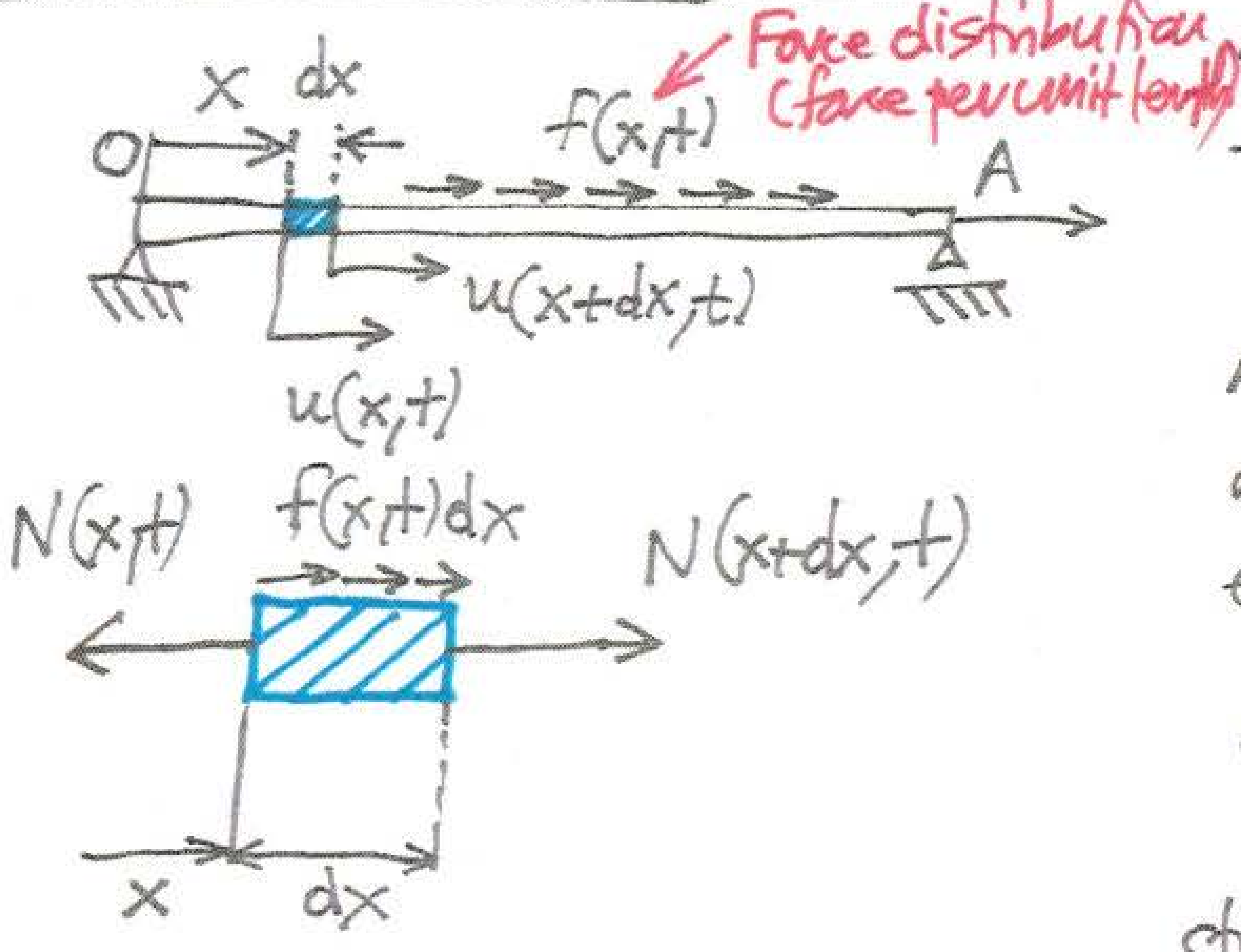
B. Axial vibrations of elastic rods



Assume modulus of elasticity E and cross section $A(x)$. Assume $m(x)=A(x)\cdot\rho$ mass per unit length of rod, ρ is density

