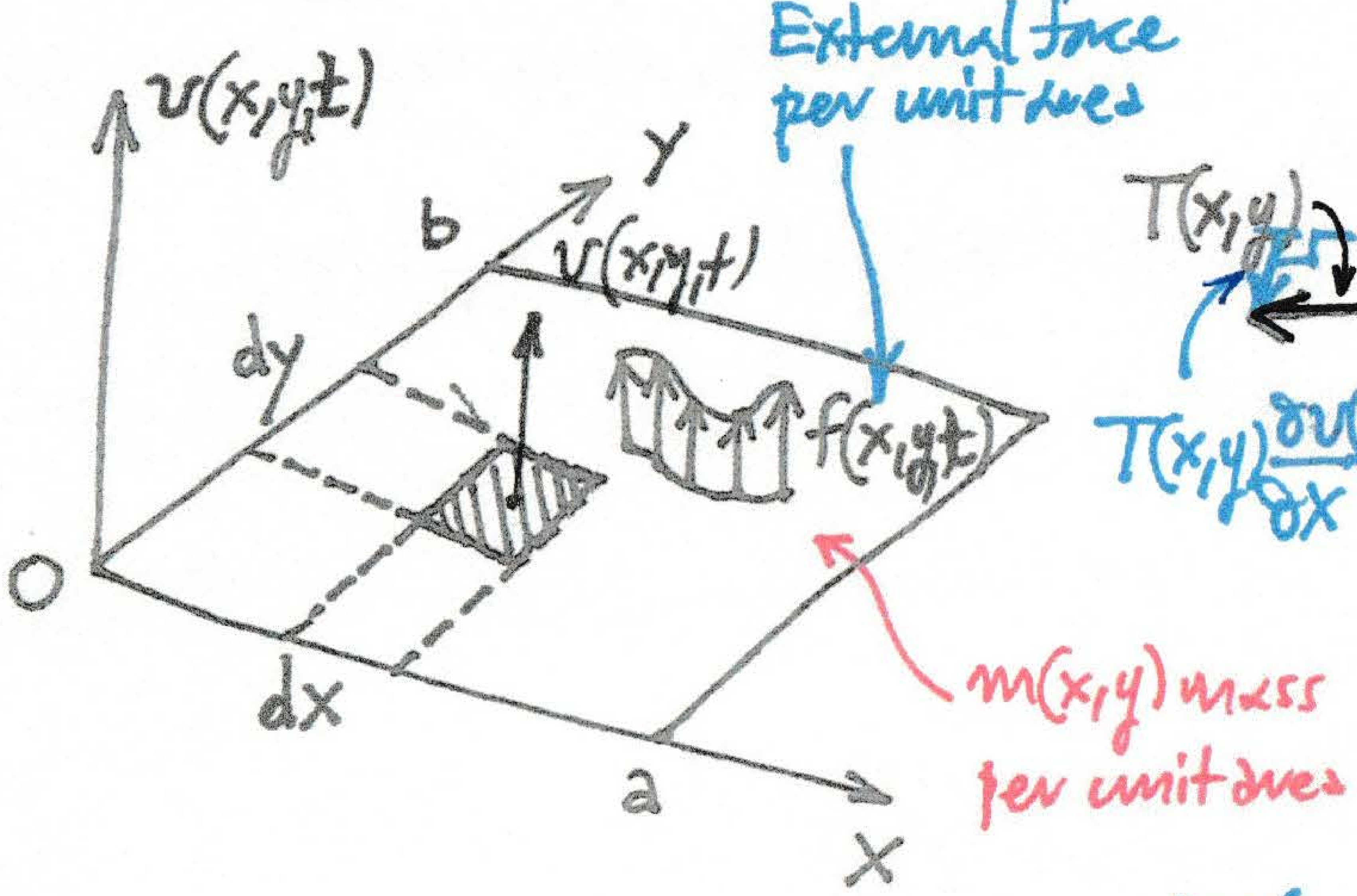
