Rayleigh Quotient (continued)

$$\frac{d}{dx}\left[A(x)\varphi'(x)\right] + \omega^2 B(x)\varphi(x) = 0, \quad 0 \le x \le L, \; t \ge 0 \qquad (1)$$

$$A(0)\varphi'(0) - (K_1 - \omega^2 M_1)\varphi(0) = 0 \qquad (1a)$$

$$A(L)\varphi'(L) + (K_2 - \omega^2 M_2)\varphi(L) = 0 \qquad (1b)$$

Two forms of RQ,

$\varphi(x)$ should be admissible functions

$$\omega^2 = R[\varphi(x)] = \frac{K_1\varphi^2(0) + K_2\varphi^2(L) + \int_0^L A(x)\varphi'^2(x)\,dx}{M_1\varphi^2(0) + M_2\varphi^2(L) + \int_0^L B(x)\varphi^2(x)\,dx} \qquad (3b)$$

or,

$\varphi(x)$ should be comparison functions

$$\omega^2 = R[\varphi(x)] = \frac{-\int_0^L \frac{d}{dx}\left[A(x)\varphi'(x)\right]\varphi(x)\,dx}{\int_0^L B(x)\varphi^2(x)\,dx} \qquad (3a)$$

error in the eigenfunction

<u>Rayleigh's principle:</u> If $\varphi(x) = \varphi_r(x) + \underbrace{\sum_{\substack{i=1 \\ i\ne r}}^{\infty} \varepsilon_i \varphi_i(x)}_{\text{'error'}}$, $\quad \varepsilon_i = O(\epsilon)$, $0 < \epsilon \ll 1$, $i = 1, 2, \dots$

Then, $\omega^2 = R[\varphi(x)] = \omega_r^2 + \sum_{i=1}^{\infty}(\omega_i^2 - \omega_r^2)\varepsilon_i^2 \qquad (4)$

Error in estimated nat. frequency

Hence, $\boxed{\omega^2 = R[\varphi(x)] \geq \omega_1^2}$ (5)

So, RQ is always greater or equal than the first natural frequency squared.
Moreover, assuming that the test function that we use is orthogonal to the first k modes of the system, $\int_0^L B(x)\,\varphi(x)\,\varphi_i(x)\,dx = 0,\ i=1,\dots,k$

$\Rightarrow$ Then, $RQ[\varphi(x)] \geq \omega_{k+1}^2$.

Based on Rayleigh Quotient it is possible to formulate approximate numerical techniques for approximating the modes of systems for which no analytical solutions of the corresponding eigenvalue problems exist.

Rayleigh-Ritz Method

Suppose that we approximate the test function as,

$$\varphi(x) = \sum_{i=1}^{n} \alpha_i\, \psi_i(x) \qquad (1)$$

where $\psi_i(x), i=1,\dots,n$ are admissible functions $\Rightarrow$ We need to use RQ in the form (3b) to account explicitly for the BCs $\Rightarrow$

$$\Rightarrow (3b) \Rightarrow \omega^2 = R[\varphi] = \frac{K_1\varphi^2(0)+K_2\varphi^2(L)+\int_0^L A(x)\varphi'^2(x)\,dx}{M_1\varphi^2(0)+M_2\varphi^2(L)+\int_0^L B(x)\varphi^2(x)\,dx} = \frac{N(\alpha_1,\alpha_2,\dots,\alpha_n)}{D(\alpha_1,\alpha_2,\dots,\alpha_n)} \quad (2)$$

$$\varphi(x) = \sum_{i=1}^{n} \alpha_i \psi_i(x)$$

unknown (α_i); known admissible functions (ψ_i)

