$$u(x,0)=g(x) \Rightarrow \sum_{i=1}^{\infty} \eta_i(0)\varphi_i(x)=g(x) \Rightarrow \int_0^L (\cdot)\, m(x)\varphi_j(x)\,dx$$

$$\Rightarrow \quad \eta_j(0)=\int_0^L m(x)g(x)\varphi_j(x)\,dx, \quad j=1,2,\dots$$

Similarly, $\frac{\partial u}{\partial t}(x,0)=h(x) \Rightarrow \dot{\eta}_j(0)=\int_0^L m(x)h(x)\varphi_j(x)\,dx, \quad j=1,2,\dots$

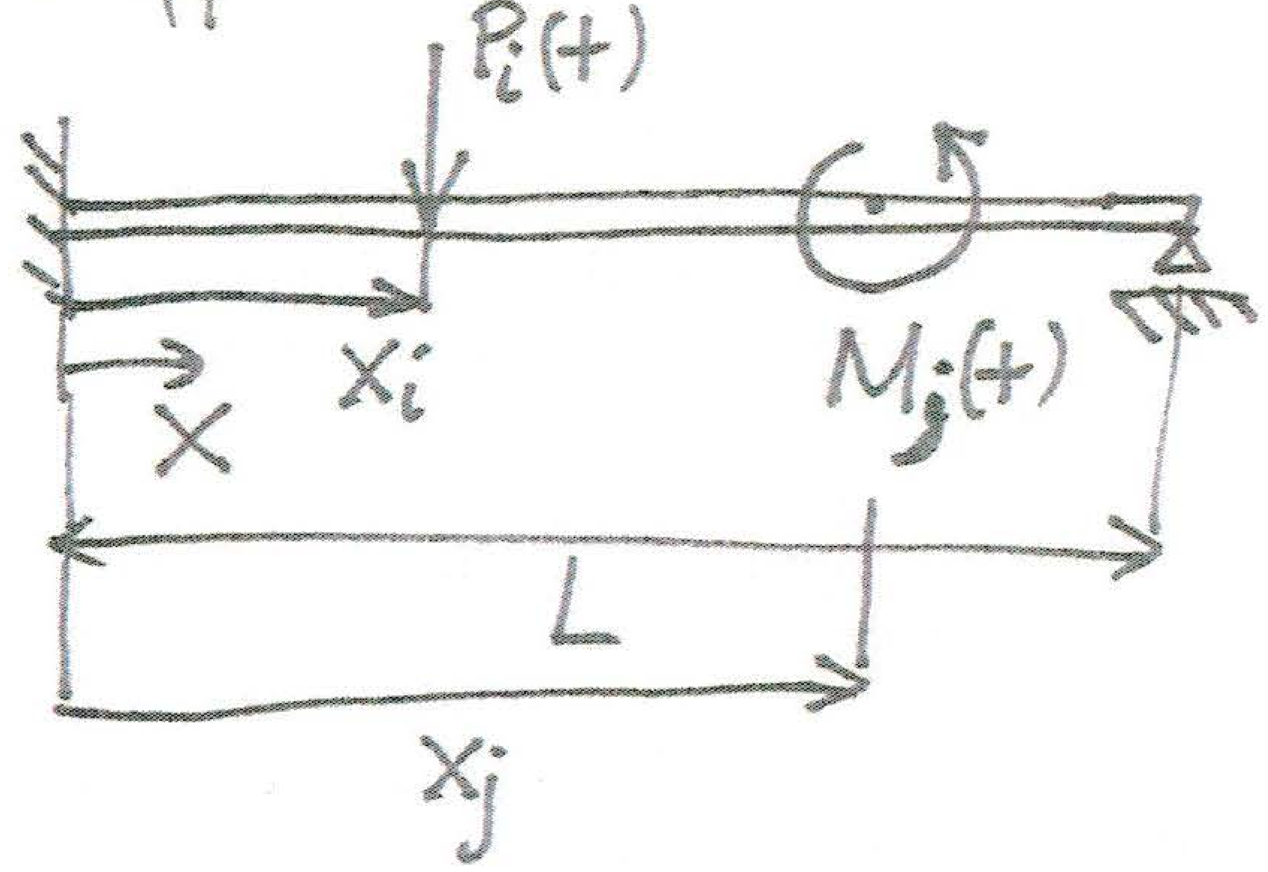
Then, the get the overall solution,

$$\boxed{v(x,t)=\sum_{i=1}^{\infty}\left[\eta_i(0)\cos\omega_i t+\frac{\dot{\eta}_i(0)}{\omega_i}\sin\omega_i t+\frac{1}{\omega_i}\int_0^t N_i(\tau)\sin\omega_i(t-\tau)\,d\tau\right]\varphi_i(x)}$$

$$\int_{-\infty}^{\infty} f(x)\delta'(a-x)\,dx=f'(a); \quad \int_{-\infty}^{\infty} f(x)\delta'(x-a)\,dx=-f'(a) \qquad (2)$$

Remark

Suppose that the beam is forced by a point force and a point moment.



