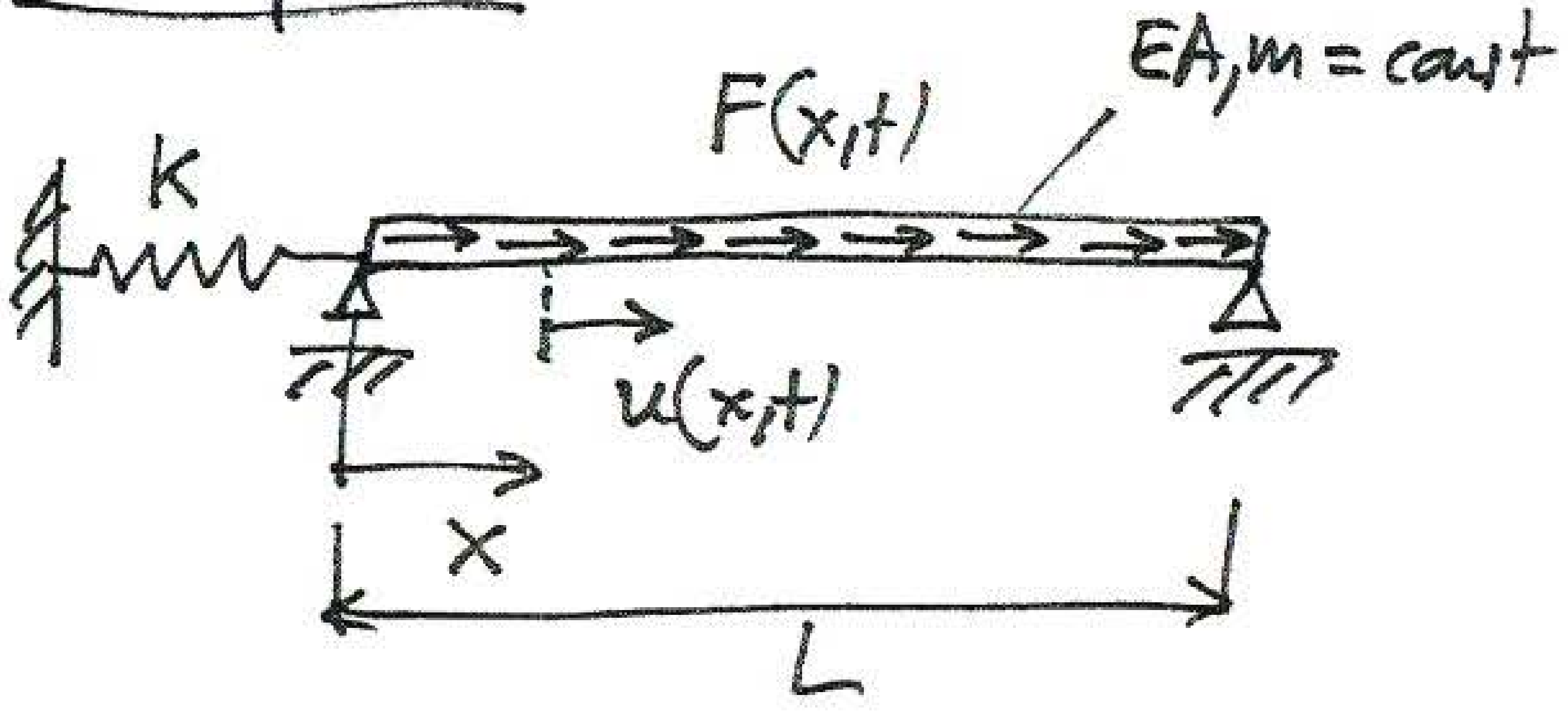
Example 3



$$EA\frac{\partial^2 u}{\partial x^2} + F(x,t) = m\frac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le L, \quad t \ge 0 \qquad (1)$$

$$\left.\begin{aligned} &EA\frac{\partial u(0,t)}{\partial x} - k u(0,t) = 0 \\ &\frac{\partial u(L,t)}{\partial x} = 0 \end{aligned}\right\} \text{BCs} \qquad (1a)$$

$$u(x,0) = g(x), \quad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad (1b)$$

first we compute the modes of this system $\Rightarrow$

$$\varphi''(x) + \left(\frac{\omega}{c}\right)^2 \varphi(x) = 0 \qquad (2)$$

$$\left.\begin{aligned} &EA\varphi'(0) - k\varphi(0) = 0 \\ &\varphi'(L) = 0 \end{aligned}\right\} \qquad (2a)$$