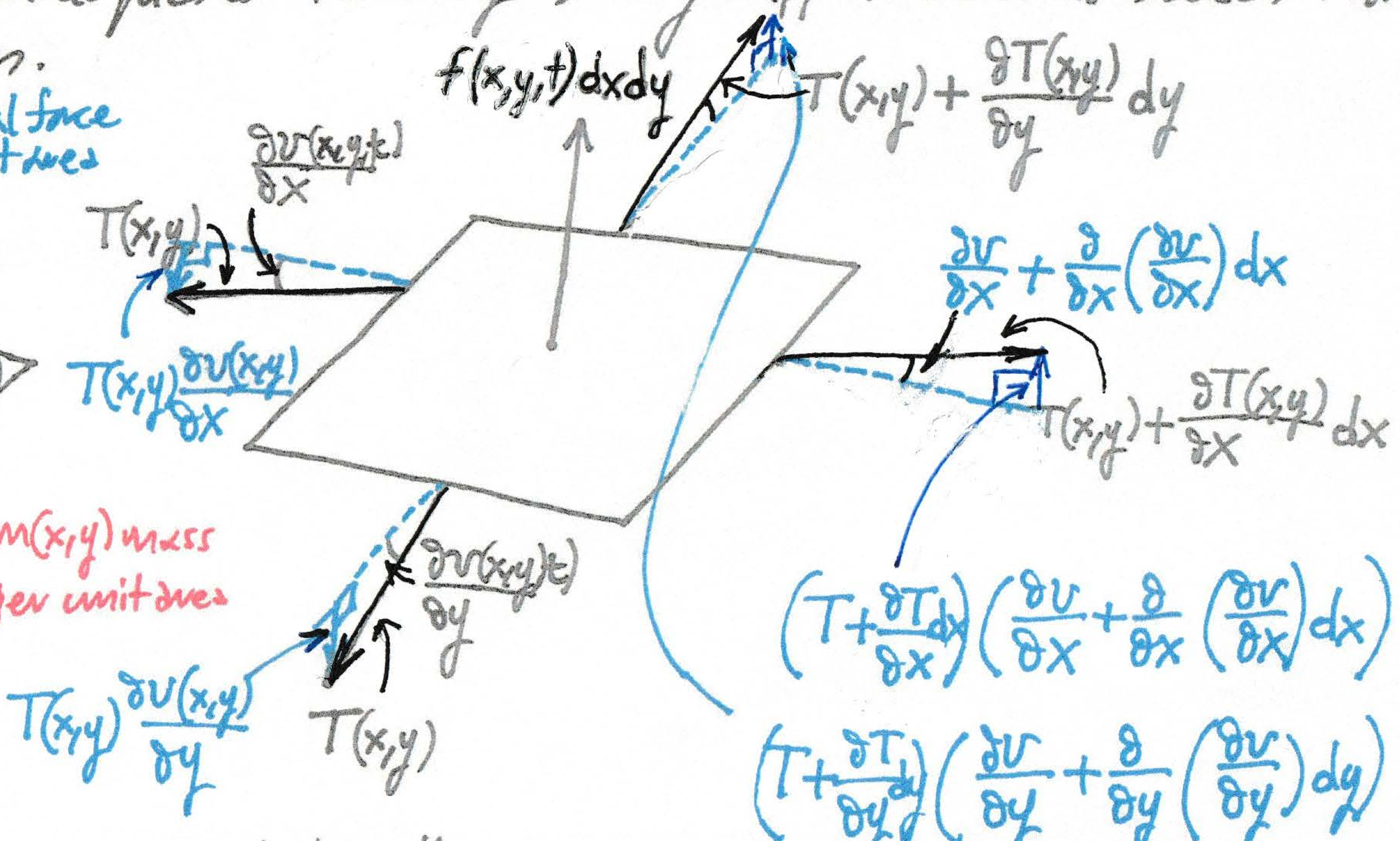
5 Higher dimensional Elastic Continua