Assuming linearly elastic material and small axial oscillations $\Rightarrow$ Infinitesimal linear elasticity in x-direction $\Rightarrow$

$$\varepsilon(x,t)=\frac{\partial u(x,t)}{\partial x}, \quad \sigma(x,t)=E\,\varepsilon(x,t) \Rightarrow$$

strain

Assume uniform distribution of stresses at each cross section

$\Rightarrow$ Local axial force at position x is equal to $N(x,t)=\sigma(x,t)A(x)=EA(x)\dfrac{\partial u(x,t)}{\partial x}$

Making at position $x+dx$, the local axial force is then given by,

$$N(x+dx,t)=EA(x+dx)\frac{\partial u(x+dx,t)}{\partial x}$$

Considering the differential element of length dx we write balance of forces in the x-direction $\Rightarrow$

$$m(x)dx\frac{\partial^2 u(x,t)}{\partial t^2}=N(x+dx,t)-N(x,t)+f(x,t)dx \Rightarrow$$

$$\Rightarrow m(x)dx\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{EA(x+dx)\frac{\partial u(x+dx,t)}{\partial x}-EA(x)\frac{\partial u(x,t)}{\partial x}}{dx}dx+f(x,t)dx \Rightarrow \text{As } dx\to 0,$$

$$m(x)\frac{\partial^2 u(x,t)}{\partial t^2}\cancel{dx} = \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]\cancel{dx} + f(x,t)\cancel{dx} + \cancel{O(dx^2)} \Rightarrow$$

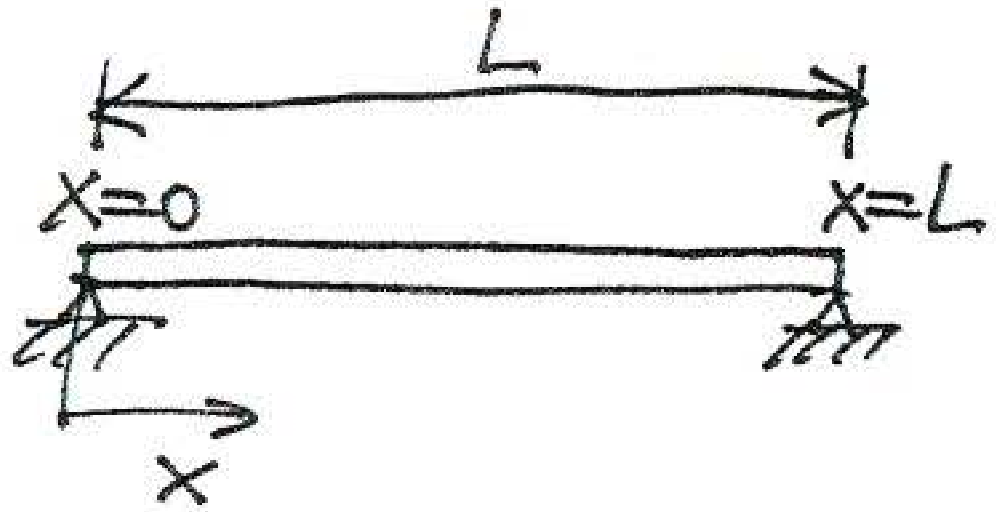
$$\Rightarrow \boxed{m(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] + f(x,t)}$$ Generalized wave equation!

