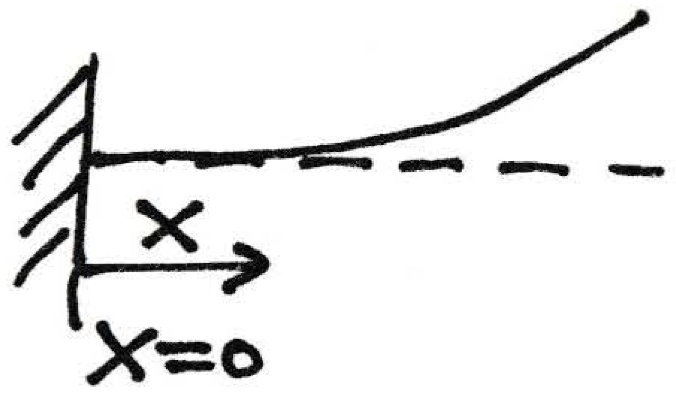
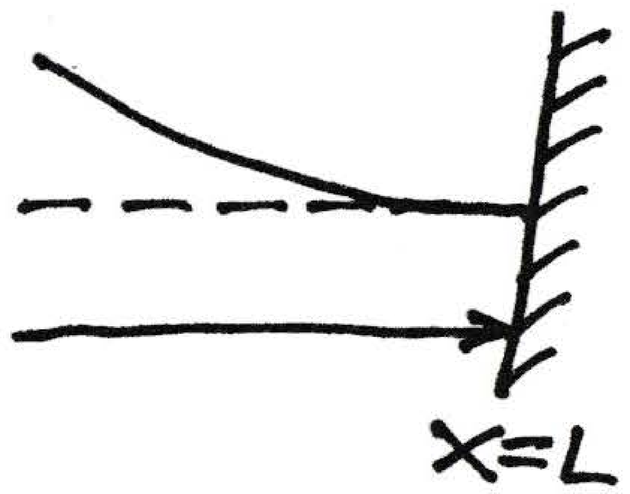
Simple BCs

1) Clamped BC:



$$v(0,t)=0$$
$$\frac{\partial v}{\partial x}(0,t)=0$$



$$v(L,t)=0$$
$$\frac{\partial v}{\partial x}(L,t)=0$$

2) Free BC:

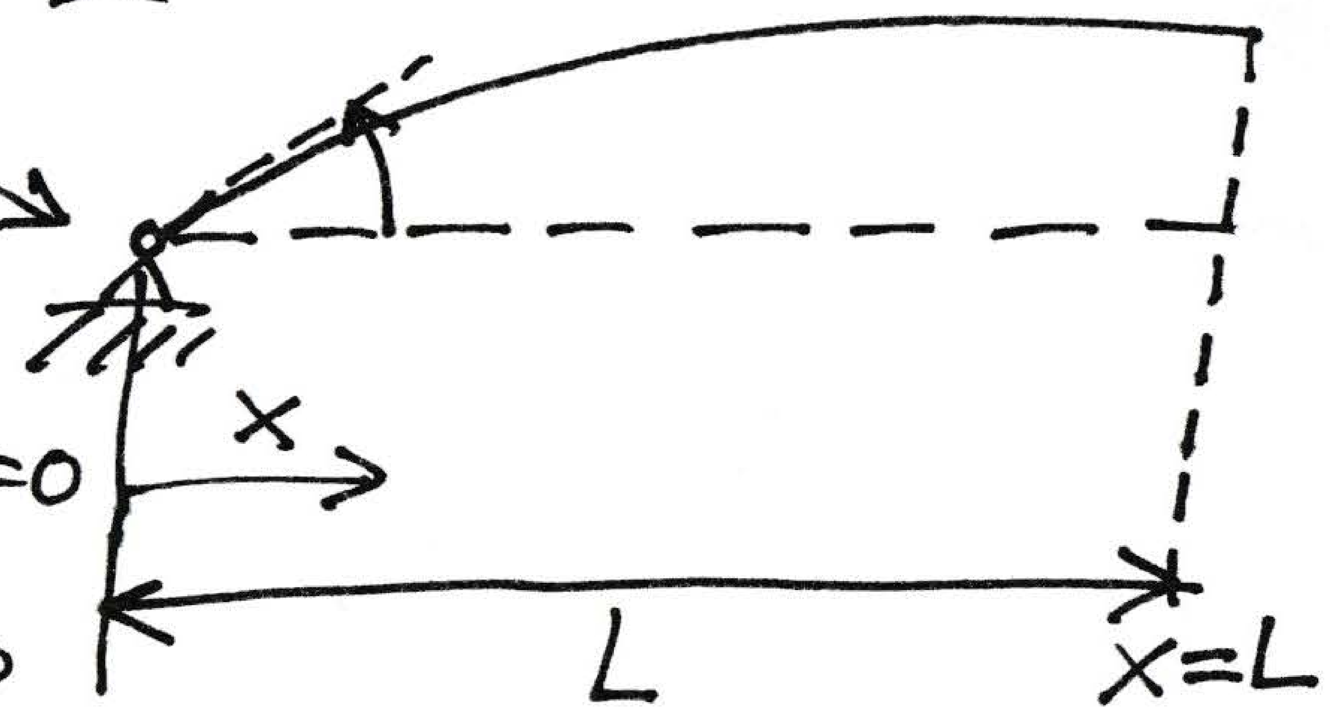
Simply-supported BC

$$v(0,t)=0$$

$$\cancel{Q}^{M}(0,t)=0 \Rightarrow$$

$$\Rightarrow \left\{-\cancel{\frac{\partial}{\partial x}}\left[EI(x)\frac{\partial^2 v}{\partial x^2}(x,t)\right]\right\}_{x=0}=0$$

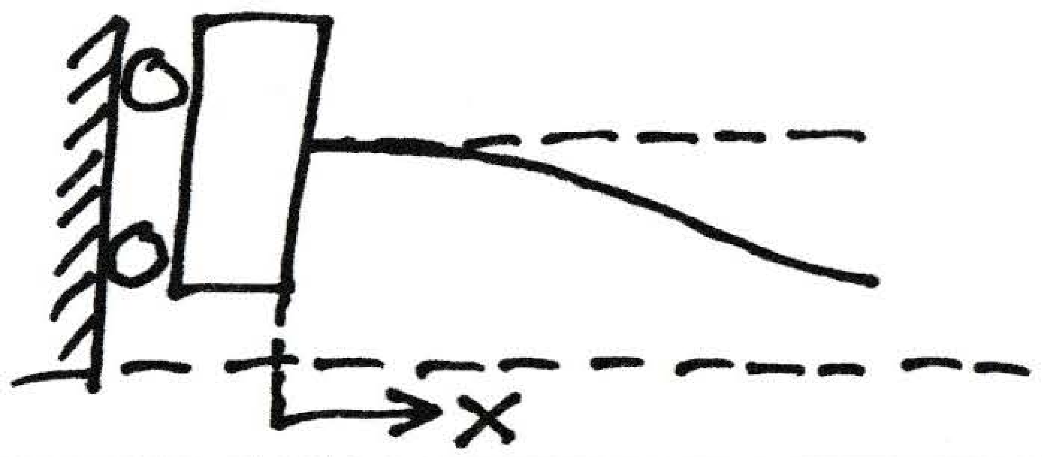
free BC



$$M(L,t)=0 \Rightarrow \left\{EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right\}_{x=L}=0 \Rightarrow \text{If } EI(L)\neq 0 \Rightarrow$$
$$\Rightarrow \frac{\partial^2 v(L,t)}{\partial x^2}=0$$

$$Q(L,t)=0 \Rightarrow \left\{-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right]\right\}_{x=L}=0 \Rightarrow \text{If } EI(L)\neq 0$$
$$\Rightarrow \frac{\partial^3 v(L,t)}{\partial x^3}=0$$

