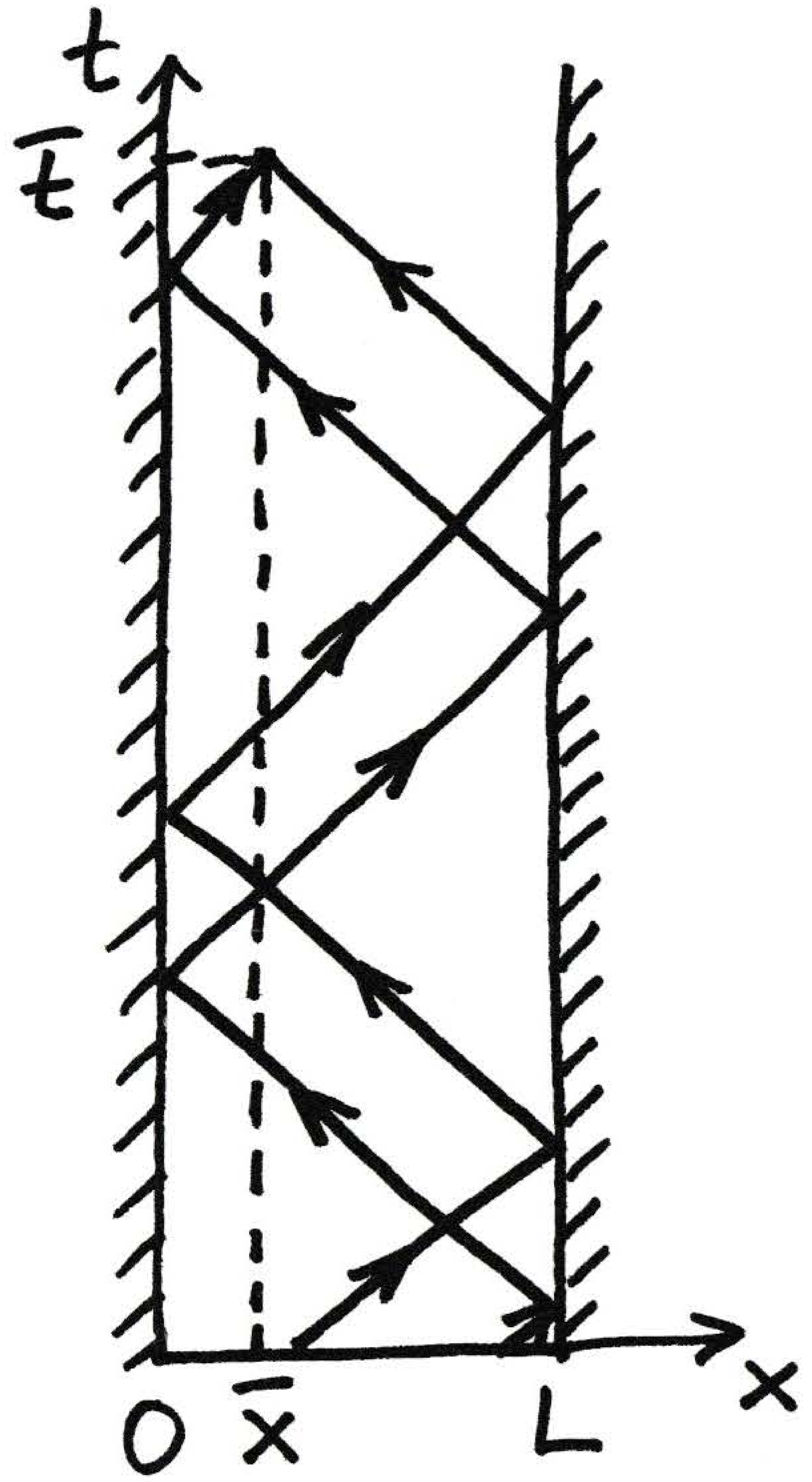
Similarly we work in domains where there are multiple "right" and "left" reflections from the boundaries. After a "large" number of reflections at the boundaries a steady state is reached $\Rightarrow$ Vibrations (or standing waves) are generated!



