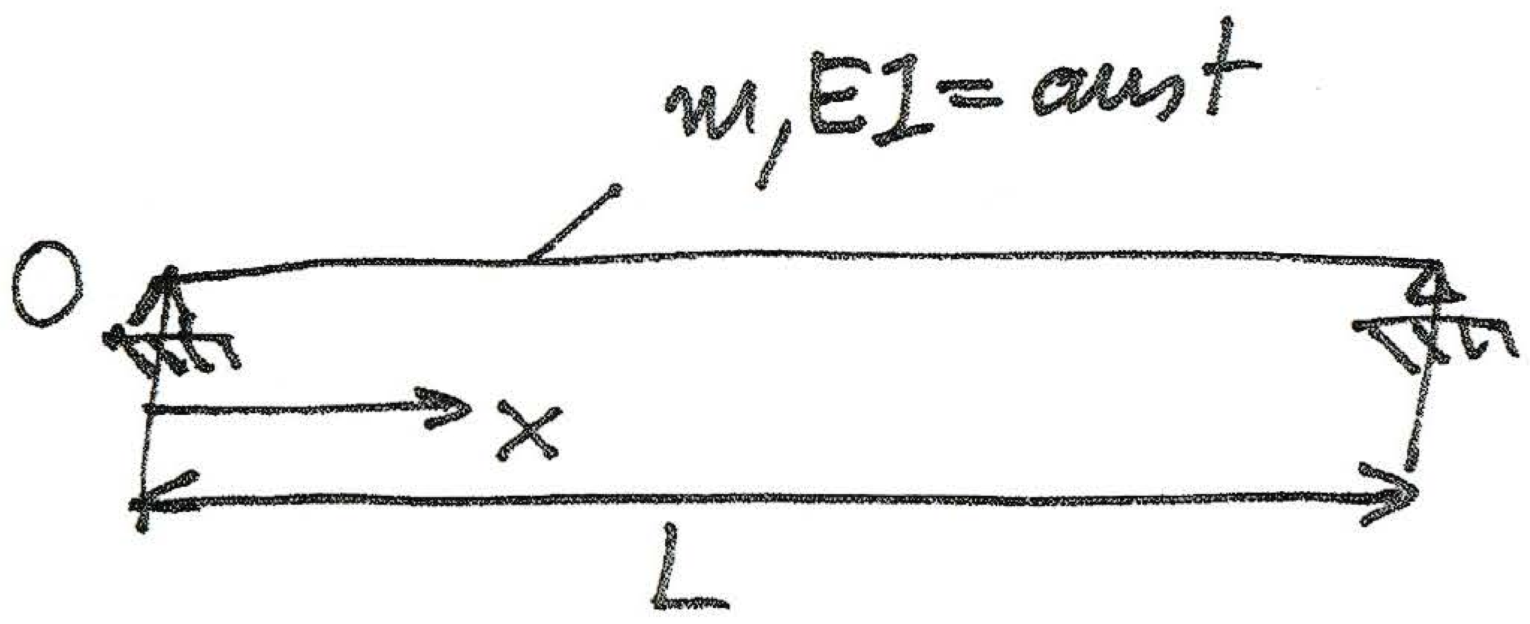
Example 1



$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v}{\partial x^2}\right]+F(x,t)=m(x)\,\frac{\partial^2 v(x,t)}{\partial t^2}\Rightarrow$$

$$\Rightarrow -EI\,\frac{\partial^4 v}{\partial x^2}+F(x,t)=m\,\frac{\partial^2 v}{\partial t^2},\quad 0\le x\le L \qquad (1)$$

$$\left.\begin{aligned} v(0,t)&=v(L,t)=0\\ v''(0,t)&=v''(L,t)=0\end{aligned}\right\}\qquad (1a)$$

$$v''=\frac{\partial^2 v}{\partial x^2}$$

$$v(x,0)=g(x),\quad \frac{\partial v}{\partial t}(x,0)=h(x)\qquad (1b)$$

As a first task we compute the linear normal modes of this system by assuming no force and disregarding the initial conditions ⇒ performing separation of variables x and t ⇒

$$\left.\begin{aligned} &EI\,\frac{d^4\varphi(x)}{dx^4}-\omega^2 m\,\varphi(x)=0,\quad 0\le x\le L\\ &\varphi(0)=\varphi(L)=0\\ &\varphi''(0)=\varphi''(L)=0\end{aligned}\right\}\qquad (2)$$

