Formal Results

Based on our heuristic results we may introduce a formal definition for the Green's function of the second-order self-adjoint operator $L[u]$, as follows.

Definition

A function $K(x,\xi)$ of the second-order differential expression $L[u] = pu'' + p'u' + qu$ is the Green's function for given homogeneous boundary conditions if the following hold:

(i) For fixed ξ, $K(x,\xi)$ is a continuous function of x and satisfies the boundary conditions.

(ii) Except at point $x=\xi$, the first and second derivatives of K with respect to x are continuous in the domain $G=[x_0,x_1]$. At point $x=\xi$ the first derivative has a jump (discontinuity) according to:

$$\left.\frac{dK(x,\xi)}{dx}\right|_{x=\xi^-}^{x=\xi^+} = -\frac{1}{p(\xi)}$$

(iii) K considered as a function of x satisfies the differential equation $L[K]=0$ in $G-\{x=\xi\}$.

The use of $K(x,\xi)$ is due to the fact that the following holds:

If $\varphi(x)$ is a continuous or piecewise continuous function of x, then the function $u(x)=\int_{x_0}^{x_1} K(x,\xi)\varphi(\xi)d\xi$ is a solution of $L[u]=-\varphi(x)$ and satisfies the boundary conditions.

Conversely, if the function $u(x)$ is the solution of $L[u]=-\varphi(x)$, then it can be represented in the form $u(x)=\int_{x_0}^{x_1}K(x,\xi)\varphi(\xi)\,d\xi$.

