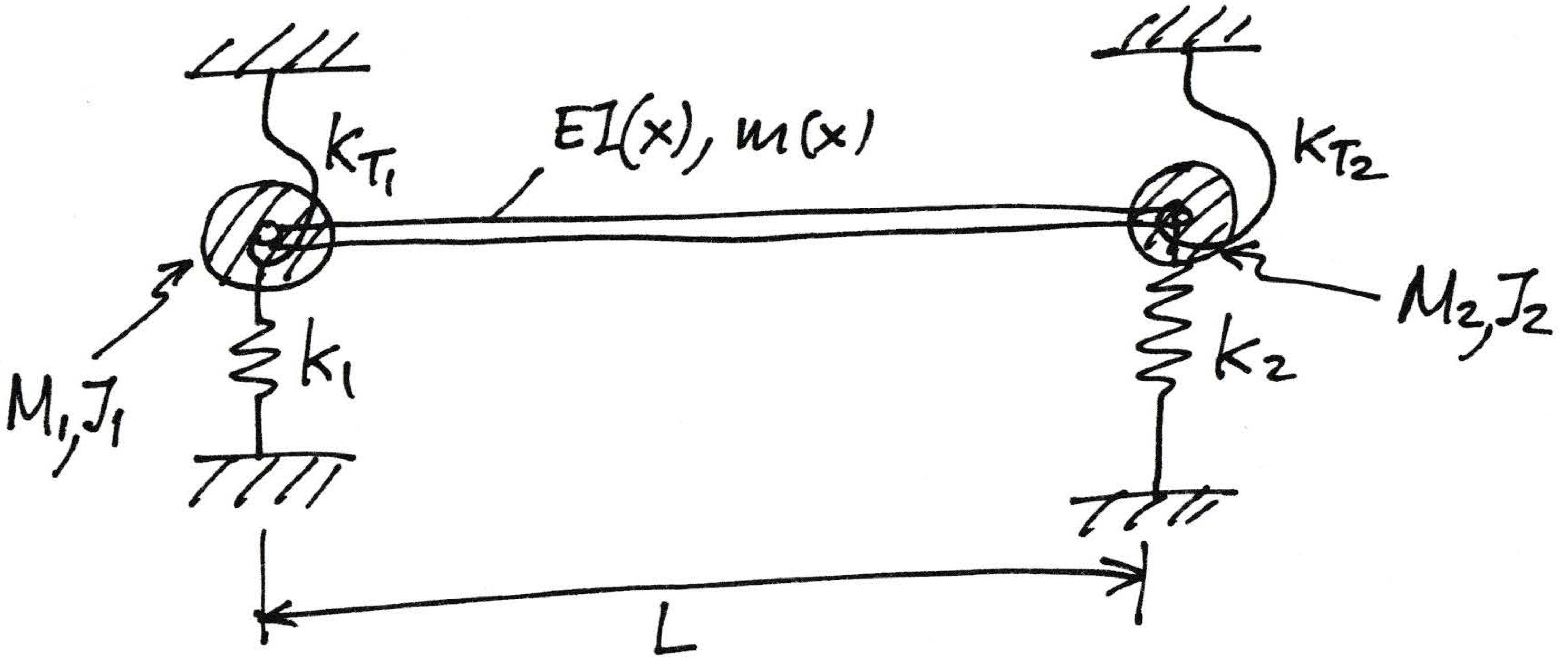
Reminder of non-simple BCs.



$$\frac{\partial}{\partial x}\left[EI(0)\frac{\partial^2 v(0,t)}{\partial x^2}\right] + k_1 v(0,t) + M_1 \frac{\partial^2 v(0,t)}{\partial t^2} = 0$$

$$EI(0)\frac{\partial^2 v(0,t)}{\partial x^2} - k_{T_1}\frac{\partial v(0,t)}{\partial x} - J_1 \frac{\partial^3 v(0,t)}{\partial x \partial t^2} = 0$$

$$\frac{\partial}{\partial x}\left[EI(L)\frac{\partial^2 v(L,t)}{\partial x^2}\right] - k_2 v(L,t) - M_2 \frac{\partial^2 v(L,t)}{\partial t^2} = 0$$

$$EI(L)\frac{\partial^2 v(L,t)}{\partial x^2} + k_{T_2}\frac{\partial v(L,t)}{\partial x} + J_2 \frac{\partial^3 v(L,t)}{\partial x \partial t^2} = 0$$

