Digression: Eigenvalue problems with nonsymmetric matrices

In many applications we encounter eigenvalue problems with nonsymmetric mass or stiffness matrices:

$$[K]\{u\} = \lambda [M]\{u\}, \quad [K] \neq [K]^T, [M] \neq [M]^T \quad , \quad u \in R^n \quad (*)$$

In such cases we cannot use the orthogonality properties and the expansion theorem that hold for systems with symmetric system matrices. Instead we will develop a special technique that will allow us to decompose an arbitrary vector in R^n using the <u>bi-orthogonality property</u>.

Considering the nonsymmetric eigenvalue problem, let's assume that all eigenvalues $\lambda_i, i=1,\dots,n$ are real and distinct $\Rightarrow$ It can be shown that the corresponding eigenvectors are linearly independent.

$$[K]\{u\} = \lambda [M]\{u\} \Rightarrow \underbrace{[M]^{-1}[K]}_{[\alpha]}\{u\} = \lambda\{u\} \Rightarrow [\alpha]\{u\} = \lambda\{u\},$$

$$[\alpha] \neq [\alpha]^T$$

Suppose that the eigenvectors of this eigenvalue problem are $\{u_i\}$, $i=1,\dots,n$ (Right eigenvectors)

Now consider the adjoint eigenvalue problem,

$$[\alpha]^T\{v\} = \lambda\{v\} \qquad \text{(Adjoint eigenvalue problem)}$$

which has the same set of eigenvalues but a different set of linearly independent eigenvectors $\{v_i\}$, $i=1,\dots,n$ (left eigenvectors).

Let $[\alpha]\{u_i\} = \lambda_i\{u_i\} \Rightarrow \{u_i\}^T[\alpha]^T = \lambda_i\{u_i\}^T \Rightarrow$

$[\alpha]^T\{v_j\} = \lambda_j\{v_j\}$, $\lambda_i \neq \lambda_j \Rightarrow$

$$\Rightarrow \{u_i\}^T[\alpha]^T\{v_j\} = \lambda_i\{u_i\}^T\{v_j\}$$

$$\Rightarrow \{u_i\}^T[\alpha]^T\{v_j\} = \lambda_j\{u_i\}^T\{v_j\}$$

$\Big| - \Rightarrow$

$$\Rightarrow (\lambda_i - \lambda_j)\{u_i\}^T\{v_j\} = 0 \Rightarrow \boxed{\{u_i\}^T\{v_j\} = 0,\ i \neq j}$$

Biorthogonality condition

So, in order to solve the nonsymmetric eigenvalue problem, it is necessary to solve also the adjoint eigenvalue problem $\Rightarrow$

$\Rightarrow$ We obtain two sets of eigenvectors $\{u_i\}$ and $\{v_j\}$, $i,j=1,\dots,n$ that satisfy the biorthogonality conditions

$$\boxed{[u]^T[v] = [v]^T[u] = [I]}$$ Biorthonormality conditions

where we normalized the right- and left-eigenvectors so that it holds that $\{u_i\}^T\{v_i\}=1$, $i=1,\dots,n$. The vectors $\{u_i\}$ and $\{v_j\}$, $i=1,\dots,n$ are called conjugates of each other. Note that for a symmetric matrix, i.e., $[a]=[a]^T \Rightarrow \{u_i\}=\{v_j\}$.

Then, any vector in the n-dim space can be decomposed as a superpositions of either $\{u_i\}$ or $\{v_j\}$ provided that both of these sets of eigenvectors are computed $\Rightarrow$ Use the dual expansion theorem,

$$\forall \{x\} \in R^n \Rightarrow \{x\} = \sum_{r=1}^{n} \alpha_r \{u_r\}, \quad \alpha_r = \{v_r\}^T\{x\}, \quad r=1,\dots,n$$

or

$$\{x\} = \sum_{s=1}^{n} \beta_s \{v_s\}, \quad \beta_s = \{u_s\}^T\{x\}, \quad s=1,\dots,n$$

where α_r and β_s can be regarded as components of $\{x\}$ in the appropriate bases.

Example where nonsymmetric system matrices can arise: The <u>collocation method</u> for solving continuous vibration problems assumes an approximate solution where it can be represented as superposition of some set of functions v_j and associated coefficients a_j. According to our selection of the basis functions v_j we have:

- The boundary method: The functions satisfy the governing dif. equations but not the boundary conditions.
- The interior method: The functions satisfy the boundary conditions but not the governing dif. equations.
- The mixed method: The functions satisfy neither the BCs nor the dif. equations.

