\ d_4 &= \frac{\xi}{6} \end{aligned} \tag{6}$$

Now taking the first two equations, we obtain:

$$\begin{aligned} c_2 &= \frac{\xi(\xi-2)(1-\xi)}{6} \\ d_2 &= \frac{(\xi^3-\xi)}{6} \end{aligned} \tag{7}$$

Thus the final form of the Green's function for case #1 is:

$$K(x,\xi)=\begin{cases}\dfrac{\xi(\xi-2)(1-\xi)}{6}x+\dfrac{(1-\xi)}{6}x^3 & x<\xi\\[2ex] \dfrac{(\xi^3-\xi)}{6}(1-x)+\dfrac{\xi}{6}(1-x)^3 & x>\xi\end{cases}\tag{8}$$

It is good practice to confirm that the green's function is symmetric. Flipping x and ξ in the first equation we see if we can obtain the second equation.

$$\begin{aligned}K(\xi,x)&=\frac{x(x-2)(1-x)}{6}\xi+\frac{(1-x)}{6}\xi^3 \qquad x<\xi\\ &=\frac{(-x^3-2x+3x^2)}{6}\xi+\frac{(1-x)}{6}\xi^3\\ &=\frac{(1-x)^3}{6}\xi+\frac{(1-x)}{6}(\xi^3-\xi)\\ &=\frac{(\xi^3-\xi)}{6}(1-x)+\frac{(1-x)^3}{6}\xi\end{aligned}\tag{9}$$

It is thus confirmed that this Green's function is symmetric. The boundary value can thus be converted to an integral equation:

(-) sign

$$u(x)=\int_0^1 K(x,\xi)\left[f(\xi)+\lambda u(\xi)\right]d\xi\tag{10}$$

Case #2 - $L[u]=u''''(x)-\lambda u(x)$

For the first case we want to solve for the following problem:

$$u''''(x)-\lambda u(x)=0\tag{11}$$

In this case the solution takes on the general form:

$$\begin{aligned}u(x)&=c_1\sin(\gamma x)+c_2\cos(\gamma x)+c_3\sinh(\gamma x)+c_4\cosh(\gamma x) && x<\xi\\ \tilde{u}(x)&=d_1\sin(\gamma x)+d_2\cos(\gamma x)+d_3\sinh(\gamma x)+d_4\cosh(\gamma x) && x>\xi\end{aligned}\tag{12}$$

Where

$$\gamma = (\lambda)^{1/4} \tag{13}$$

Taking the derivatives of each expression we get:

$$\begin{aligned}
u'(x) &= \gamma\left[c_1 \cos(\gamma x) - c_2 \sin(\gamma x) + c_3 \cosh(\gamma x) + c_4 \sinh(\gamma x)\right] && x < \xi \\
\tilde{u}'(x) &= \gamma\left[d_1 \cos(\gamma x) - d_2 \sin(\gamma x) + d_3 \cosh(\gamma x) + d_4 \sinh(\gamma x)\right] && x > \xi \\
\\
u''(x) &= \gamma^2\left[-c_1 \sin(\gamma x) - c_2 \cos(\gamma x) + c_3 \sinh(\gamma x) + c_4 \cosh(\gamma x)\right] && x < \xi \\
\tilde{u}''(x) &= \gamma^2\left[-d_1 \sin(\gamma x) - d_2 \cos(\gamma x) + d_3 \sinh(\gamma x) + d_4 \cosh(\gamma x)\right] && x > \xi \\
\\
u'''(x) &= \gamma^3\left[-c_1 \cos(\gamma x) + c_2 \sin(\gamma x) + c_3 \cosh(\gamma x) + c_4 \sinh(\gamma x)\right] && x < \xi \\
\tilde{u}'''(x) &= \gamma^3\left[-c_1 \cos(\gamma x) + c_2 \sin(\gamma x) + c_3 \cosh(\gamma x) + c_4 \sinh(\gamma x)\right] && x > \xi
\end{aligned} \tag{14}$$

We now have the following eight conditions we need to satisfy:

$$\begin{aligned}
&u(0) = \tilde{u}(1) = u''(0) = \tilde{u}''(1) = 0 \\
&u(\xi) = \tilde{u}(\xi) \\
&u'(\xi) = \tilde{u}'(\xi) \\
&u''(\xi) = \tilde{u}''(\xi) \\
&u'''(\xi^+) - u'''(\xi^-) = -1
\end{aligned} \tag{15}$$