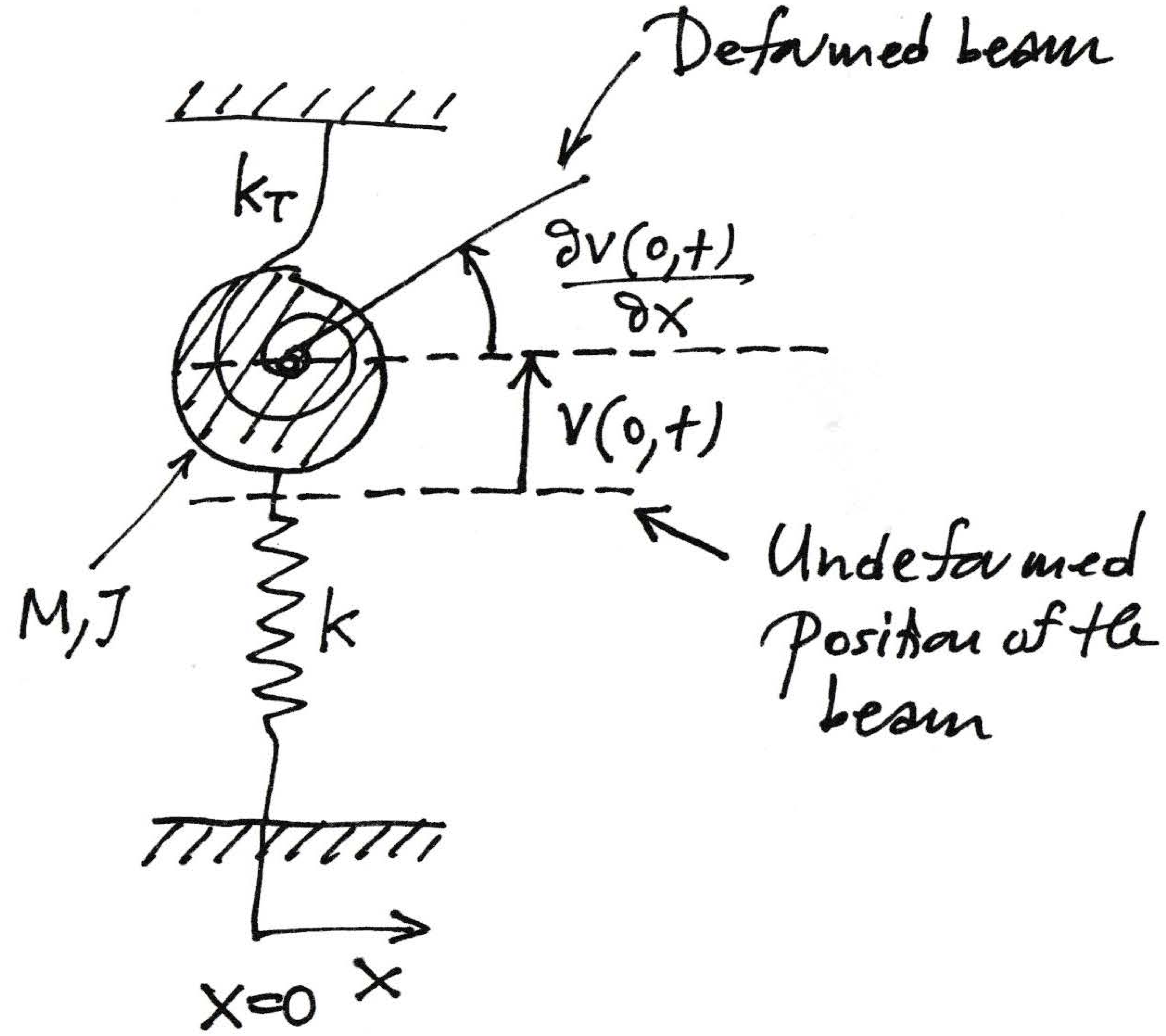
3) Semi-clamped BC:



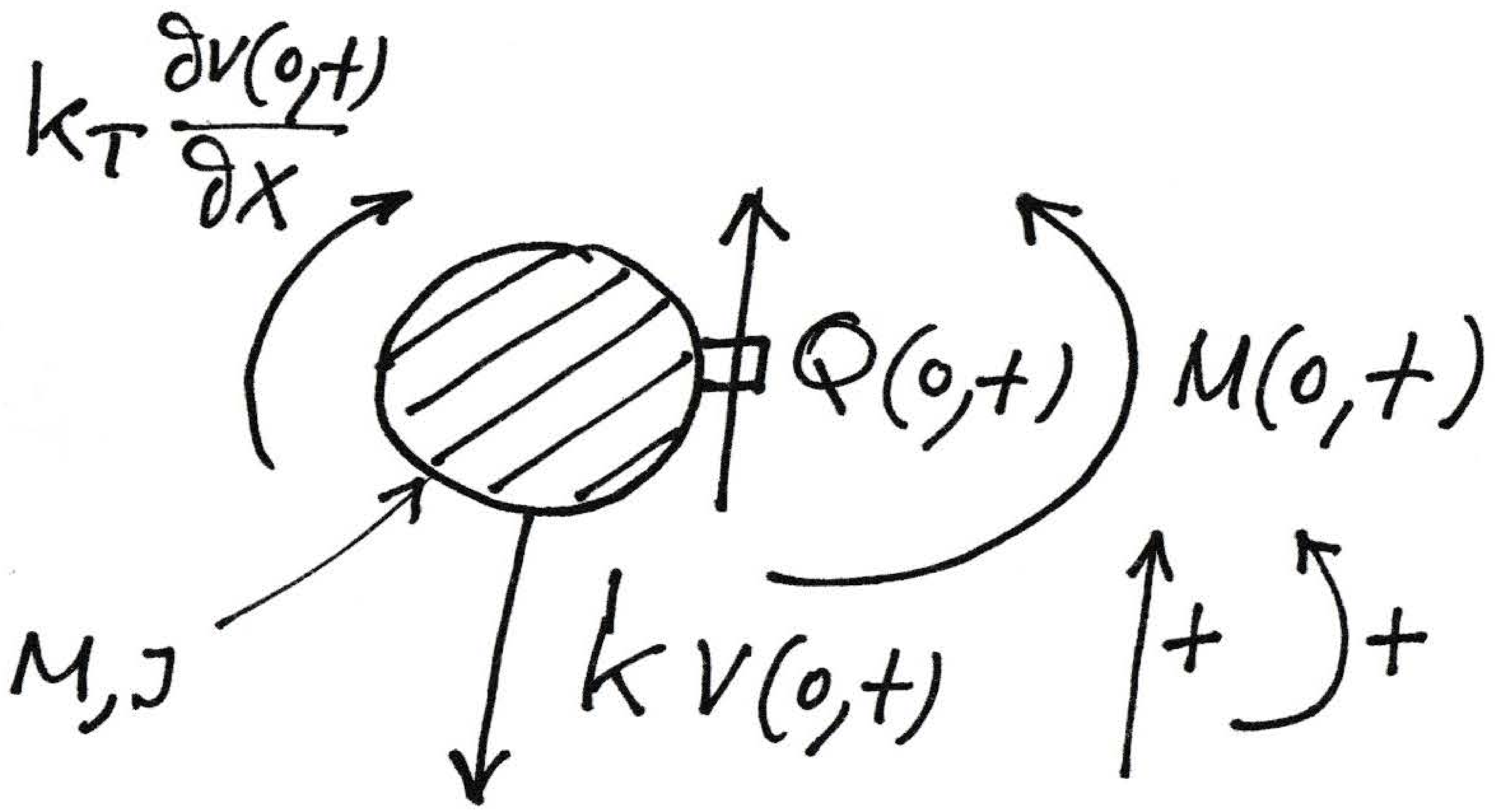
$$\frac{\partial v(0,t)}{\partial x}=0$$

$$Q(0,t)=0 \Rightarrow \left\{-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right]\right\}_{x=0}=0$$

Non-simple BCs



Perform balance of moments and forces using a free-body diagram:



Perform balance of vertical forces ⇒

$$\Rightarrow M\frac{\partial^2 v(0,t)}{\partial t^2} = Q(0,t) - k\,v(0,t) \Rightarrow M\frac{\partial^2 v(0,t)}{\partial t^2} = \left\{-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right]\right\}_{x=0} - k\,v(0,t)$$

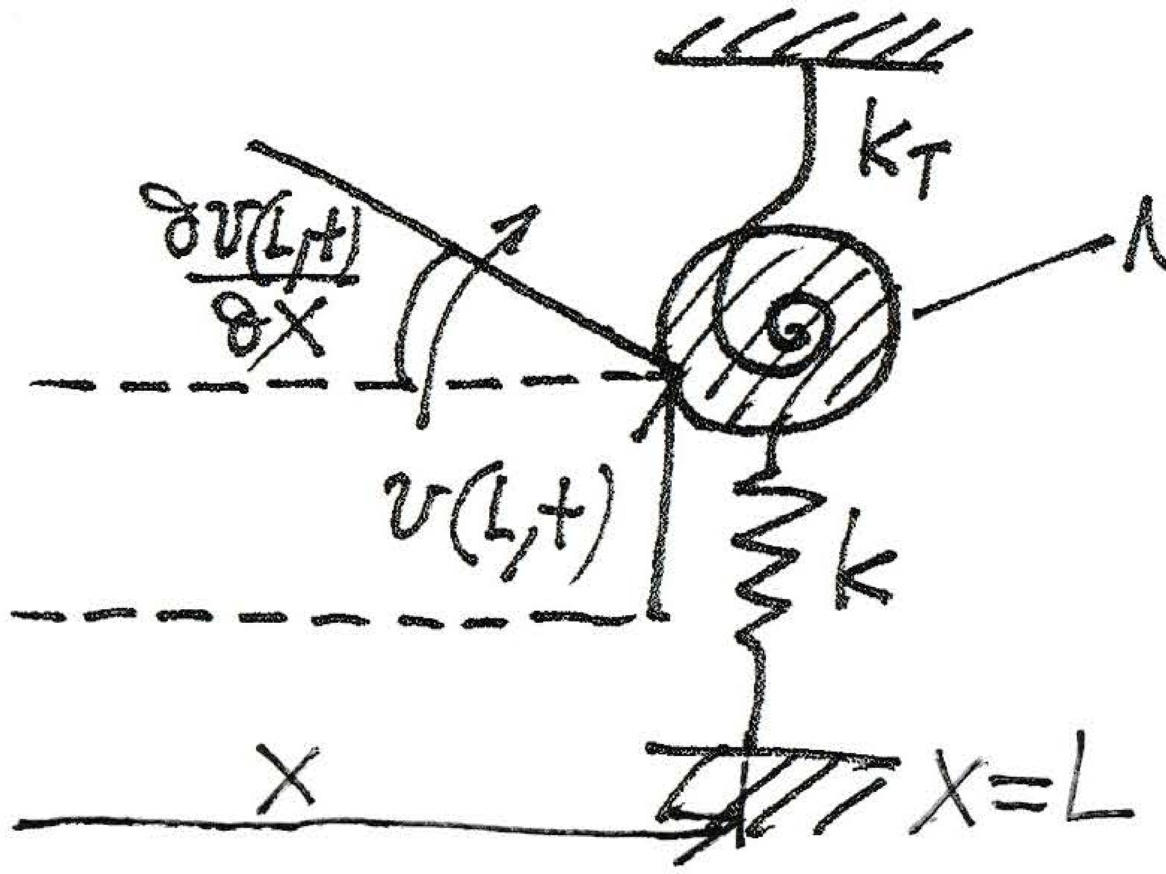
$$\Rightarrow \boxed{M\frac{\partial^2 v(0,t)}{\partial t^2} + \frac{\partial}{\partial x}\left[EI(0)\frac{\partial^2 v(0,t)}{\partial x^2}\right] + k\,v(0,t) = 0} \quad \text{BC at } x = 0+$$