Relation between vibrations and waves

Consider again the initial value problem over a finite domain,

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L \equiv \pi \\ &u(0,t) = u(\pi,t) = 0 \\ &u(x,0) = U(x), \ \frac{\partial u}{\partial t}(x,0) = V(x) \end{aligned}\right\} \quad (13)$$

Computing the normal modes of this problem and using the expansion theorem we express the solution as,

Vibrations-based solution

$$\boxed{\begin{aligned} &u(x,t) = \sum_{k=1}^{\infty} (a_k \cos kct + b_k \sin kct) \sin kx \\ &a_k = \frac{2}{\pi}\int_0^{\pi} U(x) \sin kx \, dx, \quad b_k = \frac{2}{kc\pi}\int_0^{\pi} V(x) \sin kx \, dx \end{aligned}} \quad (14)$$

We wish to show that the same solution can be re-written using a waves-based methodology. To this end, we use the trigonometric identities,

$$\left.\begin{aligned} \cos\alpha \sin\beta &= \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right] \\ \sin\alpha \sin\beta &= \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \end{aligned}\right\} \quad (15)$$

Then, the solution (14) can be expressed as,

$$\left.\begin{aligned} u(x,t) = \frac{1}{2}\sum_{k=1}^{\infty}\left[a_k \sin k(x+ct) + a_k \sin k(x-ct)\right] + \\ + \frac{1}{2}\sum_{k=1}^{\infty}\left[b_k \cos k(x-ct) - b_k \cos k(x+ct)\right] \end{aligned}\right\} \Rightarrow$$

Note, however, that $u(x,0) = U(x) = \sum_{k=1}^{\infty} a_k \sin kx$

$$\Rightarrow u(x,t) = \frac{1}{2}\left[U(x+ct) + U(x-ct)\right] +$$

$$+ \frac{1}{2}\sum_{k=1}^{\infty}\left[b_k \cos k(x-ct) - b_k \cos k(x+ct)\right] \quad (16)$$

Now differentiate with respect to time the infinite summation in (16) $\Rightarrow$

$$\frac{\partial}{\partial t}\left\{ \frac{1}{2} \sum_{k=1}^{\infty} \left[b_k \cos k(x-ct) - b_k \cos k(x+ct) \right] \right\} =$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \left[b_k ck \sin k(x+ct) + b_k ck \sin k(x-ct) \right] =$$

$$= \frac{1}{2} \left[V(x+ct) + V(x-ct) \right] \quad \text{since} \quad V(x) = \sum_{k=1}^{\infty} b_k ck \sin kx$$

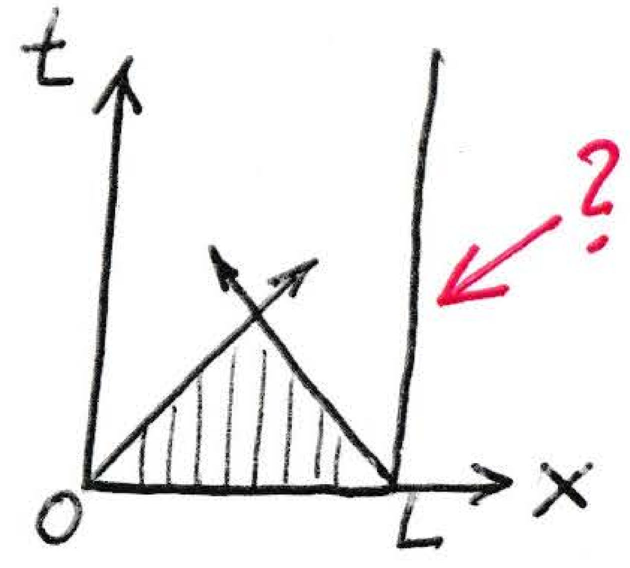
Hence, we have that,

$$\frac{1}{2} \sum_{k=1}^{\infty} \left[b_k \cos k(x-ct) - b_k \cos (x+ct) \right] = \frac{1}{2c} \int_{x-ct}^{x+ct} V(\lambda)\, d\lambda$$

Combining these results,

$$\boxed{u(x,t) = \frac{1}{2} \left[U(x+ct) + U(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} V(\lambda)\, d\lambda} \quad (17)$$

D'Alembert's solution or waves-based solution



$u(x,t) = A\cos(kx-\omega t)$

<u>Time</u>

T – period

ω – frequency (rad/sec)

$\omega = \frac{2\pi}{T}$

Dispersion

relation $\omega = \omega(k)$

Phase velocity

$\frac{\omega}{k}$

Group velocity

$\frac{d\omega}{dk}$

<u>Space</u>

λ – wavelength

k – wavenumber

$\lambda = \frac{2\pi}{k}$

Group velocity (dispersive systems)

