Problem 1

(i) Since the motion of the pivot A is prescribed and only the mass m is allowed to move in the horizontal direction, the system only has 1 DOF. (5 points)

(ii) We assume that the positive direction of the horizontal motion of the mass m is pointing to the right side. Assume that the displacement of the mass is x. Therefore, x is expressed by the prescribed motion $u(t)$ and the angle $\theta(t)$,

$$x = u + L\tan\theta$$

Similarly, the (horizontal) velocity is given by,

$$\dot{x} = \dot{u} + L\sec^2\theta\,\dot{\theta} = \dot{u} + \frac{L\dot{\theta}}{\cos^2\theta}$$

The kinetic energy of the system is given by,

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left(\dot{u} + L\sec^2\theta\,\dot{\theta}\right)^2$$

The deformation of the spring k_2 is x, and therefore the potential energy of k_2 is $\frac{1}{2}k_2x^2$

The unstretched length of the spring k_1 is L, and the current length of k_1 (at an arbitrary time instant) is $\frac{L}{\cos\theta}$, and therefore the potential energy of the spring k_1 is given by, $\frac{1}{2}k_1\left(\frac{L}{\cos\theta} - L\right)^2$

The potential energy is given by,

$$V = \frac{1}{2}k_1\left(\frac{L}{\cos\theta} - L\right)^2 + \frac{1}{2}k_2(u + L\tan\theta)^2$$

The Lagrangian is given by (10 points),

$$L = T - V = \frac{1}{2}m\left(\dot{u} + L\sec^2\theta\,\dot{\theta}\right)^2 - \frac{1}{2}k_1(L\sec\theta - L)^2 - \frac{1}{2}k_2(u + L\tan\theta)^2$$

The Lagrange's equation is given by,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = Q_\theta$$

Q_θ is the generalized force given by the virtual work principle. Assume that there is a virtual displacement $\delta\theta$ on the angle. The virtual displacement δx is given by,

$$\delta x = \delta(u + L\tan\theta) = \frac{\partial x}{\partial\theta}\delta\theta = L\sec^2\theta\,\delta\theta$$

Now we compute the non-conservative forces. There are two parts: (i) the friction force and (ii) the external force. The first step is to compute the normal force between the mass and the base. The spring k_1 is in tension and the tension force is,

$$T_{k1} = k_1\left(\frac{L}{\cos\theta} - L\right)$$

The vertical component of the tension force is $k_1\left(\frac{L}{\cos\theta}-L\right)\cos\theta = k_1 L(1-\cos\theta)$

Therefore, the contact force between the mass and the base is given by,

$$N = mg - k_1 L(1-\cos\theta)$$

The friction force is given by $f = -\mu|N|\operatorname{sgn}(\dot{x})$

If gravity is not included, no point will be deducted. However, the friction force should be $-\mu k_1 L(1-\cos\theta)\operatorname{sgn}(\dot{x})$. *If they answered* $\mu k_1 L(1-\cos\theta)\operatorname{sgn}(\dot{x})$, *then one point will be deducted. The reason is that the friction force should always be opposite to the direction of motion.*

If students use the formula $f = \mu N \operatorname{sgn}(\dot{x})$, *i.e. they only discuss the case where the gravity is larger than the tension force, no point is deducted.*

The virtual work of the external force and the friction is given by,

$$\begin{aligned}\delta W &= -F\delta x - \mu|N|\operatorname{sgn}(\dot{x})\,\delta x \\ &= -FL\sec^2\theta\,\delta\theta - \mu|mg - k_1L(1-\cos\theta)|\operatorname{sgn}\left(\dot{u}+\frac{L\dot{\theta}}{\cos^2\theta}\right)L\sec^2\theta\,\delta\theta\end{aligned}$$

Since $\delta W = Q_\theta\delta\theta$, Q_θ is given by,

$$Q_\theta = -FL\sec^2\theta - \mu|mg - k_1L(1-\cos\theta)|\operatorname{sgn}\left(\dot{u}+\frac{L\dot{\theta}}{\cos^2\theta}\right)L\sec^2\theta$$

