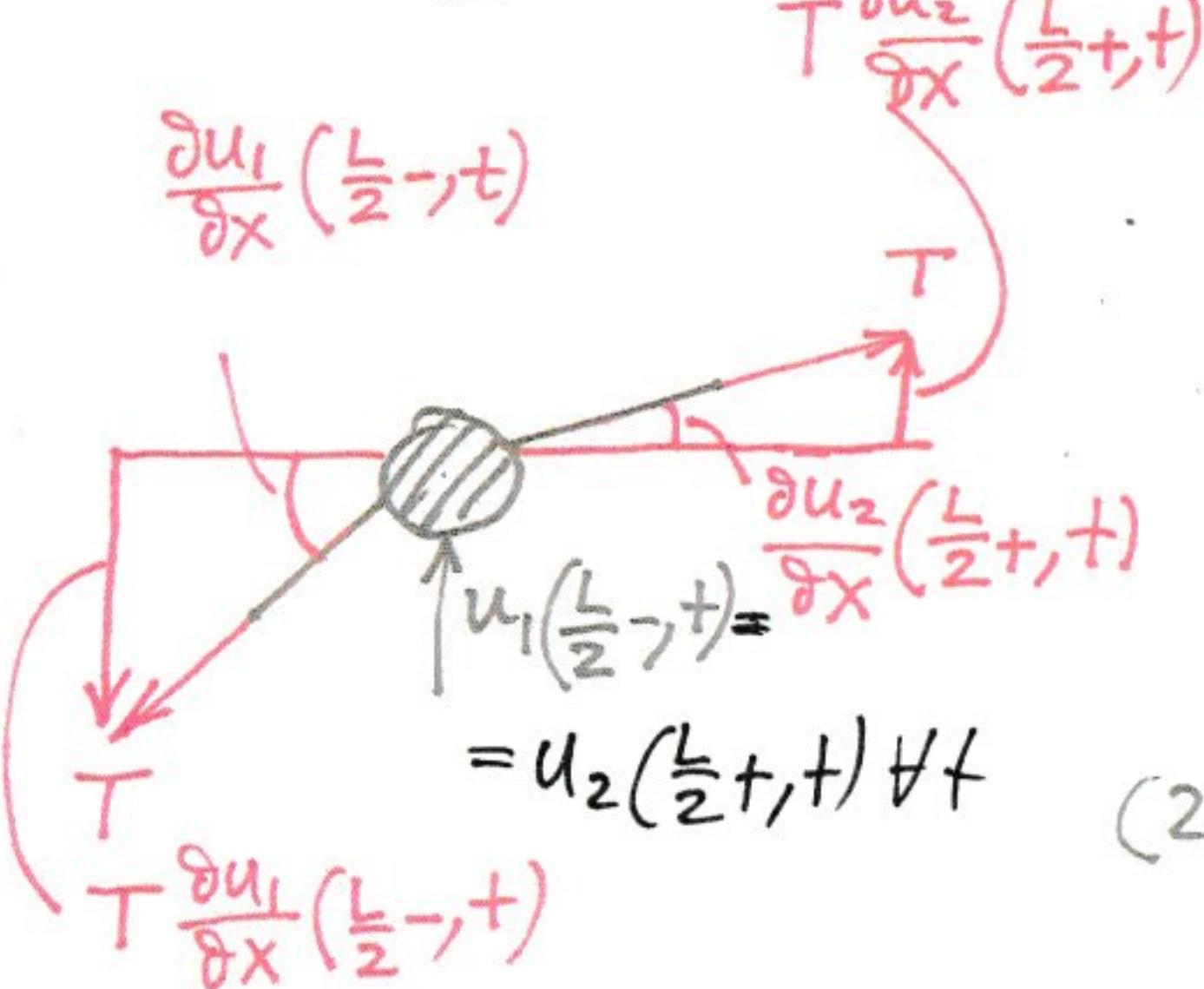
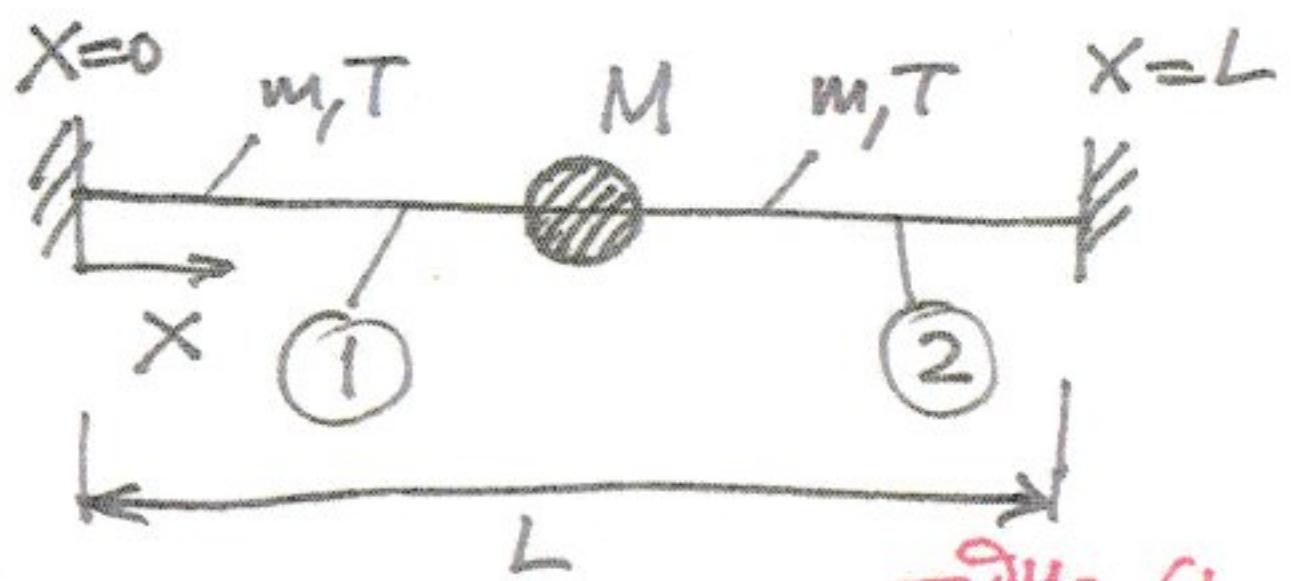