Using Rayleigh's principle, we seek a stationary point (a minimum actually) in the n-dim parameter space $\underline{\alpha} = [\alpha_1, \dots, \alpha_n]^T$, such that $R[\varphi]$ becomes minimum $\Rightarrow$ $\frac{\partial R[\varphi]}{\partial \alpha_i} = 0,\ i=1,\dots,n \Rightarrow$

$$\Rightarrow \frac{\partial}{\partial \alpha_i}\left[\frac{N(\underline{\alpha})}{D(\underline{\alpha})}\right] = 0,\ i=1,\dots,n \Rightarrow \frac{\frac{\partial N}{\partial \alpha_i}D - \frac{\partial D}{\partial \alpha_i}N}{D^2} = 0,\ i=1,\dots,n \Rightarrow$$

$$\Rightarrow \frac{1}{D}\frac{\partial N}{\partial \alpha_i} - \frac{1}{D^2}\frac{\partial D}{\partial \alpha_i}N = 0 \Rightarrow \frac{1}{D}\left[\frac{\partial N}{\partial \alpha_i} - \frac{N}{D}\frac{\partial D}{\partial \alpha_i}\right] = 0 \Rightarrow \boxed{\frac{\partial N}{\partial \alpha_i} - \Omega^2 \frac{\partial D}{\partial \alpha_i} = 0,\quad i=1,\dots,n} \quad (3)$$

But this is RQ, so it provides the estimate for the nat. freq. squared, call it Ω^2

This represents the discretization of the original eig. pr.

Hence, $D(\underline{a}) = M_1\left(\sum_{i=1}^{n} a_i \psi_i(0)\right)\left(\sum_{j=1}^{n} a_j \psi_j(0)\right) +$

$$+M_2\left(\sum_{i=1}^{n} a_i \psi_i(L)\right)\left(\sum_{j=1}^{n} a_j \psi_j(L)\right) +$$

$$+\int_0^L B(x)\left[\sum_{i=1}^{n} a_i \psi_i(x)\right]\left[\sum_{j=1}^{n} a_j \psi_j(x)\right] dx =$$

$$= M_1\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \psi_i(0)\psi_j(0)\right) + M_2\left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \psi_i(L)\psi_j(L)\right) +$$

$$+\int_0^L B(x) \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \psi_i(x)\psi_j(x)\, dx =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \underbrace{\left[M_1 \psi_i(0)\psi_j(0) + M_2 \psi_i(L)\psi_j(L) + \int_0^L B(x)\psi_i(x)\psi_j(x)\, dx\right]}_{m_{ij}} \Rightarrow$$

$$\Rightarrow D(\underline{a}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j m_{ij} = [a_1 \ldots a_n]\begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ \vdots & \ddots & & \vdots \\ m_{n1} & m_{n2} & \ldots & m_{nn} \end{bmatrix}\begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} =$$

$$= \{a\}^T [m] \{a\} \qquad (4a)$$

$(n \times n)$ discretised mass matrix

So, $D(\underline{a}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j m_{ij} \Rightarrow \frac{\partial D}{\partial a_r} = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\Big(\underbrace{\frac{\partial a_i}{\partial a_r}}_{\delta_{ir}} a_j + \underbrace{\frac{\partial a_j}{\partial a_r}}_{\delta_{jr}} a_i\Big) =$

$= \sum_{i=1}^{n} m_{ir} a_i + \sum_{j=1}^{n} \underbrace{m_{rj}}_{= m_{jr}} a_j = 2\sum_{i=1}^{n} m_{ir} a_i \Rightarrow$

$$\Rightarrow \frac{\partial D}{\partial a_r} = 2\sum_{i=1}^{n} m_{ir} a_i, \quad r = 1, \ldots, n \qquad (4b)$$

Taking into account that $m_{ij} = m_{ji}$ since $[m]$ is a symmetric matrix.

