Vibrations due to motions of the supports

Consider again $EA \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2}, \quad 0 \leq x \leq L, \quad t \geq 0 \qquad (1)$

Now, let's consider nonhomogeneous boundary conditions, i.e., conditions where time enters explicitly in the expressions,

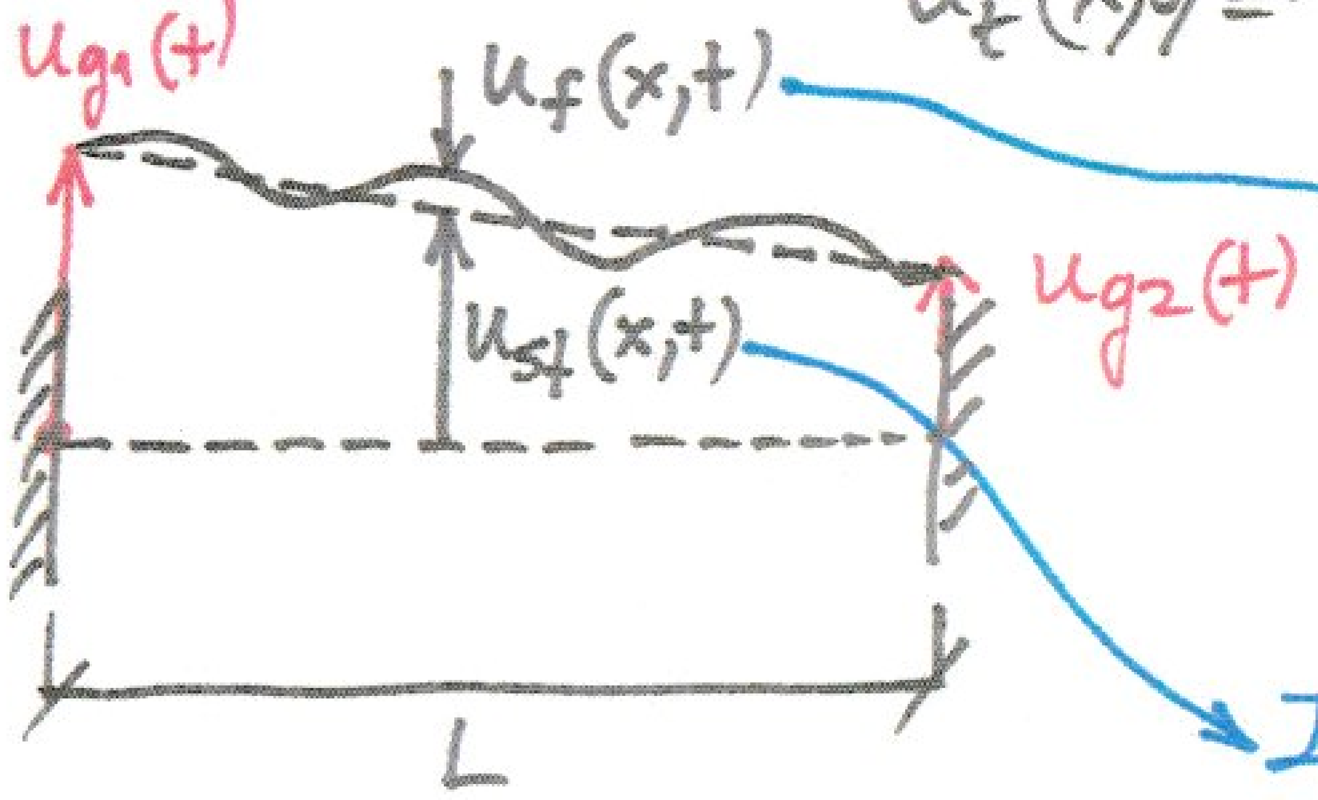
$$\left.\begin{aligned} u(0,t) &= u_{g1}(t) \\ u(L,t) &= u_{g2}(t) \end{aligned}\right\} \quad (1a)$$

subject to the initial conditions,

$$\left.\begin{aligned} u(x,0) &= g(x) \\ u_t(x,0) &= h(x) \end{aligned}\right\} \quad (1b)$$