Performing also balance of moments,

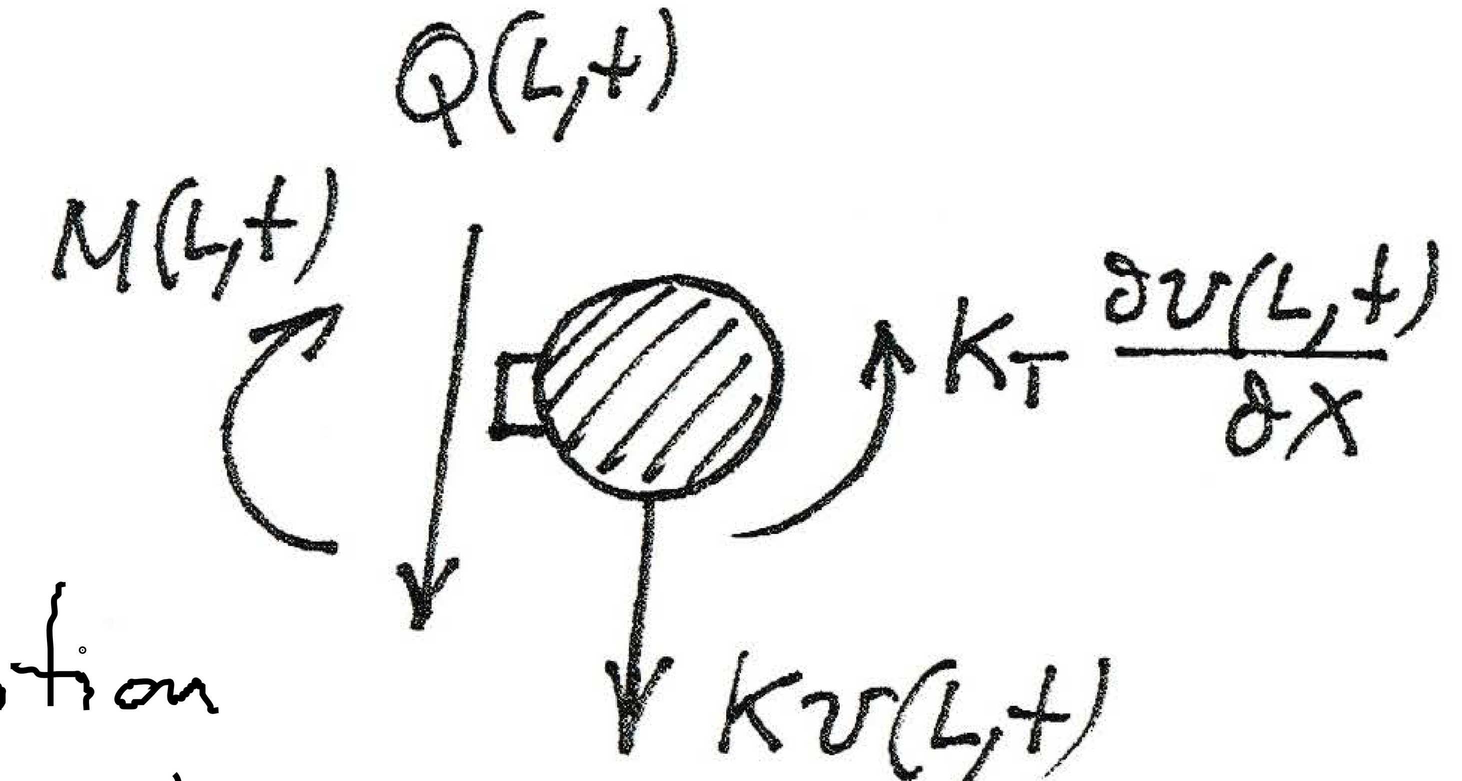
$$J\frac{\partial^2}{\partial t^2}\left[\frac{\partial v(0,t)}{\partial x}\right] = M(0,t) - K_T\frac{\partial v(0,t)}{\partial x} \Rightarrow$$

