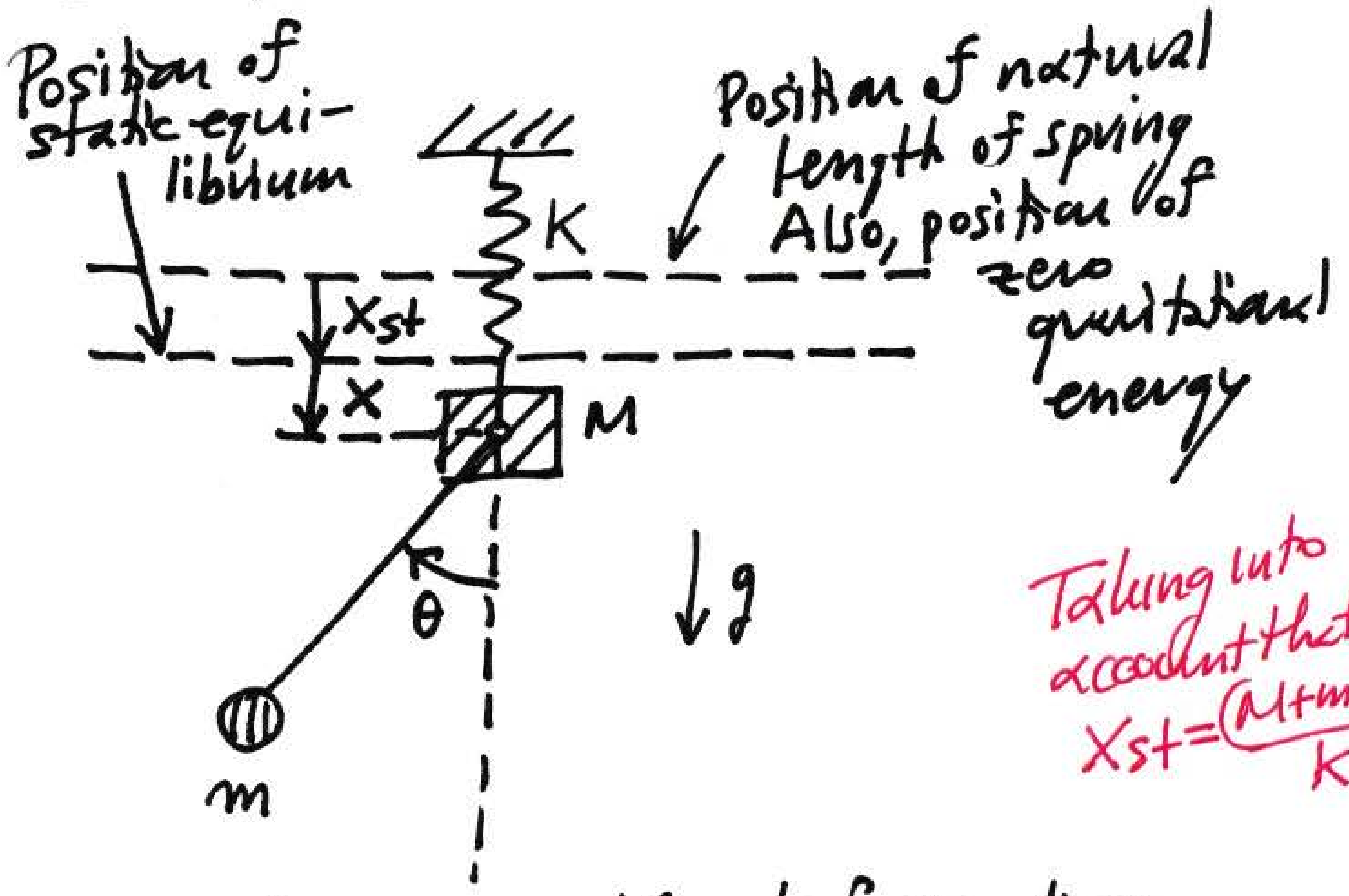
