TAM514HW4 Yichen Shi

Problem 1

i)

$$m\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u}{\partial x^2}\right]=0 \text{ with BC:}$$

$$EI\frac{\partial^2 u(0,t)}{\partial x^2}=0,\ -\frac{\partial}{\partial x}\left[EI\frac{\partial^2 u(0,t)}{\partial x^2}\right]=0, u(L,t)=0, \frac{\partial u(L,t)}{\partial x}=0$$

assume $u(x,t)=\varphi(x)f(t), \ddot{f}=-\omega^2 f$, substitute in,

$$-\omega^2 m\varphi+EI\varphi''''=0, \varphi''(0)=0, \varphi'''(0)=0, \varphi(L)=0, \varphi'(L)=0.$$

Let $\beta^4=EI/m$, $-\frac{\omega^2}{\beta^4}\varphi+\varphi''''=0$

$$\varphi=C_1\cosh(r(x-L))+C_2\sinh(r(x-L))+C_3\cos(r(x-L))+C_4\sin(r(x-L))$$

Where $r=\sqrt{\omega}/\beta$. Substitute into the boundary conditions,

$$\varphi(L)=0\Longrightarrow C_1+C_3=0, \varphi'(L)=0\Longrightarrow C_2+C_4=0.$$

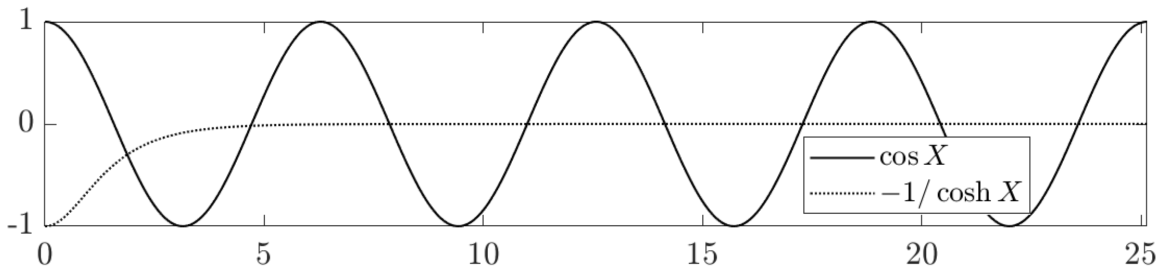
$$\varphi''(0)=0\Longrightarrow r^2[C_1\cosh(rL)-C_2\sinh(rL)-C_3\cos(rL)+C_4\sin(rL)]=0$$

$$\varphi'''(0)=0\Longrightarrow r^3[-C_1\sinh(rL)+C_2\cosh(rL)-C_3\sin(rL)-C_4\cos(rL)]=0$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cosh(rL) & -\sinh(rL) & -\cos(rL) & \sin(rL) \\ -\sinh(rL) & \cosh(rL) & -\sin(rL) & -\cos(rL) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow 2+2\cosh(rL)\cos(rL)=0\Longrightarrow\cos(rL)=-\frac{1}{\cosh(rL)}$$

Let $X=rL=\frac{\sqrt{\omega}}{\sqrt[4]{EI/m}}L$, $\cos X=-\frac{1}{\cosh X}$, the roots are plotted as follows.



Code:

```
xs=linspace(0,8*pi,501);

plot(xs,cos(xs),'k-',xs,-1./cosh(xs),'k:','LineWidth',1);

legend({"$\cos X$","$-1/\cosh X$"},'Interpreter','latex');

xlim([0 8*pi]);

set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
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the eigenfunctions are:

$$\varphi_j(x) = A_j\left[\cosh(r(x-L)) + \frac{\cosh(rL)+\cos(rL)}{\sinh(rL)+\sin(rL)}\sinh(r(x-L)) - \cos(r(x-L)) - \frac{\cosh(rL)+\cos(rL)}{\sinh(rL)+\sin(rL)}\sin(r(x-L))\right]$$

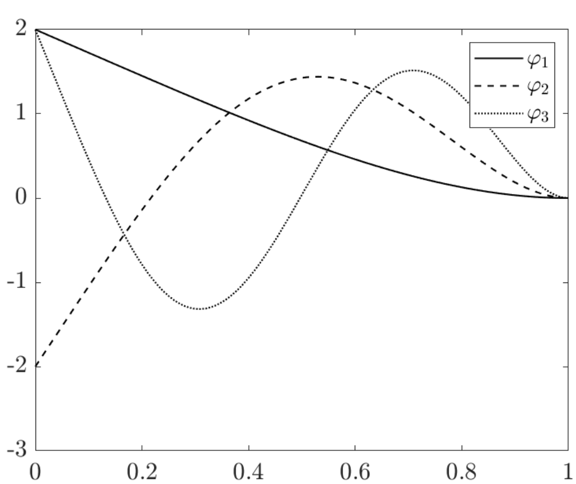
$$= A_j\left[\cosh(X(\xi-1)) + \frac{\cosh X+\cos X}{\sinh X+\sin X}\sinh(X(\xi-1)) - \cos(X(\xi-1)) - \frac{\cosh X+\cos X}{\sinh X+\sin X}\sin(X(\xi-1))\right]$$

$= A_j\psi_j(\xi)$, where $\xi = x/L$.

Because the boundary conditions are simple, the stiffness-orthogonality condition is:

$$\int_0^L m\varphi_r(x)\varphi_s(x)\,\mathrm{dx} = \delta_{rs} \Longrightarrow \int_0^1 mA_j^2\psi_j^2(\xi)L\,\mathrm{d}\xi = 1\,, A_j = \frac{1}{\sqrt{mL}}\left(\int_0^1 \psi_j^2(\xi)\,\mathrm{d}\xi\right)^{-1/2}$$

for L=1, m=1, numerically we plot the first three mass-orthogonalized eigenfunctions as:



Code:

```
r1=fzero(@(x) cos(x)+1/cosh(x),[1.6 2]);

r2=fzero(@(x) cos(x)+1/cosh(x),[4.17 5.42]);

r3=fzero(@(x) cos(x)+1/cosh(x),[7.28 8.4]);

L=1;

phi=@(r,x) cosh(r*(x-L))+(cosh(r*L)+cos(r*L))/(sinh(r*L)+sin(r*L))*sinh(r*(x-L))-
cos(r*(x-L))-(cosh(r*L)+cos(r*L))/(sinh(r*L)+sin(r*L))*sin(r*(x-L));

A1=(integral(@(x) phi(r1,x).^2,0,1))^(-1/2);

A2=(integral(@(x) phi(r2,x).^2,0,1))^(-1/2);

A3=(integral(@(x) phi(r3,x).^2,0,1))^(-1/2);

xs=linspace(0,1,101);

ys1=phi(r1,xs)*A1;

ys2=phi(r2,xs)*A2;

ys3=phi(r3,xs)*A3;

plot(xs,ys1,'k-',xs,ys2,'k--',xs,ys3,'k:','LineWidth',1);

legend({"$\varphi_1$","$\varphi_2$","$\varphi_3$"},'Interpreter','latex');

set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

Assume $u(x,t)=\sum_{n=1}^{\infty}\eta_n(t)\varphi_n(x)$ satisfy the forced equation $m\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u}{\partial x^2}\right]=P(t)\delta(x-x_1)-M(t)\delta'(x-x_2)$ with initial conditions $u(x,0)=u_t(x,0)=0$.

Substitute into the equation, multiply by φ_m and integrate from 0 to L,

$$\sum_{n=1}^{\infty}m\{\ddot{\eta}_n(t)+\omega_n^2\eta_n(t)\}\varphi_n(x)=P(t)\delta(x-x_1)-M(t)\delta'(x-x_2)$$

$$\sum_{n=1}^{\infty}\{\ddot{\eta}_n(t)+\omega_n^2\eta_n(t)\}\delta_{mn}=\int_0^L[P(t)\delta(x-x_1)-M(t)\delta'(x-x_2)]\varphi_m\mathrm{dx}$$

$$=P(t)\varphi_m(x_1)+M(t)\varphi_m{}'(x_2)$$

$\ddot{\eta}_m(t) + \omega_m^2 \eta_m(t) = P(t)\varphi_m(x_1) + M(t)\varphi_m{}'(x_2)$

So $\eta_m(t) = \dfrac{1}{\omega_m}\displaystyle\int_0^t [P(t)\varphi_m(x_1) + M(t)\varphi_m{}'(x_2)]\sin\omega_n(t-\tau)d\tau$

$$u(x,t) = \sum_{m=1}^{\infty}\left\{\frac{1}{\omega_m}\int_0^t [P(t)\varphi_m(x_1) + M(t)\varphi_m{}'(x_2)]\sin\omega_n(t-\tau)d\tau\right\}\varphi_m(x)$$

ii)

the necessary conditions for the second mode of the beam to be not excited is to let the modal forcing term $P(t)\varphi_m(x_1) + M(t)\varphi_m{}'(x_2) = 0$. The sufficient conditions for this mode to not be excited is to place x_1 on the node of the mode φ_2, and place x_2 on the place where $\varphi'_m(x_2) = 0$.

The response is then

$$u(x,t) = \sum_{m\neq 2}\left\{\frac{1}{\omega_m}\int_0^t [P(t)\varphi_m(x_1) + M(t)\varphi_m{}'(x_2)]\sin\omega_n(t-\tau)d\tau\right\}\varphi_m(x).$$

iii)

To have resonance, $P(t)\varphi_m(x_1) + M(t)\varphi_m{}'(x_2)$ must match frequency ω_m. So the condition for resonance to happen is $P(t)$ has a frequency Ω_m and $\varphi_m(x_1) \neq 0$, or $M(t)$ has a frequency Ω_m and $\varphi_m{}'(x_2) \neq 0$.