$$\frac{d^4\varphi(x)}{dx^4}-\frac{\omega^2}{c^4}\varphi(x)=0$$

$$c^4=EI/m$$

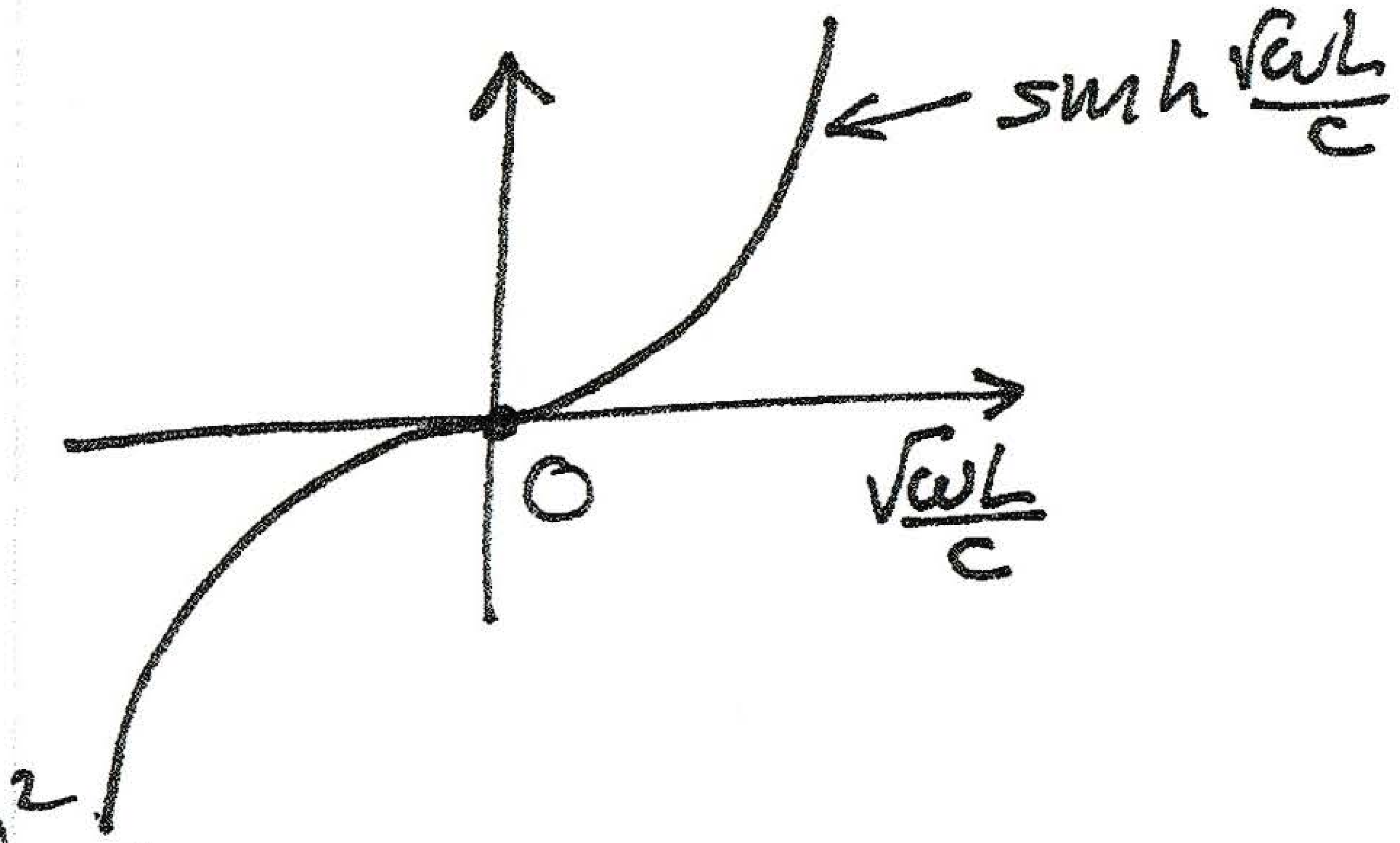
The general solution of this equation is

$$\varphi(x)=C_1\cos\frac{\sqrt{\omega}}{c}x+C_2\sin\frac{\sqrt{\omega}}{c}x+C_3\cosh\frac{\sqrt{\omega}}{c}x+C_4\sinh\frac{\sqrt{\omega}}{c}x\qquad (3)$$

$$\varphi''(x) = -\frac{\omega}{c^2} C_1 \cos\frac{\sqrt{\omega}}{c}x - \frac{\omega}{c^2} C_2 \sin\frac{\sqrt{\omega}}{c}x + \frac{\omega}{c^2} C_3 \cosh\frac{\sqrt{\omega}}{c}x + \frac{\omega}{c^2} C_4 \sinh\frac{\sqrt{\omega}}{c}x$$

$$\left.\begin{aligned} \varphi(0)=0 &\Rightarrow C_1 + C_3 = 0 \\ \varphi''(0)=0 &\Rightarrow -C_1 + C_3 = 0 \end{aligned}\right\} \Rightarrow C_1 = C_3 = 0$$

$$\left.\begin{aligned} \varphi(L)=0 &\Rightarrow C_2 \sin\frac{\sqrt{\omega}}{c}L + C_4 \sinh\frac{\sqrt{\omega}}{c}L = 0 \\ \varphi''(L)=0 &\Rightarrow -C_2 \sin\frac{\sqrt{\omega}}{c}L + C_4 \sinh\frac{\sqrt{\omega}}{c}L = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} & C_2 \sin\frac{\sqrt{\omega}L}{c} = 0 \\ & C_4 \sinh\frac{\sqrt{\omega}L}{c} = 0 \Rightarrow C_4 = 0 \end{aligned}$$



Hence, for nontrivial solutions we require that $\boxed{\sin\frac{\sqrt{\omega}L}{c} = 0}$ (frequency equation)

$$\frac{\sqrt{\omega_k}L}{c} = k\pi \Rightarrow \sqrt{\omega_k} = \frac{k\pi c}{L} \Rightarrow \omega_k = \left(\frac{k\pi c}{L}\right)^2, \quad k=1,2,\dots$$

Hence we derive the modes,

$$\left\{\omega_k = \left(\frac{k\pi c}{L}\right)^2, \quad \varphi_k(x) = C_k \sin\frac{\sqrt{\omega_k}}{c}x\right\}, \quad k=1,2,\dots$$

We mass-orthonormalize $\Rightarrow$

$$\Rightarrow \int_0^L m(x)\,\varphi_k^2(x)\,dx = 1\,, \quad k=1,\dots,\infty \Longrightarrow m C_k^2 \int_0^L \sin^2 \frac{k\pi x}{c}\,dx = 1 \Rightarrow$$