Dynamics of Rectangular Membranes

These are two-dimensional analogues of the string $\Rightarrow$ They support internal axial forces but not moments or shear forces.





Given that the membrane cannot support bending moments, the only balance of forces that can be considered is in the vertical direction:

$$m(x,y)\,\cancel{dxdy}\,\frac{\partial^2 v(x,y)}{\partial t^2} = f(x,y,t)\,\cancel{dxdy} + \left[\frac{\partial}{\partial x}\left(T(x,y)\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(T(x,y)\frac{\partial v}{\partial y}\right)\right]\cancel{dxdy} + O\left((dxdy)^2\right)$$

In the limit as $dxdy \to 0$ we get,

$$\boxed{m(x,y)\frac{\partial^2 v}{\partial t^2} = f(x,y,t) + \left[\frac{\partial}{\partial x}\left(T(x,y)\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(T(x,y)\frac{\partial v}{\partial y}\right)\right]}$$

If we assume that $m(x,y) = m$, $T(x,y) = T \Rightarrow$

$$\Rightarrow T\underbrace{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{\nabla^2 v \text{ (Laplace operator in 2D Cartesian coordinates)}} + f(x,y,t) = m\frac{\partial^2 v}{\partial t^2} \Rightarrow T\nabla^2 v + f(x,y,t) = m\frac{\partial^2 v}{\partial t^2} \Rightarrow$$

$$\Rightarrow \boxed{c^2 \nabla^2 v + \tilde{f}(x,y,t) = \frac{\partial^2 v}{\partial t^2}}$$

$c^2 = \frac{T}{m}$ is the speed of sound in the material of the rectangular membrane

Classical wave equation in 2D

This equation is complemented by boundary conditions,

$$\left.\begin{array}{ll} v(0,y,t) = 0, \quad 0 \le y \le b, & v(x,0,t) = 0, \quad 0 \le x \le a \\ v(a,y,t) = 0, \quad 0 \le y \le b, & v(x,b,t) = 0, \quad 0 \le x \le a \end{array}\right\} \text{fixed boundary conditions}$$