If $m(x)=m$, $EA(x)=EA$, $f(x,t)=0 \Rightarrow$ We get the classical wave equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}, \quad c^2 = \frac{E}{\rho},$$ c is the speed of sound in the rod.

classical wave equation

Discussing briefly boundary conditions,

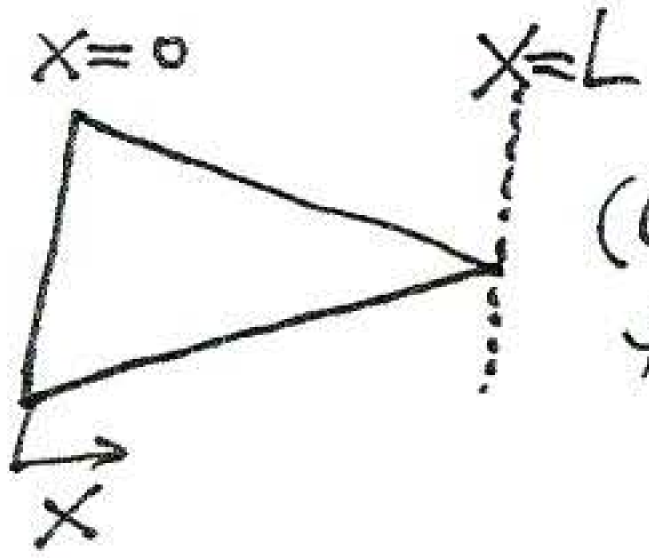


Fixed end $\Rightarrow$ $u(0,t)=0$ or $u(L,t)=0$

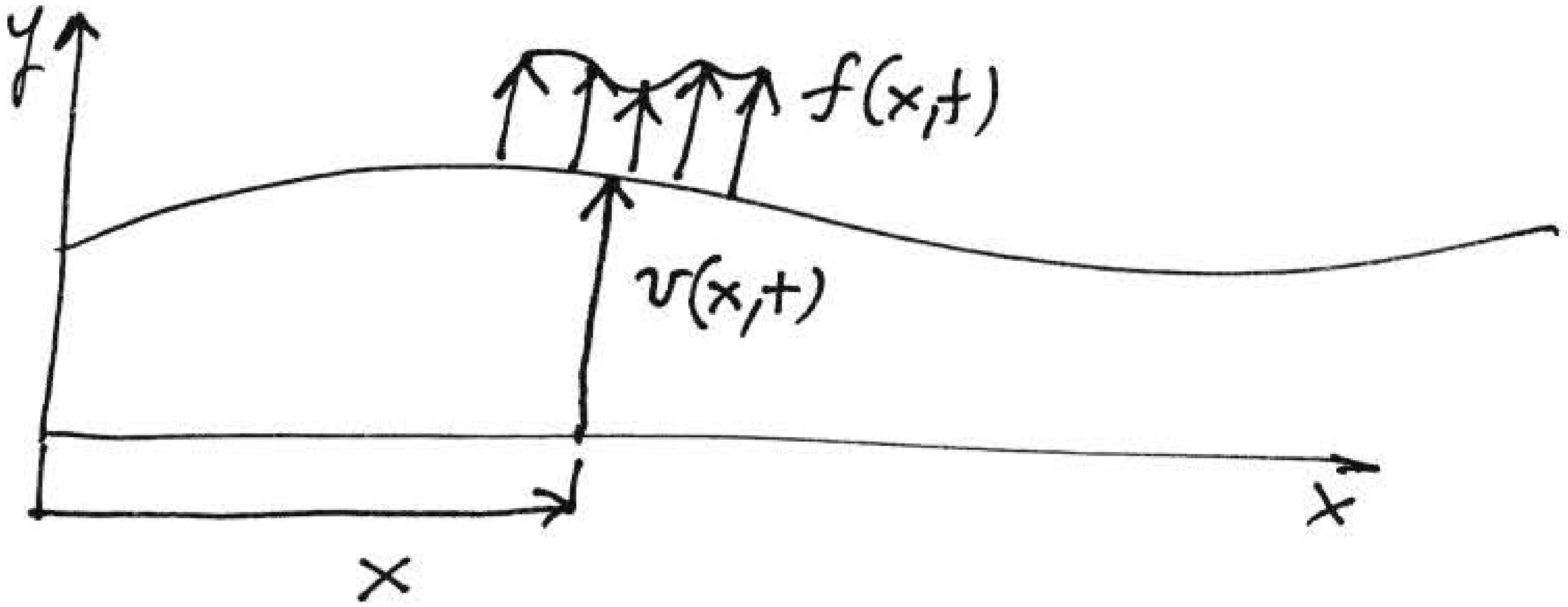
Free end $\Rightarrow$ $\lim_{x\to L}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]=0 \Rightarrow$