$$\Rightarrow C_k = \sqrt{\frac{2}{mL}}\,, \quad k=1,2,\dots$$

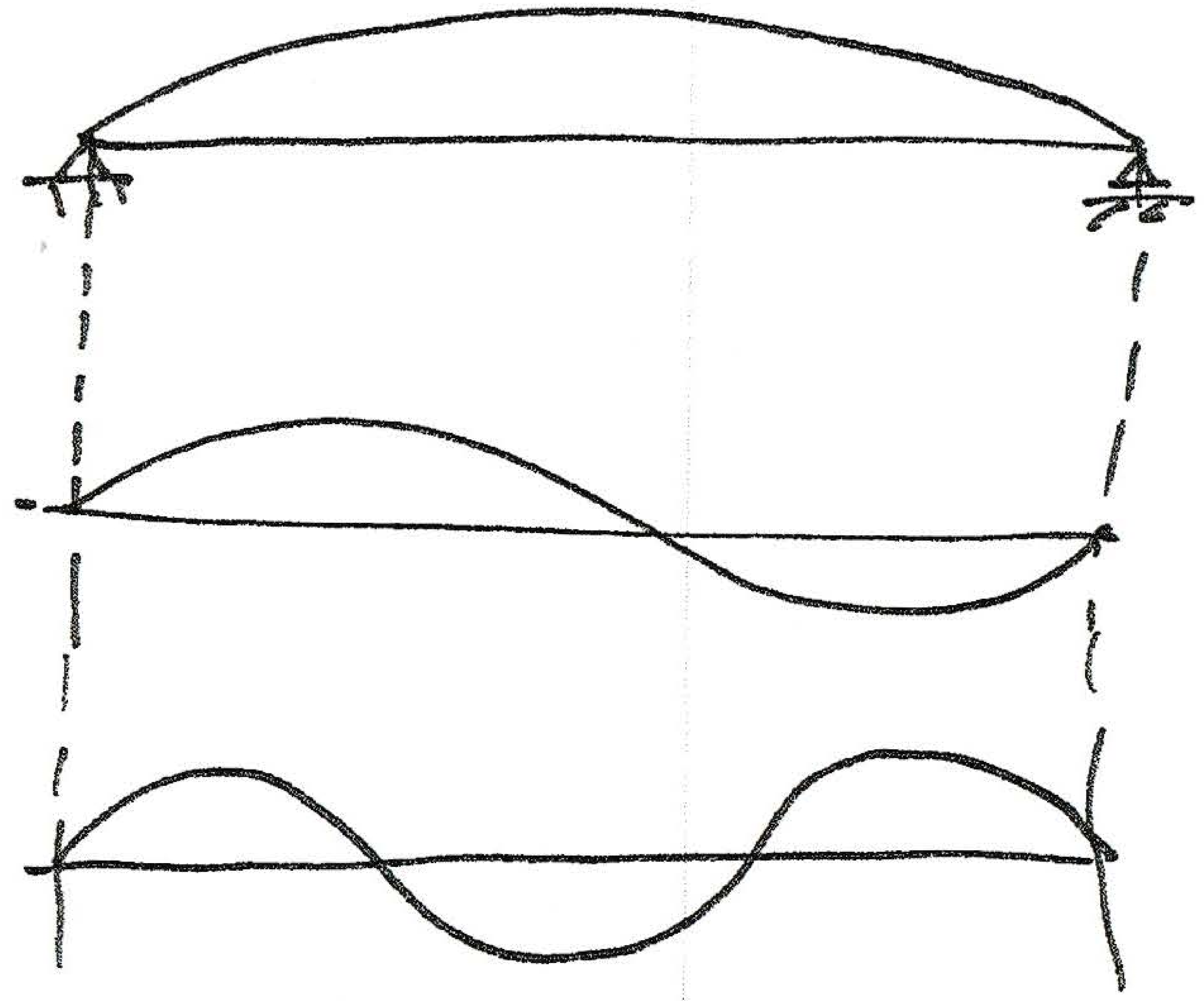
Hence, the mass-normalized eigenfunctions are,

$$\varphi_k(x) = \sqrt{\frac{2}{mL}}\,\sin\frac{\sqrt{\omega_k}}{c}x\,, \quad k=1,2,\dots$$

$\omega_1 = \left(\frac{\pi c}{L}\right)^2,\ \varphi_1(x)$

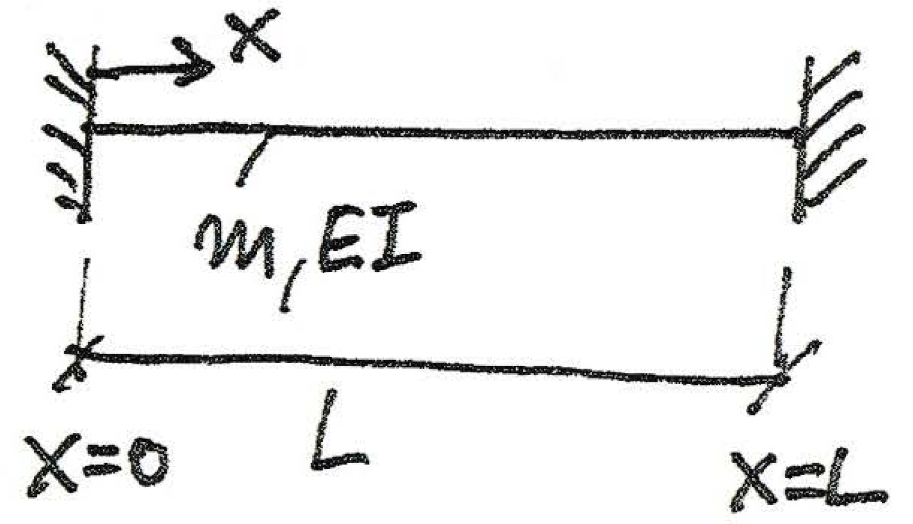
$\omega_2 = \left(\frac{2\pi c}{L}\right)^2,\ \varphi_2(x)$

$\omega_3 = \left(\frac{3\pi c}{L}\right)^2,\ \varphi_3(x)$



Then we can perform modal analysis in the usual way!

Example 2



The eigenvalue problem is as follows:

$$\varphi^{IV}(x) - \frac{\omega^2}{c^4}\varphi(x) = 0, \qquad c^4 = \frac{EI}{m} \qquad (1)$$

$$\left.\begin{array}{l} \varphi(0)=0 \\ \varphi'(0)=0 \\ \varphi(L)=0 \\ \varphi'(L)=0 \end{array}\right\} \qquad (1a)$$

The general solution of (1) is given by,

$$\varphi(x) = C_1 \cos\frac{\sqrt{\omega}}{c}x + C_2 \sin\frac{\sqrt{\omega}}{c}x + C_3 \cosh\frac{\sqrt{\omega}}{c}x + C_4 \sinh\frac{\sqrt{\omega}}{c}x \qquad (2)$$

$$\varphi'(x) = -C_1\frac{\sqrt{\omega}}{c}\sin\frac{\sqrt{\omega}}{c}x + C_2\frac{\sqrt{\omega}}{c}\cos\frac{\sqrt{\omega}}{c}x + C_3\frac{\sqrt{\omega}}{c}\sinh\frac{\sqrt{\omega}}{c}x + C_4\frac{\sqrt{\omega}}{c}\cosh\frac{\sqrt{\omega}}{c}x$$

$$\varphi(0)=0 \Rightarrow C_1 + C_3 = 0 \Rightarrow \quad C_1 = -C_3$$

$$\varphi'(0)=0 \Rightarrow C_2 + C_4 = 0 \Rightarrow \quad C_2 = -C_4$$

$$\varphi(L)=0 \Rightarrow C_1\cos\frac{\sqrt{\omega}}{c}L + C_2\sin\frac{\sqrt{\omega}}{c}L - C_1\cosh\frac{\sqrt{\omega}}{c}L - C_2\sinh\frac{\sqrt{\omega}}{c}L = 0 \Rightarrow$$

