TAM 514 /AE 551 HOMEWORK 2

Distributed: **2/12/2025**
Due: **3/5/2025 in class (for on-line students, the deadline for submission by email is 1pm CST on the due date)**

1 (150 pts). (After an idea of Richard Weaver) Consider the following suspended uniform string hanging under its own weight. Gravity produces the spatially varying tension $T(x) = \rho g x$, where x is measured from the bottom of the string. Ignore other gravitational effects. We want to study the oscillations of this string.

(i) Derive the equation of motion using an infinitesimal solid mechanics approach(as shown in class) and formulate carefully the boundary conditions.

(ii) Compute the eigenmodes through the following steps:

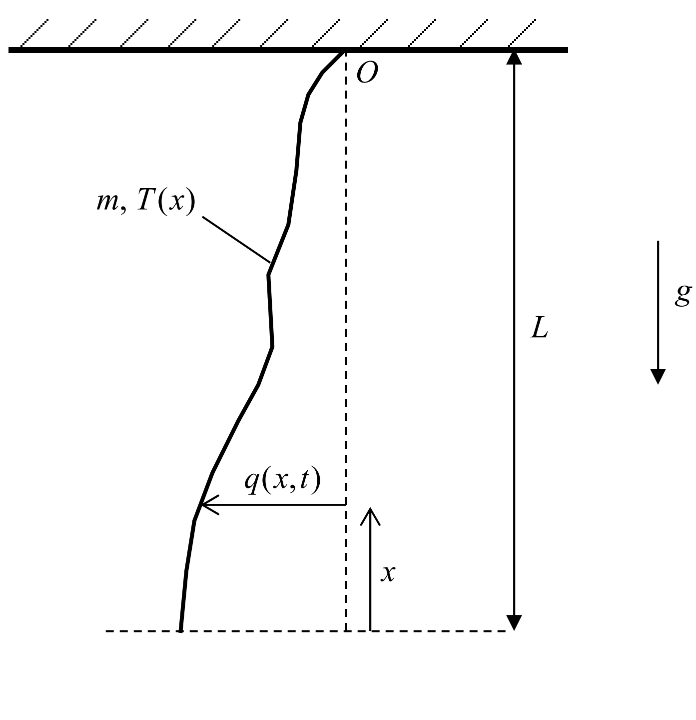
(ii1) Introduce the new independent variable $\xi = kx^n$ (determine n), and express the ordinary differential equation governing the eigenfunctions as,

$$\xi \frac{d^2 Q}{d\xi^2} + \frac{dQ}{d\xi} + \xi Q = 0$$

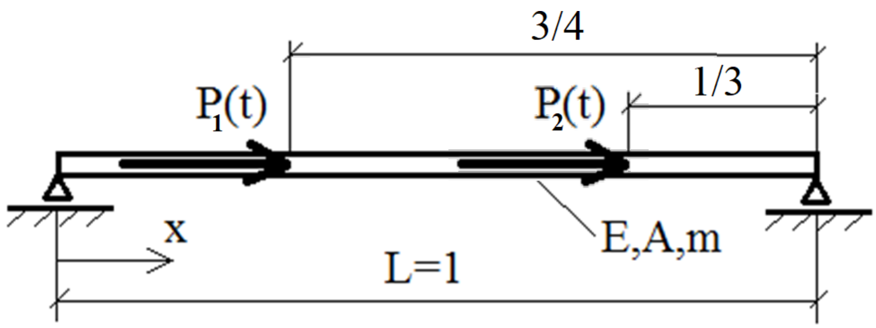
(ii2) Express the general solution of this equation as,