Now we plug the Lagrangian and Q_θ into the Lagrange's equation. Note that,

$$\frac{\partial L}{\partial\dot{\theta}} = m\left(\dot{u}+L\sec^2\theta\,\dot{\theta}\right)L\sec^2\theta = mL\sec^2\theta\,\dot{u} + mL^2\sec^4\theta\,\dot{\theta}$$

Note that $\frac{d}{dt}(\sec\theta) = \sec\theta\tan\theta\,\dot{\theta}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = mL\sec^2\theta\,\ddot{u} + 2mL\sec^2\theta\tan\theta\,\dot{\theta}\dot{u} + mL^2\sec^4\theta\,\ddot{\theta} + 4mL^2\sec^4\theta\tan\theta\,\dot{\theta}^2$$

Since

$$L = T - V = \frac{1}{2}m\left(\dot{u}+L\sec^2\theta\,\dot{\theta}\right)^2 - \frac{1}{2}k_1(L\sec\theta - L)^2 - \frac{1}{2}k_2(u+L\tan\theta)^2$$

$$\frac{\partial L}{\partial\theta} = m\left(\dot{u}+L\sec^2\theta\,\dot{\theta}\right)L\sec^2\theta\tan\theta\,\dot{\theta} - k_1(L\sec\theta - L)L\sec\theta\tan\theta - k_2(u+L\tan\theta)L\sec^2\theta$$

Therefore,

$$\begin{aligned}&\left\{mL\sec^2\theta\,\ddot{u} + 2mL\sec^2\theta\tan\theta\,\dot{\theta}\dot{u} + mL^2\sec^4\theta\,\ddot{\theta} + 4mL^2\sec^4\theta\tan\theta\,\dot{\theta}^2\right\} \\ &-\left\{m\left(\dot{u}+L\sec^2\theta\,\dot{\theta}\right)L\sec^2\theta\tan\theta\,\dot{\theta} - k_1(L\sec\theta - L)L\sec\theta\tan\theta - k_2(u+L\tan\theta)L\sec^2\theta\right\} \\ &= \left\{-FL\sec^2\theta - \mu|mg - k_1L(1-\cos\theta)|\operatorname{sgn}\left(\dot{u}+\frac{L\dot{\theta}}{\cos^2\theta}\right)L\sec^2\theta\right\}\end{aligned}$$

Problem 2

(i) Newton's method is applied. Note that the vertical position of the mass m is given by,

$$x + R\cos\omega t$$

The vertical acceleration of the mass m is given by,

$$\frac{d^2}{dt^2}(x + R\cos\omega t) = \ddot{x} - \omega^2 R\cos\omega t$$

We apply the Newton's second law in the vertical position to the machine and the rotating imbalance. The total external force is given by $-c\dot{x} - kx$. The acceleration of the machine is $\ddot{x}$, while the acceleration of the rotating imbalance is $\ddot{x} - \omega^2 R\cos\omega t$. Therefore, the Newton's equation is given by,

$$m(\ddot{x} - \omega^2 R\cos\omega t) + M\ddot{x} = -c\dot{x} - kx \Rightarrow (M+m)\ddot{x} + c\dot{x} + kx = m\omega^2 R\cos\omega t$$

(ii) We consider the complex form of the equation of motion,

$$(M+m)\ddot{x} + c\dot{x} + kx = m\omega^2 R e^{i\omega t}$$

The steady state solution is the real part of the solution in the complex form.

Assume that X_{ss} denotes the steady state. $X_{ss} = Xe^{i\omega t}$,

$$-\omega^2(M+m)X + i\omega cX + kX = m\omega^2 R$$

Therefore, we can solve for the complex amplitude X,

$$X = \frac{m\omega^2 R}{-\omega^2(M+m) + k + i\omega c} = \frac{m\omega^2 R\{-\omega^2(M+m) + k - i\omega c\}}{(-\omega^2(M+m) + k)^2 + (\omega c)^2}$$

Therefore, the solution in the real form is given by,

$$\begin{aligned} x = Re\left(Xe^{i\omega t}\right) &= Re(X)\cos\omega t - Im(X)\sin\omega t \\ &= \frac{m\omega^2 R\{-\omega^2(M+m) + k\}}{(-\omega^2(M+m) + k)^2 + (\omega c)^2}\cos\omega t + \frac{m\omega^2 R\{c\omega\}}{(-\omega^2(M+m) + k)^2 + (\omega c)^2}\sin\omega t \end{aligned}$$

(iii) When the damping is equal to zero, the natural frequency $\omega_n = \sqrt{\frac{k}{M+m}}$.

