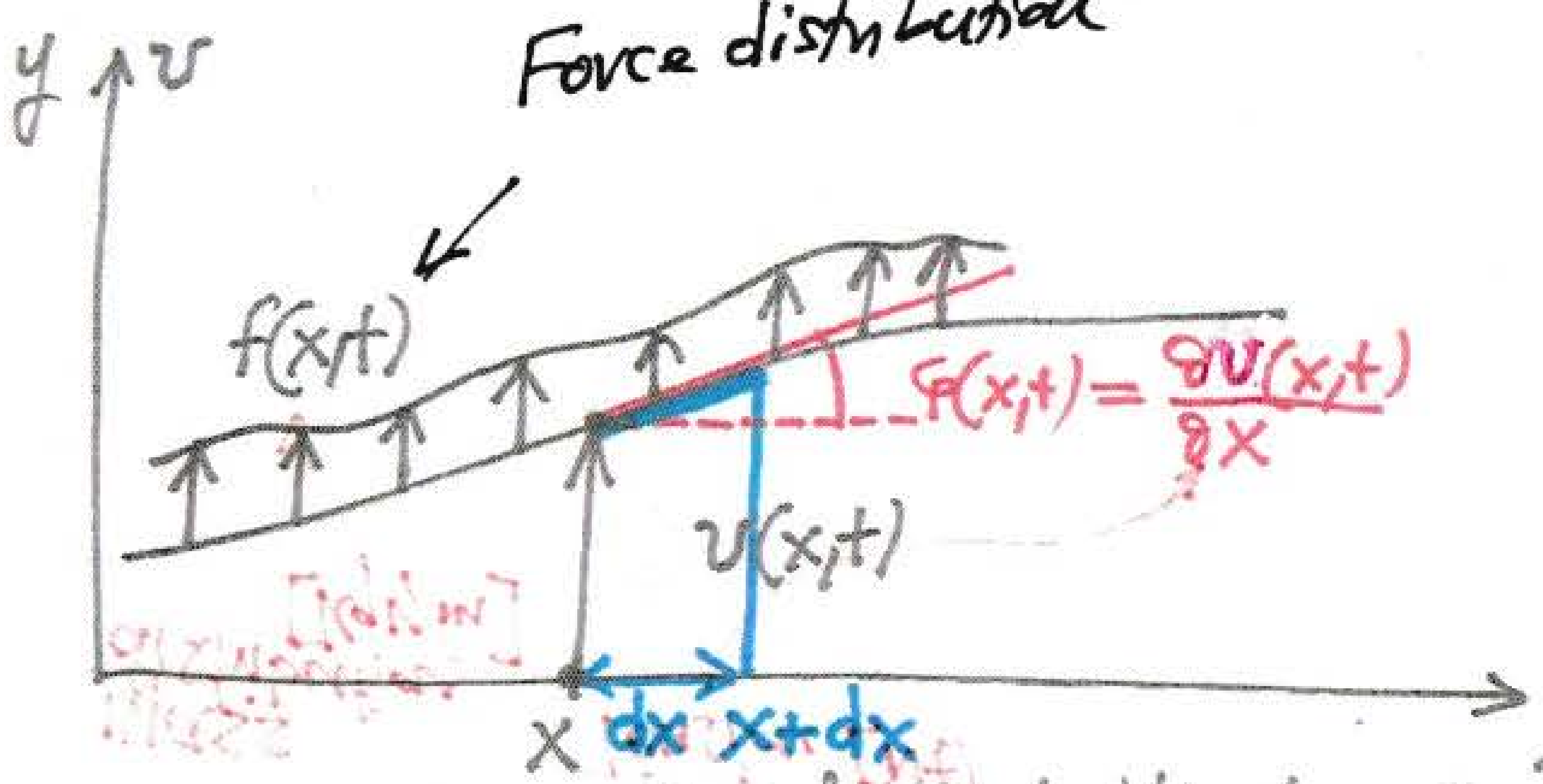
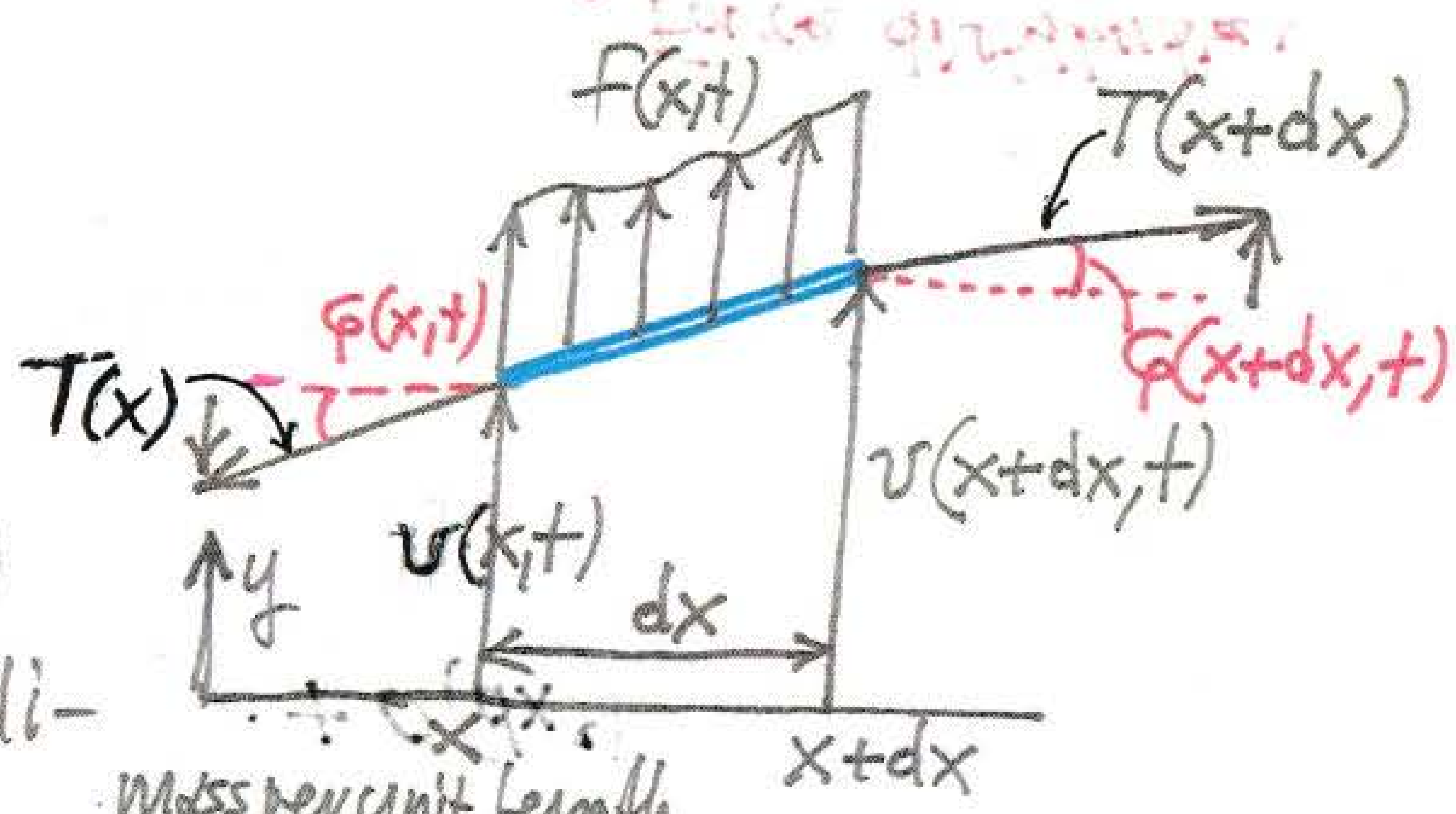
② Continuous Elastic Systems – Wave Equation