$\Rightarrow$ If $EA(L)\neq 0 \Rightarrow \frac{\partial u(L,t)}{\partial x}=0$

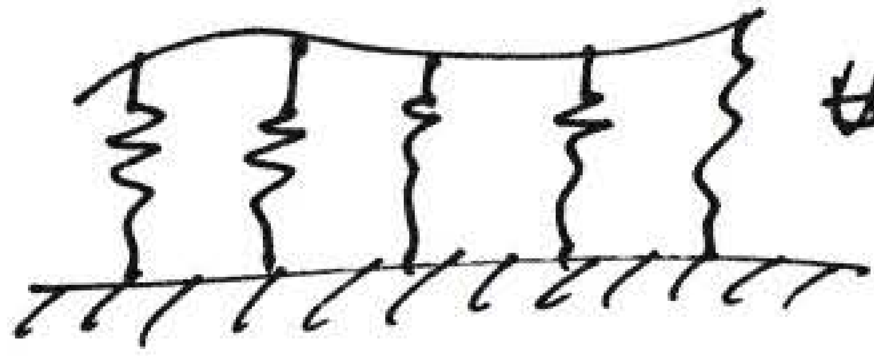
But if $\lim_{x\to L} EA(x)=0 \Rightarrow$ The full boundary condition should be taken into account



(Looking from the top)



Also dispersion



Classical wave equation with elastic foundation

Dispersive also!

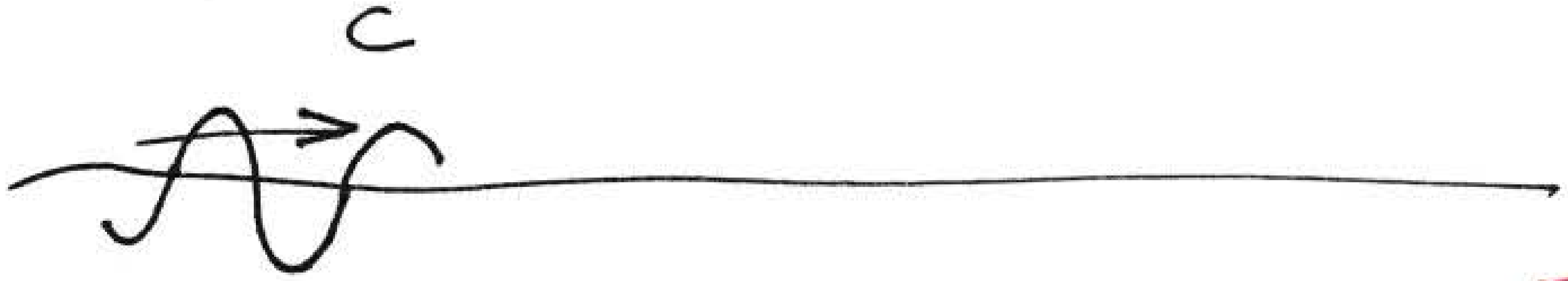
Non Dispersive medium

<u>Generalized wave equation,</u> $m(x)\dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial}{\partial x}\left[T(x)\dfrac{\partial v}{\partial x}\right] + f(x,t) = 0$

If $m(x)=m$, $T(x)=T$, $f(x,t)=0$, $\Rightarrow$ <u>Classical wave equation</u>

$$m\frac{\partial^2 v}{\partial t^2} = T\frac{\partial^2 v}{\partial x^2} \Rightarrow \frac{\partial^2 v}{\partial t^2} = \left(\frac{T}{m}\right)\frac{\partial^2 v}{\partial x^2}$$

c^2



For the classical wave equation, any applied disturbance propagates with the same velocity c.

