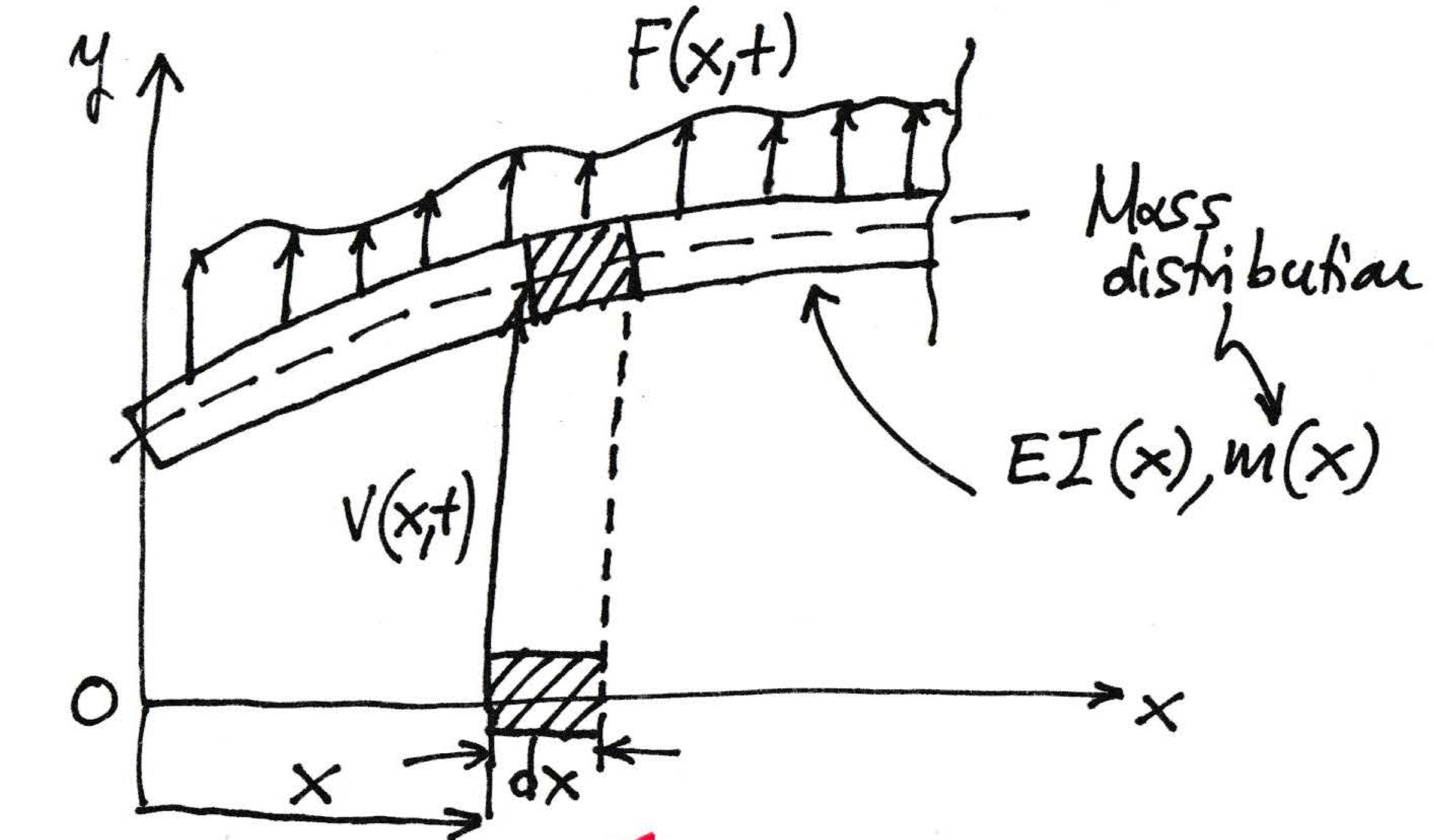
