Vibrating circular plate clamped at its boundary

Consider
$$\left.\begin{aligned} &\nabla^4 u + \frac{\gamma}{D} u_{tt} = 0 \quad \text{on} \quad \mathcal{D} = \{0 < r \leq a, \, 0 \leq \theta < 2\pi\} \\ &u(a,\theta,t) = 0 \\ &u_r(a,\theta,t) = 0 \\ &u, u_r \text{ bounded } \forall\, 0 \leq r \leq a. \end{aligned}\right\}$$

Seek solution based on separation of variables, $u(r,\theta,t) = e^{j\omega t} R(r)\Theta(\theta) \Rightarrow$

$\Rightarrow$ Should require that $\Theta(\theta) = e^{jn\theta}$, $n = 0, 1, 2, \ldots$ since otherwise the solution would not be 2π-periodic with respect to $\theta \Rightarrow$

$$\left.\begin{aligned} \Rightarrow \nabla^4(R\Theta) - \frac{\gamma}{D}\omega^2 (R\Theta) = 0 \\ \text{Let } \frac{\gamma}{D}\omega^2 \equiv k^4 \end{aligned}\right\} \Rightarrow (\nabla^2 - k^2)(\nabla^2 + k^2)(R\Theta) = 0 \Rightarrow$$

$$\Rightarrow \left\{\begin{aligned} (\nabla^2 - k^2)(R\Theta) = 0 \\ (\nabla^2 + k^2)(R\Theta) = 0 \end{aligned}\right\} \Rightarrow$$

If we satisfy these two equations we have a complete solution.

But $\nabla^2 \equiv \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}$ and $\Theta = e^{jn\theta} \Rightarrow$

$$\Rightarrow \begin{cases} \left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} - \dfrac{n^2}{r^2} - k^2\right) R = 0 \longrightarrow \text{Modified Bessel equation of } n\text{-th order} \\ \left(\dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} + \dfrac{n^2}{r^2} - k^2\right) R = 0 \longrightarrow \text{Bessel Equation of } n\text{-th order} \end{cases}$$

It follows that $R(r)$ should be in the form,

$$R_n(r) = c_1 J_n(kr) + c_2 J_n(jkr)$$

$J_n(jkr) \equiv I_n(kr)$
Modified Bessel function of the first kind

since only these functions are bounded as $r \to 0 \Rightarrow$ Let $V(r,\theta) = R(r)\Theta(\theta)$

$$\Rightarrow V_n(r,\theta) = J_n(kr)\,(a_1 \cos n\theta + b_1 \sin n\theta) + $$
$$+ J_n(jkr)\,(a_2 \cos n\theta + b_2 \sin n\theta)$$

Satisfying the boundary conditions at $r = a \Rightarrow$

$$\left.\begin{array}{ll} \Rightarrow u(a,\theta,t) = 0 \Rightarrow & J_n(ka)\,a_1 + J_n(jka)\,a_2 = 0 \quad \text{(cosine terms)} \\ & J_n(ka)\,b_1 + J_n(jka)\,b_2 = 0 \quad \text{(sine terms)} \\ u_r(a,\theta,t) = 0 \Rightarrow & J_n'(ka)\,a_1 + j J_n'(jka)\,a_2 = 0 \\ & J_n'(ka)\,b_1 + j J_n'(jka)\,b_2 = 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow$ It should be satisfied that

Frequency equation

$$\boxed{\frac{J_n'(ka)}{J_n(ka)} = \frac{j J_n'(jka)}{J_n(jka)}} \Rightarrow \text{obtain solutions } k_{n1}, k_{n2}, \ldots, k_{nm}, \ldots$$