Represents a non-dispersive waveguide

For example, consider traveling wave Ansatz

frequency   wavenumber

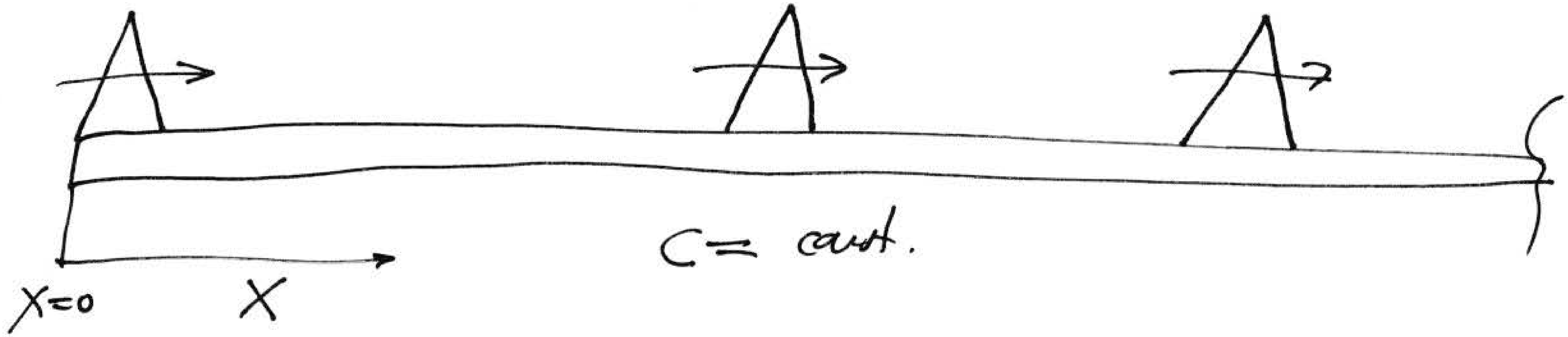
$$v(x,t) = A\cos(\omega t - kx) \Rightarrow$$

$$\Rightarrow \frac{\partial v}{\partial t} = -\omega A\sin(\omega t - kx) \Rightarrow \frac{\partial^2 v}{\partial t^2} = -\omega^2 A\cos(\omega t - kx)$$

$$\frac{\partial v}{\partial x} = kA\sin(\omega t - kx) \Rightarrow \frac{\partial^2 v}{\partial x^2} = -k^2 A\cos(\omega t - kx)$$

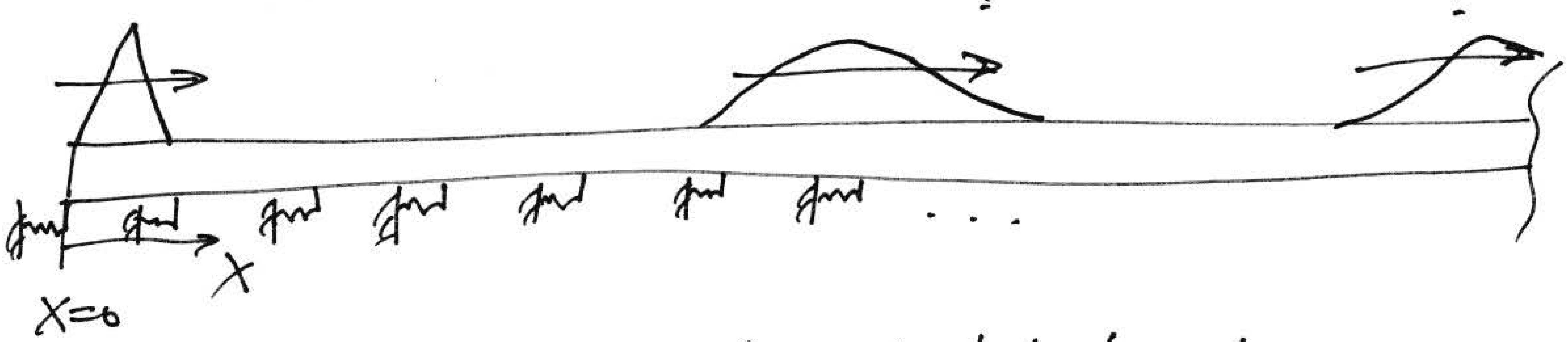
Substitute into ~~...~~ classical wave equation $\Rightarrow$ $-\omega^2 = c^2(-k^2) \Rightarrow c^2 = \dfrac{\omega^2}{k^2} \rightarrow \boxed{c = \dfrac{\omega}{k}}$