$$\Rightarrow C_1\left(\cos\frac{\sqrt{\omega}}{c}L - \cosh\frac{\sqrt{\omega}}{c}L\right) + C_2\left(\sin\frac{\sqrt{\omega}}{c}L - \sinh\frac{\sqrt{\omega}}{c}L\right) = 0$$

$$\varphi'(L)=0 \Rightarrow -C_1\frac{\sqrt{\omega}}{c}\sin\frac{\sqrt{\omega}}{c}L + C_2\frac{\sqrt{\omega}}{c}\cos\frac{\sqrt{\omega}}{c}L - C_1\frac{\sqrt{\omega}}{c}\sinh\frac{\sqrt{\omega}}{c}L - C_2\frac{\sqrt{\omega}}{c}\cosh\frac{\sqrt{\omega}}{c}L = 0 \Rightarrow$$

$$\Rightarrow C_1\frac{\sqrt{\omega}}{c}\left(-\sin\frac{\sqrt{\omega}}{c}L - \sinh\frac{\sqrt{\omega}}{c}L\right) + C_2\frac{\sqrt{\omega}}{c}\left(\cos\frac{\sqrt{\omega}}{c}L - \cosh\frac{\sqrt{\omega}}{c}L\right) = 0$$

$$\Rightarrow \begin{bmatrix} \cos\frac{\sqrt{\omega}}{c}L - \cosh\frac{\sqrt{\omega}}{c}L & \sin\frac{\sqrt{\omega}}{c}L - \sinh\frac{\sqrt{\omega}}{c}L \\ -\sin\frac{\sqrt{\omega}}{c}L - \sinh\frac{\sqrt{\omega}}{c}L & \cos\frac{\sqrt{\omega}}{c}L - \cosh\frac{\sqrt{\omega}}{c}L \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

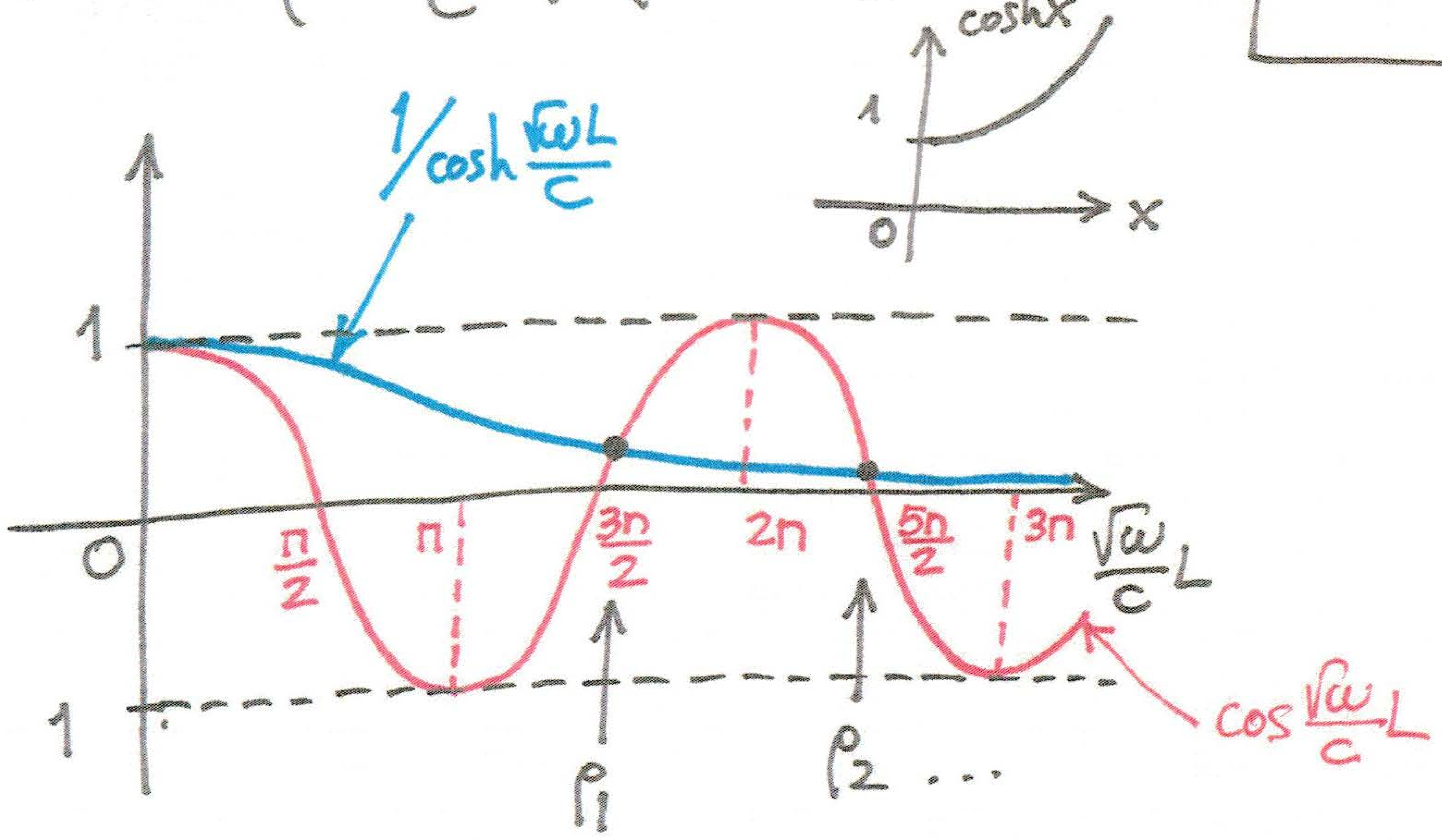
$[A]$

For nontrivial solutions we require that the determinant of coefficients should vanish $\Rightarrow \det[A]=0 \Rightarrow \left(\cos\frac{\sqrt{\omega}}{c}L - \cosh\frac{\sqrt{\omega}}{c}L\right)^2 + \left(\sin^2\frac{\sqrt{\omega}}{c}L - \sinh^2\frac{\sqrt{\omega}}{c}L\right) = 0$

