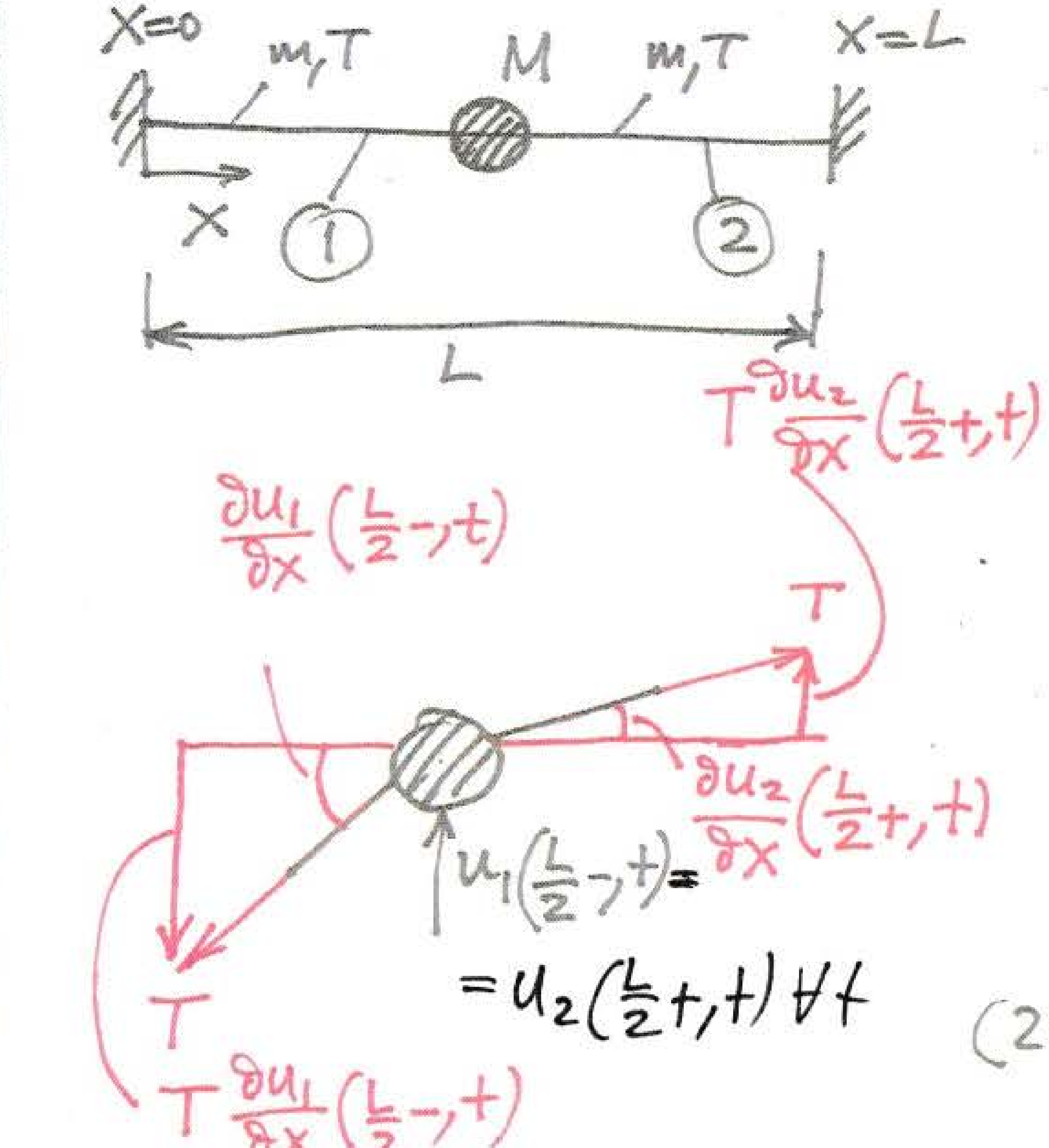
Example 4



Two ways to approach this.