We can write a MATLAB symbolic expression to solve for the 8 unknown parameters:

```
function TAM514_hw5_prob2

clear all
close all
clc

syms gm xi

A1=[sin(0),cos(0),sinh(0),cosh(0),0,0,0,0];
A2=[0,0,0,0,sin(gm),cos(gm),sinh(gm),cosh(gm)];
A3=gm^2*[-sin(0),-cos(0),sinh(0),cosh(0),0,0,0,0];
A4=gm^2*[0,0,0,0,-sin(gm),-cos(gm),sinh(gm),cosh(gm)];
A5=[sin(gm*xi),cos(gm*xi),sinh(gm*xi),cosh(gm*xi),...
    -sin(gm*xi),-cos(gm*xi),-sinh(gm*xi),-cosh(gm*xi)];
A6=gm*[cos(gm*xi),-sin(gm*xi),cosh(gm*xi),sinh(gm*xi),...
```

```
    -cos(gm*xi),sin(gm*xi),-cosh(gm*xi),-sinh(gm*xi)];
A7=gm^2*[-sin(gm*xi),-cos(gm*xi),sinh(gm*xi),cosh(gm*xi),...
    sin(gm*xi),cos(gm*xi),-sinh(gm*xi),-cosh(gm*xi)];
A8=gm^3*[-cos(gm*xi),sin(gm*xi),cosh(gm*xi),sinh(gm*xi),...
    cos(gm*xi),-sin(gm*xi),-cosh(gm*xi),-sinh(gm*xi)];

A=[A1;A2;A3;A4;A5;A6;A7;A8]
B=[0;0;0;0;0;0;0;-1]

simplify(A^(-1)*B)
```

The following answer was generated in MATLAB:

$$
\begin{aligned}
c_1 &= \frac{\sin(\gamma-\gamma\xi)}{2\gamma^2\sin(\gamma)} & d_1 &= \frac{-\sin(\gamma\xi)\cot(\gamma)}{2\gamma^3} \\
c_2 &= 0 & d_2 &= \frac{\sin(\gamma\xi)}{2\gamma^3} \\
c_3 &= \frac{-\left[\cosh(\gamma\xi)\sinh(\gamma)-\sinh(\gamma\xi)\cosh(\gamma)\right]}{2\gamma^3\sinh(\gamma)} & d_3 &= \frac{\sinh(\gamma\xi)\cosh(\gamma)}{2\gamma^3\sinh(\gamma)} \\
c_4 &= 0 & d_4 &= \frac{-\sinh(\gamma\xi)}{2\gamma^3}
\end{aligned}
\tag{16}
$$

Thus the final form of the Green's function for case #2 is:

$$
\tilde{K}(x,\xi)=\begin{cases} c_1\sin(\gamma x)+c_2\cos(\gamma x)+c_3\sinh(\gamma x)+c_4\cosh(\gamma x) & x<\xi \\ d_1\sin(\gamma x)+d_2\cos(\gamma x)+d_3\sinh(\gamma x)+d_4\cosh(\gamma x) & x>\xi \end{cases}
\tag{17}
$$

for the values provided directly above in Eq. 16.

In order to verify that the Green's function $\tilde{K}(x,\xi)$ is symmetric, the following MATLAB text was written to flip x and ξ in the first equation and then subtract that expression from the second expression:

```
function TAM514_hw5_prob2b

clear all
close all
clc

syms x gm xi c1 c2 c3 c4 d1 d2 d3 d4

c1=sin(gm - gm*xi)/(2*gm^3*sin(gm));
c2=0;
c3=-(cosh(gm*xi)*sinh(gm)-sinh(gm*xi)*cosh(gm))/(2*gm^3*sinh(gm));
```

```
c4=0;
d1=-(sin(gm*xi)*cot(gm))/(2*gm^3);
d2=sin(gm*xi)/(2*gm^3);
d3=(sinh(gm*xi)*cosh(gm))/(2*gm^3*sinh(gm));
d4=-sinh(gm*xi)/(2*gm^3);

c1=sin(gm - gm*x)/(2*gm^3*sin(gm));
c2=0;
c3=-(cosh(gm*x)*sinh(gm)-sinh(gm*x)*cosh(gm))/(2*gm^3*sinh(gm));
c4=0;

simplify(c1*sin(gm*xi)+c2*cos(gm*xi)+c3*sinh(gm*xi)+c4*cosh(gm*xi)...
    -d1*sin(gm*x)-d2*cos(gm*x)-d3*sinh(gm*x)-d4*cosh(gm*x))
```

As desired, the outcome of the last command was zero, implying that the Green's function is symmetric.

The boundary value can be converted to an integral equation using the case #2 Green's function as follows:

(−) sign

$$u(x) = \int_0^1 \tilde{K}(x,\xi) f(\xi)\, d\xi \tag{18}$$

Problem 2

By definition, a self-adjoint operator $\mathcal{L}$ on the usual Hilbert space $L^2_{[x_0,\,x_1]}$ satisfies

$$\forall\, u, v \in L^2_{[x_0,\,x_1]}, \quad \int_{x_0}^{x_1} v\mathcal{L}[u]\,\mathrm{d}x = \int_{x_0}^{x_1} u\mathcal{L}[v]\,\mathrm{d}x.$$

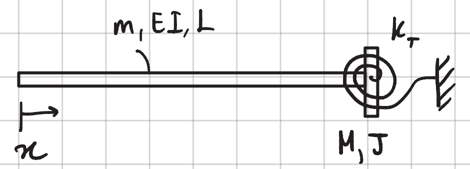
In particular, this holds for $u(x) = K(x,\xi)$ and $v(x) = K(x,\eta)$, where it is understood that the Green's function $K(x,\mu)$ satisfies $\forall\,\mu \in [x_0,\,x_1]$, $\mathcal{L}[K(x,\mu)] = \delta(x-\mu)$ (plus boundary conditions). The self-adjointness definition becomes

$$\int_{x_0}^{x_1} K(x,\eta)\mathcal{L}[K(x,\xi)]\,\mathrm{d}x = \int_{x_0}^{x_1} K(x,\xi)\mathcal{L}[K(x,\eta)]\,\mathrm{d}x$$
$$\implies \quad \int_{x_0}^{x_1} K(x,\eta)\delta(x-\xi)\,\mathrm{d}x = \int_{x_0}^{x_1} K(x,\xi)\delta(x-\eta)\,\mathrm{d}x \quad \implies \quad K(\xi,\eta) = K(\eta,\xi),$$

which establishes symmetry of the Green's function for a self-adjoint operator.