$$Q(\xi) = A\, J_0(\xi) + B\, Y_0(\xi)$$

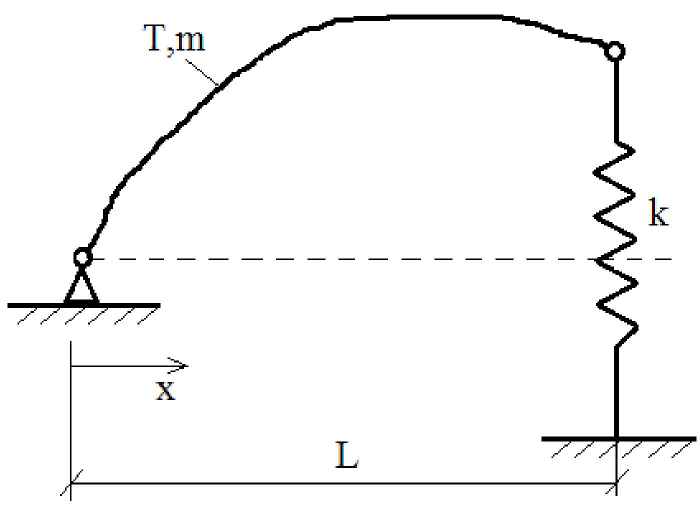
where $J_0(\xi)$, $Y_0(\xi)$ are Bessel functions of the first and second kind, respectively. Now impose the boundary conditions and obtain the equation for the eigenvalues (natural frequencies) and the eigenfunctions of the system. Compute the three leading natural frequencies and corresponding eigenfunctions.



2 (50 pts). Use modal analysis to compute the response of the free-free uniform rod performing axial vibrations subject to the axial point loads $F_1(x,t) = P_1(t)\,\delta\left(x - \frac{3}{4}\right)$, where $P_1(t) = \sin t,\ 0 \le t \le \pi$ and $P_1(t) = 0,\ t > \pi$, and $F_2(x,t) = P_2(t)\,\delta\left(x - \frac{1}{3}\right)$, where $P_2(t) = 5\sin t,\ 0 \le t \le \pi$ and $P_2(t) = 0,\ t > \pi$.



3 (50 pts). Compute the normal vibration modes of the uniform elastic string with fixed boundary condition at its left end and a transverse linear spring of constant k at its right end. Mass-orthonormalize the derived eigenfunctions. Assume that the spring can only deform in the vertical direction. Show that in the limit $k \to 0$ the modes approach those of the fixed-free string, whereas for $k \to \infty$ they approach the fixed-fixed string.



4 (50 pts). Consider the second order linear differential equation:

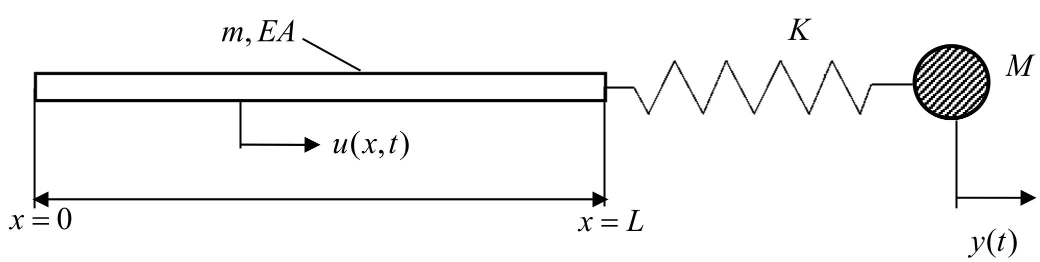
$$\ddot{x}(t) + p(t)\dot{x}(t) + q(t)x(t) = f(t)$$
$$x(0) = X,\ \dot{x}(0) = V$$

Show that if one homogeneous solution is known, then you can compute the complete solution of the initial value problem (that is, you can compute a second linearly independent homogeneous solution and a particular integral).

Hint: The attached notes could be helpful.

5 (100 pts). Consider the following rod in axial vibrations with an oscillator attached to it.
(i) Compute the normal modes (natural frequencies and eigenfunctions) and derive the orthonormality conditions satisfied by the eigenfunctions.
(ii) Solve graphically the frequency equation to show that this system has a countably infinite discrete set of natural frequencies. Is it possible to have repeated natural frequencies?
(iii) What happens in the limits $K \to \infty$, $K \to 0$ or $M \to \infty$?
(iv) Discuss how you can use modal analysis to compute the free response of the rod to initial conditions:

$$u(x,0) = U(x),\quad u_t(x,0) = V(x),\quad y(0) = \dot{y}(0) = 0$$



Linear Independence & the Wronskian of Two Functions

Recall that our derivation of the Wronskian and the idea of linearly independent functions came from considering a fundamental set of two solutions of a second order linear homogeneous ODE in standard form:

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are both continuous on some interval I.