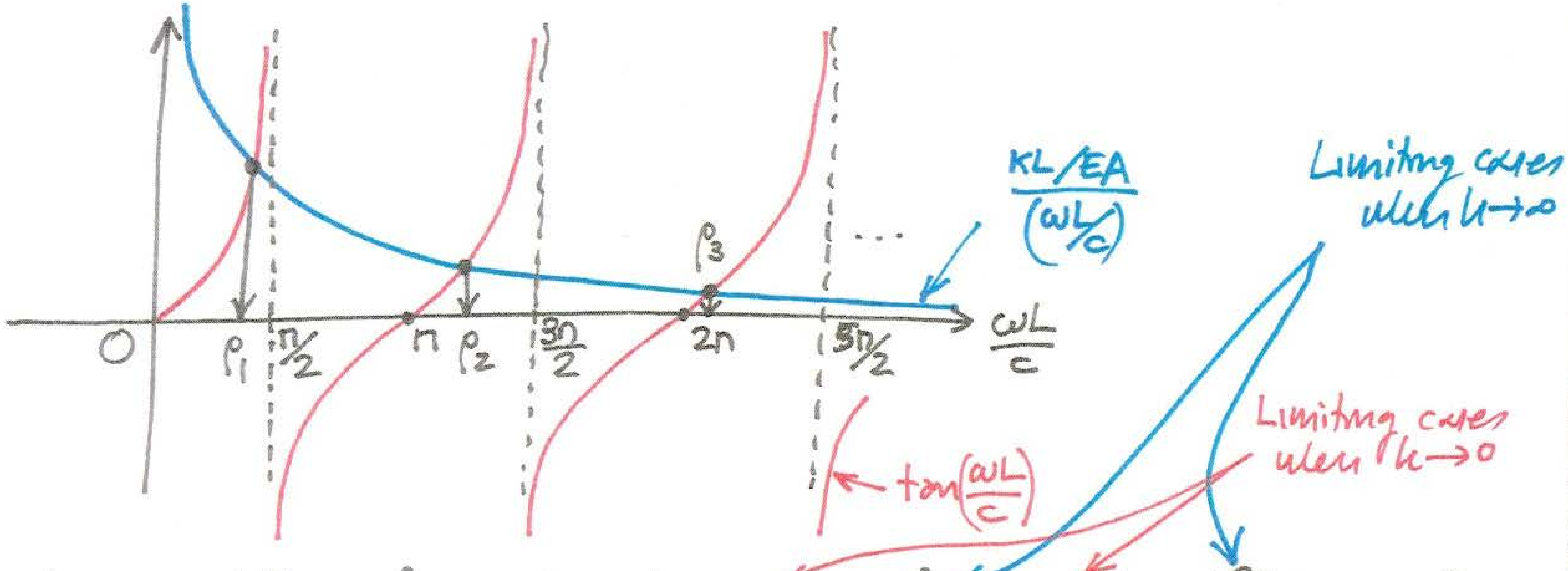
Solving for $\varphi(x) \Rightarrow \varphi(x) = C_1 \cos\frac{\omega x}{c} + C_2 \sin\frac{\omega x}{c} \Rightarrow$ Applying (2a) $\Rightarrow$

$$\Rightarrow \left.\begin{aligned} &EA\, C_2 \frac{\omega}{c} - k C_1 = 0 \\ &-C_1 \cancel{\frac{\omega}{c}} \sin\frac{\omega L}{c} + C_2 \cancel{\frac{\omega}{c}} \cos\frac{\omega L}{c} = 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} -k & EA\frac{\omega}{c} \\ -\sin\frac{\omega L}{c} & \cos\frac{\omega L}{c} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow$$

$\Rightarrow$ for nontrivial solutions we require that $\det[\quad] = 0 \Rightarrow$

$$\Rightarrow -k\cos\frac{\omega L}{c} + EA\frac{\omega}{c}\sin\frac{\omega L}{c} = 0 \Rightarrow \tan\left(\frac{\omega L}{c}\right) = \frac{k/EA}{\omega/c} = \frac{\frac{k}{EA}}{\left(\frac{\omega L}{c}\right)\frac{1}{L}} \Rightarrow \boxed{\tan\left(\frac{\omega L}{c}\right) = \frac{\frac{kL}{EA}}{\left(\frac{\omega L}{c}\right)}}$$