1) 'Global' equation for the string

$$\left.\begin{aligned} &T\frac{\partial^2 u}{\partial x^2} = \left[m + M\delta\left(x-\frac{L}{2}\right)\right]\frac{\partial^2 u}{\partial t^2} \\ &u(0,t) = u(L,t) = 0 \end{aligned}\right\} 0 \le x \le L$$

2) Write two 'local' equations, one for each section of the string.

$$T\frac{\partial^2 u_1}{\partial x^2} = m\frac{\partial^2 u_1}{\partial t^2}, \quad 0 \le x \le \frac{L}{2}- \qquad (1a)$$

$$T\frac{\partial^2 u_2}{\partial x^2} = m\frac{\partial^2 u_2}{\partial t^2}, \quad \frac{L}{2}+ \le x \le L \qquad (1b)$$

$$(2a)\quad u_1(0,t) = 0, \quad u_2(L,t) = 0 \quad (2b)$$

$$u_1\left(\frac{L}{2}-,t\right) = u_2\left(\frac{L}{2}+,t\right) \qquad (2c)$$

$$M\frac{\partial^2 u_1}{\partial t^2}\left(\frac{L}{2}-,t\right) = M\frac{\partial^2 u_2}{\partial t^2}\left(\frac{L}{2}+,t\right) =$$

$$= T\frac{\partial u_2}{\partial x}\left(\frac{L}{2}+,t\right) - T\frac{\partial u_1}{\partial x}\left(\frac{L}{2}-,t\right) \qquad (2d)$$

We'll follow the 'local' approach, and formulate two boundary value problems, one for each of the two 'sections' of the string. $\Rightarrow$

$$\Rightarrow \quad \varphi_1(x) = \cancel{c_1 \cos\frac{\omega x}{c}} + C_2 \sin\frac{\omega x}{c}, \qquad 0 \le x \le \frac{L}{2}-$$

$$\varphi_2(x) = E_1 \cos\frac{\omega x}{c} + E_2 \sin\frac{\omega x}{c} =$$

$$= \cancel{D_1 \cos\frac{\omega(L-x)}{c}} + D_2 \sin\frac{\omega(L-x)}{c}, \quad \frac{L}{2}+ \le x \le L$$

$$(2a) \Rightarrow \quad \varphi_1(0) = 0 \Rightarrow C_1 = 0$$

$$(2b) \Rightarrow \quad \varphi_2(L) = 0 \Rightarrow D_1 = 0$$

$$(2c) \Rightarrow \quad \varphi_1\left(\frac{L}{2}-\right) = \varphi_2\left(\frac{L}{2}+\right) \Rightarrow C_2 \sin\frac{\omega L}{2c} = D_2 \sin\frac{\omega L}{2c} \Rightarrow C_2 = D_2$$

$$(2d) \Rightarrow \quad T\varphi_1'\left(\frac{L}{2}-\right) - T\varphi_2'\left(\frac{L}{2}+\right) - \omega^2 M \varphi_1\left(\frac{L}{2}-\right) = 0$$