Let $v(x,t) = \varphi(x)\, y(t)$

Harmonic dependence since we have neglected the force and consider normal modes

$$\boxed{\begin{aligned}&\left[EI(L)\varphi''(L)\right]' - K_2\varphi(L) + \omega^2 M_2\varphi(L) = 0\\ &EI(L)\varphi''(L) + K_{T2}\varphi'(L) - \omega^2 J_2\varphi'(L) = 0\end{aligned}}$$

Boundary conditions at $x=L$

Then following a procedure similar to that for simple BCs (i.e., performing successive integrations by parts) we derive the following orthonormality conditions for this case:

$$\boxed{\begin{aligned}&\int_0^L m(x)\varphi_r(x)\varphi_s(x)\,dx + M_1\varphi_r(0)\varphi_s(0) + M_2\varphi_r(L)\varphi_s(L) +\\ &\qquad + J_1\varphi_r'(0)\varphi_s'(0) + J_2\varphi_r'(L)\varphi_s'(L) = \delta_{rs}, \quad r,s=1,2,\ldots\end{aligned}}$$

(Mass-orthonormality conditions)

Also

$$\boxed{\begin{aligned}&\int_0^L \left[EI(x)\varphi_r''(x)\right]''\varphi_s(x)\,dx = \omega_r^2\delta_{rs} - \omega_r^2 M_1\varphi_r(0)\varphi_s(0) - \omega_r^2 M_2\varphi_r(L)\varphi_s(L) -\\ &\qquad - \omega_r^2 J_1\varphi_r'(0)\varphi_s'(0) - \omega_r^2 J_2\varphi_r'(L)\varphi_s'(L)\end{aligned}}$$

(Stiffness-orthonormality conditions)

By performing two more integrations by parts of the first of the above terms and utilizing the non-simple boundary conditions, we get:

$$\boxed{\begin{aligned}\int_0^L [EI(x)\varphi_r''(x)]''\varphi_s(x)\,dx + [EI(0)\varphi_r''(0)]'\varphi_s(0) - [EI(L)\varphi_r''(L)]'\varphi_s(L) - \\ - EI(0)\varphi_r''(0)\varphi_s'(0) + EI(L)\varphi_r''(L)\varphi_s'(L) + \\ + k_1\varphi_r(0)\varphi_s(0) + k_2\varphi_r(L)\varphi_s(L) + k_{T1}\varphi_r'(0)\varphi_s'(0) + \\ + k_{T2}\varphi_r'(L)\varphi_s'(L) = \omega_r^2\delta_{rs}, \quad r,s=1,2,\ldots\end{aligned}}$$