Problem 3

(1)



a) GDE: $EI\,v_{x^4} + m\,v_{tt} = 0$

$$\left(v_{x^4} = \frac{\partial^4 v(x,t)}{\partial x^4} \;;\; u_{tt} = \frac{\partial^2 v(x,t)}{\partial t^2} \right)$$

BCs: $$\begin{cases} EI\,v_{xx}(0,t) = 0 \;;\; EI\,v_{x^3}(0,t) = 0 \\ M\,v_{tt}(L,t) = EI v_{x^3}(L,t) \;;\; EI\,v_{xx}(L,t) = -J\,v_{xtt}(L,t) - K_T\,v_x(L,t) \end{cases}$$

Let $v(x,t) = f(t)\,\varphi(x)$ :: GDE $\longrightarrow$ $EI f(t)\,\varphi_{x^4}(x) + m f''(t)\,\varphi(x) = 0$

. Seek $f''(t) + \omega^2 f(t) = 0 \;\rightarrow\; f(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t)$

. Then: $EI\,\varphi_{x^4}(x) - \omega^2 m\,\varphi(x) = 0$

Let $\beta^4 = \frac{EI}{m}$

general solution yields: $$\varphi(x) = C_1 \cos\frac{\sqrt{\omega}x}{\beta} + C_2 \sin\frac{\sqrt{\omega}x}{\beta} + C_3 \cosh\frac{\sqrt{\omega}x}{\beta} + C_4 \sinh\frac{\sqrt{\omega}x}{\beta}$$

. BCs become: $$\begin{cases} \varphi''(0) = 0 \;;\; \varphi'''(0) = 0 \\ EI\,\varphi'''(L) + \omega^2 M \varphi(L) = 0 \;;\; EI\,\varphi''(L) = (\omega^2 J - K_T)\,\varphi'(L) \end{cases}$$

. Applied BCs: $\varphi''(0) = 0 \Rightarrow -\frac{\omega}{\beta^2} C_1 + \frac{\omega}{\beta^2} C_3 = 0$; $\varphi'''(0) = 0 \Rightarrow \frac{-\omega^{3/2}}{\beta^3} C_2 + \frac{\omega^{3/2}}{\beta^3} C_4 = 0$

$$\Rightarrow C_1 = C_3,\quad C_2 = C_4$$

Now: $$\varphi(x) = C_1\left(\cos\frac{\sqrt{\omega}x}{\beta} + \cosh\frac{\sqrt{\omega}x}{\beta}\right) + C_2\left(\sin\frac{\sqrt{\omega}x}{\beta} + \sinh\frac{\sqrt{\omega}x}{\beta}\right)$$

$$\Rightarrow \varphi'(x) = \frac{\sqrt{\omega}}{\beta} C_1 \left[-\sin\frac{\sqrt{\omega}x}{\beta} + \sinh\frac{\sqrt{\omega}x}{\beta}\right] + \frac{\sqrt{\omega}}{\beta} C_2 \left(\cos\frac{\sqrt{\omega}x}{\beta} + \cosh\frac{\sqrt{\omega}x}{\beta}\right)$$

$$\Rightarrow \varphi''(x) = \frac{\omega}{\beta^2} C_1 \left[-\cos\frac{\sqrt{\omega}x}{\beta} + \cosh\frac{\sqrt{\omega}x}{\beta}\right] + \frac{\omega}{\beta^2} C_2 \left(-\sin\frac{\sqrt{\omega}x}{\beta} + \sinh\frac{\sqrt{\omega}x}{\beta}\right)$$

$$\Rightarrow \varphi'''(x) = \frac{\omega^{3/2}}{\beta^3} C_1 \left[\sin\frac{\sqrt{\omega}x}{\beta} + \sinh\frac{\sqrt{\omega}x}{\beta}\right] + \frac{\omega^{3/2}}{\beta^3} C_2 \left(-\cos\frac{\sqrt{\omega}x}{\beta} + \cosh\frac{\sqrt{\omega}x}{\beta}\right)$$

• $EI\,\varphi'''(L) + \omega^2 M \varphi(L) = 0 \Rightarrow \varphi'''(L) = -\frac{\omega^2 M}{EI}\varphi(L)$ $\boxed{\left(\beta^4 = \frac{EI}{m}\right)}$

$$\Rightarrow C_1\left[\sin\frac{\sqrt{\omega}L}{\beta} + \sinh\frac{\sqrt{\omega}L}{\beta}\right] + C_2\left(-\cos\frac{\sqrt{\omega}L}{\beta} + \cosh\frac{\sqrt{\omega}L}{\beta}\right)$$

$$= \frac{-\omega^2}{\beta^4}\frac{mM}{EI}\left\{C_1\left(\cos\frac{\sqrt{\omega}L}{\beta} + \cosh\frac{\sqrt{\omega}L}{\beta}\right) + C_2\left(\sin\frac{\sqrt{\omega}L}{\beta} + \sinh\frac{\sqrt{\omega}L}{\beta}\right)\right\} \quad (1)$$