Ⓐ Transverse vibrations of a string



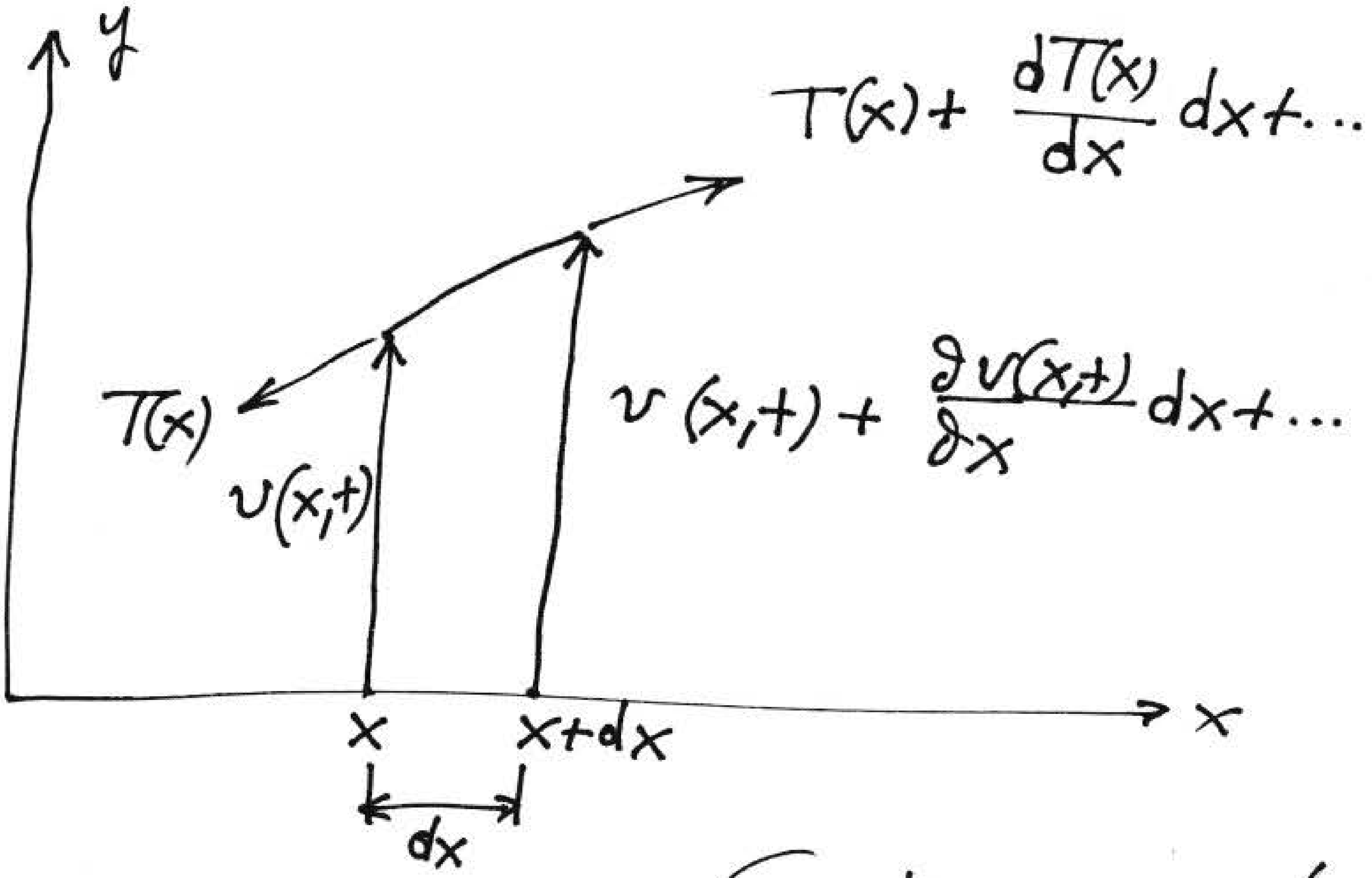
In order to derive the governing pde of motion we resort to infinitesimal theory of mechanics, take a differential element of the string and apply Newton's law in the vertical direction.



- $f(x,t)$ is the applied force distribution (Nt/m)
- Assume transverse oscillations of the string
- Assume small displacements and slopes
- There is an internal tension in the string, $T(x)$
- Each material point oscillates in the vertical direction only (approximation)
- The theory of infitesimal continuum mechanics holds (linearity)
- Elastic string does not support bending moment

