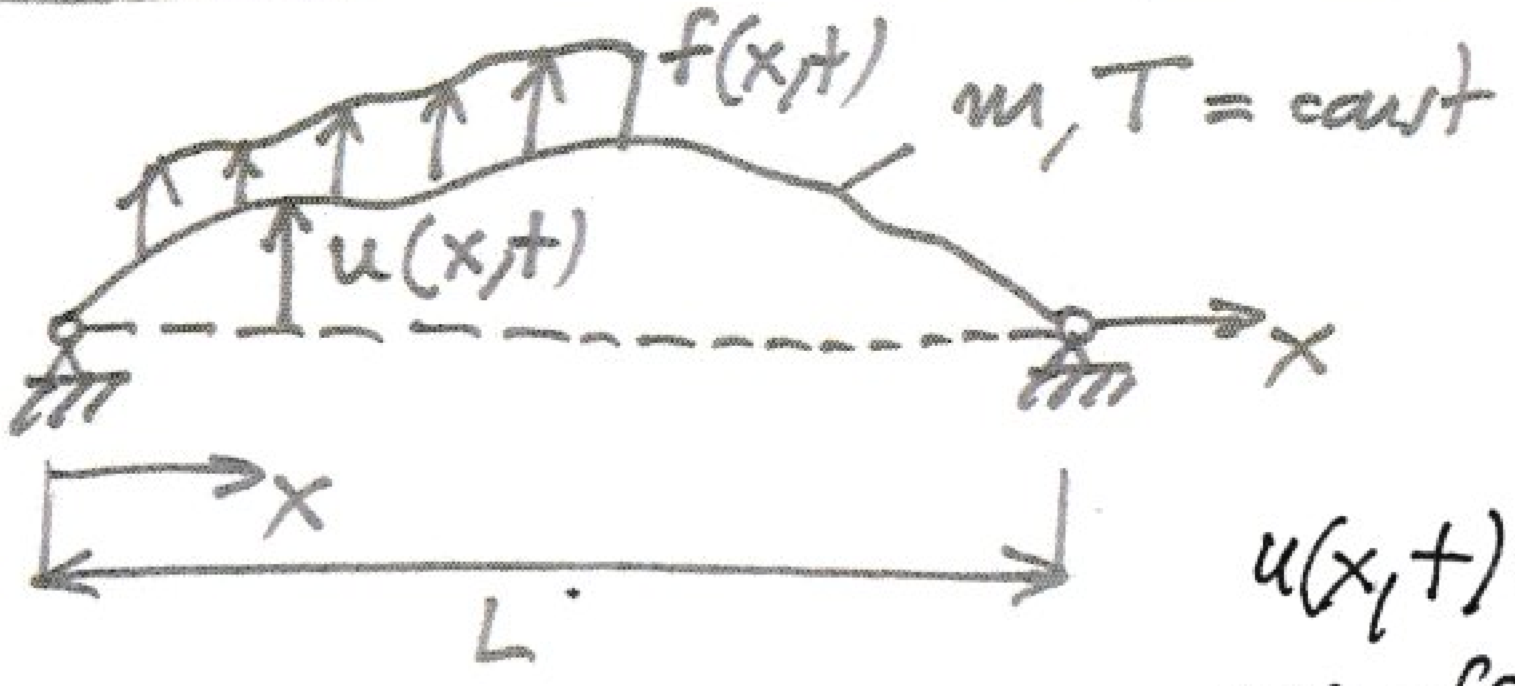
Example 1: fixed-fixed elastic string




The equation of motion in this case is,

$$u(x,t) = \varphi(x) f(t) \quad \left\{ \begin{array}{ll} T\dfrac{\partial^2 u}{\partial x^2} + F(x,t) = m\dfrac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le L, \ t \ge 0 & (1) \\ \text{BCs: } u(0,t) = u(L,t) = 0 & (1a) \\ \text{ICs: } u(x,0) = g(x), \ \dfrac{\partial u}{\partial t}(x,0) = h(x) & (1b) \end{array} \right.$$

first we consider the eigenvalue problem:

$$\left. \begin{array}{l} \dfrac{d^2\varphi(x)}{dx^2} + \left(\dfrac{\omega}{c}\right)^2 \varphi(x) = 0, \quad c = \sqrt{\dfrac{T}{m}} \\ \varphi(0) = \varphi(L) = 0 \end{array} \right\} \Rightarrow$$