$$\Rightarrow \boxed{J\frac{\partial^3 v(0,t)}{\partial x\partial t^2} = EI(0)\frac{\partial^2 v(0,t)}{\partial x^2} - k_T\frac{\partial v(0,t)}{\partial x}}$$

BC at $x = 0+$



Use same sign convention as for BCs at $x=0$



Performing balance of forces and moments before,

$$\frac{\partial}{\partial x}\left[EI(L)\frac{\partial^2 v(L,t)}{\partial x^2}\right] - kv(L,t) - M\frac{\partial^2 v(L,t)}{\partial t^2} = 0$$

$$EI(L)\frac{\partial^2 v(L,t)}{\partial x^2} + k_T\frac{\partial v(L,t)}{\partial x} + J\frac{\partial^3 v(L,t)}{\partial x\partial t^2} = 0$$

BC at $x = L-$

The boundary value problem (Ignore the forcing at this stage)

Assume the general beam equation,

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 v(x,t)}{\partial x^2}\right] + m(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} = 0 \qquad (1)$$

$$0 \le x \le L, \quad t \ge 0$$

Assume simple BCs, i.e.,

$$\left.\begin{aligned} &u(0,t) = u(L,t) = 0 \\ &\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \end{aligned}\right\} \qquad (1a)$$

Initial conditions, $u(x,0) = g(x)$, $\dfrac{\partial u(x,0)}{\partial t} = h(x)$

Again, as for the case of the generalized wave equation we assume separation of space and time in the basic solutions,

$$v(x,t) = f(t)\,\varphi(x) \qquad (2)$$

substituting back into the governing equation,

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{d^2\varphi(x)}{dx^2}\,f(t)\right] + m(x)\,\frac{d^2 f(t)}{dt^2}\,\varphi(x) = 0 \Rightarrow$$

$$\Rightarrow f(t)\,\frac{d^2}{dx^2}\left[EI(x)\,\frac{d^2\varphi(x)}{dx^2}\right] + m(x)\,\varphi(x)\,\frac{d^2 f(t)}{dt^2} = 0 \Rightarrow$$

$$\Rightarrow \frac{\ddot{f}(t)}{f(t)} = -\frac{\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi(x)}{dx^2}\right]}{m(x)\varphi(x)} = -\omega^2 \qquad (3)$$

$\dot{(\ )} \equiv d/dt$

Depends on t Depends on x

Then we get the following two subproblems in time and space:

$$\ddot{f}(t) + \omega^2 f(t) = 0, \quad t \geq 0 \Rightarrow f(t) = A_1 \cos\omega t + B_1 \sin\omega t \quad (4a)$$

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi(x)}{dx^2}\right] - \omega^2 m(x)\varphi(x) = 0, \quad 0 \leq x \leq L \quad (4b)$$

$$\varphi(0) = \varphi(L) = 0$$

$$\varphi'(0) = \varphi'(L) = 0$$

$(\ )' \equiv d/dx$

For the general problem (4b), if we can find four linearly independent solutions, then the general solution is expressed by superposition as, $\varphi(x) = C_1\varphi_1(x,\omega) + C_2\varphi_2(x,\omega) + C_3\varphi_3(x,\omega) + C_4\varphi_4(x,\omega)$ where $\varphi_i(x)$, $i = 1,2,3,4$ are the four linearly independent solutions of (4b)