For our purposes, we apply the Wronskian as a test for linear independence of two solutions to the above equation. It turns out that the idea of linear independence is more general than just checking for two fundamental solutions to the above ODE. Furthermore, a nonzero Wronskian can be used to verify the linear independence of *any* two differentiable functions, not just two solutions of the above ODE.

Linear Independence & the Wronskian for *any* two functions

Recall our definition of the linear dependence of two functions f and g on an open interval I: f and g are **linearly dependent** if there exists constants c_1 and c_2,*not both zero*, such that

$$c_1 f(t) + c_2 g(t) = 0, \quad \text{for all } t \in I$$

If we *must* choose $c_1 = 0 = c_2$, then we say f and g are **linearly independent**.

Since we are considering only *two* functions, linear dependence is **equivalent in this special case** to one function being a scalar multiple of the other:

$$f(t) = Cg(t) \quad \text{or} \quad g(t) = Cf(t) \quad \text{for some constant } C.$$

Note that C may be zero.

If two differentiable functions f and g are linearly *dependent*, then their Wronskian is zero for *all* $t \in I$, i.e.,

$$W[f,g](t) = f(t)g'(t) - g(t)f'(t) = 0, \quad \text{for all } t \in I.$$

Thus, if the Wronskian is nonzero at *any* $t \in I$, the two functions must be linearly *independent.*

Examples

1. $f(t) = 2t$ and $g(t) = 3t$ are linearly *dependent* for all $t \in \mathbb{R}$, because each of the following holds for *all* $t \in \mathbb{R}$:
   (a) $3f(t) + (-2)g(t) = 3(2t) + -2(3t) = 0$, where $c_1 = 3 \neq 0$ and $c_2 = -2 \neq 0$.
   (b) $f(t) = 2t = \frac{2}{3}g(t)$, where $C = 2/3$, i.e., one function is a scalar multiple of the other.

It follows that the Wronskian is zero for all $t \in \mathbb{R}$:

$$W[f,g](t) = f(t)g'(t) - g(t)f'(t) = 2t(3) - 3t(2) = 6t - 6t = 0.$$

2. On the interval $I = (-2, 2)$, $f(t) = 2t$ and $g(t) = 3t^2$ are linearly *independent*, because of each of the following holds for *all* $t \in (-2, 2)$:

   (a) $c_1 f(t) + c_2 g(t) = 0$ for all $t \in (-2, 2)$ implies

   $$\begin{aligned} c_1 2t + c_2 3t^2 &= 0 \\ t(2c_1 + 3c_2 t) &= 0 \end{aligned}$$

   which implies $t = 0$ or $2c_1 + 3c_2 t = 0$, neither of which hold for *all* $t \in (-2, 2)$ unless $c_1 = 0 = c_2$.

   (b) Neither function is a scalar multiple of the other (check this!).

   Not surprisingly, the Wronskian is not zero for *all* $t \in \mathbb{R}$:

   $$W[f,g](t) = f(t)g'(t) - g(t)f'(t) = 2t(6t^2) - 3t^2(2) = 6t - 6t^2 = 6t(1-t) \neq 0,$$

   as long as $t \neq 0, 1$.

   Note: The fact that the Wronskian *is* zero at two points in $I = (-2, 2)$, i.e., $W[f,g](0) = 0 = W[f,g](1)$, does *not* imply linear dependence. In fact, one can rig up two linearly *independent* functions whose Wronskian is *everywhere* zero! Take the differentiable functions $f(t) = t^2$ and $g(t) = |t|t$, for example, and consider the cases $t < 0$ and $t \geq 0$ separately.

Linear Independence & the Wronskian for two solutions to the ODE

If we are considering $f = y_1$ and $g = y_2$ to be two solutions to the ODE

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are both continuous on some interval I, then the Wronskian has some extra properties which are given by Abel's Theorem:

$$W[y_1, y_2](t) = ce^{-\int p(t)\,dt}, \quad \text{for some constant } c.$$

This theorem essentially says that if two *solutions* the the ODE are linearly independent, then the Wronskian of the two solutions is *never* zero on the interval I, i.e., $c \neq 0$. Otherwise, the Wronskian is *always* zero, i.e., $c = 0$, and the solutions are linearly dependent. This is the **key result** that we find useful for checking for a fundamental set of two solutions to a second order linear homogeneous differential equation.