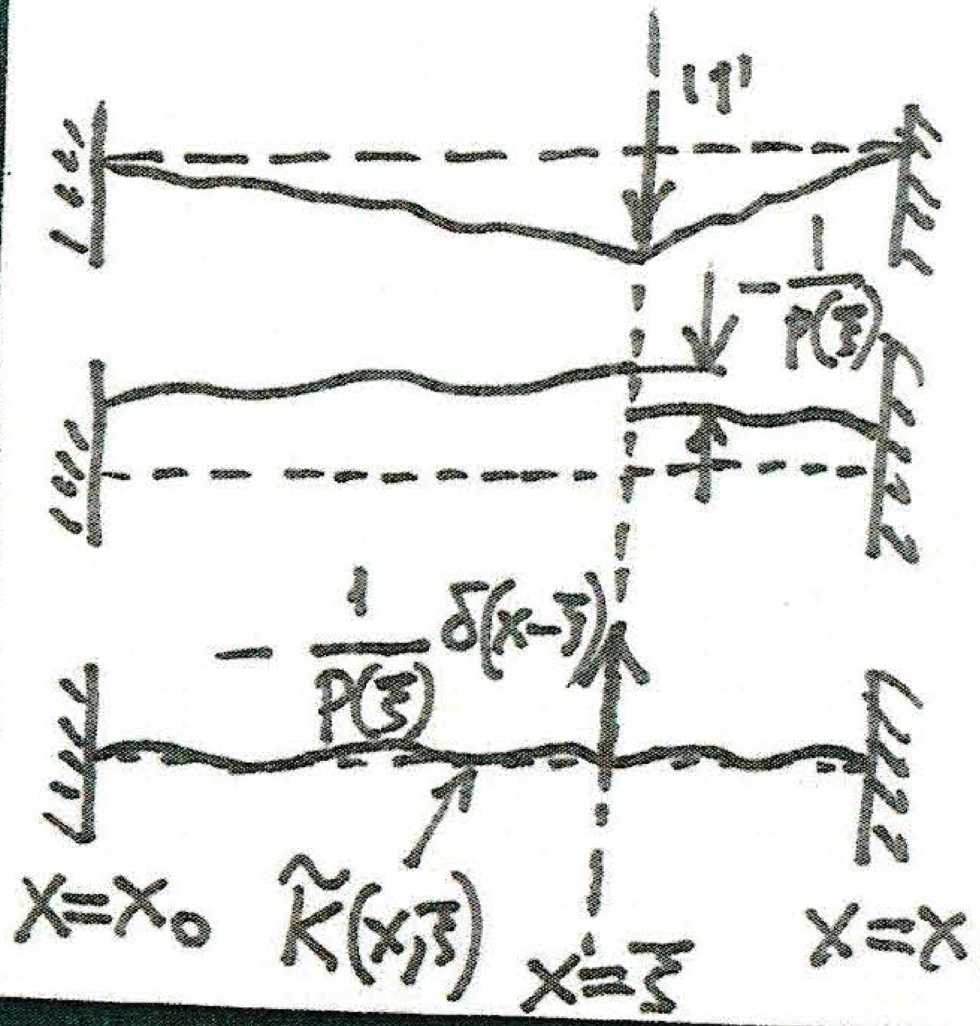
<u>Proof</u>

Consider $pu''+p'u'+qu=-\varphi(x)$ $\Rightarrow$ We want to show that $u(x)=\int_{x_0}^{x_1}K(x,\xi)\varphi(\xi)\,d\xi$ is indeed a solution of this equation. We will prove this by direct substitution, taking into account that:

$$\frac{d}{dx}\int_{A(x)}^{B(x)} f(x,\xi)\,d\xi=\int_{A(x)}^{B(x)}\frac{\partial f(x,\xi)}{\partial x}\,d\xi+f(x,B(x))\frac{dB}{dx}-f(x,A(x))\frac{dA}{dx}$$

$$u(x)=\int_{x_0}^{x_1}K(x,\xi)\varphi(\xi)\,d\xi \Rightarrow u'(x)=\int_{x_0}^{x_1}\frac{\partial K(x,\xi)}{\partial x}\varphi(\xi)\,d\xi \Rightarrow u''(x)=\int_{x_0}^{x_1}\frac{\partial^2 K(x,\xi)}{\partial x^2}\varphi(\xi)\,d\xi-\frac{1}{p(x)}\varphi(x)$$

continuous



$K(x,\xi)$

$\frac{\partial K(x,\xi)}{\partial x}$

$$\frac{\partial^2 K(x,\xi)}{\partial x^2}=\tilde{K}(x,\xi)-\frac{1}{p(\xi)}\delta(x-\xi)$$

continuous

Then, substituting into the dif. eq.:

$$\int_{x_0}^{x_1}\underbrace{\left[pK''(x,\xi)+p'K'(x,\xi)+qK(x,\xi)\right]}_{0}\varphi(\xi)\,d\xi$$

$$-p\frac{1}{p}\varphi=-\varphi \Rightarrow -\varphi=-\varphi \checkmark$$

The converse is proven using Green's formula.

<u>Important Result: Reciprocity theorem</u>

The Green's function $K(x,\xi)$ of a self-adjoint differential expression is a <u>symmetric function</u> of the parameter ξ and its argument x, i.e.,

$$K(x,\xi) = K(\xi,x) \quad \text{(Reciprocity theorem)}$$

Note that the reciprocity thm follows directly from the self-adjoindness of $L[u]$

