Boundness of solutions as the parameter increases

The absolute values of the solutions of the S-L problem,

$$u'' - ru + \lambda u = 0 \quad , \quad 0 \leq x \leq 1 \quad + \text{BCs}$$

r continuous Eigenvalue

remain bounded, i.e., less than some bound independent of λ and x, provided that the solutions are normalized according to $\int_0^1 u^2(x)\,dx = 1$, and satisfy the boundary conditions $u(0) = u(1) = 0$.

We note that this result holds even if no boundary conditions are imposed.

However, functions of more than one variable do not have an analogous boundedness property.

Asymptotic representation of solutions

Now, having established the previous result on boundedness we provide the following theorem on the asymptotic behavior of the solutions:

Consider again $u'' - ru + \lambda u = 0$, $0 \leq x \leq 1$, with $\lambda > 0$; then, there exists a solution of the simpler problem $v'' + \lambda v = 0$, such that,

$$u = v + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

, where u is a normalized solution, $\int_0^1 u^2\,dx = 1$

This means that in the limit of large λ, this result shows that the solution for u

approach trigonometric functions ⇒ Asymptotic representation of S-L solutions for $\lambda \gg 1$.

Asymptotic solutions of the S-L Eigenvalue problem

First we consider the eigenvalues of the problem

Continuous function for $0 \le t \le l$

$$(py')' - qy + \lambda \rho y = 0, \quad 0 \le x \le \pi \longrightarrow z'' - rz + \lambda z = 0, \quad 0 \le t \le l$$

$$y(0) = y(\pi) = 0 \qquad z(0) = z(l) = 0$$

$$\int_0^l z^2 dx = 1$$

Let λ_n be the n-th eigenvalue ⇒

$$\Rightarrow \lim_{n\to\infty} \frac{n^2}{\lambda_n} = \frac{1}{\pi^2}\left(\int_0^\pi \sqrt{\frac{\rho}{p}}\,dx\right)^2 \Rightarrow$$

Precisely an identical estimate may be derived for arbitrary homogeneous BCs, since the asymptotic behavior of $z'' + \lambda z = 0$ is independent of the BCs

$$\Rightarrow \boxed{\lambda_n \sim \frac{n^2\pi^2}{l^2} + O(1), \quad n \gg 1}$$

This means that the parameter dependencies $p(x), q(x), \rho(x), r(x)$ affect more the low-order eigenvalues, whereas as $n \to \infty$ the eigenvalues are not affected as much.

Consider now the eigenfunctions of the problem. We wish to compare the n-th eigenfunction $z_n(x)$ associated to the eigenvalue λ_n, with the corresponding eigenfunction $v_n(x)$ of the simpler system
$$v'' + \lambda v = 0$$
$$v(0) = v(l) = 0$$

Boundness of solutions as the parameter increases ⇒ Suffices to prove for $\lambda \gg 1$

Consider $u'' - ru + \lambda u = 0$, $u(0) = u(1) = 0$, $\int_0^1 u^2 dx = 1$, $0 \leq x \leq 1 \Rightarrow$

$\Rightarrow$ Multiply by u' and $\int_0^x (\,)\, d\xi \Rightarrow$

$$\Rightarrow u'^2(x) + \lambda u^2(x) - 2\int_0^x r u u' d\xi = u'^2(0) + \lambda u^2(0)$$

Now, integrate this equation $\int_0^1 (\,)\, dx \Rightarrow$

$$\Rightarrow \int_0^1 u'^2(x)\, dx + \lambda \underbrace{\int_0^1 u^2(x)\, dx}_{1} - 2\int_0^1 dx \int_0^x r u u' d\xi = u'^2(0) + \lambda u^2(0)$$

$\Bigg\} \Rightarrow$

$$\Rightarrow u'^2(x) + \lambda u^2(x) - 2\int_0^x r u u' dx = \int_0^1 u'^2(x)\, dx + \lambda - 2\int_0^1 dx \int_0^x r u u' d\xi \Rightarrow$$

Schwartz's inequality; $C_1 > 0$ constant

$$\Rightarrow \lambda u^2(x) \leq u'^2(x) + \lambda u^2(x) \leq \lambda + \int_0^1 u'^2(x)\, dx + C_1 \sqrt{\int_0^1 u'^2 dx}\, \sqrt{\underbrace{\int_0^1 u^2 dx}_{1}} \quad (\nabla)$$