Example 4



Two ways to approach this.

i) 'Global' equation for the string

$$\left. \begin{aligned} T \frac{\partial^2 u}{\partial x^2} &= [m + M \delta(x - \frac{L}{2})] \frac{\partial^2 u}{\partial t^2} \\ u(0, t) &= u(L, t) = 0 \end{aligned} \right\} 0 \leq x \leq L$$

ii) Write two 'local' equations, one for each section of the string.

$$T \frac{\partial^2 u_1}{\partial x^2} = m \frac{\partial^2 u_1}{\partial t^2}, \quad 0 \leq x \leq \frac{L}{2} \quad (1a)$$

$$T \frac{\partial^2 u_2}{\partial x^2} = m \frac{\partial^2 u_2}{\partial t^2}, \quad \frac{L}{2} \leq x \leq L \quad (1b)$$

$$u_1(0, t) = 0, \quad u_2(L, t) = 0 \quad (2b)$$

$$u_1(L/2-, t) = u_2(L/2+, t) \quad (2c)$$

$$M \frac{\partial^2 u_1}{\partial t^2}(L/2-, t) = M \frac{\partial^2 u_2}{\partial t^2}(L/2+, t) =$$

$$= T \frac{\partial u_2}{\partial x}(L/2+, t) - T \frac{\partial u_1}{\partial x}(L/2-, t) \quad (2d)$$

We'll follow the 'local' approach, and formulate two boundary value problems, one for each of the two 'sections' of the string, \Rightarrow

$$\Rightarrow \varphi_1(x) = C_1 \cos \frac{\omega x}{c} + C_2 \sin \frac{\omega x}{c}, \quad 0 \leq x \leq \frac{L}{2} -$$

$$\varphi_2(x) = E_1 \cos \frac{\omega x}{c} + E_2 \sin \frac{\omega x}{c} =$$

$$= D_1 \cos \frac{\omega(L-x)}{c} + D_2 \sin \frac{\omega(L-x)}{c}, \quad \frac{L}{2} + \leq x \leq L$$

$$(2a) \Rightarrow \varphi_1(0) = 0 \Rightarrow C_1 = 0$$

$$(2b) \Rightarrow \varphi_2(L) = 0 \Rightarrow D_1 = 0$$

$$(2c) \Rightarrow \varphi_1\left(\frac{L}{2}\right) = \varphi_2\left(\frac{L}{2}\right) \Rightarrow C_2 \sin \frac{\omega L}{2c} = D_2 \sin \frac{\omega L}{2c} \Rightarrow C_2 = D_2$$