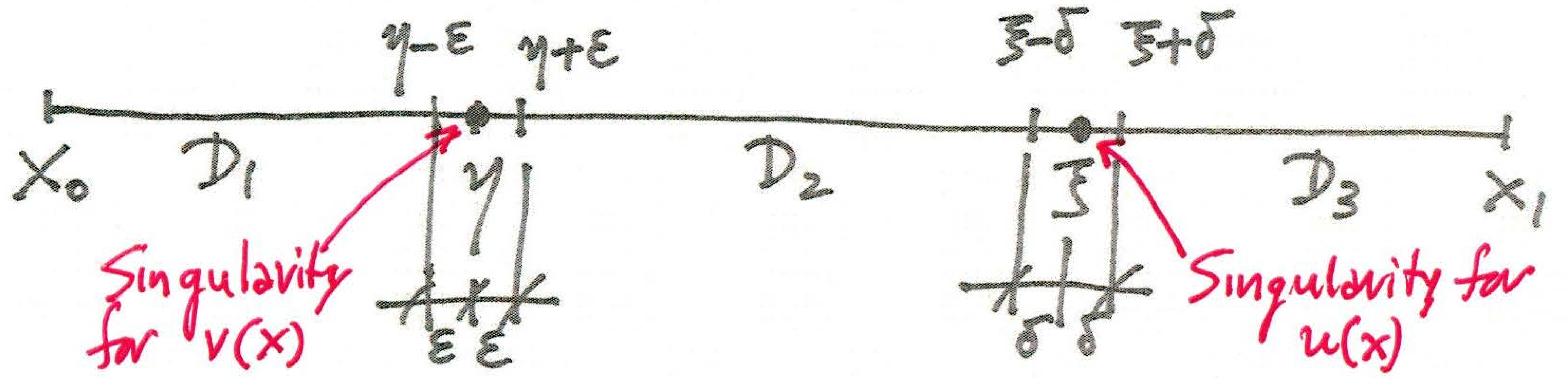
The symmetry of Green's function expresses the reciprocity theorem in mechanics: The response of an elastic medium governed by the self-adjoint operator $L[u]$ at position x due to a unit point load at position ξ, is identical to the response at position ξ due to a unit point load at position x (applied in the same direction)

<u>Proof</u>

For the self-adjoint operator $L[u] = (au')' + (b'-d)u$ the following symmetric Green's formula holds:

$$\int_{x_0}^{x_1} \left(vL[u] - uL[v]\right) dx = a\left(u'v - v'u\right)\Big|_{x_0}^{x_1} \qquad (*)$$

Apply this formula with $v(x) = K(x,y)$ and $u(x) = K(x,\xi)$; recall that $K(x,\xi)$ is continuous and satisfies $L[K(x,\xi)] = 0$ in $G - \{x=\xi\}$; also $\left.\frac{dK(x,\xi)}{dx}\right|_{\xi-}^{\xi+} = -\frac{1}{a(\xi)}$



Clearly, in the limit $\varepsilon \to 0, \delta \to 0$, $D_1 + D_2 + D_3 \to [x_0, x_1] = G$

Substitute for u and v in (*) and divide the domain of integration into 3 parts, taking the limit $\varepsilon, \delta \to 0 \Rightarrow$

$$\int_{x_0}^{\eta-\varepsilon} (vL[u] - uL[v])\,dx + \int_{\eta+\varepsilon}^{\xi-\delta} (vL[u] - uL[v])\,dx + \int_{\xi+\delta}^{x_1} (vL[u] - uL[v])\,dx =$$

$$= a\left[u'v - v'u\right]\Big|_{x_0}^{\eta-\varepsilon} + a\left[u'v - v'u\right]\Big|_{\eta+\varepsilon}^{\xi-\delta} + a\left[u'v - v'u\right]\Big|_{\xi+\delta}^{x_1}$$

in the limit $\varepsilon, \delta \to 0 \Rightarrow$

$$\Rightarrow \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \cancel{\left\{ \int_{x_0}^{\eta-\varepsilon} + \int_{\eta+\varepsilon}^{\xi-\delta} + \int_{\xi+\delta}^{x_1} \right\}} =$$

0 since $K(x, \xi)$ and $K(x, \eta)$ satisfy the dif equations $L[u] = 0$, $L[v] = 0$ and the singularities are excluded

$$= \lim_{\substack{\varepsilon \to 0 \\ \delta \to 0}} \Big\{ \underbrace{\cancel{a\left(u'v\big|_{\eta-\varepsilon} - u'v\big|_{\eta+\varepsilon}\right)}}_{0} + a\left(v'u\big|_{\eta+\varepsilon} - v'u\big|_{\eta-\varepsilon}\right) +$$

$$+ \underbrace{\cancel{\left(av'u\big|_{\xi+\delta} - av'u\big|_{\xi-\delta}\right)}}_{0} + \left(au'v\big|_{\xi-\delta} - au'v\big|_{\xi+\delta}\right) \Big\}$$