Problem 2

(i)

Assume $u = f(t)\varphi(x)$, substitute into the PDE:

$$f\varphi_{xxxx} + 4f\varphi_{xx} + 0.05f_t\varphi + f_{tt}\varphi = 0 \Longrightarrow \frac{\varphi_{xxxx}}{\varphi} + 4\frac{\varphi_{xx}}{\varphi} = -0.05\frac{f_t}{f} - \frac{f_{tt}}{f} = h$$

This is in separable form. For

$\varphi(x) = \sqrt{2/\pi}\sin kx, \varphi_{xx} = -k^2\sqrt{2/\pi}\sin kx, \varphi_{xxxx} = k^4\sqrt{2/\pi}\sin kx$, we find

$\varphi(0) = \varphi_{xx}(0) = \varphi(\pi) = \varphi_{xx}(\pi)$ satisfy boundary conditions, and

$\dfrac{\varphi_{xxxx} + 4\varphi_{xx}}{\varphi} = \dfrac{k^4\sqrt{2/\pi}\sin kx - 4k^2\sqrt{2/\pi}\sin kx}{\sqrt{2/\pi}\sin kx} = k^4 - 4k^2$ is indeed a constant, so

$\varphi_k(x) = \sqrt{2/\pi}\sin kx$ is the eigenfunction of $\dfrac{\varphi_{xxxx}}{\varphi} + 4\dfrac{\varphi_{xx}}{\varphi} = k^4 - 4k^2 = h$.

So the ansatz $u(x,t)=\sum_{k=1}^{\infty} a_k(t)\varphi_k(x)$ can completely represent the solution.

Substitute the ansatz into the PDE,

$$\sum_{k=1}^{\infty} a_k(t)\left[\varphi_k''''(x)+4\varphi_k''(x)\right]+0.05\frac{\partial a_k(t)}{\partial t}\varphi_k(x)+\frac{\partial^2 a_k(t)}{\partial t^2}\varphi_k(x)=0$$

$$\Longrightarrow \sum_{k=1}^{\infty}\left[a_k(t)\left(k^4-4k^2\right)+0.05\frac{\partial a_k(t)}{\partial t}+\frac{\partial^2 a_k(t)}{\partial t^2}\right]\varphi_k(x)=0$$

multiply by $\varphi_j(x)$ and integrate from 0 to π,

$$\sum_{k=1}^{\infty}\left[a_k(t)\left(k^4-4k^2\right)+0.05\frac{\partial a_k(t)}{\partial t}+\frac{\partial^2 a_k(t)}{\partial t^2}\right]\int_0^{\pi}\varphi_j(x)\varphi_k(x)\,\mathrm{dx}=0$$

Because $\int_0^{\pi}\varphi_j(x)\varphi_k(x)\,\mathrm{dx}=\frac{2}{\pi}\int_0^{\pi}\sin jx\sin kx\,\mathrm{dx}=\delta_{jk}$,

$$a_j(t)\left(j^4-4j^2\right)+0.05\frac{\partial a_j(t)}{\partial t}+\frac{\partial^2 a_j(t)}{\partial t^2}=0.$$

Multiply initial condition by $\varphi_j(x)$ and integrate from 0 to π,

$$\sum_{k=1}^{\infty} a_k(0)\varphi_k(x)=f(x)\Longrightarrow\int_0^{\pi}\sum_{k=1}^{\infty}a_k(0)\varphi_k(x)\varphi_j(x)\,\mathrm{dx}=\int_0^{\pi}f(x)\varphi_j(x)\,\mathrm{dx}\Longrightarrow a_j(0)=\int_0^{\pi}f(x)\varphi_j(x)\,\mathrm{dx}$$

Similarly $\frac{\partial a_j(0)}{\partial t}=\int_0^{\pi}g(x)\varphi_j(x)\,\mathrm{dx}$.

So the equations governing the modal amplitudes are:

$a_k(t)\left(k^4-4k^2\right)+0.05\frac{\mathrm{d}a_k(t)}{\mathrm{d}t}+\frac{\mathrm{d}^2 a_k(t)}{\mathrm{d}t^2}=0$, with initial conditions

$a_k(0)=\int_0^{\pi}f(x)\varphi_k(x)\,\mathrm{dx}$, $\frac{\mathrm{d}a_k(0)}{\mathrm{d}t}=\int_0^{\pi}g(x)\varphi_k(x)\,\mathrm{dx}$ with $\varphi_k=\sqrt{\frac{2}{\pi}}\sin kx$.

ii) Assume $a_k(t)=Ae^{i(\omega t-\varphi)}$, substitute in, $A\left(k^4-4k^2+0.05i\omega-\omega^2\right)e^{i(\omega t-\varphi)}=0$,
$\omega^2-0.05i\omega=k^4-4k^2$, assume $\omega=\omega_R+i\omega_I, \omega_R,\omega_I\in\mathbb{R}$, substitute into LHS,

$\omega^2-0.05i\omega=\omega_R^2-\omega_I^2+0.05\omega_I+i(2\omega_R\omega_I-0.05\omega_R)$, because LHS must be real,
$2\omega_R\omega_I-0.05\omega_R=0\Longrightarrow\omega_R(2\omega_I-0.05)=0$.

If $\omega_R = 0$, $-\omega_I^2 + 0.05\omega_I = k^4 - 4k^2$. If the mode is unstable, $\omega_I < 0, k^4 - 4k^2 = -\omega_I^2 + 0.05\omega_I < 0 \Longrightarrow 0 < k^2 < 4, k = 1$.

For k=2, $0.05\dfrac{\mathrm{d}a_2(t)}{\mathrm{d}t} + \dfrac{\mathrm{d}^2 a_2(t)}{\mathrm{d}t^2} = 0 \Longrightarrow |a_2 - C_1 t| = 20e^{(C_2 - t)/20}$, a_2 can be unbounded, which is unstable.

If $\omega_I = 0.025$, $\omega_R^2 + 0.025^2 = k^4 - 4k^2$, guarantees two real solutions of ω_R when $k \geq 3$.

So the first mode $\varphi_1(x)$ and second mode $\varphi_2(x)$ are both unstable, and $\varphi_k(x), k \geq 3$ are stable.

The physical mechanism of instability comes from the compression of the beam, which leads to buckling of the 1st mode and 2nd mode. The mode will leave the neighborhood of unstable equilibrium position and be attracted to the buckling states.

If we drop the damping term, $\omega^2 = k^4 - 4k^2$, for k=1, $\omega = \pm\sqrt{3}\,i$, leading to instability.

For k=2, $\dfrac{\mathrm{d}^2 a_2(t)}{\mathrm{d}t^2} = 0 \Longrightarrow a_2 = C_1 t + C_2$, still unstable.

So eliminating damping will not stabilize the first and second modes.

iii) to make the overall response bounded, we want the 1st and 2nd modes to have 0 initial conditions. This requires f and g to be mass-orthogonal to the first and second modes,

$\varphi_1(x) = \sqrt{2/\pi}\sin x, \varphi_2(x) = \sqrt{2/\pi}\sin 2x$.

For other modes, $a_k(t)\,(k^4 - 4k^2) + 0.05\dfrac{\mathrm{d}a_k(t)}{\mathrm{d}t} + \dfrac{\mathrm{d}^2 a_k(t)}{\mathrm{d}t^2} = 0$, let

$\omega_k^2 = k^4 - 4k^2, 2\zeta_k\omega_k = 0.05 \Longrightarrow \omega_k = \sqrt{k^4 - 4k^2}, \zeta_k = 0.025/\sqrt{k^4 - 4k^2} < 1$

$r^2 + 2\zeta_k\omega_k + \omega_k^2 = 0 \Longrightarrow r_{1,2} = -\zeta_k\omega_k \pm i\omega_k\sqrt{1-\zeta_k^2}, a_k = A_k e^{-\zeta_k\omega_k t}\cos\left(\omega_k\sqrt{1-\zeta_k^2}\,t - \varphi_k\right)$

take in initial conditions,

$a_k(0) = A_k\cos\varphi_k, \dfrac{\mathrm{d}a_k(0)}{dt} = -\zeta_k\omega_k A_k\cos\varphi_k + A_k\omega_k\sqrt{1-\zeta_k^2}\sin\varphi$

$$\Longrightarrow A_k\cos\varphi_k = a_k(0), A_k\sin\varphi_k = \frac{1}{\omega_k\sqrt{1-\zeta_k^2}}\left(\frac{\mathrm{d}a_k(0)}{dt} + \zeta_k\omega_k a_k(0)\right)$$

$$\Longrightarrow A_k = \left| a_k(0) + \frac{i}{\omega_k\sqrt{1-\zeta_k^2}}\left(\frac{da_k(0)}{dt} + \zeta_k\omega_k a_k(0)\right)\right|, \varphi_k = \angle\left\{a_k(0) + \frac{i}{\omega_k\sqrt{1-\zeta_k^2}}\left(\frac{da_k(0)}{dt} + \zeta_k\omega_k a_k(0)\right)\right\}$$