• $EI\,\varphi''(L) = (\omega^2 J - k_T)\,\varphi'(L) \Rightarrow \varphi''(L) = \dfrac{\omega^2 J - k_T}{EI}\,\varphi'(L)$

$$\Rightarrow C_1\left[-\cos\frac{\sqrt{\omega}L}{\beta} + \cosh\frac{\sqrt{\omega}L}{\beta}\right] + C_2\left(-\sin\frac{\sqrt{\omega}L}{\beta} + \sinh\frac{\sqrt{\omega}L}{\beta}\right)$$

$$= \left[\frac{\omega^{3/2}J}{\beta^3 m} - \frac{\omega^{-1/2}k_T}{\beta^3 m}\right]\left\{C_1\left[-\sin\frac{\sqrt{\omega}x}{\beta} + \sinh\frac{\sqrt{\omega}x}{\beta}\right] + C_2\left(\cos\frac{\sqrt{\omega}x}{\beta} + \cosh\frac{\sqrt{\omega}x}{\beta}\right)\right\} \quad (2)$$

let $\boxed{x = \frac{\sqrt{\omega}}{\beta}L}$ $\left(\sqrt{\omega} = \frac{\beta x}{L};\ \omega^{3/2} = \left(\frac{\beta x}{L}\right)^3\right)$; $\boxed{q = \frac{J}{mL^3};\quad \mu = \frac{k_T}{\beta^2 m L}}$

(1), (2) becomes:

$$C_1(\sin x + \sinh x) + C_2(-\cos x + \cosh x) = -\frac{x^4}{L}\left[C_1(\cos x + \cosh x) + C_2(\sin x + \sinh x)\right]$$

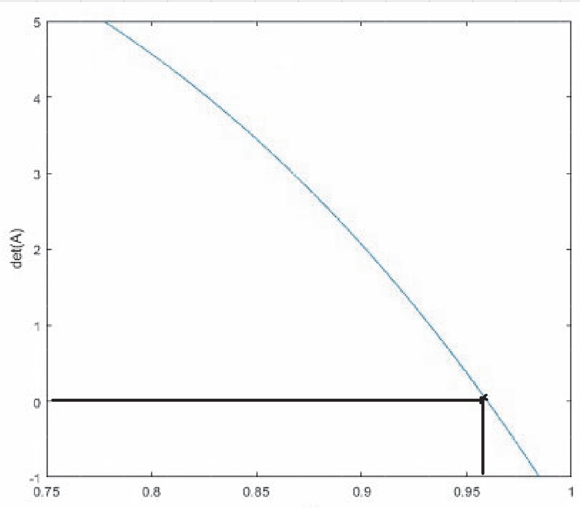
$$C_1(-\cos x + \cosh x) + C_2(-\sin x + \sinh x) = \left(qx^4 - \frac{\mu}{x}\right)\left[C_1(-\sin x + \sinh x) + C_2(\cos x + \cosh x)\right]$$

$$\Rightarrow \underbrace{\begin{bmatrix} \sin x + \sinh x + \frac{x^4\cos x}{L} + \frac{x^4\cosh x}{L} & -\cos x + \cosh x + \frac{x^4\sin x}{L} + \frac{x^4\sinh x}{L} \\ -\cos x + \cosh x - \left(qx^4 - \frac{\mu}{x}\right)(-\sin x + \sinh x) & -\sin x + \sinh x - \left(qx^4 - \frac{\mu}{x}\right)(\cos x + \cosh x) \end{bmatrix}}_{A} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To get non-trivial solution then $\det(A) = 0$, using matlab:

$$\boxed{\begin{aligned} &\{-(Lx\cos^2(x) + Lx\cosh^2(x) - \mu x\cos^2(x) - \mu x\cosh^2(x) + Lx\sin^2(x) - Lx\sinh^2(x) - \mu x\sin^2(x) \\ &\quad + \mu x\sinh^2(x) + qx^6\cos^2(x) + qx^6\cosh^2(x) + qx^6\sin^2(x) - qx^6\sinh^2(x) \\ &- 2x^2\cos(x)\sinh(x) + 2x^2\cosh(x)\sin(x) + 2qx^6\cos(x)\cosh(x) - 2Lx\cos(x)\cosh(x) \\ &- 2\mu x\cos(x)\cosh(x) - 2L\mu\cos(x)\sinh(x) - 2L\mu\cosh(x)\sin(x) + 2Lqx^5\cos(x)\sinh(x) \\ &+ 2Lqx^5\cosh(x)\sin(x)\}\cdot\frac{1}{Lx} = 0 \end{aligned}}$$

To show graphically the leading frequencies: choose: $EI = m = J = k = L = M = 1$:



The leading root frequency for this case is

$$\boxed{\omega_1 = 0.9607}$$

$$\boxed{\begin{aligned} &\text{Then: } \varphi_k(x) = C_1\left(\cos\frac{\sqrt{\omega_k}x}{\beta} + \cosh\frac{\sqrt{\omega_k}x}{\beta}\right) + C_2\left(\sin\frac{\sqrt{\omega_k}x}{\beta} + \sinh\frac{\sqrt{\omega_k}x}{\beta}\right) \\ &\text{where } C_2 = -\frac{\sin x + \sinh x + \frac{x\cos x}{L} + \frac{x\cosh x}{L}}{-\cos x + \cosh x + \frac{x\sin x}{L} + \frac{x\sinh x}{L}} \end{aligned}}$$

Orthonormality conditions is then:

$$\boxed{\begin{aligned}&\text{Mass: } \int_0^L m\,\varphi_r(x)\varphi_s(x)\,dx + M\varphi_r(L)\varphi_s(L) + J\varphi_r'(L)\varphi_s'(L) = \delta_{rs}\\ &\text{Stiffness: } \int EI\,\varphi_r''(x)\varphi_s''(x)\,dx + K_T\varphi_r'(L)\varphi_s'(L) = \omega_r^2\delta_{rs}\end{aligned}}$$

(b) Rayleigh Quotient expression that taken into account of BCs:

$$\boxed{w^2[RQ] = \frac{V_{max}}{T_{max}} = \frac{\int_0^L EI\,\varphi''^2(x)\,dx + K_T\,\varphi'^2(L)}{\int_0^L m\,\varphi^2(x)\,dx + M\varphi^2(L) + J\varphi'^2(L)}}$$

Choose a solution for a free-free beam: $\varphi(x) = \cos(\pi x)$

with the parameters of choice similar to above, then:

$$w^2 = \frac{\pi^4\int_0^1 \cos^2(\pi x)\,dx + \pi^2\sin^2(\pi)}{\int_0^1 \cos^2(\pi x)\,dx + \cos^2(\pi) + \pi^2\sin^2(\pi)} = \frac{\pi^4/2}{3/2} = 32 > 0.9 \Rightarrow \text{bad guess}$$

c) Method of Rayleigh - Ritz: (using the parameter of choice similar to above:

Function of choice: $\varphi(x) = \sum_{i=1}^{n}\alpha_i\sin(i\pi x) \rightarrow \begin{cases}\varphi'(x) = \sum_{i=1}^{n}\alpha_i(i\pi)\cos(i\pi x)\\ \varphi''(x) = -\sum_{i=1}^{n}\alpha_i(i\pi)^2\sin(i\pi x)\end{cases}$

$$w^2 = R[\varphi(x)] = \frac{N(\alpha_1,\dots,\alpha_n)}{D(\alpha_1,\dots,\alpha_n)} \qquad \begin{cases} D = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j m_{ij}\\ N = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j k_{ij}\end{cases} \quad \text{where:}$$

$$\begin{cases} m_{ij} = \sin(i\pi)\sin(j\pi) + ij\pi^2\cos(i\pi)\cos(j\pi) + \int_0^1 \sin(i\pi x)\sin(j\pi x)\,dx\\ k_{ij} = ij\pi^2\cos(i\pi)\cos(j\pi) + \int_0^1 (ij)^2\pi^4\sin(i\pi x)\sin(j\pi x)\,dx\end{cases}$$

$$\Rightarrow \boxed{\begin{cases} m_{ij} = \sin(i\pi)\sin(j\pi) + ij\pi^2\cos(i\pi)\cos(j\pi) + \dfrac{i\sin(\pi j)\cos(\pi i) - j\cos(\pi j)\sin(\pi i)}{\pi(j^2-i^2)}\\ k_{ij} = ij\pi^2\cos(i\pi)\cos(j\pi) - \dfrac{\pi^3 j^2 i^2\left(j\cos(\pi j)\sin(\pi i) - i\sin(\pi j)\cos(\pi i)\right)}{j^2-i^2}\end{cases}}$$

Rayleigh Ritz yields: $\dfrac{\partial N}{\partial \alpha_i} - \Omega^2\dfrac{\partial D}{\partial \alpha_i} = 0$