Then the coefficients a_r are computed by requiring that the approximation satisfies the dif. equation and/or the BCs at a certain set of points (sensors) or locations s_r, $r = 1, \ldots, n$. For example, taking the interior method, let's solve the eigenvalue problem

$$L[w] = \lambda M[w] \quad , \; w = w(s) \Rightarrow \text{Approximation: } w(s) \cong w_n(s) = \sum_{j=1}^{n} a_j v_j(s)$$

Substituting into the governing dif. eq. the error that we obtain is,

$$\mathcal{E} = L[W_n] - {}^{n}\lambda\, M[W_n] \Rightarrow$$ Require that $\mathcal{E}=0$ at the set of sensors or locations chosen $\Rightarrow$

Approximation to the eigenvector

Approximation of the eigenvalue

But $w_n(s) = \sum_{j=1}^{n} a_j v_j(s)$

$$\Rightarrow L[W_n(s_r)] - {}^{n}\lambda\, M[W_n(s_r)] = 0, \quad r = 1, \dots, n$$

Note that $\mathcal{E} \neq 0$ $\forall s$ in the domain of the problem, since otherwise we would have solved exactly the original problem! Now use the fact that $L[\cdot]$ and $M[\cdot]$ are linear integrodifferential operators $\Rightarrow$

$$\Rightarrow \sum_{j=1}^{n} a_j \underbrace{L[v_j(s_r)]}_{k_{rj}} - {}^{n}\lambda \sum_{j=1}^{n} a_j \underbrace{M[v_j(s_r)]}_{m_{rj}} = 0, \quad r = 1, \dots, n \Rightarrow$$

$$\Rightarrow \sum_{j=1}^{n} a_j k_{rj} - {}^{n}\lambda \sum_{j=1}^{n} a_j m_{rj} = 0 \Rightarrow \sum_{j=1}^{n} \left(k_{rj} - {}^{n}\lambda\, m_{rj}\right) a_j = 0, \quad r = 1, \dots, n \Rightarrow$$

$$\Rightarrow [K]\{a\} = {}^{n}\lambda\, [M]\{a\}$$ Eigenvalue problem, where in general $[K] \neq [K]^T$, $[M] \neq [M]^T$

Example of non-symmetric eigenvalue problems

Consider $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Let $\{x\} = \{x_1 \ x_2\}^T$

$[-A]$

$\Rightarrow \{\dot{x}\} + [-A]\{x\} = \{0\}$

where $[A] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

Let $\{x\} = \{u\}e^{\lambda t}$,

where $\{u\} = \{u_1 \ u_2\}^T$ $\Rightarrow$

$$\Rightarrow \lambda\{u\}e^{\lambda t} + [-A]\{u\}e^{\lambda t} = \{0\} \Rightarrow \lambda\{u\} = [A]\{u\}$$

where $[A] \neq [A]^T$

Non-symmetric eigenvalue problem

Right eigenvectors

First solve the eigenvalue problem

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 1 & -1-\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Computation of the eigenvalues:

$$-(1-\lambda)(1+\lambda)-2=0 \Rightarrow \lambda_{1,2}=\pm\sqrt{3}$$

For $\lambda=\lambda_1=-\sqrt{3}$ the eigenvector is computed as,

$$(1-\lambda_1)u_1+2u_2=0 \Rightarrow u_1=-\frac{2}{1+\sqrt{3}}u_2 \Rightarrow \{u_1\}=\begin{Bmatrix}-\frac{2}{1+\sqrt{3}}\\ 1\end{Bmatrix}c,$$

$c\in R$ (multiplicative const)

For $\lambda=\lambda_2=\sqrt{3}$ the eigenvector is computed as,

$$(1-\lambda_2)u_1+2u_2=0 \Rightarrow u_1=-\frac{2}{1-\sqrt{3}}u_2 \Rightarrow \{u_2\}=\begin{Bmatrix}-\frac{2}{1-\sqrt{3}}\\ 1\end{Bmatrix}d,$$