Now, $u = K(x,\xi)$, $v = K(x,y) \Rightarrow K(x,\xi)$ is continuous and continuously differentiable at $x = y$, and so is $K(x,y)$ at $x = \xi \Rightarrow$

$$\Rightarrow \underbrace{0 = a u v' \Big|_{y-\varepsilon}^{y+\varepsilon} + a u' v \Big|_{\xi+\delta}^{\xi-\delta}}_{\text{Take limit at } \varepsilon\to 0,\ \delta\to 0} = K(y,\xi)\, a(y)\left(-\frac{1}{a(y)}\right) + a(\xi) K(\xi,y)\left(\frac{1}{a(\xi)}\right) \Rightarrow$$

$-K(y,\xi)$

$K(\xi,y)$

$$\Rightarrow \boxed{K(\xi,y) = K(y,\xi)}$$

Hence, the symmetry of the Green's function and reciprocity follows directly from the self adjointness of the differential operator $L[u]$

Constructing Green's functions

To construct the Green's function of an operator $L[u]$ for given boundary conditions at $x = x_0, x_1 \in \partial G$, $G = [x_0, x_1]$, we work as follows.

Step 1: Consider a solution $u_0(x)$ of the homogeneous problem $L[u] = 0$ that satisfies the 'left' BC at $x = x_0$ only $\Rightarrow c_0 u_0(x)$ is the most general such solution.

2nd order linear dif. operator

Step 2: Similarly, construct a solution $u_1(x)$ of $L[u] = 0$ that satisfies only the 'right' BC at $x = x_1 \Rightarrow c_1 u_1(x)$ is the most general such solution. Under the assumption that $u_1(x)$ and $u_0(x)$ are linearly independent solutions, it holds that $u_0 u_1' - u_1 u_0' \neq 0 \;\forall\, x \in G$. Wronskian relation

Step 3: Match the two solutions $c_0 u_0(x)$ and $c_1 u_1(x)$ at $x=\xi$, so that,

$$c_0 u_0(\xi) = c_1 u_1(\xi) \Rightarrow \text{continuity of } K(x,\xi) \text{ at } x=\xi$$

$$c_0 \frac{du_0(\xi)}{dx} - c_1 \frac{du_1(\xi)}{dx} = -\frac{1}{p(\xi)} \Rightarrow \text{Required discontinuity of } K'(x,\xi) \text{ at } x=\xi$$

from this set of non-homogeneous algebraic equation compute the coefficients c_0 and c_1. ← provided that $u_0(x)$ and $u_1(x)$ are linearly independent!

Then, the Green's function is computed as follows:

$$K(x,\xi) = \begin{cases} -\frac{1}{C} u_1(\xi) u_0(x), & x \leq \xi \\ -\frac{1}{C} u_1(x) u_0(\xi), & x \geq \xi \end{cases}$$

$$\text{where } C = p(\xi) \underbrace{\left[u_0(\xi) u_1'(\xi) - u_0'(\xi) u_1(\xi)\right]}_{\text{Wronskian} \neq 0} = \text{const.}$$