Then taking account that $\cos^2\frac{\sqrt{\omega}}{c}L + \sin^2\frac{\sqrt{\omega}}{c}L = 1$, $\cosh^2\frac{\sqrt{\omega}}{c}L - \sinh^2\frac{\sqrt{\omega}}{c}L = 1 \Rightarrow$

$\Rightarrow \left(\cos\frac{\sqrt{\omega}}{c}L\right)\cdot\left(\cosh\frac{\sqrt{\omega}}{c}L\right) = 1 \Rightarrow \boxed{\cos\frac{\sqrt{\omega}}{c}L = \frac{1}{\cosh\frac{\sqrt{\omega}L}{c}}}$ frequency equation

$\neq 0$ for $0<\omega<\infty$



$\frac{3\pi}{2} < \rho_1 < 2\pi$, $\rho_1 = 4.73$

$2\pi < \rho_2 < \frac{5\pi}{2}$, $\rho_2 = 7.853$

$\frac{7\pi}{2} < \rho_3 < 4\pi$, $\rho_3 = 10.996$

$4\pi < \rho_4 < \frac{9\pi}{2}$, $\rho_4 = 14.137$

...

Then, $p_i = \frac{\sqrt{\omega_i}}{c} L \Rightarrow \omega_i = \left(\frac{p_i c}{L}\right)^2, \quad i = 1, 2, \cdots$

Corresponding to each root p_i we compute the ratio,

$$(\cos p_i - \cosh p_i)\, C_1^{(i)} + (\sin p_i - \sinh p_i)\, C_2^{(i)} = 0 \Rightarrow$$

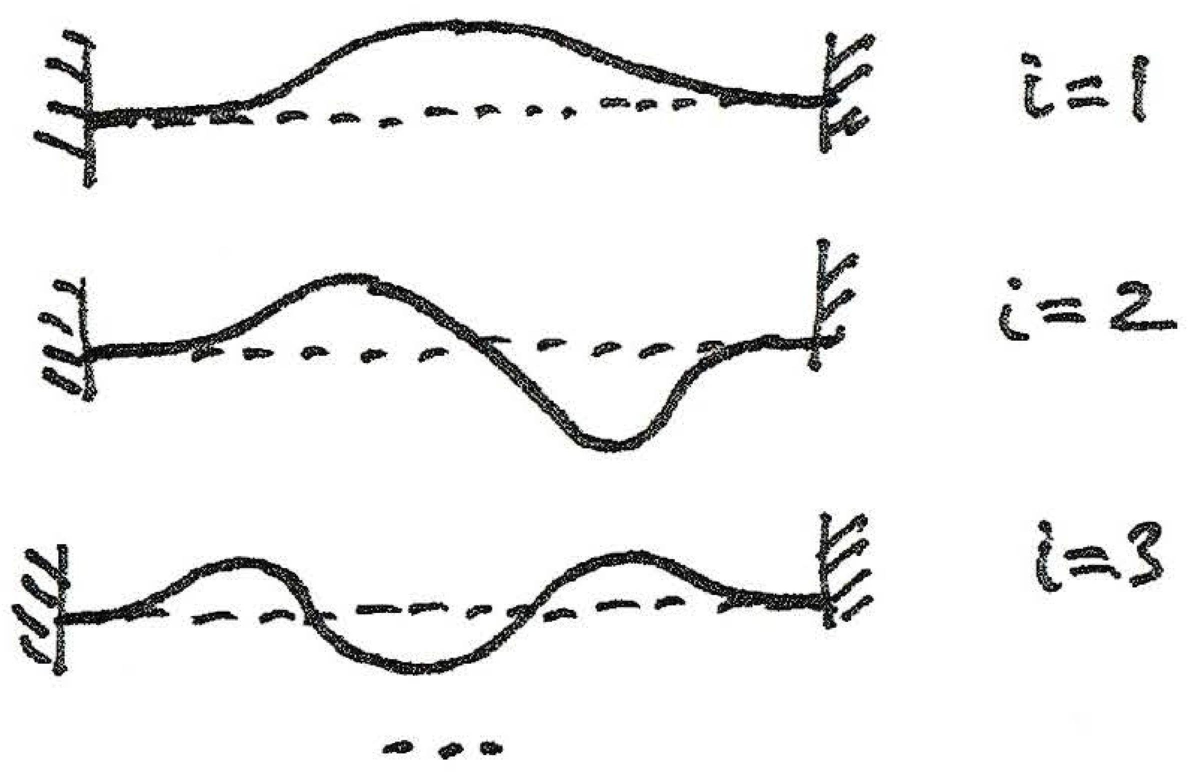
$$\Rightarrow C_2^{(i)} = -\left(\frac{\cos p_i - \cosh p_i}{\sin p_i - \sinh p_i}\right) C_1^{(i)}, \quad C_3^{(i)} = -C_1^{(i)}, \quad C_4^{(i)} = -C_2^{(i)} \Rightarrow$$

$\Rightarrow$ The i-th eigenfunction is

$$\varphi_i(x) = C_1^{(i)}\left[\cos p_i \frac{x}{L} - \cosh p_i \frac{x}{L}\right] - \left(\frac{\cos p_i - \cosh p_i}{\sin p_i - \sinh p_i}\right) C_1^{(i)} \left[\sin p_i \frac{x}{L} - \sinh p_i \frac{x}{L}\right] \Rightarrow$$