$$\Rightarrow \varphi(x) = C_1 \overbrace{\cos\frac{\omega}{c}x}^{\varphi_1(x,\omega)} + C_2 \overbrace{\sin\frac{\omega}{c}x}^{\varphi_2(x,\omega)}$$

$$\left. \begin{array}{l} \varphi(0) = 0 \Rightarrow C_1 = 0 \\ \varphi(L) = 0 \Rightarrow C_1 \cos\dfrac{\omega L}{c} + C_2 \sin\dfrac{\omega L}{c} = 0 \end{array} \right\} \rightarrow \begin{bmatrix} 1 & 0 \\ \cos\frac{\omega L}{c} & \sin\frac{\omega L}{c} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\Rightarrow$ Setting the determinant of coefficients equal to zero $\rightarrow \sin\frac{\omega L}{c} = 0 \Rightarrow$

$$\Rightarrow \frac{\omega L}{c} = k\pi, \ k = 1, 2, \ldots \Rightarrow \boxed{\omega_k = \frac{k\pi c}{L}, \ k = 1, 2, \ldots}, \quad \omega_1 = \frac{\pi c}{L} < \omega_2 = \frac{2\pi c}{L} < \omega_3 < \ldots$$

Then, the corresponding eigenfunctions are $\varphi_r(x) = C_r \sin \omega_r x \Rightarrow$

$\Rightarrow$ Mass-orthonormalizing $\Rightarrow \int_0^L \underset{\substack{\uparrow \\ m}}{\mathcal{B}(x)} \varphi_r^2(x)\,dx = 1 \Rightarrow \int_0^L C_r^2 \sin^2\omega_r x\,dx = \frac{1}{m} \Rightarrow$

$\Rightarrow C_r = \sqrt{\frac{2}{mL}} \Rightarrow \boxed{\varphi_r(x) = \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L}, \quad r = 1, 2, \dots}$ $\Leftarrow$ Mass-orthonormalized eigenfunction

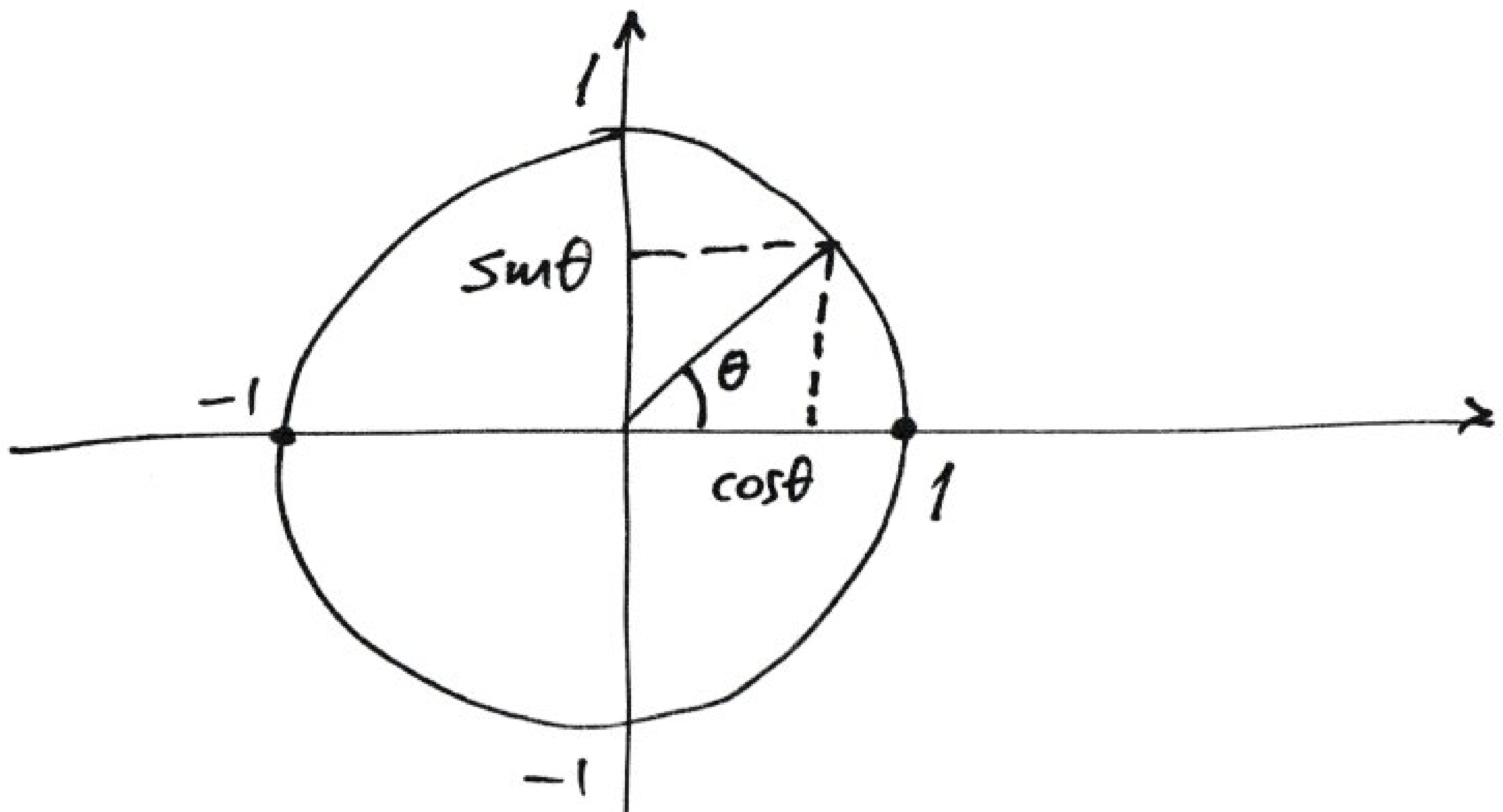
Then, we express the solution of problem (1) as,