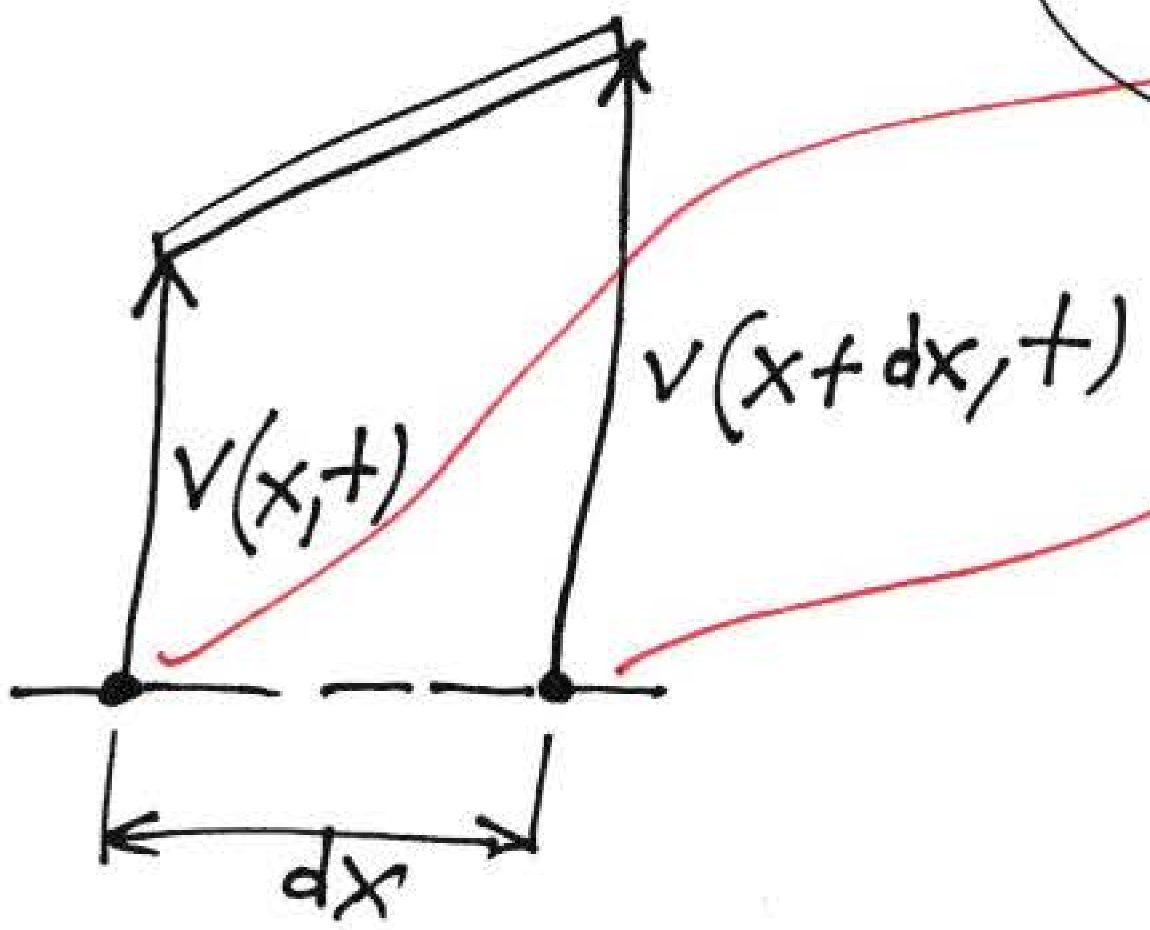
mass per unit length

$$m\,dx\,\frac{\partial^2 v}{\partial t^2} = T(x+dx)\underbrace{\frac{\partial v(x+dx,t)}{\partial x}}_{\varphi(x+dx,t)} - T(x)\underbrace{\frac{\partial v(x,t)}{\partial x}}_{\varphi(x,t)} + f(x,t)\,dx \Rightarrow 1 + O(dx^2)$$



Continuum mechanics

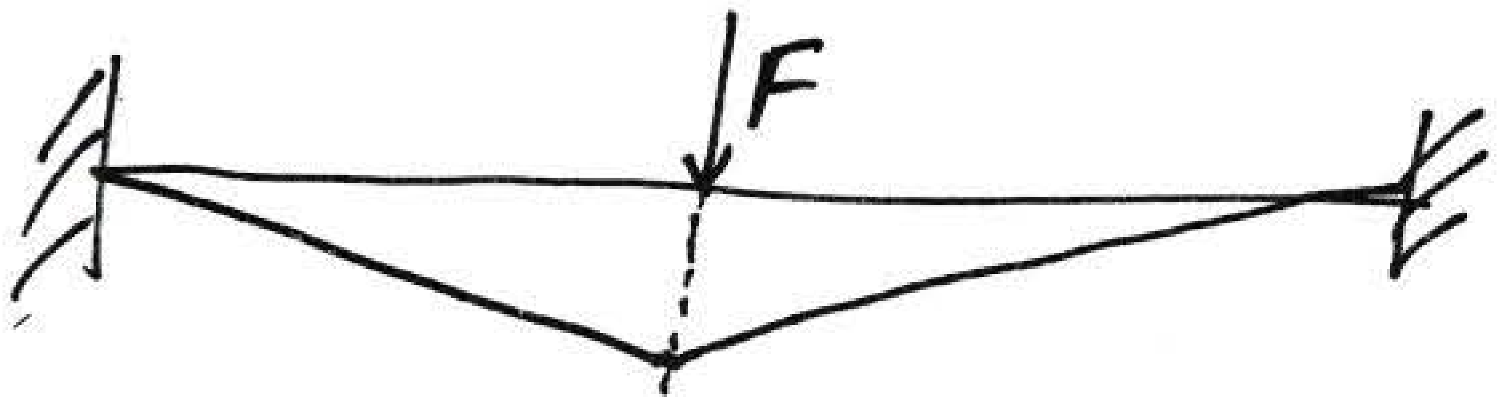
$$m\,dx\,\frac{\partial^2 v(x,t)}{\partial t^2} = \ldots$$



$$m\,dx\,\frac{\partial^2 v(x+dx,t)}{\partial t^2} =$$

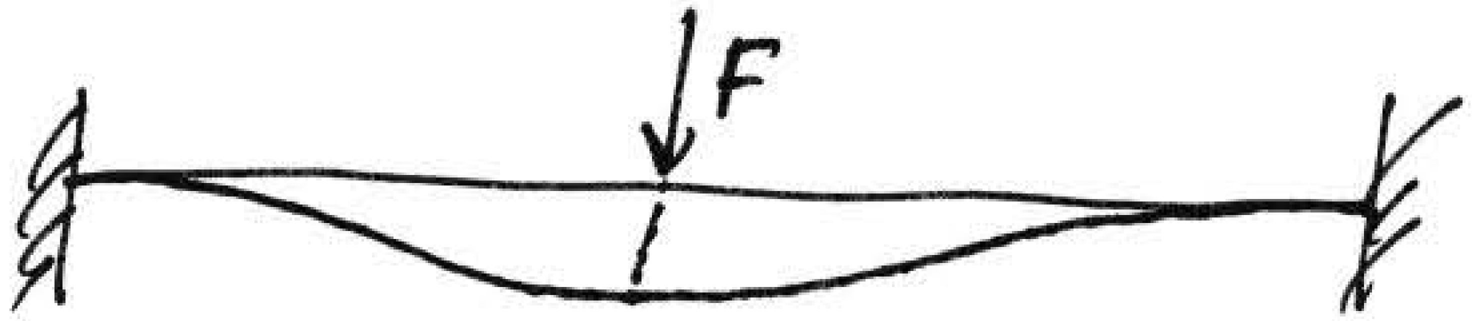
$$= m\,dx\left[\frac{\partial^2 v(x,t)}{\partial t^2} + \underbrace{\cancel{dx\,\frac{\partial^3 v(x,t)}{\partial t^2\,\partial x}+\ldots}}_{O(dx^2)}\right]$$

