

Concepts of Kinetic and Elastic Energy

Consider the unforced generalized wave equation,

$$\underbrace{B(x) \frac{\partial^2 u}{\partial t^2}}_{\text{Inertia effects}} - \underbrace{\frac{\partial}{\partial x} [A(x) \frac{\partial u}{\partial x}]}_{\text{Elastic effects}} = 0, \quad 0 \leq x \leq L, \quad t \geq 0 \quad (1)$$

No damping effects.

Consider non-simple BCs,

$$-A(0) \frac{\partial u(0,t)}{\partial x} + k_1 u(0,t) + M_1 \frac{\partial^2 u(0,t)}{\partial t^2} = 0 \quad (1a)$$

$$A(L) \frac{\partial u(L,t)}{\partial x} + k_2 u(L,t) + M_2 \frac{\partial^2 u(L,t)}{\partial t^2} = 0 \quad (1b)$$

We wish to show energy conservation in the system, in the absence of dissipative forces $\Rightarrow \int (1) \frac{\partial u}{\partial t} dx \Rightarrow$

$$\Rightarrow \int_0^L B(x) \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} dx - \int_0^L \frac{\partial}{\partial x} [A(x) \frac{\partial u}{\partial x}] \frac{\partial u}{\partial t} dx = 0 \Rightarrow \frac{\partial}{\partial t} \left(\frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 \right)$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 \right) + A(x) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \Big|_0^L - \int_0^L A(x) \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} dx = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \int_0^L \frac{1}{2} B(x) \left(\frac{\partial u}{\partial t} \right)^2 dx - A(L) \frac{\partial u(L, t)}{\partial x} \frac{\partial u(L, t)}{\partial t} + A(0) \frac{\partial u(0, t)}{\partial x} \frac{\partial u(0, t)}{\partial t} +$$

$$+ \frac{\partial}{\partial t} \int_0^L \frac{1}{2} A(x) \left(\frac{\partial u}{\partial x} \right)^2 dx = 0 \quad (2)$$

Now (1a) $\times \frac{\partial u(0, t)}{\partial t} \Rightarrow +A(0) \frac{\partial u(0, t)}{\partial x} \frac{\partial u(0, t)}{\partial t} = \underbrace{k_1 u(0, t) \frac{\partial u(0, t)}{\partial t}}_{\frac{\partial}{\partial t} (\frac{1}{2} k_1 u^2(0, t))} +$

$$+ M_1 \frac{\partial^2 u(0, t)}{\partial t^2} \frac{\partial u(0, t)}{\partial t} \Rightarrow$$

$$\underbrace{\frac{\partial}{\partial t} \left(\frac{1}{2} M_1 \left(\frac{\partial u(0, t)}{\partial t} \right)^2 \right)}$$

$$\Rightarrow A(0) \frac{\partial u(0, t)}{\partial x} \frac{\partial u(0, t)}{\partial t} = \frac{\partial}{\partial t} \left[\frac{1}{2} k_1 u^2(0, t) \right] + \frac{\partial}{\partial t} \left[\frac{1}{2} M_1 \left(\frac{\partial u(0, t)}{\partial t} \right)^2 \right] \quad (2a)$$

Similarly, (1b) $\times \frac{\partial u(L, t)}{\partial t} \Rightarrow -A(L) \frac{\partial u(L, t)}{\partial x} \frac{\partial u(L, t)}{\partial t} =$

$$= \frac{\partial}{\partial t} \left[\frac{1}{2} k_2 u^2(L, t) \right] + \frac{\partial}{\partial t} \left[\frac{1}{2} M_2 \left(\frac{\partial u(L, t)}{\partial t} \right)^2 \right] \quad (2b)$$

Substituting (2a), (2b) into (2) \Rightarrow

$$\Rightarrow \frac{\partial}{\partial t} \left\{ \frac{1}{2} \int_0^L B(x) \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^L A(x) \left(\frac{\partial u}{\partial x} \right)^2 dx + \right.$$

Kinetic energy in the elastodynamic system

$$+ \frac{1}{2} M_1 \left(\frac{\partial u(0, t)}{\partial t} \right)^2 + \frac{1}{2} M_2 \left(\frac{\partial u(L, t)}{\partial t} \right)^2 +$$

$$\left. + \frac{1}{2} k_1 u^2(0, t) + \frac{1}{2} K_2 u^2(L, t) \right\} = 0 \Rightarrow \text{Expression that shows conservation of energy in this elastodynamic system.}$$

