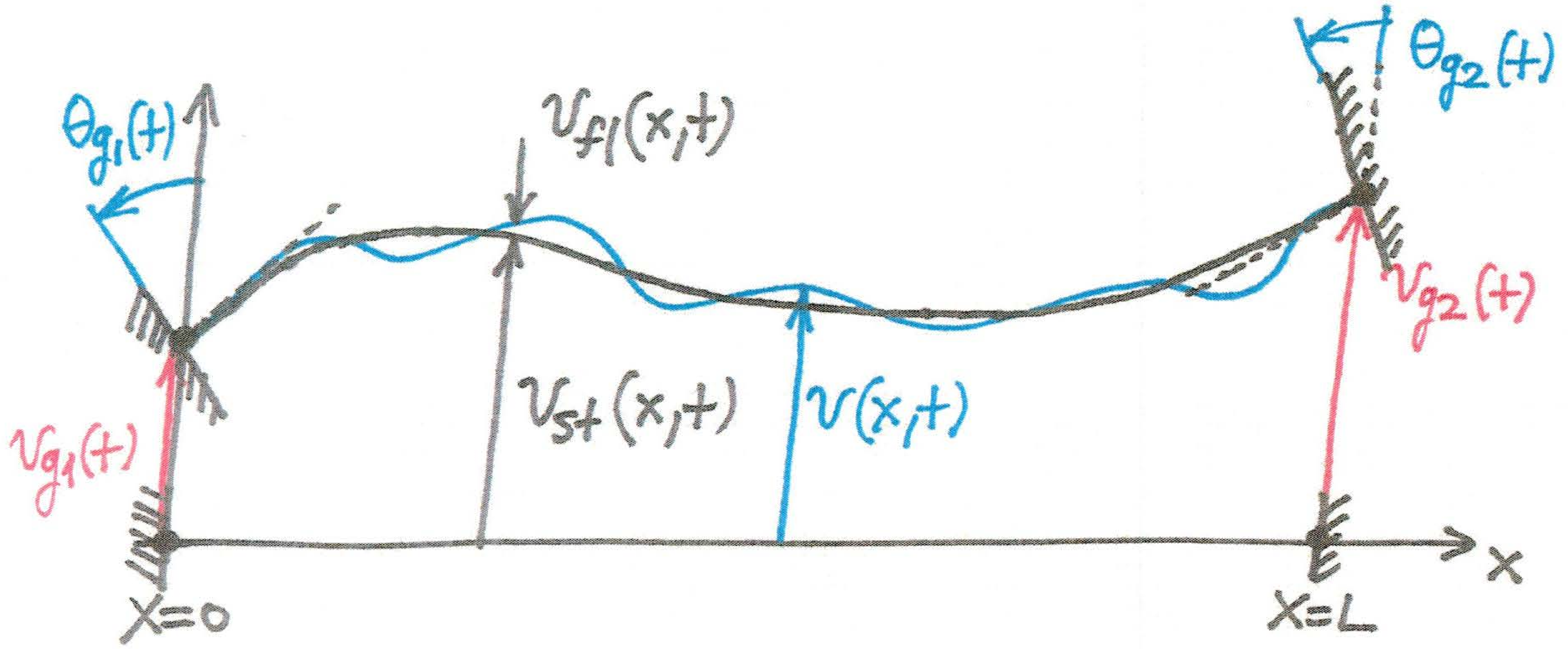


The governing equation of motion is,

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right] = m(x)\frac{\partial^2 v}{\partial t^2}, \quad 0 \le x \le L \qquad (1)$$

Boundary conditions are,

$$\left.\begin{aligned} v(0,t) &= v_{g_1}(t), & \frac{\partial v(0,t)}{\partial x} &= \theta_{g_1}(t) \\ v(L,t) &= v_{g_2}(t), & \frac{\partial v(L,t)}{\partial x} &= \theta_{g_2}(t) \end{aligned}\right\} \qquad (1a)$$

Initial conditions are,

$$v(x,0) = g(x), \quad \frac{\partial v(x,0)}{\partial t} = h(x) \qquad (1b)$$

Note that we need to have compatibility between BCs and ICs, e.g.,

$$v(0,0) = v_{g_1}(0) = g(0), \quad \frac{\partial v(0,0)}{\partial x} = \theta_{g_1}(0) = h(0), \ldots$$

Now decompose the response as $v(x,t) = v_{st}(x,t) + v_{fl}(x,t)$ (2)

Compute $v_{st}(x,t)$

This is governed by the pseudo-static problem,

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v_{st}}{\partial x^2}\right] = 0 \Rightarrow \text{Assuming that } EI(x) = EI$$

$$\Rightarrow \quad \frac{\partial^4 v_{st}}{\partial x^4} = 0 \Rightarrow v_{st}(x,t) = c_1(t)\frac{x^3}{6} + c_2(t)\frac{x^2}{2} + c_3(t)x + c_4(t) \quad (3)$$

To evaluate the four unknown coefficients consider the BCs that, by definition, are satisfied fully by the pseudo-static response,

$$\left.\begin{array}{ll} v_{st}(0,t) = v_{g1}(t), & v_{st}'(0,t) = \theta_{g1}(t) \\ v_{st}(L,t) = v_{g2}(t), & v_{st}'(L,t) = \theta_{g2}(t) \end{array}\right\} \quad (3a)$$

Hence $v_{st}(x,t)$ is fully determined.