As long as the excitation frequency is equal to the natural frequency, i.e., $\omega = \sqrt{\frac{k}{M+m}}$ the resonance occurs.

Problem 3

Denote by $\mathcal{L}(\cdot)$ the Laplace transform, so the governing equation in the Laplace domain is given by,

$$\mathcal{L}\big(\ddot{x}(t) + x(t)\big) = \mathcal{L}\big(f(t)\big)$$

Denote $X(s) = \mathcal{L}\big(x(t)\big)$ and $F(s) = \mathcal{L}\big(f(t)\big)$,

We note that

$$\mathcal{L}\big(\ddot{x}(t)\big) = s^2X(s) - sx(0) - \dot{x}(0)$$

Therefore,

$$(s^2+1)X(s) - sx(0) - \dot{x}(0) = F(s)$$

$$X(s) = \frac{sx(0) + \dot{x}(0)}{s^2+1} + \frac{F(s)}{s^2+1}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+1} + \frac{F(s)}{s^2+1}\right\}$$

Now we compute the $F(s)$. Denote $U(t)$ as the unit step function,

$$f(t) = tU(t) - (t-1)U(t-1) - (t-2)U(t-2) + (t-3)U(t-3)$$

Note that $\mathcal{L}\big(tU(t)\big) = \frac{1}{s}$. We apply the shift theorem to the other terms and therefore,

$$F(s) = \frac{1}{s^2} - e^{-s}\frac{1}{s^2} - e^{-2s}\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}$$

Now we do partial fractions. Note that

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Therefore,

$$\frac{F(s)}{s^2+1} = (1 - e^{-s} - e^{-2s} + e^{-3s})\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$$

We notice that,

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+1}\right\} = (\cos t - 2\sin t)U(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = tU(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t\, U(t)$$

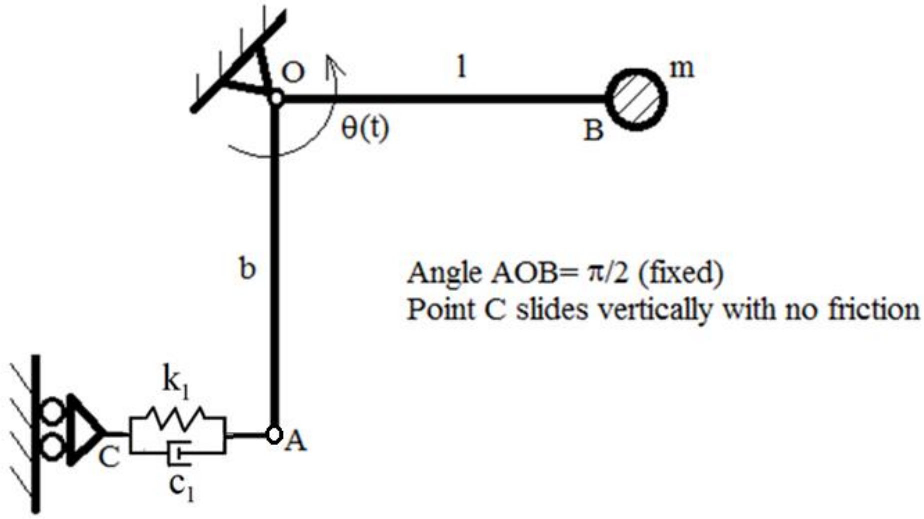
$$\mathcal{L}^{-1}\{e^{-as}G(s)\} = g(t-a)U(t-a)$$

where $G(s) = \mathcal{L}\big(g(t)\big)$

Therefore,

$$
\begin{aligned}
x(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+1} + \frac{F(s)}{s^2+1}\right\} \\
&= (\cos t - 2\sin t)U(t) \\
&+ \{tU(t) - (t-1)U(t-1) - (t-2)U(t-2) + (t-3)U(t-3)\} \\
&- \{\sin t\, U(t) - \sin(t-1)\, U(t-1) - \sin(t-2)\, U(t-2) + \sin(t-3)\, U(t-3)\}
\end{aligned}
$$

