Consider $\left.\begin{array}{l} z'' - rz + \lambda z = 0 \\ z(0) = z(\ell) = 0 \end{array}\right\} \Rightarrow$ We can show that the solutions of this S-L differential equation satisfy the following Volterra integral equation,

$$z(t) = \alpha \sin\sqrt{\lambda}\,t + \frac{1}{\sqrt{\lambda}}\int_0^t r(\tau)\, z(\tau) \sin\sqrt{\lambda}(t-\tau)\, d\tau \qquad (**)$$

But $z(\ell)=0$ is not yet satisfied

$z(0)=0$ is satisfied by this form.

Note that this is in the form of a linear map in infinite dimensional space, by iterating,

$z_0(t)$,

$z_1(t) = M[z_0(t)]$

$z_2(t) = M[z_1(t)]$

...

There are powerful theorems for integral equations (e.g., the contraction theorem) that prove that under certain conditions this map converges to a unique solution $z(t)$ in infinite-dimensional space.

Remark

$z'' + \lambda z = f(t)$

$z'' + \lambda z = \overbrace{r(t)z(t)}$

'pseudo forcing' (not true force)

Then the eigenvalues are obtained by imposing on (**) the second boundary condition,

$$z(\ell)=0 \Rightarrow \quad 0 = \alpha \sin\sqrt{\lambda}\,\ell + \frac{1}{\sqrt{\lambda}}\int_0^{\ell} r(\tau)z(\tau)\sin\sqrt{\lambda}(t-\tau)\,d\tau \Rightarrow$$

Compute λ_n, $n=1,2,\ldots$ and corresponding eigenfunctions $z_n(t)$

It can be proven that the eigenfunctions are bounded for all $\lambda_n \Rightarrow$ Normalizing $\int_0^{\ell} z^2\,dt = 1 \Rightarrow$

$$\left.\begin{array}{l}\Rightarrow \alpha = \sqrt{\frac{2}{\ell}} + O\left(\frac{1}{\sqrt{\lambda}}\right) \Rightarrow z(t) = \sqrt{\frac{2}{\ell}}\sin\sqrt{\lambda}\,t + O\left(\frac{1}{\sqrt{\lambda}}\right) \\ \text{However, we have seen that } \lambda_n \sim \frac{n^2\pi^2}{\ell^2} + O(1) \text{ as } n \gg 1\end{array}\right\} \Rightarrow$$

$$\Rightarrow z_n(t) = \sqrt{\frac{2}{\ell}}\sin\sqrt{\lambda_n}\,t + \frac{1}{\sqrt{\lambda_n}}O(1) \Rightarrow \boxed{z_n(t) \sim \sqrt{\frac{2}{\ell}}\sin\frac{n\pi t}{\ell} + O\left(\frac{1}{n}\right)}$$

In terms of the original eigenvalue problem, these estimates read:

$$u_n(x) = c_n\,\frac{\sin\left(\frac{n\pi}{\ell}\int_0^x\sqrt{\frac{\rho}{p}}\,dx\right)}{\sqrt[4]{p\rho}} + O\left(\frac{1}{n}\right)$$

$$\frac{1}{c_n^2} = \int_0^{\pi}\frac{\sin^2\left(\frac{n\pi}{\ell}\int_0^{\pi}\sqrt{\frac{\rho}{p}}\,dx\right)}{\sqrt{p\rho}}\,dx, \quad \ell = \int_0^{\pi}\sqrt{\frac{\rho}{p}}\,dx$$

Similar asymptotic estimates can be derived for other types of homogeneous BCs, e.g., $z''-rz+\lambda z=0$, $z'(0)-hz(0)=0$, $z(\ell)=0 \Rightarrow$

$$\Rightarrow z_n(t)\sim\sqrt{\frac{2}{\ell}}\cos\frac{n\pi}{\ell}t+O\left(\frac{1}{n}\right),\quad h \text{ is bounded}$$

Finally, let's show how the Volterra integral equation approach can be used to derive approximations for the eigensolutions:

$$z(t)=\alpha\sin\sqrt{\lambda}t+\frac{1}{\sqrt{\lambda}}\int_0^t r(\tau)z(\tau)\sin\sqrt{\lambda}(t-\tau)\,d\tau,\quad z(0)=0\ \checkmark$$