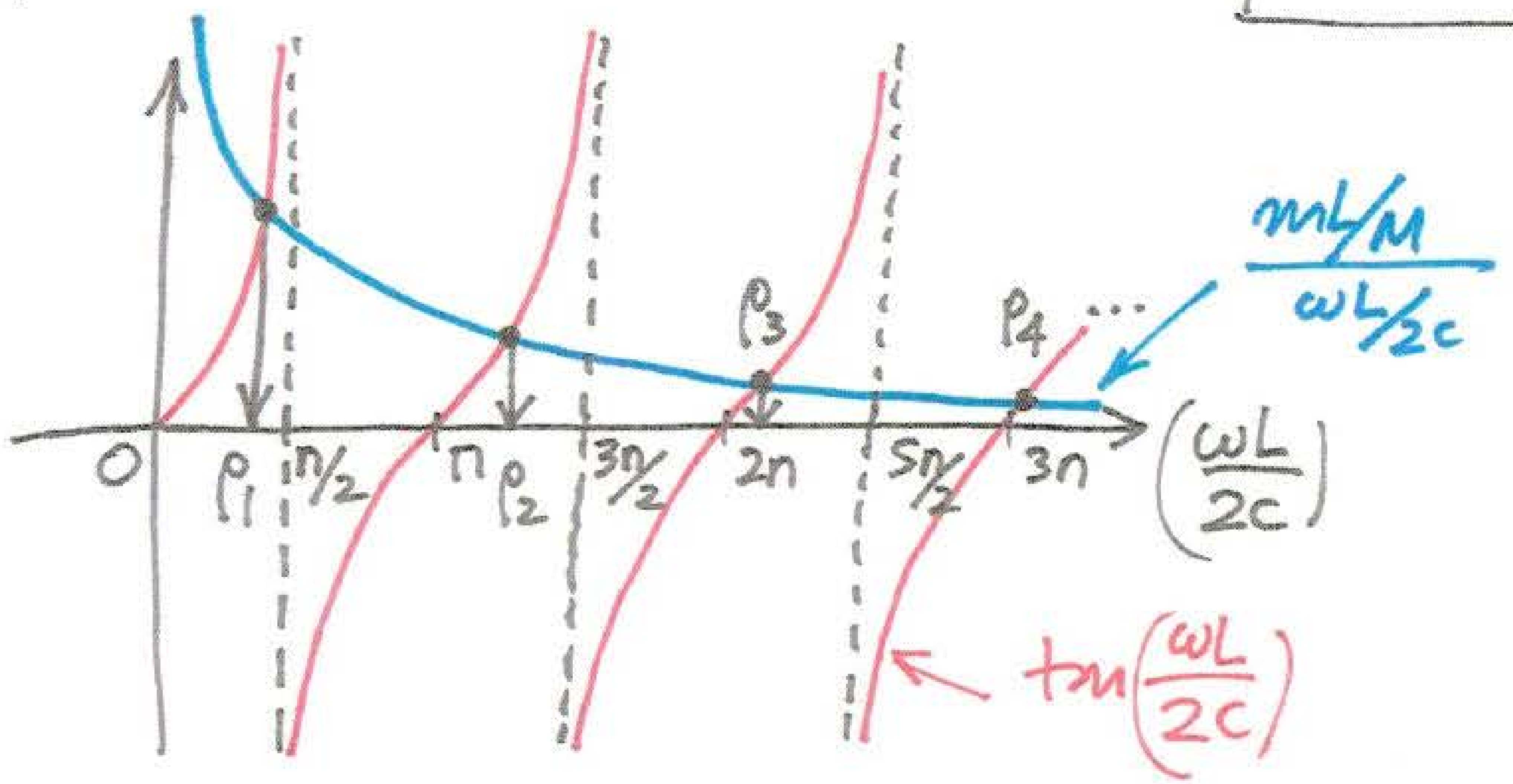
$$\Rightarrow \quad \Rightarrow$$

$$\Rightarrow T c_2 \frac{\omega}{c} \cos\frac{\omega L}{2c} - T c_2\left(-\frac{\omega}{c}\right) \cos\frac{\omega L}{2c} - \omega^2 M c_2 \sin\frac{\omega L}{2c} = 0 \Rightarrow$$

$$\Rightarrow 2Tc_2 \frac{\omega}{c} \cos\frac{\omega L}{2c} - \omega^2 M c_2 \sin\frac{\omega L}{2c} = 0 \Rightarrow c_2\left[2T\frac{\omega}{c}\cos\frac{\omega L}{2c} - \omega^2 M \sin\frac{\omega L}{2c}\right] = 0$$

$\Rightarrow$ For nontrivial solutions we require that $[\ \ ] = 0 \Rightarrow$

⇒ Taking into account that $c^2 = T/m$ ⇒ $\boxed{\tan\left(\frac{\omega L}{2c}\right) = \frac{mL/M}{(\omega L/2c)}}$



