Notes for Problem 1

The following relations might be of use:

A. $J_0'(\xi) = -J_1(\xi) = -\left[\frac{1}{2}\xi \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\xi^2\right)^k}{k!\,\Gamma(k+2)}\right]$

where $\Gamma(x)$ is the gamma function (bounded for finite integer values), $\Gamma(k+2) = (k+1)!$

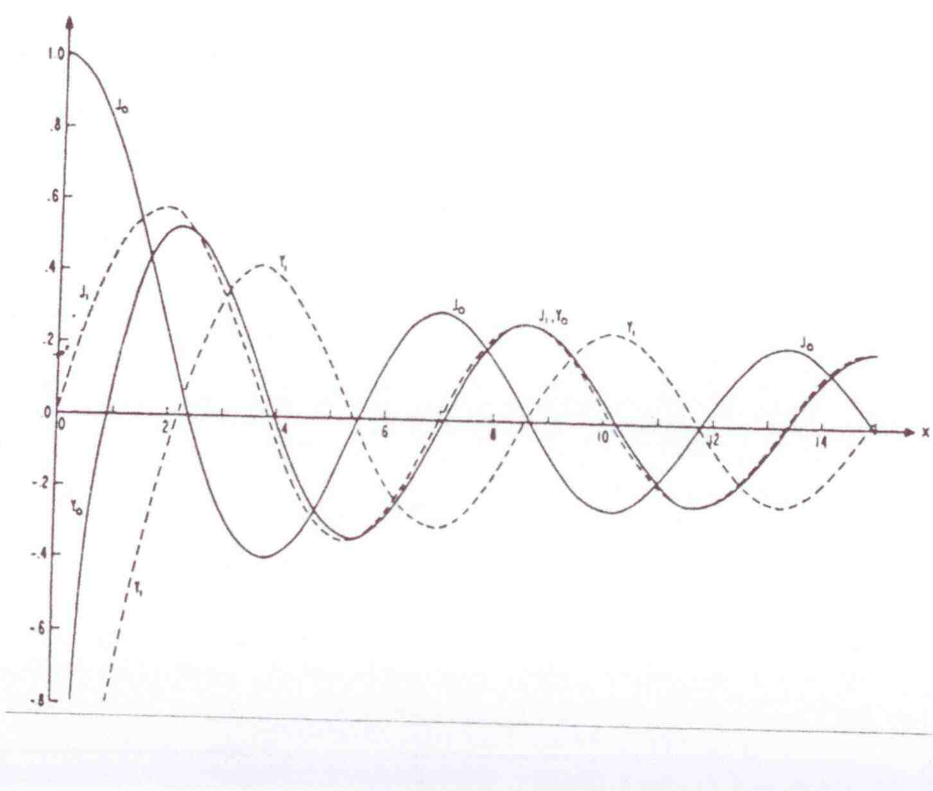
B. $Y_0'(\xi) = -Y_1(\xi) =$

$$= -\frac{2}{\xi\pi} + \frac{2}{\pi}\ln\left(\frac{1}{2}\xi\right) J_1(\xi) - \frac{\xi}{2\pi}\sum_{k=0}^{\infty}\left\{\psi(k+1)+\psi(k+2)\right\}\frac{\left(-\frac{1}{4}\xi^2\right)^k}{k!\,(1+k)!}$$

where $\psi(x)$ is the Psi (Digamma) function, $\psi(1) = -\gamma$, $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$, $n \geq 2$, $\gamma \cong 0.5772$ is Euler's constant

C. $\lim_{x\to 0} x^{\alpha}\ln x = 0$ if $Re(\alpha) > 0$

D. The Bessel functions $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$ are plotted below.



Problem 1

1. Balance of forces imparted to the infinitesimal element of length $\mathrm{d}x$ requires that

$$m\,\mathrm{d}x\frac{\partial^2 v(x,t)}{\partial t^2} = T(x+\mathrm{d}x)\frac{\partial v(x+\mathrm{d}x,t)}{\partial x} - T(x)\frac{\partial v(x,t)}{\partial x}.$$

Expanding in Taylor series around x leads to

$$\begin{aligned} m\,\mathrm{d}x\frac{\partial^2 v(x,t)}{\partial t^2} &= \left[T(x) + \frac{\partial T(x)}{\partial x}\mathrm{d}x\right]\left[\frac{\partial v(x,t)}{\partial x} + \frac{\partial^2 v(x,t)}{\partial x^2}\,\mathrm{d}x\right] - T(x)\frac{\partial v(x,t)}{\partial x} + \mathcal{O}(\mathrm{d}x^2) \\ &= \left[T(x)\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial T(x)}{\partial x}\frac{\partial v(x,t)}{\partial x}\right]\mathrm{d}x + \mathcal{O}(\mathrm{d}x^2). \end{aligned}$$

which gives

$$m\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x)\frac{\partial v(x,t)}{\partial x}\right].$$

Recognizing that $T(x) = \rho g x$, we end up with

$$m\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[\rho g x\frac{\partial v(x,t)}{\partial x}\right].$$

The fixed end at $x = L$ requires that $v(L,t) = 0$ and the free end at $x = 0$ that

$$\lim_{x\to 0}\rho g x\frac{\partial v(x,t)}{\partial x} = 0.$$

Because the quantity $\rho g x$ is exactly zero at $x = 0$, we require that $\partial v(x,t)/\partial x$ be bounded at that point.