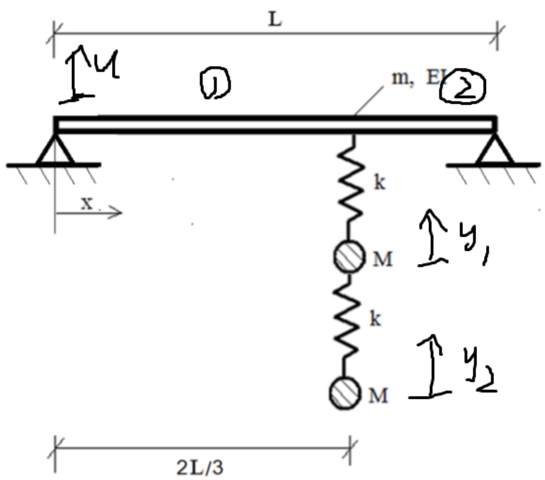
The response is: $u(x,t) = \sum_{k=3}^{\infty}\sqrt{\frac{2}{\pi}}\sin kx A_k e^{-\zeta_k\omega_k t}\cos\left(\omega_k\sqrt{1-\zeta_k^2}\,t - \varphi_k\right)$ where,

$\omega_k = \sqrt{k^4 - 4k^2}, \zeta_k = 0.025/\sqrt{k^4 - 4k^2}$

$$a_k(0) = \int_0^{\pi}\sqrt{\frac{2}{\pi}}f(x)\sin kx\,\mathrm{dx}, \frac{\mathrm{d}a_k(0)}{\mathrm{d}t} = \int_0^{\pi}\sqrt{\frac{2}{\pi}}g(x)\sin kx\,\mathrm{dx}$$

$$A_k = \left| a_k(0) + \frac{i}{\omega_k\sqrt{1-\zeta_k^2}}\left(\frac{\mathrm{d}a_k(0)}{dt} + \zeta_k\omega_k a_k(0)\right)\right|, \varphi_k = \angle\left\{a_k(0) + \frac{i}{\omega_k\sqrt{1-\zeta_k^2}}\left(\frac{\mathrm{d}a_k(0)}{dt} + \zeta_k\omega_k a_k(0)\right)\right\}$$

Problem 3



The governing equations are: $m\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 u}{\partial x^2}\right] = 0$

$$M\frac{\mathrm{d}^2 y_1}{\mathrm{d}t^2} = k(u(2L/3) - y_1) - k(y_1 - y_2), M\frac{\mathrm{d}^2 y_2}{\mathrm{d}t^2} = k(y_1 - y_2)$$

With boundary conditions:

$$u(0) = 0, EI\frac{\partial^2 u(0)}{\partial x^2} = 0, u(L) = 0, EI\frac{\partial^2 u(L)}{\partial x^2} = 0.$$

Force balance at $x = 2L/3$:

$$-\frac{\partial}{\partial x}\left[EI\frac{\partial^2 u_1}{\partial x^2}\right]_{2L/3}+k[u(2L/3)-y_1]=-\frac{\partial}{\partial x}\left[EI\frac{\partial^2 u_2}{\partial x^2}\right]_{2L/3}$$

Continuity of displacement at 2L/3: $u_1(2L/3)=u_2(2L/3)$.

Continuity of deflection angle at 2L/3: $\frac{\partial u_1}{\partial x}|_{(2L/3)}=\frac{\partial u_2}{\partial x}|_{(2L/3)}$.

Continuity of moment at 2L/3: $\frac{\partial^2 u_1}{\partial x^2}|_{(2L/3)}=\frac{\partial^2 u_2}{\partial x^2}|_{(2L/3)}$.

Assume synchronize motion $u_1=f(t)\varphi_1(x), u_2=f(t)\varphi_2(x), \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=f(t)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$,

$\ddot{f}=-\omega^2 f$, substitute in,

$-\omega^2 m\varphi_1+EI\varphi_1''''(x)=0$, $-\omega^2 m\varphi_2+EI\varphi_2''''(x)=0$, take $\beta^4=EI/m$,

$-\omega^2 Mz_1=k(\varphi_1(2L/3)-z_1)-k(z_1-z_2), -\omega^2 Mz_2=k(z_1-z_2)$

$\varphi_1(0)=0, \varphi_1''(0)=0, \varphi_2(L)=0, \varphi_2''(L)=0$,
$[EI\varphi_2''(2L/3)]'-[EI\varphi_1''(2L/3)]'=-k[\varphi(2L/3)-z_1], \varphi_1(2L/3)=\varphi_2(2L/3)$,
$\varphi_1'(2L/3)=\varphi_2'(2L/3), \varphi_1''(2L/3)=\varphi_2''(2L/3)$.

Assume $\varphi_1(x)=C_1\sinh\lambda x+C_2\cosh\lambda x+C_3\sin\lambda x+C_4\cos\lambda x$ for $0\le x\le(2L/3)$,
$\varphi_2(x)=C_5\sinh\lambda(x-L)+C_6\cosh\lambda(x-L)+C_7\sin\lambda(x-L)+C_8\cos\lambda(x-L)$ for $(2L/3)\le x\le L$, where $\lambda=\sqrt{\omega}/\beta$.

Take in boundary conditions,

$\varphi_1(0)=0\Longrightarrow C_2+C_4=0$, $\varphi_1''(0)=0\Longrightarrow C_2\lambda^2-C_4\lambda^2=0\Longrightarrow C_2=C_4=0$.

Similarly,

$\varphi_2(L)=0\Longrightarrow C_6+C_8=0, \varphi_2''(L)=0\Longrightarrow\lambda^2 C_6-\lambda^2 C_8=0\Longrightarrow C_6=C_8=0$

Continuity of displacement:
$C_1\sinh(\lambda\cdot 2L/3)+C_3\sin(\lambda\cdot 2L/3)=-C_5\sinh(\lambda\cdot L/3)-C_7\sin(\lambda\cdot L/3)$

Continuity of deflection angle:

$C_1\lambda\cosh(\lambda\cdot 2L/3)+C_3\lambda\cos(\lambda\cdot 2L/3)=C_5\lambda\cosh(\lambda\cdot L/3)+C_7\lambda\cos(\lambda\cdot L/3)$

Continuation of moment:

$$C_1\lambda^2\sinh^2(\lambda\cdot 2L/3)-C_3\lambda^2\sin(\lambda\cdot 2L/3)=-C_5\lambda^2\sinh(\lambda\cdot L/3)+C_7\lambda^2\sin(\lambda\cdot L/3)$$

Force balance at 2L/3:

$$C_1\lambda^3 EI\cosh(\lambda\cdot 2L/3)-C_3\lambda^3 EI\cos(\lambda\cdot 2L/3)-k[\varphi_1(2L/3)-z_1]=C_5\lambda^3 EI\cosh(\lambda\cdot L/3)-C_7\lambda^3 EI\cos(\lambda\cdot L/3)$$

Equations of motions of point masses:

$$-\omega^2 Mz_1=k(\varphi_1(2L/3)-z_1)-k(z_1-z_2),-\omega^2 Mz_2=k(z_1-z_2)$$

$$\Longrightarrow\begin{bmatrix}2k-\omega^2M & -k\\ -k & k-\omega^2M\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\begin{bmatrix}k\varphi_1(2L/3)\\ 0\end{bmatrix}\Longrightarrow\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\frac{k\varphi_1(2L/3)}{k^2-3kM\omega^2+M^2\omega^4}\begin{bmatrix}k-M\omega^2\\ k\end{bmatrix}$$

$$k[\varphi_1(2L/3)-z_1]=k\left[\varphi_1(2L/3)-\frac{k\varphi_1(2L/3)\,(k-M\omega^2)}{k^2-3kM\omega^2+M^2\omega^4}\right]=k\varphi_1(2L/3)\,\frac{-2kM\omega^2+M^2\omega^4}{k^2-3kM\omega^2+M^2\omega^4}$$

Let $q=\dfrac{-2kM\omega^2+M^2\omega^4}{k^2-3kM\omega^2+M^2\omega^4}$, substitute into the equation of force balance,

$$C_1\lambda^3\cosh(\lambda\cdot 2L/3)-C_3\lambda^3\cos(\lambda\cdot 2L/3)-kq\varphi(2L/3)=C_5\lambda^3\cosh(\lambda\cdot L/3)-C_7\lambda^3\cos(\lambda\cdot L/3)$$

Rearrange,

$$C_1[\lambda^3\cosh(\lambda\cdot 2L/3)-kq\sinh(\lambda\cdot 2L/3)]+C_3[-\lambda^3\cos(\lambda\cdot 2L/3)-kq\sin(\lambda\cdot 2L/3)]$$

$$-C_5\lambda^3\cosh(\lambda\cdot L/3)+C_7\lambda^3\cos(\lambda\cdot L/3)=0$$

Assemble all continuity equations and force balance into matrix form:

$$\begin{bmatrix}\sinh(\lambda\cdot 2L/3) & \sin(\lambda\cdot 2L/3) & \sinh(\lambda\cdot L/3) & \sin(\lambda\cdot L/3)\\ \lambda\cosh(\lambda\cdot 2L/3) & \lambda\cos(\lambda\cdot 2L/3) & -\lambda\cosh(\lambda\cdot L/3) & -\lambda\cos(\lambda\cdot L/3)\\ \lambda^2\sinh(\lambda\cdot 2L/3) & -\lambda^2\sin(\lambda\cdot 2L/3) & \lambda^2\sinh(\lambda\cdot L/3) & -\lambda^2\sin(\lambda\cdot L/3)\\ \lambda^3 EI\cosh(\lambda\cdot 2L/3)-kq\sinh(\lambda\cdot 2L/3) & -\lambda^3 EI\cos(\lambda\cdot 2L/3)-kq\sin(\lambda\cdot 2L/3) & -\lambda^3 EI\cosh(\lambda\cdot L/3) & \lambda^3 EI\cos(\lambda\cdot L/3)\end{bmatrix}\begin{bmatrix}C_1\\ C_3\\ C_5\\ C_7\end{bmatrix}=0$$

With $q=\dfrac{-2kM\omega^2+M^2\omega^4}{k^2-3kM\omega^2+M^2\omega^4}=\dfrac{-2kM(\beta^2\lambda^2)^2+M^2(\beta^2\lambda^2)^4}{k^2-3kM(\beta^2\lambda^2)^2+M^2(\beta^2\lambda^2)^4}$.