Problem 4



(i) Newton's method (omitted here)

Lagrange's method,

The kinetic energy is given by $T = \frac{1}{2}ml^2\dot{\theta}^2$

The potential energy is given by $V = \frac{1}{2}k_1(b\sin\theta)^2$

The Lagrangian is given by $L = T - V$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_\theta$$

Denote the point O as the origin. Note that the horizontal position of the point A is $x = b\sin\theta$. Therefore, the horizontal velocity of A is $b\cos\theta\,\dot{\theta}$. The damping force is $-c_1 b\cos\theta\,\dot{\theta}$

$$Q_\theta\delta\theta = -c_1 b\cos\theta\,\dot{\theta}\cdot b\delta x \Rightarrow Q_\theta = -c_1 b^2\cos^2\theta\,\dot{\theta}$$

Substitute T, V and Q_θ into the Lagrange's equation,

$$ml^2\ddot{\theta} + k_1 b^2\sin\theta\cos\theta = -c_1 b^2\cos^2\theta\,\dot{\theta}$$

Apply the small angle approximation, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, the linearized equation of motion is given by,

$$ml^2\ddot{\theta} + c_1 b^2\dot{\theta} + k_1 b^2\theta = 0$$

(ii) The natural frequency and damping ratio are given by,

$$\omega_n = \sqrt{\frac{k_1 b^2}{ml^2}};\zeta = \frac{c_1 b^2}{2\sqrt{k_1 b^2 ml^2}} = \frac{c_1 b}{2l\sqrt{k_1 m}}$$

(iii) The characteristic equation is given by,

$$ml^2s^2 + c_1b^2s + k_1b^2 = 0 \Rightarrow s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The condition of overdamped response is that the roots of the characteristic equation are real and distinct, which is equivalent to $\zeta > 1$ and therefore $\frac{c_1 b}{2l\sqrt{k_1 m}} > 1$

The roots of the characteristic equations are given by,

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

The general solution is given by,

$$\theta = C_1e^{s_1t} + C_2e^{s_2t} = C_1e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t} + C_2e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t}$$

where C_1 and C_2 are coefficients determined by the initial conditions.

The initial conditions are given by $\theta(0) = \theta_0$ and $\dot{\theta}_0 = \omega_0$. Plug the initial conditions into the general solutions,

$$C_1 + C_2 = \theta_0$$

$$s_1C_1 + s_2C_2 = \omega_0$$

Solve the equations and we obtain,

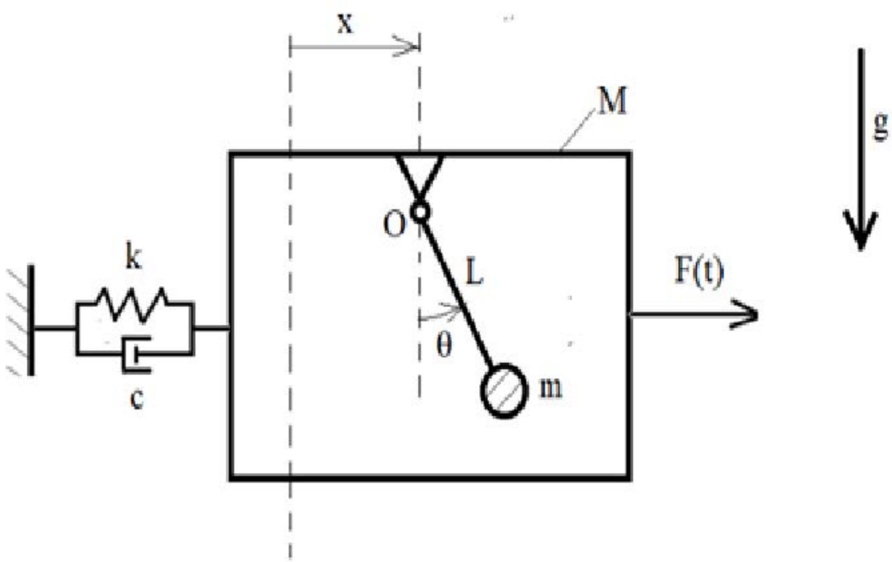
$$C_1 = \frac{\omega_0 - s_2\theta_0}{s_1 - s_2}, C_2 = \frac{\omega_0 - s_1\theta_0}{s_2 - s_1}$$