Compute $v_{fl}(x,t)$ $\Rightarrow$ Substitute (2) into (1), taking into account that $v_{st}(x,t)$ is fully determined in the previous step.

In general form,

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v_{st}}{\partial x^2}\right]^{\nearrow 0} + \left\{-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v_{fl}}{\partial x^2}\right]\right\} =$$

$$= m(x)\left\{\frac{\partial^2 v_{st}}{\partial t^2} + \frac{\partial^2 v_{fl}}{\partial t^2}\right\} \Rightarrow -\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v_{fl}}{\partial x^2}\right] \underbrace{- m(x)\frac{\partial^2 v_{st}}{\partial t^2}}_{F(x,t)} = m(x)\frac{\partial^2 v_{fl}}{\partial t^2} \quad (4)$$

Since the nonhomogeneous BCs are fully accounted for by the computation of $v_{st}(x,t)$, the corresponding BCs for problem (4) are simply,

$$\underbrace{v_{fl}(0,t) = 0, \quad \frac{\partial v_{fl}}{\partial x}(0,t) = 0, \quad v_{fl}(L,t) = 0, \quad \frac{\partial v_{fl}}{\partial x}(0,t) = 0}_{\text{Note, homogeneous BCs!}} \quad (4a)$$

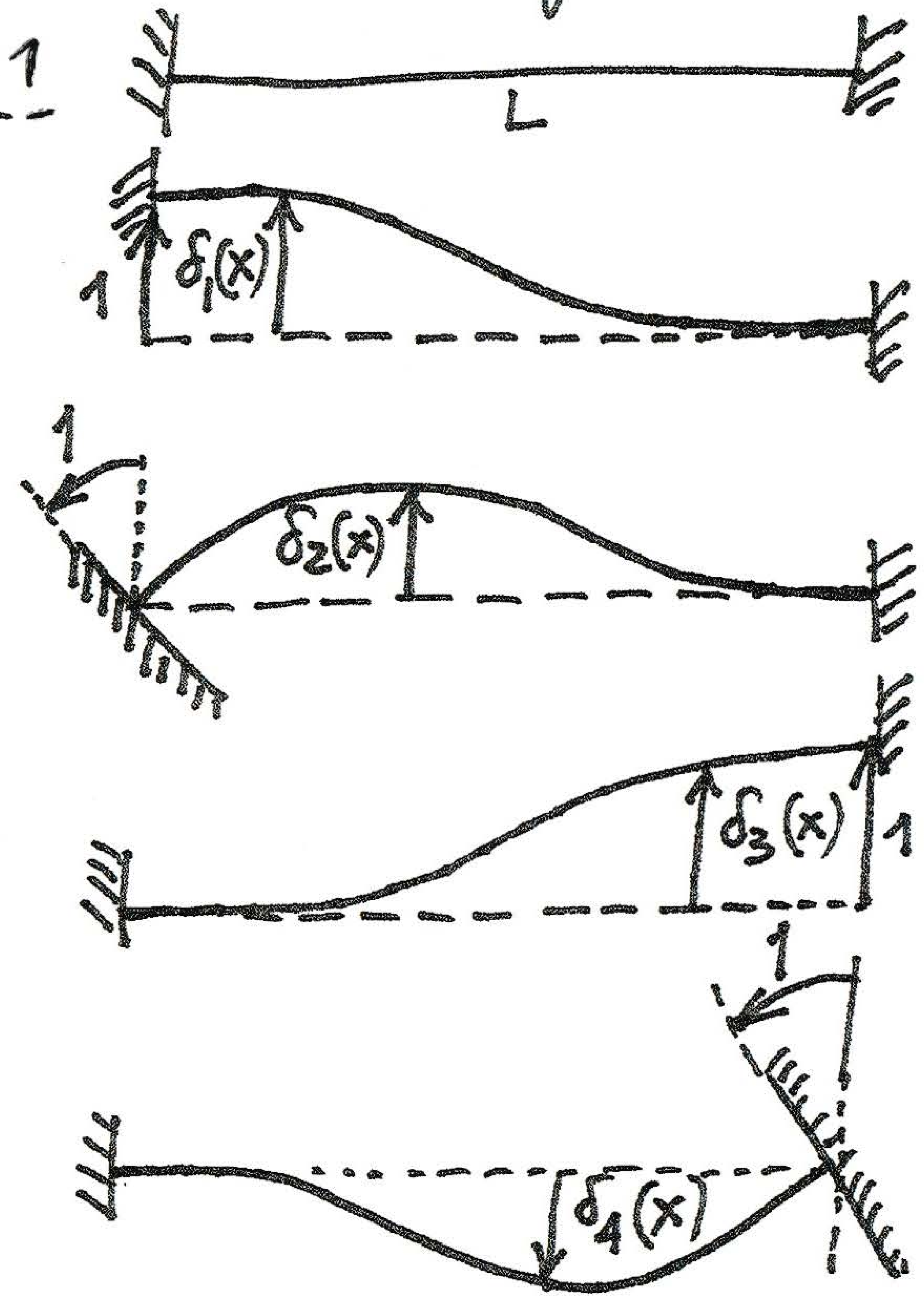
What about the ICs for (4)? These were not accounted for in the computation of $v_{st}(x,t)$ since we were solving a static problem in that case; for flexible displacements,

