Hence we can conclude that it holds:

Orthonormality with respect to inertia distribution

$$\boxed{\int_0^L \beta(x)\,\varphi_r(x)\,\varphi_s(x)\,dx = \delta_{rs}} \quad \text{(Orthonormality condition)}\ r,s=1,2,\dots \qquad (10)$$

where $\delta_{rs}=1$ if $r=s$, and $\delta_{rs}=0$ if $r\neq s$. Now let's go back to either one of relations (9a) or (9b), say (9a), multiply it by $\varphi_s(x)$ and integrate from 0 to L with respect to $x \Rightarrow$

$$\int_0^L \frac{d}{dx}\left[A(x)\frac{d\varphi_r(x)}{dx}\right]\varphi_s(x)\,dx + \omega_r^2 \underbrace{\int_0^L \beta(x)\,\varphi_s(x)\,\varphi_r(x)\,dx}_{\delta_{rs}} = 0 \Rightarrow$$

Orthonormality with respect to elastic distribution

$$\Rightarrow \boxed{\int_0^L \frac{d}{dx}\left[A(x)\frac{d\varphi_r(x)}{dx}\right]\varphi_s(x)\,dx = -\omega_r^2\delta_{rs}}, \quad r=1,2,\dots,\ s=1,2,\dots \qquad (11)$$

We can simply this more by integrating by parts the left-hand-side, $\Rightarrow$

$$\Rightarrow \underbrace{A(x)\frac{d\varphi_r(x)}{dx}\varphi_s(x)\Big|_0^L}_{0 \text{ for 'simple' BCs (4c)}} - \int_0^L A(x)\frac{d\varphi_r(x)}{dx}\frac{d\varphi_s(x)}{dx}\,dx = -\omega_r^2\delta_{rs} \Rightarrow$$

Orthonormality with respect to stiffness distribution

$$\Rightarrow \boxed{\int_0^L A(x)\,\varphi_r'(x)\,\varphi_s'(x)\,dx = \omega_r^2\delta_{rs}}, \quad r,s=1,2,\dots \qquad (12)$$

Note that these orthogonality conditions are for simple BCs only!

By conditions (10) and (11) or (12) the eigenfunctions are determined uniquely, so there are no arbitrary multiplicative constants in them $\Rightarrow$ Eigenfunctions for systems with 'simple' BCs that are orthonormalized according to (10), (11) or (12) will be called <u>mass-orthonormalized modes.</u>

So now let's reconsider the general solution of (1) and let's assume also that there is external distributed forcing:

$$\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] + F(x,t) = B(x)\frac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le L \tag{13}$$

with boundary conditions (12) and initial conditions (16).

The first thing we do is to neglect for the moment the applied force and the initial conditions and formulate and solve the resulting boundary value problem $\Rightarrow$ Obtain the countable infinity of natural modes of vibration $\{\omega_r^2, \varphi_r(x)\}$, $r = 1, 2, \ldots$

mass orthonormalized

Then, we use the principle of linear superposition and express the general forced response of (13) in terms of modal responses:

Coordinates' or amplitudes of modes

$$u(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\,\varphi_i(x) \tag{14}$$

Form a basis in functional space

Let's substitute (14) into (13) $\Longrightarrow$

$$\Rightarrow B(x)\sum_{i=1}^{\infty}\ddot{\eta}_i(t)\varphi_i(x)=\sum_{i=1}^{\infty}\eta_i(t)\frac{d}{dx}\left[A(x)\frac{d\varphi_i(x)}{dx}\right]+F(x,t)$$

We pick an arbitrary ~~by~~ but fixed index j and multiply both sides of the above equation by $\varphi_j(x)$, and then integrate both sides with respect to x from 0 to L $\Longrightarrow$

$$\Rightarrow \int_0^L B(x)\sum_{i=1}^{\infty}\ddot{\eta}_i(t)\varphi_i(x)\varphi_j(x)\,dx=\int_0^L\sum_{i=1}^{\infty}\eta_i(t)\frac{d}{dx}\left[A(x)\frac{d\varphi_i(x)}{dx}\right]\varphi_j(x)\,dx+$$
$$+\int_0^L F(x,t)\varphi_j(x)\,dx \Longrightarrow \qquad (15)$$