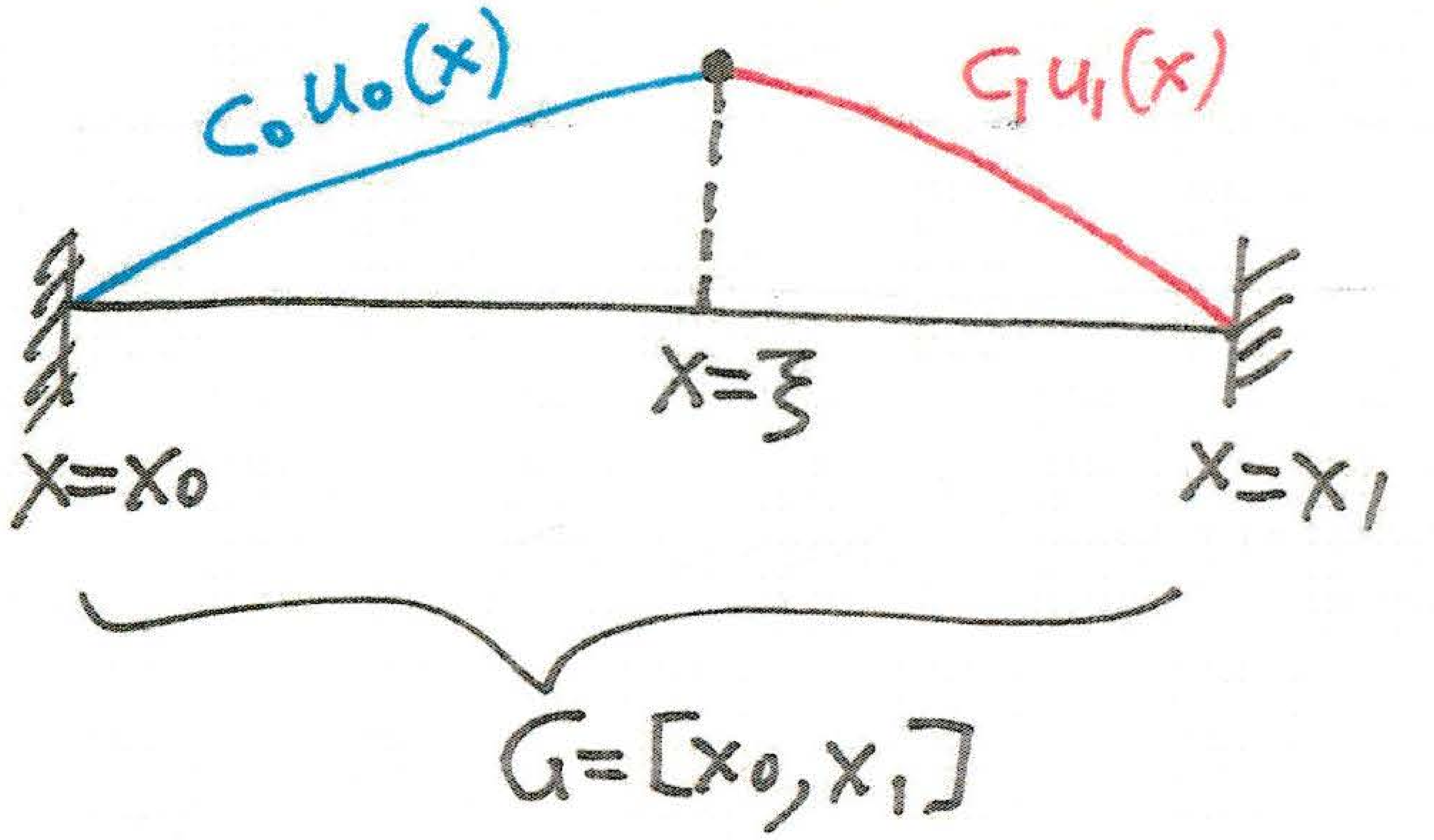
Remark 1

Under what conditions will $u_0(x)$ and $u_1(x)$ be linearly dependent? (i.e., $u_0(x) = u_1(x)$) ⇒ This is the case of rigid body modes, i.e., when $L[u]=0$ has nontrivial solutions $u=u_n$ (Note that the eigenvalue problem is formulated as $L[u] + \cancel{\lambda}^{\,0} u = 0$ ⇒ ⇒ $L[u]=0$ has nontrivial solution, which is equivalent to saying that $\lambda=0$ is an eigenvalue).

Hence, our derivation of $K(x,\xi)$ assumes that no rigid body mode(s) exist, or that $\lambda>0$. ← So, interesting corollary: Either $L[u]=0$ has a trivial solution, in which case a Green's function exists; or $L[u]=0$ has a nontrivial solution (the eigenvector of $\lambda=0$) in which case, no Green's function exists.