2. To compute the mode shapes, we separate space and time according to $v(x,t) = \phi(x)f(t)$, which leads to the following two subproblems

$$\begin{cases} \ddot{f}(t) + \omega^2 f(t) &= 0 \\ \left[T(x)\phi'(x)\right]' + m\omega^2\phi(x) &= 0. \end{cases}$$

The spatial problem is solved by first introducing a new variable, $\xi = kx^n$. In particular, we have that

$$x = \left[\frac{\xi}{k}\right]^{\frac{1}{n}} \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}x} = n\xi\left[\frac{\xi}{k}\right]^{-\frac{1}{n}}\frac{\mathrm{d}}{\mathrm{d}\xi}.$$

Substituting back in the spatial equation and simplifying, we obtain

$$\xi\phi^{**} + \phi^* + \frac{\omega^2}{n^2 g k^{1/n}}\xi^{\frac{1}{n}-1}\phi = 0,$$

where we have introduced the notation $(\ )^* = \mathrm{d}(\ )/\mathrm{d}\xi$ and we have made use of the fact that $m = \rho$ (both denote the mass per unit length of the string). We now have to determine the values of k and n that bring the system into the form $\xi\phi^{**} + \phi^* + \xi\phi = 0$. This holds when $(1/n) - 1 = 1$ and $\omega^2/(n^2 g k^{1/n}) = 1$ or, equivalently,

$$n = \frac{1}{2} \quad\text{and}\quad k^2 = \frac{4\omega^2}{g}.$$

We express the solution ϕ using Bessel functions as $\phi(\xi) = AJ_0(\xi) + BY_0(\xi)$. The boundary condition at $x = 0$ becomes

$$\lim_{\xi\to 0}\rho g\xi\left[AJ_0^*(\xi) + BY_0^*(\xi)\right] = 0.$$

Note that, because $\lim_{\xi\to 0}\xi Y_0^*(\xi) = +\infty$, we must pick $B = 0$. This is consistent with our original assumption of boundedness for $\phi'(x)$. We turn to the fixed boundary condition at $x = L$, which is now given by

$AJ_0(k\sqrt{L}) = 0$. The constant A cannot be zero — we are looking for non-trivial solutions — and we are left with $J_0(k\sqrt{L}) = 0$. This equation entirely determines the natural frequencies ω_n, which are such that

$$J_0(2\omega_n\sqrt{L/g}) = 0.$$

The characteristic equation is in the form $J_0(\rho_n) = 0$, where $\rho_n = 2\omega_n\sqrt{L/g}$ are the roots of the Bessel function of the first kind. The associated mode shapes are given by

$$\phi_n(x) = A_n J_0(\rho_n\sqrt{x/L}),$$

some of which are shown in Figure 1 below.

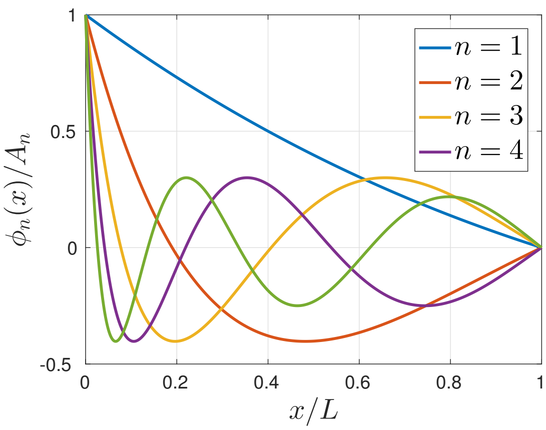


Figure 1: First four mode shapes.

Problem 2

The axial vibrations of a rod is governed by the following balance equation

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u(x,t)}{\partial x}\right] + F(x,t) = m\ddot{u}(x,t) \qquad \text{for} \qquad 0 \le x \le L$$

the boundary conditions the rod is subjected to are given as

$$EA\frac{\partial u}{\partial x}\bigg|_{x=0,t} = EA\frac{\partial u}{\partial x}\bigg|_{x=L,t} = 0$$

with arbitray initial conditions

$$u(x,0) = g(x), \qquad \dot{u}(x,0) = h(x)$$

The modes for the above system are obtain by disregarding the forcing term and instead solving a homogenous problem. The solution for the homogenous problem can be represented by the seperation of variables such that $u(x,t) = \phi(x)\eta(t)$. Furthermore, we consider constant E, A, m.Accordingly the above governing equation is reduced to the following form:

$$EA\phi''(x)\eta(t) = m\phi(x)\ddot{\eta}(t)$$

We can see from the above expression that

$$\begin{cases} \dfrac{\ddot{\eta}(t)}{\eta(t)} = -\omega^2 \\ \dfrac{EA\phi''(x)}{m\phi(x)} = -\omega^2 \end{cases}$$

This governing equation can thus be seperated to two equations in each variable space. Spatially we obtain $EA\phi'' + m\omega^2\phi = 0$ and temporally $m\ddot{\eta} + k\eta = 0$. First we consider the spatial problem:

$$EA\phi''(x) + m\omega^2\phi(x) = 0$$

This is equivalent to $\phi'' + \omega^2/c^2\phi = 0$, where c is the speed of sound $c = \sqrt{EA/m}$. This ordinary differential equation permits a solution of the form

$$\phi = a_1\cos(\omega x/c) + b_1\sin(\omega x/c)$$

considering the first derivative we obtain

$$\phi' = -a_1\omega/c\sin(\omega x/c) + b_1\omega/c\cos(\omega x/c)$$

we note that due to the seperation of variable at all times $\phi(x)$ has to satisfy the boundary conditions thus

$$\phi'(0) = b_1\omega/c\cos(\omega x/c) = 0 \quad \rightarrow \quad b_1 = 0$$

At the other end we have

$$\phi'(0) = -a_1\omega/c\sin(\omega/c) = 0$$

thus for non-trivial solution we require that $\sin(\omega/c) = 0$ and obtain the eigenfrequencies of the system as:

$$\omega_n = n\pi c \quad \text{, where } n = 0, 1, 2, \ldots$$

Here, we the rigid body mode induced by $n = 0$ would also satisfy the governing equation, due to the absence of any Dirichlet boundary conditions. Thus the spatial solution of the problem can be represented as $\phi_n(x) = a_{1n}\cos(n\pi x)$, we rely on the orthogonality condition to obtain the coefficient a_1 which states the following