Frequency equation

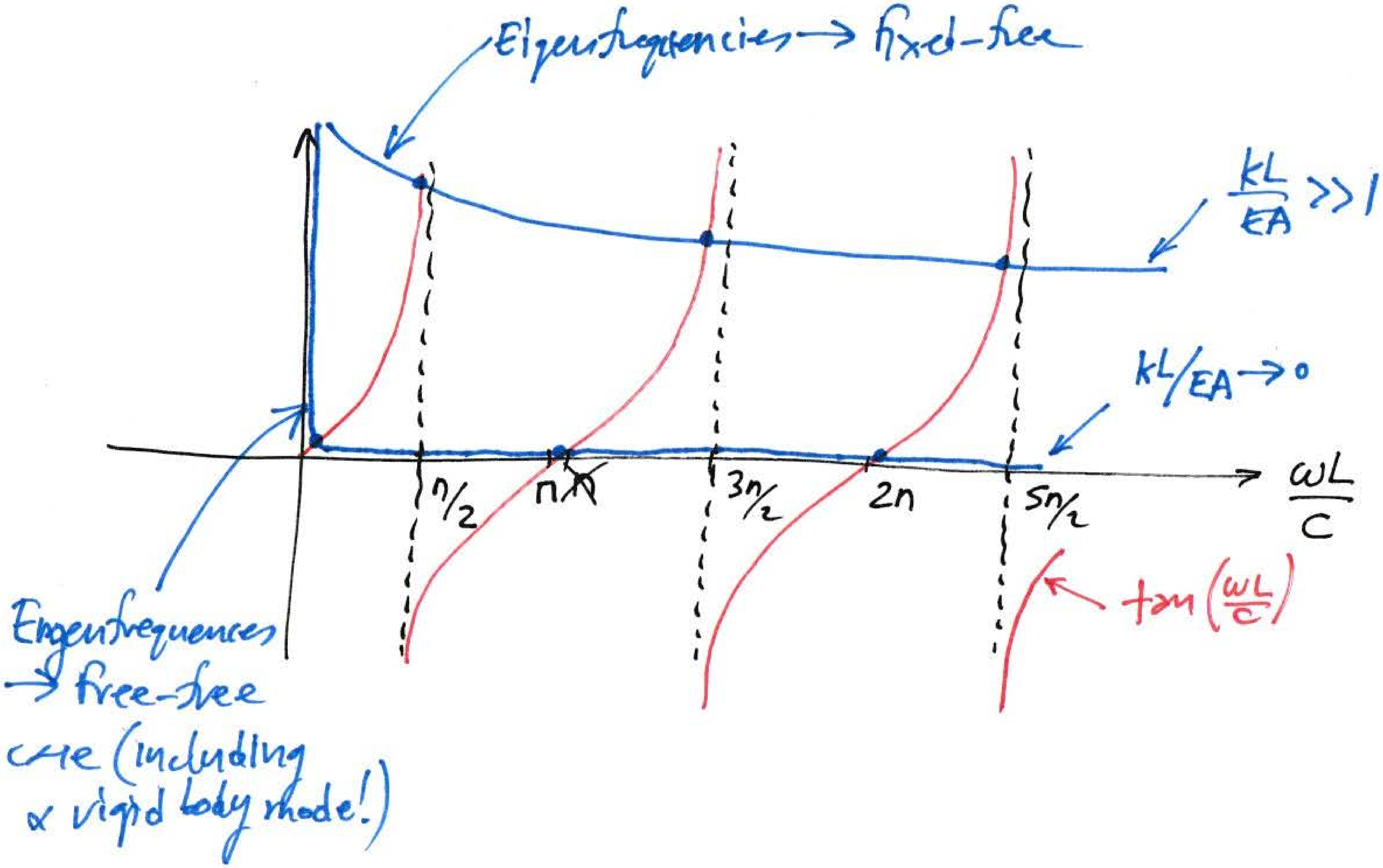


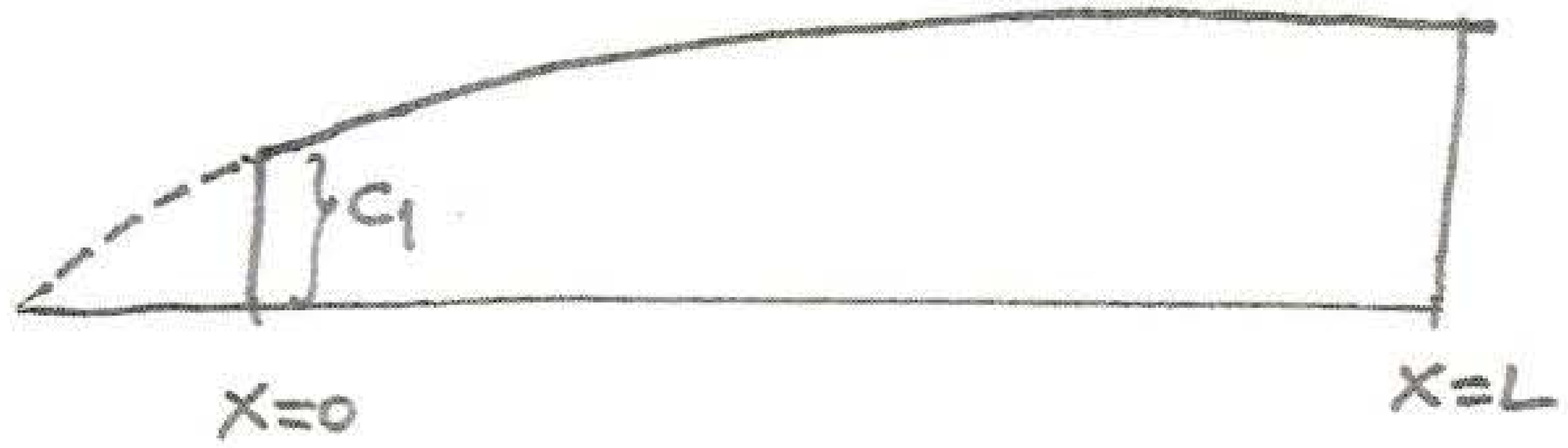
So we get a countable infinity of roots $0<\rho_1<\pi/2$, $\pi<\rho_2<3\pi/2$, $\ldots \Rightarrow$

$\Rightarrow$ Then we obtain the corresponding natural frequencies as $\frac{\omega_i L}{c}=\rho_i \Rightarrow$

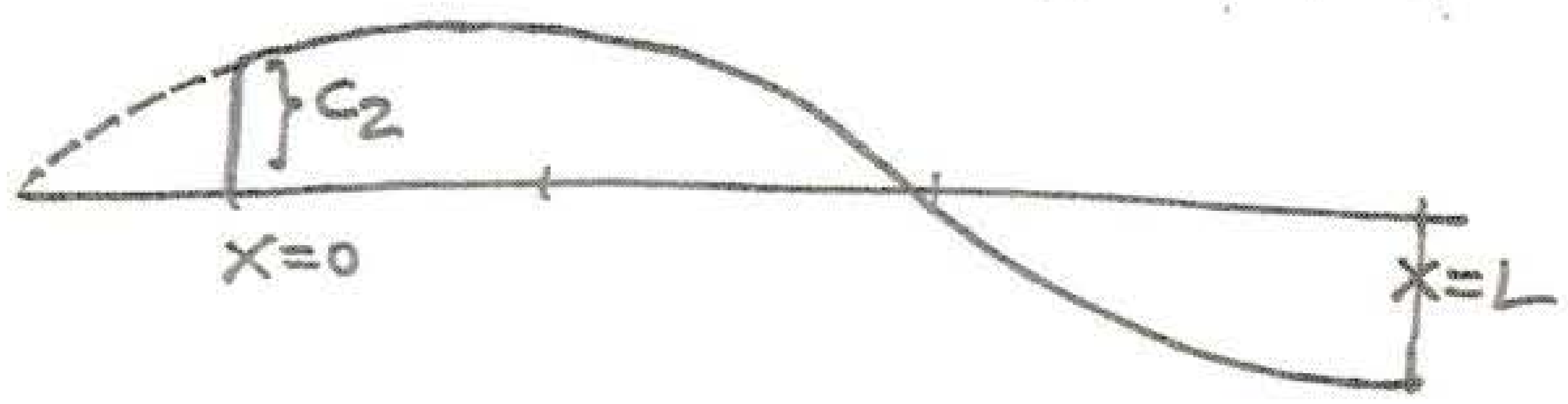
$\Rightarrow \omega_i = \frac{c\rho_i}{L},\ i=1,2,\ldots \Rightarrow$ Then we can compute the corresponding eigenfunction by noting that $c_2=c_1\tan\frac{\omega_i L}{c} \Rightarrow \varphi_i(x)=c_i\left(\cos\frac{\omega_i x}{c}+\tan\frac{\omega_i L}{c}\sin\frac{\omega_i x}{c}\right)\Rightarrow$

$\Rightarrow \varphi_i(x)=c_i\,\frac{\cos\left(\rho_i\frac{L-x}{L}\right)}{\cos\rho_i}$ Eigenfunctions for $i=1,2,\ldots$ Of course, we may orthonormalize the eigenfunctions $\varphi_i(x)$ using the formulas for 'non-simple' BCs derived in earlier lecture.


