Note that the mass-orthonormalized eigenfunctions satisfy the orthogonality conditions:

$$\int_0^a\int_0^b m\,\varphi_{ij}(x,y)\,\varphi_{rs}(x,y)\,dx\,dy = \cancel{\frac{m4}{mab}}\underbrace{\int_0^a \sin\frac{i\pi x}{a}\sin\frac{r\pi x}{a}\,dx}_{\delta_{ir}\cancel{a/2}}\cdot\underbrace{\int_0^b \sin\frac{j\pi y}{b}\sin\frac{s\pi y}{b}\,dy}_{\delta_{js}\cancel{b/2}} = \delta_{ir}\delta_{js} \Rightarrow$$

$$\Rightarrow \boxed{\int_0^a\int_0^b m\,\varphi_{ij}(x,y)\,\varphi_{rs}(x,y)\,dx\,dy = \delta_{ir}\delta_{js},\quad i,j,r,s=1,2,\ldots}$$

And a similar stiffness orthonormality condition...

Now we consider modal analysis of the forced problem,

$$\left.\begin{aligned} &c^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{f}{m}(x,y,t)=\frac{\partial^2 v}{\partial t^2}\\ &\text{Let } v(x,y,t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\eta_{ij}(t)\,\varphi_{ij}(x,y)\end{aligned}\right\}\Rightarrow$$

Mass-normalized modes

modal amplitudes

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \eta_{ij}(t)\, c^2 \underbrace{\left[-\left(\frac{i\pi}{a}\right)^2-\left(\frac{j\pi}{b}\right)^2\right]}_{-\left(\frac{\omega_{ij}}{c}\right)^2} \varphi_{ij}(x,y) + \frac{f(x,y,t)}{m} =$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \ddot{\eta}_{ij}(t)\, \varphi_{ij}(x,y) \Longrightarrow \int_0^a\int_0^b (\ )\, m\, \varphi_{rs}(x,y)\, dx\,dy \Longrightarrow$$

$$\Rightarrow \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \left[\ddot{\eta}_{ij}(t) + \omega_{ij}^2\, \eta_{ij}(t)\right] \delta_{ir}\, \delta_{js} = \underbrace{\int_0^a\int_0^b f(x,y,t)\, \varphi_{rs}(x,y)\, dx\,dy}_{N_{rs}(t)} \Rightarrow$$

$$\Rightarrow \boxed{\ddot{\eta}_{rs}(t) + \omega_{rs}^2\, \eta_{rs}(t) = N_{rs}(t)\,, \quad r,s=1,2,\ldots}$$

The initial conditions for the modal oscillators are computed using the orthonality properties of the eigenfunctions,

$$v(x,y,0) = g(x,y) \Rightarrow \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \eta_{ij}(0)\, \varphi_{ij}(x,y) = g(x,y) \Longrightarrow \int_0^a\int_0^b (\ )\, m\, \varphi_{rs}\, dx\,dy \Rightarrow$$

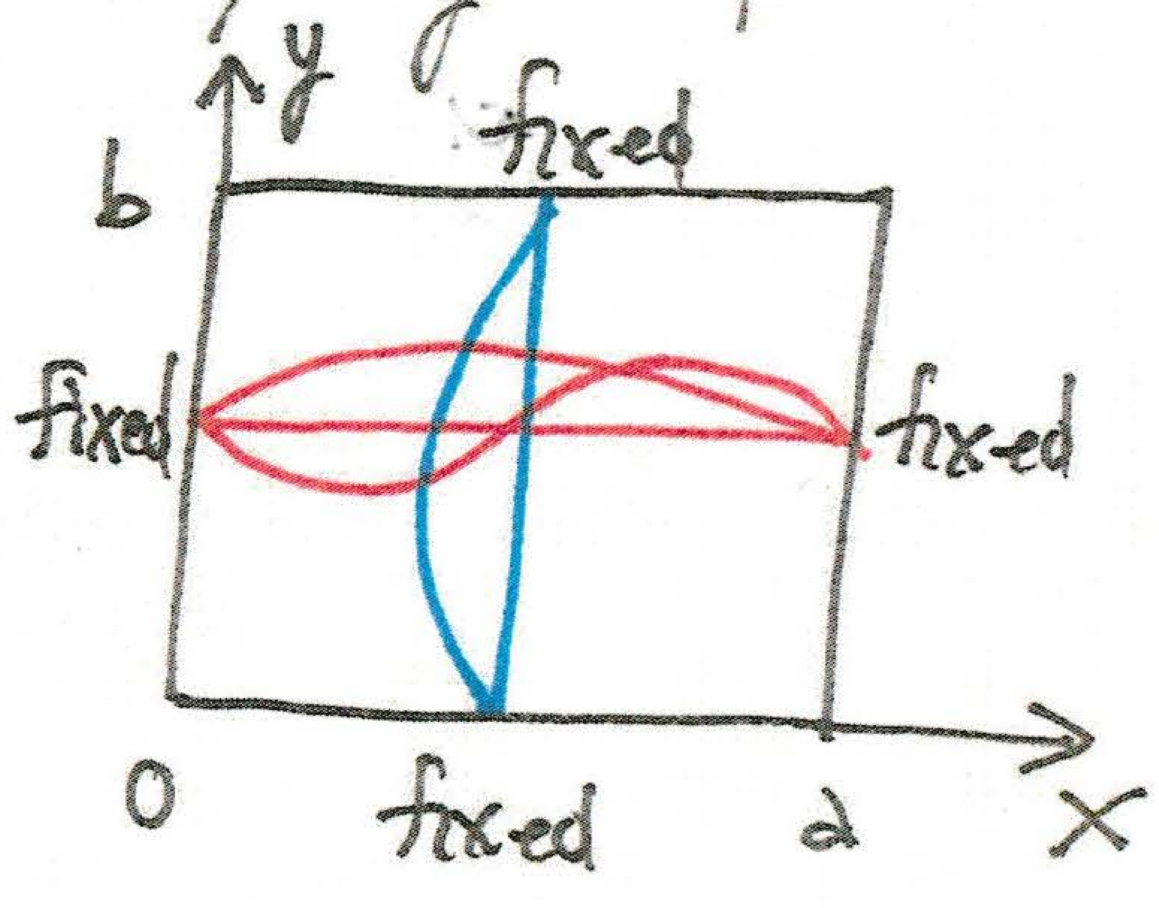
$$\Rightarrow \boxed{\eta_{rs}(0) = \int_0^a\int_0^b m\, g(x,y)\, \varphi_{rs}(x,y)\, dx\,dy}$$