$$\left.\begin{aligned} v_{st}(x,0) + v_{fl}(x,0) = g(x) \Rightarrow v_{fl}(x,0) &= g(x) - v_{st}(x,0) \\ \partial v_{fl}/\partial t\,(x,0) &= h(x) - \partial v_{st}/\partial t\,(x,0) \end{aligned}\right\} \quad (4b)$$

Note: We can more conveniently express $v_{st}(x,t)$ using the idea of 'influence' functions $\Rightarrow$

$$\Rightarrow v_{st}(x,t) = v_{g_1}(t)\,\delta_1(x) + \theta_{g_1}(t)\,\delta_2(x) + v_{g_2}(t)\,\delta_3(x) + \theta_{g_2}(t)\,\delta_4(x)$$

Example 1



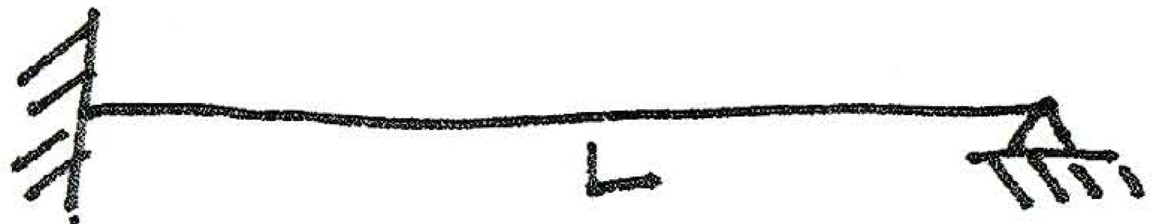
$$\delta_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

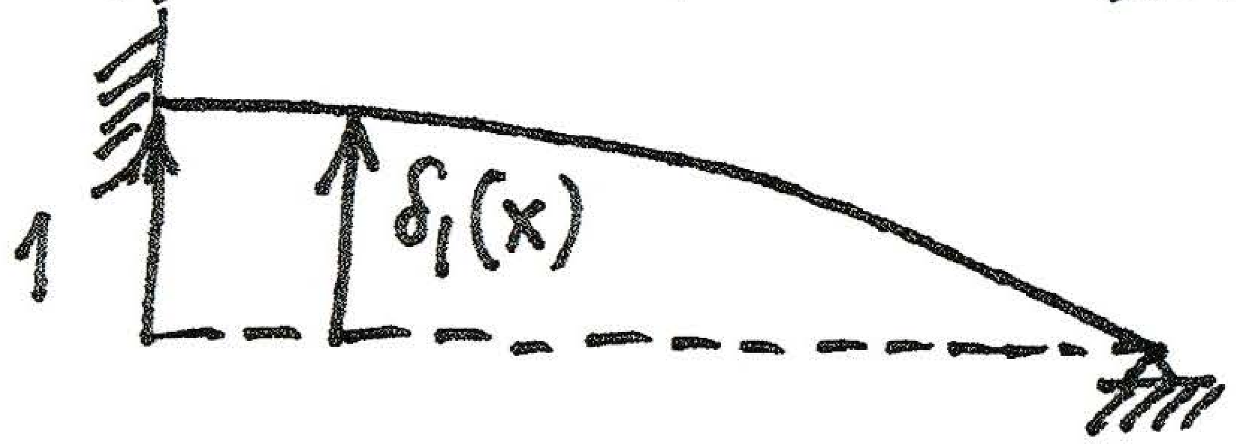
$$\delta_2(x) = x\left[1 - 2\frac{x}{L} + \left(\frac{x}{L}\right)^2\right]$$