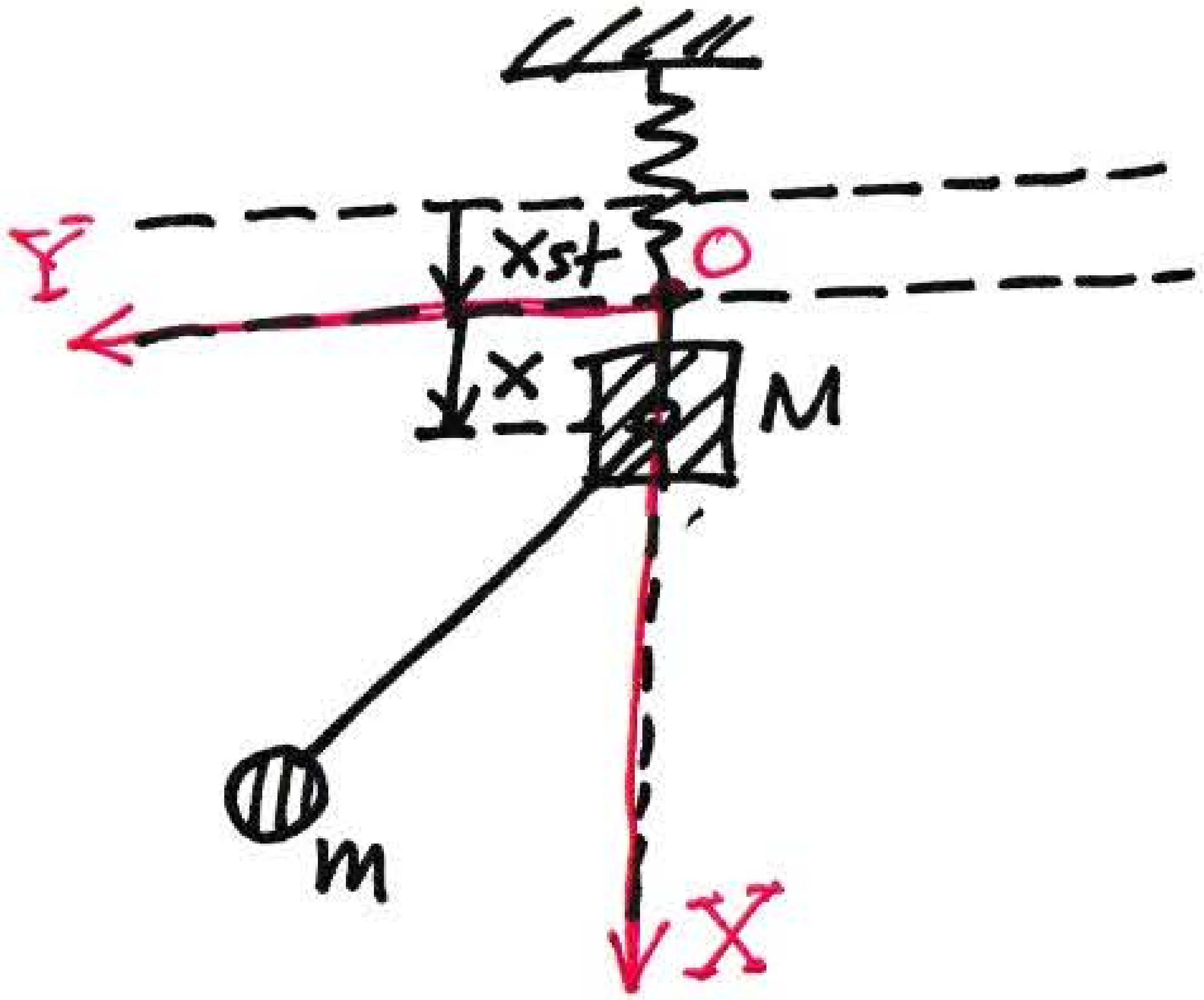
Example