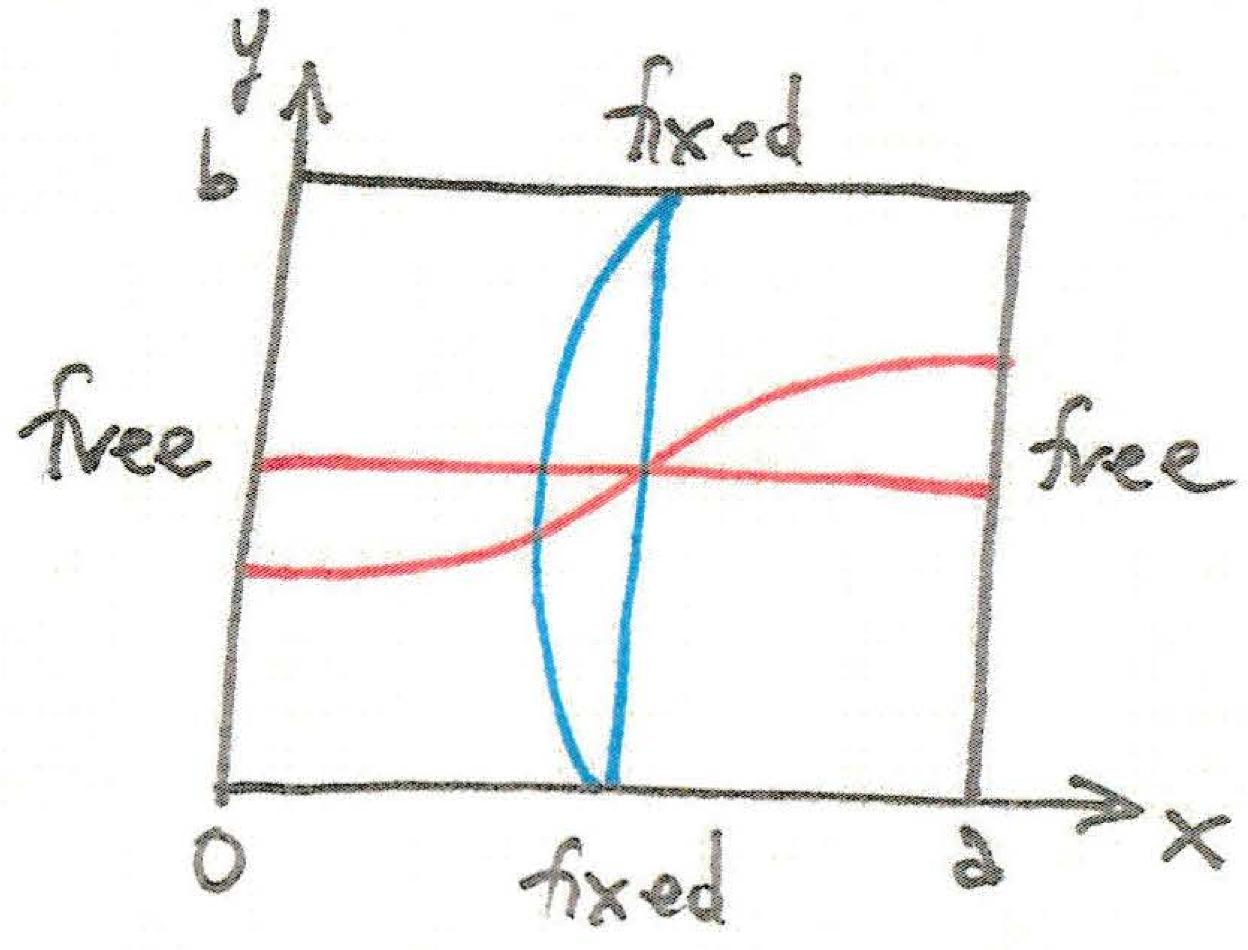
Similarly,

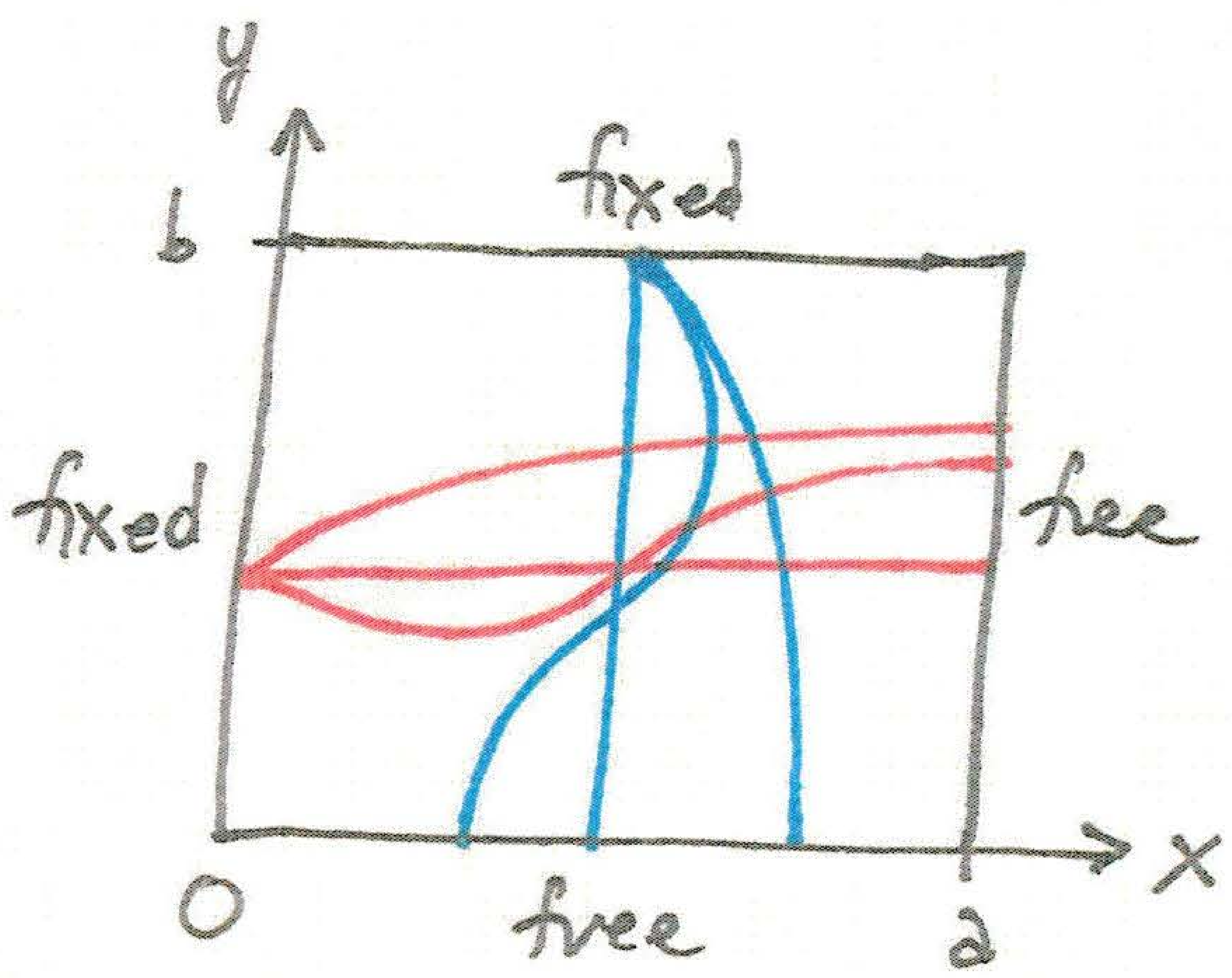
$$\boxed{\dot{\eta}_{rs}(0) = \int_0^a\int_0^b m\, h(x,y)\, \varphi_{rs}(x,y)\, dx\,dy}$$