Non-dispersive waveguide: E.g., governed by classic wave equation



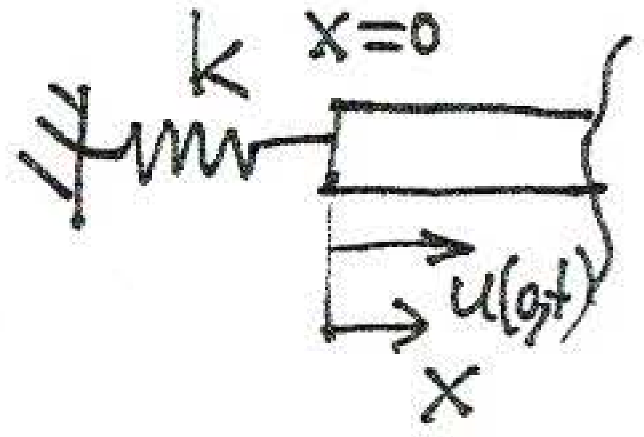
$c =$ const.

Dispersive waveguides



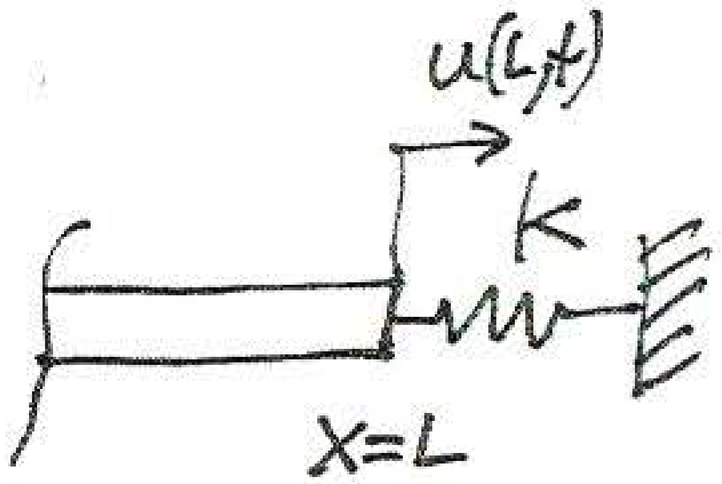
$c \neq$ const but depends on wavenumber or frequency

Stiffness at the boundary



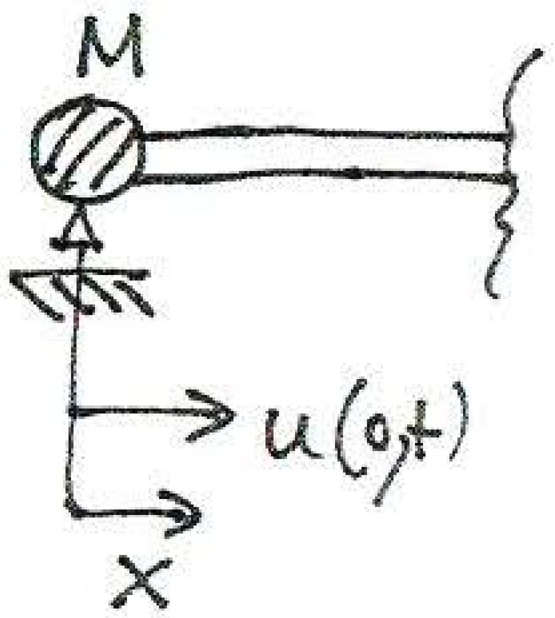
Performing balance of axial forces at $x=0 \Rightarrow N(0,t) - k u(0,t) = 0 \Rightarrow$