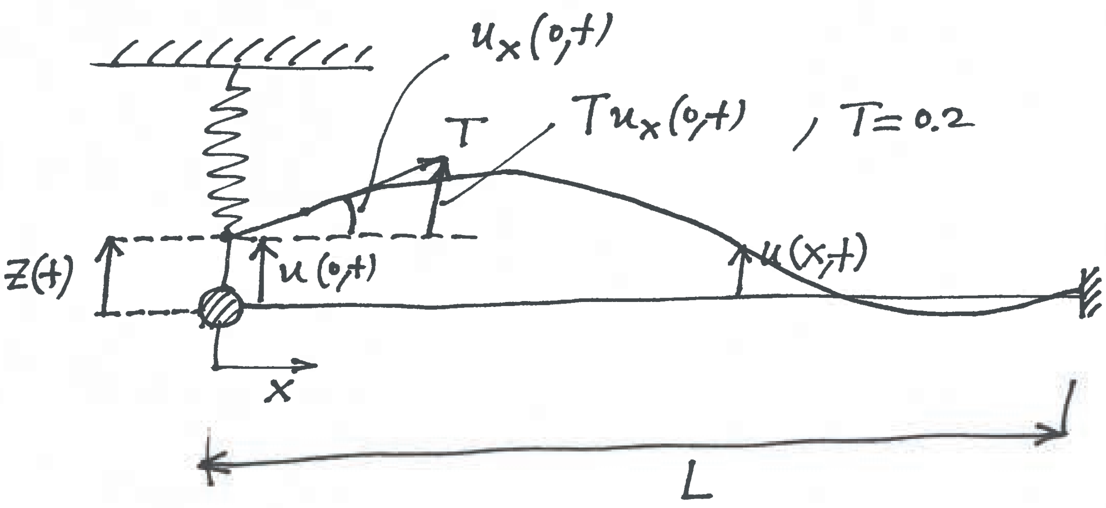
So, $\omega_{ni} = \sqrt{\frac{D}{\gamma}}\, k_{ni}^2 \,,\quad n = 0,1,3\ldots$
$i = 1,2,3,\ldots$

Hence, $V_{ij}(r,\theta) = \left[J_i\left(\frac{\lambda_{ij} r}{a}\right) - \frac{J_i(\lambda_{ij})}{I_i(\lambda_{ij})} I_i\left(\frac{\lambda_{ij} r}{a}\right) \right] (A_i \cos i\theta + B_i \sin i\theta)$

Where, $\lambda_{ij} = k_{ij} a$, with corresponding eigenfrequency,

$$\omega_{ij} = \frac{\lambda_{ij}^2}{a^2} \sqrt{\frac{D}{\gamma}}$$

and $i = 0, 1, 2, \ldots$, $j = 1, 2, 3, \ldots$



$$\left.\begin{array}{l} \dfrac{d^2z(t)}{dt^2} + z = 0.2\, u_x(0,t) \\ z(0+) = 0, \quad \dfrac{dz}{dt}(0+) = V \end{array}\right\} (1)$$

$$\left.\begin{array}{l} u_{xx} - u_{tt} = 0, \quad 0<x<L \\ u(x,0) = u_t(x,0) = 0 \\ u(0,t) = z(t) \\ u(L,t) = 0 \end{array}\right\} (2)$$

$\downarrow$

Laplace-transform with respect to time $\Rightarrow$ $\tilde{Z}(s) = \mathcal{L}[z(t)] \Rightarrow$

Take (1) $\Rightarrow \mathcal{L} \Rightarrow s^2\tilde{Z}(s) - s\cancelto{0}{z(0+)} - \overset{V}{\dot{z}(0+)} + \tilde{Z}(s) = 0.2\,\dfrac{d\tilde{U}(0,s)}{dx}$,

where $\tilde{U}(x,s) = \mathcal{L}[u(x,t)] \Rightarrow$

$$\Rightarrow (s^2+1)\tilde{Z}(s) = 0.2\,\frac{d\tilde{U}(0,s)}{dx} + V \qquad (3)$$

Then $\mathcal{L}$-transform (2) $\Rightarrow$ $\dfrac{d^2\tilde{U}(x,s)}{dx^2} - s^2\tilde{U}(x,s) + s\underbrace{\cancel{u(x,0)}}_{0} + \underbrace{\cancel{u_t(x,0)}}_{0} = 0 \Rightarrow$

$\Rightarrow \dfrac{d^2\tilde{U}(x,s)}{dx^2} - s^2\tilde{U}(x,s) = 0$ (4) $\Rightarrow$ Solving (4) $\Rightarrow$ $\tilde{U}(x,s) = \underbrace{\cancel{A}}_{0}\cosh[s(L-x)] + B\sinh[s(L-x)]$

$u(L,t) = 0 \overset{\mathcal{L}}{\Longrightarrow} \tilde{U}(L,s) = 0$ (5)

$u(0,t) = \phi Z(t) \overset{\mathcal{L}}{\Longrightarrow} \tilde{U}(0,s) = Z(s)$ (6) $\Rightarrow$