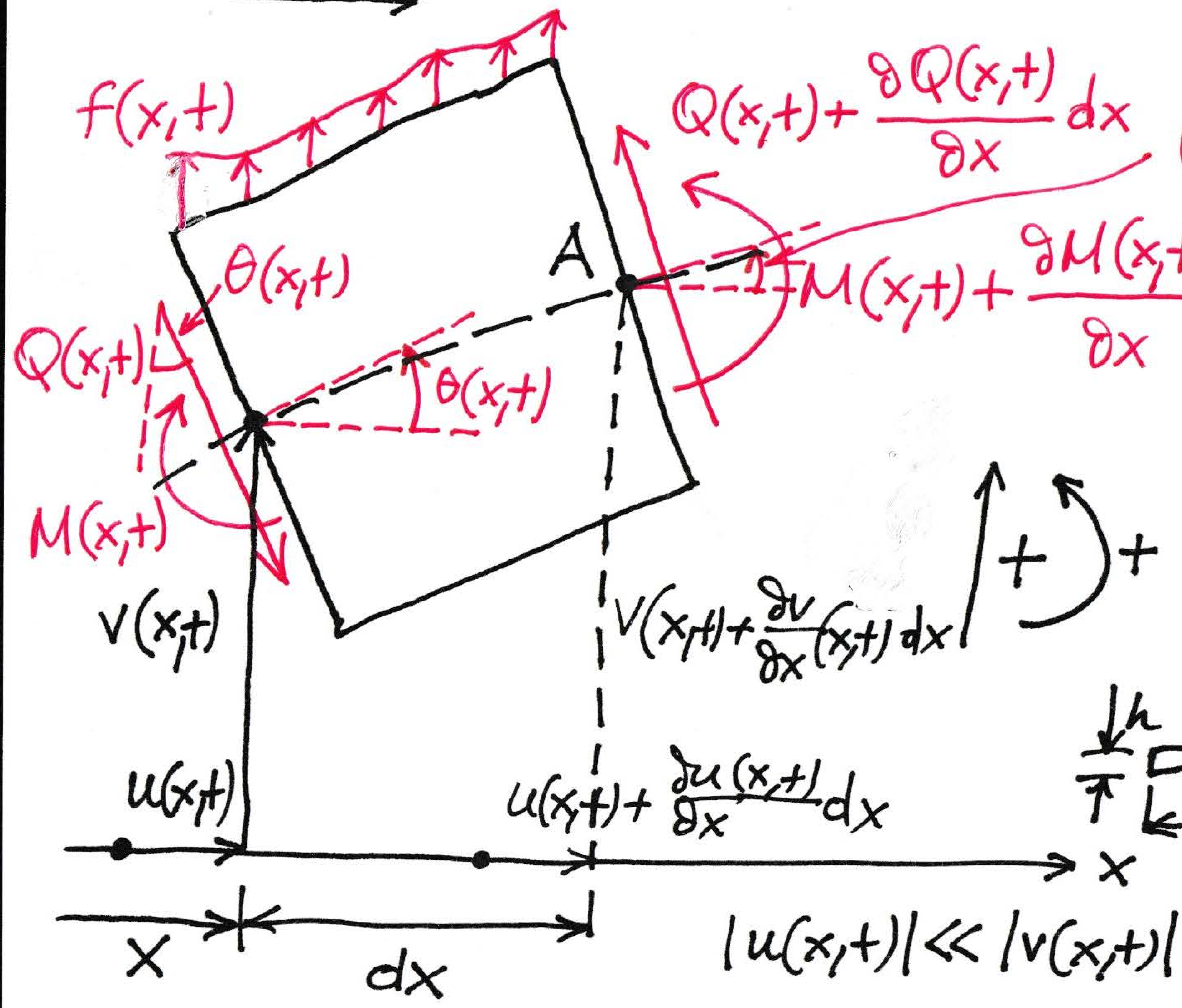
3. The Beam equation