Phase velocity is the velocity with which an individual wave propagates. A wavepacket of multiple waves propagates with a group velocity. To show this, consider two traveling waves with closely spaced wave numbers (a wavenumber is the equivalent of frequency in space → So it can be considered as a "spatial frequency"). Then, through the dispersion relation $\omega = \omega(k)$, which is assumed to be smooth, the corresponding frequencies are also assumed to be close ⇒ Consider

$$\varphi(x,t) = \sin(kx - \omega t) + \sin[(k+\Delta k)x - (\omega + \Delta\omega)t] =$$

wavepacket

$$= 2 \sin \frac{1}{2}[(kx - \omega t) + (k+\Delta k)x - (\omega + \Delta\omega)t] \cdot$$

$$\cos \frac{1}{2}[(kx - \omega t) - (k+\Delta k)x + (\omega + \Delta\omega)t] =$$

$$= 2\cos\left(\frac{\Delta k\, x - \Delta\omega t}{2}\right) \cdot \sin \frac{1}{2}[kx - \omega t + (k+\Delta k)x - (\omega+\Delta\omega)t] =$$

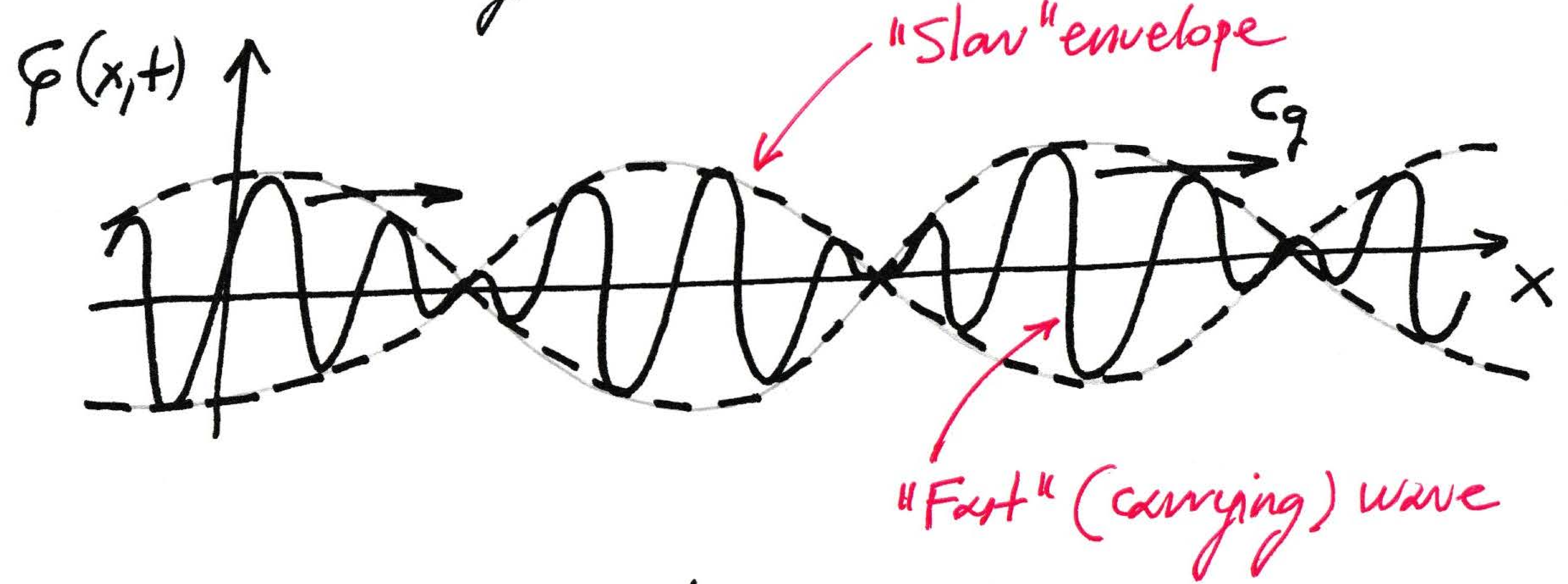
"slow" envelope "fast" oscillation

$$= 2\cos \frac{\Delta k}{2}\left(x - \frac{\Delta\omega}{\Delta k}t\right) \cdot \sin\left(kx - \omega t + \frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right) \Rightarrow$$

$$\Rightarrow \varphi(x,t) = 2\cos\left[\frac{\Delta k}{2}\left(x - \frac{\Delta\omega}{\Delta k}t\right)\right]\sin\left[\left(k+\frac{\Delta k}{2}\right)x - \left(\omega + \frac{\Delta \omega}{2}\right)t\right]$$

Group velocity $c_g = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$

Note that the phase velocities of the two individual waves are $\frac{\omega}{k}$ and $\frac{\omega+\Delta\omega}{k+\Delta k}$ $\Rightarrow$ Group velocity results due to the proximity of the two phase velocities (similar to beat phenomenon between two modes with closely spaced frequencies!).



Wavepacket shown at t = fixed

Energy of the wavepacket propagates with group velocity $\Rightarrow$
$\Rightarrow$ When $c_g \to 0$ energy cannot propagate (standing waves!)