Then, $F(x,t)=-P_i(t)\underbrace{\delta(x-x_i)}_{\text{Dirac's function}}+M_j(t)\underbrace{\delta'(x-x_j)}_{\text{Doublet function}}$

Aside

Definition of $\delta(t)$: $\int_{-\infty}^{\infty}\delta(t)\,dt=1, \quad \delta(t)=0, \ t\neq 0 \Rightarrow$

$$\Rightarrow \int_{-\infty}^{\infty} F(x)\delta(x)\,dx=F(0)$$

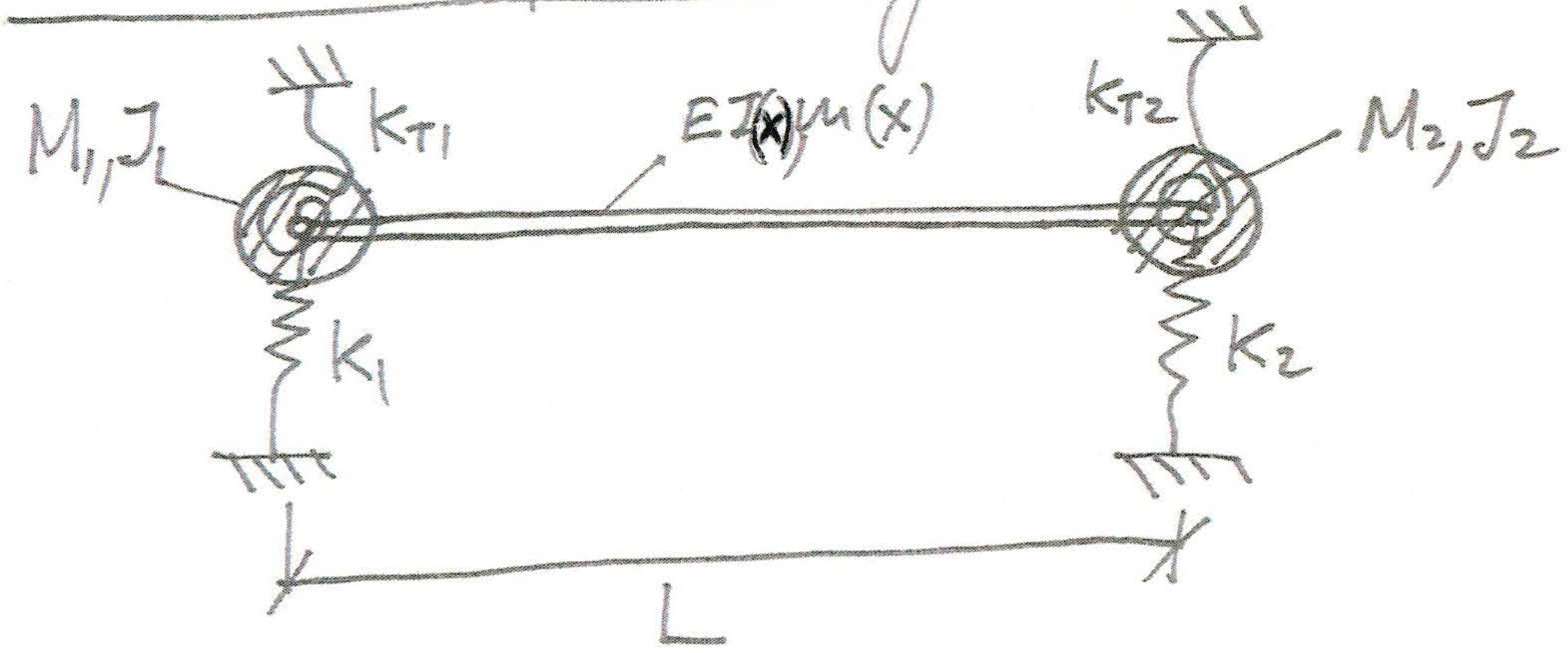
$$\int_{-\infty}^{\infty} F(x)\delta'(x)\,dx=\overset{0}{\cancel{F(x)\delta(x)\Big|_{-\infty}^{+\infty}}}-\int_{-\infty}^{\infty} F'(x)\delta(x)\,dx=0=-F'(0)$$

But $N_p(t) = \int_0^L F(x,t)\,\varphi_p(x)\,dx \Rightarrow$

$$\Rightarrow N_p(t) = \int_0^L \left[-P_i(t)\,\delta(x-x_i) + M_j(t)\,\delta'(x-x_j)\right]\varphi_p(x)\,dx \Rightarrow$$

$$\Rightarrow N_p(t) = \underbrace{-P_i(t)\,\varphi_p(x_i)}_{\substack{\text{If } P_i(t) \text{ is applied} \\ \text{at a node of } \varphi_p(x) \\ \text{then, this} \\ \text{term is equal to zero}}} \underbrace{-M_j(t)\,\varphi_p'(x_j)}_{\substack{\text{If } M_j(t) \text{ is applied} \\ \text{at a point of zero} \\ \text{slope of } \varphi_p(x) \\ \text{then this term} \\ \text{is equal to zero}}}$$

Case of nonsimple boundary conditions



At $x=0$: Balance of vertical forces:

$$\boxed{[EI(0)\varphi''(0)]' + k_1\varphi(0) - \omega^2 M_1\varphi(0) = 0}$$

Balance of moments:

$$\boxed{EI(0)\varphi''(0) - k_{T1}\varphi'(0) + \omega^2 J_1\varphi'(0) = 0}$$