$$\theta(x,t)=\frac{\partial v(x,t)}{\partial x}$$

$$|u(x,t)| \ll |v(x,t)|$$

Main assumptions, Bernoulli beam theory

Consider a beam in transverse vibrations. We make the following assumptions

- In-plane motions
- Sections perpendicular to the neutral axis remain perpendicular to it after deformation.
- Neglect shear deformations
- Assume small axial deformations, and neglect them.
- Assume small displacements, curvatures and slopes
- There is no axial stretching of the beam after deformation so the length of the beam after deformation is nearly equal to the undeformed length.
- No rotary inertia effects in the differential element are assumed.
- Linearly elastic beam, slender beam, $L \gg h$.

Balance of forces of the differential element in the vertical direction:

$$\left(Q+\frac{\partial Q}{\partial x}dx\right)\underbrace{\cos\left(\theta+\frac{\partial \theta}{\partial x}dx\right)}_{\cos\theta-\frac{\partial\theta}{\partial x}\sin\theta\, dx+O(dx^2)}-Q\cos\theta+f(x,t)dx=m(x)\,dx\,\frac{\partial^2 v}{\partial t^2}\Rightarrow$$

mass distribution per unit length

$0\le x\le L,\ t\ge 0$

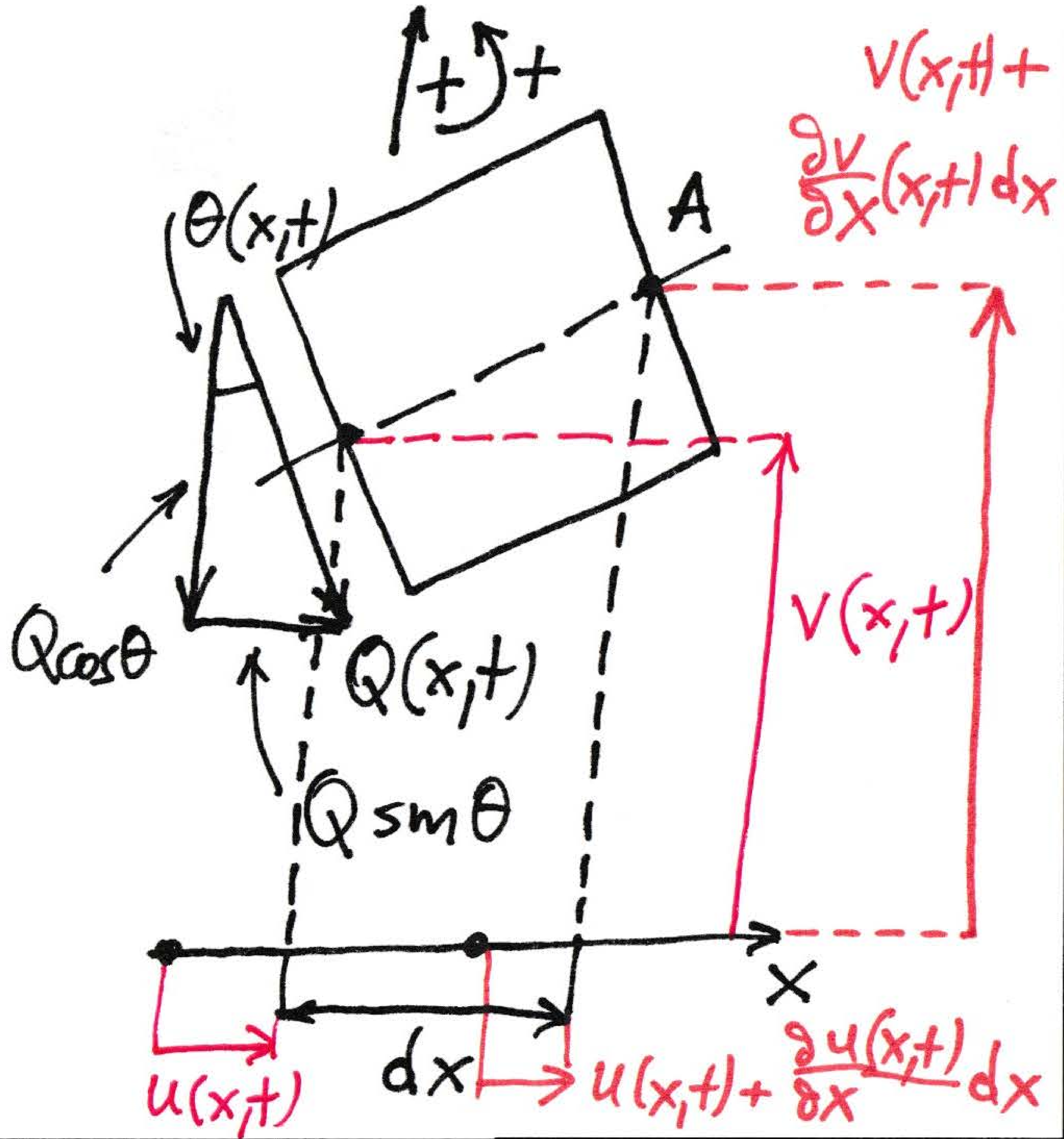
$|\theta|\ll 1$

$$\Rightarrow \frac{\partial}{\partial x}(Q\cos\theta)+f(x,t)=m(x)\frac{\partial^2 v}{\partial t^2}\Rightarrow \boxed{\frac{\partial Q(x,t)}{\partial x}+f(x,t)=m(x)\frac{\partial^2 v(x,t)}{\partial t^2}}\quad(1)$$

Due to the previous assumptions, there is no balance of forces in the axial direction. So, consider balance of moments about point A of the differential element:

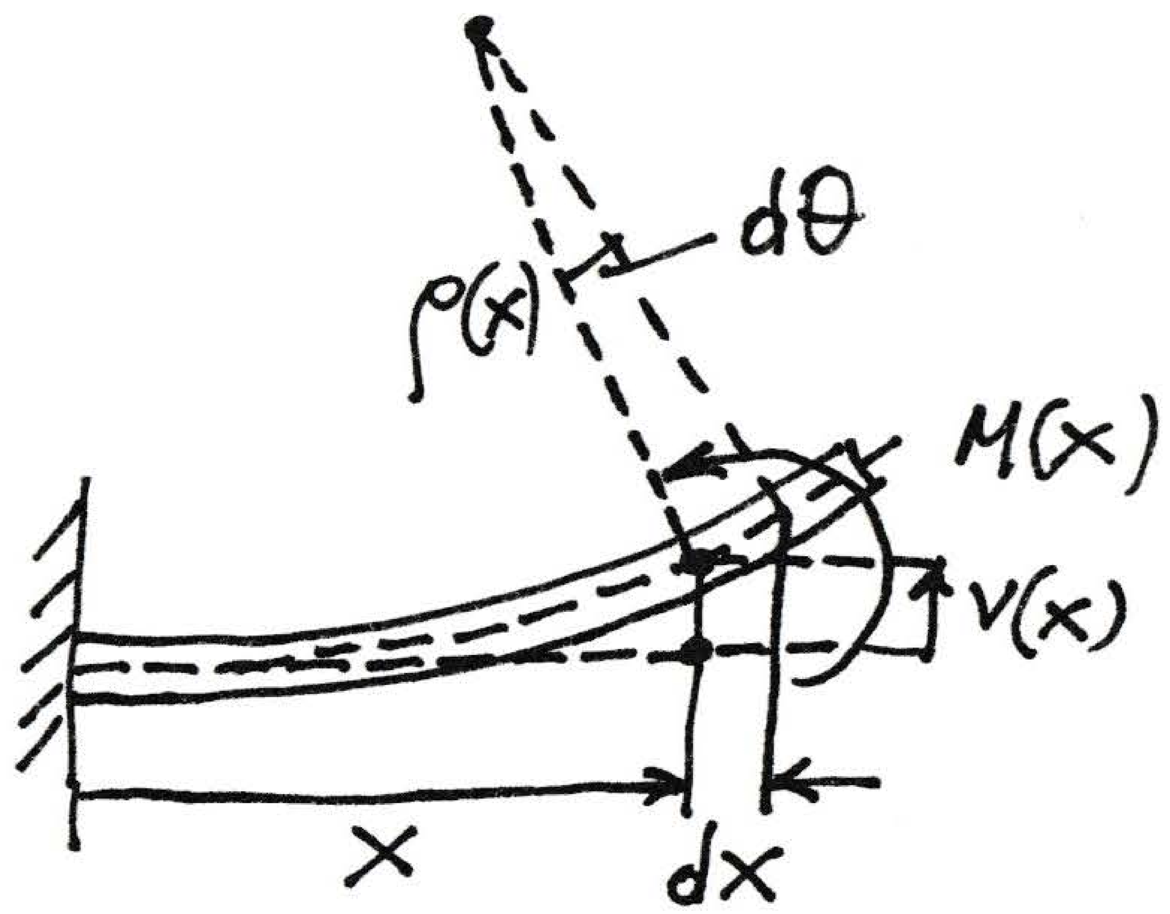
$$J\,dx\,\frac{\partial^2\theta}{\partial t^2}=-M+\left(M+\frac{\partial M}{\partial x}dx\right)+f(x,t)\,dx\,\frac{dx}{2}+$$
$$+Q\cos\theta\left[u(x,t)+\frac{\partial u(x,t)}{\partial x}dx+dx-u(x,t)\right]+$$
$$+Q\sin\theta\left[v(x,t)+\frac{\partial v(x,t)}{\partial x}dx-v(x,t)\right]+$$
$$+O(dx^2)\Rightarrow \text{Assuming that } J=0\Rightarrow$$