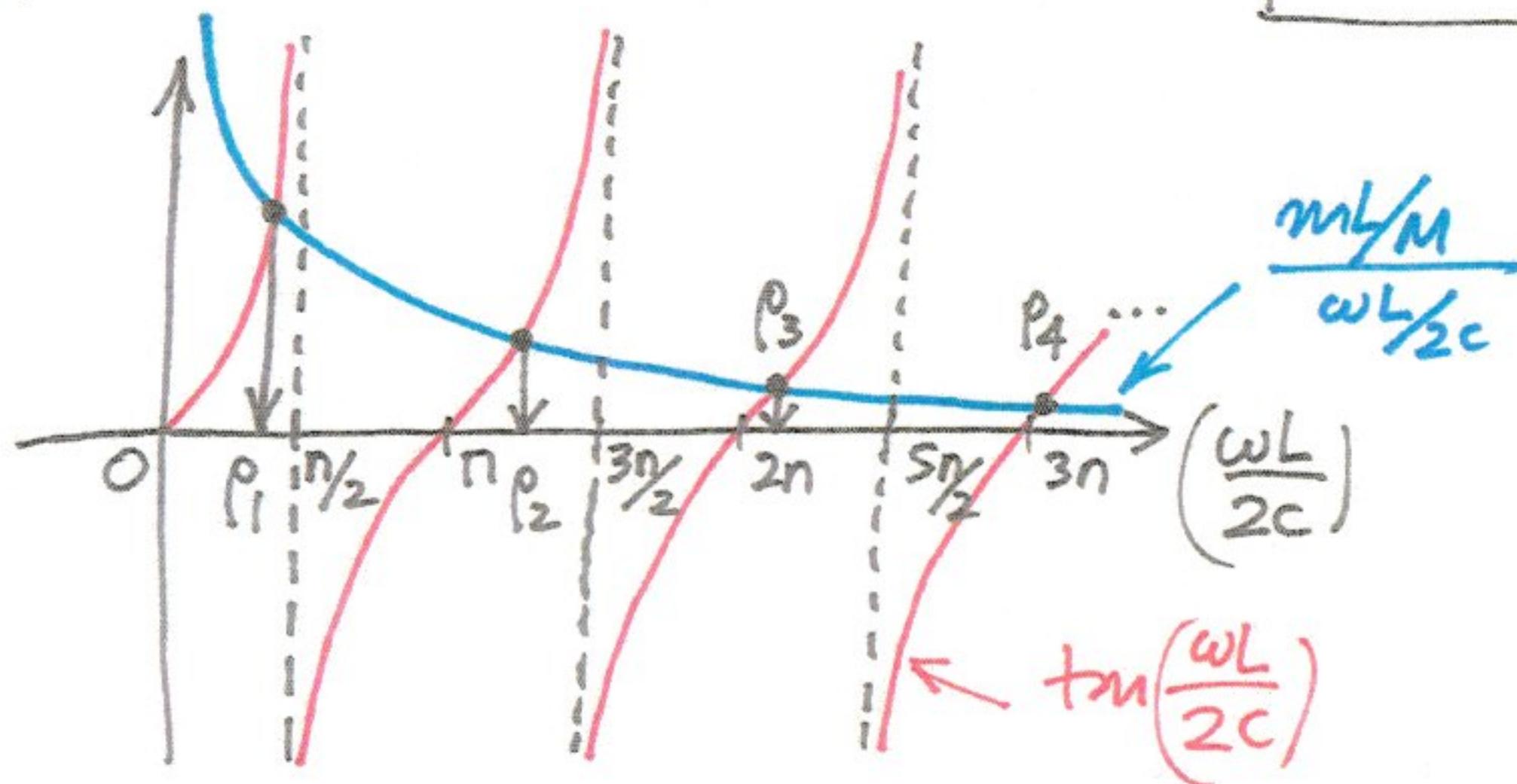
$$(2d) \Rightarrow T\varphi'_1\left(\frac{L}{2}\right) - T\varphi'_2\left(\frac{L}{2}\right) - \omega^2 M \varphi_1\left(\frac{L}{2}\right) = 0$$