$$\int_0^1 m\phi_n^2 dx = ma_{1n}^2 \int_0^1 \cos(n\pi x)^2 dx = 1$$

For the rigid body mode, $\phi_0(x) = a_{10}$. Since there is no dependancy on x we could intuitively reach the conclusion that the orthognality condition would result in $ma_{10}^2 = 1$ and a corresponding coefficient $a_{10} = \sqrt{1/m}$. For the other modes we can demonstrate through trignometric identities that the above integral is equivalent to

$$ma_{1n}^2 \int_0^1 \left(\cos(2n\pi x)/2 + 1\right) dx$$

$$ma_{1n}^2 \left[\frac{\sin(2n\pi x)}{4\pi n} + x/2\right]_0^1 = 1$$

$$ma_{1n}^2/2 = 1$$

The coefficients then becomes $a_{1n} = \sqrt{2/m} \quad \forall n > 0$, and $a_{10} = \sqrt{1/m}$. The total response of the system thus require the superposition of all modes and we obtain

$$u = \sum_{n=0}^{\infty} \phi_n(x)\eta_n(t)$$

We substitute back to the partial differential equation

$$EA\sum_{n=0}^{\infty} \phi_n''(x)\eta_n(t) + F(x,t) = m\sum_{n=0}^{\infty} \phi_n(x)\ddot{\eta}_n(t)$$

Utilizing the orthogonality conditions in Lecture 7, we can simplify the above term by multiplying with $\phi_s(x)$ and integrating over the domain length to obtain

$$EA\sum_{k=0}^{\infty} \int_0^1 \phi_n''(x)\phi_s(x)dx\eta_n(t) + \int_0^1 F\phi_s(x)dx = m\sum_{k=0}^{\infty} \int_0^1 \phi_n(x)\phi_s(x)dx\ddot{\eta}(t)$$

The orthogonality conditions state for simple boundary conditions are the following

$$\int_0^1 m\phi_i(x)\phi_j(x)dx = \delta_{ij}, \qquad \int_0^1 EA\phi_i''\phi_j dx = -\omega_i^2\delta_{ij}$$

The system thus reduce to the following

$$\ddot{\eta}_n(t) + \omega_n^2\eta_n(t) = \int_0^1 F(x,t)\phi_s(x)dx$$

We note that $F(x,t) = \sum_{i=1}^2 P^{(i)}(t)D^{(i)}(x)$, plug that in $\int_0^1 P^{(i)}(t)D^{(i)}(x)\phi_s(x)dx$, with $D^{(i)}(x) = \delta(x - x^{(i)})$. The dirac delta function within the integral will result in

$$N^{(i)}(t) = P^{(i)}(t)\phi_s(x^{(i)}) \rightarrow \ddot{\eta}_n(t) + \omega_n^2\eta_n(t) = \sum_{i=1}^2 P^{(i)}(t)\phi_n(x^{(i)})$$

Since $\phi_n(x^{(i)})$ is just a constant value $C_n^{(i)}$ that depends on the mode number we can simplify the expression above to

$$\ddot{\eta}_n(t) + \omega_n^2\eta_n(t) = \sum_{i=1}^2 P^{(i)}(t)\mathcal{C}_n^{(i)}$$

We realize that the $P^{(i)}(t) = 0 \quad \forall \quad t > \pi$ thus the system above can be decomposed into to ordinary differential equations with different limits one homogenous and the other being non-homoegenous

$$\ddot{\eta}_{1n}(t) + \omega_n^2\eta_{1n}(t) = \sum_{i=1}^2 P^{(i)}(t)\mathcal{C}_n^{(i)}, \qquad 0 \le t \le \pi$$

$$\ddot{\eta}_{2n}(t) + \omega_n^2\eta_{2n}(t) = 0, \qquad \pi < t$$

The initial conditions of this problem are $u(x,0) = g(x)$, and $\dot{u}(x,0) = h(x)$, we thus use the mapping of the orthogonal basis such that we start from $u(x,t) = \sum_{n=0}^{\infty}\phi_n(x)\eta_n(t)$ then multiply by $m\phi_s$ and integrate over x to obtain the inverse map and the following relation:

$$\int_0^1 u(x,t)m\phi_s(x)dx = \sum_{n=0}^{\infty}\int_0^1 m\phi_n(x)\phi_s(x)dx\eta_n(t)$$

The orthognolity condition then dicates that $\int_0^1 u(x,t)m\phi_s(x)dx = \eta_s(t)$. The initial conditions are then obtained by:

$$\int_0^1 mu(x,0)\phi_s(x)dx = \eta_s(0)$$

and we obtain the following expression for the initial conditions $\eta_{1n}(0) = \int_0^1 mg(x)\phi_n(x)dx$, and $\dot{\eta}_{1n}(0) = \int_0^1 mh(x)\phi_n(x)dx$. This gives the modal boundary conditions for the first problem. For the second system response we then enforce continuity of the displacements and velocity such that $\eta_{1n}(\pi^-) = \eta_{2n}(\pi^+)$ and $\dot{\eta}_{1n}(\pi^-) = \dot{\eta}_{2n}(\pi^+)$. The ordinary differential equations in time allow for a general solution of the form:

$$\eta_n = \eta_n(0)\cos(\omega_n t) + \dot{\eta}_n(0)/\omega_n \sin(\omega_n t) + 1/\omega_n \int_0^t p(\tau)\sin(t-\tau)d\tau$$

Lets consider the response in the duration $0 \le t \le \pi$

$$\eta_{1n} = \eta_{1n}(0)\cos(\omega_n t) + \dot{\eta}_{1n}(0)/\omega_n \sin(\omega_n t) + 1/\omega_n \int_0^t \sum_{i=1}^{2} P^{(i)}(\tau)\mathcal{C}_n^{(i)} \sin(t-\tau)d\tau$$

since $P^{(i)}(\tau) = A_i \sin(\tau)$, we can swap the constant $\mathcal{C}$ to another one $\mathcal{A}$ in which we incorporate the constant amplitude of the force.