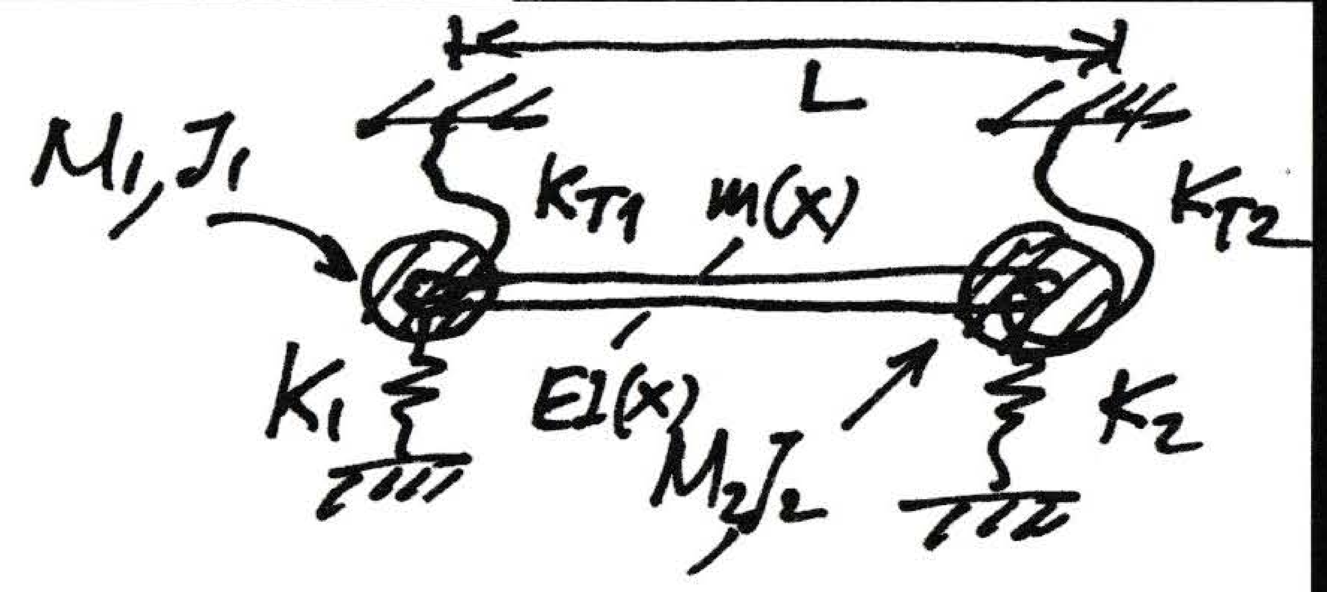
(Alternative form of stiffness orthonormality condition)

$$\Big\downarrow \text{Integration by parts}$$

$$\boxed{\begin{aligned}\int_0^L EI(x)\varphi_r''(x)\varphi_s''(x)\,dx + k_1\varphi_r(0)\varphi_s(0) + k_2\varphi_r(L)\varphi_s(L) + \\ + k_{T1}\varphi_r'(0)\varphi_s'(0) + k_{T2}\varphi_r'(L)\varphi_s'(L) = \omega_r^2\delta_{rs}, \quad r,s=1,2,\ldots\end{aligned}}$$

(stiffness-orthonormality condition)

Modal Analysis of the System with non-simple BCs



$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right] + f(x,t) = m(x)\frac{\partial^2 v}{\partial t^2} \qquad (1)$$

Express the solution in terms of superposition of modal responses $\Rightarrow$

$$\Rightarrow v(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\,\varphi_i(x) \qquad (2)$$

Mass-orthonormalized eigenfunctions

Substituting (2) into (1) and multiplying by $\varphi_j(x)$, where j is fixed but otherwise arbitrary, while integrating with respect to x from 0 to $L \Rightarrow$

$$\Rightarrow \sum_{i=1}^{\infty} \ddot{\eta}_i(t) \int_0^L m(x)\,\varphi_i(x)\,\varphi_j(x)\,dx = -\sum_{i=1}^{\infty} \eta_i(t) \int_0^L \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\varphi_i(x)}{dx^2}\right]\varphi_j(x)\,dx$$

$$+ \int_0^L f(x,t)\,\varphi_j(x)\,dx \qquad (3)$$

At this point we need to form the mass-orthonormalization condition $\Rightarrow$

$\Rightarrow$ Work with the <u>original boundary conditions</u>.

Consider the first BC $\Rightarrow$ $M_1 \dfrac{\partial^2 v(0,t)}{\partial t^2} + \dfrac{\partial}{\partial x}\left[EI(0)\dfrac{\partial^2 v(0,t)}{\partial x^2}\right] + k_1 v(0,t) = 0$ at $x=0$

Substitute (2) $\Rightarrow$ $M_1 \sum_{i=1}^{\infty} \ddot{\eta}_i(t)\varphi_i(0) + EI(0)\sum_{i=1}^{\infty} \eta_i(t)[\varphi_i''(0)]' + k_1 \sum_{i=1}^{\infty} \eta_i(t)\varphi_i(0) = 0 \Rightarrow$