Hence, we get the countable infinity of eigenfrequencies,

$$\frac{\omega_i L}{2c} = \rho_i \Rightarrow \omega_i = \frac{2c\rho_i}{L}, \quad i = 1, 2, 3, \dots$$

Then, the corresponding mode shapes are,

$\varphi_{11}(x) = C_{21}\sin\rho_1 \frac{x}{L/2}$, $\quad \varphi_{21}(x) = C_{21}\sin\rho_1 \frac{L-x}{L/2}$ $\quad$ 1st eigenfunction

$\varphi_{12}(x) = C_{22}\sin\rho_2 \frac{x}{L/2}$, $\quad \varphi_{22}(x) = C_{22}\sin\rho_2 \frac{L-x}{L/2}$ $\quad$ 2nd eigenfunction

...

$\varphi_{1i}(x) = C_{2i}\sin\rho_i \frac{x}{L/2}$, $\quad \varphi_{2i}(x) = C_{2i}\sin\rho_i \frac{L-x}{L/2}$ $\quad$ i-th eigenfunction

...

Section 1 $\qquad$ Section 2

To orthonormalize, require that

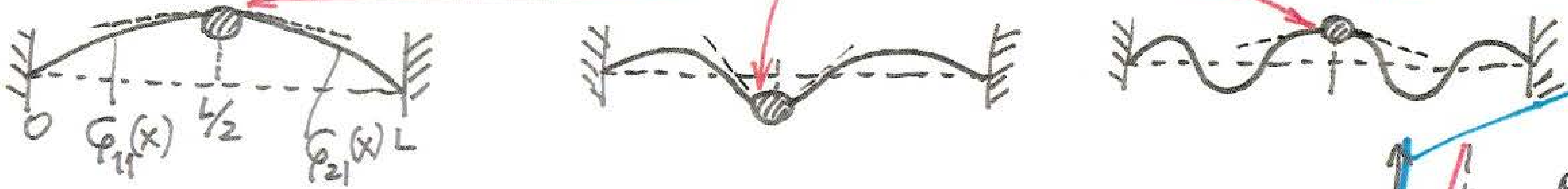
$$\int_0^L m(x)\,\varphi_i^2(x)\,dx=1 \Rightarrow \int_0^{L/2^-} m\, C_{2i}^2 \sin^2\rho_i \frac{x}{L/2}\,dx + \underbrace{\int_{L/2^-}^{L/2^+} M\delta\left(x-\frac{L}{2}\right) C_{2i}^2 \sin^2\rho_i \frac{x}{L/2}\,dx}_{M C_{2i}^2 \sin^2\rho_i} +$$