$$\eta_{1n} = \eta_{1n}(0)\cos(\omega_n t) + \dot{\eta}_{1n}(0)/\omega_n \sin(\omega_n t) + 1/\omega_n \int_0^t \sum_{i=1}^{2} \mathcal{A}_n^{(i)} \sin(\tau)\sin(\omega_n(t-\tau))d\tau$$

the evaluation of the integral using [1] yields:

$$\eta_{1n} = \eta_{1n}(0)\cos(\omega_n t) + \dot{\eta}_{1n}(0)/\omega_n \sin(\omega_n t) + 1/\omega_n \sum_{i=1}^{2} \mathcal{A}_n^{(i)} \frac{\omega_n \sin(t) - \sin(\omega_n t)}{w_n^2 - 1}$$

from the above expression we can sum up the equivalent terms such that the modal response reduces to:

$$\boxed{\eta_{1n} = \eta_{1n}(0)\cos(\omega_n t) + \left[\dot{\eta}_{1n}(0) - \frac{\sum_{i=1}^{2} \mathcal{A}_n^{(i)}}{w_n^2 - 1}\right] \frac{\sin(\omega_n t)}{w_n} + \sum_{i=1}^{2} \mathcal{A}_n^{(i)} \frac{\sin(t)}{w_n^2 - 1}}$$

where, $\eta_{1n}(0) = \int_0^1 mg(x)\phi_n(x)dx$, and $\dot{\eta}_{1n}(0) = \int_0^1 mh(x)\phi_n(x)dx$. The expressions for $\mathcal{A}_n^{(i)}$ can also be obtained $\mathcal{A}_n^{(1)} = \mathcal{C}_n^{(1)}$, and $\mathcal{A}_n^{(2)} = 5\mathcal{C}_n^{(2)}$.

The above expression is for $n \ge 1$. The rigid body mode we need to consider the limiting case that $\omega_n \to 0$ using [2] we obtain the following expression for η_{10}:

$$\boxed{\eta_{10} = \eta_{10}(0) + \left[\dot{\eta}_{10}(0) + \sum_{i=1}^{2} \mathcal{A}_0^{(i)}\right] t - \sum_{i=1}^{2} \mathcal{A}_0^{(i)} \sin(t)}$$

where, $\eta_{10}(0) = \int_0^1 \sqrt{m}g(x)dx$, $\dot{\eta}_{1n}(0) = \int_0^1 \sqrt{m}h(x)dx$, and $\sum_{i=1}^{2} \mathcal{A}_0^{(i)} = 6/\sqrt{m}$

Lets consider the response in the duration $t > \pi$

The solution during this period is such that we satisfy the boundary conditions mentioned earlier and without any forced response, this yields:

$$\eta_{2n} = \eta_{2n}(\pi^+)\cos(\omega_n(t-\pi)) + \dot{\eta_{2n}}(\pi^+)/\omega_n \sin(\omega_n(t-\pi))$$

From continuity these correspond to the modal response of the earlier time history

$$\boxed{\eta_{2n} = \eta_{1n}(\pi)\cos(\omega_n(t-\pi)) + \dot{\eta_{1n}}(\pi)/\omega_n \sin(\omega_n(t-\pi))}$$

This expression is for $n \ge 1$. The rigid body mode is computed in a similar fashion and corresponds to

$$\boxed{\eta_{20} = \eta_{10}(\pi) + \dot{\eta_{10}}(\pi)(t-\pi)}$$

[1]
$$I = \int_0^t \sin(\tau)\sin(\omega_n(t-\tau))d\tau = \frac{\omega_n \sin(t) - \sin(\omega_n t)}{w_n^2 - 1}$$

[2]
$$\lim_{\omega_n \to 0} \frac{\sin(\omega_n t)}{\omega_n} = t$$

Problem 3

The transverse displacement of the string is governed by the classical wave equation

$$m\frac{\partial^2 v(x,t)}{\partial t^2} = T\frac{\partial^2 v(x,t)}{\partial x^2},$$

subject to boundary conditions

$$v(0,t) = 0 \quad \text{and} \quad \left[T\frac{\partial v(x,t)}{\partial x} + kv(x,t)\right]_{x=L} = 0.$$

Assuming an ansatz of the form $v(x,t) = \phi(x)f(t)$ and substituting in the equation of motion leads to

$$\begin{cases} \ddot{f}(t) + \omega^2 f(t) & = 0 \\ T\phi''(x) + m\omega^2\phi(x) & = 0. \end{cases}$$

The modes are obtained by solving the spatial problem

$$T\phi''(x) + m\omega^2\phi(x) = 0 \quad \text{subject to} \quad \begin{cases} \phi(0) = 0 \\ T\phi'(L) + k\phi(L) = 0. \end{cases}$$

We introduce the speed of sound $c = \sqrt{T/m}$ and write the solution as $\phi(x) = c_1\cos(\omega x/c) + c_2\sin(\omega x/c)$. The boundary conditions demand that

$$\begin{cases} c_1 = 0 \\ \tan(wL/c) = -\omega T/(ck). \end{cases}$$

The characteristic equation is in the form $\tan(z) = \alpha z$ and has a countable infinity of roots (ω_i, $i \in \mathbb{N}^+$) which may be determined numerically or graphically. To each of these natural frequencies corresponds an eigenfunction, $\phi_i(x) = b_i\sin(\omega_i x/c)$. Mass-normalization of the modes requires that

$$\int_0^L m\phi_i(x)^2 \mathrm{d}x = 1 \implies b_i = \left\{m\left[\frac{L}{2} - \frac{\sin(2\omega_i L/c)}{4\omega_i/c}\right]\right\}^{-1/2}.$$

Limiting cases:

- When $k \to 0$, the characteristic equation becomes $\tan(\omega L/c) \to \infty$, which has solutions $\omega_k \to k\pi c/(2L)$ ($k \in \mathbb{N}^+$), and the mode shapes therefore approach $\phi_k(x) = b_k\sin[k\pi x/(2L)]$, which are the mode shapes for the fixed-free system. This can be easily seen by solving the problem with boundary conditions $\phi(0) = 0$ and $T\phi'(L) = 0$.