Remarks :

1) 'Easy way' to perform modal analysis for simple BCs.

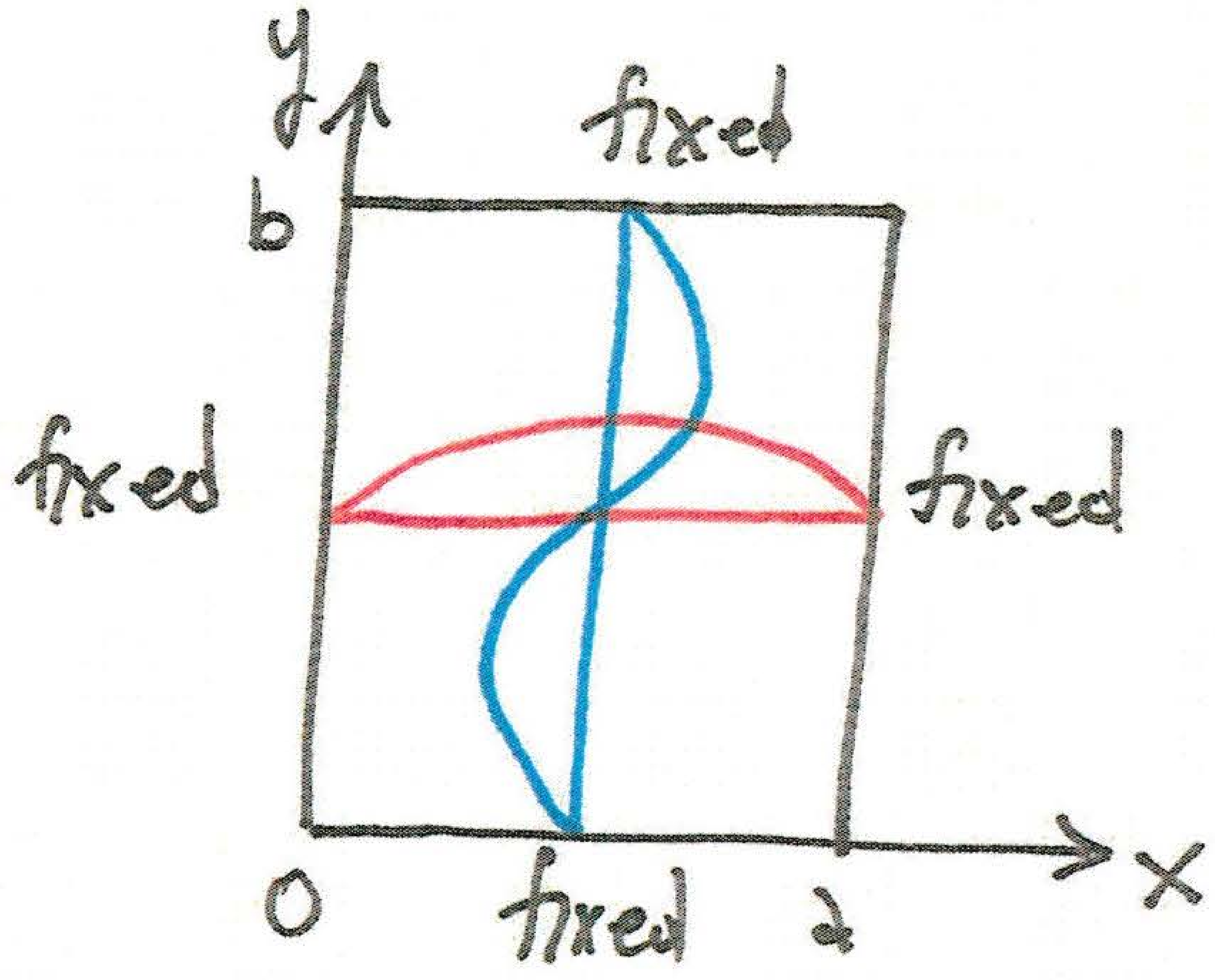


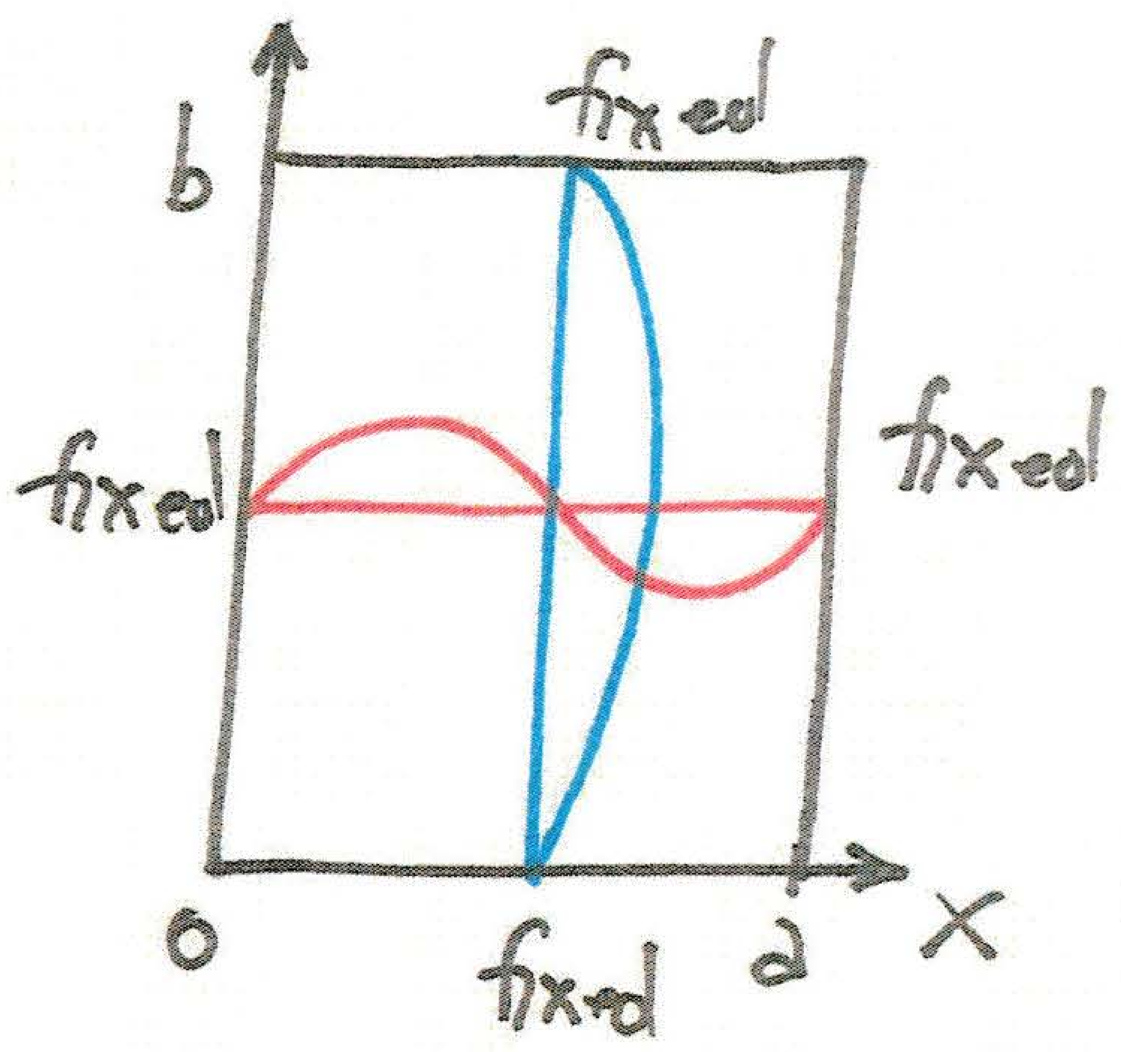




$$\varphi_{ij}(x,y) = c_{ij} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}$$

2) Consider degeneracies of modes in the rectangular membrane