and initial conditions,

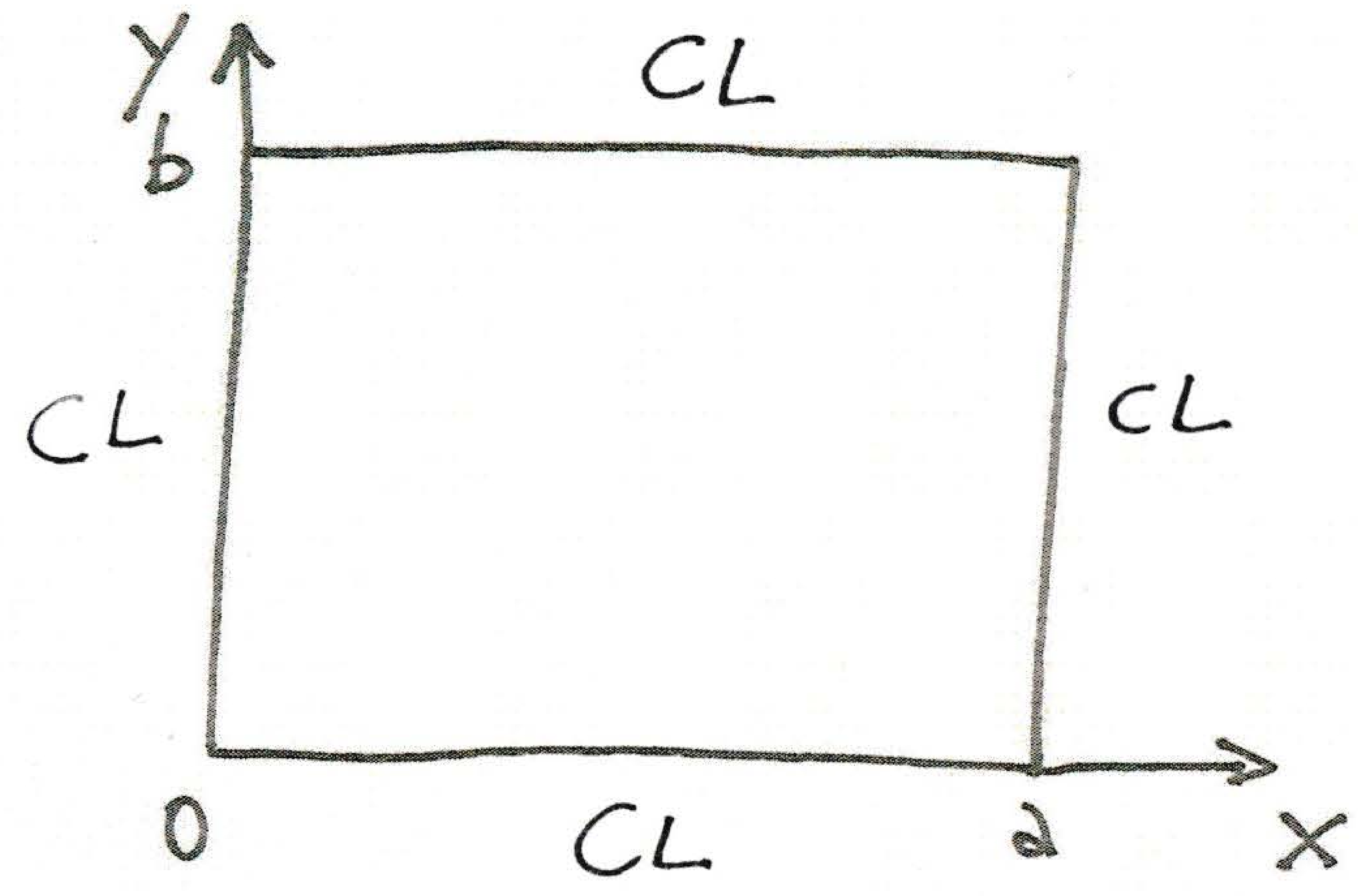
$$\left.\begin{array}{l} v(x,y,0) = g(x,y) \\ \frac{\partial v}{\partial t}(x,y,0) = h(x,y) \end{array}\right\} \begin{array}{l} 0 \le x \le a \\ 0 \le y \le b \end{array}$$

Remark

For free boundary conditions at the edge $\{x=0,\ 0 \le y \le b\}$,

$$\frac{\partial v}{\partial x}(0,y,t) = 0, \quad 0 \le y \le b$$

To solve this problem, first we consider the unforced problem and compute the modes using space-time reperation. Then we orthonormalize these modes and solve the original forced problem using modal analysis.



$$T\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)=m\frac{\partial^2 v}{\partial t^2},\quad 0\le x\le a,\ 0\le y\le b$$

Space-time separation: $v(x,y,t)=\Phi(x,y)\tilde{T}(t)$ $\Rightarrow$

$$\Rightarrow\ \ddot{\tilde{T}}(t)+\omega^2\tilde{T}(t)=0,\quad t\ge 0 \Rightarrow$$ Harmonic time dependence

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\left(\frac{\omega}{c}\right)^2\Phi=0,\quad 0\le x\le a,\ 0\le y\le b$$

$$\Phi(0,y)=0,\ 0\le y\le b,\quad \Phi(x,0)=0,\ 0\le x\le a$$

$$\Phi(a,y)=0,\ 0\le y\le b,\quad \Phi(x,b)=0,\ 0\le x\le a$$

Seperate again variables $\Rightarrow$

$$\Rightarrow \Phi(x,y)=\varphi(x)\psi(y)\Rightarrow$$

$$\Rightarrow \varphi''(x)\psi(y)+\varphi(x)\psi''(y)+\left(\frac{\omega}{c}\right)^2\varphi(x)\psi(y)=0 \Rightarrow \frac{\varphi''(x)}{\varphi(x)}+\frac{\psi''(y)}{\psi(y)}=-\left(\frac{\omega}{c}\right)^2\Rightarrow$$