Applying (5) $\Rightarrow$ $\tilde{U}(L,s) = 0 \Rightarrow A = 0$

Applying (6) $\Rightarrow$ $\tilde{U}(0,s) = Z(s) \Rightarrow B\sinh sL = Z(s) \Rightarrow B = \dfrac{Z(s)}{\sinh sL}$

$$\Rightarrow \boxed{\tilde{U}(x,s) = \frac{Z(s)}{\sinh sL}\sinh[s(L-x)]} \quad (7a)$$

Now, substituting (7a) into (3) $\Rightarrow$ $(s^2+1)\tilde{Z}(s) = -0.2s\tilde{Z}(s)\coth(sL) + V \Rightarrow$

$$\Rightarrow [s^2 + 0.2s\coth(sL) + 1]\tilde{Z}(s) = V \Rightarrow \boxed{\tilde{Z}(s) = \frac{V}{s^2 + 0.2s\coth(sL) + 1}} \quad (7b)$$

Effect of the string interaction!

Unfortunately, no known inversion of (7b) exists...

So, $$\tilde{Z}(s) = \frac{V}{s^2 + 0.2\, s \coth sL + 1}$$

But $$\coth(sL) = \frac{e^{sL} + e^{-sL}}{e^{sL} - e^{-sL}} = \frac{1 + e^{-2sL}}{1 - e^{-2sL}} =$$

$$= \left(1 + e^{-2sL}\right)\left(1 + e^{-2sL} + e^{-4sL} + e^{-6sL} + \dots\right) =$$

$$= 1 + 2e^{-2sL} + 2e^{-4sL} + 2e^{-6sL} + \dots$$

Hence, $$\tilde{Z}(s) = \frac{V}{s^2 + 1 + 0.2s\left[1 + 2e^{-2sL} + 2e^{-4sL} + 2e^{-6sL} + \dots\right]} =$$

$$= \frac{V}{(s^2 + 1 + 0.2s) + 0.4\, s\, e^{-2sL}\left[1 + e^{-2sL} + e^{-4sL} + \dots\right]} =$$

$$\frac{1}{1+x} = 1 - x + x^2 \dots$$

$$= \frac{V}{s^2 + 1 + 0.2s}\left[\frac{1}{1 + \frac{0.4\, s\, e^{-2sL}}{s^2 + 0.2s + 1}\left\{1 + e^{-2sL} + e^{-4sL} + \dots\right\}}\right] \Rightarrow$$

$$\Rightarrow \tilde{Z}(s) = \frac{V}{s^2+0.2s+1}\left[1 - \frac{0.4se^{-2sL}}{s^2+0.2s+1}\left\{1+e^{-2sL}+e^{-4sL}+\dots\right\} + \left(\frac{0.4se^{-2sL}}{s^2+0.2s+1}\right)^2\left\{1+e^{-2sL}+e^{-4sL}+\dots\right\}^2 + \dots\right] \Rightarrow$$

$$\Rightarrow \tilde{Z}(s) = \frac{V}{s^2+0.2s+1} - \frac{0.4Vse^{-2sL}}{(s^2+0.2s+1)^2} - \frac{0.4Vse^{-4sL}}{(s^2+0.2s+1)^3} + \frac{V(0.4)^2s^2e^{-4sL}}{(s^2+0.2s+1)^3} + \dots \Rightarrow z(t) = \mathcal{L}^{-1}\left[Z(s)\right] \Rightarrow$$

Response of a damped SDOF oscillator

This term contributes to the response after time $t=\frac{2L}{c}$, $c=1$, i.e., after the first wave gets reflected from the fixed boundary at $x=L$.

$$\Rightarrow z(t) = \mathcal{L}^{-1}\left[\frac{V}{s^2+0.2s+1}\right] - \mathcal{L}^{-1}\left[\frac{0.4Vse^{-2sL}}{(s^2+0.2s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{\left[(0.4s)^2-0.4s\right]Ve^{-4sL}}{(s^2+0.2s+1)^3}\right] + \dots$$

This term contributes to the response after time $t=\frac{4L}{c}$, $c=1$, after it is reflected twice from the boundary at $x=L$

Note that $\mathcal{L}^{-1}\left[\dfrac{V}{s^2+0.2s+1}\right] = \dfrac{V e^{-0.1t}\sin 0.99t}{0.99}$

$$\mathcal{L}^{-1}\left[\frac{0.4 V s\, e^{-2sL}}{(s^2+0.2s+1)^2}\right] = 0.4V\,\mathcal{U}(t-2L)\,\mathcal{L}^{-1}\left[\frac{s}{(s^2+0.2s+1)^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{[(0.4s)^2-0.4s]\,V e^{-4sL}}{(s^2+0.2s+1)^3}\right] = V\,\mathcal{U}(t-4L)\,\mathcal{L}^{-1}\left[\frac{(0.4s)^2-0.4s}{(s^2+0.2s+1)^3}\right]$$

...