$$\Rightarrow \sum_{i=1}^{\infty}\ddot{\eta}_i(t)\underbrace{\int_0^L B(x)\varphi_i(x)\varphi_j(x)\,dx}_{\delta_{ij}}=\sum_{i=1}^{\infty}\eta_i(t)\underbrace{\int_0^L\frac{d}{dx}\left[A(x)\frac{d\varphi_i(x)}{dx}\right]\varphi_j(x)\,dx}_{-\omega_i^2\delta_{ij}}+$$
$$+\int_0^L F(x,t)\varphi_j(x)\,dx\;,\quad j \text{ arbitrary but fixed.}$$

jth Modal oscillator

j-th modal forcing

Hence, $$\boxed{\ddot{\eta}_j(t)+\omega_j^2\eta_j(t)=\int_0^L F(x,t)\varphi_j(x)\,dx\equiv N_j(t),\quad j=1,2,\dots} \qquad (16)$$

But what about initial conditions?
Consider the initial conditions of the problem, relations (1b).

$$u(x,0) = g(x) = \sum_{i=1}^{\infty} \eta_i(0)\varphi_i(x)$$

$$\frac{\partial u}{\partial t}(x,0) = h(x) = \sum_{i=1}^{\infty} \dot{\eta}_i(0)\varphi_i(x)$$

Now consider the first of the above relations and multiply it by $\varphi_j(x)$ where j is arbitrary but fixed $\Rightarrow$

$$\Rightarrow \quad g(x)\varphi_j(x) = \sum_{i=1}^{\infty} \eta_i(0)\varphi_i(x)\varphi_j(x) \Rightarrow$$

$\int_0^L B(x)\,dx$ Multiply both sides by $B(x)$ Integrate wrt x

$$\Rightarrow \int_0^L B(x) g(x)\varphi_j(x)\,dx = \sum_{i=1}^{\infty} \eta_i(0) \underbrace{\int_0^L B(x)\varphi_i(x)\varphi_j(x)\,dx}_{\delta_{ij}} = \eta_j(0) \Rightarrow$$

$$\Rightarrow \boxed{\eta_j(0) = \int_0^L B(x) g(x)\varphi_j(x)\,dx}, \quad j=1,2,\dots$$

Similarly we can compute the j-th initial modal velocity as

$$\boxed{\dot{\eta}_j(0) = \int_0^L B(x) h(x)\varphi_j(x)\,dx}, \quad j=1,2\dots$$

Then we can solve completely for the j-th modal response:

$$\eta_j(t)=\eta_j(0)\cos\omega_j t+\frac{\dot{\eta}_j(0)}{\omega_j}\sin\omega_j t+\frac{1}{\omega_j}\int_0^t N_j(\tau)\sin\omega_j(t-\tau)\,d\tau,$$

$$j=1,2,\dots$$

For the forced solution of our original problem (1) we get,

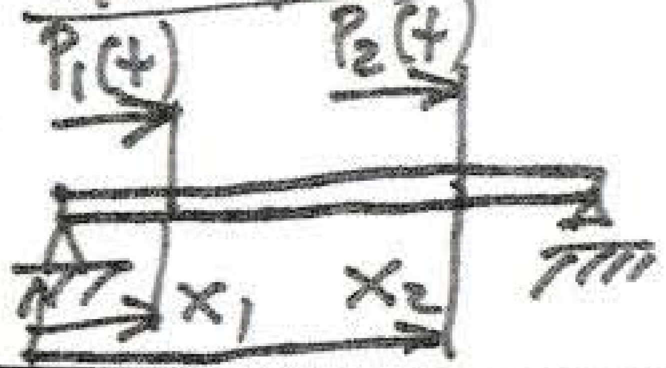
$$\boxed{u(x,t)=\sum_{i=1}^{\infty}\left[\eta_i(0)\cos\omega_i t+\frac{\dot{\eta}_i(0)}{\omega_i}\sin\omega_i t+\frac{1}{\omega_i}\int_0^t N_i(\tau)\sin\omega_i(t-\tau)\,d\tau\right]\varphi_i(x)} \tag{18}$$

Inertia orthonormalized eigenfunctions.