$$\left.\begin{array}{l} u(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\varphi_r(x) = \sum_{i=1}^{\infty} \eta_i(t)\sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L} \\ \text{where } \ddot{\eta}_i(t) + \omega_i^2 \eta_i(t) = N_i(t) = \int_0^L m F(x,t)\varphi_i(x)\,dx \end{array}\right\} \Rightarrow$$

$$\eta_i(0) = \sqrt{\frac{2}{mL}} \int_0^L m\, g(x) \sin\left(\frac{i\pi x}{L}\right) dx$$

$$\dot{\eta}_i(0) = \sqrt{\frac{2}{mL}} \int_0^L m\, h(x) \sin\left(\frac{i\pi x}{L}\right) dx$$

$$\Rightarrow \boxed{u(x,t) = \sum_{i=1}^{\infty} \left[ \eta_i(0) \cos\left(\frac{i\pi c t}{L}\right) + \frac{\dot{\eta}_i(0)}{\frac{i\pi c}{L}} \sin\left(\frac{i\pi c}{L} t\right) + \frac{1}{\left(\frac{i\pi c}{L}\right)} \int_0^t N_i(\tau) \sin\left[\frac{i\pi c}{L}(t-\tau)\right] d\tau \right] \cdot \sqrt{\frac{2}{mL}} \sin(i\pi x/L)}$$