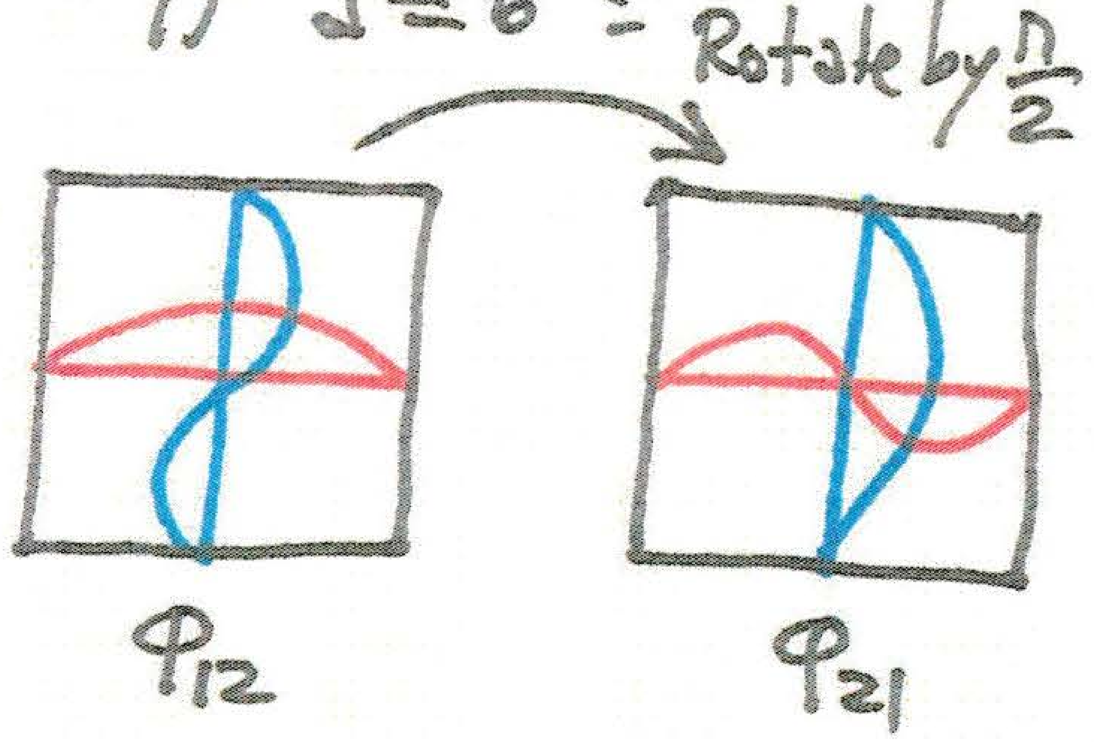



Mode $\varphi_{12}$ with frequency $\omega_{12}$




Mode $\varphi_{21}$ with frequency $\omega_{21} \neq \omega_{12}$ if $a \neq b$

But what happens if $a = b$?




$\omega_{21} = \omega_{12}$ and mode shapes differ by rotation of $\pi/2$!