estimate of natural frequency

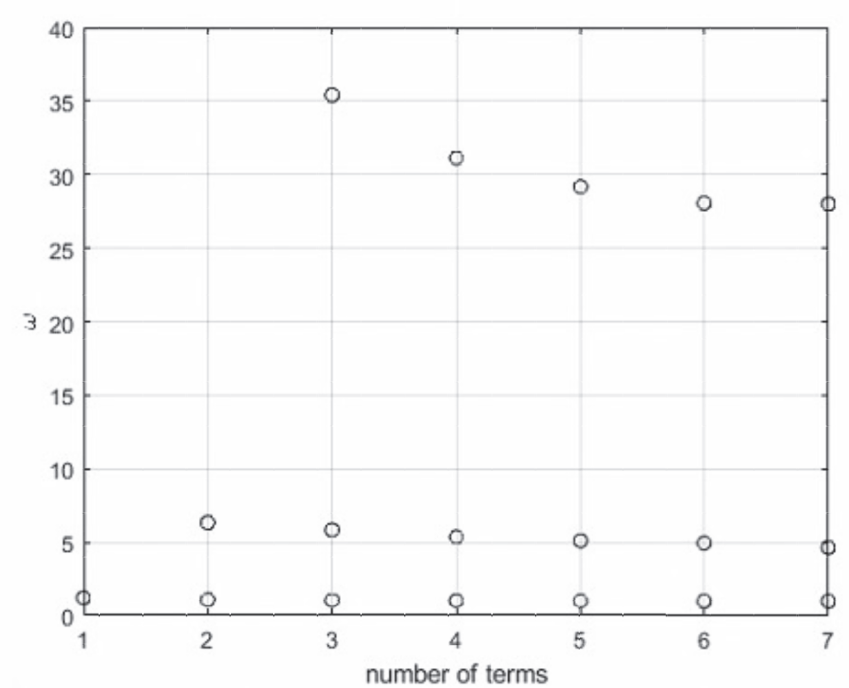
$$\begin{cases}\dfrac{\partial N}{\partial \alpha_i} = 2\sum_{r=1}^{n} k_{ir}\alpha_r\\ \dfrac{\partial D}{\partial \alpha_i} = 2\sum_{r=1}^{n} m_{ir}\alpha_r\end{cases} \Rightarrow \sum_{r=1}^{n} k_{ir}\alpha_r - \Omega^2\sum_{r=1}^{n} m_{ir}\alpha_r = 0$$

In matrix form: $[K]\underline{\alpha} - \Omega^2 [M]\underline{\alpha} = 0$ where:

$$K = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \\ k_{n1} & \cdots & k_{nn} \end{bmatrix} ; \quad M = \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix} ; \quad \underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Then: $\left([K] - \Omega^2 [M]\right)\underline{\alpha} = 0 \Rightarrow$ find Ω^2 by set $\det([K] - \Omega^2[M]) = 0$

Using eigs formula in matlab, the 1st 3 modes are: $\boxed{1.01,\ 4.85,\ 28.7}$



Problem 4

$$u_{xx} - [1 + \varepsilon \alpha(x)] u_{tt} = 0 \quad , \quad 0 \le x \le 1$$

$$u(0,t) = u(1,t) = 0 \quad ; \quad \alpha(x) = A \sin(\pi \tfrac{x}{2})$$

(i) Let $u(x,t) = \varphi(x) e^{j\omega t} \Rightarrow \begin{cases} u_{xx} = \varphi''(x) e^{j\omega t} \\ u_{tt} = -\omega^2 \varphi(x) e^{j\omega t} \end{cases}$

$$\Rightarrow \varphi''(x) e^{j\omega t} + [1 + \varepsilon \alpha(x)] \omega^2 \varphi(x) e^{j\omega t} = 0$$

$$\Rightarrow \boxed{\varphi''(x) + (1 + \varepsilon \alpha(x)) \omega^2 \varphi(x) = 0}$$

(ii) Let $\varphi(x) = \varphi_0(x) + \varepsilon \varphi_1(x) + \varepsilon^2 \varphi_2(x) + \dots$ and $\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2$

then:

$$\varphi_0''(x) + \varepsilon \varphi_1''(x) + \varepsilon^2 \varphi_2''(x) + \dots + (1 + \varepsilon \alpha(x))(\omega_0 + \omega_1 \varepsilon + \varepsilon^2 \omega_2 + \dots)^2 (\varphi_0(x) + \varepsilon \varphi_1(x) + \varepsilon^2 \varphi_2(x) + \dots) = 0$$

$$\Rightarrow \varphi_0'' + \varepsilon \varphi_1'' + \varepsilon^2 \varphi_2'' + \dots + (1 + \varepsilon \alpha(x))(\omega_0^2 + 2\omega_0\omega_1 \varepsilon + 2\omega_0 \omega_2 \varepsilon^2 + \omega_1^2 \varepsilon^2 + \dots)(\varphi_0(x) + \varepsilon \varphi_1(x) + \varepsilon^2 \varphi_2(x) + \dots) = 0$$

$$\Rightarrow \varphi_0'' + \varepsilon \varphi_1'' + \varepsilon^2 \varphi_2'' + \dots + (1 + \varepsilon \alpha(x))(\varphi_0 \omega_0^2 + \varepsilon \varphi_1 \omega_0^2 + \varepsilon^2 \omega_0^2 \varphi_2 + 2\omega_0 \omega_2 \varepsilon^2 \varphi_1 + 2\omega_0 \omega_1 \varepsilon \varphi_0 + 2\omega_0\omega_1 \varepsilon^2 \varphi_1 + 2\omega_0 \omega_2 \varphi_0 \varepsilon^2) = 0$$

$$\Rightarrow \varphi_0'' + \varepsilon \varphi_1'' + \dots + \varphi_0 \omega_0^2 + \varepsilon \varphi_1 \omega_0^2 + 2\omega_0 \omega_1 \varepsilon \varphi_0 + \varepsilon \alpha(x) \varphi_0 \omega_0^2 + O(\varepsilon^2) = 0$$