$$\delta_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

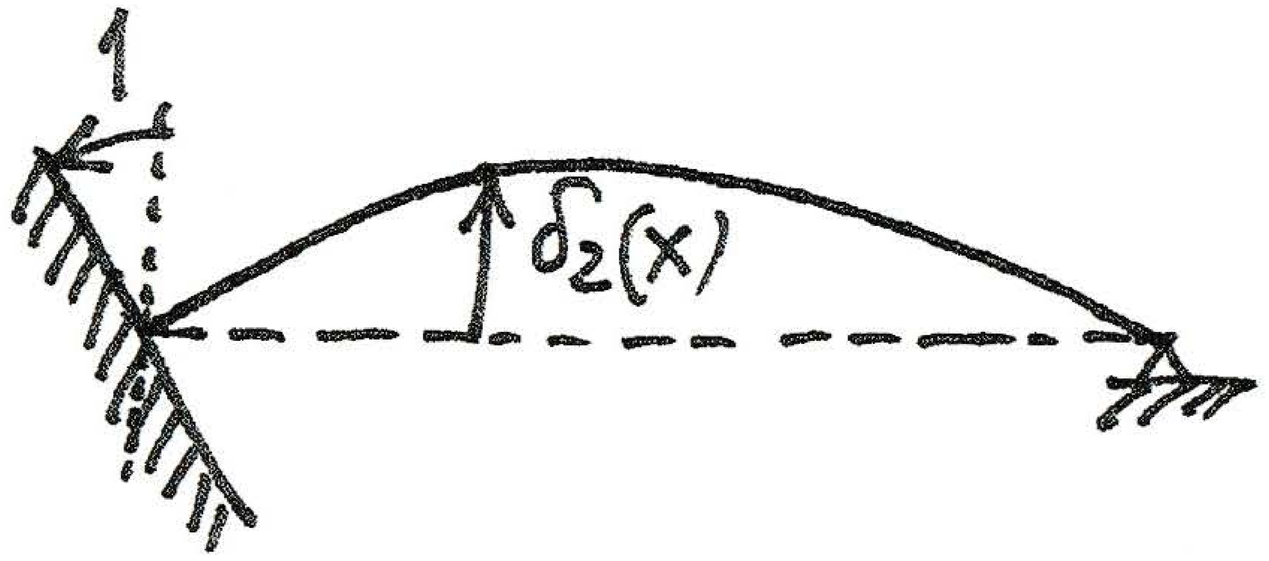
$$\delta_4(x) = -x\left[\frac{x}{L} - \left(\frac{x}{L}\right)^2\right]$$

Example 2

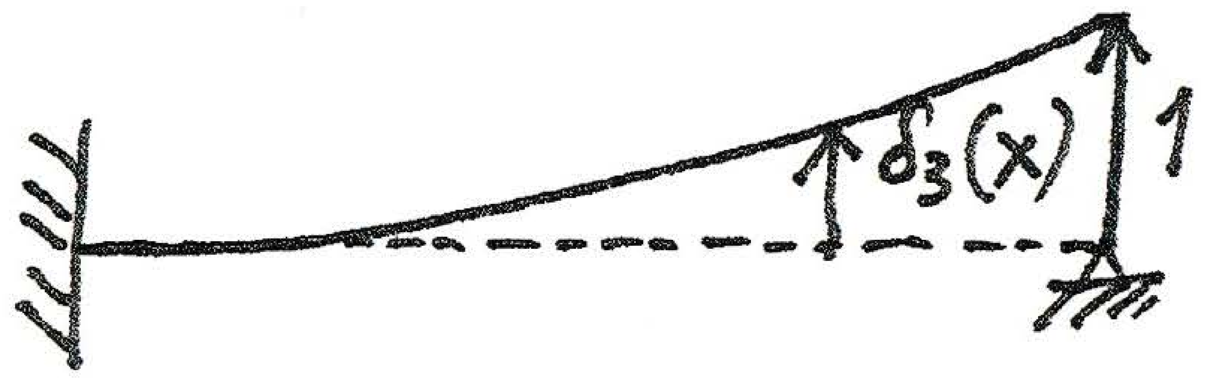




$$\delta_1(x) = 1 - \frac{3}{2}\left(\frac{x}{L}\right)^2 + \frac{1}{2}\left(\frac{x}{L}\right)^3$$



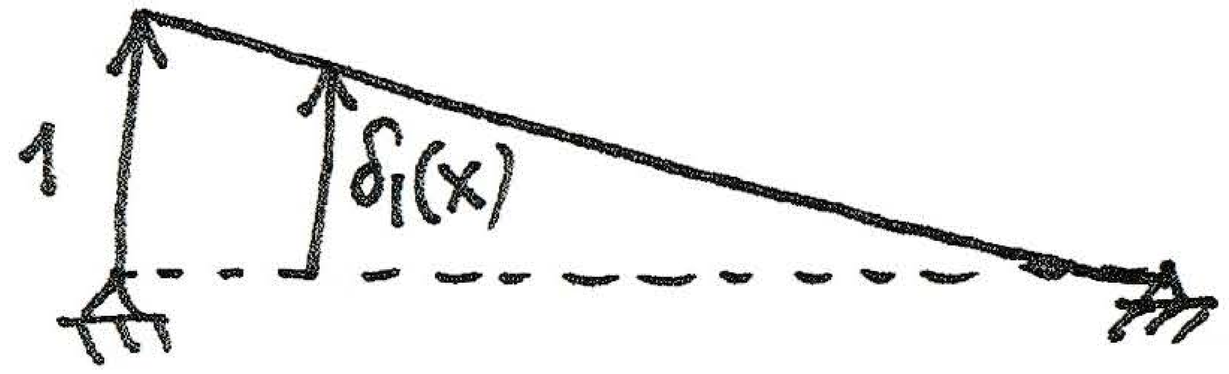
$$\delta_2(x) = x\left[1 - \frac{3}{2}\frac{x}{L} + \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$