$$\Rightarrow \boxed{EA(0)\,\frac{\partial u(0,t)}{\partial x} - k\,u(0,t) = 0}$$



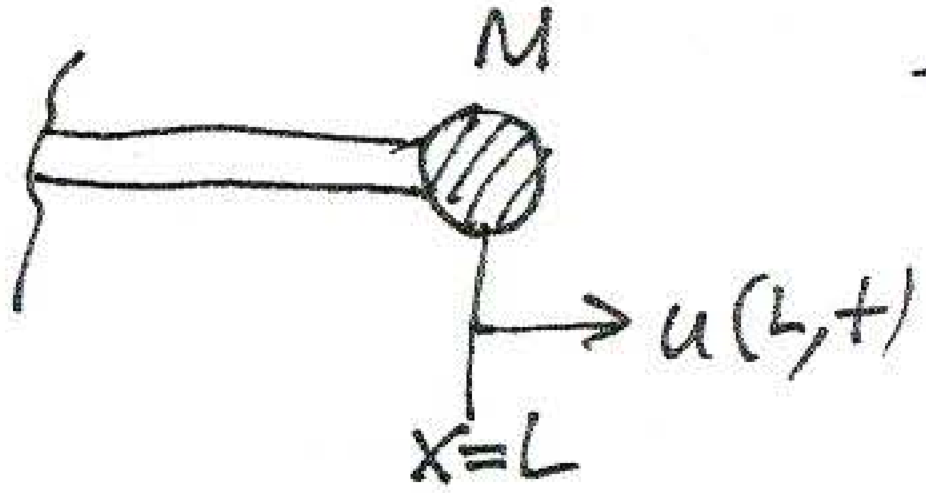
At the other end we can show that $\boxed{EA(L)\,\frac{\partial u(L,t)}{\partial x} + k\,u(L,t) = 0}$