$\Rightarrow$ We may normalize the eigenfunction according to,

$$\int_0^L m\, \varphi_i^2(x)\, dx = 1 \Rightarrow \text{Compute } C_1^{(i)}$$

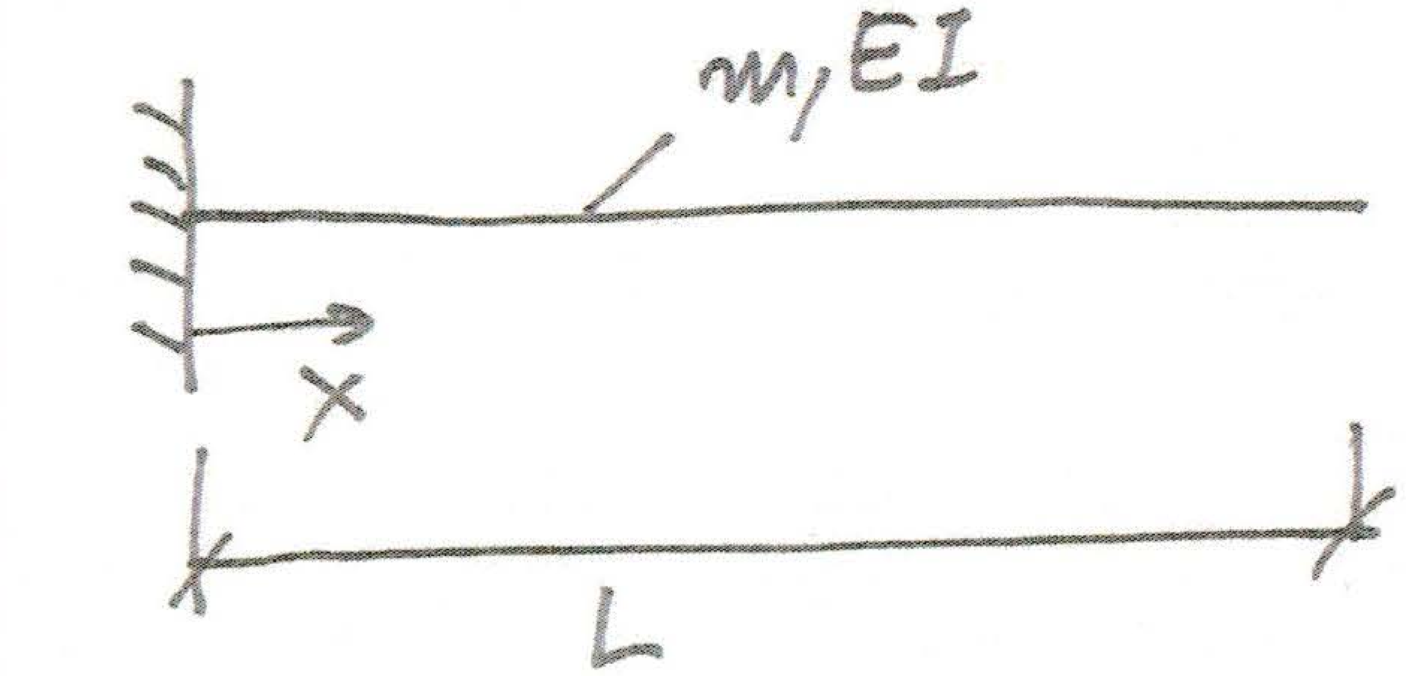


To compute the free response due to initial conditions, or the forced response use modal analysis.

$$u(x,t) = \sum_{i=1}^{\infty} \alpha_i(t)\, \varphi_i(x)$$

mass-orthonormalized eigenfunctions

Example 3

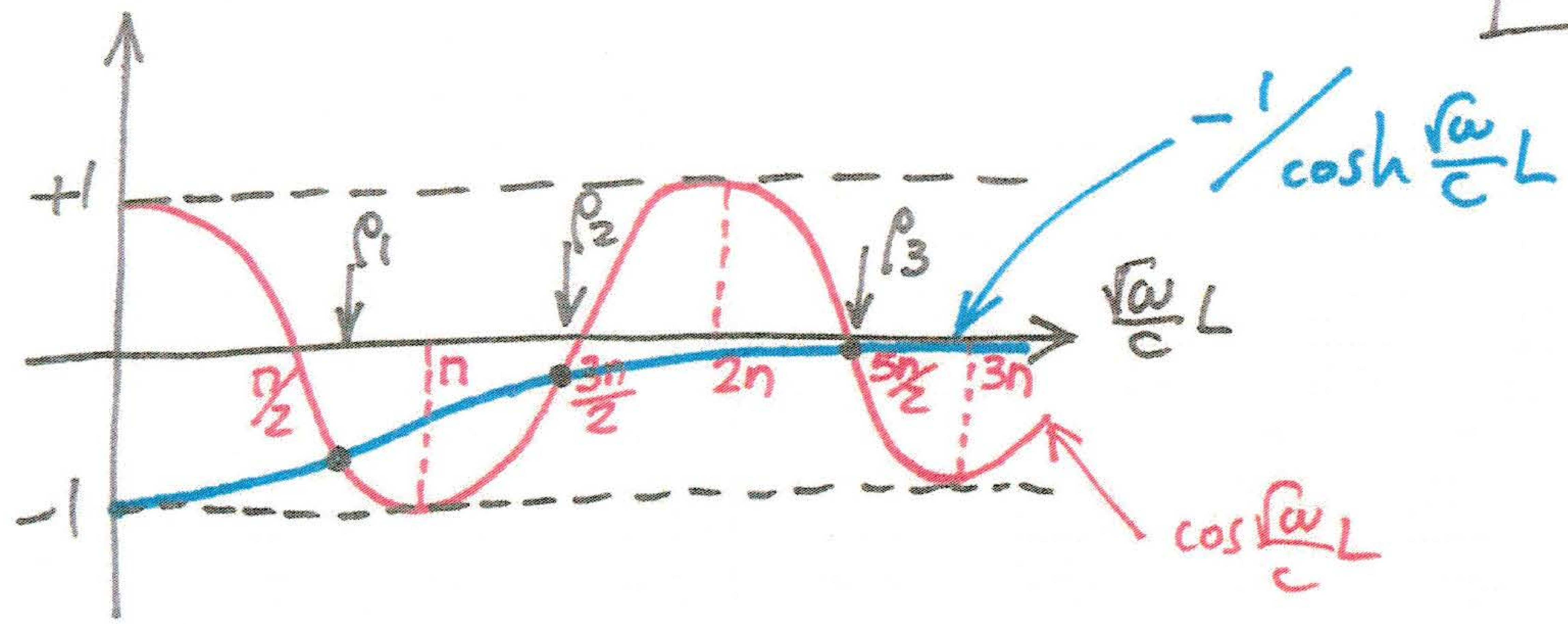


$$\left.\begin{aligned} &\varphi^{IV}(x) - \frac{\omega^2}{c^4}\varphi(x) = 0 \\ &\varphi(0) = 0 \\ &\varphi'(0) = 0 \\ &\varphi'''(L) = 0 \\ &\varphi''(L) = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \varphi(x) = C_1 \cos\frac{\sqrt{\omega}}{c}x + C_2 \sin\frac{\sqrt{\omega}}{c}x + C_3 \cosh\frac{\sqrt{\omega}}{c}x + C_4 \sinh\frac{\sqrt{\omega}}{c}x \Rightarrow$$