Potential energy stored at the boundaries

Note, that if the boundary conditions are 'simple', i.e., there are no inertial or stiffness elements, then the conservation of energy expression simplifies to,

$$\frac{\partial}{\partial t} \left\{ \frac{1}{2} \int_0^L B(x) \left(\frac{\partial u}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^L A(x) \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} = 0$$

Rayleigh's Quotient

We now formulate approximate methods that allow us to solve a broad class of problems governed by the generalized wave equation, even when analytical solutions are not available. To this end, consider the following general eigenvalue problem in space:

$$\frac{d}{dx} [A(x) \varphi'(x)] + \omega^2 B(x) \varphi(x) = 0 \quad (1)$$

$$A(0) \varphi'(0) - (k_1 - \omega^2 M_1) \varphi(0) = 0 \quad (1a)$$

$$A(L) \varphi'(L) + (k_L - \omega^2 M_L) \varphi(L) = 0 \quad (1b)$$

Suppose that $\varphi(x)$ is a trial function \Rightarrow

$$\Rightarrow \int_0^L (1) \varphi(x) dx \Rightarrow \int_0^L \frac{d}{dx} [A(x) \varphi'(x)] \varphi(x) dx + \omega^2 \int_0^L B(x) \varphi^2(x) dx \approx 0 \Rightarrow$$

$$\Rightarrow \omega^2 \approx \frac{- \int_0^L \frac{d}{dx} [A(x) \varphi'(x)] \varphi(x) dx}{\int_0^L B(x) \varphi^2(x) dx} = R[\varphi(x)] \quad \text{Functional of } \varphi(x)$$

Rayleigh quotient $(3a)$
(does not incorporate the $\varphi(s)$)

Now, let's try to include the boundary conditions into Rayleigh's quotient

\Rightarrow Reconsider (2) \Rightarrow Perform integration by parts \Rightarrow

$$\Rightarrow [A(x)\varphi'(x)\varphi(x)] \Big|_0^L - \int_0^L A(x)\varphi'^2(x)dx + \tilde{\omega}^2 \int_0^L B(x)\varphi^2(x)dx = 0 \Rightarrow$$

$$\Rightarrow \underbrace{A(L)\varphi'(L)\varphi(L)}_{-(k_2 - \tilde{\omega}^2 M_2)\varphi^2(L)} - \underbrace{A(0)\varphi'(0)\varphi(0)}_{-(k_1 - \tilde{\omega}^2 M_1)\varphi^2(0)} - \int_0^L A(x)\varphi'^2(x)dx + \tilde{\omega}^2 \int_0^L B(x)\varphi^2(x)dx = 0$$

$$\Rightarrow \tilde{\omega}^2 [M_1\varphi^2(0) + M_2\varphi^2(L) + \int_0^L B(x)\varphi^2(x)dx] =$$

$$= k_1\varphi^2(0) + k_2\varphi^2(L) + \int_0^L A(x)\varphi'^2(x)dx \Rightarrow$$

Maximum potential of
elastic energy, V_{max}

$$\Rightarrow \left[\frac{\tilde{\omega}^2 = \frac{k_1\varphi^2(0) + k_2\varphi^2(L) + \int_0^L A(x)\varphi'^2(x)dx}{M_1\varphi^2(0) + M_2\varphi^2(L) + \int_0^L B(x)\varphi^2(x)dx}}{R[\varphi(x)]} \right] = R[\varphi(x)]$$

Maximum kinetic energy, T_{max}

(3b)
Alternative form
of Rayleigh's
quotient
(does incorporate
the BCs)

Remarks

- 1) Expressions (3a) and (3b) are of the form $\omega^2 = R[\varphi(x)]$, so these are functionals.
- 2) If $\varphi(x) = \varphi_r(x)$, $r=1, 2, \dots \Rightarrow$ By construction (3a) and (3b) will compute the corresponding exact natural frequencies ω_r^2 , $r=1, 2, \dots$
- 3) Since in the RQ (3a) the boundary conditions are not taken into account, the trial functions should satisfy the BCs as well; also the trial functions $\varphi(x)$ should be as differentiable as needed in order to compute the RQ. We will designate such trial functions as comparison functions.
- 4) However, considering the RQ (3b), the BCs are fully accounted for in the Quotient, so the trial functions do not need to satisfy them. In addition we note that there are less smoothness requirements for the trial functions in this case (just continuously differentiable). We will designate such trial functions as admissible functions.
- 5) Hence, $\{\text{natural modes}\} \subset \{\text{comparison functions}\} \subset \{\text{admissible functions}\}$

(3a)

(3b)

6) Rayleigh's principle: It can be shown that if ϕ that function is $O(\epsilon)$ close to the eigenfunction $\phi_r(x)$, i.e., $\phi(x) = \phi_r(x) + \sum_{\substack{i=1 \\ i \neq r}}^{\infty} \epsilon_i \phi_i(x)$,