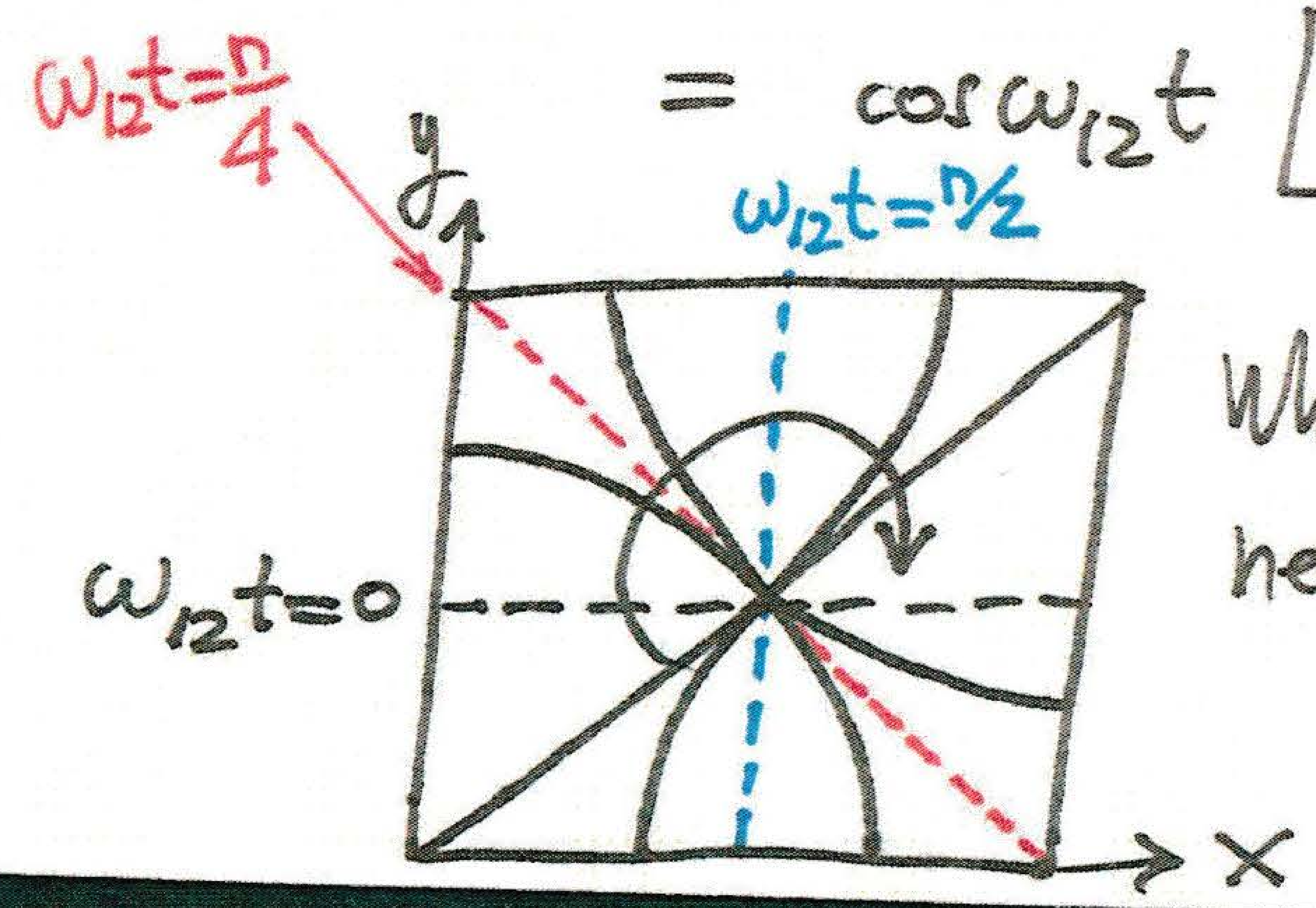
Hence in square membranes with constant properties we have mode degeneracies $\Rightarrow$ $\varphi_{ij} = \varphi_{ji}$ $\forall\, i,j = 1,2,\ldots \Rightarrow$ Then every combination of degenerate modes $\varphi_{ij}$ and $\varphi_{ji}$ will also be a mode with frequency $\omega_{ij} \Rightarrow$ $\tilde{\varphi}_{ij} = a\,\varphi_{ij} + b\,\varphi_{ji}$ is also a mode! We can even induce travelling waves with rotating nodal diameters by suitable choise of initial conditions.

for example, consider the square membrane with initial conditions:

$$\left.\begin{aligned} u(x,y,0) &= \sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} \\ \frac{\partial u}{\partial t}(x,y,0) &= \omega_{12}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow u(x,y,t) = \sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\cos\omega_{12}t + \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\sin\omega_{12}t =$$

$$= \cos\omega_{12}t\left[\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + \tan\omega_{12}t\,\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right]$$




Where there are nodal lines or curves (points where instantaneasly there is no motion of the membrane).

Hence in square membranes with constant properties we have mode degeneracies $\Rightarrow$ $\varphi_{ij} = \varphi_{ji}$ $\forall\ i,j = 1,2,\dots$ $\Rightarrow$ Then every combination of degenerate modes $\varphi_{ij}$ and $\varphi_{ji}$ will also be a mode with frequency $\omega_{ij}$ $\Rightarrow$ $\tilde{\varphi}_{ij} = a\varphi_{ij} + b\varphi_{ji}$ is also a mode! We can even induce travelling waves with rotating nodal diameters by suitable choise of initial conditions.

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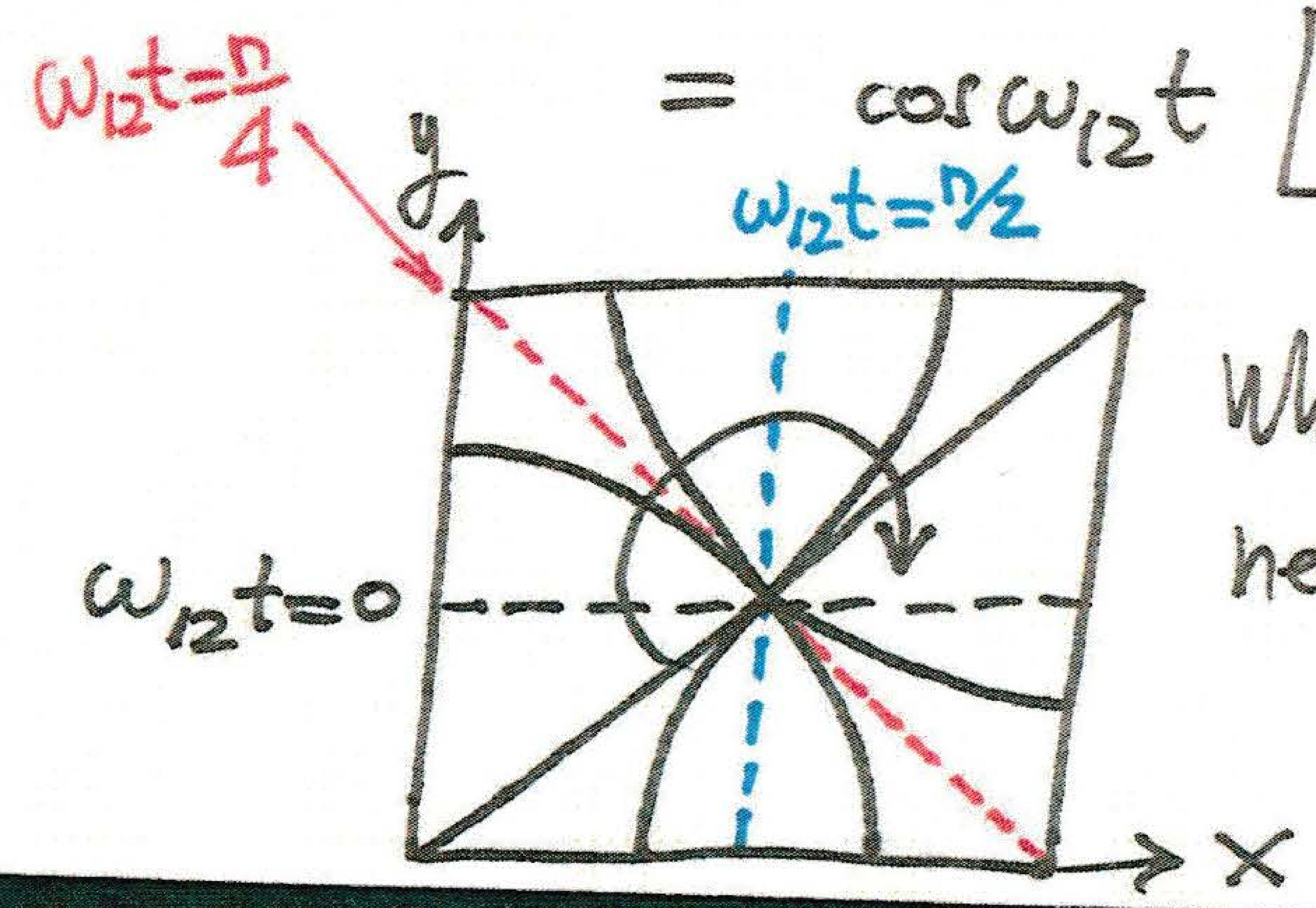
$$\Rightarrow u(x,y,t) = \sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\cos\omega_{12}t + \sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\sin\omega_{12}t =$$