To start the iteration let's assume that $z(t)=\alpha\sin\sqrt{\lambda}t$, $a=1\Rightarrow$

$$\Rightarrow z_0(t)=\sin\sqrt{\lambda}t\equiv v(t)$$

$$z_1(t)=v(t)+\frac{1}{\sqrt{\lambda}}\int_0^t r(\tau_1)v(\tau_1)\sin\sqrt{\lambda}(t-\tau_1)\,d\tau_1$$

$$z_2(t)=v(t)+\frac{1}{\sqrt{\lambda}}\int_0^t r(\tau_1)\sin\sqrt{\lambda}(t-\tau_1)\overbrace{\left\{v(\tau_1)+\frac{1}{\sqrt{\lambda}}\int_0^{\tau_1} r(\tau_2)v(\tau_2)\sin\sqrt{\lambda}(\tau_1-\tau_2)\,d\tau_2\right\}}^{z_1(\tau_1)}\times d\tau_1$$

$$=v(t)+\frac{1}{\sqrt{\lambda}}\int_0^t r(\tau_1)\sin\sqrt{\lambda}(t-\tau_1)v(\tau_1)\,d\tau_1+$$

$$+\frac{1}{\lambda}\int_0^t\int_0^{\tau_1} r(\tau_1)r(\tau_2)v(\tau_2)\sin\sqrt{\lambda}(t-\tau_1)\sin\sqrt{\lambda}(\tau_1-\tau_2)\,d\tau_1\,d\tau_2$$

$$\cdots$$

Then, to compute the estimate for the ~~eigenvalue~~, set $z_2(\ell)=0 \Rightarrow$

$$\Rightarrow \quad 0 = v(\ell) + \frac{1}{\sqrt{\lambda}}\int_0^{\ell} r(\tau_1)\sin\sqrt{\lambda}(\ell-\tau_1)\, v(\tau_1)\, d\tau_1 +$$
$$+ \frac{1}{\lambda}\int_0^{\ell}\int_0^{\tau_1} r(\tau_1)\, r(\tau_2)\, v(\tau_2)\sin\sqrt{\lambda}(\ell-\tau_1)\sin\sqrt{\lambda}(\tau_1-\tau_2)\, d\tau_1\, d\tau_2$$

This is the equation for determining λ!

Remark: Eigenvalue problems with a continuous spectrum

Up to now we considered eigenvalue problems with discrete spectra, where the eigenvalues formed a countable infinity of real numbers. However, if the coefficients of the S-L eigenvalue problem are singular at the boundaries, or if the domain is infinite, then the spectrum (or the set of eigenvalues) may be continuous. That means that $\lambda \in [a,b]$, $a,b \in \mathbb{R}$.

In this case the Fourier integral theorem replaces the eigenfunction expansion theorem $\Rightarrow f(x) = \frac{1}{2\pi}\int_0^{\infty} du \int_{-\infty}^{\infty} f(t)\, e^{-ju(t-x)}\, dt$, provided $\int_{-\infty}^{\infty} |f(x)|\, dx$ exists, $f(x)$ is piecewise smooth in every finite interval.

Example

$u'' + \lambda u = 0$, $-\infty < x < \infty \Rightarrow$ Clearly every pair $\{\cos\sqrt{\lambda}x, \sin\sqrt{\lambda}x\}$ is a solution for $\lambda > 0 \Rightarrow \lambda \in [0, \infty) \Rightarrow$ Continuous spectrum.

Example

Bessel functions $\Rightarrow$ Consider $(xu')' + (\lambda x - \frac{n^2}{x})u = 0$, $0 \leq x < \infty$ with the boundary condition that u is finite at $x = 0$ and as $x \to \infty \Rightarrow$

$\Rightarrow$ All Bessel functions $u = J_n(\sqrt{\lambda}x)$, $\lambda \geq 0$ are solutions $\Rightarrow$

$\Rightarrow \lambda \in [0, \infty) \Rightarrow$ Continuous spectrum. Here also the expansion theorem is replaced by an integral theorem, in which the domain of integration is the spectrum $[0, \infty) \Rightarrow$

$$f(x) = \int_0^\infty t\, J_n(tx)\, g(t)\, dt, \quad g(t) = \int_0^\infty \xi\, J_n(\xi t)\, f(\xi)\, d\xi$$

This representation is valid if $f(x)$ is piecewise smooth for $x \geq 0$, so that $\int_0^\infty x|f(x)|dx$ exists and $f(0) = 0$.