Distributed rotary inertia per unit length



$$\Rightarrow \frac{\partial M}{\partial x} + Q\cos\theta\left(1+\frac{\partial u}{\partial x}\right) + Q\sin\theta\,\frac{\partial v}{\partial x} = 0 \Rightarrow$$ Assuming that $u \sim 0$ and that the slope θ is small $\Rightarrow$ $$\boxed{\frac{\partial M(x,t)}{\partial x} + Q(x,t) = 0} \qquad (2)$$

The plan is to obtain a partial differential equation governing the transverse deformation $V(x,t)$. Considering the Bernoulli assumption for beam vibrations, we obtain the following relation between the local curvature $K(x)$ and the local bending moment $M(x)$ for the corresponding static problem (i.e., without time dependence): $K(x) = \frac{M(x)}{EI} = \frac{1}{\rho(x)}$ (Note that if $M(x) \equiv M$, then $\rho(x) \equiv \rho$)



But the geometric relation holds,

$$\left.\begin{array}{l} K(x) = \dfrac{V''(x)}{\left\{1+V'^2(x)\right\}^{3/2}} \\ \text{Assuming that } |y'(x)| \ll 1 \end{array}\right\} \Rightarrow K(x) \sim V''(x)$$

Hence we get the approximate relation,

$$M(x) \sim EI\,V''(x) + O(V''V'^2)$$

Now moving onto elastodynamics we get

$$\boxed{M(x,t) \sim EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}} \quad (3)$$

(Euler–Bernoulli constitutive relation)

Key relation as it relates the bending moment to the transverse deformation

from (2) $\Rightarrow$ $Q(x,t) = -\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right]$ (4)

Substituting (4) into (1) $\Rightarrow$

$0 \le x \le L,\ t \ge 0$

$$\boxed{-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right] + f(x,t) = m(x)\frac{\partial^2 v(x,t)}{\partial t^2}} \quad (5)$$

Note that this is a fourth-order pde in terms of x and second-order in terms of t, reflecting the fact that beams can support bending moments (compare to the wave equation).

Complementing this equation we two initial conditions, $v(x,0) = g(x)$ and $\frac{\partial v}{\partial t}(x,0) = h(x)$, and boundary conditions $\rightarrow$ Then get well-posed problem.