- When $k \to \infty$, the characteristic equation becomes $\tan(\omega L/c) \to 0$, which has solutions $\omega_k \to k\pi c/L$ ($k \in \mathbb{N}^+$), and the mode shapes therefore approach $\phi_k(x) = b_k\sin(k\pi x/L)$, which are the mode shapes for the fixed-fixed system. This can be easily seen by solving the problem with boundary conditions $\phi(0) = 0$ and $\phi(L) = 0$.

Problem 4

Consider the problem

$$\begin{cases} \ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = f(t) \\ x(0) = X,\ \dot{x}(0) = V, \end{cases}$$

where $p(t)$, $q(t)$ and $f(t)$ are continuous on $[0, \infty[$. Consider first the homogeneous problem $\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = 0$ and assume that a non-zero solution $x_1(t)$ of this problem is known. We know that there must exist a second linearly independent homogenous solution, $x_2(t)$, and we show that this solution can be constructed from the original homogeneous solution $x_1(t)$. To do so, assume that $x_2(t)$ can be written in the form $x_2(t) = v(t)x_1(t)$, where $v(t)$ is a continuously differentiable function. Substitute in the homogeneous ODE and find

$$\ddot{v}x_1 + (2\dot{x}_1 + px_1)\dot{v} = 0 \qquad \Longrightarrow \qquad \dot{v} = C\exp\left\{-\int\left[p + 2\frac{\dot{x}_1}{x_1}\right]\mathrm{d}t\right\} = C\frac{\exp\left\{-\int p(t)\mathrm{d}t\right\}}{x_1^2(t)}$$

Absorbing the constant C in $v(t)$, we obtain the expression for the second fundamental solution $x_2(t)$,

$$x_2(t) = x_1(t)\int\frac{\exp\left\{-\int p(t)\mathrm{d}t\right\}}{x_1^2(t)}\mathrm{d}t.$$

Any solution $x_\mathrm{h}(t)$ of the homogenous equation can be written as a linear combination of the two fundamental solution, that is, $x_\mathrm{h}(t) = C_1x_1(t) + C_2x_2(t)$. The particular solution $x_\mathrm{p}(t)$ can be obtained by the method of variation of parameters. We seek a solution of the form $x_\mathrm{p}(t) = A(t)x_1 + B(t)x_2$, where $A(t)$ and $B(t)$ are unknown. Substituting in the original inhomogeneous ODE and enforcing $\dot{A}x_1 + \dot{B}x_2 = 0$, we obtain

$$\begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}\begin{bmatrix} \dot{A} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

which can be inverted as follows:

$$\begin{bmatrix} \dot{A} \\ \dot{B} \end{bmatrix} = \frac{1}{W}\begin{bmatrix} \dot{x}_2 & -x_2 \\ -\dot{x}_1 & x_1 \end{bmatrix}\begin{bmatrix} 0 \\ f \end{bmatrix},$$

where W is the Wronskian of x_1 and x_2 (W is never zero since x_1 and x_2 are linearly independent). This immediately gives

$$A(t) = -\int\frac{1}{W}x_2(t)f(t)\mathrm{d}t, \quad B(t) = \int\frac{1}{W}x_1(t)f(t)\mathrm{d}t.$$

The solution to the inhomogeneous equation is given by $x(t) = x_\mathrm{h}(t) + x_\mathrm{p}(t)$, *i.e.*,

$$x(t) = [C_1 + A(t)]x_1(t) + [C_2 + B(t)]x_2(t),$$

where C_1 and C_2 are chosen so as to satisfy the initial conditions. In particular, because $\dot{A}x_1 + \dot{B}x_2 = 0$ for all t, we have that

$$\begin{bmatrix} x_1(0) & x_2(0) \\ \dot{x}_1(0) & \dot{x}_2(0) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} X - A(0)x_1(0) - B(0)x_2(0) \\ V - A(0)\dot{x}_1(0) - B(0)\dot{x}_2(0) \end{bmatrix},$$

from which we get C_1 and C_2.

Problem 5

Compute the normal modes and derive orthonormality conditions

The axial motions in the rod is governed by:

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right] = m\frac{\partial^2 u}{\partial t^2}$$

for a uniform rod the equation simplifies to:

$$EA\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2}$$

The boundary conditions on this rod are:

$$EA\frac{\partial u}{\partial x}\bigg|_{x=0} = 0, \qquad EA\frac{\partial u}{\partial x}\bigg|_{x=L} = -F_s$$

here, F_s is the force due to the spring motions and can be expressed as $F_s = K(u(L,t) - y(t))$. The equation of motion for the mass is given as $M\ddot{y}(t) + Ky(t) = Ku(L,t)$. Lets first consider synchronous motions. This implies that each point in space behaves similarily in time such that $u(x,t) = \phi(x)\eta(t)$. Another implication is that mass M will move in time in a synchronous fashion such that $y(t) = \mathcal{Y}\eta(t)$, where, $\mathcal{Y}$ represent the transmition of motion from the end of the rod to the mass which depends on neither space or time.