Perturbation Theory

Suppose that the eigensolution of the eigenvalue problem $\boxed{L(u_n)+\lambda_n u_n=0}$ is known, i.e., we can compute $\{\lambda_n, u_n\}$, $n=1,2,\dots$ for prescribed domain g and boundary conditions. Can we then establish a perturbation theory that would approximately compute the eigensolution of the perturbed system

$$\boxed{L(\bar{u}_n) - \varepsilon r \bar{u}_n + \bar{\lambda}_n \bar{u}_n = 0}$$

defined in the same domain g, with the same boundary conditions, with $r=r(x)$ being a continuous function in g, and $|\varepsilon| \ll 1$ being the small parameter of the problem $\Longrightarrow$ Based on the known eigensolution $\{\lambda_n, u_n\} \Longrightarrow$ $\Rightarrow$ Approximate the 'perturbed' solution $\{\bar{\lambda}_n; \bar{u}_n\}$. We assume that $\lambda_i \neq \lambda_j$, if $i \neq j$ (the case where there are multiple eigenvalues λ_i is a degenerate one and its treatment can be found in Courant + Hilbert).

Assuming that $|\varepsilon| \ll 1$, we make the hypothesis that the perturbed solutions are 'close' to the unperturbed ones, and consider a regular perturbation approach. Of course we should check if the solution complies with this important hypothesis.

Then, assume that the sought solutions can be expressed in the following series expansions:

$$\left.\begin{aligned} \bar{u}_n &= u_n + \overbrace{\varepsilon v_n}^{\text{1st order correction}} + \overbrace{\varepsilon^2 w_n}^{\text{2nd order correction}} + \dots \\ \bar{\lambda}_n &= \lambda_n + \varepsilon \mu_n + \varepsilon^2 \nu_n + \dots \end{aligned}\right\} \Rightarrow$$

Clearly, these expressions should satisfy the perturbed eigenproblem:

$$L(\bar{u}_n) - \varepsilon r \bar{u}_n + \bar{\lambda}_n \bar{u}_n = 0$$

Linear operator

$$\Rightarrow L(u_n + \varepsilon v_n + \varepsilon^2 w_n + \dots) - \varepsilon r (u_n + \varepsilon v_n + \varepsilon^2 w_n + \dots) + (\lambda_n + \varepsilon \mu_n + \varepsilon^2 \nu_n + \dots) \times (u_n + \varepsilon v_n + \varepsilon^2 w_n + \dots) = 0 \Rightarrow$$

$$\Rightarrow L(u_n) + \varepsilon L(v_n) + \varepsilon^2 L(w_n) - \varepsilon r u_n - \varepsilon^2 r v_n + \lambda_n u_n + \varepsilon \lambda_n v_n + \varepsilon^2 \lambda_n w_n + \varepsilon \mu_n u_n + \varepsilon^2 \mu_n v_n + \varepsilon^2 \nu_n u_n + \dots = 0$$

Matching terms at different orders of ε we then derive an hierarchy of problems at successive orders of approximation.

O(1) Terms: $L(u_n) + \lambda_n u_n = 0$ ✓ (We recover the unperturbed problem).

Note orthonormality property $\int u_i u_j \, dy = \delta_{ij}$

$\underline{O(\varepsilon)\ \text{Terms}}$: $\underbrace{L(v_n)+\lambda_n v_n}_{\substack{\text{We recover the structure}\\ \text{of the } O(1) \text{ approximation,}\\ \text{i.e., the unperturbed problem}}} = r u_n - \mu_n u_n \qquad (*)$

But $v_n \in g \Rightarrow v_n$ can be expanded in terms of the eigenfunctions of the unperturbed problem $\Rightarrow$ $v_n = \sum_{j=1}^{\infty} a_{nj} u_j$, $\quad a_{nj} = \int v_n u_j \, dg \Rightarrow$

$\Rightarrow$ Substituting into $(*) \Rightarrow$ $L\left(\sum_{j=1}^{\infty} a_{nj} u_j\right) + \lambda_n \sum_{j=1}^{\infty} a_{nj} u_j = r u_n - \mu_n u_n \Rightarrow$

$$\Rightarrow \sum_{j=1}^{\infty} a_{nj} L(u_j) + \lambda_n \sum_{j=1}^{\infty} a_{nj} u_j = r u_n - \mu_n u_n \xrightarrow{\int_g (\cdot) u_\ell \, dg}$$

$$\Rightarrow \sum_{j=1}^{\infty} a_{nj} \underbrace{\int_g L(u_j) u_\ell \, dg}_{\substack{-\lambda_\ell \delta_{j\ell} \\ \text{(Stiffness orthogonality)}}} + \lambda_n \sum_{j=1}^{\infty} a_{nj} \underbrace{\int_g u_j u_\ell \, dg}_{\substack{\delta_{j\ell} \\ \text{(Mass orthogonality)}}} = \underbrace{\int_g r u_n u_\ell \, dg}_{d_{n\ell}} - \mu_n \underbrace{\int_g u_n u_\ell \, dg}_{\delta_{n\ell}} \Rightarrow$$