$$\Rightarrow T C_2 \frac{\omega}{c} \cos \frac{\omega L}{2c} - T C_2 \left(-\frac{\omega}{c}\right) \cos \frac{\omega L}{2c} - \omega^2 M C_2 \sin \frac{\omega L}{2c} = 0 \Rightarrow$$

$$\Rightarrow 2T C_2 \frac{\omega}{c} \cos \frac{\omega L}{2c} - \omega^2 M C_2 \sin \frac{\omega L}{2c} = 0 \Rightarrow C_2 \left[2T \frac{\omega}{c} \cos \frac{\omega L}{2c} - \omega^2 M \sin \frac{\omega L}{2c} \right] = 0$$

\Rightarrow For nontrivial solutions we require that $\left[\quad \right] = 0 \Rightarrow$

\Rightarrow Taking into account that $C = \frac{I}{m} \Rightarrow \tan\left(\frac{\omega L}{2C}\right) = \frac{mL/M}{(\omega L/2c)}$



Hence, we got the countable infinity of eigenfrequencies,

$$\frac{\omega_i \cdot L}{2c} = p_i \Rightarrow \omega_i = \frac{2c p_i}{L}, \quad i=1, 2, 3, \dots$$

Then, the corresponding mode shapes are,

$$\varphi_{11}(x) = C_{21} \sin p_1 \frac{x}{L}, \quad \varphi_{21}(x) = C_{21} \sin p_1 \frac{L-x}{L}$$

1st eigenfunction

$$\varphi_{12}(x) = C_{22} \sin p_2 \frac{x}{L}, \quad \varphi_{22}(x) = C_{22} \sin p_2 \frac{L-x}{L}$$

2nd eigenfunction

$$\varphi_{1i}(x) = C_{2i} \sin p_i \frac{x}{L}, \quad \varphi_{2i}(x) = C_{2i} \sin p_i \frac{L-x}{L}$$