With that in mind we can seperate the equation of motion for the rod into two seperate ODEs given as:

$$\begin{cases} \ddot{\eta}(t) + \omega^2\eta(t) = 0 \\ EA\phi''(x) + \omega^2 m\phi(x) = 0 \end{cases}$$

The spatial ODE $EA\phi''(x) + \omega^2 m\phi(x) = 0$ admits solutions of the form $\phi = a_1\cos(\omega x/c) + b_1\sin(\omega x/c)$. Imposing the boundary condition at $x = 0$ leads to $EA\frac{\partial u}{\partial x}\Big|_{x=0} = \eta(t)EA\frac{\partial \phi}{\partial x}\Big|_{x=0} = 0 \rightarrow b_1 = 0$. Next we explore the implications of the other boundary conditions namely:

$$\eta(t)EA\frac{\partial \phi}{\partial x}\bigg|_{x=L} = -F_s$$

$$-\eta(t)EA\frac{a_1\omega}{c}\sin(\omega L/c) = -F_s \quad \rightarrow \quad -\eta(t)EA\frac{a_1\omega}{c}\sin(\omega L/c) = -K(u(L,t) - y(t)) = -K(\phi(L)\eta(t) - \mathcal{Y}\eta(t))$$

$$\boxed{-\eta(t)EA\frac{a_1\omega}{c}\sin(\omega L/c) = -K(\phi(L)\eta(t) - \mathcal{Y}\eta(t)) \xrightarrow[\text{solutions}]{\text{Non-trivial}} -EA\frac{a_1\omega}{c}\sin(\omega L/c) = -K(\phi(L) - \mathcal{Y})}$$

The expression for $\mathcal{Y}$ can be derived as in[4] and we can then further simplify the boundary condition to the following:

$$-EA\frac{a_1\omega}{c}\sin(\omega L/c) = -K\phi(L)(1 - \mathcal{T})$$

[4] $M\ddot{y}(t) + Ky(t) = Ku(L,t)$ synchronous motions $y(t) = \mathcal{Y}\eta(t)$, and $\ddot{\eta}/\eta = -\omega^2$ implies that $-M\omega^2\mathcal{Y}\eta(t) + K\mathcal{Y}\eta(t) = K\phi(L)\eta(t)$ for non-trivial expressions this leads to:

$$-M\omega^2\mathcal{Y} + K\mathcal{Y} = K\phi(L) \quad \rightarrow \quad \mathcal{Y} = \frac{K\phi(L)}{K - M\omega^2} = \mathcal{T}\phi(L)$$

where, $\mathcal{T}$ represent the transmissibilty of the rod excitation to the mass

Finally to find the characteristic equation for the rod behavior we can plug in $\phi(L)$ and assume the non-trivial solution of $a_1 \neq 0$ leading to:

$$-EA\frac{a_1\omega}{c}\sin(\omega L/c) = -Ka_1\cos(\omega L/c)(1-\mathcal{T}) \rightarrow \tan(\omega L/c) = cK(1-\mathcal{T})/EA\omega$$

we can do one more simplification step and expand $(1-\mathcal{T})$ to arrive at a final form of the characterisitc equation:

$$\boxed{\tan(\omega L/c) = \frac{-Kc}{EA\omega}\left(\frac{M\omega^2}{K-M\omega^2}\right) = \frac{-Kc}{EA\omega}\left(\frac{\omega^2}{\Omega^2-\omega^2}\right), \quad \Omega^2 = K/M}$$

The roots of the characterisitic equation will provide a countable infinite solutions for ω_k, and the eigenfunctions can then be represented as:

$$\phi_k(x) = a_{1k}\cos(\omega_k x/c)$$

The coefficients a_{1k} are obtain throught the orthogonality conditions. Which will start deriving now.

<u>Orthogonality conditions:</u>

Prior to the the derivation I would like to introduce some definitions that will assist us in the derivation using inner product standard notation on vector space for continuous functions. First we define mass scaled inner product [5], "a" inner product [6], and "b" inner product [7]. Starting from the spatial ODE:

$$EA\phi'' + m\omega^2\phi = 0$$

The governing equation should be satisfied for all modes, thus we can rewrite it as:

$$EA\phi_k'' + m\omega_k^2\phi_k = 0$$

$$\int_x (\quad)\phi_s dx \Rightarrow EA\phi_k'' + m\omega_k^2\phi_k = 0$$

using the definitions above we can rewrite the equation as

$$b(\phi_k,\phi_s) + \omega_k^2(\phi_k,\phi_s) = 0 \xrightarrow[\text{by parts}]{\text{Integration}} EA\phi_k'\phi_s\Big|_{x=0}^{x=L} - a(\phi_k,\phi_s) + \omega_k^2(\phi_k,\phi_s) = 0$$

from the free boundary condition at $x=0$ we know that $EA\phi_k'(0) = 0$, we can reduce the above expression to the following for modes k:

$$EA\phi_k'(L)\phi_s(L) - a(\phi_k,\phi_s) + \omega_k^2(\phi_k,\phi_s) = 0$$

We first swap indicies k and s to obtain another equation for modes s

$$EA\phi_s'(L)\phi_k(L) - a(\phi_s,\phi_k) + \omega_s^2(\phi_s,\phi_k) = 0$$

5
$$(v,w) = \int_V mv(x)w(x)dx = (w,v) \rightarrow symmetric$$

6
$$a(v,w) = EA\int_V v'w'dx = a(w,v) \rightarrow symmetric$$

7
$$b(v,w) = EA\int_V v''wdx \neq b(w,v) \rightarrow non-symmetric$$

Noting the symmetry of the linear operators we can rewrite this equation as:

$$EA\phi_s'(L)\phi_k(L) - a(\phi_k, \phi_s) + \omega_s^2(\phi_k, \phi_s) = 0$$

subtracting the two equations for modes s and k we obtain

$$EA\phi_k'(L)\phi_s(L) - EA\phi_s'(L)\phi_k(L) + (\omega_k^2 - \omega_s^2)(\phi_k, \phi_s) = 0$$

we can use the boundary condition $EA\phi'(L) = -K\phi(L)(1-\mathcal{T})$, which in each mode is reduced to $EA\phi_k'(L) = K\phi_k(L)(\mathcal{T}_k - 1)$ and rewrite the above expression as:

$$K\phi_k(L)(\mathcal{T}_k - 1)\phi_s(L) - K\phi_s(L)(\mathcal{T}_s - 1)\phi_k(L) + (\omega_k^2 - \omega_s^2)(\phi_k, \phi_s) = 0$$

$$K\phi_k(L)\phi_s(L)\left[\mathcal{T}_k - \mathcal{T}_s\right] + (\omega_k^2 - \omega_s^2)(\phi_k, \phi_s) = 0$$

we can show [8] that $\mathcal{T}_k - \mathcal{T}_s = \dfrac{\Omega^2(\omega_k^2 - \omega_s^2)}{(\Omega^2 - \omega_k^2)(\Omega^2 - \omega_s^2)}$ and we can simplify the above expression such that we obtain the following equation:

$$(\omega_k^2 - \omega_s^2)\left[\phi_k(L)\phi_s(L)\frac{K\Omega^2}{(\Omega^2 - \omega_k^2)(\Omega^2 - \omega_s^2)} + (\phi_k, \phi_s)\right] = 0$$

Accordingly, we can arrive to the orthogonality conditions.

Lets consider $k \neq s$

$$\left[\phi_k(L)\phi_s(L)\frac{K\Omega^2}{(\Omega^2 - \omega_k^2)(\Omega^2 - \omega_s^2)} + (\phi_k, \phi_s)\right] = 0$$

Lets consider $k = s$, and mass-orthonormalize by enforcing:

$$\left[\phi_k(L)\phi_s(L)\frac{K\Omega^2}{(\Omega^2 - \omega_k^2)(\Omega^2 - \omega_s^2)} + (\phi_k, \phi_s)\right] = 1$$

in which we arrive to the mass-orthogonality condition which can be written as:

$$\left[\frac{K\Omega^2\phi_k(L)\phi_s(L)}{(\Omega^2 - \omega_k^2)(\Omega^2 - \omega_s^2)} + (\phi_k, \phi_s)\right] = \delta_{ks}$$

To obtain the stiffness orthogonality condition we use $EA\phi_s'(L)\phi_k(L) - a(\phi_k, \phi_s) + \omega_s^2(\phi_k, \phi_s) = 0$ and rewrite it as

$$K\phi_s(L)(\mathcal{T}_s - 1)\phi_k(L) + \omega_s^2(\phi_k, \phi_s) = a(\phi_k, \phi_s)$$

[8]

$$\mathcal{T}_k = \frac{K}{K - M\omega_k^2} = \frac{\Omega^2}{\Omega^2 - \omega_k^2}$$

$$\mathcal{T}_k - \mathcal{T}_k = \frac{\Omega^2}{\Omega^2 - \omega_k^2} - \frac{\Omega^2}{\Omega^2 - \omega_s^2} = \frac{\Omega^2(\omega_k^2 - \omega_s^2)}{(\Omega^2 - \omega_k^2)(\Omega^2 - \omega_s^2)}$$

$$K\phi_s(L)\left[\frac{\omega_s^2}{(\Omega^2-\omega_s^2)}\right]\phi_k(L)+\omega_s^2(\phi_k,\phi_s)=a(\phi_k,\phi_s)$$

$$K\phi_s(L)\left[\frac{\omega_s^2\Omega^2-\omega_s^2\omega_k^2}{(\Omega^2-\omega_s^2)(\Omega^2-\omega_k^2)}\right]\phi_k(L)+\omega_s^2(\phi_k,\phi_s)=a(\phi_k,\phi_s)$$

We can then observe that:

$$\omega_s^2\left[\frac{K\Omega^2\phi_s(L)\phi_k(L)}{(\Omega^2-\omega_s^2)(\Omega^2-\omega_k^2)}+(\phi_k,\phi_s)\right]=a(\phi_k,\phi_s)+K\phi_s(L)\left[\frac{\omega_s^2\omega_k^2}{(\Omega^2-\omega_s^2)(\Omega^2-\omega_k^2)}\right]\phi_k(L)$$

the LHS correspond to mass orthogonality thus we can write the stiffness orthogonality as:

$$a(\phi_k,\phi_s)+\frac{K\omega_s^2\omega_k^2\phi_s(L)\phi_k(L)}{(\Omega^2-\omega_s^2)(\Omega^2-\omega_k^2)}=\omega_s^2\delta_{sk}$$

Solve graphically and is it possible to have repeated natural frequencies

We go back to the expression we derived earlier

$$\boxed{\tan(\omega L/c)=\frac{-Kc}{EA\omega}\left(\frac{M\omega^2}{K-M\omega^2}\right)=\frac{-Kc}{EA\omega}\left(\frac{\omega^2}{\Omega^2-\omega^2}\right),\quad \Omega^2=K/M}$$

prior to graphical representation we need to manipulate this equation slightly such that on the RHS we obtain $\omega L/c$ term

$$\boxed{\tan(\omega L/c)=\frac{KL}{EA}\left(\frac{\alpha^2(\omega L/c)}{\alpha^2(\omega L/c)^2-\Omega^2}\right)=\frac{KL}{EA}\left(\frac{(\omega L/c)}{(\omega L/c)^2-(\Omega/\alpha)^2}\right)=\frac{\gamma(\omega L/c)}{(\omega L/c)^2-\gamma\beta}}$$

This equation would depend on the behavior of $\gamma = KL/EA$ representing the contributions of the different stiffnesses and $\beta = mL/M$ representing the contributions of the different masses. Fig.4 illustrates a sample of the results, showing that there are countable infinite solutions. There exist a possibility in which we can get very close frequencies in case the width of the function $\frac{\gamma(\omega L/c)}{(\omega L/c)^2-\gamma\beta}$ decrease near singularity location. However, not repeated ones. At resonance, when $\mathcal{T}\to\infty$ could represent a case when the eigenvalues are extremely close.