Requirement of distinct eigenvalues

$$\Rightarrow a_{n\ell}(\lambda_n - \lambda_\ell) = d_{n\ell} - \mu_n \delta_{n\ell} \Rightarrow \begin{cases} \text{If } n \neq \ell \Rightarrow \boxed{a_{n\ell} = d_{n\ell}/(\lambda_n - \lambda_\ell)} \\ \text{If } n = \ell \Rightarrow \boxed{\mu_n = d_{nn}} \end{cases}$$

Digression with an example

$$L(v_n)+\lambda_n v_n = \frac{d}{dx}\left[A(x)\, v_n'\right] + m\,\omega_n^2 v_n =$$

Let $v_n = \sum_{j=1}^{\infty} a_{nj} u_j$

$$\int_0^\pi (\cdot)\, u_\ell \, dx$$

$$= \sum_{j=1}^{\infty} \frac{d}{dx}\left[A(x)\, a_{nj} u_j'\right] + \omega_n^2 \sum_{j=1}^{\infty} a_{nj}\, m\, u_j \;\Rightarrow_{g=[0,\pi]}$$

$$\Rightarrow \int_0^\pi \left(L(v_n)+\lambda_n v_n\right) u_\ell \, dx = \sum_{j=1}^{\infty} a_{nj} \underbrace{\int_0^\pi \frac{d}{dx}\left[A(x)\, u_j'\right] u_\ell \, dx}_{-\omega_\ell^2 \delta_{j\ell}} + \omega_n^2 \sum_{j=1}^{\infty} a_{nj} \underbrace{\int_0^\pi m\, u_j u_\ell \, dx}_{\delta_{j\ell}}$$

$$= -a_{n\ell}\,\omega_\ell^2 + \omega_n^2 a_{n\ell} = a_{n\ell}\left(\omega_n^2 - \omega_\ell^2\right)$$

To compute the coefficients a_{nn} we impose the orthonormality property to the perturbed eigenfunctions $\Rightarrow$

$$\Rightarrow \int_g (u_n + \varepsilon v_n)^2 dg = 1 \Rightarrow \underbrace{\int_g u_n^2 dg}_{1} + 2\varepsilon \int_g u_n v_n dg + \ldots = 1 \Rightarrow$$

$$\Rightarrow \int_g u_n v_n dg = 0 \Rightarrow \boxed{a_{nn} = 0}$$

Summarising, at this order of approximation the solution is

$$v_n = \sum_{\substack{j=1 \\ j \neq n}}^{\infty} \frac{\int_g r u_n u_j dg}{\lambda_n - \lambda_j} u_j \quad , \quad \mu_n = \int_g r u_n^2 dg \quad , n = 1, 2, 3, \ldots$$

The derived eigenfunctions $(u_n + \varepsilon v_n)$ are mass orthonormalized.

Remark 1: We are consistent with our hypothesis that the perturbed eigensolutions are 'close' to the unperturbed ones in the limit $\varepsilon \to 0$.

Remark 2: Since this is a regular perturbation analysis, we were able to completely determine the $O(\varepsilon)$ solution by solving the $O(\varepsilon)$ subproblem. In singular perturbation expansions this is not the case (terms at $O(\varepsilon)$ are left undetermined until higher order terms are considered)

$O(\varepsilon^2)$ terms: $L(w_n) + \lambda_n w_n = r v_n - \mu_n v_n - \nu_n u_n$

Similarly we express $w_n = \sum_{j=1}^{\infty} b_{nj} u_j$ and we can show that $b_{nl} = \frac{1}{\lambda_n - \lambda_l} \left\{ \sum_{j=1}^{\infty} a_{nj} d_{jl} - \mu_n a_{nl} \right\}, \ n \neq l$

$$\nu_n = \sum_{j=1}^{\infty} a_{nj} d_{jn}$$

To determine the coefficients b_{nn} impose again orthonormality condition:

$$\int (u_n + \varepsilon v_n + \varepsilon^2 w_n)^2 dy = 1 \Rightarrow b_{nn} = -\frac{1}{2} \sum_{j=1}^{\infty} a_{nj}^2$$