$\Rightarrow$ Applying the boundary conditions we get the frequency equation:

$$\left(\cos\frac{\sqrt{\omega}}{c}L\right)\left(\cosh\frac{\sqrt{\omega}}{c}L\right) = -1 \Rightarrow \boxed{\cos\frac{\sqrt{\omega}}{c}L = -\frac{1}{\cosh\frac{\sqrt{\omega}}{c}L}}$$



$$\frac{\pi}{2} < \rho_1 = 1.873 < \pi$$

$$\pi < \rho_2 = 4.694 < 3\pi/2$$

$$\frac{5\pi}{2} < \rho_3 = 7.855 < 3\pi$$

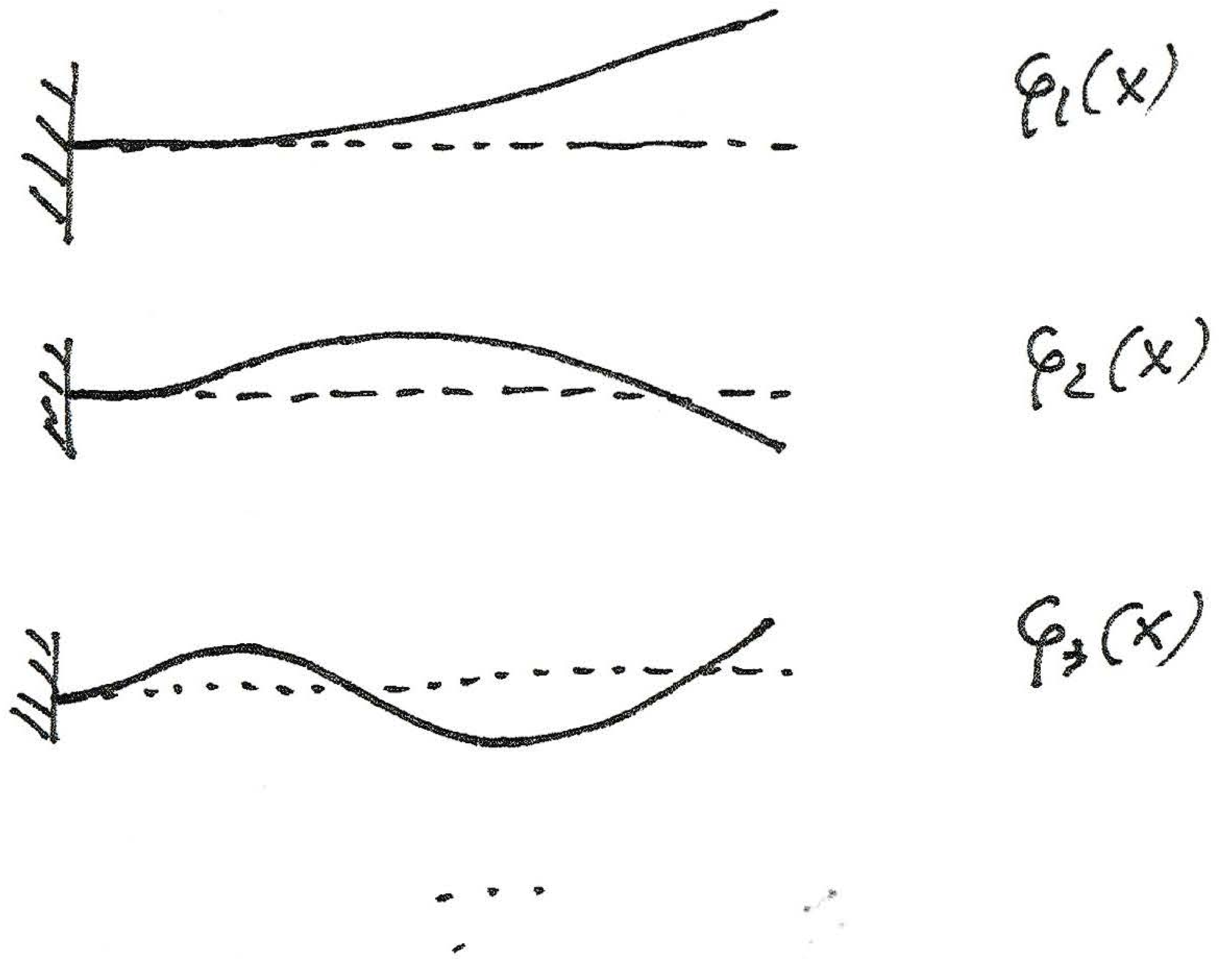
$$3\pi < \rho_4 = 10.996 < 7\pi/2$$

...

We note that the natural frequencies have decreased compared to the clamped-clamped beam of example 2; this is due to the fact that by relaxing one of the boundary conditions we decreased the stiffness of the system. To compute the modes in this case we use the relations