Remark 2

Graphic depiction of the construction of $K(x,\xi)$.



Hence, considering the nonhomogeneous differential equation $L[u]=-\varphi(x)$, there exist the following two possibilities:

1) $L[u]=0$ has only the trivial solution $u=0$ ⇒ The operator has no zero eigenvalue ⇒ Operator is invertible ⇒ $u(x)=\int_{x_0}^{x_1} K(x,\xi)\,\varphi(\xi)\,d\xi$ unique solution.

Remark This is analogous to the following situation at finite dimensions:

$$Ax=f \Rightarrow \text{If } A \text{ is invertible matrix} \Rightarrow x=A^{-1}f \text{ unique solution.}$$

$$Ax=0 \Rightarrow x=0$$

2) $L[u]=0$ has a nontrivial solution $\Rightarrow$ $\lambda=0$ is an eigenvalue $\Rightarrow$ L is non-invertible $\Rightarrow$ Question: What is the solution of $L[u]=-\varphi(x)$? If $u_0(x)$ is the eigenfunction corresponding to the zero eigenvalue $\Rightarrow$ It should be that $\int_{x_0}^{x_1}\varphi(x)u_0(x)dx=0$

Fredholm's alternative

Digression-remark.

Consider the analogous problem in finite dimensions

$Ax=f$, A singular $\Rightarrow$ In order for this equation to have a solution, f must be in the range of A $\Rightarrow$ f must be orthogonal to the eigenvector of A corresponding to the zero eigenvalue. $\leftarrow$ Fredholm's alternative

$x\in R^n \to Ax \in M \subset R^n$

If the Fredholm's alternative holds then $L[u]=-\varphi(x)$ has solutions (non-unique). $\leftarrow$ Actually, an infinity of solutions!

Hence, the very existence of Green's function is equivalent to the existence of a unique solution of the problem $L[u]=-\varphi(x)$ with homogeneous boundary conditions (since otherwise 'rigid body' modes exist and Fredholm's alternative is involved leading to nonunique solutions). Therefore the following alternative exists: Under given homogeneous boundary conditions either $L[u]=-\varphi(x)$ has a unique solution $u(x)$ $\forall\, \varphi(x)$, or $L[u]=0$ has a nontrivial solution (since this would imply that $\lambda=0$ is an eigenvalue and L is singular). In the second one a solution exists only iff $\int_{x_0}^{x_1} \varphi(x) u_0(x)\,dx = 0$ where $u_0(x)$ is the eigenfunction of $\lambda = 0$, or equivalently the nontrivial solution of $L[u]=0$.