i-th eigenfunction

Section ①

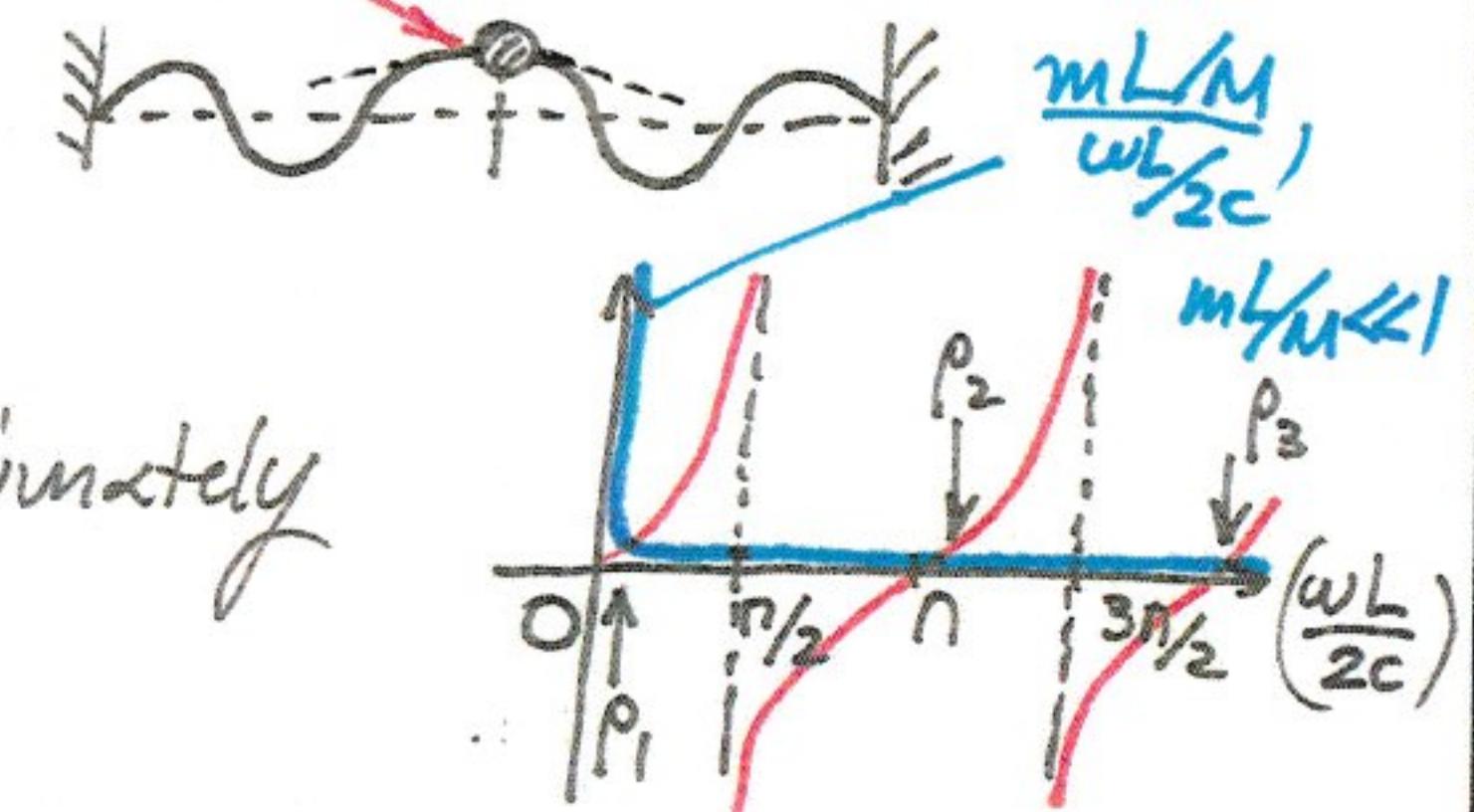
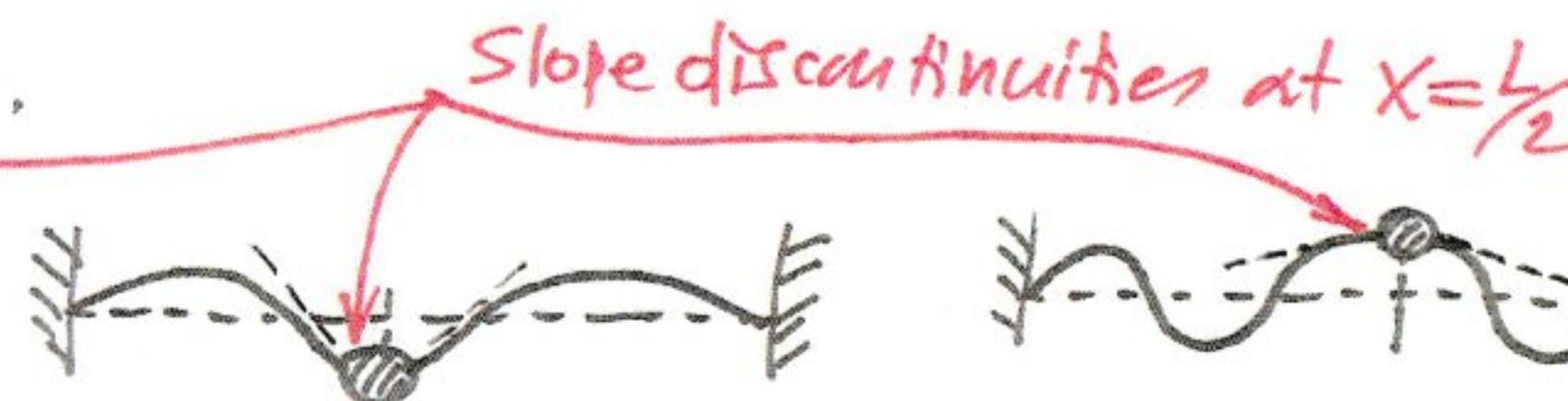
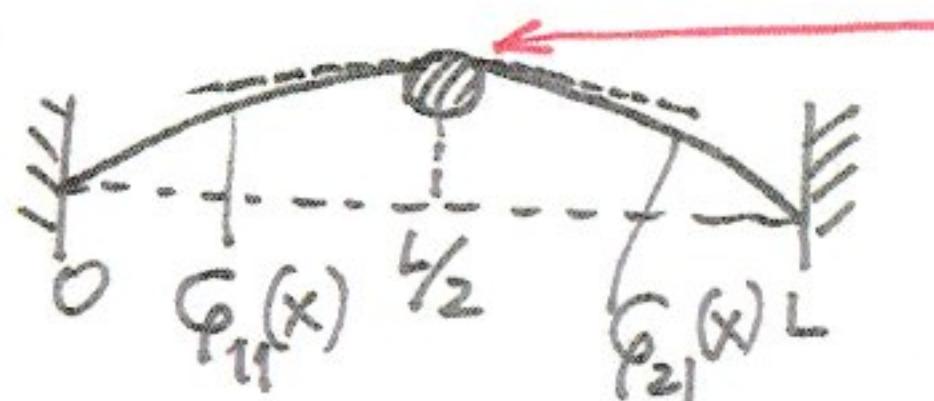
Section ②

