Orthogonality properties of normal modes

Suppose that we have two modes of the generalized wave equation,

$$\{\omega_r, \varphi_r(x)\}, \quad \{\omega_s, \varphi_s(x)\}, \quad \omega_s \neq \omega_r$$

These are modes derived by solving the eigenvalue problem

$$\frac{d}{dx}\left[A(x)\frac{d\varphi(x)}{dx}\right] + \omega^2 B(x)\varphi(x) = 0, \quad 0 \leq x \leq L \qquad (4b)$$

with boundary conditions are of (4c). It follows that for each of these modes we can write,

$$\left.\begin{aligned} &\frac{d}{dx}\left[A(x)\frac{d\varphi_r(x)}{dx}\right] + \omega_r^2 B(x)\varphi_r(x) = 0 \qquad (9a)\ (\times \varphi_s(x)) \\ &\frac{d}{dx}\left[A(x)\frac{d\varphi_s(x)}{dx}\right] + \omega_s^2 B(x)\varphi_s(x) = 0 \qquad (9b)\ (\times \varphi_r(x)) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \quad \frac{d}{dx}\left[A(x)\frac{d\varphi_r(x)}{dx}\right]\varphi_s(x) + \omega_r^2 B(x)\varphi_r(x)\varphi_s(x) = 0 \Rightarrow \int_0^L dx \Rightarrow$$

$$\frac{d}{dx}\left[A(x)\frac{d\varphi_s(x)}{dx}\right]\varphi_r(x) + \omega_s^2 B(x)\varphi_s(x)\varphi_r(x) = 0 \Rightarrow \int_0^L dx \Rightarrow$$

$\int u\,dv = uv - \int v\,du$

$$\int_0^L \underbrace{\frac{d}{dx}\left[A(x)\frac{d\varphi_r(x)}{dx}\right]}_{dv} \underset{\uparrow\, u}{\varphi_s(x)}\,dx + \omega_r^2\int_0^L B(x)\varphi_r(x)\varphi_s(x)\,dx = 0 \Rightarrow$$

$\Rightarrow$ Perform integration by parts of the first term $\left(\int u\,dv = uv - \int v\,du\right) \Rightarrow$

$$\Rightarrow \underbrace{A(x)\frac{d\varphi_r(x)}{dx}\varphi_s(x)\Big|_0^L}_{\to\, 0} - \int_0^L A(x)\frac{d\varphi_r(x)}{dx}\frac{d\varphi_s(x)}{dx}\,dx + \omega_r^2\int_0^L B(x)\varphi_r(x)\varphi_s(x)\,dx = 0$$

For the 'simple' boundary conditions (4c) that we considered, either φ_r or φ_s or their derivatives are equal to zero at the boundaries!

We can do the identical operation for the second equation $\Rightarrow$

$$\Rightarrow -\int_0^L A(x)\frac{d\varphi_s(x)}{dx}\frac{d\varphi_r(x)}{dx}\,dx + \omega_s^2\int_0^L B(x)\varphi_s(x)\varphi_r(x)\,dx = 0$$

$(-) \Rightarrow$

$$\Rightarrow (\omega_r^2 - \omega_s^2)\int_0^L B(x)\varphi_r(x)\varphi_s(x)\,dx = 0 \Rightarrow \text{But, since } \omega_r \neq \omega_s \Rightarrow$$

$\Rightarrow \int_0^L B(x)\varphi_r(x)\varphi_s(x)\,dx = 0,\ r \neq s.$ But recall normalization $\int_0^L B(x)\varphi_r^2(x)\,dx = 1$