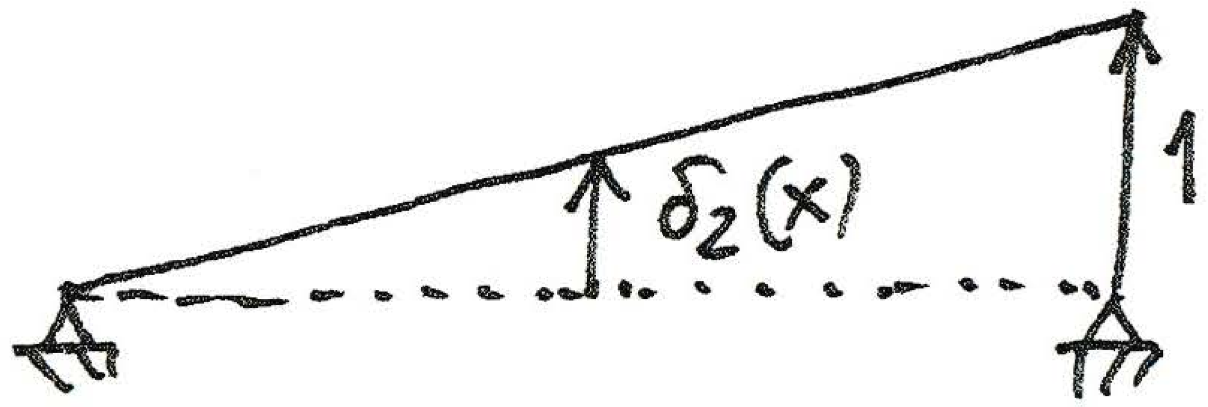
$$\delta_3(x) = \frac{3}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{2}\left(\frac{x}{L}\right)^3$$

$$\delta_4(x) = 0$$

Example 3

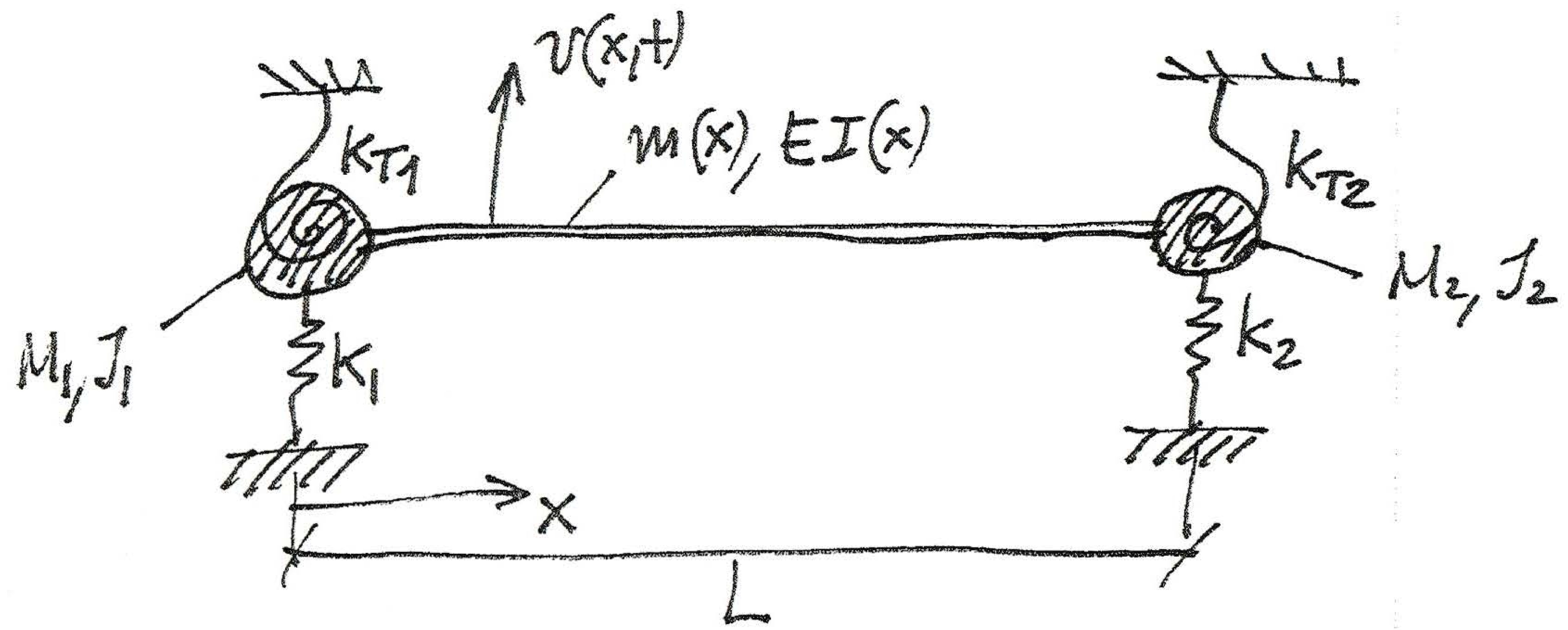


$$\delta_1(x) = 1 - \frac{x}{L} \quad , \quad \delta_2(x) = 0$$



$$\delta_3(x) = \frac{x}{L} \quad , \quad \delta_4(x) = 0$$

Kinetic and Potential (Elastic) Energy of the Beam



$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right] + m(x)\frac{\partial^2 v}{\partial t^2} = 0, \qquad 0 \le x \le L \tag{1}$$