Substitute the coefficients back to the equations,

$$\begin{aligned}\theta = C_1e^{s_1t} + C_2e^{s_2t} &= \frac{\omega_0 - s_2\theta_0}{s_1 - s_2}e^{s_1t} + \frac{\omega_0 - s_1\theta_0}{s_2 - s_1}e^{s_2t} \\ &= -\frac{\omega_0 - \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\theta_0}{2\omega_n\sqrt{\zeta^2 - 1}}e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t} \\ &\quad + \frac{\omega_0 - \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)\theta_0}{2\omega_n\sqrt{\zeta^2 - 1}}e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t}\end{aligned}$$

where $\omega_n = \sqrt{\frac{k_1b^2}{ml^2}}$ and $\zeta = \frac{c_1b}{2l\sqrt{k_1m}}$

Problem 5



(i) A cartesian coordinate is shown in the schematic. Set the position of O with the springs unstretched as the origin of the coordinate. O is also the zero level of gravitational energy. The positive directions are shown in the schematics.

The position of the pendulum is given by,

$$X_p = x + L\sin\theta$$
$$Y_p = -L\cos\theta$$

Therefore, the horizontal and vertical velocity of the pendulum is given by,

$$\dot{X}_p = \dot{x} + L\cos\theta\,\dot{\theta}$$
$$\dot{Y}_p = L\sin\theta\,\dot{\theta}$$

The kinetic energy of the pendulum is given by,

$$T_p = \frac{1}{2}m\left(\dot{X}_p^2 + \dot{Y}_p^2\right) = \frac{1}{2}m\left(\dot{x}^2 + 2L\cos\theta\,\dot{x}\dot{\theta} + L^2\dot{\theta}^2\right)$$

The potential energy of the pendulum is given by,

$$V_p = mgY_p = -mgL\cos\theta$$

The kinetic energy of the rectangular frame is given by,

$$T_f = \frac{1}{2}M\dot{x}^2$$

The potential energy of the spring connecting the rectangular frame is given by,

$$V_f = \frac{1}{2}kx^2$$

Therefore, the Lagrangian $\mathcal{L}$ is given by,

$$\mathcal{L} = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2L\cos\theta\,\dot{x}\dot{\theta} + L^2\dot{\theta}^2\right) - \frac{1}{2}kx^2 + mgL\cos\theta$$

The Lagrange's equations are given by,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} = Q_x$$
$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = Q_\theta$$

Note that $Q_x = F(t) - c\dot{x}$ and $Q_\theta = 0$,

$$(M+m)\ddot{x} + mL\cos\theta\,\ddot{\theta} - mL\sin\theta\,\dot{\theta}^2 + kx = F(t) - c\dot{x}$$
$$mL^2\ddot{\theta} + mL\cos\theta\,\ddot{x} + mgL\sin\theta = 0$$

Assume that $F(t) = 0$, therefore the governing equations of motion are,

$$(M+m)\ddot{x} + c\dot{x} + mL\cos\theta\,\ddot{\theta} - mL\sin\theta\,\dot{\theta}^2 + kx = 0$$
$$mL^2\ddot{\theta} + mL\cos\theta\,\ddot{x} + mgL\sin\theta = 0$$

At the equilibrium positions, $\ddot{\theta} = \ddot{x} = \dot{\theta} = \dot{x} = 0$. Therefore, the equations of motion are governed by,

$$kx = 0$$
$$mgL\sin\theta = 0$$

Therefore $x = \theta = 0$ is a fixed point.

Linearization:

Apply small-angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, neglect the nonlinear terms and we obtain the following equations,

$$(M+m)\ddot{x} + c\dot{x} + mL\ddot{\theta} + kx = 0$$
$$mL^2\ddot{\theta} + mL\ddot{x} + mgL\theta = 0$$

Simplify the equations and we obtain,

$$M\ddot{x} + c\dot{x} + kx - mg\theta = 0$$
$$L\ddot{\theta} + \ddot{x} + g\theta = 0$$