$\Rightarrow$ Multiply by $\varphi_j(0) \Rightarrow$ $\sum_{i=1}^{\infty} \ddot{\eta}_i(t) M_1 \varphi_i(0)\varphi_j(0) = -\sum_{i=1}^{\infty} \eta_i(t)\left\{[EI(0)\varphi_i''(0)]'\varphi_j(0) + k_1 \varphi_i(0)\varphi_j(0)\right\}$ (4)

Similarly we work for the other BCs:

$$J_1 \frac{\partial^3 v(0,t)}{\partial x \partial t^2} = EI(0)\frac{\partial^2 v(0,t)}{\partial x^2} - k_{T_1}\frac{\partial v(0,t)}{\partial x} \Rightarrow$$ Substitute (2) into it and then multiply by $\varphi_j'(0) \Rightarrow$

$$\Rightarrow \sum_{i=1}^{\infty} J_1 \ddot{\eta}_i(t)\varphi_i'(0)\varphi_j'(0) = \sum_{i=1}^{\infty} EI(0)\eta_i(t)\varphi_i''(0)\varphi_j'(0) - \sum_{i=1}^{\infty} k_{T_1}\eta_i(t)\varphi_i'(0)\varphi_j'(0) \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{\infty} \ddot{\eta}_i(t) J_1\varphi_i'(0)\varphi_j'(0) = \sum_{i=1}^{\infty} \eta_i(t)\left\{EI(0)\varphi_i''(0)\varphi_j'(0) - k_{T_1}\varphi_i'(0)\varphi_j'(0)\right\} \quad (5)$$

$$M_2\frac{\partial^2 v(L,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[EI(L)\frac{\partial^2 v(L,t)}{\partial x^2}\right] - k_2 v(L,t) \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{\infty} \ddot{\eta}_i(t) M_2\varphi_i(L)\varphi_j(L) = \sum_{i=1}^{\infty} \eta_i(t)\left\{\left[EI(L)\varphi_i''(L)\right]'\varphi_j(L) - k_2\varphi_i(L)\varphi_j(L)\right\} \quad (6)$$

$$-EI(L)\frac{\partial^2 v(L,t)}{\partial x^2} - k_{T2}\frac{\partial v(L,t)}{\partial x} = J_2\frac{\partial^3 v(L,t)}{\partial x \partial t^2} \Rightarrow$$

$$\Rightarrow \sum_{i=1}^{\infty} \ddot{\eta}_i(t) J_2\varphi_i'(L)\varphi_j'(L) = \sum_{i=1}^{\infty} \eta_i(t)\left\{-EI(L)\varphi_i''(L)\varphi_j'(L) - k_{T2}\varphi_i'(L)\varphi_j'(L)\right\} \quad (7)$$

Now add (3)+(4)+(5)+(6)+(7) $\Rightarrow$

$$\sum_{i=1}^{\infty} \ddot{\eta}_i(t) \underbrace{\Big\{ \int_0^L m(x)\varphi_i(x)\varphi_j(x)\,dx + M_1 \varphi_i(0)\varphi_j(0) + J_1 \varphi_i'(0)\varphi_j'(0) + M_2 \varphi_i(L)\varphi_j(L) + J_2 \varphi_i'(L)\varphi_j'(L) \Big\}}_{\delta_{ij}} =$$

$$= -\sum_{i=1}^{\infty} \eta_i(t) \underbrace{\Big\{ \int_0^L \frac{d^2}{dx^2}\Big[EI(x)\frac{d^2\varphi_i(x)}{dx^2}\Big]\varphi_j(x)\,dx + \big[EI(0)\varphi_i''(0)\big]'\varphi_j(0) + k_1\varphi_i(0)\varphi_j(0) - EI(0)\varphi_i''(0)\varphi_j'(0) + k_{T_1}\varphi_i'(0)\varphi_j'(0) - \big[EI(L)\varphi_i''(L)\big]'\varphi_j(L) + k_2\varphi_i(L)\varphi_j(L) + EI(L)\varphi_i''(L)\varphi_j'(L) + k_{T_2}\varphi_i'(L)\varphi_j'(L) \Big\}}_{\omega_i^2\delta_{ij}} + \underbrace{\int_0^L F(x,t)\varphi_j(x)\,dx}_{N_j(t)}$$

$$\Rightarrow \quad \boxed{\ddot{\eta}_j(t)+\omega_j^2\,\eta_j(t)=N_j(t), \quad j=1,2,3,\dots} \qquad (8)$$