Inertia at the boundary



Perform again balance of axial forces at $x=0 \Rightarrow$

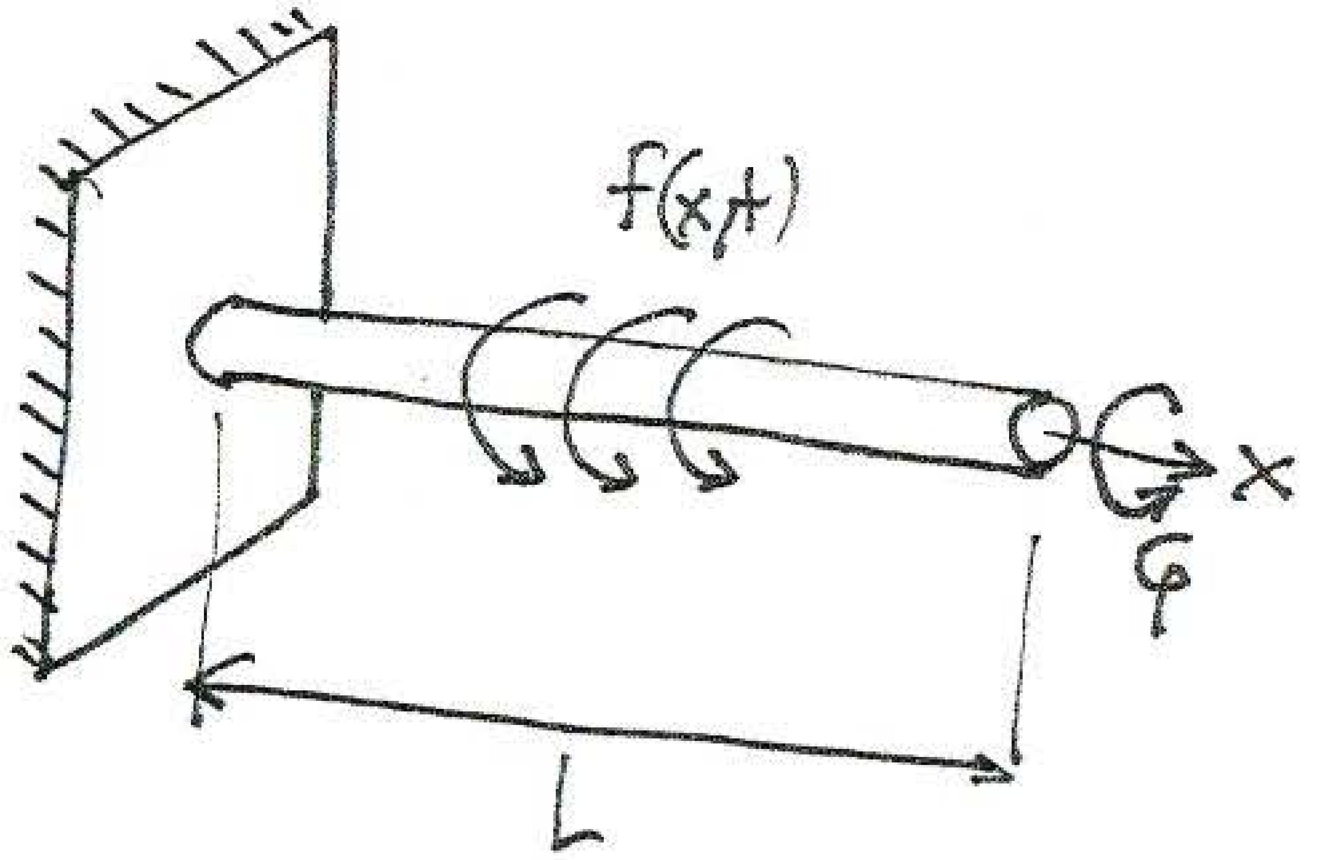
$$\Rightarrow \boxed{EA(0)\,\frac{\partial u(0,t)}{\partial x} - M\,\frac{\partial^2 u(0,t)}{\partial t^2} = 0}$$



Then the BC becomes $\boxed{EA(L)\,\frac{\partial u(L,t)}{\partial x} + M\,\frac{\partial^2 u(L,t)}{\partial t^2} = 0}$

<u>Suggestion</u>: Please try to recover these boundary conditions!

C. Torsional vibrations of circular shafts



By performing balance of moments on a differential element of this system we can show that the torsional oscillations are governed by the generalized wave equation in the form:

$$\boxed{\rho J_p(x)\,\frac{\partial^2 \varphi(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[G J_p(x)\,\frac{\partial \varphi(x,t)}{\partial x}\right] + f(x,t)}$$

Polar moment of inertia of the cross section at x

Modulus of torsional rigidity

Assuming a shaft with constant properties $\Rightarrow \rho J_p(x) = \rho J_p \Rightarrow$ Equation becomes the classic wave equation $\boxed{\frac{\partial^2 \varphi(x,t)}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 \varphi(x,t)}{\partial t^2}}$, $c = \sqrt{\frac{G}{\rho}}$ speed of sound in this medium

Brief overview of boundary conditions:

fixed boundary: $\varphi(0,t) = 0$ or $\varphi(L,t) = 0$

free boundary: $\frac{\partial \varphi(0,t)}{\partial x} = 0$ or $\frac{\partial \varphi(L,t)}{\partial x} = 0$