Chord



Wave equation

Beam

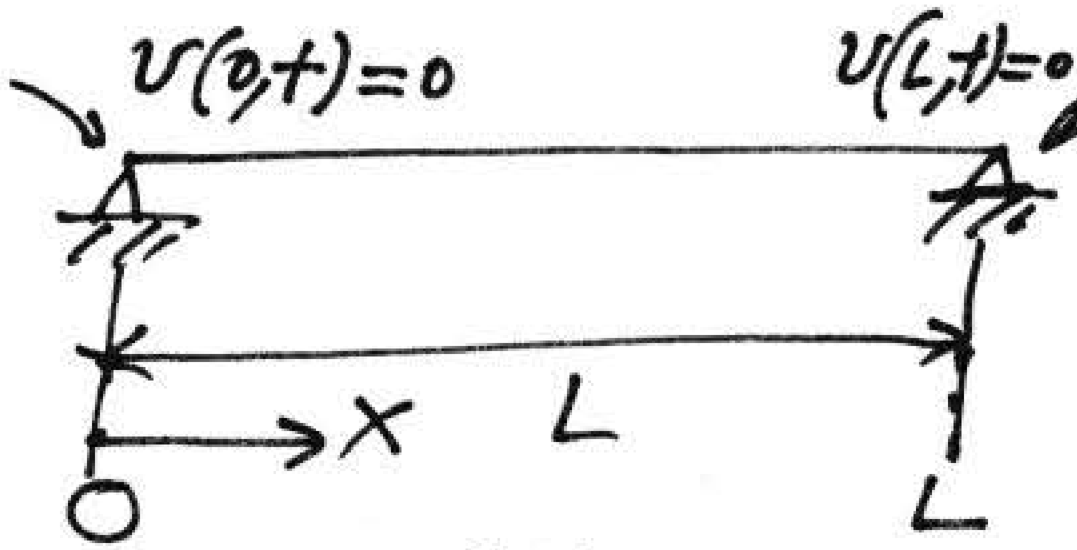


Beam equation

Bending moments

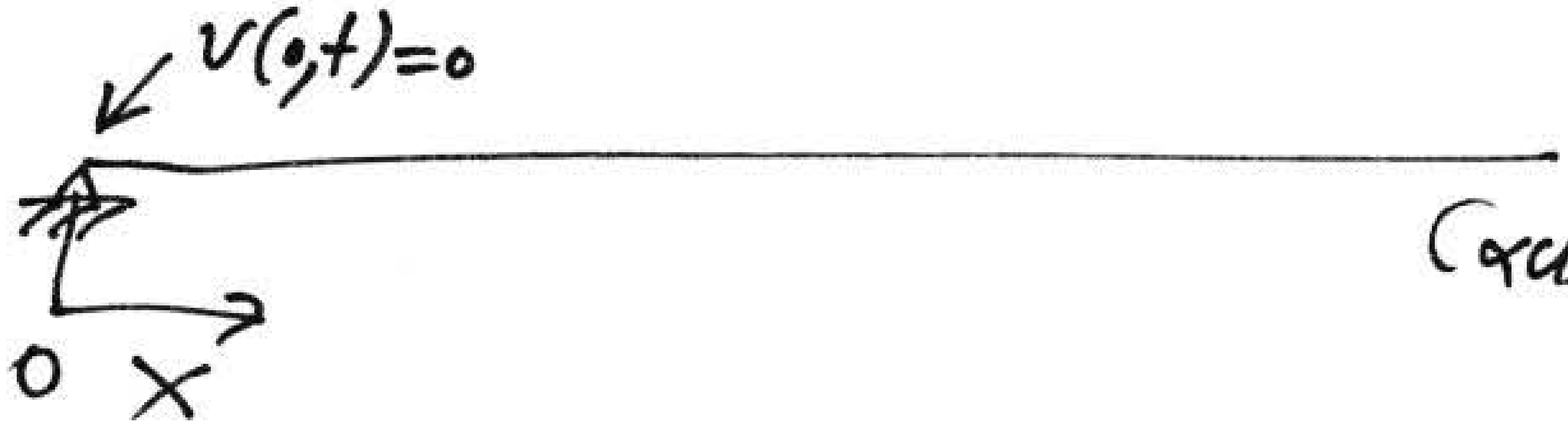
finite string

BC



BC

Semi-infinite string



Causality condition
at $x \to \infty$

$$\Rightarrow m(x)\,dx\,\frac{\partial^2 v(x,t)}{\partial t^2} = \underbrace{\left[T(x) + \frac{\partial T(x)}{\partial x}dx\right]}_{\sim T(x+dx)} \underbrace{\left[\frac{\partial v(x,t)}{\partial x} + \frac{\partial^2 v(x,t)}{\partial x^2}dx\right]}_{\sim \frac{\partial v}{\partial x}(x+dx,t)} - \quad + O(dx^2)$$

$$- T(x)\,\frac{\partial v(x,t)}{\partial x} + f(x,t)\,dx + O(dx^2) \Rightarrow$$

$$\Rightarrow m(x)\,\cancel{dx}\,\frac{\partial^2 v(x,t)}{\partial t^2} = \cancel{T(x)\frac{\partial v(x,t)}{\partial x}} + T(x)\,\frac{\partial^2 v(x,t)}{\partial x^2}\,\cancel{dx} + \frac{\partial T(x)}{\partial x}\,\cancel{dx}\,\frac{\partial v(x,t)}{\partial x} + O(dx^2)$$

$$- \cancel{T(x)\frac{\partial v(x,t)}{\partial x}} + f(x,t)\,\cancel{dx} \Rightarrow$$

mass distribution [kgr/m]

Tension [Nt]

Force distribution [Nt/m]

$$\Rightarrow \boxed{m(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x)\,\frac{\partial v}{\partial x}(x,t)\right] + f(x,t)}$$

Generalized wave equation

If we assume that $m(x) = m$ and $T(x) = T$, $f(x,t) = 0 \Rightarrow$

$\rightarrow$ Equation reduces to $\quad m\,\dfrac{\partial^2 v(x,t)}{\partial t^2} = T\,\dfrac{\partial^2 v(x,t)}{\partial x^2} \Rightarrow$

speed of sound squared

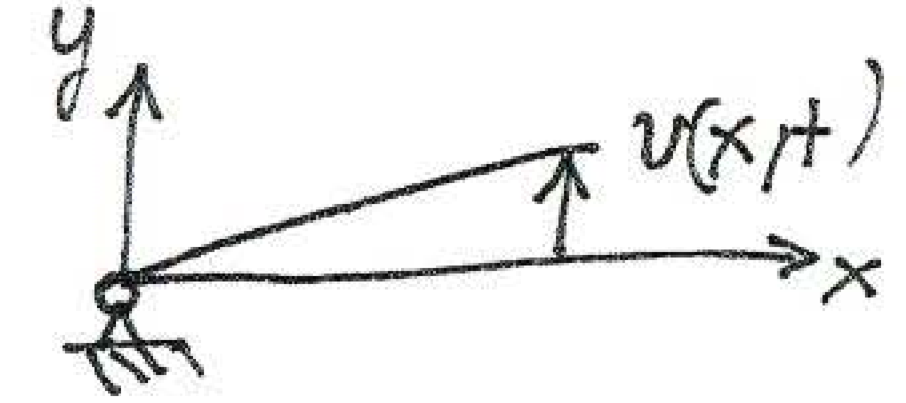
$$\boxed{\frac{\partial^2 v(x,t)}{\partial t^2} = \frac{T}{m}\,\frac{\partial^2 v(x,t)}{\partial x^2}}$$

Wave Equation

To solve this problem, in addition to the governing pde we need:

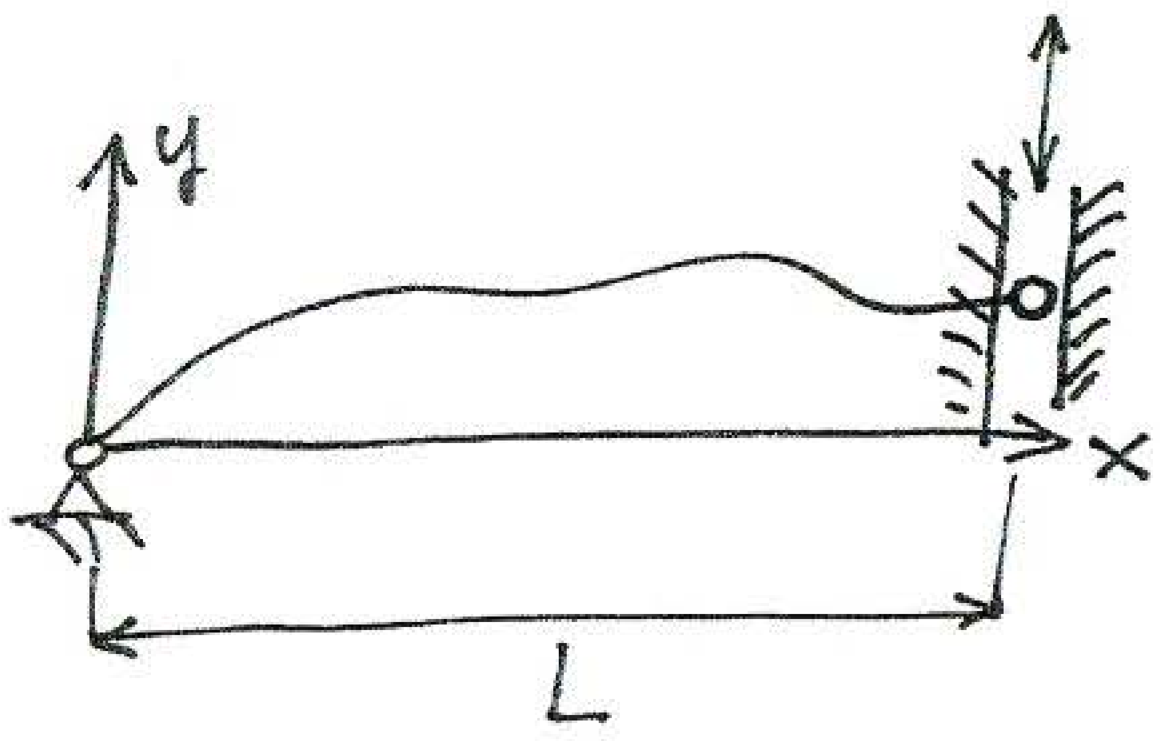
- Initial conditions: What are the initial displacement and velocity distributions at $t=0$?
- Boundary conditions: Need to specify two boundary conditions, one at each of the two ends (boundaries) of the string.

Fixed end:



$v(0,t)=0$ or $v(L,t)=0$ at the other boundary, where L is the length of the string. (Deformed length is nearly equal to undeformed length)

Free end:

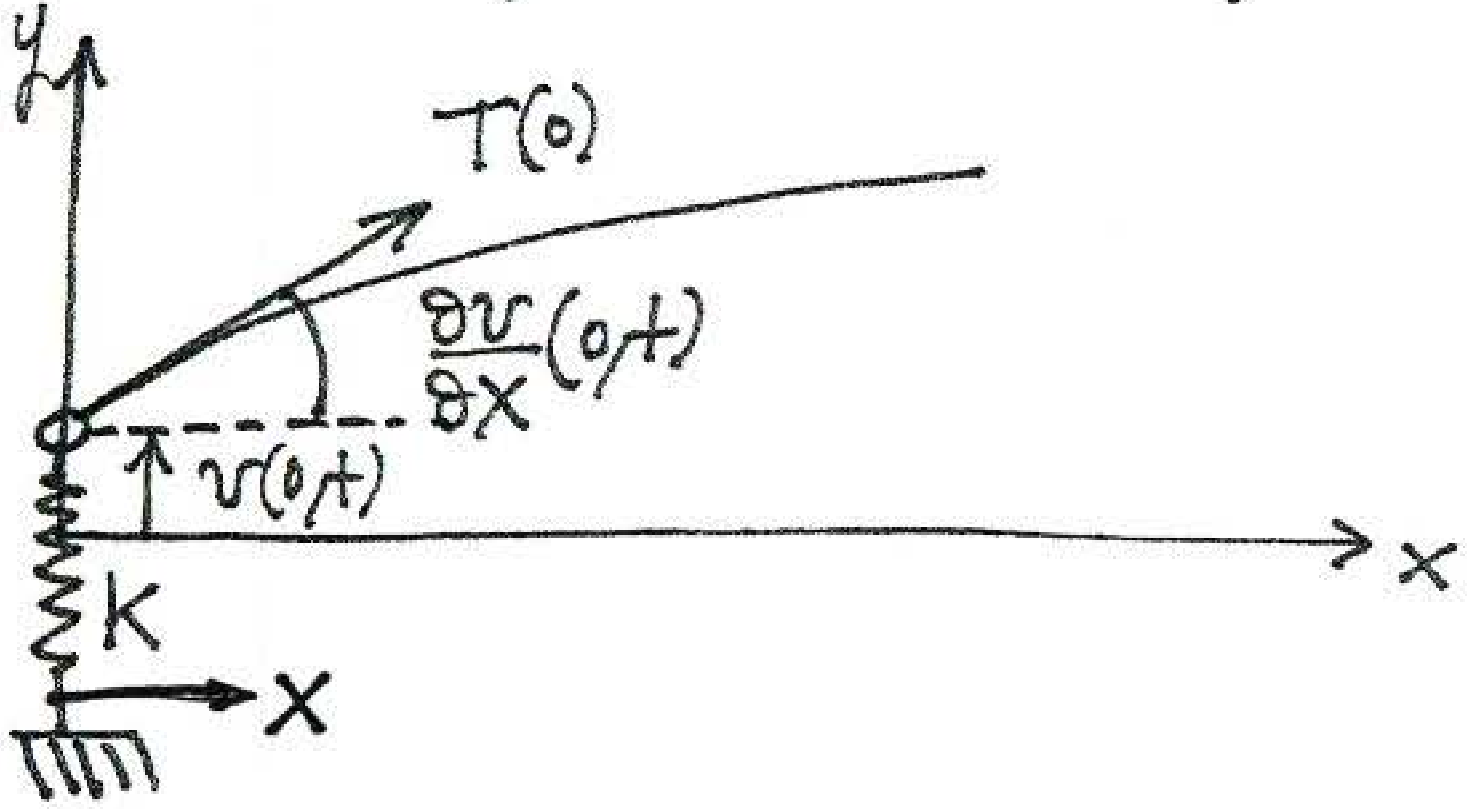


At the free end there is no vertical force applied $\Rightarrow$ $T(L)\dfrac{\partial v(L,t)}{\partial x}=0$ $\Rightarrow$

$\Rightarrow$ Assuming that $T(L)\neq 0$ $\Rightarrow$

$\Rightarrow$ $\dfrac{\partial v(L,t)}{\partial x}=0$ or $\dfrac{\partial v}{\partial x}(0,t)=0$ at the other free end