Equate the determinant to 0 and simplify,

$$2\lambda^6\left(\begin{matrix}\sin(\lambda L)\sinh\left(\frac{\lambda L}{3}\right)\left(\beta^4k\lambda M\sinh\left(\frac{2\lambda L}{3}\right)(\beta^4\lambda^4M-2k)-2EI\left(2\cosh\left(\frac{2\lambda L}{3}\right)+1\right)(k^2-3\beta^4k\lambda^4M+\beta^8\lambda^8M^2)\right)\\ +\beta^4k\lambda M\sin\left(\frac{\lambda L}{3}\right)\sin\left(\frac{2\lambda L}{3}\right)\sinh(\lambda L)(2k-\beta^4\lambda^4M)\end{matrix}\right)=0$$

No rigid body mode, divide $2\lambda^6$ and define non-dimensional parameters

$\xi=\dfrac{mL}{M}, \eta=\dfrac{EI}{kL^3}$, let $\omega_0^2=k/M$, also define $\alpha=\left(\dfrac{\omega^2\xi}{\omega_0^2\eta}\right)^{1/4}$.

$$\xi/\eta=\frac{mL}{M}\cdot\frac{kL^3}{EI}=\frac{k}{M}\frac{L^4}{EI/m}=\frac{\omega_0^2L^4}{EI/m}, \lambda\cdot L/3=\frac{1}{3}\left(\frac{\omega^2L^4}{EI/m}\right)^{1/4}=\frac{1}{3}\left(\frac{\omega^2\xi}{\omega_0^2\eta}\right)^{1/4}=\frac{1}{3}\alpha$$

$$\Longrightarrow \lambda=\alpha/L\,,\ \beta^4=EI/m=\omega_0^2L^4\eta/\xi\,,$$

substitute in and simplify,

$$\alpha\sin(\alpha)\sinh\left(\frac{\alpha}{3}\right)\sinh\left(\frac{2\alpha}{3}\right)(\alpha^4\eta-2\xi)-\sinh(\alpha)\left(\alpha\sin\left(\frac{\alpha}{3}\right)\sin\left(\frac{2\alpha}{3}\right)(\alpha^4\eta-2\xi)+2\sin(\alpha)(\alpha^8\eta^2-3\alpha^4\eta\xi+\xi^2)\right)=0$$

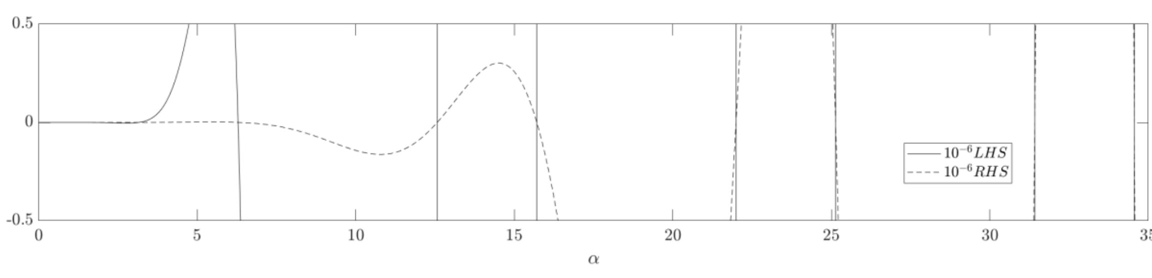
When $\sin(\alpha/3)=0$, the frequency equation above is automatically satisfied, where $x=2L/3$ is a node and the two spring-mass oscillators both stay motionless, because $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\dfrac{k\varphi_1(2L/3)}{k^2-3kM\omega^2+M^2\omega^4}\begin{bmatrix} k-M\omega^2 \\ k \end{bmatrix}=0$. This is the first type of modes.

Let $\alpha=3n\pi$, substitute back to the matrix and find $C_3=(-1)^n, C_7=1$, so the eigenfunctions are $\varphi_{1i}=A_i(-1)^n\sin 3n\pi, \varphi_{2i}=A_i\sin(3n\pi(x-L))$, $z_1=z_2=0$, where A_i are multiplicative constants.

Divide the frequency equation by $\sin(\alpha/3)\sinh(\alpha/3)$, the second type of modes satisfy:

$$\alpha\sin\left(\frac{2\alpha}{3}\right)(2\xi-\alpha^4\eta)-2\left(2\cos\left(\frac{2\alpha}{3}\right)+1\right)(\alpha^8\eta^2-3\alpha^4\eta\xi+\xi^2)=-\frac{\alpha\left(2\cos\left(\frac{2\alpha}{3}\right)+1\right)\sinh\left(\frac{2\alpha}{3}\right)(\alpha^4\eta-2\xi)}{2\cosh\left(\frac{2\alpha}{3}\right)+1}$$

Choose $\xi=\eta=1$, we plot the two curves indicated by LHS and RHS.



The second type of modes have nontrivial interactions between the beam and the attachments. Assume $\alpha_i, [C_{1i}, C_{3i}, C_{5i}, C_{7i}]^\top$ are the eigenvalues and eigenvectors of the matrix above, respectively, the eigenfunctions are: $\varphi_{1i}(x)=A_iC_{1i}\sinh\lambda x+A_iC_{3i}\sin\lambda x$

$\varphi_{2i}(x) = C_{5i}\sinh\lambda(x-L) + C_{7i}\sin\lambda(x-L)$ with

$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \dfrac{k\varphi_1(2L/3)}{k^2 - 3kM\omega^2 + M^2\omega^4}\begin{bmatrix} k - M\omega^2 \\ k \end{bmatrix}$. Where A_i are multiplicative constants.

Now we find mass and stiffness orthogonality conditions for the beam.

$-\omega^2 m\varphi_{1i} + EI\varphi_{1i}''''(x) = 0$, $-\omega^2 m\varphi_{2i} + EI\varphi_{2i}''''(x) = 0$

Multiply by $\varphi_{1j}, \varphi_{2j}$ respectively and integrate in their domain,

$$\omega_i^2\int_0^{2L/3}\varphi_{1j}(x)m\varphi_{1i}(x)\mathrm{dx} = \int_0^{2L/3}\varphi_{1j}(x)EI\varphi_{1i}''''(x)\mathrm{dx} = [\varphi_{1j}(x)EI\varphi_{1i}'''(x)]_0^{2L/3} - \int_0^{2L/3}\varphi_{1j}'(x)EI\varphi_{1i}'''(x)\mathrm{dx}$$

$$= \varphi_{1j}(2L/3)EI\varphi_{1i}'''(2L/3) - [\varphi_{1j}'(x)EI\varphi_{1i}''(x)]_0^{2L/3} + \int_0^{2L/3}\varphi_{1j}''(x)EI\varphi_{1i}''(x)\mathrm{dx}$$

$$= \varphi_{1j}(2L/3)EI\varphi_{1i}'''(2L/3) - \varphi_{1j}'(2L/3)EI\varphi_{1i}''(2L/3) + \int_0^{2L/3}\varphi_{1j}''(x)EI\varphi_{1i}''(x)\mathrm{dx}\ (1)$$

$$\omega_i^2\int_{2L/3}^{L}\varphi_{2j}(x)m\varphi_{2i}(x)\mathrm{dx} = \int_{2L/3}^{L}\varphi_{2j}(x)EI\varphi_{2i}''''(x)\mathrm{dx} = [\varphi_{2j}(x)EI\varphi_{2i}'''(x)]_{2L/3}^{L} - \int_{2L/3}^{L}\varphi_{2j}'(x)EI\varphi_{2i}'''(x)\mathrm{dx}$$

$$= -\varphi_{2j}(2L/3)EI\varphi_{2i}'''(2L/3) - [\varphi_{2j}'(x)EI\varphi_{2i}''(x)]_{2L/3}^{L} + \int_{2L/3}^{L}\varphi_{2j}''(x)EI\varphi_{2i}''(x)\mathrm{dx}$$

$$= -\varphi_{2j}(2L/3)EI\varphi_{2i}'''(2L/3) + \varphi_{2j}'(2L/3)EI\varphi_{2i}''(2L/3) + \int_{2L/3}^{L}\varphi_{2j}''(x)EI\varphi_{2i}''(x)\mathrm{dx}\ (2)$$

Since the slope and moment are continuous at 2L/3, (1)+(2),

$$\omega_i^2\int_0^{2L/3}\varphi_{1j}(x)m\varphi_{1i}(x)\mathrm{dx} + \omega_i^2\int_{2L/3}^{L}\varphi_{2j}(x)m\varphi_{2i}(x)\mathrm{dx} = \varphi_{1j}(2L/3)EI\varphi_{1i}'''(2L/3) - \varphi_{2j}(2L/3)EI\varphi_{2i}'''(2L/3)$$

$$+\int_0^{2L/3}\varphi_{1j}''(x)EI\varphi_{1i}''(x)\mathrm{dx} + \int_{2L/3}^{L}\varphi_{2j}''(x)EI\varphi_{2i}''(x)\mathrm{dx}\ \ (3)$$