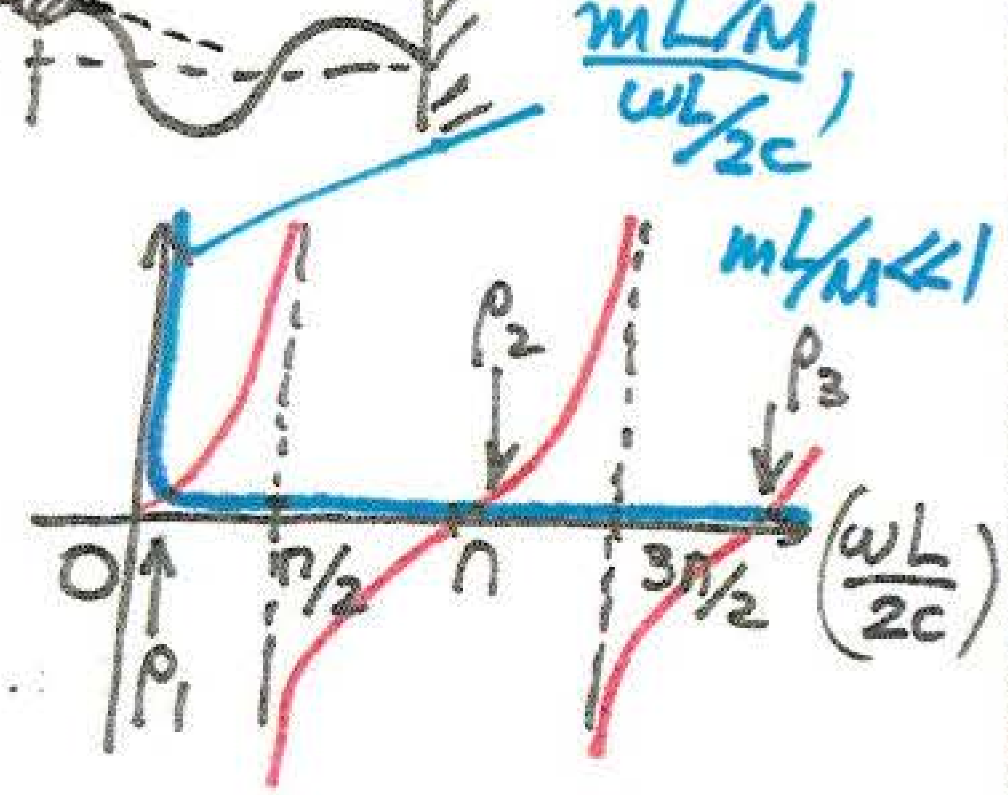
$$+\int_{L/2^+}^{L} m\, C_{2i}^2 \sin^2\rho_i \frac{L-x}{L/2}\,dx=1 \Rightarrow$$

$$\Rightarrow C_{2i}^2\left\{\int_0^{L/2^-} m \sin^2\rho_i \frac{x}{L/2}\,dx + M\sin^2\rho_i + \int_{L/2^+}^{L} m\sin^2\rho_i \frac{L-x}{L/2}\,dx\right\}=1 \Rightarrow$$

$\Rightarrow$ from here we may compute the unknown coefficients C_{2i}^2, $i=1,3\ldots$ for orthonormalization.

Slope discontinuities at $x=L/2$

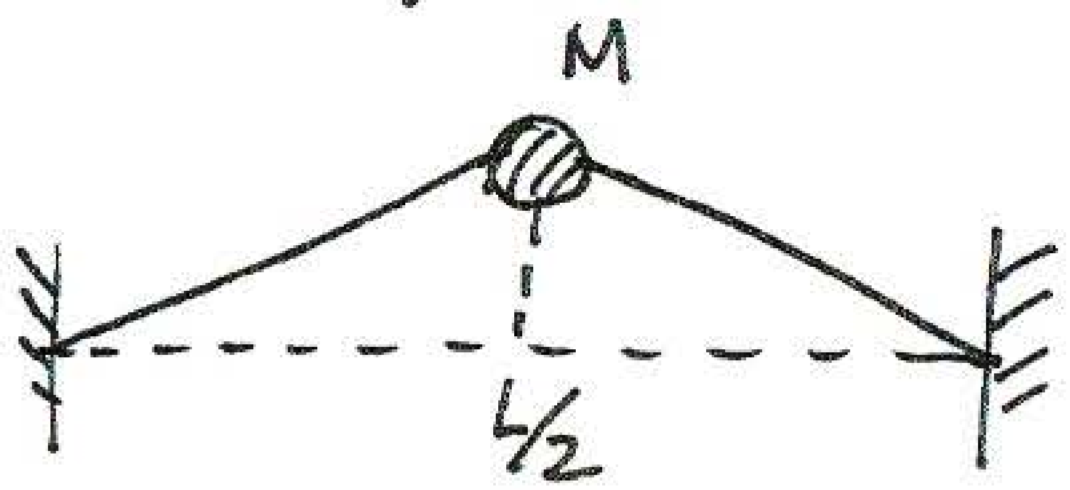




Remark: When $mL/M \ll 1 \Rightarrow$ The first root is approximately computed as $\rho_1=\frac{\omega_1 L}{2c} \ll 1 \Rightarrow \tan\frac{\omega_1 L}{2c} \sim \frac{\omega_1 L}{2c} \Rightarrow$

$$\Rightarrow \frac{\omega_1 L}{2c} \sim \frac{mL/M}{\omega_1 L/2c} \Rightarrow \omega_1 \sim \sqrt{\frac{2(T/L/2)}{M}} \equiv \sqrt{\frac{K}{M}},\quad K=T/\tfrac{L}{2}$$

In that case the string acts as a pure spring and its inertia does not affect the dynamics!