$d\in R$

So, the right eigenvector matrix is,

$$[u]=[\{u_1\}\ \{u_2\}]=\begin{bmatrix}-\frac{2c}{1+\sqrt{3}} & -\frac{2d}{1-\sqrt{3}}\\ 1\cdot c & 1\cdot d\end{bmatrix}$$

Right Eigenvector for $\lambda=\lambda_1=-\sqrt{3}$

Right Eigenvector for $\lambda=\lambda_2=\sqrt{3}$

Left eigenvectors

Second, solve the eigenvalue problem

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^T \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \lambda \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \Rightarrow \begin{bmatrix} 1-\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This adjoint eigenvalue problem has the same eigenvalues,

$\lambda = \pm\sqrt{3} \Rightarrow \lambda_1 = -\sqrt{3}$ and $\lambda_2 = \sqrt{3}$.

For $\lambda = \lambda_1 = -\sqrt{3}$, the left eigenvector is computed as,

$$(1-\lambda_1) v_1 + v_2 = 0 \Rightarrow v_1 = -\frac{1}{1+\sqrt{3}} v_2 \Rightarrow \{v_1\} = \begin{Bmatrix} -\frac{1}{1+\sqrt{3}} \\ 1 \end{Bmatrix} f, \quad f \in R$$

For $\lambda = \lambda_2 = \sqrt{3}$, the left eigenvector is computed as,

$$(1-\lambda_2) v_1 + v_2 = 0 \Rightarrow v_1 = -\frac{1}{1-\sqrt{3}} v_2 \Rightarrow \{v_2\} = \begin{Bmatrix} -\frac{1}{1-\sqrt{3}} \\ 1 \end{Bmatrix} g, \quad g \in R$$

So, the left eigenvector matrix is,

$$[V] = [\{v_1\} \; \{v_2\}] = \begin{bmatrix} -\frac{1 \cdot f}{1+\sqrt{3}} & -\frac{1 \cdot g}{1-\sqrt{3}} \\ 1 \cdot f & 1 \cdot g \end{bmatrix}$$

Hence, $[V]^T[U] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $[V]^T$ is orthogonal to $[U]$; similarly, $[U]^T$ is orthogonal to $[V]$ (bi-orthogonality)

Compute multiplicative constants c, d, f, g: $\{U_i\}^T\{V_i\} = 1$, $i = 1, 2$

So, to solve the original problem we express the solution as,

$$\boxed{\{X\} = C_1\{U_1\}e^{\lambda_1 t} + C_2\{U_2\}e^{\lambda_2 t}},$$

where C_1, C_2 are computed by initial conditions,

$$\left.\begin{aligned} X_1(0) &= X_{10}, \\ X_2(0) &= X_{20} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \{X(0)\} = \begin{Bmatrix} X_{10} \\ X_{20} \end{Bmatrix}$$

or

$$\{X\} = D_1\{V_1\}e^{\lambda_1 t} + D_2\{V_2\}e^{\lambda_2 t},$$

$D_1, D_2 \in R$

Hence, $\{X(0)\} = C_1\{U_1\} + C_2\{U_2\} \Rightarrow$

$$\Rightarrow \{V_1\}^T\{X(0)\} = C_1\{V_1\}^T\{U_1\} + C_2\underbrace{\cancel{\{V_1\}^T\{U_2\}}}_{0} \Rightarrow$$

$$\Rightarrow \quad C_1 = \frac{\{V_1\}^T\{X(0)\}}{\underbrace{\{V_1\}^T\{U_1\}}_{1}}; \text{ similarly, } C_2 = \frac{\{V_2\}^T\{X(0)\}}{\underbrace{\{V_2\}^T\{U_2\}}_{1}}$$

Remarks:

1) Note that we can also express the solution as,

$$\boxed{\{x\} = D_1 \{v_1\} e^{\lambda_1 t} + D_2 \{v_2\} e^{\lambda_2 t}}$$

where D_1 and D_2 are computed in terms of the bi-orthogonality condition using the initial conditions.