Linear spring at the boundary:



$$m\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial v}{\partial x}\right) - K v(0+,t)\,\delta(x) \Rightarrow \int_{0-}^{0+} dx$$

$$\Rightarrow \int_{0-}^{0+} m \underbrace{\cancel{\frac{\partial^2 v}{\partial t^2}}\, dx}_{0} = \int_{0-}^{0+} \frac{\partial}{\partial x}\left(T\frac{\partial v}{\partial x}\right) dx - K v(0+,t) \Rightarrow$$

$$\Rightarrow 0 = T(0+)\frac{\partial v}{\partial x}(0+,t) - \overbrace{\cancel{T(0-)\frac{\partial v}{\partial x}(0-,t)}}^{0} - K v(0+,t) \Rightarrow$$

$$\Rightarrow T(0+)\frac{\partial v}{\partial x}(0+,t) = K v(0+,t)$$

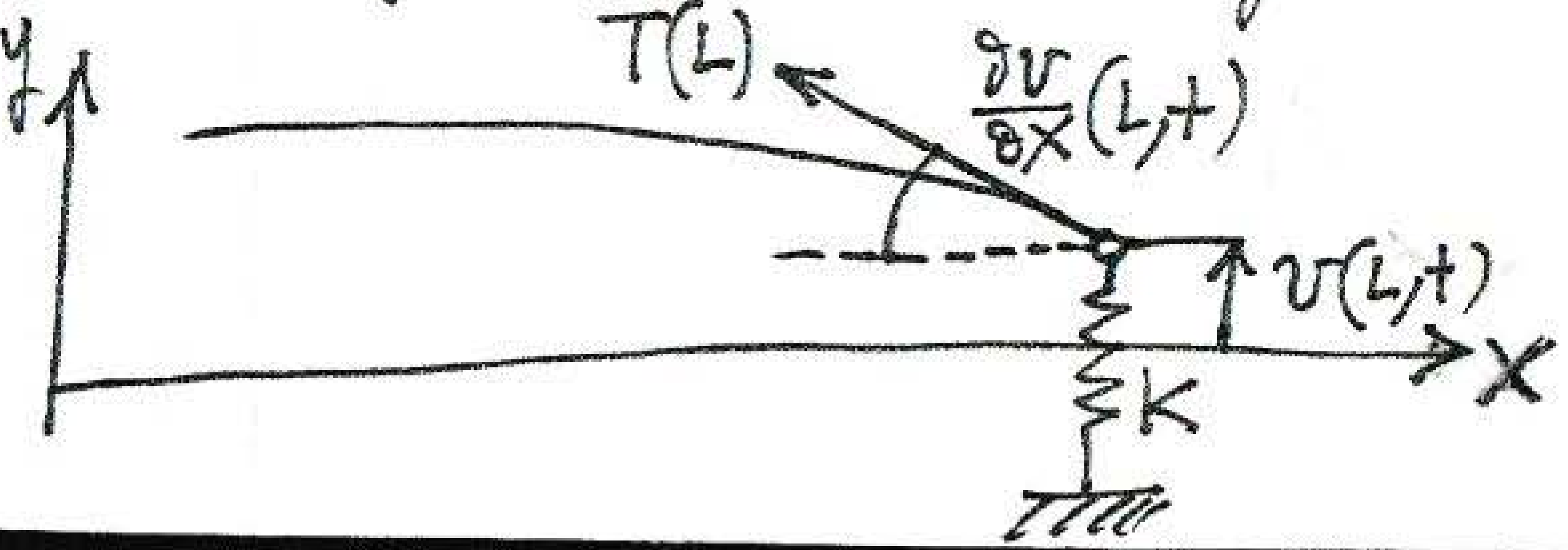
[Aside: Definition of the Delta function

$$\delta(x) = 0\ ,\ x \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$
]

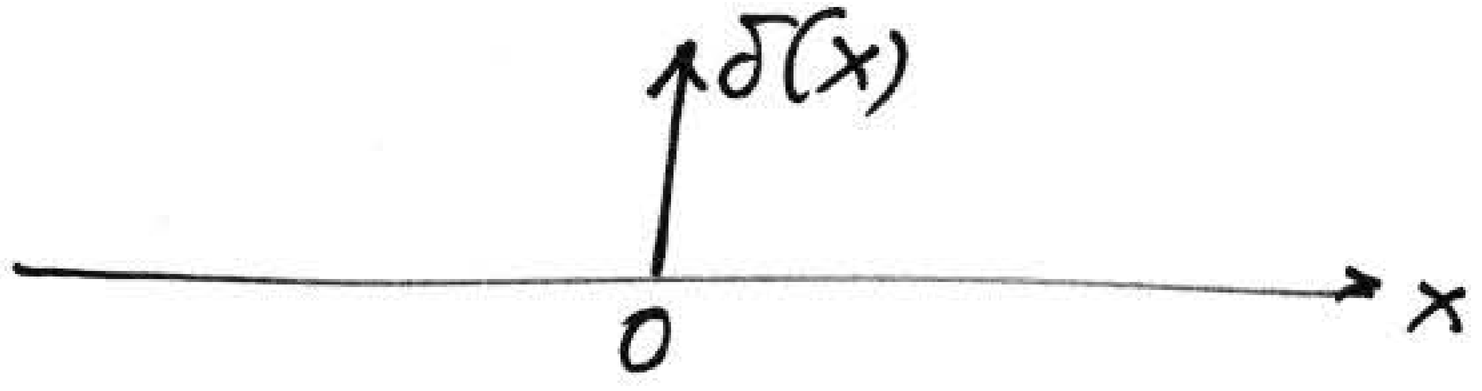
Now suppose that the spring is at the boundary $x=L$

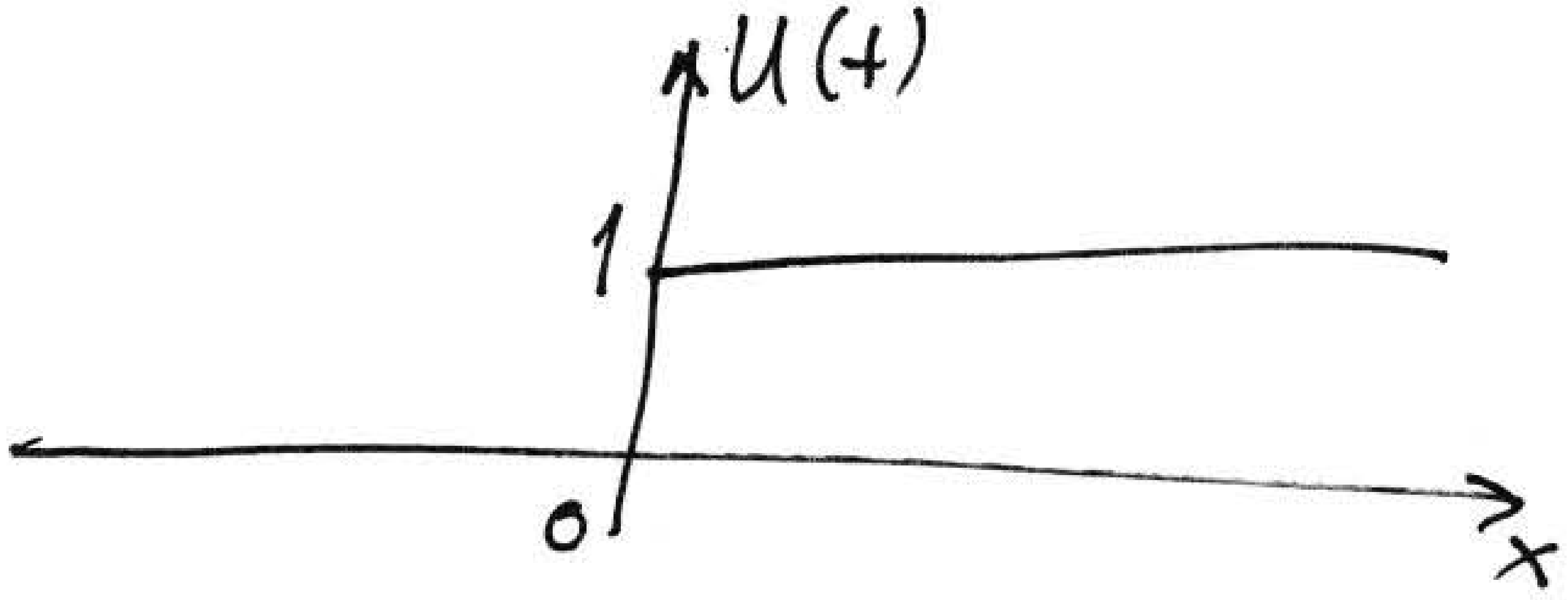


Then, $m\frac{\partial^2 v}{\partial t} = \frac{\partial}{\partial x}\left(T\frac{\partial v}{\partial x}\right) - k v(L-,t)\,\delta(x-L) \Rightarrow$