$$\Rightarrow \underbrace{\frac{\varphi''(x)}{\varphi(x)}}_{\text{Depends on } x}=\underbrace{-\frac{\psi''(y)}{\psi(y)}-\left(\frac{\omega}{c}\right)^2}_{\text{depends on } y}\Rightarrow \frac{\varphi''(x)}{\varphi(x)}=-k^2,\quad \frac{\psi''(y)}{\psi(y)}+\left(\frac{\omega}{c}\right)^2=k^2\Rightarrow$$

Hence, we obtain the following two subproblems,

$$\varphi''(x)+k^2\varphi(x)=0 \Rightarrow \varphi(x)=c_1\cos kx+c_2\sin kx,\quad \varphi(0)=0,\ \varphi(a)=0$$

$$\psi''(y)+l^2\psi(y)=0 \Rightarrow \psi(y)=c_3\cos ly+c_4\sin ly,\quad \psi(0)=0,\ \psi(b)=0$$

$$l^2=\left(\frac{\omega}{c}\right)^2-k^2\geq 0 \Rightarrow \left(\frac{\omega}{c}\right)^2=l^2+k^2$$

Now we separate the boundary conditions as well, e.g., $\varphi(0,y)=0 \Rightarrow$

$$\Rightarrow \varphi(0)\psi(y)=0,\quad 0\leq y\leq b \Rightarrow$$

$$\Rightarrow \varphi(0)=0$$

$$\Rightarrow \varphi_i(x)=c_{2i}\sin\frac{i\pi x}{a},\quad k_i=\frac{i\pi}{a},\quad i=1,2,\ldots$$

$$\psi_j(y)=c_{4j}\sin\frac{j\pi y}{b},\quad l_j=\frac{j\pi}{b},\quad j=1,2,\ldots$$

But, $\left(\frac{\omega}{c}\right)^2=l^2+k^2$

$$\Rightarrow \varphi_{ij}(x,y)=\varphi_i(x)\psi_j(y),$$

$$\left(\frac{\omega_{ij}}{c}\right)^2=l_j^2+k_i^2 \Rightarrow$$

$$\Rightarrow \omega_{ij}=c\pi\sqrt{\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2},\quad i,j=1,2,\ldots$$

$$\varphi_{ij}(x,y)=c_{ij}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$$

Further, we mass-~~normalize~~ orthonormalize by requiring that,

$$\int_0^a\int_0^b m\,\varphi_{ij}^2(x,y)\,dx\,dy = 1 \Rightarrow m\,c_{ij}^2 \underbrace{\int_0^a \sin^2\frac{i\pi}{a}x\,dx}_{a/2}\underbrace{\int_0^b \sin^2\frac{j\pi}{b}y\,dy}_{b/2} = 1 \Rightarrow$$

$$\Rightarrow c_{ij} = \frac{2}{\sqrt{mab}} \Rightarrow$$

$\Rightarrow$ The mass-~~normalized~~ orthonormalized eigenfunctions are given by,

$$\boxed{\begin{aligned} \varphi_{ij}(x,y) &= \frac{2}{\sqrt{mab}}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}, \quad i,j=1,2,\dots \\ \omega_{ij} &= c\pi\sqrt{\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2} \end{aligned}}$$

Then, the response of the membrane will be the superposition of all modes,

$$v(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\underbrace{\left(A_{ij}\cos\omega_{ij}t + B_{ij}\sin\omega_{ij}t\right)}_{\tilde{T}_{ij}(t)}\varphi_{ij}(x,y)$$

where the coefficients A_{ij}, B_{ij} are computed by imposing the initial conditions.