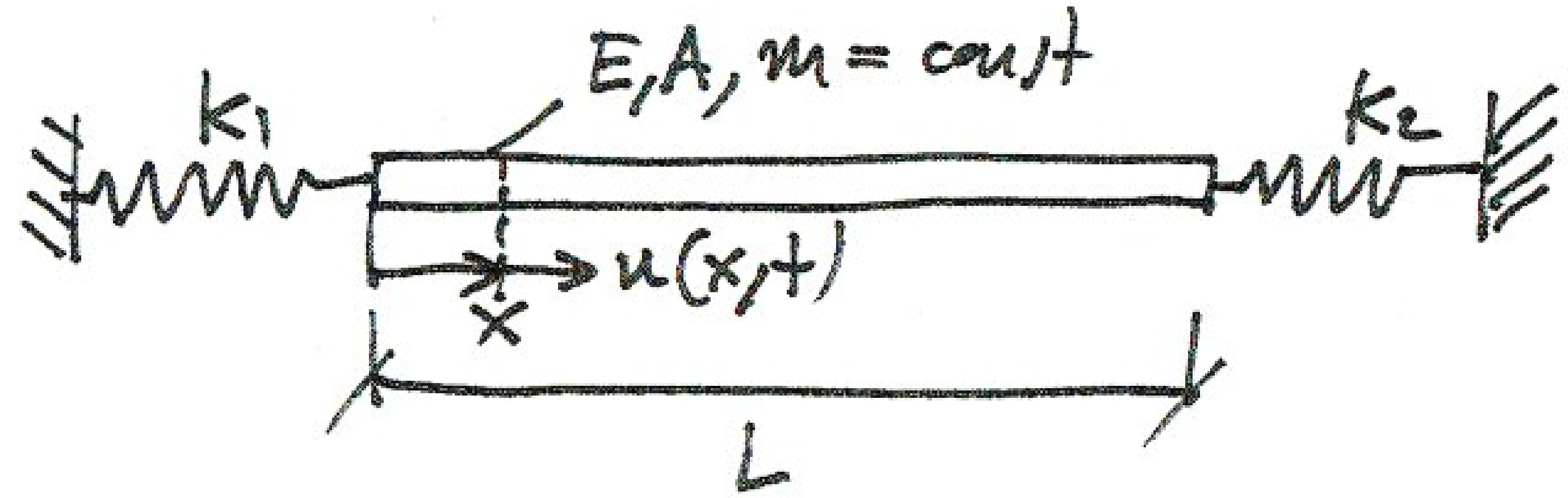

$\sin\theta = 0 \Rightarrow \theta = k\pi, \ k = 0, 1, 2, \dots$

$\cos\theta = 0 \Rightarrow \theta = \frac{(2k+1)\pi}{2}, \ k = 0, 1, 2, \dots$

$\tan\theta = 0 \Rightarrow \dots$

$\cot\theta = 0 \Rightarrow \dots$

## Example 2




the governing pde is,

$$EA \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2} \qquad (1)$$

BCs are:

$$\left.\begin{aligned} EA \frac{\partial u(0,t)}{\partial x} - k_1 u(0,t) &= 0 \\ EA \frac{\partial u(L,t)}{\partial x} + k_2 u(L,t) &= 0 \end{aligned}\right\} \qquad (1a)$$

ICs are: $u(x,0) = g(x)$, $\frac{\partial u(x,0)}{\partial t} = h(x)$ (1c)

Working similarly we formulate a linear eigenvalue problem:

$$\varphi''(x) + \left(\frac{\omega}{c}\right)^2 \varphi(x) = 0, \quad c = \sqrt{\frac{E}{\rho}} \qquad (2)$$

$$\left.\begin{aligned} EA\,\varphi'(0) - k_1 \varphi(0) &= 0 \\ EA\,\varphi'(L) + k_2 \varphi(L) &= 0 \end{aligned}\right\} \qquad (2a)$$

$$(2) \Rightarrow \varphi(x) = C_1 \cos \frac{\omega}{c} x + C_2 \sin \frac{\omega}{c} x \Rightarrow$$

$$\Rightarrow \varphi'(x) = -\frac{\omega}{c} C_1 \sin \frac{\omega x}{c} + \frac{\omega}{c} C_2 \cos \frac{\omega x}{c}$$

$$\Rightarrow EA \frac{\omega}{c} C_2 \overbrace{\cos \frac{\omega 0}{c}}^{1} - k_1 C_1 \cdot 1 = 0$$

$$EA\left[-\frac{\omega}{c} C_1 \sin \frac{\omega L}{c} + \frac{\omega}{c} C_2 \cos \frac{\omega L}{c}\right] + k_2\left[C_1 \cos \frac{\omega L}{c} + C_2 \sin \frac{\omega L}{c}\right] = 0$$