Remark

Suppose that we have zero initial conditions and the applied force distribution is a series of point loads, i.e., $F(x,t)=\sum_{i=1}^{K}P_i(t)\,\delta(x-x_i)\Rightarrow$

Example



$$\Rightarrow u(x,t)=\sum_{i=1}^{\infty}\left[\frac{1}{\omega_i}\int_0^t N_i(\tau)\sin\omega_i(t-\tau)\,d\tau\right]\varphi_i(x)$$

But $N_j(t)=\int_0^L\sum_{i=1}^{K}P_i(t)\,\delta(x-x_i)\,\varphi_j(x)\,dx=\sum_{i=1}^{K}P_i(t)\underbrace{\int_0^L\delta(x-x_i)\,\varphi_j(x)\,dx}_{\varphi_j(x_i)}$

$$N_j(x) = \int_0^L \sum_{i=1}^{k} P_i(t)\, \delta(x - x_i)\, \varphi_j(x)\, dx =$$

$$= \sum_{i=1}^{k} P_i(t) \underbrace{\int_0^L \delta(x - x_i)\, \varphi_j(x)\, dx}_{\varphi_j(x_i)} \Rightarrow$$

$$\int_{-\infty}^{+\infty} G(x)\, \delta(x - a)\, dx = G(a)$$

$$\Rightarrow N_j(x) = \sum_{i=1}^{k} P_i(t)\, \varphi_j(x_i)$$

Solution process

$$\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] + F(x,t) = B(x)\frac{\partial^2 u}{\partial t^2} + BCs + ICs$$

$$0 \leq x \leq L$$

Discrete spectrum of eigenvalues

$$\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] = B(x)\frac{\partial^2 u}{\partial t^2} + BCs \Rightarrow \{\omega_r^2, \varphi_r(x)\}$$

Remarks

1) Finite system

2) BCs should give distinct modes $\omega_r \neq \omega_s$ if $r \neq s$

$$\boxed{u(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\,\varphi_i(x)}$$

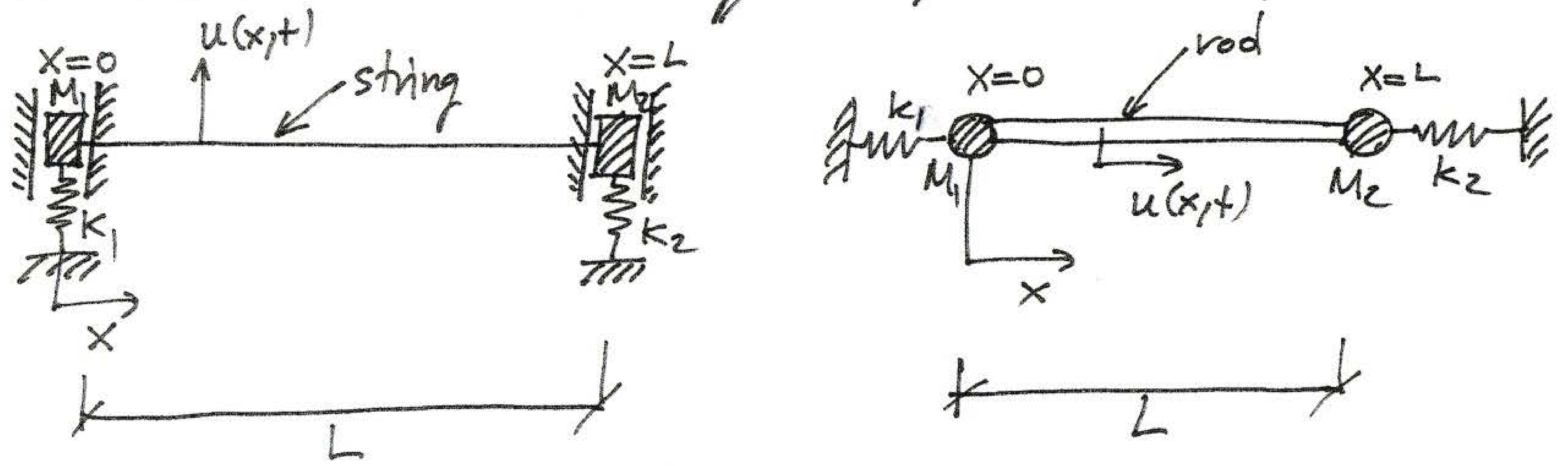
Infinity of uncoupled modal oscillators

ICs $\rightarrow$ Derive the initial conditions for these modal oscillators

$\Rightarrow$ Solve!

Free oscillations of elastodynamic system with 'complex' boundary conditions

We will demonstrate the methodology through an example.