Similarly we deal with $N(\underline{a})$,

$N(\underline{a}) = K_1\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \psi_i(0)\psi_j(0)\Big) + K_2\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \psi_i(L)\psi_j(L)\Big) +$

$+ \int_0^L A(x) \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \psi_i'(x)\psi_j'(x)\, dx \Rightarrow$

$\Rightarrow N(\underline{a}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \underbrace{\Big[K_1 \psi_i(0)\psi_j(0) + K_2 \psi_i(L)\psi_j(L) + \int_0^L A(x)\psi_i'(x)\psi_j'(x)\, dx\Big]}_{k_{ij}} \Rightarrow N(\underline{a}) = \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} a_i a_j \Rightarrow$

$$\Rightarrow N(\underline{a}) = \{a\}^T \underbrace{\begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix}}_{[k]} \{a\}, \quad [k]=[k]^T \tag{5a}$$

Then, working similarly we can compute,

$$\boxed{\frac{\partial N}{\partial a_r} = 2\sum_{i=1}^{n} k_{ir} a_i, \quad r=1,\dots,n} \tag{5b}$$

So, substituting (4b) and (5b) into our stationarity condition (3), we get the discretized n-dim eigenvalue problem

$$\sum_{i=1}^{n} k_{ri} a_i - \Omega^2 \sum_{i=1}^{n} m_{ri} a_i = 0, \quad r=1,\dots,n \Rightarrow$$

$$\Rightarrow -\Omega^2 \begin{bmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} + \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \quad \boxed{-\Omega^2 [m]\{a\} + [k]\{a\} = 0} \tag{6}$$

Hence we reduced the original infinite-dimensional elastodynamic problem (1) to an n-dim discretised eigenvalue problem.

The plan is to estimate the n-values $\Omega_1^2, \dots, \Omega_n^2$ by setting

$$\det\left[[k] - \Omega^2[m]\right] = 0$$

Then, for each estimated nat. frequency Ω_r^2, to solve

$$\left([k] - \Omega^2[m]\right)\{a^{(r)}\} = \{0\}$$

and to obtain the corresponding eigenvector $\{a^{(r)}\} = [a_1^{(r)} \dots a_n^{(r)}]^T$.

Then, the r-th eigenfunction of the original problem (1) is estimated according to the definition of the test function,

$$\varphi^{(r)}(x) \sim \sum_{i=1}^{n} a_i^{(r)} \psi_i(x), \quad r = 1, \dots, n$$

When we have 'good' initial guesses for the admissible functions $\psi_i(x)$, it turns out that,

$$\begin{array}{ccccccccc} \Omega_1 < & \Omega_2 < & \Omega_3 < & \dots & < \Omega_n & & \\ \vee & \vee & \vee & & \vee & & \\ \omega_1 < & \omega_2 < & \omega_3 < & \dots & < \omega_n & < \omega_{n+1} & < \dots \end{array}$$

To accurately approximate the leading natural frequencies and mode shapes, we need to consider much more many 'modes'

Galerkin procedure

Considering again the infinite-dimensional eigenvalue problem (1), (1a), (1b), we again approximate the eigenfunction by,

(7) $\frac{d}{dx}\left[A(x)\frac{d\varphi(x)}{dx}\right] = -\omega^2 B(x)\varphi(x)$

$$\varphi(x) = \sum_{i=1}^{n} a_i \psi_i(x)$$

where $\psi_i(x)$ are now comparison functions, i.e., they need to satisfy the boundary conditions $\Rightarrow$ Substituting into the original problem,