Note that x is the deformation of the spring from its static equilibrium.



Remark:
To compute T, refer the motions of the mass to an inertial coordinate frame, e.g., (OXY).
Then, $T=\frac{1}{2}mV_{m\,abs}^2+\frac{1}{2}MV_{M\,abs}^2$

This system has 2DOF with its static equilibrium at $x=x_{st}$, $\theta=0$.
Gravity is taken into account.

Potential energy is:

$$V=\frac{1}{2}K(x_{st}+x)^2-Mg\,(x_{st}+x)-$$
$$-mg\,(x_{st}+x+l\cos\theta)=$$

Taking into account that $x_{st}=\frac{(M+m)g}{k}$

$$=\frac{1}{2}kx^2+mgl(1-\cos\theta)+\text{Const}$$

Not needed

Kinetic energy is:

$$T=\frac{1}{2}(M+m)\dot{x}^2+\frac{1}{2}m\,(l\sin\dot{\theta})^2+$$
$$+\frac{1}{2}m\left(x\left(\dot{l}-l\cos\theta\right)\right)^2$$

So, $V=V(x,\theta)$ and $T=T(\theta,\dot{x},\dot{\theta})$ and this is a nonlinear system.

But, what if we assume small vibrations and linearize close to the position of static equilibrium?

Then, $V \sim \frac{1}{2}kx^2 + \frac{1}{2}mgl\theta^2$, since $1-\cos\theta \sim \frac{\theta^2}{2} + \cdots$

$T \sim \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2$, since $\sin\theta \sim \theta + \cdots$

So, both V and T become quadratic forms,

$$V \sim \frac{1}{2}\begin{Bmatrix} x \\ \theta \end{Bmatrix}^T \begin{bmatrix} K & 0 \\ 0 & mgl \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix}, \quad T \sim \frac{1}{2}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix}^T \begin{bmatrix} M+m & 0 \\ 0 & ml^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix}$$

Stiffness matrix [K] is positive definite

Mass matrix [M] is positive definite

Then, the linearized system of equations becomes uncoupled (i.e., both the stiffness and mass matrices are diagonal). Note that the original nonlinear equations of motion are coupled since both x and θ appear in both equations of motion.

For linear vibrating systems, the stiffness matrix $[K]$ is <u>positive semi-definite</u>, whereas $[M]$ is <u>positive definite</u> $\Rightarrow$ We can express the n equations of motion governing the dynamics as (assuming n-DOF):

$$[M]\{\ddot{q}\}+[K]\{q\}=\{Q\}$$ These equations will be <u>coupled</u>, in general!

Now, we may introduce the change of coordinates, $\{q\}_{n\times1}=[B]_{n\times n}\{\eta\}_{n\times1}$ where $[B]$ is a constant non-singular matrix $\Rightarrow$

$$\Rightarrow [M][B]\{\ddot{\eta}\}+[K][B]\{\eta\}=\{Q\} \Rightarrow \text{Pre-multiply by } [B]^{-1} \Rightarrow$$

$$\Rightarrow \underbrace{[B]^{-1}[M][B]}_{[M_B]}\{\ddot{\eta}\}+\underbrace{[B]^{-1}[K][B]}_{[K_B]}\{\eta\}=\underbrace{[B]^{-1}\{Q\}}_{\{F\}}$$

<u>Then, we'll denote</u> $[B]\equiv[\Psi]$

Now, if $[B]$ is such that <u>both</u> $[M_B]$ and $[K_B]$ become diagonal matrices $\Rightarrow$ The transformed equations of motion become <u>uncoupled</u>:

$$\lceil m_b \rfloor\{\ddot{\eta}\}+\lceil k_b \rfloor\{\eta\}=\{F\}\Rightarrow$$

$$\Rightarrow \lceil I \rfloor\{\ddot{\eta}\}+\underbrace{\lceil m_b^{-1}\rfloor\lceil k_b\rfloor}_{\lceil k \rfloor}\{\eta\}=\underbrace{\lceil m_b^{-1}\rfloor\{F\}}_{\{N\}}\Rightarrow \lceil I\rfloor\{\ddot{\eta}\}+\lceil k\rfloor\{\eta\}=\{N\}$$

Then, we have n odes of the form,

$$\ddot{\eta}_r + k_r \eta_r = N_r(t), \quad r = 1, \dots, n$$

and assuming initial conditions $\eta_r(0) = D_r$, $\dot{\eta}_r(0) = V_r$, we may Duhamel's integral (or convolution integral) to write the solution of each of these equations as,

$$\eta_r(t) = D_r \cos \omega_r t + \frac{V_r}{\omega_r} \sin \omega_r t + \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r (t-\tau) \, d\tau$$

where $k_r = \omega_r^2$, $r = 1, \dots, n$.

<u>Remark</u>: The initial conditions for $\{y\}$ are expressed in terms of the initial conditions of the problem through the relations

$$\{y(0)\} = [\Psi]^{-1} \{q(0)\}, \quad \{\dot{y}(0)\} = [\Psi]^{-1} \{\dot{q}(0)\}$$

The procedure just outlined that enables us to decouple the equations of motion of a linear vibrating system is termed <u>modal decomposition</u>.