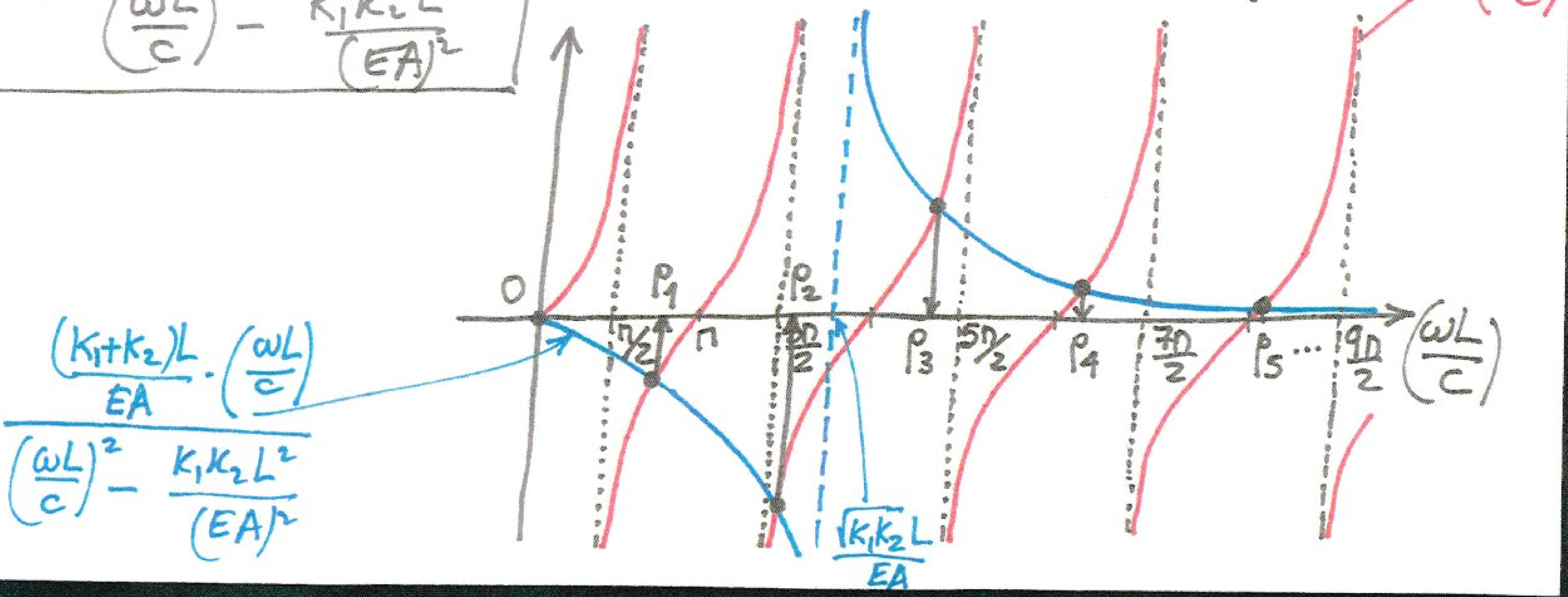
Hence, we get:

$$\begin{bmatrix} -k_1 & \frac{EA\omega}{c} \\ -EA\frac{\omega}{c}\sin\frac{\omega L}{c}+k_2\cos\frac{\omega L}{c} & EA\frac{\omega}{c}\cos\frac{\omega L}{c}+k_2\sin\frac{\omega L}{c} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow$$

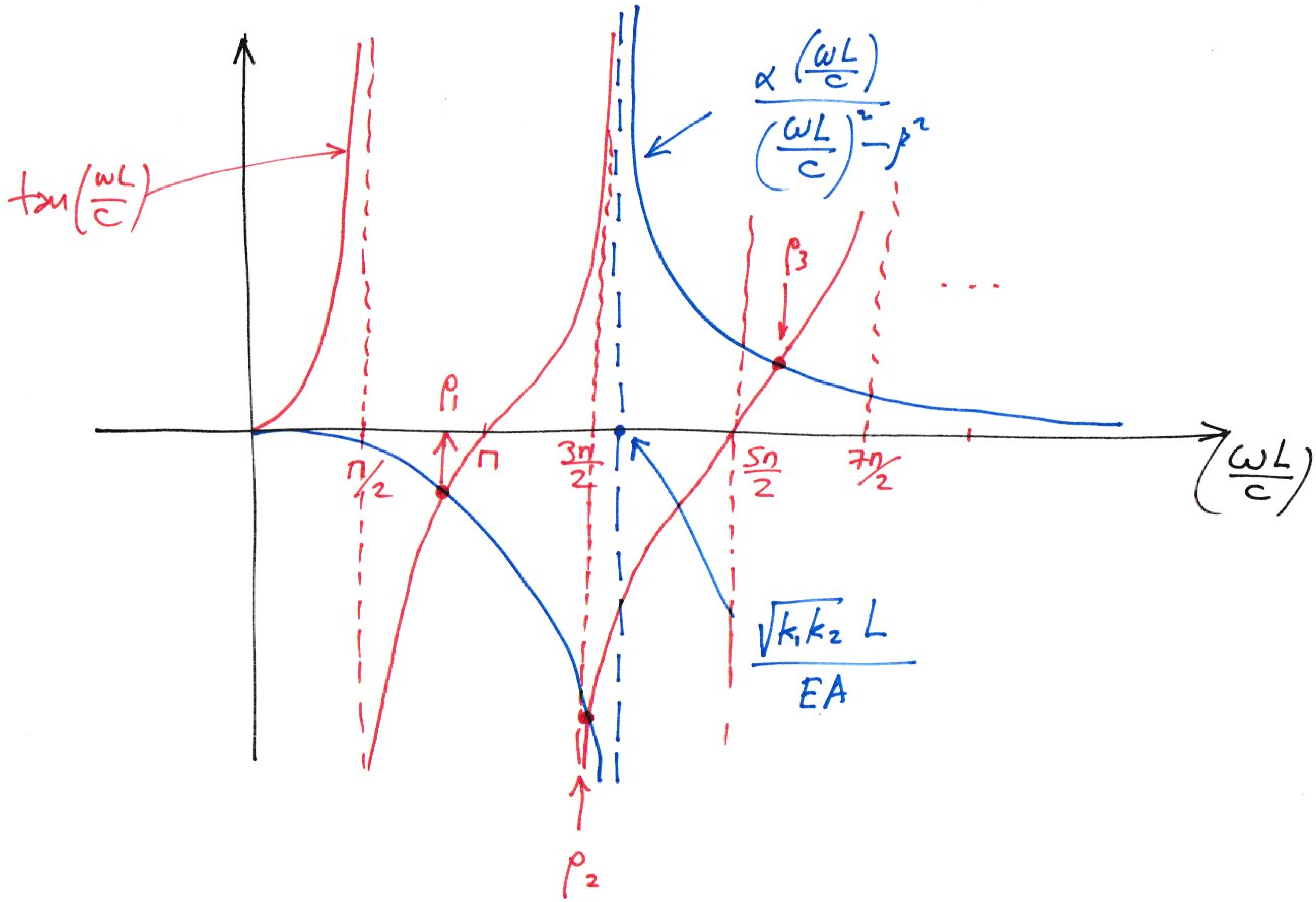
$\Rightarrow$ for nontrivial solutions we require that $\det[\ ]=0 \Rightarrow$