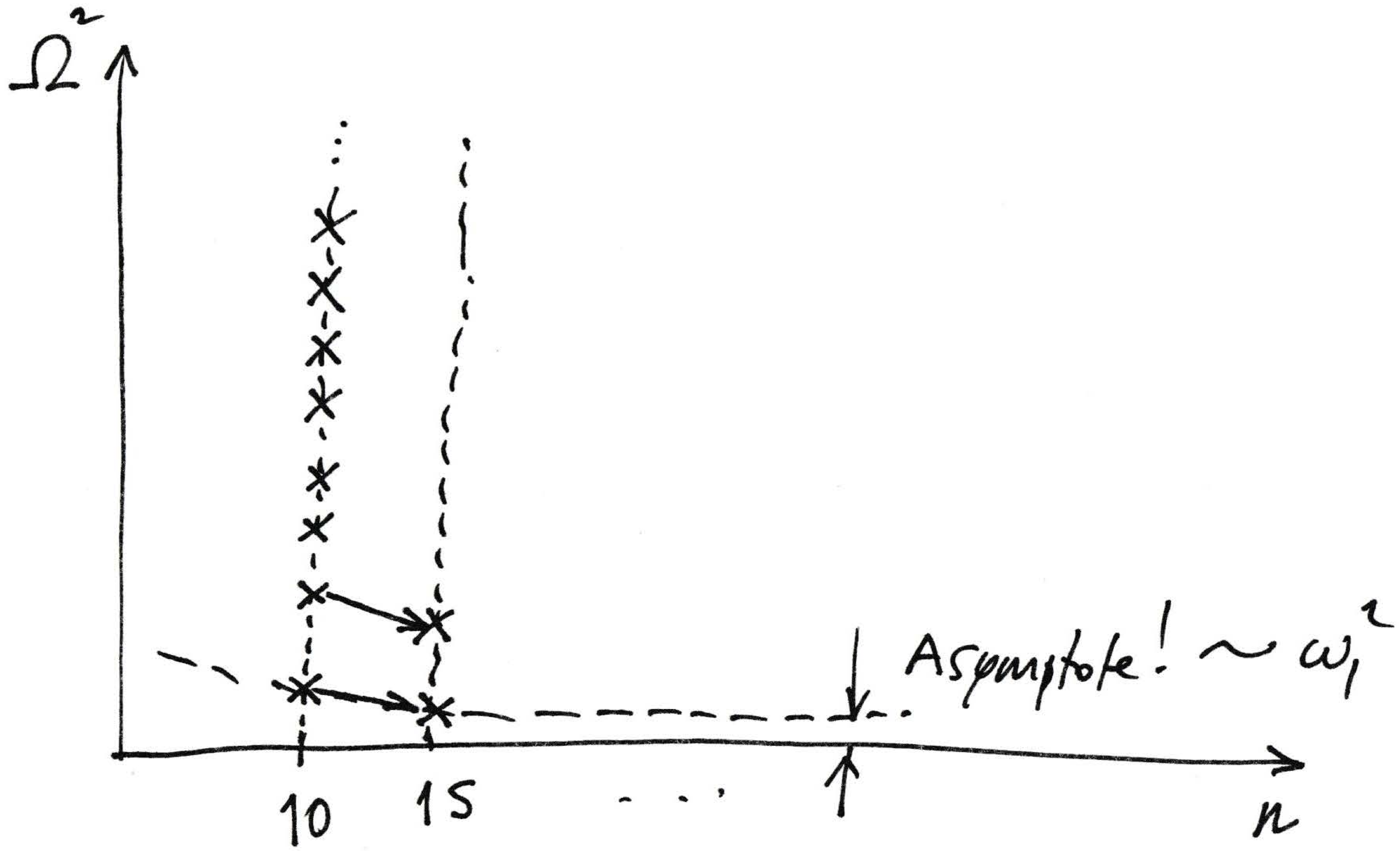
$$\sum_{i=1}^{n} \frac{d}{dx}\left[A(x) a_i \psi_i'(x)\right] + \omega^2 \sum_{i=1}^{n} B(x) a_i \psi_i(x) \cong \cancel{0} = \varepsilon(x)$$

Small error due to approximation

Since $\varepsilon(x) \neq 0$, the best we can do it to request that $\varepsilon(x)$ be orthogonal to each of the test functions $\psi_i(x) \Rightarrow$

$$\Rightarrow \boxed{\int_0^L \varepsilon(x)\psi_i(x)\,dx = 0, \quad i = 1, \dots, n} \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{n} \int_0^L \frac{d}{dx}\left[A(x) a_i \psi_i'(x)\right]\psi_j(x)\,dx + \omega^2 \sum_{i=1}^{n} \int_0^L B(x) a_i \psi_i(x)\psi_j(x)\,dx = 0 \Rightarrow$$



$$\frac{\Omega_1^{(n+1)} - \Omega_1^{(n)}}{\Omega_1^{(n)}} < 10^{-3}$$

$$\Rightarrow \sum_{i=1}^{n}\left[A(x)a_i\psi_i'(x)\psi_j(x)\right]_0^L - \sum_{i=1}^{n}\int_0^L A(x)a_i\psi_i'(x)\psi_j'(x)\,dx +$$

$$+\omega^2\sum_{i=1}^{n}\int_0^L B(x)a_i\psi_i(x)\psi_j(x)\,dx = 0 \Rightarrow$$