$$= \cos\omega_{12}t\left[\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + \tan\omega_{12}t\,\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right]$$




Where there are nodal lines or curves (points where instantaneously there is no motion of the membrane).

Hence, by suitable choice of initial conditions we can introduce rotating waves in the degenerate membrane with $a=b$, whereas in the nondegenerate membrane (with $a \neq b$) only standing waves are possible.

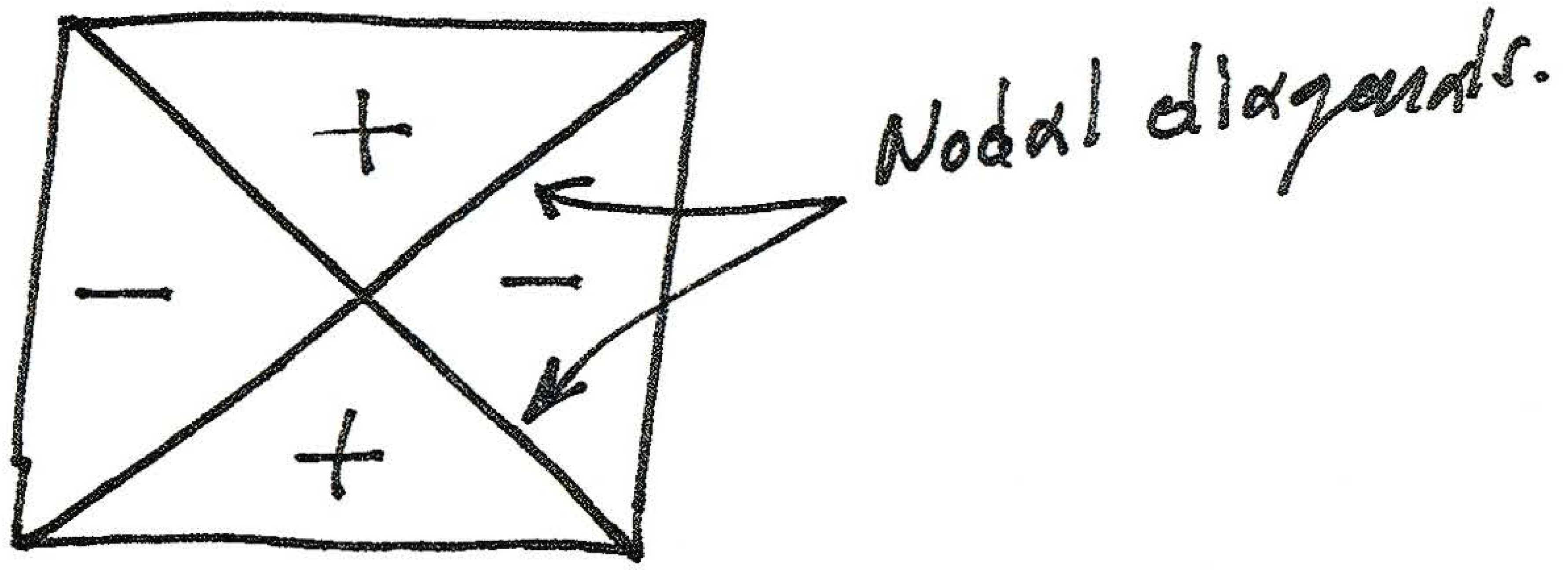
In addition we can get very interesting mode shapes in the degenerate membrane by linear superposition of modes.

Example: Consider modes $\varphi_{1,2N}(x,y)$ and $\varphi_{2N,1}(x,y) \Rightarrow$ Form the composite mode

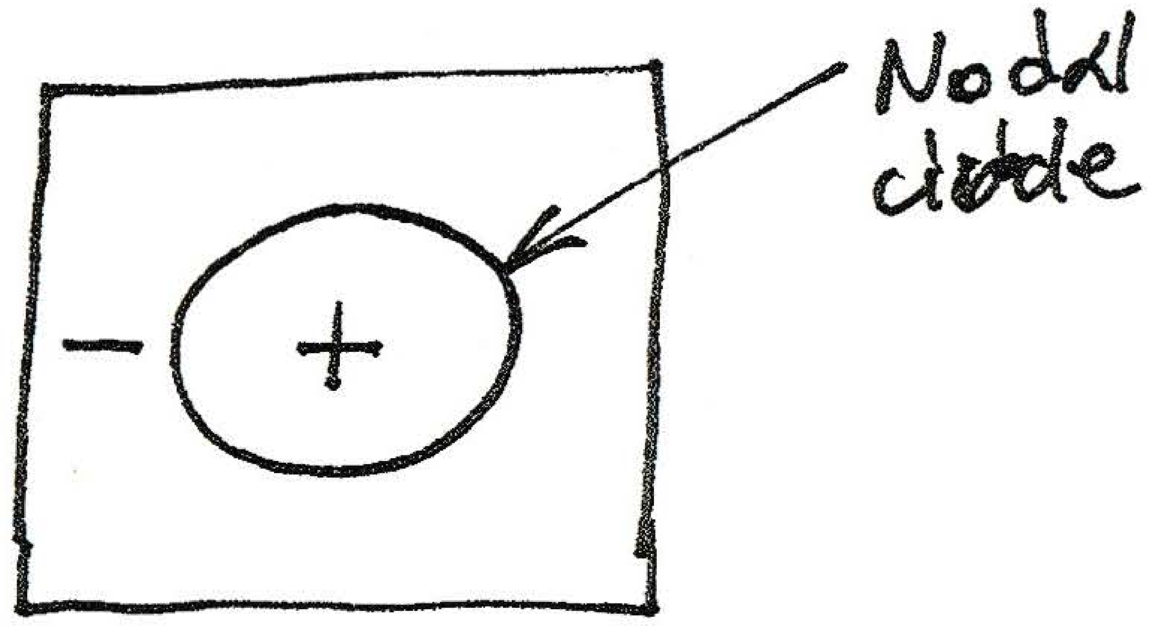
$$\tilde{\varphi}_{1,2N}(x,y) = \varphi_{1,2N}(x,y) + (1-\varepsilon)\varphi_{2N,1}, \quad 0<\varepsilon\ll 1 \Rightarrow \text{find a}$$

very complicated picture of the mode shape!