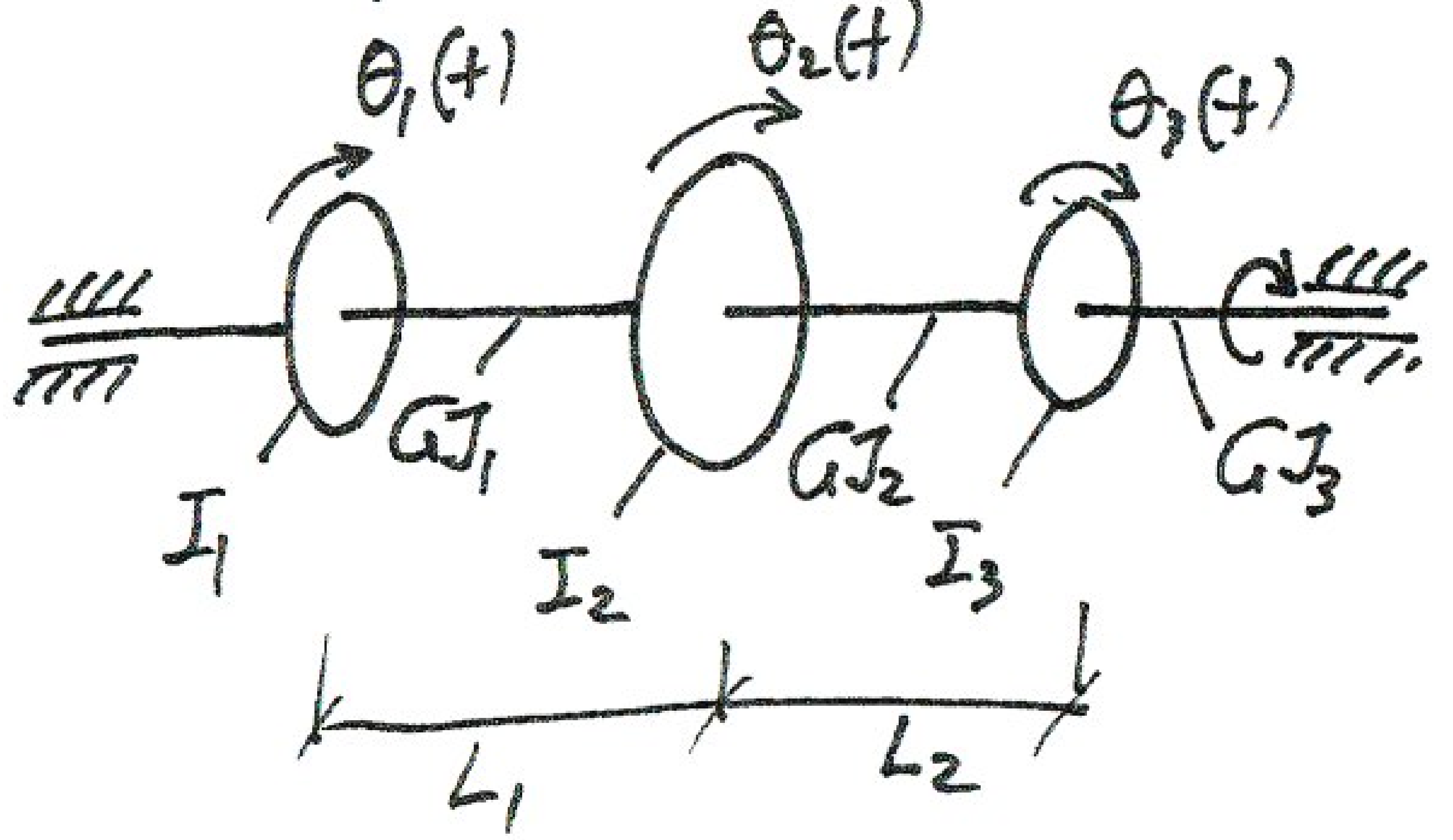
2) We can also not bi-orthonormalize the right- and left-eigenvectors so that,