Switch i and j,

$$\omega_j^2\int_0^{2L/3}\varphi_{1j}(x)m\varphi_{1i}(x)\mathrm{dx} + \omega_j^2\int_{2L/3}^{L}\varphi_{2j}(x)m\varphi_{2i}(x)\mathrm{dx} = \varphi_{1i}(2L/3)EI\varphi_{1j}'''(2L/3) - \varphi_{2i}(2L/3)EI\varphi_{2j}'''(2L/3)$$

$$+\int_0^{2L/3}\varphi_{1j}''(x)EI\varphi_{1i}''(x)\mathrm{dx} + \int_{2L/3}^{L}\varphi_{2j}''(x)EI\varphi_{2i}''(x)\mathrm{dx}\ \ (4)$$

(3)-(4),

$$(\omega_i^2-\omega_j^2)\int_0^{2L/3}\varphi_{1j}(x)m\varphi_{1i}(x)\mathrm{dx}+(\omega_i^2-\omega_j^2)\int_{2L/3}^{L}\varphi_{2j}(x)m\varphi_{2i}(x)\mathrm{dx}=\varphi_{1j}(2L/3)EI\varphi_{1i}'''(2L/3)-\varphi_{2j}(2L/3)EI\varphi_{2i}'''(2L/3)$$

$$-\varphi_{1i}(2L/3)EI\varphi_{1j}'''(2L/3)+\varphi_{2i}(2L/3)EI\varphi_{2j}'''(2L/3)$$

Using $[EI\varphi_2''(2L/3)]'-[EI\varphi_1''(2L/3)]'=-k[\varphi(2L/3)-z_1]=-kq(\omega)\varphi(2L/3)$

$$\varphi_{2i}(2L/3)EI\varphi_{2j}'''(2L/3)-\varphi_{2j}(2L/3)EI\varphi_{2i}'''(2L/3)=\varphi_{2i}(2L/3)EI\varphi_{1j}'''(2L/3)-kq(\omega_j)\varphi_{2i}(2L/3)\varphi_{2j}(2L/3)$$

$$-\varphi_{2j}(2L/3)EI\varphi_{1i}'''(2L/3)+\varphi_{2j}(2L/3)kq(\omega_i)\varphi_{1i}(2L/3)$$

$$RHS=\varphi_{2j}(2L/3)k[q(\omega_i)-q(\omega_j)]\varphi_{1i}(2L/3)$$

Let $\omega_0^2=k/M$,

$$s=q(\omega_i)-q(\omega_j)=-\frac{\omega_0^2(\omega_i^2-\omega_j^2)(-\omega_0^2(\omega_i^2+\omega_j^2)+\omega_i^2\omega_j^2+2\omega_0^4)}{(-3\omega_0^2\omega_i^2+\omega_i^4+\omega_0^4)(-3\omega_0^2\omega_j^2+\omega_j^4+\omega_0^4)}.$$

So $(\omega_i^2-\omega_j^2)\left[\int_0^{2L/3}\varphi_{1j}(x)m\varphi_{1i}(x)\mathrm{dx}+\int_{2L/3}^{L}\varphi_{2j}(x)m\varphi_{2i}(x)\mathrm{dx}\right]$

$=-\dfrac{\omega_0^2(\omega_i^2-\omega_j^2)(-\omega_0^2(\omega_i^2+\omega_j^2)+\omega_i^2\omega_j^2+2\omega_0^4)}{(-3\omega_0^2\omega_i^2+\omega_i^4+\omega_0^4)(-3\omega_0^2\omega_j^2+\omega_j^4+\omega_0^4)}\varphi_{2j}(2L/3)k\varphi_{1i}(2L/3)$,plug in $\omega_0^2=k/M$, then let

$$\int_0^{2L/3}\varphi_{1j}(x)m\varphi_{1i}(x)\mathrm{dx}+\int_{2L/3}^{L}\varphi_{2j}(x)m\varphi_{2i}(x)\mathrm{dx}$$

$$+\frac{\omega_0^4(-\omega_0^2(\omega_i^2+\omega_j^2)+\omega_i^2\omega_j^2+2\omega_0^4)}{(-3\omega_0^2\omega_i^2+\omega_i^4+\omega_0^4)(-3\omega_0^2\omega_j^2+\omega_j^4+\omega_0^4)}\varphi_{2j}(2L/3)M\varphi_{1i}(2L/3)=\delta_{ij}$$

This is mass orthogonality. It is symmetric because $\varphi_{1i}(2L/3)=\varphi_{1j}(2L/3)$.

Substitute in

$$\omega_i^2\int_0^{2L/3}\varphi_{1j}(x)m\varphi_{1i}(x)\mathrm{dx}=\varphi_{1j}(2L/3)EI\varphi_{1i}'''(2L/3)-\varphi_{1j}'(2L/3)EI\varphi_{1i}''(2L/3)+\int_0^{2L/3}\varphi_{1j}''(x)EI\varphi_{1i}''(x)\mathrm{dx}$$

$$\omega_i^2\int_{2L/3}^{L}\varphi_{2j}(x)m\varphi_{2i}(x)\mathrm{dx}=-\varphi_{2j}(2L/3)EI\varphi_{2i}'''(2L/3)+\varphi_{2j}'(2L/3)EI\varphi_{2i}''(2L/3)+\int_{2L/3}^{L}\varphi_{2j}''(x)EI\varphi_{2i}''(x)\mathrm{dx}$$

To get

$$\int_0^{2L/3}\varphi_{1j}''(x)EI\varphi_{1i}''(x)\mathrm{dx}+\int_{2L/3}^{L}\varphi_{2j}''(x)EI\varphi_{2i}''(x)\mathrm{dx}-\varphi_{2j}(2L/3)EI\varphi_{2i}'''(2L/3)+\varphi_{2j}'(2L/3)EI\varphi_{2i}''(2L/3)$$

$$+\int_0^{2L/3}\varphi_{1j}''(x)EI\varphi_{1i}''(x)\mathrm{dx}+\varphi_{1j}(2L/3)EI\varphi_{1i}'''(2L/3)-\varphi_{1j}'(2L/3)EI\varphi_{1i}''(2L/3)$$

$$=\omega_i^2\delta_{ij}-\frac{\omega_i^2\omega_0^2(-\omega_0^2(\omega_i^2+\omega_j^2)+\omega_i^2\omega_j^2+2\omega_0^4)}{(-3\omega_0^2\omega_i^2+\omega_i^4+\omega_0^4)(-3\omega_0^2\omega_j^2+\omega_j^4+\omega_0^4)}\varphi_{2j}(2L/3)k\varphi_{1i}(2L/3)$$

This is stiffness orthogonality.

To reduce the dynamics to modal oscillators, suppose the solution is in the form $u_1(x,t)=\sum_{i=1}^{\infty}\eta_i(t)\varphi_{1i}(x), u_2(x,t)=\sum_{i=1}^{\infty}\eta_i(t)\varphi_{2i}(x)$, substitute into the governing equations of the two parts, multiply by $\varphi_{1j},\varphi_{2j}$, respectively, and integrate from their appropriate domains, [0,2L/3] for φ_1, [2L/3,L] for φ_2.

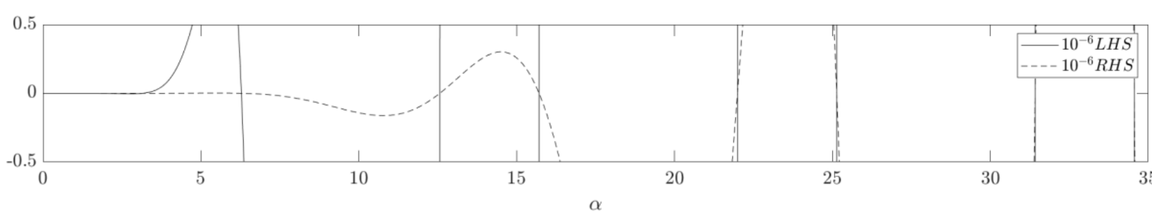
$$\int_0^{2L/3}\sum_{i=1}^{\infty}\left(\ddot{\eta}_i(t)+\omega_i^2\eta_i(t)\right)\varphi_{1j}m\varphi_{1i}\mathrm{dx}=0,\int_{2L/3}^{L}\sum_{i=1}^{\infty}\left(\ddot{\eta}_i(t)+\omega_i^2\eta_i(t)\right)\varphi_{2j}m\varphi_{2i}\mathrm{dx}=0$$

We add the two equations together, apply mass orthogonality condition and get closed ODEs of modal oscillators. For initial conditions $u_1(x,0)=g_1(x), u_2(x,0)=g_2(x), \frac{\partial}{\partial t}u_1(x,0)=h_1(x), \frac{\partial}{\partial t}u_2(x,0)=h_2(x)$, multiply by $m\varphi_{1j}, m\varphi_{2j}$ respectively and integrate from [0,2L/3] for φ_1, [2L/3,L] for φ_2 and use mass-orthogonality to obtain closed algebraic equations to solve for the initial conditions of the modal oscillators.

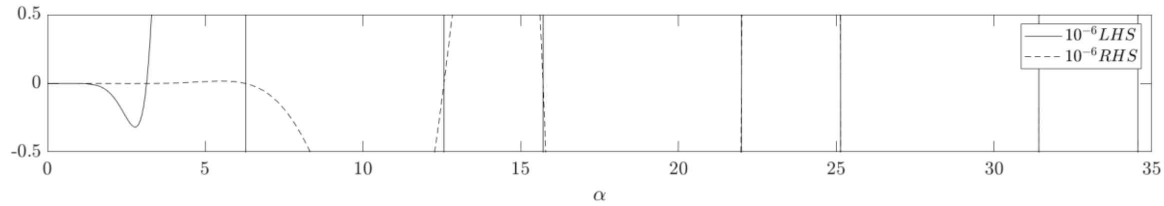
When $k\to 0,\eta\to\infty$, the first type of modes are not affected. For the second type, α will converge to $\alpha=n\pi$, where n is not multiples of 3.