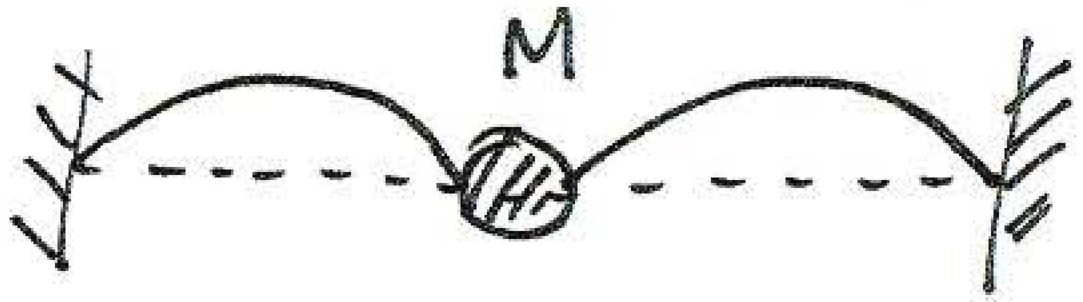


$mL \ll M$, 1st mode

for the second natural frequency we can find that $p_2 = \frac{\omega_2 L}{2c} \sim \pi \Rightarrow$

$$\Rightarrow \frac{\omega_2}{c} \sim \frac{\pi}{L/2} \Rightarrow \varphi_{12}(x) \sim C_2 \sin\frac{\pi x}{L/2}$$

$$\varphi_{22}(x) \sim C_2 \sin\frac{\pi(L-x)}{L/2}$$



$mL \ll M$, 2nd mode

Also we can show that for the third natural frequency it holds that

$$\frac{\omega_3}{c} \sim 2\pi/(L/2)$$

$mL/M \ll 1$, 3rd mode

Additional class of solutions

Recall the boundary value problem:

$$\left.\begin{aligned} &\varphi_1''(x)+\left(\frac{\omega}{c}\right)^2\varphi_1(x)=0, \quad 0\leq x\leq \frac{L}{2}- \\ &\varphi_2''(x)+\left(\frac{\omega}{c}\right)^2\varphi_2(x)=0, \quad \frac{L}{2}+\leq x\leq L \end{aligned}\right\}\Rightarrow \begin{aligned} &\varphi_1(x)=C_1\cos\frac{\omega x}{c}+C_2\sin\frac{\omega x}{c} \\ &\varphi_2(x)=D_1\cos\frac{\omega(L-x)}{c}+D_2\sin\frac{\omega(L-x)}{c} \end{aligned}$$

$$\varphi_1(0)=0\Rightarrow C_1=0$$

$$\varphi_2(L)=0\Rightarrow D_1=0$$

$$\varphi_1\left(\frac{L}{2}-\right)=\varphi_2\left(\frac{L}{2}+\right)\Rightarrow \boxed{C_2\sin\frac{\omega L}{c2}=D_2\sin\frac{\omega L}{c2}} \quad (*)$$

$$T\varphi_1'\left(\frac{L}{2}-\right)-T\varphi_2'\left(\frac{L}{2}+\right)-\omega^2M\varphi_1\left(\frac{L}{2}-\right)=0 \quad (**)$$