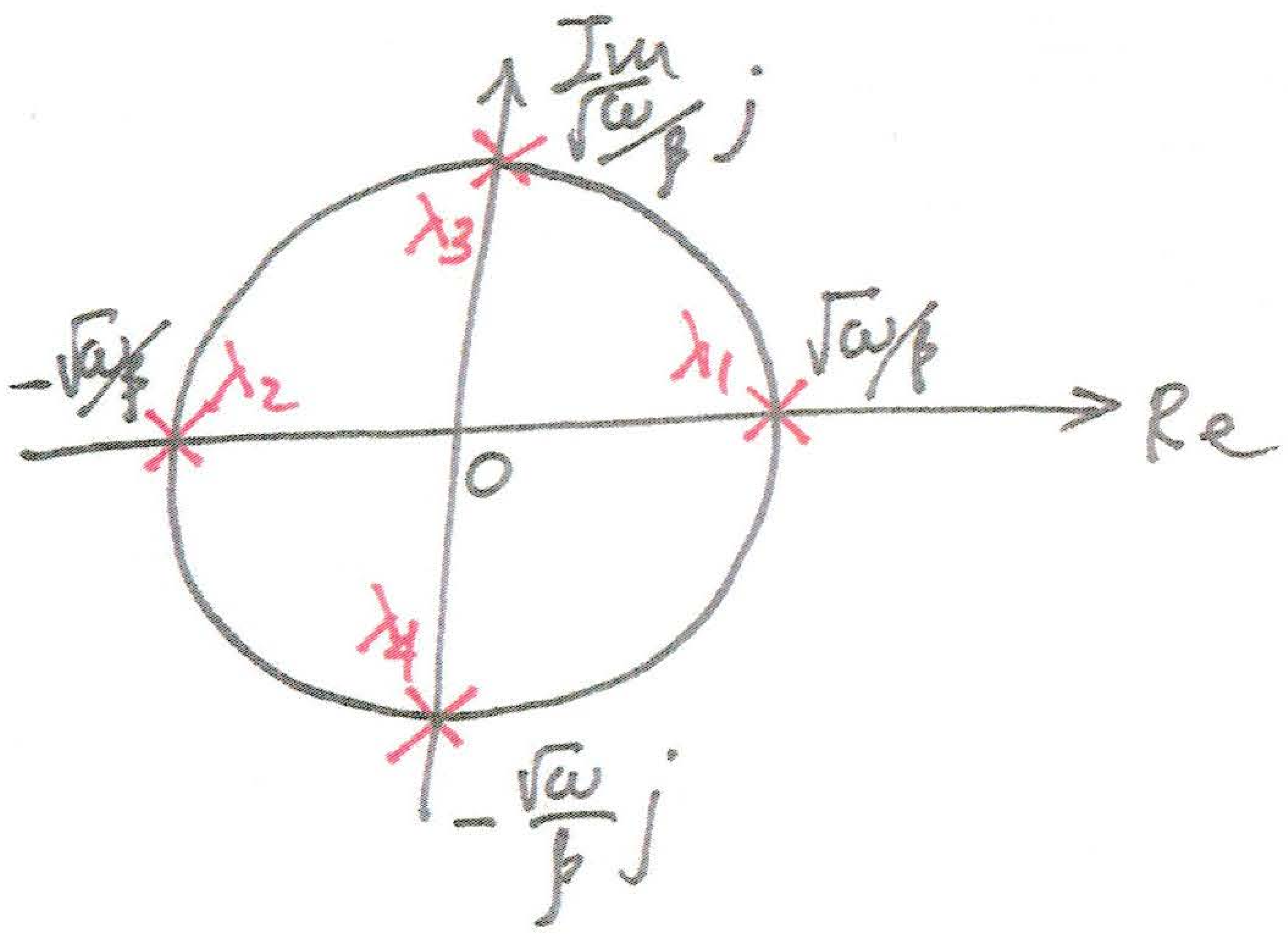
To obtain analytical solutions we examine the uniform beam with $m(x)=m$, $EI(x)=EI \Rightarrow \quad EI\dfrac{d^4\varphi(x)}{dx^4} - \omega^2 m\,\varphi(x) = 0 \Rightarrow$

$$\Rightarrow \quad \frac{d^4\varphi(x)}{dx^4} - \frac{\omega^2}{\beta^4}\varphi(x) = 0, \quad \beta^4 = \frac{EI}{m}$$