Examples

1. $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ are linearly *independent* solutions to

$$y'' - 5y' + 6 = 0$$

for *all* $t \in \mathbb{R}$ because of each of the following equivalences holds:

(a) $c_1 y_1(t) + c_2 y_2(t) = 0$ for all $t \in \mathbb{R}$ implies

$$\begin{aligned} c_1 e^{2t} + c_2 e^{3t} &= 0 \\ e^{2t}(c_1 + c_2 e^t) &= 0 \end{aligned}$$

which implies $c_1 = 0 = c_2$.

(b) Neither function is a scalar multiple of the other (check this!).

(c) $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = e^{2t}(3e^{3t}) - e^{3t}(2e^{2t}) = 3e^{5t} - 2e^{5t} = e^{5t} \neq 0$ for *all* $t \in \mathbb{R}$. Here $p(t) = -5$ and $c = 1$ in Abel's Theorem.

2. $y_1(t) = e^{2t}$ and $y_2(t) = e^{\ln 3 + 2t}$ are linearly *dependent* solutions to

$$y'' - 5y' + 6 = 0$$

for *all* $t \in \mathbb{R}$ because of each of the following equivalences holds for *all* $t \in \mathbb{R}$:

(a) $-3y_1(t) + y_2(t) = -3e^{2t} + e^{\ln 3 + 2t} = -3e^{2t} + e^{\ln 3}e^{2t} = -3e^{2t} + 3e^{2t} = 0$, where $c_1 = -3 \neq 0$ and $c_2 = 1 \neq 0$.

(b) $y_2(t) = e^{\ln 3 + 2t} = e^{\ln 3}e^{2t} = 3e^{2t} = 3y_1(t)$, where $C = 3$, i.e., one function is a scalar multiple of the other.

(c) $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = e^{2t}(2e^{\ln 3 + 2t}) - e^{\ln 3 + 2t}(2e^{2t}) = 0$. Here, $p(t) = -5$ and $c = 0$ in Abel's Theorem.

In each of the above two examples we see that the Wronskian of the two solutions is *everywhere* zero or *nowhere* zero on the interval $I = \mathbb{R}$. This is guaranteed by Abel's Theorem.

Considering our earlier example, it follows that $f(t) = 2t$ and $g(t) = 3t^2$ cannot be a fundamental set of solutions to *any* second order linear homogeneous ODE on the interval $I = (-2, 2)$. Can you figure out why not?!?

LECTURE 14

More On The Wronskian

Last lecture, we introduced the Wronskian of two functions y_1 and y_2,

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2'(t)y_1(t).$$

We saw that if $W(y_1,y_2)(t) \neq 0$, then y_1 and y_2 are linearly independant, *i.e.*, the only constants c_1 and c_2 that satisfy

$$c_1y_1(t) + c_2y_2(t) = 0$$

are $c_1 = c_2 = 0$. In other words, two functions are linearly independent if they aren't constant multiples of each other.

We also saw that, in the context where y_1 and y_2 are solutions to the linear homogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = 0,$$

$W(y_1,y_2)(t) \neq 0$ is precisely the condition for the general solution of the differential equation to be

$$y(t) = c_1y_1(t) + c_2y_2(t),$$

i.e., for y_1 and y_2 to be a fundamental set of solutions. Thus, if y_1 and y_2 are a fundamental set of solutions, they are linearly independent. If we have two solutions y_1 and y_2 which are linearly dependent, on the other hand, then they cannot possibly be a fundamental set of solutions, as they have a zero Wronskian.

There was one point last lecture that we should clear up now. We know that if our initial data is at t_0, y_1 and y_2 will be a fundamental set of conditions if and only if $W(y_1,y_2)(t_0) \neq 0$. But this is a condition only at one point. What happens if y_1 and y_2 have a nonzero Wronskian only at t_0 but not at nearby points? This would be problematic, since then our condition would tell us that y_1 and y_2 are a fundamental set of solutions for initial dat at t_0, but not for any initial data near t_0. The possibility of this should seem odd, since we know there should be a unique solution on any interval around t_0 where we have continuity. So how do we know that this can't happen? The answer is something called Abel's Theorem.