The system then degenerates to a simply supported beam uncoupled from any attachment.

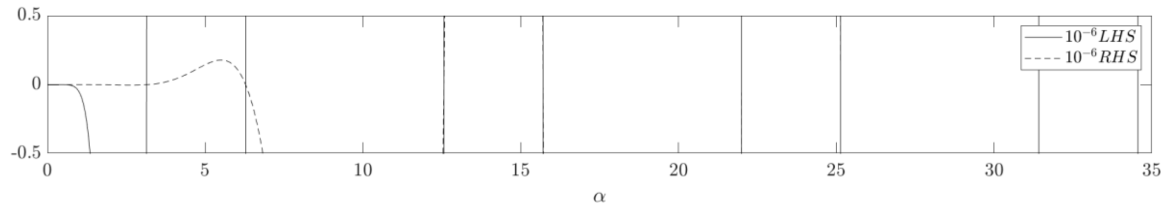
$\eta=1$



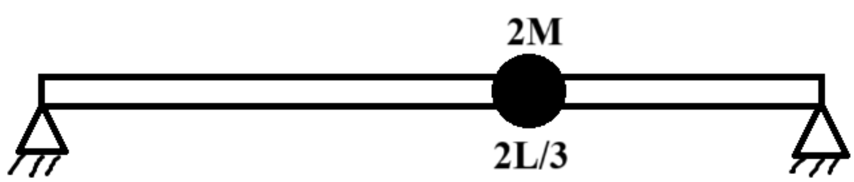
$\eta=10$



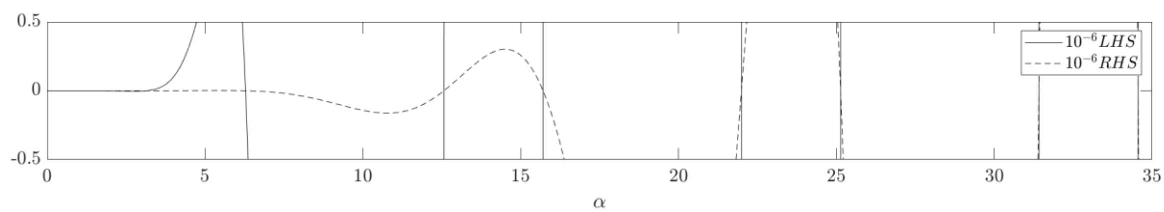
$\eta = 100$



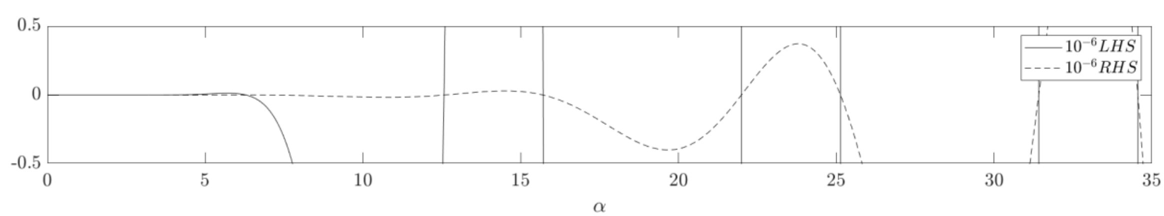
When $k \to \infty, \eta \to 0$, the first type of modes are not affected. α of modes of the second type will converge to certain values as shown below. The whole system will converge to simply supported on the left and right end, at x=2L/3, the masses merge and being attached to the beam, depicted as follows:



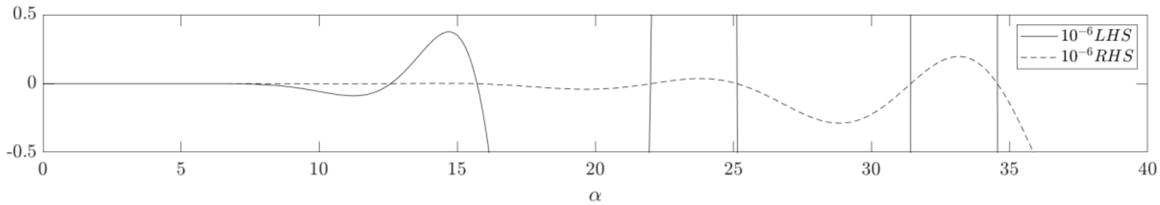
$\eta = 1$



$\eta = 0.1$



$\eta = 0.01$



Mathematica code:

```
r1={Sinh[λ*2L/3],Sin[λ*2*L/3],Sinh[λ*L/3],Sin[λ*L/3]};

r2={λ*Cosh[λ*2L/3],λ*Cos[λ*2L/3],-λ*Cosh[λ*L/3],-λ*Cos[λ*L/3]};

r3={λ^2*Sinh[λ*2L/3],-λ^2*Sin[λ*2L/3],λ^2*Sinh[λ*L/3],-λ^2*Sin[λ*L/3]};

r4={λ^3*E1*I1*Cosh[λ*2L/3]-k*q*Sinh[λ*2L/3],-λ^3*E1*I1*Cos[λ*2L/3]-
k*q*Sin[λ*2L/3],-λ^3*E1*I1*Cosh[λ*L/3],λ^3*E1*I1*Cos[λ*L/3]};

R={r1,r2,r3,r4}/.{q->(-2*k*M*ω^2+M^2*ω^4)/(k^2-3*k*M*ω^2+M^2*ω^4)};

s=Det[R]/.{ω->β^2*λ^2};

sq=FullSimplify[s]

sd2=FullSimplify[Numerator[sq]]

sd3=FullSimplify[sd2/(2 λ^6 )]

sd4=FullSimplify[sd3/.{m->M*ξ/L,λ->α/L,β->(k/M*L^4*η/ξ)^(1/4),E1->η*k*L^3/I1}]

sd5=FullSimplify[sd4*ξ^2/(k^3*L^3*η)]

sd6=FullSimplify[TrigExpand[(sd5/(Sin[α/3]*Sinh[α/3]))/.{α->3*q}]/.{q->α/3}];
```

Matlab code

```
alphas=linspace(0,40,1001);

xi=1;

eta=0.01;

lhs=alphas.*sin(2*alphas/3).*(2*xi-alphas.^4*eta)-
2*(2*cos(2*alphas/3)+1).*(alphas.^8*eta^2-3*alphas.^4*eta*xi+xi^2);

rhs=-alphas.*(2*cos(2*alphas/3)+1).*sinh(2*alphas/3).*(alphas.^4*eta-
2*xi)./(2*cosh(2*alphas/3)+1);

plot(alphas,1e-6*lhs,'k-',alphas,1e-6*rhs,'k--')

ylim([-0.5 0.5])
```

```
legend({'$10^{-6} LHS$','$10^{-6} RHS$'},'Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
xlabel('$\alpha$','Interpreter','latex')
```

Problem 4

RQ of the first system is:

$$R\{\tilde{\varphi}\}=\frac{\int_0^{L/2} EI\left(\tilde{\varphi}''(x)\right)^2 \mathrm{dx}+\int_{L/2}^{L} EI\left(\tilde{\varphi}''(x)\right)^2 \mathrm{dx}+k_T\tilde{\varphi}'(L/2)^2+k\tilde{\varphi}(L/2)^2}{\int_0^{L/2} m\tilde{\varphi}(x)^2\mathrm{dx}+\int_{L/2}^{L} m\tilde{\varphi}(x)^2\mathrm{dx}}$$

With BC $EI\tilde{\varphi}''(0)=EI\tilde{\varphi}''(L)=0$, $-[EI\tilde{\varphi}''(0)]'=-[EI\tilde{\varphi}''(L)]'=0$, no geometric boundary conditions.

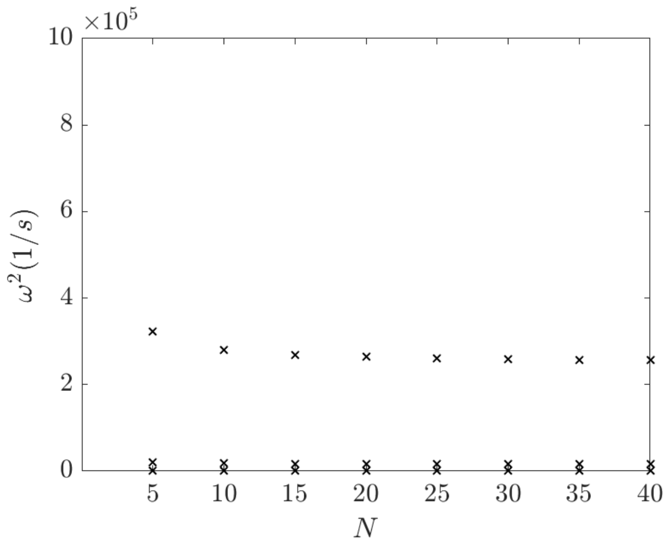
Use Rayleigh-Ritz.