$$[v]^T [u] \neq [I], \text{ but diagonal matrix}$$

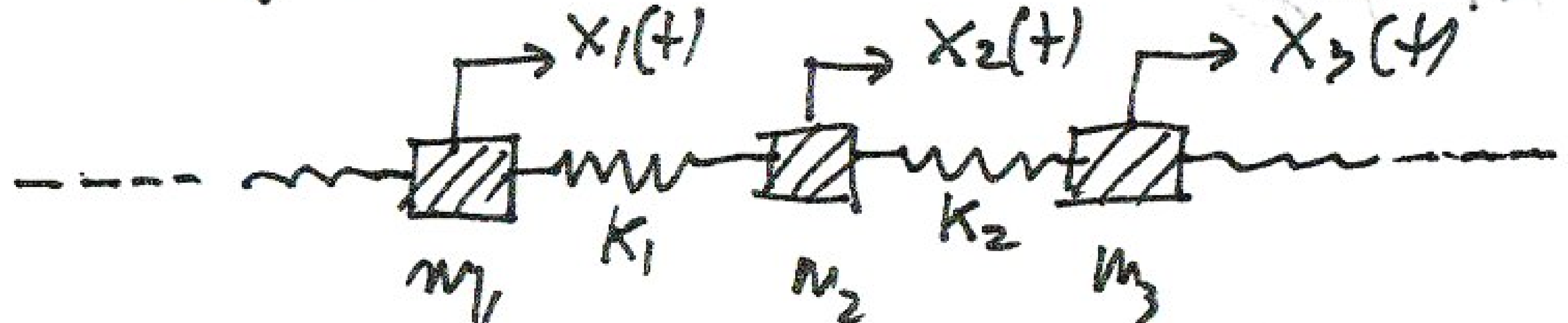
That, however, would not change the form of the solutions.

Digression: Systems with Rigid-Body modes

There are cases where the stiffness matrix $[K]$ is <u>positive-semidefinite</u>, i.e., is <u>singular</u>. We will demonstrate a technique for decomposing the rigid body modes from the elastic modes (i.e., the modes corresponding to nonzero nat. frequencies) by an example.



Equivalent translational system



The kinetic energy of this system is

$$T = \frac{1}{2}\left\{I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2\right\} =$$

$$= \frac{1}{2}\{\dot{\theta}\}^T[I]\{\dot{\theta}\},$$

$$\{\theta\} = \begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}, \quad [I] = \begin{bmatrix} I_1 & & 0\\ & I_2 & \\ 0 & & I_3\end{bmatrix}$$

The potential energy is

$$V = \frac{1}{2}\left[k_1(\theta_1-\theta_2)^2 + k_2(\theta_2-\theta_3)^2\right] =$$

$$= \frac{1}{2}\{\theta\}^T[K]\{\theta\},$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0\\ -k_1 & k_1+k_2 & -k_2\\ 0 & -k_2 & k_2\end{bmatrix}, \quad k_i = \frac{GJ_i}{L_i}, \; i=1,2$$

Assuming synchronous torsional oscillations we require that $\theta_i(t) = \Theta_i\, f(t)$, $i = 1, 2, 3$, $f(t)$ is harmonic function $\Rightarrow$ Derive the eigenvalue problem,

$$\omega^2 [I]\{\Theta\} = [k]\{\Theta\}$$

I could denote the rigid body mode as $\{\Theta_0\} = \begin{Bmatrix} A_0 \\ A_0 \\ A_0 \end{Bmatrix} = A_0 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

We can easily show that one of the eigenvalues is $\omega = 0$ corresponding to the eigenvector $\{\Theta_0\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$. This fact is consistent with the singularity of the stiffness matrix and the corresponding mode is the rigid body mode of this system $\Rightarrow$ The other flexible modes of the system should be orthogonal to the rigid body mode $\Rightarrow$

$$\{\Theta_0\}^T [I] \{\Theta\} = 0 \Rightarrow A_0 (I_1 \Theta_1 + I_2 \Theta_2 + I_3 \Theta_3) = 0 \Rightarrow$$

$\{\Theta_0\} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} A_0$

Denotes a flexible mode corresponding to nat. freq. > 0

$$\Rightarrow \underbrace{I_1 \Theta_1 + I_2 \Theta_2 + I_3 \Theta_3}_{\text{for the flexible modes}} \underset{\forall t \geq 0}{=} 0 \Rightarrow \boxed{I_1 \dot{\theta}_1 + I_2 \dot{\theta}_2 + I_3 \dot{\theta}_3 = 0}$$

Satisfied by the flexible modes

<u>This implies that for all flexible modes the angular momentum of the system is zero.</u>

The general motion of the system would be a combination of the rigid body and the flexible modes $\Rightarrow$ How could we separate the flexible modes?
But, due to conservation (actually nullification) of the angular momentum of the system in a flexible mode, it holds that,

$$\theta_3(t) = -\frac{I_1}{I_3}\theta_1(t) - \frac{I_2}{I_3}\theta_2(t)$$

$\Rightarrow$ We reduce the number of independent coordinates by one in a flexible mode.