$$\frac{\partial}{\partial x}\left[EI(0)\frac{\partial^2 v(0,t)}{\partial x^2}\right] + K_1 v(0,t) + M_1\frac{\partial^2 v(0,t)}{\partial t^2} = 0 \tag{1a}$$

$$EI(0)\frac{\partial^2 v(0,t)}{\partial x^2} - k_{T1}\frac{\partial v(0,t)}{\partial x} - J_1\frac{\partial^3 v(0,t)}{\partial x\,\partial t^2} = 0 \tag{1b}$$

$$\frac{\partial}{\partial x}\left[EI(L)\frac{\partial^2 v(L,t)}{\partial x^2}\right] - K_2 v(L,t) - M_2\frac{\partial^2 v(L,t)}{\partial t^2} = 0 \tag{1c}$$

$$EI(L)\frac{\partial^2 v(L,t)}{\partial x^2} + k_{T2}\frac{\partial v(L,t)}{\partial x} + J_2\frac{\partial^3 v(L,t)}{\partial x\,\partial t^2} = 0 \tag{1d}$$

$$\int_0^L (1)\times\frac{\partial v}{\partial t}\,dx = 0 \Rightarrow$$

$$\Rightarrow \int_0^L \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right]\frac{\partial v}{\partial t}\,dx + \int_0^L m(x)\underbrace{\frac{\partial^2 v}{\partial t^2}\frac{\partial v}{\partial t}}_{\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2\right)}\,dx = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right]\frac{\partial v}{\partial t}\Bigg|_0^L - \int_0^L \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right]\frac{\partial^2 v}{\partial x\partial t}\,dx + \int_0^L m(x)\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2\right)dx = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right]\frac{\partial v}{\partial t}\Bigg|_0^L - EI(x)\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial x\partial t}\Bigg|_0^L + \int_0^L EI(x)\underbrace{\frac{\partial^2 v}{\partial x^2}\frac{\partial^3 v}{\partial x^2\partial t}}_{\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right)}\,dx +$$

$$+ \int_0^L m(x)\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2\right)dx = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right]\frac{\partial v}{\partial t}\Bigg|_0^L - EI(x)\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial x\partial t}\Bigg|_0^L + \frac{\partial}{\partial t}\Bigg\{\underbrace{\int_0^L EI(x)\frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx}_{\text{Elastic energy stored in the beam itself at time } t}\Bigg\} +$$

$$+ \frac{\partial}{\partial t}\Bigg\{\underbrace{\int_0^L m(x)\frac{1}{2}\left(\frac{\partial v}{\partial t}\right)^2 dx}_{\text{Kinetic energy stored in the beam itself at time } t}\Bigg\} = 0 \qquad (2)$$

So, now we consider kinetic and potential energy due to the boundaries $\Rightarrow$

$\Rightarrow$ For example, we will get terms of the form,

$$\underbrace{\frac{\partial}{\partial x}\left[EI(0)\frac{\partial^2 v(0,t)}{\partial x^2}\right]}_{\text{Appears in (1a)}}\frac{\partial v(0,t)}{\partial t}, \quad \underbrace{EI(0)\frac{\partial^2 v(0,t)}{\partial x^2}}_{\text{Appears in (1b)}}\frac{\partial^2 v(0,t)}{\partial x\partial t}, \ldots, \ldots$$

Hence, to evaluate these terms we consider the following products:

$$(1c)\times\frac{\partial v(L,t)}{\partial t} - (1a)\times\frac{\partial v(0,t)}{\partial t} \Rightarrow \text{obtain relation (2a)}$$

$$(1b)\times\frac{\partial^2 v(0,t)}{\partial x\partial t} - (1d)\times\frac{\partial^2 v(L,t)}{\partial x\partial t} \Rightarrow \text{obtain relation (2b)}$$