Let $\varphi(x)=\sum_{i=1}^{N} a_i\psi_i(x), \psi_i(x)=\frac{x^2}{2}+\frac{L^2\cos(2n\pi x/L)}{4n^2\pi^2}$

$$K_{ij}=\int_0^{L/2} EI\psi_i''(x)\psi_j''(x)\mathrm{dx}+\int_{L/2}^{L} EI\psi_i''(x)\psi_j''(x)\mathrm{dx}+k_T\psi_i'(L/2)\psi_j'(L/2)+k\psi_i(L/2)\psi_j(L/2)$$

$$M_{ij}=\int_0^{L/2} m\psi_i(x)\psi_j(x)\mathrm{dx}+\int_{L/2}^{L} m\psi_i(x)\psi_j(x)\mathrm{dx}$$

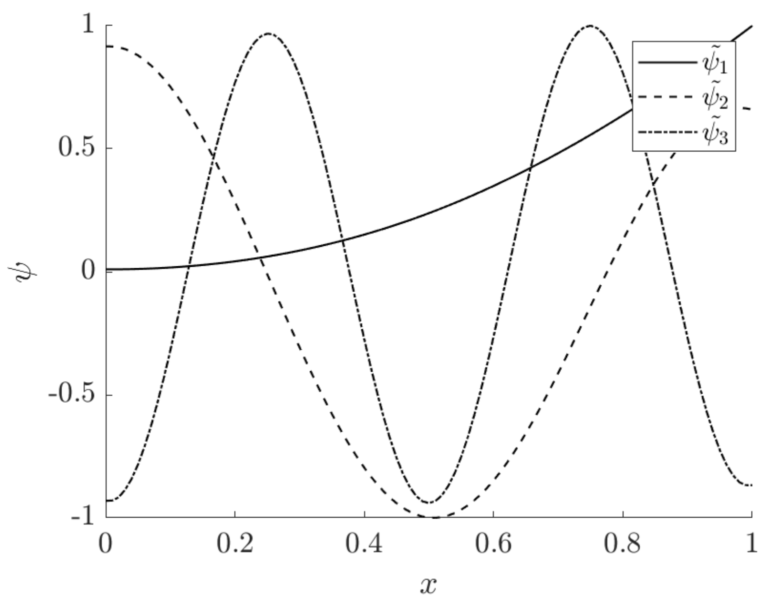
Choose $L=1m, E=10^5 Pa, I=10^{-4}m^4, m=1kg, k=100N/m, k_T=100Nm$



The first three modes are:

$\omega_1 = 27.03Hz, \omega_2 = 129.11Hz, \omega_3 = 506.48Hz$

Mode shapes are:



RQ of the second system is:

$$R\{\tilde{\varphi}\} = \frac{\int_0^a EI\tilde{\varphi}''(x)^2 \mathrm{dx} + \int_a^{L-b} EI\tilde{\varphi}''(x)^2 \mathrm{dx} + \int_{L-b}^{L} EI\tilde{\varphi}''(x)^2 \mathrm{dx} + k\tilde{\varphi}(a)^2 + k_T\tilde{\varphi}'(L-b)^2}{\int_0^a m\tilde{\varphi}(x)^2 \mathrm{dx} + \int_a^{L-b} m\tilde{\varphi}(x)^2 \mathrm{dx} + \int_{L-b}^{L} m\tilde{\varphi}(x)^2 \mathrm{dx} + M_1\tilde{\varphi}(a)^2 + M_2\tilde{\varphi}(L-b)^2}$$

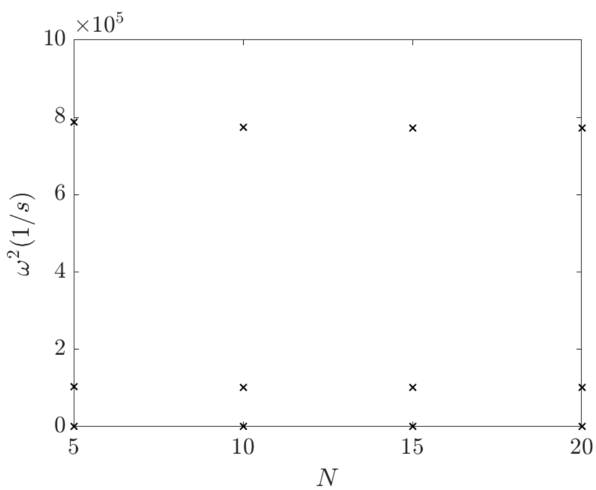
With BC $\tilde{\varphi}(0) = \tilde{\varphi}'(0) = \tilde{\varphi}(L) = \tilde{\varphi}'(L) = 0$.

Let $\varphi(x) = \sum_{i=1}^{N} a_i \psi_i(x), \psi_i(x) = 1 - \cos\frac{2n\pi x}{L}$,

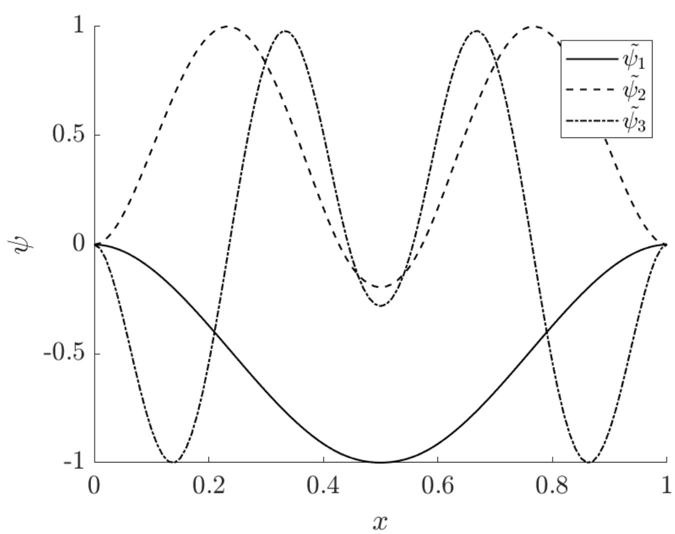
$$M_{ij} = \int_0^a m\psi_i(x)\psi_j(x)\mathrm{dx} + \int_a^{L-b} m\psi_i(x)\psi_j(x)\mathrm{dx} + \int_{L-b}^{L} m\psi_i(x)\psi_j(x)\mathrm{dx} + M_1\psi_i(a)\psi_j(a) + M_2\psi_i(L-b)\psi_j(L-b)$$

$$K_{ij} = \int_0^a EI\psi_i''(x)\psi_j''(x)\mathrm{dx} + \int_a^{L-b} EI\psi_i''(x)\psi_j''(x)\mathrm{dx} + \int_{L-b}^{L} EI\psi_i''(x)\psi_j''(x)\mathrm{dx} + k\psi_i(a)\psi_j(a) + k_T\psi_i'(L-b)\psi_j'(L-b)$$

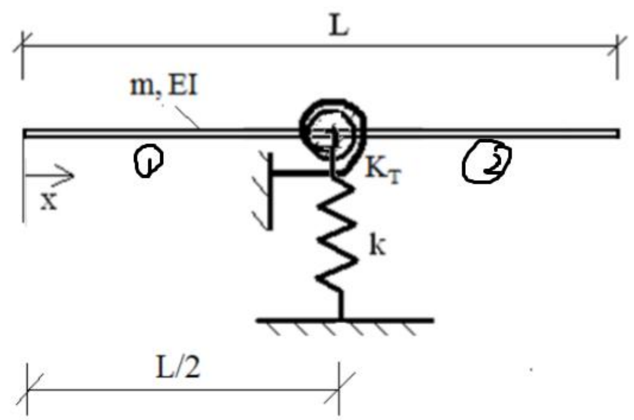
Choose
$L = 1m, E = 10^5 Pa, I = 10^{-4} m^4, k_T = 100Nm, k = 100N/m, m = 1kg, M_1 = 1kg, M_2 = 1kg, a = b = 0.45m$



Eigen modes are: $\omega_1 = 30.37Hz, \omega_2 = 319.52Hz, \omega_3 = 878.67Hz$, eigenfunctions are:



To know what conditions for system 1 will resemble a 2DOF system, we find the frequency equation.



Assume $u_1 = \varphi_1 f(t), u_2 = \varphi_2 f(t), \ddot{f} = -\omega^2 f$, $\beta^4 = EI/m$,

$-\omega^2 \varphi_1 + \beta^4 \varphi_1'''' = 0$, $-\omega^2 \varphi_2 + \beta^4 \varphi_2'''' = 0$

$[EI\varphi_2''(L/2)]' - [EI\varphi_1''(L/2)]' = -k\varphi_1(L/2)$

$EI\varphi_1''(L/2) + K_T \varphi_1'(L/2) = EI\varphi_2''(L/2)$

With BCs $EI\varphi_1''(0) = EI\varphi_2''(L) = 0$, $-[EI\varphi_1''(0)]' = -[EI\varphi_1''(L)]' = 0$

Assume $\varphi_1 = C_1 \sinh \lambda x + C_2 \cosh \lambda x + C_3 \sin \lambda x + C_4 \cos \lambda x$,
$\varphi_2 = C_5 \sinh \lambda (x-L) + C_6 \cosh \lambda (x-L) + C_7 \sin \lambda (x-L) + C_8 \cos \lambda (x-L)$

Where $\lambda = \sqrt{\omega}/\beta$.