Because of compatibility we require,

$$u(0,0) = g(0) = u_{g1}(0)$$
$$u(L,0) = g(L) = u_{g2}(0)$$
$$u_t(0,0) = h(0) = u_{g1}'(0)$$
$$u_t(L,0) = h(L) = u_{g2}'(0)$$
$$(\;)' \equiv d/dt, \quad (\;)_t \equiv \frac{\partial}{\partial t}$$



Now we take into account the inertia forces and compute the flexible deformation from the pseudostatic displacement ⇒ ⇒ 'Flexible' response $u_f(x,t)$

Is only due to the motion of the supports without taking into account the inertia forces of the string ⇒ 'Pseudo-static' response $u_{st}(x,t)$

Summarizing, we express the total response $u(x,t)$ as,

$$u(x,t) = u_{st}(x,t) + u_f(x,t) \qquad (2)$$

where, $u_{st}(0,t) = u_{g1}(t)$ and $u_{st}(L,t) = u_{g2}(t)$

$u_f(0,t) = 0$ and $u_f(L,t) = 0$

<u>Solving for $u_{st}(x,t)$</u>

Reconsider the wave equation, $EA\dfrac{\partial^2 u}{\partial x^2} = m\dfrac{\partial^2 u}{\partial t^2} \Rightarrow$ Neglecting the inertia forces (i.e., treating the string as non-flexible), $EA\dfrac{\partial^2 u_{st}}{\partial x^2} = 0 \Rightarrow$

$\Rightarrow u_{st}(x,t) = c_1(t)x + c_2(t)$ $\Rightarrow u_{g1}(t) = c_2(t)$

But $u_{st}(0,t) = u_{g1}(t)$

Also, $u_{st}(L,t) = u_{g2}(t) \Rightarrow u_{g2}(t) = c_1(t)L + u_{g1}(t) \Rightarrow$

$$\Rightarrow c_1(t) = \frac{u_{g2}(t) - u_{g1}(t)}{L}$$

$\Rightarrow$

$$\boxed{u_{st}(x,t) = u_{g1}(t)\left(1 - \frac{x}{L}\right) + u_{g2}(t)\frac{x}{L}}, \quad 0 \le x \le L$$

Nonhomogeneous BCs are fully accounted for in $u_{st}(x,t)$

Solving for $u_f(x,t)$

Substitute (2) into the governing equation (1) $\Rightarrow$

$$\underbrace{EA\frac{\partial^2 u_{st}}{\partial x^2}}_{0} + EA\frac{\partial^2 u_f}{\partial x^2} = m\frac{\partial^2 u_{st}}{\partial t^2} + m\frac{\partial^2 u_f}{\partial t^2} \Rightarrow$$

$$\Rightarrow EA\frac{\partial^2 u_f}{\partial x^2} - m\frac{\partial^2 u_f}{\partial t^2} = \underbrace{m\frac{\partial^2 u_{st}}{\partial t^2}}_{\text{known function of } x \text{ and } t} \equiv f(x,t) \quad (3)$$

$u_f(0,t)=0,\ u_f(L,t)=0 \Rightarrow$ (3a)

$\Rightarrow$ we get 'simple' boundary conditions!

$\Rightarrow$ We have transformed the problem for $u_f(x,t)$ to a forced standard problem with simple BCs.

Note on the initial conditions: $u(x,0)=g(x) \Rightarrow u_{st}(x,0)+u_f(x,0)=g(x) \Rightarrow$

$\Rightarrow u_f(x,0)=g(x)-u_{st}(x,0)$

Similarly we compute the other initial condition, $u_{f,t}(x,0)=h(x)-u_{st,t}(x,0)$ (3b)

Hence, (3), (3a) and (3b) form a standard problem that we can solve.