Then, the governing partial differential equation would be,

$$\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] = B(x)\frac{\partial^2 u}{\partial t^2}, \quad 0 < x \leq L$$

with boundary conditions:

$$A(0)\frac{\partial u(0,t)}{\partial x} - k_1 u(0,t) - M_1\frac{\partial^2 u(0,t)}{\partial t^2} = 0 \qquad (1a)$$

$$A(L)\frac{\partial u(L,t)}{\partial x} + k_2 u(L,t) + M_2\frac{\partial^2 u(L,t)}{\partial t^2} = 0 \qquad (1b)$$

Seek again synchronous vibrations where $u(x,t) = f(t)\varphi(x) \Rightarrow$
Then $\ddot{f}(t) + \omega^2 f(t) = 0$

$\Rightarrow$ Substituting into the first of the boundary conditions,

$$A(0)\varphi'(0)f(t) - K_1\varphi(0)f(t) - M_1\varphi(0)\underset{-\omega^2 f(t)}{\ddot{f}(t)} = 0 \Rightarrow$$

$$\Rightarrow A(0)\varphi'(0) - K_1\varphi(0) + \omega^2 M_1\varphi(0) = 0 \Rightarrow$$

$$\Rightarrow \boxed{A(0)\varphi'(0) - (K_1 - \omega^2 M_1)\varphi(0) = 0} \quad (2a)$$

Similarly at $x=L$ we get the boundary condition,

$$\boxed{A(L)\varphi'(L) + (K_2 - \omega^2 M_2)\varphi(L) = 0} \quad (2b)$$

Remarks:
1) If $M_1 = M_2 = 0$ we get stiffness BCs
2) If $K_1 \to \infty \Rightarrow$ $\Rightarrow \varphi(0) \to 0$, so we recover fixed BC.
3) If $K_1 \to 0$ and $M_1 \to 0 \Rightarrow A(0)\varphi'(0) \to 0$ recover free BC.

Recall general eigenvalue analysis in the system with 'simple' boundary conditions; then we got the eigenvalue problem

$$\frac{d}{dx}\left[A(x)\frac{d\varphi(x)}{dx}\right] = -\omega^2 B(x)\varphi(x) \Rightarrow \varphi(x,\omega) = c_1\varphi_1(x,\omega) + c_2\varphi_2(x,\omega)$$

Two linearly independent solutions

Then substituting this solution into (2a) and (2b) we get:

(2a) $\Rightarrow \quad A(0)\left[c_1\varphi_1'(0,\omega)+c_2\varphi_2'(0,\omega)\right]-(k_1-\omega^2M_1)\left[c_1\varphi_1(0,\omega)+c_2\varphi_2(0,\omega)\right]=0$

(2b) $\Rightarrow \quad A(L)\left[c_1\varphi_1'(L,\omega)+c_2\varphi_2'(L,\omega)\right]+(k_2-\omega^2M_2)\left[c_1\varphi_1(L,\omega)+c_2\varphi_2(L,\omega)\right]=0$

Hence we have that

$$\underbrace{\begin{bmatrix} A(0)\varphi_1'(0,\omega)-(k_1-\omega^2M_1)\varphi_1(0,\omega) & A(0)\varphi_2'(0,\omega)-(k_1-\omega^2M_1)\varphi_2(0,\omega) \\ A(L)\varphi_1'(L,\omega)+(k_2-\omega^2M_2)\varphi_1(L,\omega) & A(L)\varphi_2'(L,\omega)+(k_2-\omega^2M_2)\varphi_2(L,\omega) \end{bmatrix}}_{\text{Set determinant of coefficients}=0\Rightarrow}\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\Rightarrow G(\omega)=0 \Rightarrow \omega_1,\omega_2,\ldots$ and order them such that $0<\omega_1<\omega_2<\omega_3<\ldots$

Then, substitute $\omega=\omega_r$, say, back into matrix-equation and compute the ratio $c_1^{(r)}/c_2^{(r)}$ which determines the r-th eigenfunction $\varphi_r(x)$.

Now look at orthogonality conditions. Suppose that we pick two modes

$$\{\omega_r,\varphi_r(x)\},\quad \{\omega_s,\varphi_s(x)\},\quad \omega_r\neq\omega_s,\ r\neq s$$

Then, we get:

No! $\cancel{\int_0^L B(x)\varphi_r^2(x)\,dx=1}$