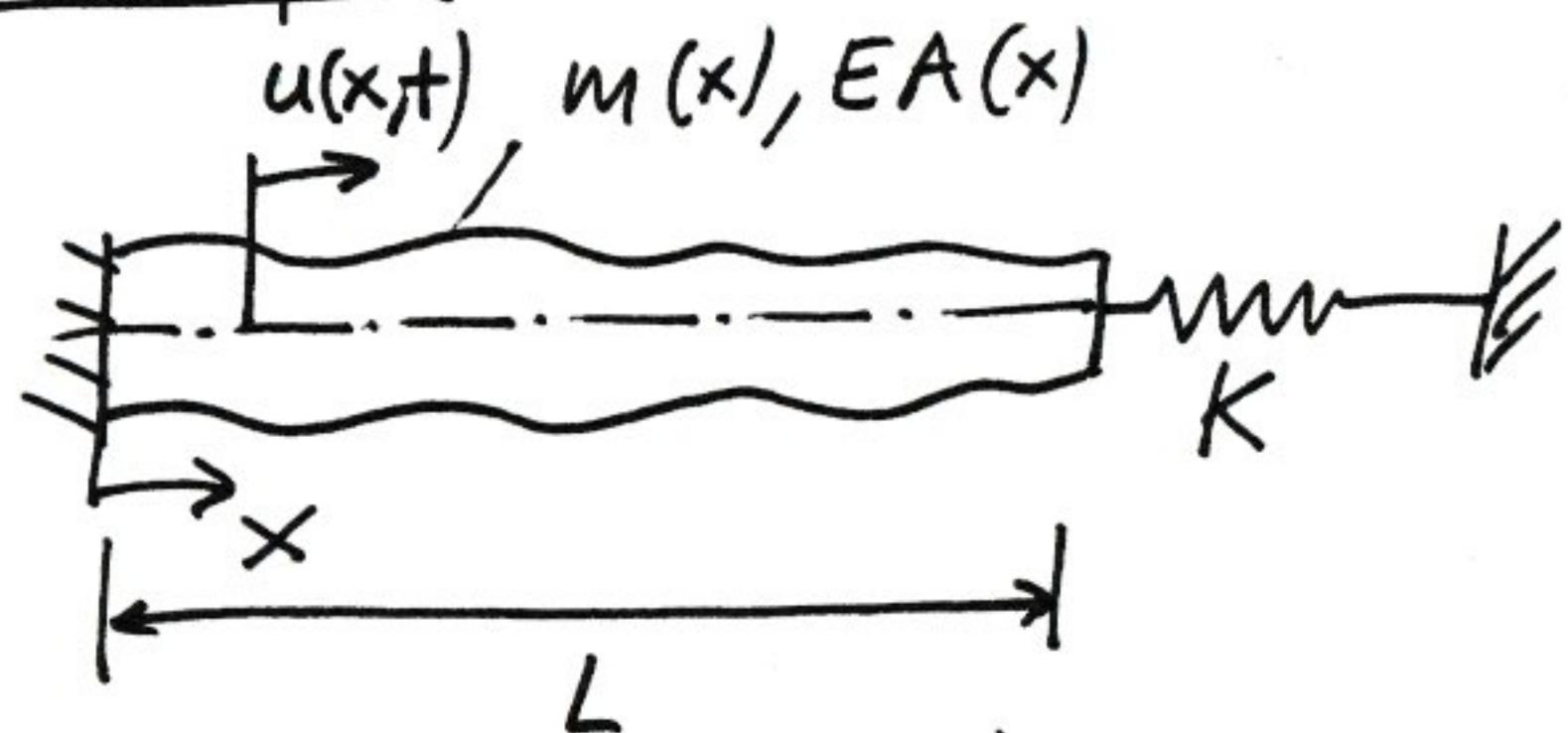
$\epsilon_i = O(\epsilon)$, $0 < \epsilon \ll 1$, then the estimate for the r -th natural frequency that we get using RQ can be expected to,

$$\omega^2 = R[\phi(x)] = \omega_r^2 + \sum_{i=1}^{\infty} (\omega_i^2 - \omega_r^2) \epsilon_i^2 \quad (4)$$