1. Abel's Theorem

You may notice that throughout our entire discussion of the Wronskian, we have yet to actually use the differential equation (beyond deriving the formula for the Wronskian assuming that y_1 and y_2 satisfied some differential equation). Fortunately, when y_1 and y_2 are solutions to a linear homogeneous differential equation, we can say a bit more about their Wronskian.

THEOREM 14.1 (Abel's Theorem). *Suppose $y_1(t)$ and $y_2(t)$ solve the linear homogeneous equation*

$$y''(t) + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous on some interval (a,b). Then, for $a < t < b$, their Wronskian is given by

$$W(y_1,y_2)(t) = W(y_1,y_2)(t_0)e^{-\int_{t_0}^{t} p(x)\,dx},$$

where t_0 is in (a,b).

If $W(y_1, y_2)(t_0) \neq 0$ at some point t_0 in the interval (a, b), then Abel's Theorem tells us that the Wronskian can't be zero for any t in (a, b), since exponentials are never zero. This assures us that we can change our initial data (without crossing points of discontinuity of the coefficient functions) without worry that our general solution will change.

Another advantage of Abel's Theorem is that it lets us compute the general form of the Wronskian of any two solutions to the differential equation without knowing them explicitly. This is useful, for example, with regard to reduction of order, where we only begin by knowing a single solution. The formulation given in the statement of the theorem isn't so computationally useful, however, because we might not have a precise t_0 in mind, let alone knowing the value of the Wronskian there. But if we apply the Fundamental Theorem of Calculus, things simplify nicely.

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)e^{-\int_{t_0}^{t} p(x)\,dx} = ce^{-\int p(t)\,dt}$$

What is this constant c? Well, it doesn't really end up mattering. If we know the value of the Wronskian at one point, we can compute it, but our general interest in the Wronskian mostly involves knowing its general form. As long as we know $c \neq 0$, that's all that matters to us.

EXAMPLE 14.1. *Compute, up to a constant, the Wronskian of two solutions y_1 and y_2 of the differential equation*

$$t^4 y'' - 2t^3 y' - t^8 y = 0.$$

First, we need to put the equation in the form specified in Abel's Theorem. We do this by dividing by the leading coefficient.

$$y'' - \frac{2}{t}y' - t^4 y = 0.$$

So, Abel's Theorem tells us

$$W = ce^{-\int -\frac{2}{t}\,dt} = ce^{2\ln t} = ct^2.$$

□

Ok, great...but the main virtue of this is that it gives us a second way to compute the Wronskian. A general rule in mathematics is that whenever you can compute something in two different ways, something good will happen. In this case, we know by Abel's Theorem that

$$W(y_1, y_2)(t) = ce^{-\int p(t)\,dt}.$$

On the other hand, by definition,

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

Setting these equal, if we know one solution $y_1(t)$, we're left with a first order differential equation for y_2 that we can then solve.

Let's see this with an example of reduction of order we did the traditional way.

EXAMPLE 14.2. Suppose we want to find the general solution to $2t^2y'' + ty' - 3y = 0$ and we're given that $y_1(t) = t^{-1}$ is a solution. We need to find a second solution that will form a fundamental set of solutions with y_1. Let's compute the Wronskian both ways.

$$ce^{-\int \frac{1}{2t}\,dt} = W(t^{-1}, y_2)(t) = y_2't^{-1} + y_2t^{-2}$$

$$y_2't^{-1} + y_2t^{-2} = ce^{-\frac{1}{2}\ln(t)} = ct^{-\frac{1}{2}}$$

This is a first order linear equation with integrating factor $\mu(t) = e^{\int t^{-1}\,dt} = e^{\ln(t)} = t$. Thus

$$\begin{aligned} [ty_2]' &= ct^{\frac{3}{2}} \\ ty_2 &= \frac{2}{5}ct^{\frac{5}{2}} + k \\ y_2(t) &= \frac{2}{5}ct^{\frac{3}{2}} + kt^{-1} \end{aligned}$$

Now, we can choose constants c and k. Notice that k is the coefficient of t^{-1}, which is just $y_1(t)$. So we don't have to worry about that term, and we can take $k = 0$. We can similarly take $c = \frac{5}{2}$, and so we'll get $y_2(t) = t^{\frac{3}{2}}$, which is precisely what we had gotten when we did reduction of order the traditional way. $\square$