Remark 1: What happens as $L\to\infty$?

Recall that $\tilde{Z}(s) = \dfrac{V}{s^2+0.2s\coth(sL)+1}$

But $\coth(sL) = \dfrac{1+e^{-2sL}}{1-e^{-2sL}} \Rightarrow$ As $L\to\infty \Rightarrow \coth(sL)\to 1$ $\Bigg\} \Rightarrow$

$\Rightarrow \lim\limits_{\text{as } L\to\infty} \tilde{Z}(s) = \dfrac{V}{s^2+0.2s+1} \Rightarrow$ Effect of semi-infinite string (causes radiation) is exactly a "viscous damping" term in the oscillator, strength of "damping" is related to internal tension T in string.

But what about the response of the string as $L \to \infty$? Recall that,

$$\tilde{V}(x,s) = \frac{Z(s)}{\sinh(sL)} \sinh[s(L-x)]$$

But,
$$\frac{\sinh[s(L-x)]}{\sinh(sL)} = \frac{e^{s(L-x)} - e^{-s(L-x)}}{e^{sL} - e^{-sL}} = \frac{e^{-sx} - e^{-2sL} e^{sx}}{1 - e^{-2sL}} \Rightarrow$$

$$\Rightarrow \text{As } L \to \infty, \quad \frac{\sinh[s(L-x)]}{\sinh(sL)} \sim e^{-sx} \Rightarrow$$

$\Rightarrow$ Combining with the previous result, as $L \to \infty$ we get

$$u(x,t) \sim \mathcal{L}^{-1}\left[\frac{V e^{-sx}}{s^2 + 0.2s + 1}\right] = \frac{V e^{-0.1(t-x)} \sin 0.99(t-x)}{0.99} \, u(t-x)$$

Note that the effect of lack of dispersion in the semi-infinite string is evident!

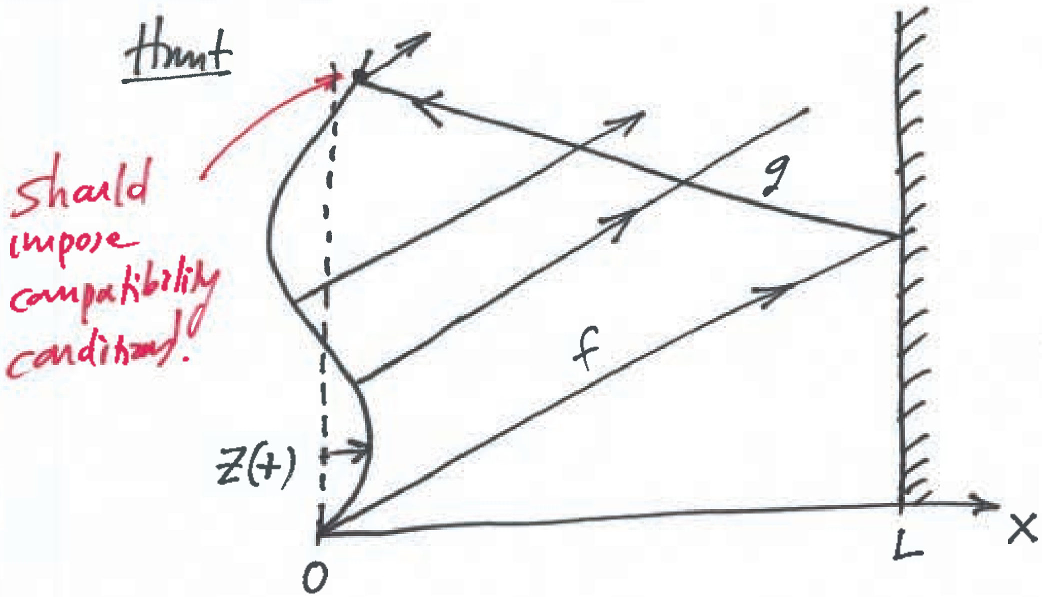
Farewell note: Open problems

1) What are the effects of viscous damping and/or elastic foundation on the string?

2) What happens when you replace the string by a beam?

3) Could you solve the previous problem using the method of characteristics?

Hint



4) Study condition(s) of resonance when the oscillator is forced by a periodic force.