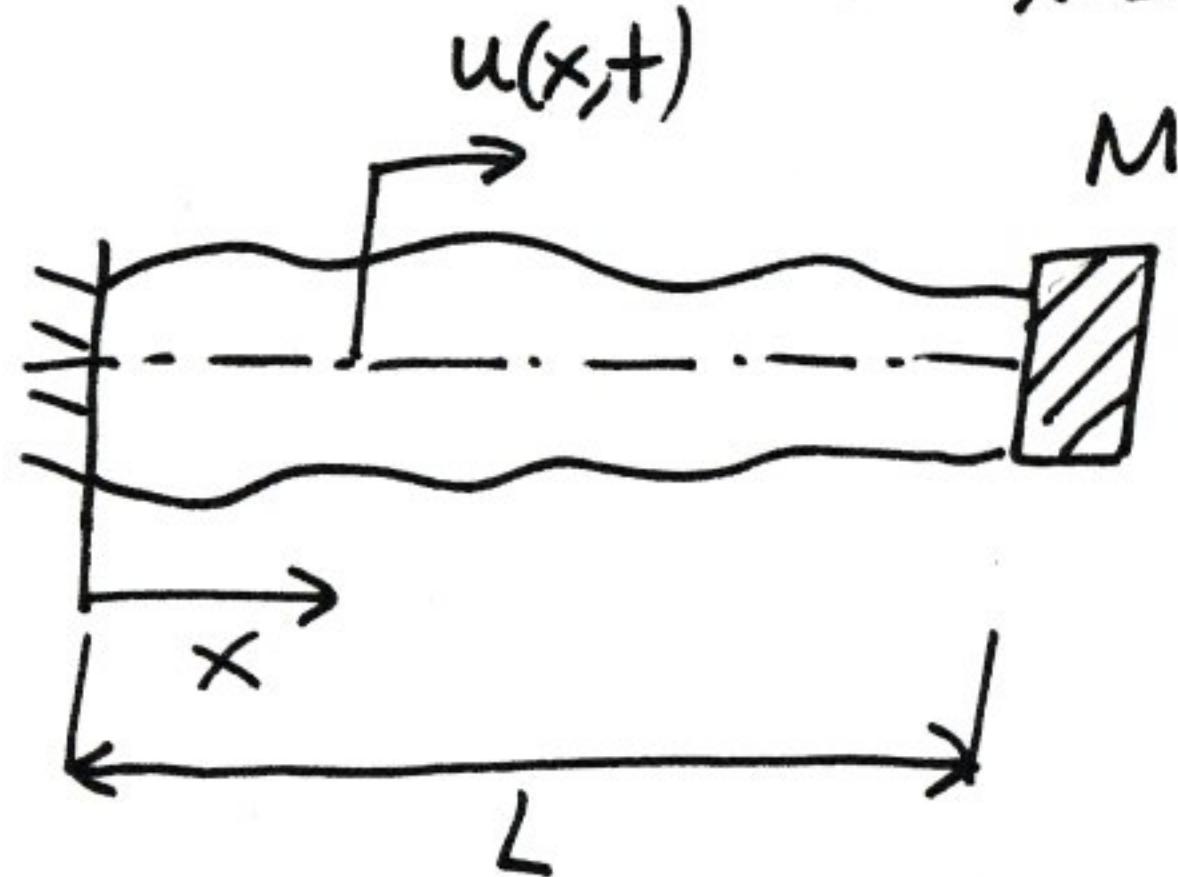
So the estimate for the natural frequency is $O(\epsilon)$ close to the true value! Note that if we have $\omega_1 < \omega_2 < \omega_3 < \dots < \omega_r < \dots \Rightarrow$

$$\Rightarrow \boxed{\omega^2 = R[\phi(x)] \geq \omega_1^2} \quad (5)$$

Examples



$$\varphi(0) = 0, EA(x) \frac{d\varphi}{dx} \Big|_{x=L} = -K\varphi(L)$$



$$\varphi(0) = 0$$

$$EA \frac{d\varphi}{dx} \Big|_{x=L} = \omega^2 M \varphi(L)$$

$$R[\varphi(x)] = \frac{\int_0^L EA(x) \left(\frac{d\varphi(x)}{dx}\right)^2 dx + K\varphi^2(L)}{\int_0^L m(x) \varphi^2(x) dx}$$

$\varphi(x)$ is an admissible function

Term from integration by parts

$$\omega^2 = R[\varphi(x)] = \frac{-\omega^2 M \varphi^2(L) + \int_0^L EA(x) \left(\frac{d\varphi(x)}{dx}\right)^2 dx}{\int_0^L m(x) \varphi^2(x) dx} \Rightarrow$$

$$\Rightarrow \omega^2 = \frac{\int_0^L EA(x) \left(\frac{d\varphi(x)}{dx}\right)^2 dx}{\int_0^L m(x) \varphi^2(x) dx + M \varphi^2(L)}$$

$\varphi(x)$ is an admissible function