$\Rightarrow$ Taking into account that $\psi_i(x)$ are comparison functions $\Rightarrow$

$$\Rightarrow \sum_{i=1}^{n}\Big[-(k_2-\omega^2M_2)a_i\psi_i(L)\psi_j(L) -$$

$$-(k_1-\omega^2M_1)a_i\psi_i(0)\psi_j(0)\Big] - \sum_{i=1}^{n}\int_0^L A(x)a_i\psi_i'(x)\psi_j'(x)\,dx +$$

$$+\omega^2\sum_{i=1}^{n}\int_0^L B(x)a_i\psi_i(x)\psi_j(x)\,dx = 0, \quad j=1,2,\dots,n \Rightarrow$$

$$\Rightarrow -\sum_{i=1}^{n}a_i\overbrace{\left[K_1\psi_i(0)\psi_j(0)+K_2\psi_i(L)\psi_j(L)+\int_0^L A(x)\psi_i'(x)\psi_j'(x)\,dx\right]}^{k_{ij}} +$$

$$+\omega^2\sum_{i=1}^{n}a_i\underbrace{\left[M_1\psi_i(0)\psi_j(0)+M_2\psi_i(L)\psi_j(L)+\int_0^L B(x)\psi_i(x)\psi_j(x)\,dx\right]}_{m_{ij}} = 0 \Rightarrow$$

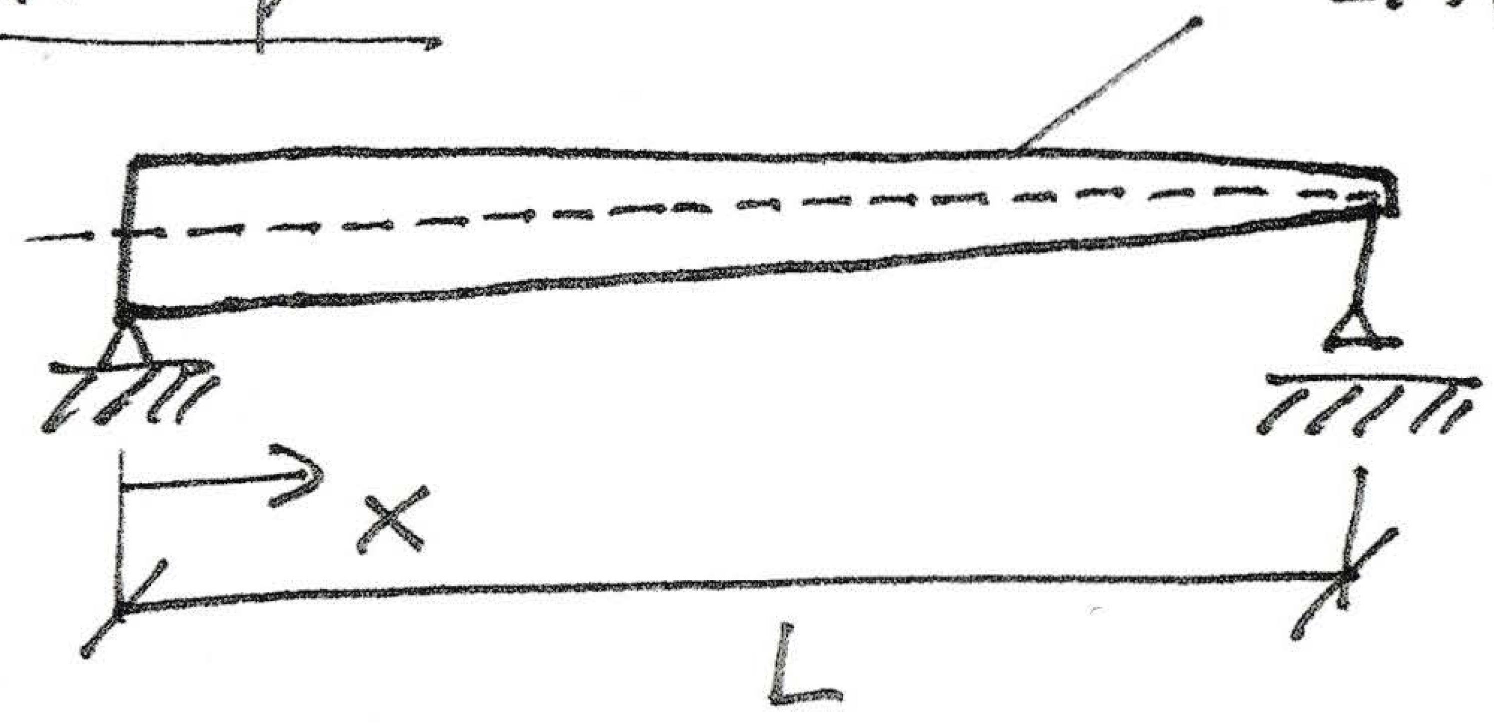
$$\Rightarrow -\sum_{i=1}^{n} a_i k_{ij} + \omega^2 \sum_{i=1}^{n} a_i m_{ij} = 0, \quad j=1,\dots,n \Rightarrow$$