The original response of the system is computed simply as,

$$\{q\} = [\Psi] \{y\}$$

1.1 Normal Modes of Vibration

Let's reconsider the unforced n-DOF linear vibrating system,

$$[M]\{\ddot{q}\}+[K]\{q\}=\{0\}$$

Seek synchronous motions of this system, i.e., oscillations where all DOFs move in-unison, they pass through zero at the same instant of time and they reach their extremum values at the same instant of time (oscillations with trivial phase differences between DOFs, 0 or π) $\Rightarrow$ In that case the motions will be separable in time, $\boxed{\{q(t)\}=\{C\}f(t)}$

$\{C\}$ constant vector; $f(t)$ scalar function of time

Example: $\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} f_1(t) \Rightarrow$

$\Rightarrow q_1=f_1(t)=q_2$

or

$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}=\begin{Bmatrix} 0.5 \\ -1 \end{Bmatrix} f_2(t) \Rightarrow$

$\Rightarrow q_1=0.5\, f_2(t)$

$q_2=-f_2(t)$

Remark:
Can such motions exist?

Remark: Note that we have omitted damping from this discussion.

Then, we get:

$[M]\{C\}\ddot{f}(t) + [K]\{C\}f(t) = \{0\} \Rightarrow$ Obtain equations of the form,

$$\sum_{j=1}^{n}(\overset{(i,j)\text{-th element of }[M]}{M_{ij}}\, c_j)\ddot{f}(t) + \sum_{j=1}^{n}(K_{ij}\, c_j) f(t) = 0, \quad i=1,\dots,n \Rightarrow$$

$$\Rightarrow \underbrace{-\frac{\ddot{f}(t)}{f(t)}}_{\text{Depends on } t} = \underbrace{\frac{\sum_{j=1}^{n} K_{ij} c_j}{\sum_{j=1}^{n} M_{ij} c_j}}_{\text{Does not depend on } t}, \quad i=1,\dots,n, \quad \forall t \geq 0 \Rightarrow$$

Necessarily it must hold that

$$-\frac{\ddot{f}(t)}{f(t)} \equiv \omega^2 > 0$$

We require this to be a positive number (we exclude at this point the case $\omega = 0$), since otherwise $f(t)$ would not be an oscillating function (actually it would be unbounded).

Then, $\ddot{f}(t) + \omega^2 f(t) = 0 \Rightarrow f(t) = \overset{?}{A} \cos(\overset{?}{\omega} t - \overset{?}{\varphi})$

But then, $\sum_{j=1}^{n}(K_{ij}-\omega^2 M_{ij})\,c_j=0,\ i=1,\ldots,n \Rightarrow$

$$\Rightarrow \boxed{\underbrace{([K]-\omega^2[M])}_{[A(\omega^2)]}\{C\}=\{0\}}$$

Either $[A(\omega^2)]$ is nonsingular $\Rightarrow \{C\}=\{0\}$

or $[A(\omega^2)]$ is singular $\Rightarrow$ Problem may have an infinity of solutions!

This outlines the solution of a linear eigenvalue problem with ω^2 being the eigenvalue. For nontrivial solutions it must hold that $\det[A(\omega^2)]=0 \Rightarrow$ $\Rightarrow$ We can compute ω^2! Once ω^2 is computed, the vector $\{C\}$ can be computed as well. In general, we obtain n roots $\omega_1^2 \le \omega_2^2 \le \ldots \le \omega_n^2$ for the eigenvalue, $\omega_i^2>0,\ i=1,\ldots,n$ for $[K]$ positive definite. Moreover, all roots are real, but it is possible that two or more roots are equal, e.g., $\omega_1^2=\omega_2^2$. We will provide a detailed example of the solution of the eigenvalue problem in the next lecture!

Notes

Newton's law — D'Alembert — Dynamical equilibrium

$$m\underline{\ddot{x}} = \underline{F} \longrightarrow m\underline{\ddot{x}} - \underline{F} = \underline{0}$$

$\underline{A}\,\underline{x} = 0$

- Either $\underline{A}$ singular $\Rightarrow$ $\det \underline{A} = 0$
  ∞ solutions
- Or $\underline{A}$ nonsingular $\Rightarrow$ $\underline{x} = \underline{A}^{-1}\underline{0} = 0$

$\underline{A}\,\underline{x} = \underline{b}$

- Either $\underline{A}$ nonsingular $\Rightarrow$ $\underline{x} = \underline{A}^{-1}\underline{b}$
- Or $\underline{A}$ singular
  - If $\underline{b}$ is in the range of $\underline{A}$ $\Rightarrow$ ∞ solutions
    Fredholm's alternative!
  - If $\underline{b}$ is not in the range of $\underline{A}$ $\Rightarrow$ No solution.