We expressed the elastodynamics of the system with non-simple BCs in terms of an infinite set of uncoupled modal oscillators, exactly as in the case of simple BCs!

One last step involves the computation of the required initial conditions for the set (8).

Recall,
$$v(x,0)=g(x)$$
$$\frac{\partial v(x,0)}{\partial t}=h(x)$$

But
$$v(x,t)=\sum_{i=1}^{\infty}\eta_i(t)\,\varphi_i(x)$$

$$\Rightarrow \quad \sum_{i=1}^{\infty}\eta_i(0)\,\varphi_i(x)=g(x) \quad \xRightarrow{\int_0^L (\cdot)\, m(x)\,\varphi_j(x)\,dx}$$
$$\sum_{i=1}^{\infty}\dot{\eta}_i(0)\,\varphi_i(x)=h(x)$$

$$\Rightarrow \sum_{i=1}^{\infty}\eta_i(0)\int_0^L \varphi_i(x)\,m(x)\,\varphi_j(x)\,dx=\int_0^L m(x)\,g(x)\,\varphi_j(x)\,dx \qquad (9)$$

We formulate the mass-orthonormality condition in the ~~R~~LHS exactly as previously using the boundary conditions:

$$\sum_{i=1}^{\infty} \eta_i(0)\varphi_i(0) = g(0) \underset{(\cdot)M_1\varphi_j(0)}{\Rightarrow} \sum_{i=1}^{\infty} \eta_i(0) M_1 \varphi_i(0)\varphi_j(0) = M_1 \varphi_j(0) g(0) \quad (10)$$

Consider at $t=0$, $x=0$

$$\sum_{i=1}^{\infty} \eta_i(0)\varphi_i(L) = g(L) \underset{(\cdot)M_2\varphi_j(L)}{\Rightarrow} \sum_{i=1}^{\infty} \eta_i(0) M_2 \varphi_i(L)\varphi_j(L) = M_2 \varphi_j(L) g(L) \quad (11)$$

Consider at $t=0$, $x=L$

$$\sum_{i=1}^{\infty} \eta_i(0)\varphi_i(x) = g(x) \Rightarrow \sum_{i=1}^{\infty} \eta_i(0)\varphi_i'(x) = g'(x) \underset{(\cdot)J_1\varphi_j'(0)}{\Rightarrow} \sum_{i=1}^{\infty} \eta_i(0) J_1 \varphi_i'(0)\varphi_j'(0) =$$

$$= J_1 \varphi_j'(0) g(0) \quad (12)$$

Consider at $t=0$, $x=0$

Similarly, considering the above relation at $t=0$, $x=L$ and multiplying by $J_2\varphi_j'(L) \Rightarrow$

$$\sum_{i=1}^{\infty} \eta_i(0) J_2 \varphi_i'(L)\varphi_j'(L) = J_2 \varphi_j'(L) g'(L) \quad (13)$$

Adding (9), (10), (11), (12) and (13) $\Rightarrow$

$$\underbrace{\sum_{i=1}^{\infty} \eta_i(0)\delta_{ij}}_{\eta_j(0)} = \int_0^L m(x) g(x) \varphi_j(x)\,dx + M_1\varphi_j(0) g(0) + M_2\varphi_j(L) g(L) + J_1 \varphi_j'(0) g'(0) + J_2\varphi_j'(L) g'(L) \quad (14)$$