To orthonormalize, require that

$$\int_0^L m(x) \tilde{\rho}_i^2(x) dx = 1 \Rightarrow \int_0^{L/2} m C_{2i}^2 \sin^2 \tilde{\rho}_i \frac{x}{L/2} dx + \underbrace{\int_{L/2}^{L+} M \delta(x - L/2) C_{2i}^2 \sin^2 \tilde{\rho}_i \frac{x}{L/2} dx}_{M C_{2i}^2 \sin^2 \tilde{\rho}_i} + \int_{L/2+}^L m C_{2i}^2 \sin^2 \tilde{\rho}_i \frac{L-x}{L/2} dx = 1 \Rightarrow$$

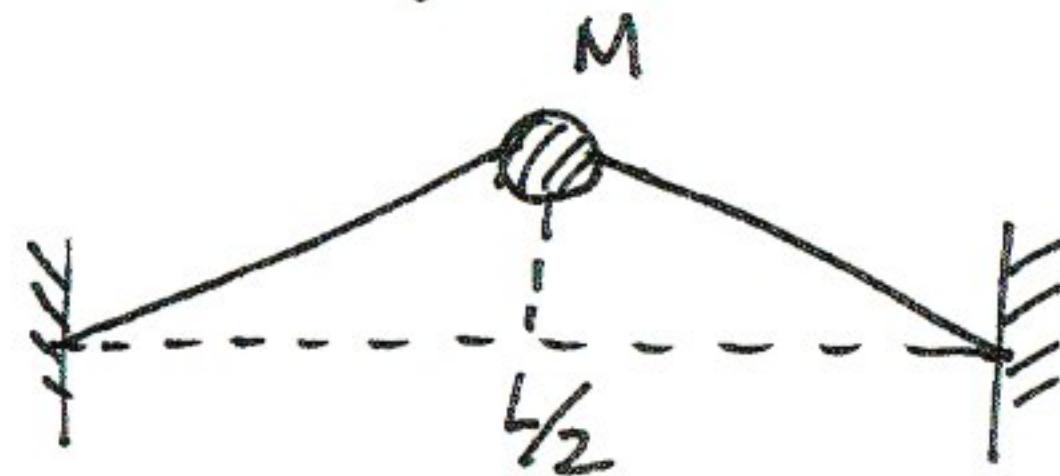
$$\Rightarrow C_{2i}^2 \left\{ \int_0^{L/2} m \sin^2 \tilde{\rho}_i \frac{x}{L/2} dx + M \sin^2 \tilde{\rho}_i + \int_{L/2+}^L m \sin^2 \tilde{\rho}_i \frac{L-x}{L/2} dx \right\} = 1 \Rightarrow$$

\Rightarrow From here we may compute the unknown coefficients $C_{2i}^2, i=1, 3, \dots$ for orthonormalization.