$$\Rightarrow \int_{L-}^{L+} dx \quad \cancel{T(L+)\frac{\partial v}{\partial x}(L+,t)} - T(L-)\frac{\partial v}{\partial x}(L-,t) - k v(L-,t) = 0$$

Generalized function (distribution)





Definition

$$\left.\begin{aligned} &\delta(x)=0, \quad x\neq 0 \\ &\int_{-\infty}^{+\infty} \delta(x)\,dx = 1 \end{aligned}\right\}$$

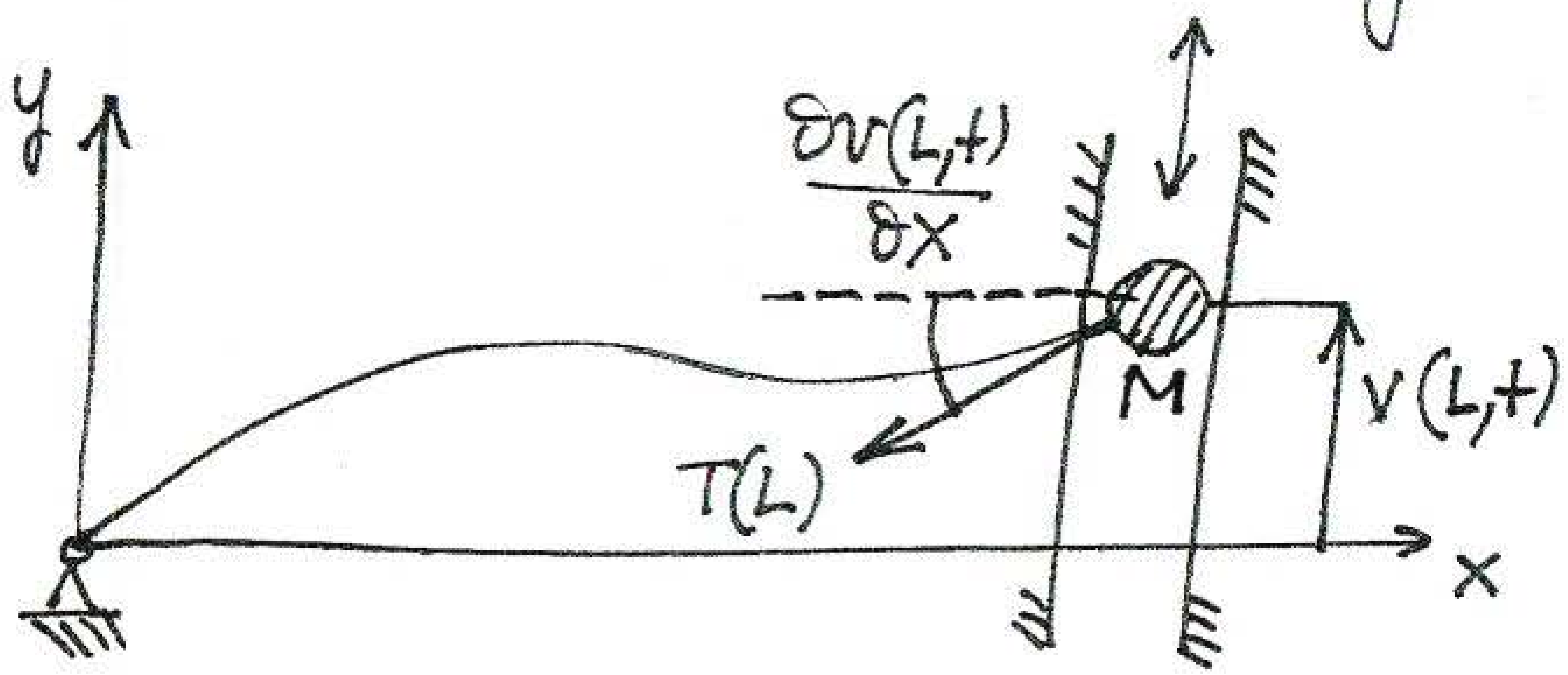
Property

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

Richtmeyer

Concentrated mass M at the boundary:



We can perform balance of vertical forces at the boundary ⇒

$$\Rightarrow \quad M\,\frac{\partial^2 v(L,t)}{\partial t^2} = -\,T(L)\,\frac{\partial v(L,t)}{\partial x}$$

On the other hand, if we have a mass at the end $x=0$ we can show that we get the boundary condition

$$M\,\frac{\partial^2 v(0,t)}{\partial t^2} = T(0)\,\frac{\partial v(0,t)}{\partial x}$$

Do it!