From $(*)$ there are two possibilities:
- $C_2=D_2$ if $\sin\frac{\omega L}{c2}\neq 0\Rightarrow$
  $\Rightarrow$ Obtain the class of solutions discussed previously
- However, there is the additional possibility that
  $\sin\frac{\omega L}{c2}=0\Rightarrow \frac{\omega_k}{c}\frac{L}{2}=k\pi\Rightarrow$

$$\Rightarrow \boxed{\omega_k = \frac{2k\pi c}{L} = \frac{k\pi c}{(L/2)}, \ k = 1, 2, \ldots}$$

Then, considering the remaining boundary condition (**) $\Rightarrow$

$$\Rightarrow T C_2 \frac{\omega_k}{c} \underbrace{\cos \frac{\omega_k L}{2c}}_{(-1)^k} + T D_2 \frac{\omega_k}{c} \underbrace{\cos \frac{\omega_k L}{2c}}_{(-1)^k} - \omega_k^2 M C_2 \underbrace{\sin \frac{\omega_k L}{2c}}_{0} = 0 \Rightarrow$$

$$\Rightarrow C_2 + D_2 = 0 \Rightarrow C_2 = -D_2 \Rightarrow \boxed{\begin{aligned} \varphi_{1k}(x) &= C_2 \sin \frac{\omega_k x}{c} \\ \varphi_{2k}(x) &= -C_2 \sin \frac{\omega_k (L-x)}{c} \end{aligned}}$$

Additional class of modes

Note that this additional class of modes corresponds to motionless mass M, which acts as a rigid boundary condition at $x = L/2$.

$k=1$

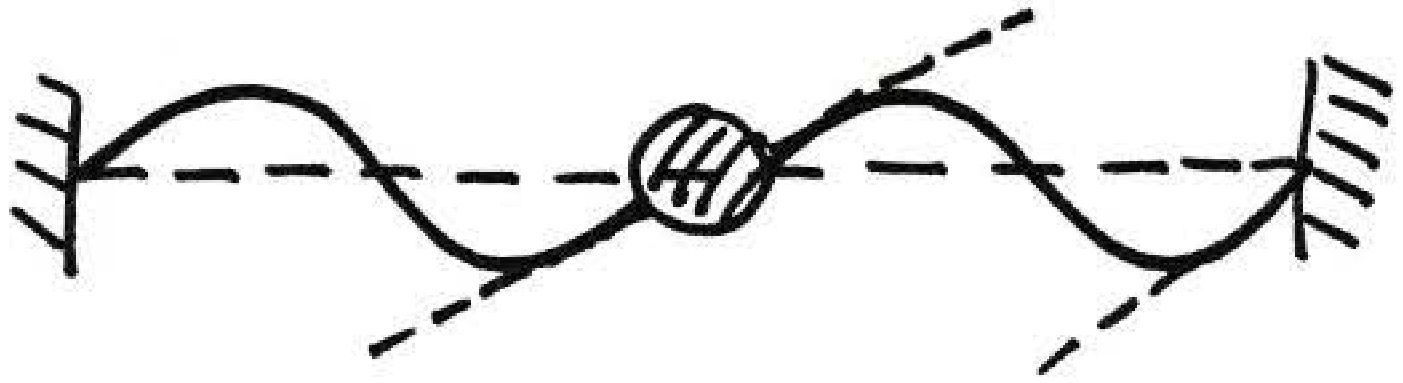
$$\varphi_{21}'(L) = -C_2\left(-\frac{\omega_1}{c}\right)\cos 0 = C_2\,\frac{\omega_1}{c} > 0 \quad \text{if } C_2>0$$

Slope continuity at $x = L/2$

$$\varphi_{21}'\left(\frac{L}{2}+\right) = -C_2\left(-\frac{\omega_1}{c}\right)\cos\frac{\omega_1}{c}\left(L-\frac{L}{2}\right) =$$

$$= \frac{C_2\omega_1}{c}\cos\underbrace{\frac{\omega_1 L}{2c}}_{\pi} = -\frac{C_2\omega_1}{c} < 0, \text{ if } C_2>0$$

$k=2$



Remark

This example shows the importance of not missing any solutions by assuming non-zero divisors!

$k=3$