Eigenfrequencies → fixed-free
kL/EA >> 1
kL/EA → 0
π/2
π
3π/2
2π
5π/2
ωL/c
tan(ωL/c)
Eigenfrequencies → free-free case (including a rigid body mode!)



$\varphi_1(x)$



$\varphi_2(x)$

$\varphi_3(x)$

$\vdots$

To perform orthonormalization require that for each eigenfunction $\varphi_i(x)$ it holds that,

$$\int_0^L m\,\varphi_i^2(x)\,dx = 1,\ i=1,2,\ldots \Rightarrow$$

In simple form since the mass is uniform

$$\Rightarrow \int_0^L m\,c_i^2\,\frac{\cos^2\!\left(\rho_i\frac{L-x}{L}\right)}{\cos^2\rho_i}\,dx = 1 \Rightarrow$$

$$\Rightarrow c_i = 2\cos\rho_i\sqrt{\frac{\rho_i}{Lm(2\rho_i+\sin 2\rho_i)}}$$

$i=1,2,\ldots$

Then, the orthonormalized eigenfunctions are,

$$\boxed{\varphi_i(x) = 2\sqrt{\frac{\rho_i}{Lm(2\rho_i+\sin 2\rho_i)}}\,\cos\rho_i\frac{L-x}{L},\quad \omega_i = \rho_i\frac{c}{L},\quad i=1,2,\ldots}$$

Note that the stiffness-orthogonality condition won't be simple due to the local stiffness at $x=0$.

Remark: What happens when $k \to 0$? Suppose that $\frac{kL}{EA} \ll 1 \Rightarrow$

$$\Rightarrow \tan\left(\frac{\omega L}{c}\right) = \frac{kL/EA}{\left(\frac{\omega L}{c}\right)} \Rightarrow \tan\frac{\omega L}{c} \ll 1 \Rightarrow \tan\frac{\omega L}{c} \sim \frac{\omega L}{c} \Rightarrow$$

For the first branch of tan

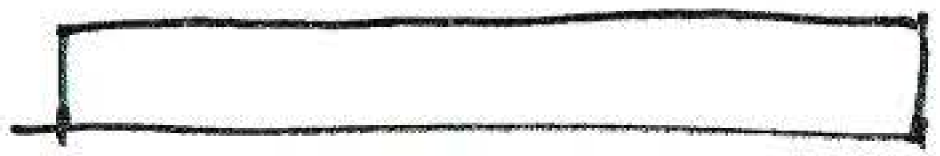
$$\Rightarrow \frac{\omega_1 L}{c} \sim \frac{kL/EA}{\frac{\omega_1 L}{c}} \Rightarrow \left(\frac{\omega_1 L}{c}\right)^2 \sim \frac{kL}{EA} \Rightarrow \omega_1^2 \sim \frac{c^2}{L^2}\frac{kL}{EA} = \frac{\cancel{EA}}{mL}\frac{k}{\cancel{EA}} = \frac{k}{mL} \Rightarrow$$

$$\Rightarrow \omega_1^2 \sim \frac{k}{mL}$$

Then we get the approximation that the rod moves as a rigid body

$M = mL$

So as $k \to 0 \Rightarrow \omega_1^2 \to 0 \Rightarrow \varphi_1(x) = \text{const}$



0 L

Hence the first mode becomes the rigid-body mode.

But what happens to higher modes? As $k \to 0 \Rightarrow \tan\frac{\omega L}{c} \to 0 \Rightarrow$

For other branches of tan

$\Rightarrow \frac{\omega_i L}{c} \to \pi, 2\pi, 3\pi, \ldots$ (the natural frequencies of free-free rod)

What happens when $k \to \infty$? In that case, $\rho_i \to \frac{(2(i-1)+1)\pi}{2} = \frac{(2i-1)\pi}{2}$, $i = 1, 2, 3, \ldots$

Natural frequencies of fixed-free rod!