Remark

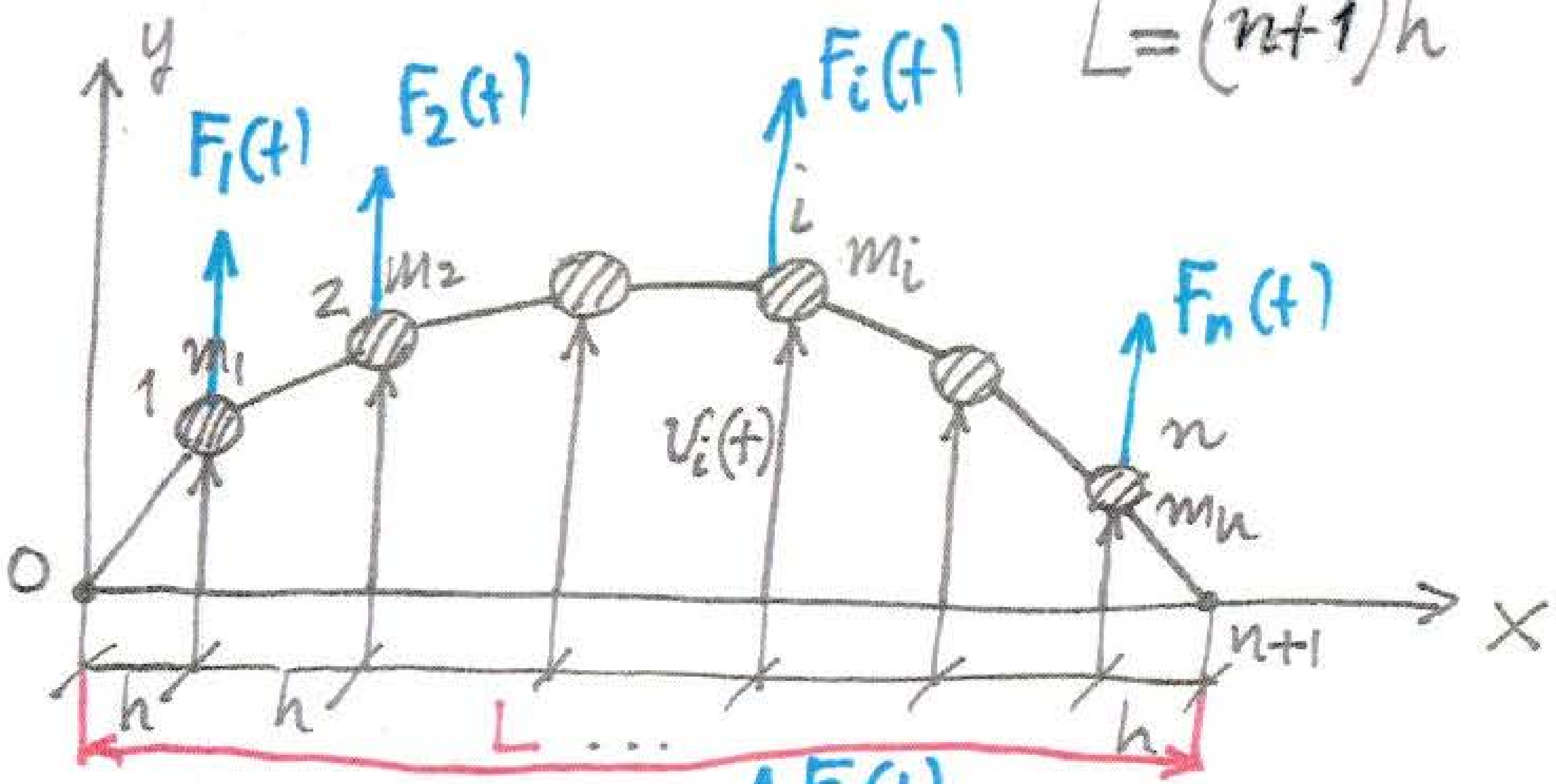
We can recover these relations by performing the same limiting process as in the previous case for the equation

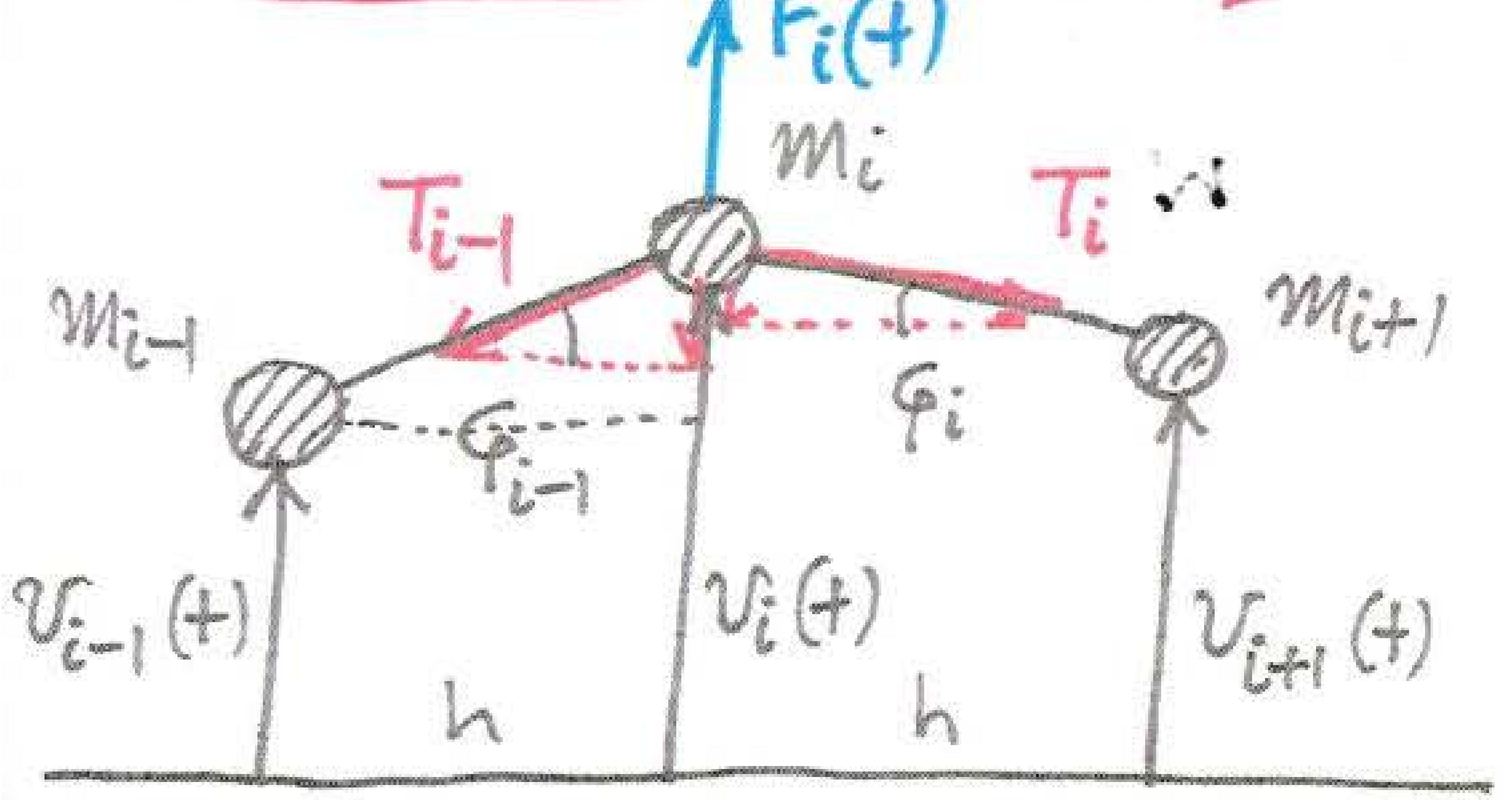
$$\left[m(x) + M\,\delta(x)\right]\frac{\partial^2 v}{\partial t^2}(x,t) = \frac{\partial}{\partial x}\left[T(x)\frac{\partial v}{\partial x}(x,t)\right] \Rightarrow$$

$$\Rightarrow \int_{0-}^{0+} dx \Rightarrow \text{Recover boundary condition at } x=0.$$

Remark: Alternative way for deriving the generalized wave equation

We will show how under certain assumption we can replace a set of n ordinary dif. equations by a single partial dif. eq., or how to make the transition from a n-DOF discrete system to a continuum (this process is refered to as 'continuum approximation').





Consider the n-DOF system composed of discrete masses connected by massless linear strings and performing vertical vibrations.

Considering the i-th mass and applying Newton's force law in the vertical direction

$$m_i \ddot{v}_i(t) = F_i(t) - T_{i-1}\frac{v_i(t)-v_{i-1}(t)}{h} - T_i \frac{v_i(t)-v_{i+1}(t)}{h}, \quad i=1,\dots,n$$

BCs: $v_0(t)=0,\ v_{n+1}(t)=0$