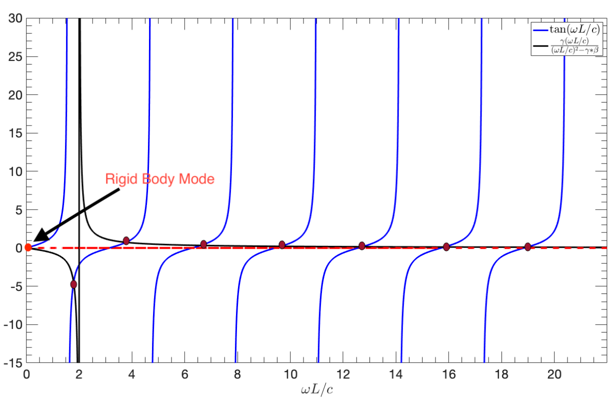


Figure 4: Roots of the characteristic equations.

Limiting cases

Lets consider $K \to \infty$:

Physically, this implies that the stiffness acts like a rigid link and displacement of the attached mass must be equal to the end point of the rod at $x = L$. This means that this case is similar to a rod with a mass boundary condition, note the emergence of a rigid body mode in this case.

$$\lim_{K\to\infty} \mathcal{T} = \lim_{K\to\infty} \frac{K}{K - M\omega^2} = 1$$

Thus, we obtain:

$$-EA\frac{\omega}{c}\sin(\omega L/c) = -M\omega^2\cos(\omega L/c)$$

This implies that $\gamma \to \infty$ with the roots presented in Fig.5 varing linearly in ω.

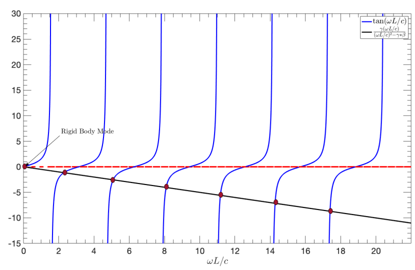


Figure 5: Roots of the characteristic equations as $K \to \infty$.

Lets consider $K \to 0$:

Physically, this implies that there is no coupling between the rod and the mass. This correspond to a free boundary condition. Thus, we obtain:

$$-EA\frac{\omega}{c}\sin(\omega L/c) = 0$$

This implies that $\gamma \to 0$ with the roots presented in Fig.7 i.e the roots correspond to $\sin(\omega L/c) = 0$ and presented by a horizontal line here, note the emergence of a rigid body mode in this case.

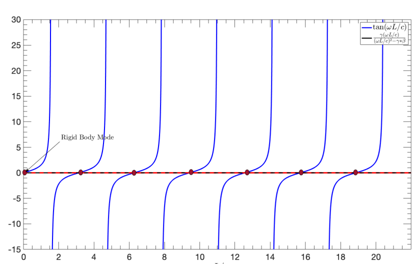


Figure 6: Roots of the characteristic equations as $K \to \infty$.

Lets consider $M \to \infty$:

Physically, this implies that mass acts as a fixed support. Then we recover a stiffness boundary condition at $x = L$. This

$$\lim_{M\to\infty} \mathcal{T} = \lim_{M\to\infty} \frac{K}{K - M\omega^2} = 0$$

Thus, we obtain:

$$-EA\frac{\omega}{c}\tan(\omega L/c) = K$$

This implies that $\beta \to 0$ with the roots presented in Fig.7 i.e the roots correspond to $\tan(\omega L/c) = \dfrac{\gamma}{(\omega L/c)}$ and presented by a decaying function of ω, the clamping meant that there is no rigid body modes.

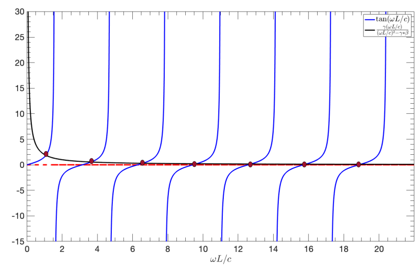


Figure 7: Roots of the characteristic equations as $M \to \infty$.

Discuss how you can use modal analysis

We started the problem with a decomposition of space and time, and we have identified using the eigenvalue problem that we have countable infinity solutions. we can use superposition to obtain the overall response

$$u(x,t) = \sum_{i=0}^{\infty} \phi_k(t)\eta_k(t)$$

similarly we know that

$$\ddot{\eta}_k(t) + \omega_k^2\eta_k(t) = 0$$

since the governing equations need to be satisfied for all modes. An alternative way to prove that is from the original problem, replace $u(x,t)$ with the summation, then use the orthogonality conditions to arrive to the same conclusion.

$$EA\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2} \to \sum_{k=0}^{\infty} \eta_k(t) EA\frac{\partial^2 \phi_k}{\partial x^2} = \sum_{k=0}^{\infty} \phi_k(x) m\frac{\partial^2 \eta}{\partial t^2}$$

Integrate and multiple by ϕ_s:

$$\sum_{k=0}^{\infty} \eta_k(t)\int_0^L EA\frac{\partial^2 \phi_k}{\partial x^2}\phi_s dx = \int_0^L \sum_{k=0}^{\infty} \phi_k(x) m\phi_s(x) dx\frac{\partial^2 \eta}{\partial t^2}$$

This leads to system posed ealier. A general solution can be found that satisfy each mode independently such that:

$$\eta_k(t) = \begin{cases} a_{20} + b_{20}t & k = 0 \\ a_{2k}\cos(\omega_k t) + b_{2k}\sin(\omega_k t) & k \geq 1 \end{cases}$$

The coefficients of the system a_{2k}, and b_{2k} can be obtained through the orthogonality conditions such that they satisfy $u(x,0) = U(x)$ and $\dot{u}(x,0) = V(x)$. Since we know the mapping from $u(x,0) \rightarrow \eta(0)$ through $\phi(x)$ we can then obtain the values of the coefficients.