(iii)
$$\boxed{\begin{array}{l} O(1) \text{ term}: \ \varphi_0''(x) + \omega_0^2 \varphi_0(x) \\ O(\varepsilon) \text{ term}: \ \varphi_1''(x) + \omega_0^2 \varphi_1(x) + (2\omega_0\omega_1 + \alpha(x)\omega_0^2)\varphi_0(x) \end{array}}$$

(iv) $O(1)$ term recovers the unpertubed problem : $\varphi_0''(x) + \omega_0^2 \varphi_0(x)$

$$\Rightarrow \varphi_0(x) = C_1 \cos(\omega_0 x) + C_2 \sin(\omega_0 x)$$

$u(0,t) = u(1,t) = 0 \Rightarrow \varphi_0(0) = 0$ and $\varphi_1(1) = 0 \Rightarrow C_1 = 0$ and $C_2 \sin(\omega_0) = 0$

For non trivial solution: $\sin(\omega_0) = 0 \Rightarrow \omega_0 = k\pi \quad (k = 0, 1, \dots)$

$\Rightarrow \varphi_{0_k}(x) = C_k \sin(k\pi x)$ :: orthogonormalized: $C_k^2 \int_0^1 \sin^2(k\pi x) = 1$

$\Rightarrow C_k^2 = 2 \Rightarrow C_k = \sqrt{2} \Rightarrow \boxed{\varphi_{0_k} = \sqrt{2} \sin(k\pi x)}$

•) $O(\varepsilon)$ terms give: $\varphi_1''(x) + \omega_0^2 \varphi_1(x) + (2\omega_0\omega_1 + \alpha(x)\omega_0^2)\varphi_0(x) = 0$

Since $\varphi_1(x)$ part of the domain g, then: $\varphi_{1_n}(x) = \sqrt{2} \sum_{k=1}^{\infty} A_{nk} \sin(k\pi x)$

Then: $-\sqrt{2}(k\pi)^2\sum_{i=1}^{\infty} A_{nk}\sin(k\pi x) + (n\pi)^2\sqrt{2}\sum_{k=1}^{\infty} A_{nk}\sin(k\pi x) + (2\omega_0\omega_1 + \alpha(x)\omega_0^2)\Psi_0(x) = 0$

$$\Rightarrow \sum_{k=0}^{\infty} A_{nk}\underbrace{\int_0^1 -\sqrt{2}(k\pi)^2\sin(k\pi x)\sin(\ell\pi x)\,dx}_{-(k\pi)^2\delta_{k\ell}} + (n\pi)^2\sum_{k=0}^{\infty} A_{nk}\underbrace{\int_0^1 \sqrt{2}\sin(k\pi x)\sin(\ell\pi x)\,dx}_{\delta_{k\ell}}$$

$$2(n\pi)\omega_1\underbrace{\int_0^1 \sin(n\pi x)\sin(\ell\pi x)\,dx}_{\delta_{n\ell}} + \underbrace{(n\pi)^2\int_0^1 \alpha(x)\sqrt{2}\sin(n\pi x)\sin(\ell\pi x)\,dx}_{d_{n\ell}} = 0$$

$$\Rightarrow -(\ell\pi)^2 A_{n\ell} + (n\pi)^2 A_{n\ell} = d_{n\ell} - 2n\pi\,\omega_1\,\delta_{n\ell}$$

where $d_{n\ell} = (n\pi)^2\int_0^1 A\sin\left(\frac{\pi}{2}x\right)\sqrt{2}\sin(n\pi x)\sin(\ell\pi x)\,dx$

$$\Rightarrow \boxed{d_{n\ell} = \frac{-16\ell A\sqrt{2}}{n\pi^3(16\ell^4 + 32\ell^2n^2 + 8\ell^2 - 16n^4 + 8n^2 - 1)}}$$

If $n \neq \ell$: $A_{n\ell} = \dfrac{d_{n\ell}}{\pi^2(n^2-\ell^2)}$; If $n = \ell$: $d_{nn} = 2n\pi\,\omega_1 \Rightarrow \boxed{\omega_1 = \dfrac{d_{nn}}{2n\pi}}$

A_{nn} can be compute via:

$$\sqrt{2}\int_0^1 \left(\sin(n\pi x) + \varepsilon A_{nn}\sin(n\pi x)\right)^2 dx = 1 \Rightarrow A_{nn} = 0$$

Then $$\boxed{\varphi_{1n}(x) = \sum_{\substack{k=1\\k\neq n}}^{\infty} \frac{\sqrt{2}d_{n\ell}}{\pi^2(n^2-\ell^2)}\sin(k\pi x)}$$

Putting everything together, the approximation solution is:

$$\boxed{u(x,t) = \sum_{n=0}^{\infty}\left(\sqrt{2}\sin(n\pi x) + \sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{\sqrt{2}d_{n\ell}}{\pi^2(n^2-\ell^2)}\sin(k\pi x)\right)e^{j\left(n\pi + \frac{d_{nn}}{2\pi n}\right)t}}$$

where $d_{n\ell}$ is defined above.