$$\Rightarrow \begin{bmatrix} k_{11} & \dots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & \dots & k_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix} = \omega^2 \begin{bmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_n \end{Bmatrix}$$

Again we were able to discretize the continuous eigenvalue problem!

Example

$EA(x) = \frac{6}{5} EA\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right], \; m(x) = \frac{6}{5} m\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$



We need to find an approximation for the first natural frequency $\Rightarrow$ Since we only need to approximate ω_1 we can use Rayleigh's quotient,

$$\omega^2 = R[\varphi(x)] = \frac{\int_0^L EA(x)\, \varphi'(x)^2\, dx}{\int_0^L m(x)\, \varphi^2(x)\, dx}, \quad \varphi(x) \text{ are admissible functions.}$$

A good choice for a test function would be $\varphi(x) = \sin\frac{\pi x}{2L}$ which is the first eigenfunction of the corresponding problem with uniform properties $m(x)=m$, $EA(x)=EA$

Hence, $\int_0^L EA(x)\,\varphi'^2(x)\,dx = \frac{6}{5} EA\left(\frac{\pi}{2L}\right)^2 \int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right] \cos^2\frac{\pi x}{2L}\,dx = \frac{EA}{40L}(5\pi^2+6)$

$$\int_0^L m(x)\,\varphi^2(x)\,dx = \frac{6}{5} m \int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right] \sin^2\frac{\pi x}{2L}\,dx = \frac{mL}{10\pi^2}(5\pi^2-6)$$

Hence, $\omega_{1e}^2 \sim \dfrac{\frac{EA}{40L}(5\pi^2+6)}{\frac{mL}{10\pi^2}(5\pi^2-6)} = 3.1504\,\dfrac{EA}{mL^2} \Rightarrow \omega_{1e} \sim 1.7749\sqrt{\dfrac{EA}{mL^2}}$ rad/sec

Note that $\omega_{1e} \geq \omega_1$; also note that the corresponding first eigen-frequency of the rod with uniform properties is $\omega_{1u} = \frac{\pi}{2}\sqrt{\frac{EA}{mL^2}} =$

$$= 1.5708\sqrt{\frac{EA}{mL^2}}$$

So we see that $\omega_{1e} > \omega_{1u}$. This makes sense, since the non-uniform rod is much stiffer at the fixed end.