The characteristic equation for this problem is $\lambda^4 - \dfrac{\omega^2}{\beta^4} = 0 \Rightarrow$

$$\Rightarrow \lambda^4 = \frac{\omega^2}{\beta^4} e^{j2n\pi}, \quad n = 0,1,2,3, \quad j = (-1)^{1/2} \Rightarrow \lambda_i = \sqrt[4]{\frac{\omega^2}{\beta^4}}\, e^{j\frac{2i\pi}{4}} =$$

$$= \frac{\sqrt{\omega}}{\beta} e^{j\frac{i\pi}{2}}, \quad i = 0,1,2,3 \Rightarrow$$



$$\Rightarrow \left.\begin{aligned} &\lambda_1 = \frac{\sqrt{\omega}}{\beta}, \quad \lambda_2 = -\frac{\sqrt{\omega}}{\beta}, \\ &\lambda_3 = j\frac{\sqrt{\omega}}{\beta}, \quad \lambda_4 = -j\frac{\sqrt{\omega}}{\beta} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} \varphi_1(x,\omega) &= e^{\frac{\sqrt{\omega}}{\beta}x} \\ \varphi_2(x,\omega) &= e^{-\frac{\sqrt{\omega}}{\beta}x} \\ \varphi_3(x,\omega) &= e^{j\frac{\sqrt{\omega}}{\beta}x} \\ \varphi_4(x,\omega) &= e^{-j\frac{\sqrt{\omega}}{\beta}x} \end{aligned}\right\}$$

Should take into account that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

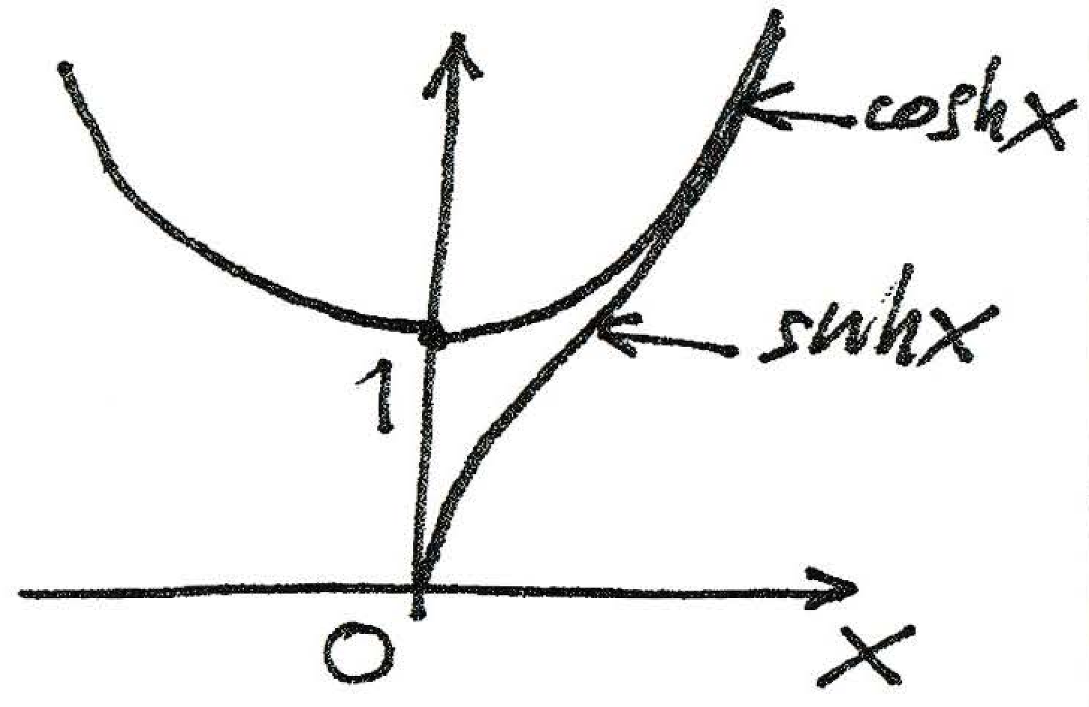
$$e^{j\theta} = \cos\theta + j\sin\theta$$

Then, the general solution is

$$\varphi(x) = A_1\,\varphi_1(x,\omega) + A_2\,\varphi_2(x,\omega) + A_3\,\varphi_3(x,\omega) + A_4\,\varphi_4(x,\omega) \Rightarrow$$

$$\Rightarrow \varphi(x) = C_1 \cos\frac{\sqrt{\omega}x}{\beta} + C_2 \sin\frac{\sqrt{\omega}x}{\beta} + C_3 \cosh\frac{\sqrt{\omega}x}{\beta} + C_4 \sinh\frac{\sqrt{\omega}x}{\beta}$$

Note that $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$



BCs: $\varphi(0) = \varphi(L) = \varphi'(0) = \varphi'(L) = 0 \Rightarrow$

$\Rightarrow$ Substituting into $\varphi(x)$ we get a condition

$\det[A(\omega)] = 0 \Rightarrow$ Compute the eigenvalues (nat. frequencies)

and then the modes of the problem.