Then, if we consider the expression $(2)-(2a)-(2b)=0 \Rightarrow$

$$\Rightarrow \frac{\partial}{\partial t}\underbrace{\left[\frac{1}{2}\int_0^L EI(x)\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx + \frac{1}{2}k_1 v^2(0,t) + \frac{1}{2}k_{T_1}\left(\frac{\partial v(0,t)}{\partial x}\right)^2 + \frac{1}{2}k_2 v^2(L,t) + \frac{1}{2}k_{T_2}\left(\frac{\partial v(L,t)}{\partial x}\right)^2\right]}_{\text{Total potential (elastic) energy of beam + boundaries at time } t,\ V(t)} +$$

$$+\frac{\partial}{\partial t}\underbrace{\left[\frac{1}{2}\int_0^L m(x)\left(\frac{\partial^2 v}{\partial t^2}\right)^2 dx + \frac{1}{2}M_1\left(\frac{\partial v(0,t)}{\partial t}\right)^2 + \frac{1}{2}J_1\left(\frac{\partial^2 v(0,t)}{\partial x\partial t}\right)^2 + \frac{1}{2}M_2\left(\frac{\partial v(L,t)}{\partial t}\right)^2 + \frac{1}{2}J_2\left(\frac{\partial^2 v(L,t)}{\partial x\partial t}\right)^2\right]}_{\text{Total kinetic energy of beam + boundaries at time } t,\ T(t)} = 0$$

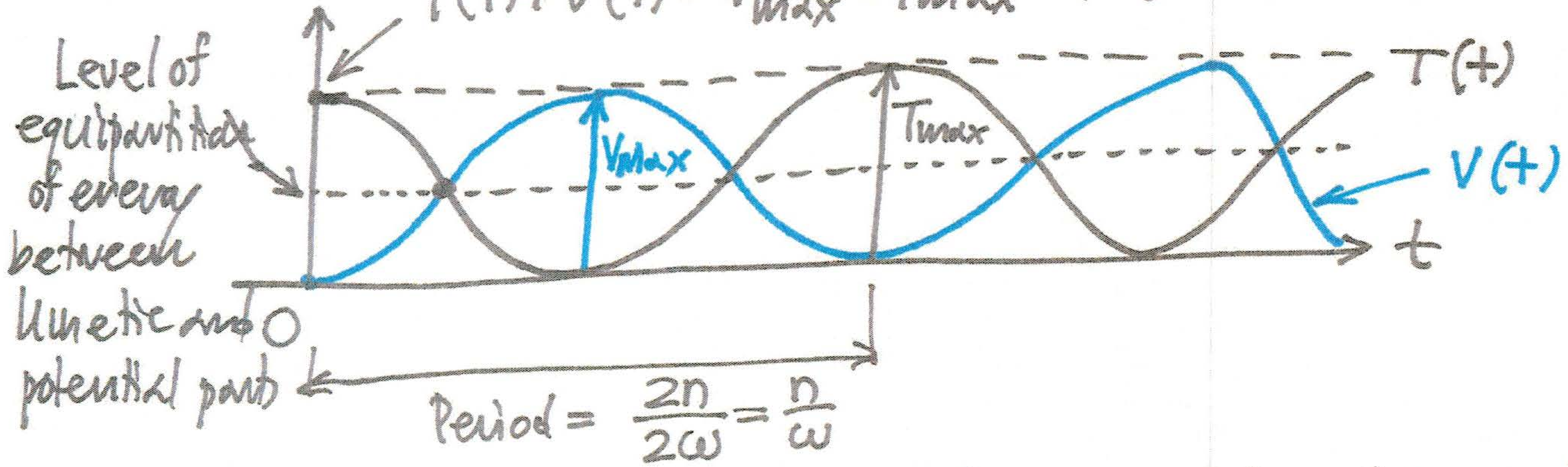
Similarly we work for $T(t) \Rightarrow$

$$\Rightarrow T(t) = \dot{f}^2(t)\left[\frac{1}{2}\int_0^L m(x)\,\varphi^2(x)\,dx + \frac{1}{2}M_1\varphi^2(0) + \frac{1}{2}J_1\left(\varphi'(0)\right)^2 + \frac{1}{2}M_2\varphi^2(L) + \frac{1}{2}J_2\left(\varphi'(L)\right)^2\right]$$

$\omega^2 A^2 \sin^2(\omega t+\varphi)$

Again $T(t)$ varies harmonically with twice the frequency of $f(t)$.

$$T(t)+V(t) = V_{max} = T_{max} \quad \forall\, t \geq 0$$



$$\left(\varphi = \frac{\pi}{2}\right)$$

Hence, $V_{max} = A^2\left[\frac{1}{2}\int_0^L EI(x)\,\varphi''(x)^2\,dx + \frac{1}{2}k_1\varphi^2(0) + \frac{1}{2}k_{T_1}\varphi'(0)^2 + \frac{1}{2}k_2\varphi^2(L) + \frac{1}{2}k_{T_2}\varphi'(L)^2\right]$

$$T_{max} = \omega^2 A^2\left[\frac{1}{2}\int_0^L m(x)\,\varphi^2(x)\,dx + \frac{1}{2}M_1\varphi^2(0) + \frac{1}{2}J_1\varphi'(0)^2 + \frac{1}{2}M_2\varphi^2(L) + \frac{1}{2}J_2\varphi'(L)^2\right]$$

But $V_{max} = T_{max} \Rightarrow$ Then we derive Rayleigh's quotient!