Problem 5

Dynamics of a long bridge due to a crossing locomotive

$$EI\, u_{xxxx} + c\, u_t + m\, u_{tt} = (M_1 + \omega^2 M_2 \cos \omega t)\, \delta(x - Vt)$$

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0$$

$$0 \le x \le L \qquad t \ge 0$$

a) Solve :

Assume normal modes: $\quad u(x,t) = \phi(x)\, \eta(t)$

where $\quad \ddot{\eta}(t) + \frac{c}{m}\dot{\eta}(t) + \omega^2 \eta(t) = 0$

$\frac{c}{m} \to 0$

Rewriting equations (no forcing) :

↳ $EI\, \eta(t)\, \phi''''(x) + m\, \ddot{\eta}(t)\, \phi(x) =$

↳ $\eta(t)\, EI\, \phi''''(x) - \eta(t)\, m\, \omega^2 \phi(x) = 0$

↳ $\boxed{EI\, \phi''''(x) - m\, \omega^2\, \phi(x) = 0}$

BC: → $\boxed{\phi(0) = \phi(L) = \phi''(0) = \phi''(L) = 0}$

Solution form :

$$\phi(x) = C_1 \cos \Omega x + C_2 \sin \Omega x + C_3 \cosh \Omega x + C_4 \sinh \Omega x$$

where $\quad \Omega = \frac{\sqrt{\omega}}{EI/m}$

$$\phi''(x) = \Omega^2 \left[-C_1 \cos \Omega x - C_2 \sin \Omega x + C_3 \cosh \Omega x + C_4 \sinh \Omega x \right]$$

Applying BC:

$$\left.\begin{aligned}\phi(0) &= C_1 + C_3 = 0\\ \phi''(0) &= -C_1 + C_3 = 0\end{aligned}\right\} \rightarrow C_1 = C_3 = 0$$

$$\phi(L) = C_2 \sin \Omega L + C_4 \sinh \Omega L = 0$$

$$\phi''(L) = \Omega^2 \left[-C_2 \sin \Omega L + C_4 \sinh \Omega L \right] = 0$$

Solving for frequency relation:

$$\underbrace{\begin{bmatrix} \sin \Omega L & \sinh \Omega L \\ -\sin \Omega L & \sinh \Omega L \end{bmatrix}}_{[A]} \begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\rightarrow \det [A] = 0$

$$2 \sin \Omega L \sinh \Omega L = 0$$

For $\Omega L \neq 0 \rightarrow$

$\sinh \Omega L \neq 0$

$\sin \Omega L = 0 \rightarrow \Omega_K L = K\pi \quad K = 1, 2, \ldots$

$$\Omega_K = \frac{K\pi}{L}$$

$$\boxed{\omega_K = \left[K \frac{EI}{mL} \pi \right]^2} \; ; \quad K = 1, 2, \ldots$$

Mode Shapes: $\boxed{\phi(x) = C_K \sin \Omega_K x}$

Orthonormalizing:

- Since we have simple BCs $\rightarrow \int_0^L m \, \phi_r \phi_s \, dx = \delta_{rs}$

$r, s = 1, 2, \ldots$

$$\rightarrow \quad m C_K^2 \int_0^L \sin^2 \Omega_K x \, dx = 1$$

$$m C_K^2 \left[\frac{L}{2} - \frac{1}{4\Omega_K} \underbrace{\sin 2\Omega_K L}_{0} \right] = 1$$

$$C_K = \left[\frac{m}{2} \left(L - \cancel{\frac{1}{2\Omega_K} \sin 2\Omega_K L} \right) \right]^{-1/2} \rightarrow \boxed{C_K = \left(\frac{2}{mL} \right)^{1/2}}$$

Assuming zero initial conditions: $\eta(0) = \dot{\eta}(0) = 0$

→ Solving forcing solution

$$\ddot{\eta}(t) + \omega_K^2 \eta(t) = N_K(t)$$

$$\rightarrow \quad N_K(t) = \int_0^L \left(M_1 + \omega_K^2 M_2 \cos \omega_K t \right) \delta(x - v t) \, \phi_K(x) \, dx$$

$$= \left(M_1 + \omega_K^2 M_2 \cos \omega_K t \right) \phi_K(v t)$$

$$= C_K \Big[M_1 \sin \Omega_K v t + \frac{\omega_K^2 M_2}{2} \left[\sin(\omega_K t + \Omega_K v t) + \sin(\Omega_K v t - \omega_K t) \right] \Big]$$

Complete solution:

$$\boxed{v(x,t) = \sum_{i=1}^{\infty} \left[\frac{1}{\omega_i} \int_0^t N_i(\tau) \sin \omega_i (t - \tau) \, d\tau \right] \phi_i(x)}$$

(b) Under what conditions will resonance occur?

→ Resonance occurs when:

$$\omega_K = \Omega_K V$$

$$\omega_K = \Omega_K V - \omega_K \rightarrow \omega_K = \frac{\Omega_K V}{2}$$

since it corresponds to frequencies of forcing function:

$$\rightarrow \omega_K = \frac{\sqrt{\omega_K}}{EI/m} V \rightarrow \boxed{\omega_K = \left(\frac{mV}{EI}\right)^2}$$

$$\rightarrow \omega_K = \frac{\sqrt{\omega_K}}{2EI/m} V \longrightarrow \boxed{\omega_K = \left(\frac{mV}{2EI}\right)^2}$$