Remark: When $ml/M \ll 1 \Rightarrow$ The first root is approximately computed $\Rightarrow \rho_1 = \frac{\omega_1 L}{2c} \ll 1 \Rightarrow \tan \frac{\omega_1 L}{2c} \sim \frac{\omega_1 L}{2c} \Rightarrow$
 $\Rightarrow \frac{\omega_1 L}{2c} \sim \frac{ml/M}{\omega_1 L/2c} \Rightarrow (\omega_1 \sim \sqrt{\frac{2(T/L/2)}{M}} = \sqrt{\frac{k}{M}}), k = T_{L/2}$

In that case the string acts as a pure spring and its inertia does not affect the dynamics!



$ML \ll M$, 1st mode

For the second natural frequency we can find that $\rho_2 = \frac{\omega_2 L}{2c} \sim n \Rightarrow$

$$\Rightarrow \frac{\omega_2}{c} \sim \frac{n}{L/2} \Rightarrow \varphi_{12}(x) \sim c_2 \sin \frac{nx}{L/2}$$

$$\varphi_{22}(x) \sim c_2 \sin \frac{n(L-x)}{L/2}$$



$ML \ll M$, 2nd mode

Also we can show that for the third natural frequency it holds that

$$\frac{\omega_3}{c} \sim \frac{2n}{L/2}$$



$ML \ll M$, 3rd mode

Additional class of solutions

Recall the boundary value problem:

$$\left. \begin{array}{l} \varphi_1''(x) + \left(\frac{\omega}{c}\right)^2 \varphi_1(x) = 0, \quad 0 \leq x \leq \frac{L}{2} \\ \varphi_2''(x) + \left(\frac{\omega}{c}\right)^2 \varphi_2(x) = 0, \quad \frac{L}{2} < x \leq L \end{array} \right\} \Rightarrow \begin{array}{l} \varphi_1(x) = C_1 \cos \frac{\omega x}{c} + C_2 \sin \frac{\omega x}{c} \\ \varphi_2(x) = D_1 \cos \frac{\omega(L-x)}{c} + D_2 \sin \frac{\omega(L-x)}{c} \end{array}$$

$$\varphi_1(0) = 0 \Rightarrow C_1 = 0$$

$$\varphi_2(L) = 0 \Rightarrow D_1 = 0$$

$$\varphi_1\left(\frac{L}{2}^-\right) = \varphi_2\left(\frac{L}{2}^+\right) \Rightarrow \boxed{C_2 \sin \frac{\omega L}{c^2} = D_2 \sin \frac{\omega L}{c^2}} \quad (*)$$

$$T\varphi_1'\left(\frac{L}{2}^-\right) - T\varphi_2'\left(\frac{L}{2}^+\right) - \omega^2 M \varphi_1\left(\frac{L}{2}^-\right) = 0 \quad (**)$$

From (*) there are two possibilities: - $C_2 = D_2$ if $\sin \frac{\omega L}{c^2} \neq 0 \Rightarrow$
 \Rightarrow obtain the class of solutions
discussed previously

- However, there is the additional possibility that

$$\sin \frac{\omega L}{c^2} = 0 \Rightarrow \frac{\omega k}{c} \frac{L}{2} = kn \Rightarrow$$

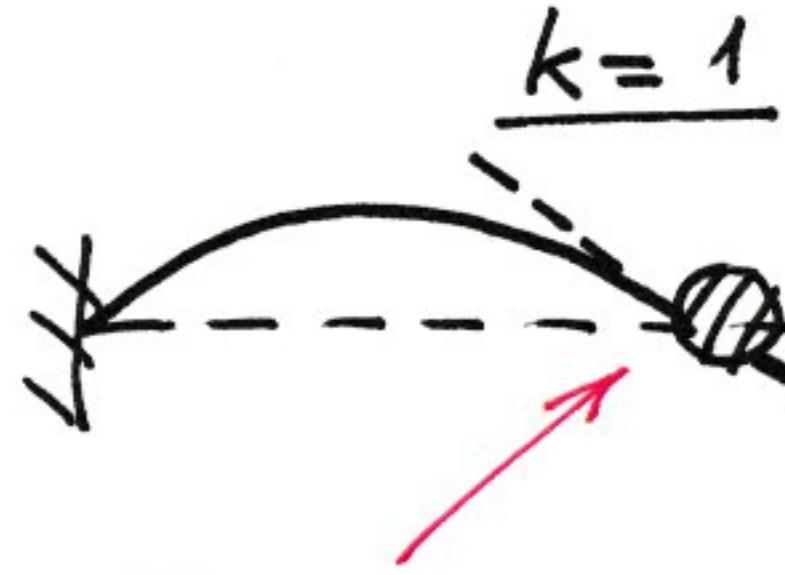
$$\Rightarrow \boxed{\omega_k = \frac{2kn\pi c}{L} = \frac{k\pi c}{(L/2)}, k=1, 2, \dots}$$