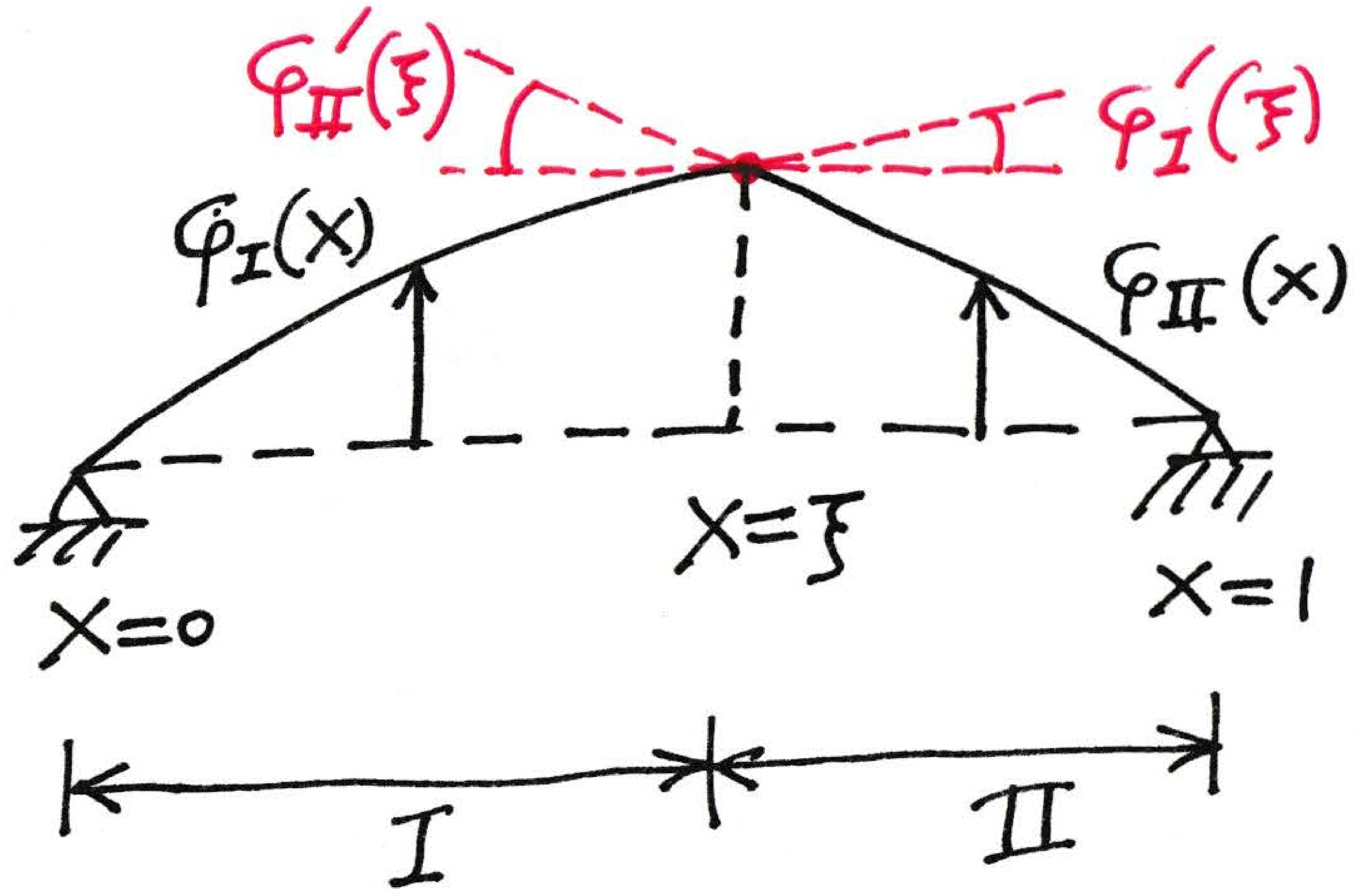
Example of Green's function computation

Consider $\left.\begin{array}{l} u_{xx} - u_{tt} = 0, \quad 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0 \end{array}\right\} \Rightarrow$ Let $u(x,t) = \varphi(x)\, T(t)$

where $\ddot{T}(t) + \omega^2 T(t) = 0$

Hence, obtain the eigenvalue problem, $\left.\begin{array}{l} \varphi''(x) + \omega^2 \varphi(x) = 0, \quad 0 \leq x \leq 1 \\ \varphi(0) = \varphi(1) = 0 \end{array}\right\}$

To compute the Green's function, consider $L[u] \equiv \dfrac{d^2u}{dx^2} + \omega^2 u$,



and separate the domain $G = [0, 1]$ to two subdomains, $0 \leq x \leq \xi$ and $\xi \leq x \leq 1$.