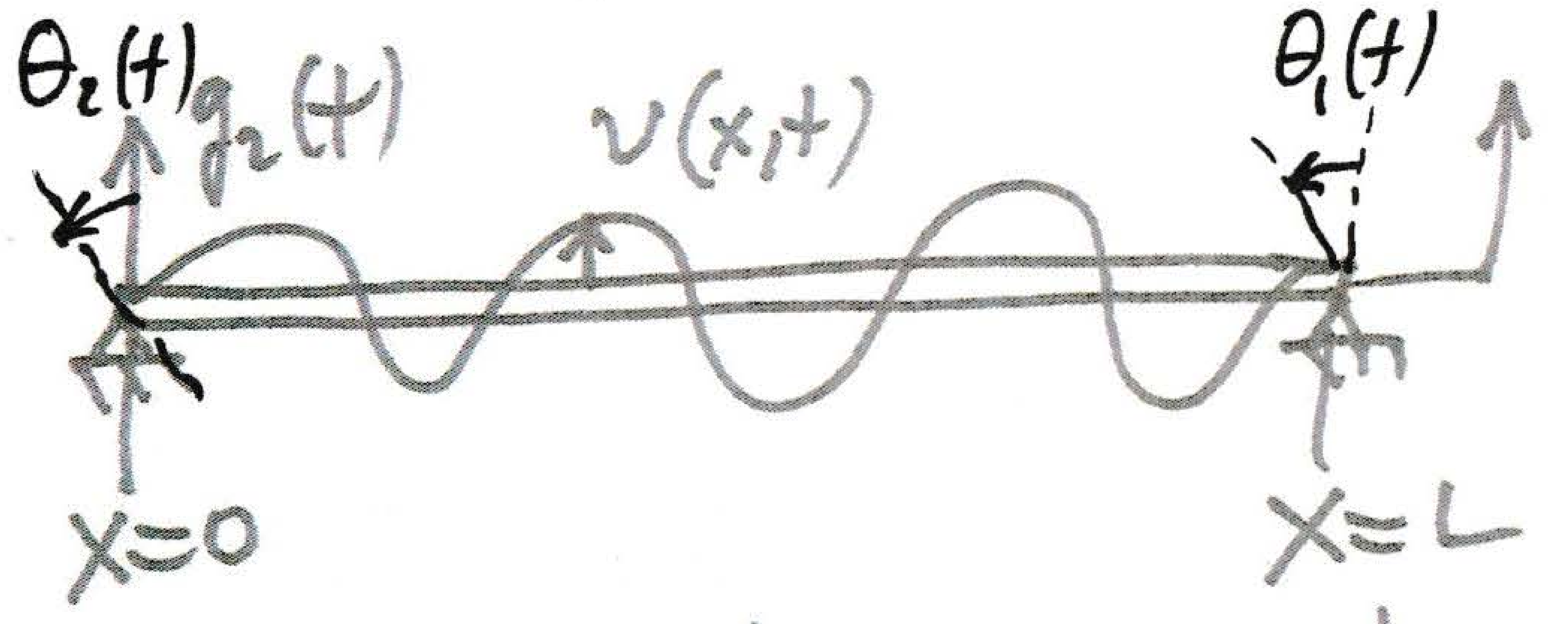
Similarly, the second set of initial conditions are,

$$\dot{\eta}_j(0) = \int_0^L m(x)\, h(x)\varphi_j(x)\,dx + M_1\varphi_j(0)h(0) + M_2\varphi_j(L)h(L) +$$

$$+ J_1\varphi_j'(0)\,h'(0) + J_2\,\varphi_j'(L)\,h'(L)\ , \ j=1,2,\dots \qquad (15)$$

Case of nonhomogeneous BCs

This is the case where time enters explicitly in the BCs (motions of the support). To solve this problem we follow the exact same methodology developed for the generalized wave equation ⇒



Model of seismic excitation of a bridge!

$$\Rightarrow\ v(x,t) = v_{st}(x,t) + v_{fl}(x,t)$$

Pseudo-static motion exclusively due to the motion of the supports without considering the inertia effects of the beam

Residual flexible motion due to inertia effects