TAM 514 /AE 551 HOMEWORK 5

Distributed: **4/14/2025**
Due: **5/5/2025 in class (for on-line students, the deadline for submission by email is 1pm CST on the due date)**

1 (100 pts). Compute the Green's function for the eigenvalue problem of the simply supported Euler-Bernoulli beam:

$$u''''(x) - \lambda u = f(x), \quad 0 \le x \le 1$$
$$u(0) = u''(0) = u(1) = u''(1) = 0$$

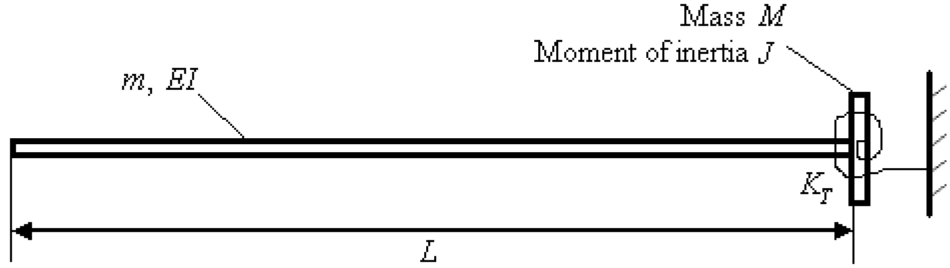
You should use two alternative definitions for the linear operator and construct two different Green's function formulations. In each case verify that the Green's function is symmetric with respect to its arguments. Then, use the derived Green's functions to convert the boundary value problem to an integral equation (do that for each of the two Green's functions).
<u>Hint</u>: In the first case define $L[u] = u''''(x)$, and in the second $L[u] = u''''(x) - \lambda u$.

2 (100 pts). Prove that for a second order, linear time-invariant self-adjoint operator, the Green's function is symmetric with respect to its arguments,

$$K(x,\xi) = K(\xi,x)$$

for $x,\xi \in G = [x_0, x_1]$ (the domain of independent variable). This proves reciprocity in systems governed by time-invariant self-adjoint operators.

3 (200 points). Consider the following homogeneous beam supported only by a torsional spring mounted on a disk at its end. Ignore gravity effects and select your own (reasonable) values for the system parameters.
(i) Formulate the eigenvalue problem and determine the frequency equation. Show graphically the leading natural frequencies. Show the orthonormality conditions satisfied by the normal modes.
(ii) Study the limiting case when this system behaves as a single-degree-of-freedom oscillator.
(iii) Formulate the Rayleigh quotient for this system and estimate its first natural frequency. Compare with the exact value determined in (a).
(iv) Derive a reduced-order model for this system by the Rayleigh-Ritz approach and provide approximations for the three leading modes.



4 (200 points). Given,

$$u_{xx} - [1+\varepsilon\alpha(x)]u_{tt} = 0, \quad 0 \le x \le 1$$
$$u(0,t) = u(1,t) = 0$$

where $0 < \varepsilon << 1$ is a small parameter, and $\alpha(x) = A\sin(\pi x/2),\ x \in [0,1]$ a smooth function. Compute the first order analytical approximation of the eigensolution of this problem by means of a perturbation approach.

(i) Introduce a space-time separation by expressing the solution as $u(x,t) = \varphi(x)\ e^{j\omega t}$

(ii) Solve approximately the resulting eigenvalue problem governing $\varphi(x)$ by expressing the solution in the form, $\varphi(x) = \varphi_0(x) + \varepsilon\varphi_1(x) + \ldots$, expanding also the frequency in the form $\omega = \omega_0 + \varepsilon\omega_1 + \ldots$, and substituting into the eigenvalue problem

(iii) By grouping terms of $O(1)$ and $O(\varepsilon)$ obtain the two eigenvalue subproblems that govern the leading-order approximations $\varphi_0(x)$ and $\varphi_1(x)$.

(iv) Solve these eigenvalue subproblems to obtain an analytic approximation for the eigensolution of the original problem correct to $O(\varepsilon)$. Orthonormalize the approximations derived for the eigenfunctions.

5. (T.K. Caughey) (100 points) The dynamics of a long bridge due to a crossing locomotive is described approximately by the following equation,

$$EIu_{xxxx} + cu_t + mu_{tt} = (M_1 + \omega^2 M_2 cos\omega t)\delta(x - Vt), \ \ 0 \le x \le L, \qquad t \ge 0$$
$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0$$

where M_1 is the mass of the locomotive, M_2 the mass of the unbalanced rotating and reciprocating parts with ω being the frequency of rotation in (rad/sec) of the drive wheels, and V is the forward velocity of the locomotive. Assume that the bridge is underdamped.

(i) Solve this problem assuming that the bridge is at rest when the train enters.

(ii) Under what conditions can resonance occur, and the bridge be in danger?