Subdomain I: $\left.\begin{array}{l} \varphi_I''(x) + \omega^2 \varphi_I(x) = 0 \\ \varphi_I(0) = 0 \end{array}\right\} \Rightarrow$

$\Rightarrow \varphi_I(x) = c_0 \sin\omega x + d_0 \cos\omega x \Rightarrow$

$\Rightarrow \quad \varphi_I(x) = c_0 \sin\omega x, \quad 0 \leq x \leq \xi$

Subdomain II:

$$\left.\begin{aligned}&\varphi_{II}''(x)+\omega^2\varphi_{II}(x)=0\\&\varphi_{II}(1)=0\end{aligned}\right\}\Rightarrow \varphi_{II}(x)=c_1\sin\omega x+d_1\cos\omega x$$

$$\Rightarrow \varphi_{II}(1)=0\Rightarrow$$

$$\Rightarrow c_1\sin\omega+d_1\cos\omega=0\Rightarrow$$

$$\Rightarrow d_1=-c_1\tan\omega\Rightarrow$$

$$\Rightarrow \varphi_{II}(x)=c_1\left[\sin\omega x-\tan\omega\cos\omega x\right],$$

$$\xi\le x\le 1$$

Matching at $x=\xi$ (discontinuity)

$$\left.\begin{aligned}&\varphi_I(\xi)=\varphi_{II}(\xi)\Rightarrow c_0\sin\omega\xi=c_1\left[\sin\omega\xi-\tan\omega\cos\omega\xi\right]\\&\frac{d\varphi_{II}(\xi)}{dx}-\frac{d\varphi_I(\xi)}{dx}=-1\Rightarrow c_1\omega\left[\cos\omega\xi+\tan\omega\sin\omega\xi\right]-c_0\omega\cos\omega\xi=-1\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow\begin{bmatrix}\sin\omega\xi & -\sin\omega\xi+\tan\omega\cos\omega\xi\\ -\omega\cos\omega\xi & \omega\left[\cos\omega\xi+\tan\omega\sin\omega\xi\right]\end{bmatrix}\begin{Bmatrix}c_0\\c_1\end{Bmatrix}=\begin{Bmatrix}0\\-1\end{Bmatrix}\Rightarrow$$

$$\Rightarrow c_0=\frac{1}{\omega\tan\omega}\left[\tan\omega\cos\omega\xi-\sin\omega\xi\right],\quad c_1=-\frac{\sin\omega\xi}{\omega\tan\omega}$$

Hence, we compute the Green's function as follows:

$$K(x,\xi) = \begin{cases} \varphi_{I}(x,\xi) = \dfrac{\sin\omega x}{\omega\tan\omega}\left[\tan\omega\cos\omega\xi - \sin\omega\xi\right], & 0\leq x\leq\xi \\ \varphi_{II}(x,\xi) = \dfrac{\sin\omega\xi}{\omega\tan\omega}\left[\tan\omega\cos\omega x - \sin\omega x\right], & \xi\leq x\leq 1 \end{cases}$$

Note the symmetry of $K(x,\xi) \Rightarrow K(x,\xi) = K(\xi,x),\ x,\xi\in[0,1]$

<u>Remark</u>

This result is associated with the definition of the self-adjoint operator $L[u]\equiv \dfrac{d^2u}{dx^2}+\omega^2u$. Suppose, however, that we define differently the operator, e.g., $\tilde{L}[u]\equiv\dfrac{d^2u}{dx^2} \Rightarrow$ Then, the eigenvalue problem is formulated as $\left.\begin{matrix}\tilde{L}[u]=-\omega^2u(x)\equiv\varphi(x)\\ u(0)=u(1)=0\end{matrix}\right\}\Rightarrow$

$\Rightarrow$ Suppose that the Green's function for $\tilde{L}[u]$ is $\tilde{K}(x,\xi)\Rightarrow$

⇒ Then, the solution of the eigenvalue problem is expressed as,

$$\tilde{L}[u] = -\varphi(x) \Rightarrow \quad u(x) = \int_0^1 \tilde{K}(x,\xi)\,\varphi(\xi)\,d\xi \;, \text{ where}$$

↖ But note that then, $\varphi(\xi)$ would depend on $u(\xi)$!

$\varphi(x)$ is now considered as the "non-homogeneous" term.

So, the way we define the operator in the first place, determines not only the expression of the Green's function, but also, the form of the solution itself.