Then, considering the remaining boundary condition (**) \Rightarrow

$$\Rightarrow T C_2 \underbrace{\frac{\omega_k}{c} \cos \frac{\omega_k L}{2c}}_{(-1)^k} + T D_2 \underbrace{\frac{\omega_k}{c} \cos \frac{\omega_k L}{2c}}_{(-1)^k} - \omega_k^2 M C_2 \underbrace{\sin \frac{\omega_k L}{2c}}_0 = 0 \Rightarrow$$

$$\Rightarrow C_2 + D_2 = 0 \Rightarrow \boxed{C_2 = -D_2} \Rightarrow \boxed{\begin{aligned} \varphi_{1k}(x) &= C_2 \sin \frac{\omega_k x}{c} \\ \varphi_{2k}(x) &= -C_2 \sin \frac{\omega_k (L-x)}{c} \end{aligned}} \quad \boxed{\text{Additional class of modes}}$$

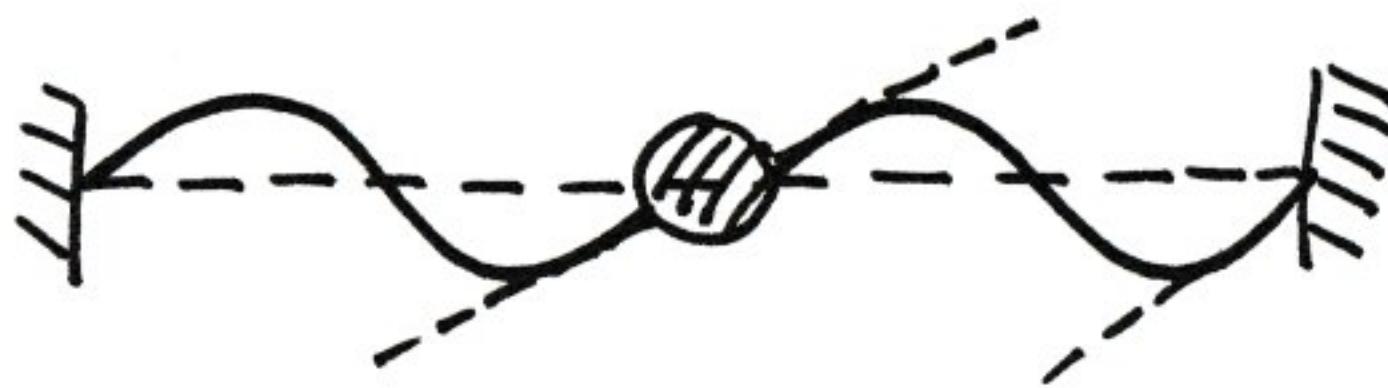
Note that this additional class of modes corresponds to motionless mass M , which acts as a rigid boundary condition at $x=L/2$.



$$\varphi_{21}'(L) = -C_2 \left(-\frac{\omega_1}{c}\right) \cos 0 = C_2 \frac{\omega_1}{c} > 0 \quad \text{if } C_2 > 0$$

Slope
continuity at
 $x=L/2$

$k=2$



$k=3$



$$\begin{aligned} \varphi_{21}'\left(\frac{L}{2}+\right) &= -C_2 \left(-\frac{\omega_1}{c}\right) \cos \frac{\omega_1}{c} \left(L - \frac{L}{2}\right) = \\ &= \underbrace{\frac{C_2 \omega_1}{c} \cos \frac{\omega_1 L}{2c}}_{n} = -\frac{C_2 \omega_1}{c} < 0, \text{ if } C_2 > 0 \end{aligned}$$

Remark

This example shows the importance of not missing any solutions by assuming non-zero divisors!