But $u'' - ru + \lambda u = 0 \Rightarrow (\cdot)\, u',\ \int_0^1 (\,)\, dx \Rightarrow \int_0^1 u'^2 dx + \int_0^1 r u^2 dx = \lambda \Rightarrow$

$$\Rightarrow \int_0^1 u'^2 dx = \lambda - \int_0^1 \underset{r>0}{r} u^2 dx \leq \underset{\lambda>0}{\lambda} + \underset{C_2>0}{C_2} \underbrace{\int_0^1 u^2 dx}_{1} = \lambda + C_2 \quad (\nabla\nabla)$$

$\Bigg\} \Rightarrow$

$$\Rightarrow \lambda u^2(x) \leq \lambda + \lambda + C_2 + C_1\sqrt{\lambda + C_2} \Rightarrow u^2(x) \leq 2 + \frac{C_2}{\lambda} + \frac{C_1\sqrt{\lambda + C_2}}{\lambda} = 2 + \frac{C_2}{\lambda} + \frac{C_1}{\sqrt{\lambda}}\left(1 + \frac{C_2}{\lambda}\right)^{1/2} \Rightarrow$$

$$\Rightarrow \text{As } \lambda \gg 1 \Rightarrow u^2(x) \leq 2 + C_3/\sqrt{\lambda} + C_4/\lambda, \quad \lambda \gg 1$$

$$u'' - ru + \lambda u = 0 \Rightarrow \underbrace{u''u'}_{\frac{d}{dx}\left(\frac{1}{2}u'^2\right)} - ruu' + \underbrace{\lambda u u'}_{\frac{d}{dx}\left(\frac{\lambda}{2}u^2\right)} = 0 \Rightarrow \int_0^x (\;)\,d\xi \Rightarrow$$

$$\Rightarrow u'^2(x) - u'^2(0) + \lambda u^2(x) - \lambda u^2(0) - 2\int_0^x ruu' = 0 \Rightarrow$$

$$\Rightarrow u'^2(x) + \lambda u^2(x) - 2\int_0^x ru'u\,dx = u'^2(0) \Rightarrow \int_0^1 (\;)\,dx \Rightarrow$$

$$\Rightarrow \int_0^1 u'^2(x)\,dx + \lambda \underbrace{\int_0^1 u^2(x)\,dx}_{1} - 2\int_0^1 dx \int_0^x ruu'\,d\xi = u'^2(0) \Rightarrow$$

$$\Rightarrow u'^2(x) + \lambda u^2(x) - 2\int_0^x ruu'\,dx = \int_0^1 u'^2(x)\,dx + \lambda - 2\int_0^1 dx \int_0^x ruu'\,d\xi$$

Now, use Schwartz's inequality $\Rightarrow$

$$\Rightarrow \int_0^x ruu'\,dx \le \int_0^x |r||u||u'|\,dx \le r_{max}\int_0^x |u||u'|\,dx \le$$

Let $r \le r_{max}$

$$\le r_{max}\int_0^1 |u||u'|\,dx \le$$

$$\le r_{max}\underbrace{\sqrt{\int_0^1 |u|^2 dx}}_{1}\sqrt{\int_0^1 |u'|^2 dx} \Rightarrow$$

$$\Rightarrow \int_0^x ruu'\,dx \le r_{max}\sqrt{\int_0^1 |u'|^2 dx}$$

So, $\lambda u^2(x) \leqslant u'^2(x) + \lambda u^2(x) \leqslant$

$$\leqslant \lambda + \int_0^1 u'^2(x)\,dx + 2r_{max}\sqrt{\int_0^1 |u'|^2 dx} + $$

$$+ \underbrace{2\int_0^1 dx \int_0^x r|u||u'|\,dx}_{\leqslant 2r_{max}\sqrt{\int_0^1 |u'|^2 dx}} \Rightarrow$$

$$\Rightarrow \lambda u^2(x) \leqslant \lambda + \int_0^1 u'^2(x)\,dx + C_1\sqrt{\int_0^1 |u'|^2 dx} \quad (*)$$

$C_1 > 0$

But, $u'' - ru + \lambda u = 0 \Rightarrow$

$\Rightarrow u''u' - ruu' + \lambda uu' = 0$

$$\int_0^1 \frac{d}{dx}(u')^2 dx - \int_0^1 ruu' dx + \lambda \int_0^1 \frac{d}{dx}(u^2) dx = 0 \Rightarrow$$

$$\Rightarrow \frac{d}{dx}\left[\int_0^1 u'^2 dx\right] - \int_0^1 ruu' dx + \frac{d}{dx}\left[\lambda \underbrace{\int_0^1 u^2 dx}_{1}\right] = 0$$