$$\Rightarrow -k_1 EA\frac{\omega}{c}\cos\frac{\omega L}{c}-k_1k_2\sin\frac{\omega L}{c}+\left(EA\frac{\omega}{c}\right)^2\sin\frac{\omega L}{c}-k_2EA\frac{\omega}{c}\cos\frac{\omega L}{c}=0\Rightarrow$$

$$\Rightarrow \boxed{\tan\left(\frac{\omega L}{c}\right)=\frac{\frac{(k_1+k_2)L}{EA}\cdot\left(\frac{\omega L}{c}\right)}{\left(\frac{\omega L}{c}\right)^2-\frac{k_1k_2L^2}{(EA)^2}}} \Rightarrow \text{In the form } \tan x=\frac{\alpha x}{x^2-\beta}$$



$$\tan\left(\frac{\omega L}{c}\right) = \frac{\alpha\left(\frac{\omega L}{c}\right)}{\left(\frac{\omega L}{c}\right)^2 - \beta^2}, \quad \alpha = \frac{(k_1+k_2)L}{EA}, \quad \beta = \frac{k_1 k_2 L^2}{(EA)^2}$$



Then we see that $\rho_i$ stays within certain limits:

$$\frac{\pi}{2}<\rho_1<\pi,\quad \frac{3\pi}{2}<\rho_2<2\pi,\quad \overset{5\pi/2}{\cancel{2\pi}}<\rho_3<\overset{3\pi}{\cancel{\frac{5\pi}{2}}},\ \ldots$$

Once we have computed $\rho_i \Rightarrow \rho_i=\dfrac{\omega_i L}{c} \Rightarrow \omega_i=\dfrac{c\rho_i}{L}$, $i=1,2,\ldots$

Then we can show that the corresponding eigenfunction is,

$$\boxed{\varphi_i(x)= C_i\,\frac{\sin\left(\rho_i \frac{x}{L}+\psi_i\right)}{\cos\psi_i},\quad i=1,2,\ldots},\quad \tan\psi_i=\frac{EA\rho_i}{K_1 L} \Rightarrow$$

$$\Rightarrow 0<\psi_i<\pi/2,\ i=1,2,\ldots$$

Then we can mass-orthonormalize, $\int_0^L m\,\varphi_i^2(x)\,dx=1 \Rightarrow$

$\Rightarrow m\,\dfrac{C_i^2}{\cos^2\psi_i}\displaystyle\int_0^L \sin^2\left(\rho_i\frac{x}{L}+\psi_i\right)dx=1 \Rightarrow$ from this we may compute the constant $C_i$, $i=1,2,\ldots \Rightarrow$

$$\boxed{C_i = 2\cos\psi_i\sqrt{\frac{\rho_i}{mL\left(2\rho_i-\sin 2(\rho_i+\psi_i)+2\sin 2\psi_i\right)}}}$$