$EI\varphi_1''(0) \Longrightarrow C_2 = C_4, -[EI\varphi_1''(0)]' = 0 \Longrightarrow C_1 = C_3$

$EI\varphi_1''(L) \Longrightarrow C_6 = C_8, -[EI\varphi_1''(L)]' = 0 \Longrightarrow C_5 = C_7$

Continuity of displacement $\varphi_1(L/2) = \varphi_2(L/2)$

Continuity of deflection $\varphi_1'(L/2) = \varphi_2'(L/2)$

Combine all equations, let $\xi = \dfrac{EI}{kL^3}, \eta = \dfrac{EI}{k_T L}, \alpha = L\lambda$, and set determinant to 0.

$$-4\alpha^2\left(2\alpha\eta\cos\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)+\cosh\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right)-2\alpha\eta\sin\left(\frac{\alpha}{2}\right)\right)+1\right)$$

$$\cdot\left(2\alpha^3\xi\cos\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)+\cosh\left(\frac{\alpha}{2}\right)\left(2\alpha^3\xi\sin\left(\frac{\alpha}{2}\right)-\cos\left(\frac{\alpha}{2}\right)\right)-1\right)=0$$

Because we want 2DOF, get rid of rigid body mode and we have

Case 1, $2\alpha\eta\cos\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)+\cosh\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right)-2\alpha\eta\sin\left(\frac{\alpha}{2}\right)\right)+1=0$

Case 2, $2\alpha^3\xi\cos\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)+\cosh\left(\frac{\alpha}{2}\right)\left(2\alpha^3\xi\sin\left(\frac{\alpha}{2}\right)-\cos\left(\frac{\alpha}{2}\right)\right)-1=0$

Case 1 do not have ξ, which means $x=L/2$ is a node. Case 2 does not have η, which means $x=L/2$ have 0 moment. Solve η,ξ back from the equation, and take the limit of $\alpha\to 0$ gives:

$$\eta=\frac{\cos\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right)+1}{2\alpha\left(\sin\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right)-\cos\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)\right)}\to\infty$$

$$\xi=\frac{\cos\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right)+1}{2\alpha^3\left(\cos\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\alpha}{2}\right)+\sin\left(\frac{\alpha}{2}\right)\cosh\left(\frac{\alpha}{2}\right)\right)}\to\infty$$

So when $EI\gg k$ or $EI\gg k_T$ one mode will reduce to a 2DOF system.

Matlab code

```
L=1;

psi=@(n,x) x.^2/2+L^2*cos(2*n*pi*x/L)/(4*n^2*pi^2);

psip=@(n,x) x-L*sin(2*n*pi*x/L)/(4*n*pi);

psipp=@(n,x) 1-cos(2*n*pi*x/L);

E=1e5;

I=1e-4;

kt=100;

k=100;

mass=1;
```

```
for N=5:5:40
    K=zeros(N,N);
    M=K;
    for m=1:N
        for n=1:N
            K(m,n)=integral(@(x)
E*I*psipp(m,x).*psipp(n,x),0,L)+kt*psip(m,1/2)*psip(n,L/2)+k*psi(m,L/2)*psi(n,L/2);
            M(m,n)=integral(@(x) mass*psi(m,x).*psi(n,x),0,L);
        end
    end
    [V,D]=eig(K,M);
    plot(N,diag(D),'kx','LineWidth',1);
    hold on;
end
xticks(5:5:40);
xlabel('$N$','Interpreter','latex')
ylabel('$\omega^2(1/s)$','Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
%%
[D,I]=sort(diag(D));
hold on;
xs=linspace(0,1,101);
ys1=0*xs;
ys2=ys1;
ys3=ys1;
for l=1:40
    ys1=ys1+psi(l,xs)*V(l,I(1));
```

```
    ys2=ys2+psi(l,xs)*V(l,I(2));
    ys3=ys3+psi(l,xs)*V(l,I(3));
end
plot(xs,ys1/max(abs(ys1)),'k-','LineWidth',1);
plot(xs,ys2/max(abs(ys2)),'k--','LineWidth',1);
plot(xs,ys3/max(abs(ys3)),'k-.','LineWidth',1);
legend({'$\tilde{\psi_1}$','$\tilde{\psi_2}$','$\tilde{\psi_3}$'},'Interpreter','latex')
xlabel('$x$','Interpreter','latex')
ylabel('$\psi$','Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');

L=1;
psi=@(n,x) 1-cos(2*n*pi*x/L);
psip=@(n,x) 2*n*pi*sin(2*n*pi*x/L)/L;
psipp=@(n,x) 4*n^2*pi^2*cos(2*n*pi*x/L)/(L^2);
E=1e5;
I=1e-4;
kt=100;
k=100;
mass=1;
M1=1;
M2=1;
a=0.45;
b=0.45;
for N=5:5:20
    K=zeros(N,N);
    M=K;
```

```
    for m=1:N
        for n=1:N
            K(m,n)=E*I*integral(@(x) psipp(m,x).*psipp(n,x),0,L,AbsTol=1e-4)+kt*psip(m,L-b)*psip(n,L-b)+k*psi(m,a)*psi(n,a);
            M(m,n)=mass*integral(@(x) psi(m,x).*psi(n,x),0,L,AbsTol=1e-4)+M1*psi(m,a)*psi(n,a)+M2*psi(m,L-b)*psi(n,L-b);
        end
    end
    [V,D]=eig(K,M);
    plot(N,diag(D),'kx','LineWidth',1);
    hold on;
end
xticks(5:5:20);
xlabel('$N$','Interpreter','latex')
ylabel('$\omega^2(1/s)$','Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
%%
[D,I]=sort(diag(D));
hold on;
xs=linspace(0,1,101);
ys1=0*xs;
ys2=ys1;
ys3=ys1;
for l=1:20
    ys1=ys1+psi(l,xs)*V(l,I(1));
    ys2=ys2+psi(l,xs)*V(l,I(2));
    ys3=ys3+psi(l,xs)*V(l,I(3));
end
```

```
plot(xs,ys1/max(abs(ys1)),'k-','LineWidth',1);

plot(xs,ys2/max(abs(ys2)),'k--','LineWidth',1);

plot(xs,ys3/max(abs(ys3)),'k-.','LineWidth',1);

legend({'$\tilde{\psi_1}$','$\tilde{\psi_2}$','$\tilde{\psi_3}$'},'Interpreter','latex')

xlabel('$x$','Interpreter','latex')

ylabel('$\psi$','Interpreter','latex')

set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

Mathematica code:

```
ϕ1=C1*Sinh[λ*x]+C2*Cosh[λ*x]+C3*Sin[λ*x]+C4*Cos[λ*x];

ϕ2=C5*Sinh[λ*(x-L)]+C6*Cosh[λ*(x-L)]+C7*Sin[λ*(x-L)]+C8*Cos[λ*(x-L)];

q1=(ϕ1/.{x->L/2})-(ϕ2/.{x->L/2})/.{C3->C1,C4->C2,C7->C5,C8->C6};

q2=(D[ϕ1,x]/.{x->L/2})-(D[ϕ2,x]/.{x->L/2})/.{C3->C1,C4->C2,C7->C5,C8->C6};

q3=(E1*I1*D[ϕ1,{x,2}]/.{x->L/2})+kT*(D[ϕ1,x]/.{x->L/2})-
(E1*I1*D[ϕ2,{x,2}]/.{x->L/2})/.{C3->C1,C4->C2,C7->C5,C8->C6};

q4=-
(E1*I1*D[ϕ1,{x,3}]/.{x->L/2})+k*(ϕ1/.{x->L/2})+(E1*I1*D[ϕ2,{x,3}]/.{x->L/2})/.{C3-
>C1,C4->C2,C7->C5,C8->C6};

R1={Coefficient[q1,C1],Coefficient[q1,C2],Coefficient[q1,C5],Coefficient[q1,C6]};

R2={Coefficient[q2,C1],Coefficient[q2,C2],Coefficient[q2,C5],Coefficient[q2,C6]};

R3={Coefficient[q3,C1],Coefficient[q3,C2],Coefficient[q3,C5],Coefficient[q3,C6]};

R4={Coefficient[q4,C1],Coefficient[q4,C2],Coefficient[q4,C5],Coefficient[q4,C6]};

R={R1,R2,R3,R4};

sd1=Det[R];

sd2=(sd1/.{k->E1*I1/(ξ*L^3),kT->E1*I1/(η*L),λ->α/L})/(E1*I1)^2;

sd3=Simplify[-sd2*(L^6 η ξ)]

-4 α^2 (1+Cosh[α/2] (Cos[α/2]-2 α η Sin[α/2])+2 α η Cos[α/2] Sinh[α/2]) (-1+Cosh[α/2] (-
Cos[α/2]+2 α^3 ξ Sin[α/2])+2 α^3 ξ Cos[α/2] Sinh[α/2])
```

```
Solve[1+Cosh[α/2] (Cos[α/2]-2 α η Sin[α/2])+2 α η Cos[α/2] Sinh[α/2]==0,η]

{{η->(1+Cos[α/2] Cosh[α/2])/(2 α (Cosh[α/2] Sin[α/2]-Cos[α/2] Sinh[α/2]))}}

Limit[(1+Cos[α/2] Cosh[α/2])/(2 α (Cosh[α/2] Sin[α/2]-Cos[α/2] Sinh[α/2])),α->0]

∞

Solve[-1+Cosh[α/2] (-Cos[α/2]+2 α^3 ξ Sin[α/2])+2 α^3 ξ Cos[α/2] Sinh[α/2]==0,ξ]

{{ξ->(1+Cos[α/2] Cosh[α/2])/(2 α^3 (Cosh[α/2] Sin[α/2]+Cos[α/2] Sinh[α/2]))}}

Limit[(1+Cos[α/2] Cosh[α/2])/(2 α^3 (Cosh[α/2] Sin[α/2]+Cos[α/2] Sinh[α/2])),α->0]

∞
```