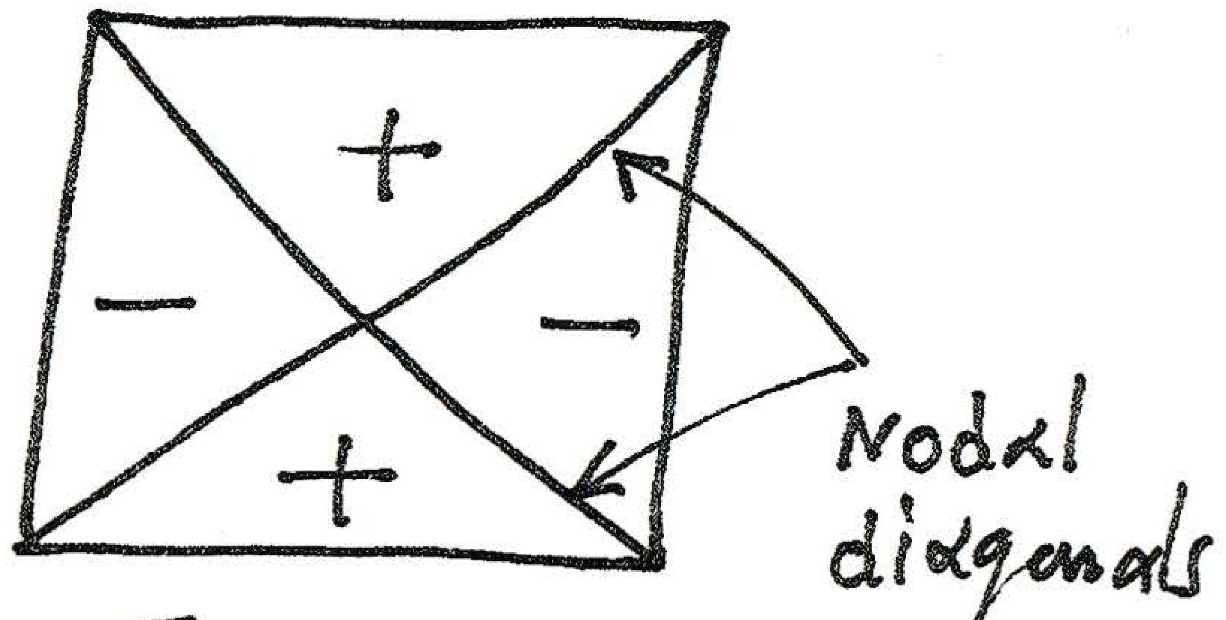
Example By combining $\varphi_{ij}(x,y)$ and $\varphi_{ji}(x,y)$ we can get a composite mode with two nodal diagonals.



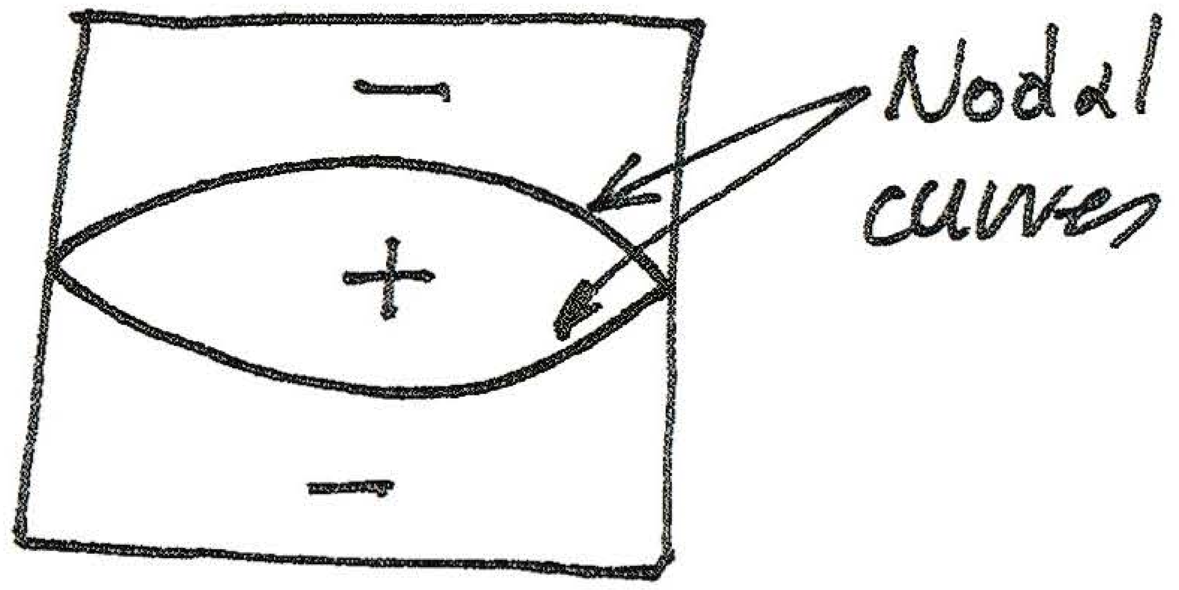
**Example** By combining $\varphi_{13}$ and $\varphi_{31}$ we can get interesting shapes, all of which correspond to modes of the degenerate system!