But $\displaystyle\int_0^1 ruu' dx = \underbrace{ru^2\Big|_0^1}_{0} - \int_0^1 \frac{d}{dx}(ru^2) dx \Rightarrow$

$$\Rightarrow \int_0^1 ruu' dx = -\frac{d}{dx}\left[\int_0^1 ru^2 dx\right]$$

$$\Rightarrow \int_0^1 u'^2 dx + \int_0^1 ru^2 dx + \lambda = 0 \Rightarrow$$

$$\Rightarrow \int_0^1 u'^2 dx = -\lambda - \int_0^1 ru^2 dx \leq \lambda + r_{max} \underbrace{\int_0^1 u^2 dx}_{1} =$$

$$= \lambda + r_{max} \quad (**)$$

$(*), (**) \Rightarrow \lambda u^2(x) \leq \lambda + \lambda + r_{max} + C_1 \sqrt{\lambda + r_{max}} \Rightarrow$

$$\Rightarrow u^2(x) \leq 2 + \frac{r_{max}}{\lambda} + \frac{C_1\sqrt{\lambda + r_{max}}}{\lambda} \Rightarrow$$

$\Rightarrow$ As $\lambda \gg 1 \Rightarrow u^2(x) \leq 2 + \dfrac{r_{max}}{\lambda} + \dfrac{C_1}{\sqrt{\lambda}}$ ✓

$\lambda > 0$

Asymptotic representation of the solutions of the S-L problem for $\lambda \gg 1$

We consider the solution of $v'' + \lambda v = 0$, with $v(0) = u(0)$, $v'(0) = u'(0)$.

Now let $w = u - v \Rightarrow v = u - w \Rightarrow u'' - w'' + \lambda u - \lambda w = 0$
But $u'' - ru + \lambda u = 0$ $\Big\} \Rightarrow$

Solution of S-L problem with BCs $u(0) = u(1) = 0$

$$\Rightarrow w'' + \lambda w = ru, \quad w(0) = 0, \quad w'(0) = 0 \Rightarrow \text{Multiply by } 2w' \text{ and } \int_0^x (\,)\,d\xi \Rightarrow$$

$$\Rightarrow w'^2(x) - \underbrace{w'^2(0)}_{0} + \lambda w^2(x) - \underbrace{\lambda w^2(0)}_{0} = 2\int_0^x r u w' \, d\xi \Rightarrow$$

$$\Rightarrow w'^2(x) + \lambda w^2(x) = 2\int_0^x r u w' \, d\xi$$

$$\left|\int_a^b \psi_1(x)\psi_2(x)\,dx\right|^2 \leq \int_a^b [\psi_1(x)]^2 dx \int_a^b [\psi_2(x)]^2 dx$$

Now let $M' = \max\limits_{0 \leq x \leq 1} |w'(x)| \Rightarrow$ Apply Schwartz inequality $\Rightarrow$

$$\Rightarrow 2\int_0^x r u w' \, d\xi \leq 2\int_0^x |r||u||w'| \, dx \leq 2\int_0^1 |r||u||w'|\,dx \leq 2 r_{max} \underbrace{\sqrt{\int_0^1 |u|^2 dx}}_{1} \sqrt{\int_0^1 |w'|^2 dx} \leq \underbrace{2 r_{max}}_{C} M' = CM'$$

Hence, $\lambda w^2(x) \leq w'^2(x) + \lambda w^2(x) \leq CM' \Rightarrow w^2(x) \leq \dfrac{CM'}{\lambda} \Rightarrow w(x) \leq \dfrac{\sqrt{CM'}}{\sqrt{\lambda}}$, $0 \leq x \leq 1$

Consider $2\int_0^x r u w' d\xi$

Schwartz inequality: $\left| \int_a^b \psi_1(x)\psi_2(x)\,dx \right|^2 \leq \int_a^b \psi_1^2(x)\,dx \int_a^b \psi_2^2(x)\,dx$

Let $M' = \max\limits_{0 \leq x \leq 1} |w'(x)| \Rightarrow$

$$\Rightarrow 2\int_0^x r u w' d\xi \leq 2\int_0^x |r||u||w'|\,dx \leq$$

$$\leq 2\int_0^1 |r||u||w'|\,dx \leq$$

$$\leq 2 r_{max} \int_0^1 |u||w'|\,dx =$$

$$= 2 r_{max} \left(\sqrt{\int_0^1 |u||w'|\,dx} \right)^2 \leq$$

$$\leq 2 r_{max} \underbrace{\sqrt{\int_0^1 |u|^2 dx}}_{1} \sqrt{\int_0^1 |w'|^2 dx} \leq$$

$$\leq \underbrace{2 r_{max}}_{C} M' = CM'$$