Remark

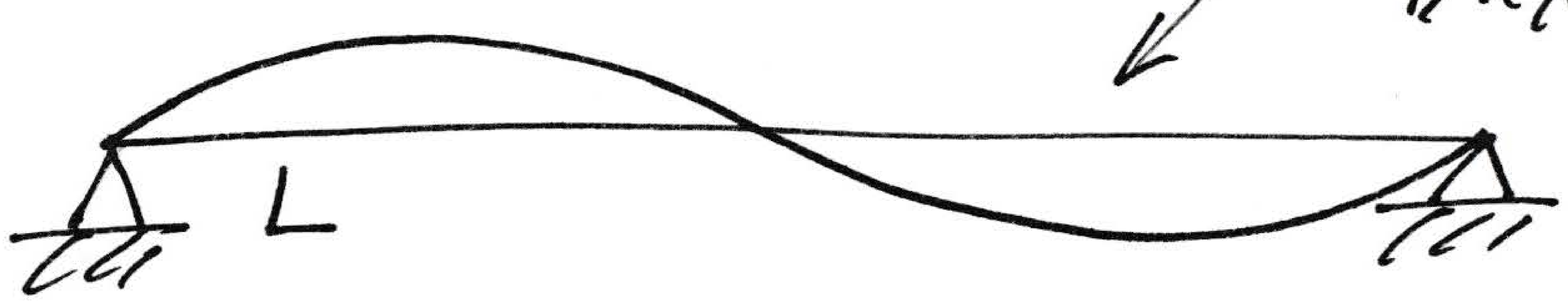
Note that if $EI(x) = EI$, $m(x) = m$, then the equation of motion becomes,

$$m\frac{\partial^2 v}{\partial t^2} + EI\frac{\partial^4 v}{\partial x^4} = f(x,t)$$

$$\Rightarrow \frac{\partial^4 v}{\partial x^4} + b^4\frac{\partial^2 v}{\partial t^2} = 0,$$

If $f(x,t) \equiv 0$

$$b^4 \equiv \frac{m}{EI}\left(\frac{sec^2}{m^4}\right)$$



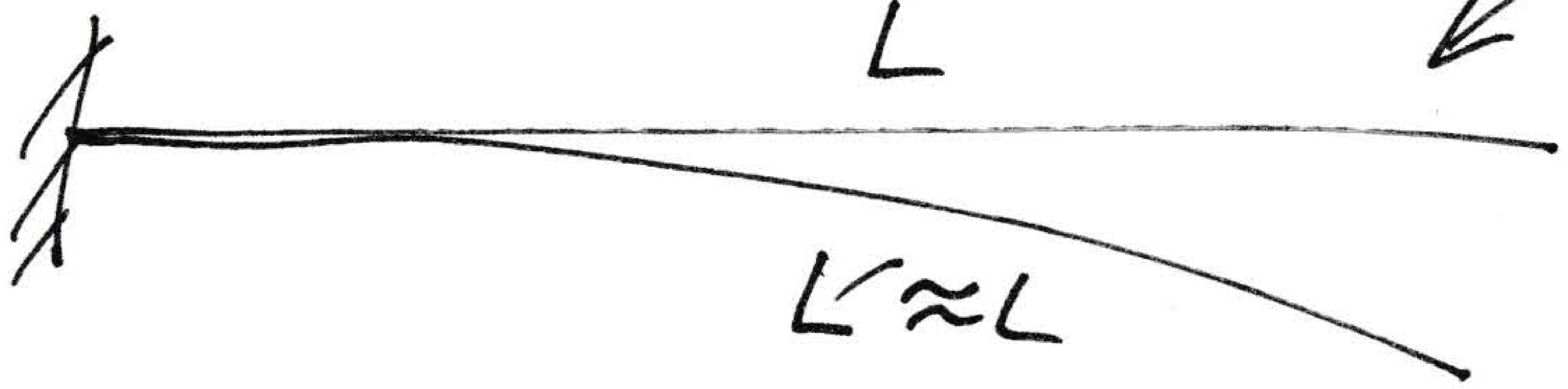
Source of nonlinearity is axial stretching since $L' > L$

Extensional Beam theory

Euler-Bernoulli theory $L' \approx L$

$L' > L \longrightarrow$ One type of Nonlinear beam theory

Source of nonlinearity is large curvature, inertia effects, etc.



Inextensional beam theory

Another type of nonlinear beam theory