Then,

$$\underbrace{\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix}_f}_{\{\theta\}_f} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -\frac{I_1}{I_3} & -\frac{I_2}{I_3} & 0\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\\ -\frac{I_1}{I_3} & -\frac{I_2}{I_3}\end{bmatrix}}_{[C]}\underbrace{\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}}_{\{\theta\}} \Rightarrow$$

$$\Rightarrow \{\theta\}_f = [C]\{\theta\}$$ Applies only for a flexible mode!

where $[C]$ plays the role of a constraint matrix.

Then I can express the kinetic and potential energy of the system for motion on a flexible mode $\Rightarrow$ Then I can reduce the dimensionality of the problem by extracting the rigid body mode out of the problem and working only with flexible modes $\Rightarrow$

$$T = \frac{1}{2}\{\dot{\theta}\}_f^T[I]\{\theta\}_f = \frac{1}{2}\{\dot{\theta}\}^T\underbrace{[C]^T[I][C]}_{[I']}\{\theta\} = \frac{1}{2}\{\dot{\theta}\}^T\underbrace{[I']}_{\text{Reduced mass matrix}}\{\theta\}$$

Do the same for elastic matrix,

$$V = \frac{1}{2}\{\theta\}_f^T[K]\{\theta\}_f = \frac{1}{2}\{\theta\}^T\underbrace{[C]^T[K][C]}_{[K']}\{\theta\} = \frac{1}{2}\{\theta\}^T[K']\{\theta\}$$

Reduced stiffness matrix.

Explicitly, $[I'] = \frac{1}{I_3}\begin{bmatrix} I_1(I_1+I_3) & I_1 I_2 \\ I_1 I_2 & I_2(I_2+I_3) \end{bmatrix}$

$$[K'] = \frac{1}{I_3^2}\begin{bmatrix} k_1 I_3^2 + k_2 I_1^2 & -k_1 I_3^2 + k_2 I_1(I_2+I_3) \\ -k_1 I_3^2 + k_2 I_1 (I_2+I_3) & (k_1+k_2) I_3^2 + k_2 I_2 (2I_2+I_3) \end{bmatrix}$$

which are (2×2) positive definite matrices $\Rightarrow$ The reduced eigenvalue problem for the flexible modes is,

$$\omega^2[I']\{\Theta\} = [K']\{\Theta\} \Rightarrow \text{Compute the flexible modes.}$$

for example, if $k_1=k_2=k$, $I_1=I_2=I_3=I \Rightarrow$

$$[I'] = \frac{1}{I}\begin{bmatrix}2I^2 & I^2\\ I^2 & 2I^2\end{bmatrix} = I\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

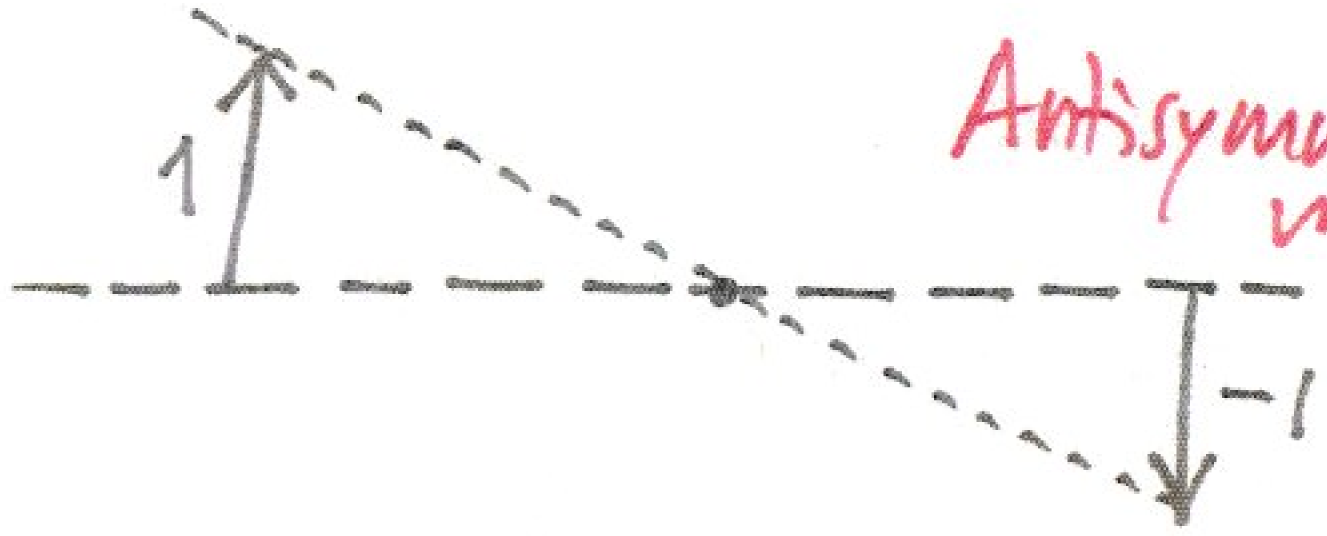
$$[k'] = k\begin{bmatrix}2 & 1\\ 1 & 5\end{bmatrix}$$