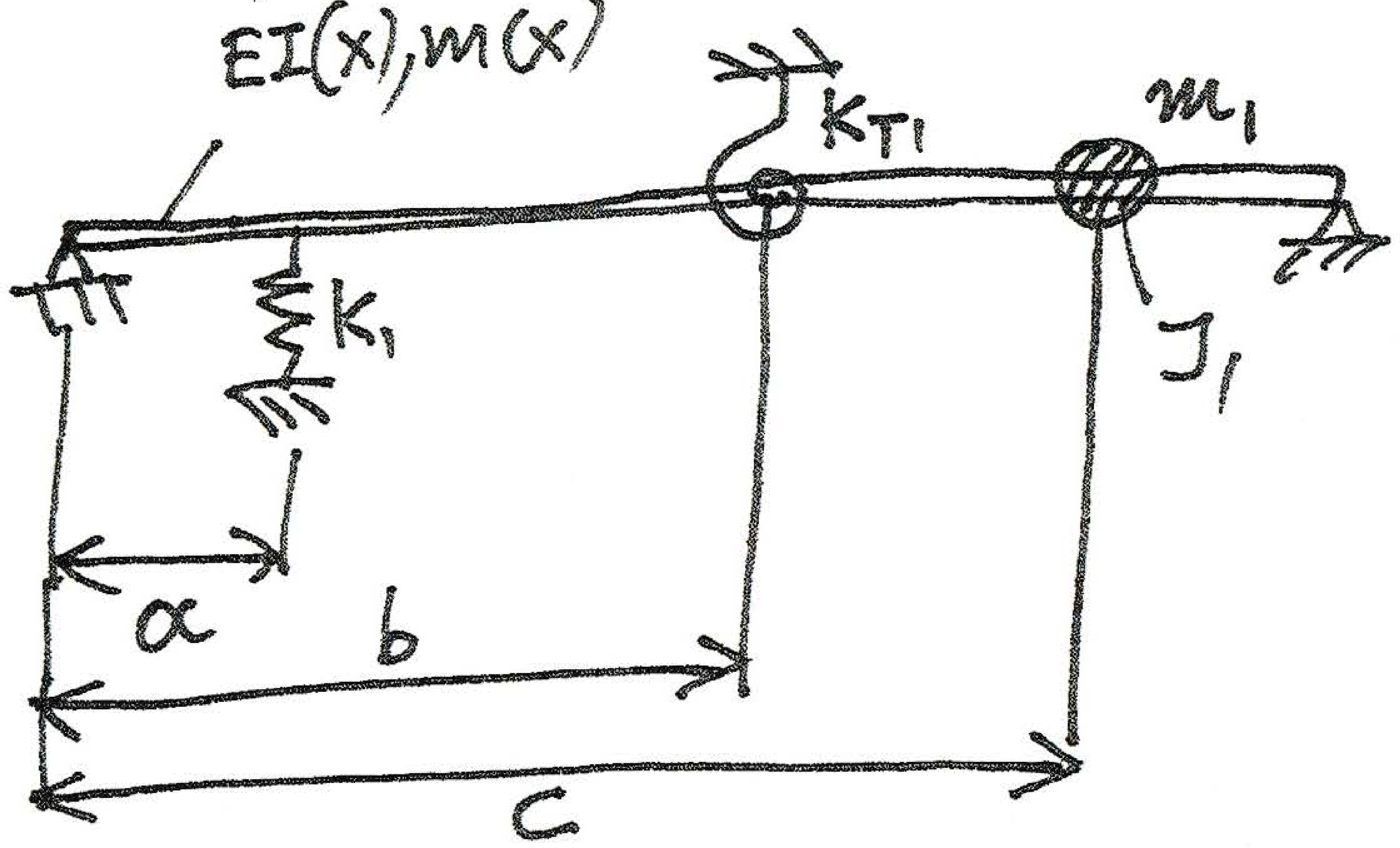
Use admissible test functions since the BCs are accounted explicitly in the RQ

$$\omega^2 = \frac{V_{max}}{T^*_{max}}, \quad T_{max} = \omega^2 T^*_{max} \Rightarrow$$

$$\Rightarrow \boxed{\omega^2 = \frac{\int_0^L EI(x)\,\varphi''(x)^2 dx + K_1 \varphi^2(0) + K_{T1}\varphi'(0)^2 + K_{T2}\varphi'(L)^2 + K_2 \varphi^2(L)}{\int_0^L m(x)\,\varphi^2(x)dx + M_1 \varphi^2(0) + J_1 \varphi'(0)^2 + M_2 \varphi^2(L) + J_2 \varphi'(L)^2}} \quad (4)$$

Remark

Hence we can immediately formulate Rayleigh's quotient for more complicated beams,



Use comparison functions

$$\omega^2 = \frac{\int_0^L EI(x)\varphi''(x)^2 dx + K_1 \varphi^2(a) + K_{T1}\varphi'(b)^2}{\int_0^L m(x)\varphi^2(x)dx + m_1 \varphi^2(c) + J_1 \varphi'(c)^2}$$

Note that the test functions should satisfy the geometric BCs (these do not appear in the RQ) and be differentiable at least two times.