$\varphi_{13} + \varphi_{31}$




$\varphi_{13} - \varphi_{31}$




$\varphi_{13} + \frac{1}{3}\varphi_{31}$

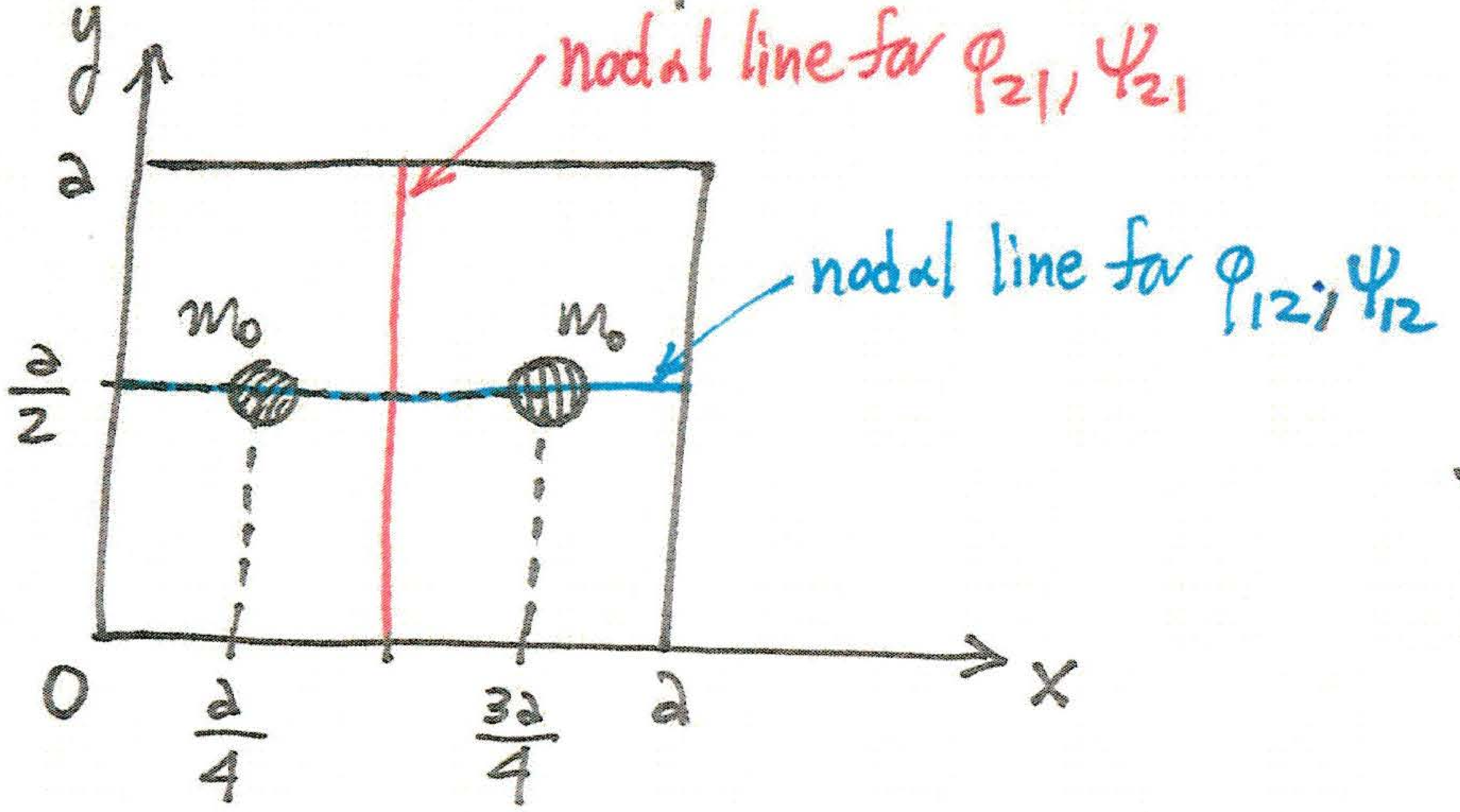
All these are standing waves.

## Example of RQ for rectangular membrane

Suppose that we have a problem that does not admit analytical solution in a simple form - for example,

$$\nabla^2 v = [1+m(x,y)]\, v_{tt}, \quad \text{on } D=\{0\le x\le a,\ 0\le y\le b=a\},\quad v=0 \text{ on } \partial D$$

Assume that $m(x,y) = m_0\, \delta\!\left(x-\frac{a}{4}\right)\delta\!\left(y-\frac{a}{2}\right) + m_0\, \delta\!\left(x-\frac{3a}{4}\right)\delta\!\left(y-\frac{a}{2}\right)$




Note that due to the break of symmetry the modes $\varphi_{12}$ and $\varphi_{21}$ won't be degenerate any more; this can be seen from the fact that the discrete masses lie on a nodal line of $\varphi_{12}$!

To formulate the RQ for this problem $\Rightarrow$ Assume that $v(x,y,t) = \varphi(x,y)e^{j\omega t} \Rightarrow$

$$\Rightarrow \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) = [1+m(x,y)](-\omega^2)\varphi \Rightarrow \int_0^a\!\!\int_0^a (\cdot)\varphi\, dx\,dy \Rightarrow$$

$$\Rightarrow \omega^2 = \frac{-\int_0^a\!\int_0^a \varphi\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right]dx\,dy}{\int_0^a\!\int_0^a [1+m(x,y)]\,\varphi^2\,dx\,dy} = \frac{\cancel{-\varphi\frac{\partial\varphi}{\partial x}\Big|_0^a\Big|_0^a} + \int_0^a\!\int_0^a\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]dx\,dy - \cancel{\varphi\frac{\partial\varphi}{\partial y}\Big|_0^a\Big|_0^a}}{\int_0^a\!\int_0^a [1+m(x,y)]\,\varphi^2\,dx\,dy} \Rightarrow$$

$$\Rightarrow \omega^2 = \frac{\int_0^a \int_0^a \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right] dx\,dy}{\int_0^a \int_0^a \varphi^2 dx\,dy + \varphi^2\left(\frac{a}{4}, \frac{a}{2}\right) m_0 + \varphi^2\left(\frac{3a}{4}, \frac{a}{2}\right) m_0}$$ (Rayleigh-quotient)

To formulate a Rayleigh-Ritz procedure $\Rightarrow \varphi(x,y) = a_1 \psi_{12}(x,y) + a_2 \psi_{21}(x,y) + a_3 \psi_{11}(x,y)$

where $\psi_{ij}(x,y)$ is the eigenfunction of the uniform membrane with $m_0 = 0$

$$\left.\begin{aligned} \psi_{11}(x,y) &= \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \\ \psi_{12}(x,y) &= \sin\frac{\pi x}{a} \sin\frac{2\pi y}{a} \\ \psi_{21}(x,y) &= \sin\frac{2\pi x}{a} \sin\frac{\pi y}{a} \end{aligned}\right\}$$

Substituting into the RQ and introducing the stationarity conditions with respect to $a_1, a_2$ and $a_3 \Rightarrow$

$$\left(\omega^2[M] - [K]\right)\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad k_{ij} = \int_0^a \int_0^a \left[\frac{\partial \psi_{ij}}{\partial x}\frac{\partial \psi_{ji}}{\partial x} + \frac{\partial \psi_{ij}}{\partial y}\frac{\partial \psi_{ji}}{\partial y}\right] dx\,dy$$

$$m_{ij} = \int_0^a \int_0^a \psi_{ij}\, \psi_{ji}\, [1 + m(x,y)]\, dx\,dy$$