$\Rightarrow$

Flexible mode 1

$$\omega_1=\sqrt{\frac{k}{I}}, \quad \{\Theta_1\}=\begin{Bmatrix}1\\0\end{Bmatrix} \Rightarrow$$

$$\Rightarrow \{\Theta_1\}_f = [C]\{\Theta_1\} = \begin{Bmatrix}1\\0\\-1\end{Bmatrix}$$



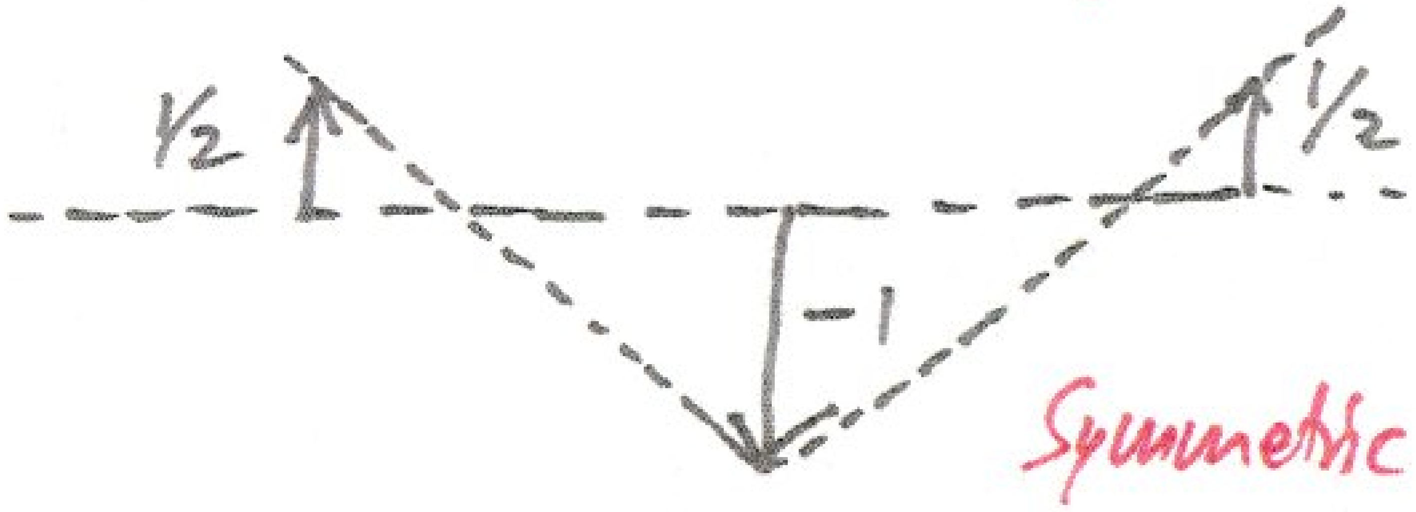
Antisymmetric mode

Flexible mode 2

$$\omega_2=\sqrt{\frac{2k}{I}}, \quad \{\Theta_2\}=\begin{Bmatrix}1/2\\-1\end{Bmatrix} \Rightarrow$$

$$\Rightarrow \{\Theta_2\}_f = [C]\{\Theta_2\} = \begin{Bmatrix}1/2\\-1\\1/2\end{Bmatrix}$$

Symmetric mode



Rigid body mode

$$\omega_0=0, \quad \{\Theta\}=\begin{Bmatrix}1\\1\\1\end{Bmatrix}$$

Symmetric mode