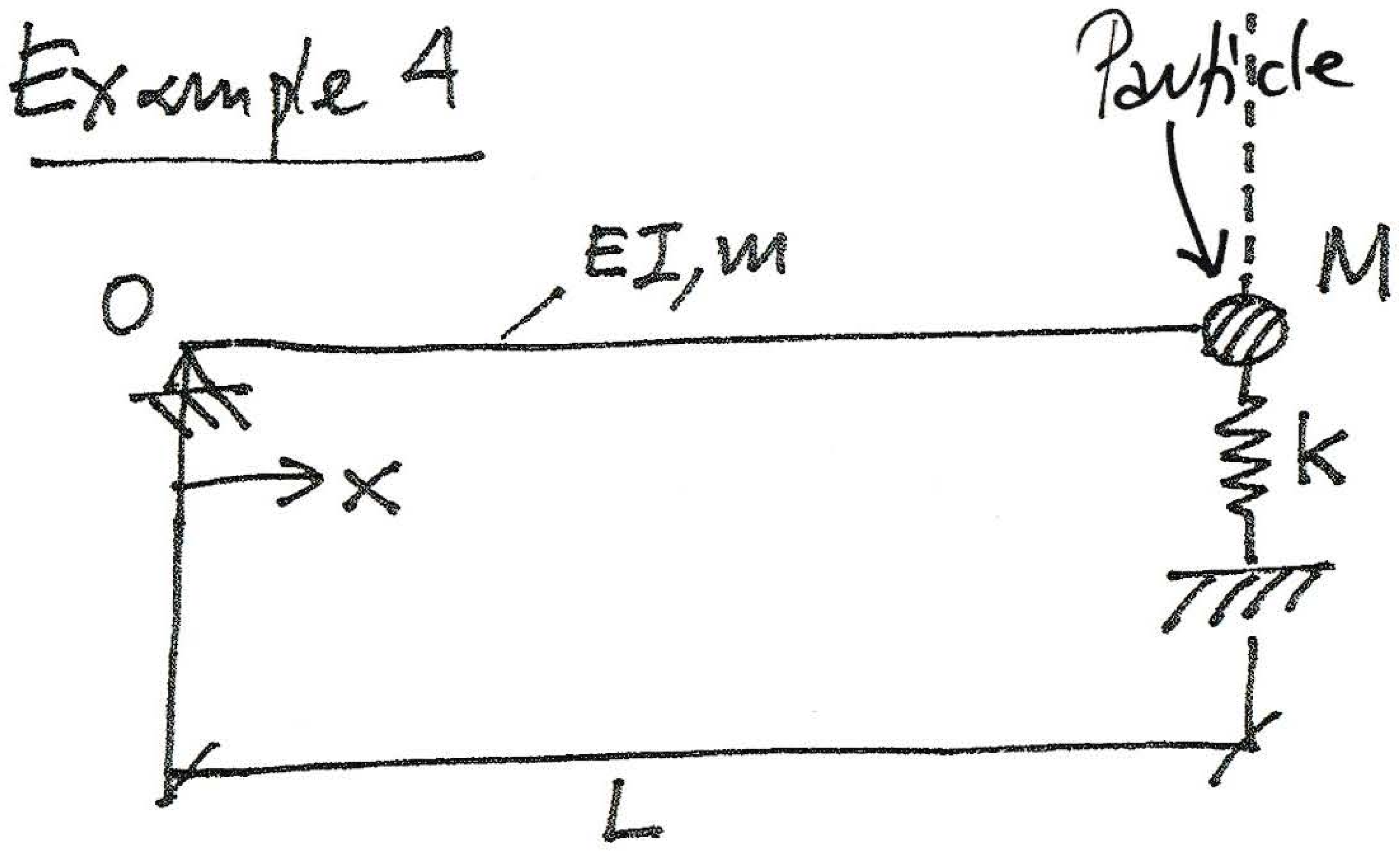
$$c_1 = -c_3, \quad c_2 = -c_4$$

$$\begin{bmatrix} \cos\frac{\sqrt{\omega}}{c}L + \cosh\frac{\sqrt{\omega}}{c}L & \sin\frac{\sqrt{\omega}}{c}L + \sinh\frac{\sqrt{\omega}}{c}L \\ \sin\frac{\sqrt{\omega}}{c}L - \sinh\frac{\sqrt{\omega}}{c}L & -\cos\frac{\sqrt{\omega}}{c}L - \cosh\frac{\sqrt{\omega}}{c}L \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



We note that as the order of the modes increases, there is an increasing number of nodes in the corresponding mode shapes. However, there is no node of any mode at the free end $\Rightarrow$ By forcing a cantilever beam by a point force at its end we excite all modes!

Example 4



We want to study the free oscillations of this system.

Equation of motion is

$$-EI\frac{\partial^4 v}{\partial x^4} = m\frac{\partial^2 v}{\partial t^2}, \quad 0\le x\le L, \; t\ge 0$$

$v(0,t)=0$ (Geometric BC)

$$\left.\begin{aligned} &\frac{\partial^2 v}{\partial x^2}(0,t)=0 \\ &\frac{\partial^2 v}{\partial x^2}(L,t)=0 \\ &EI\frac{\partial^3 v(L,t)}{\partial x^3} - k v(L,t) - M\frac{\partial^2 v(L,t)}{\partial t^2}=0 \end{aligned}\right\} \text{(Natural BCs)}$$

Separating space and time dependencies we formulate the following eigenvalue problem:

$$\left.\begin{aligned} &\varphi^{IV}(x) - \frac{\omega^2}{c^4}\varphi(x)=0, \quad c^4 = EI/m \\ &\varphi(0)=0 \\ &\varphi''(0)=0 \\ &\varphi''(L)=0 \\ &EI\varphi'''(L) - k\varphi(L) + \omega^2 M\varphi(L)=0 \end{aligned}\right\} \quad 0\le x\le L$$

$v(L,t) = f(t)\varphi(L) \Rightarrow$

$$\Rightarrow \frac{\partial^2 v(L,t)}{\partial t^2} = -\omega^2 f(t)\varphi(L)$$

To have a mode!

The general solution of the governing equation is

$$\varphi(x) = \cancel{C_1 \cos \frac{\sqrt{\omega}}{c} x} + C_2 \sin \frac{\sqrt{\omega}}{c} x + \cancel{C_3 \cosh \frac{\sqrt{\omega}}{c}} x + C_4 \sinh \frac{\sqrt{\omega}}{c} x$$

$$\left.\begin{aligned} \varphi(0)=0 &\Rightarrow C_1 + C_3 = 0 \\ \varphi''(0)=0 &\Rightarrow -C_1 + C_3 = 0 \end{aligned}\right\} \Rightarrow C_1 = C_3 = 0$$

$$\varphi''(L)=0 \Rightarrow -C_2 \sin \frac{\sqrt{\omega}}{c} L + C_4 \sinh \frac{\sqrt{\omega}}{c} L = 0$$

$$EI\varphi'''(L) - k\varphi(L) + \omega^2 M \varphi(L) = 0 \Rightarrow$$

$$\Rightarrow EI\left(\frac{\sqrt{\omega}}{c}\right)^3 \left(-C_2 \cos \frac{\sqrt{\omega}}{c} L + C_4 \cosh \frac{\sqrt{\omega}}{c} L\right) - (k - \omega^2 M)\left(C_2 \sin \frac{\sqrt{\omega}}{c} L + C_4 \sinh \frac{\sqrt{\omega}}{c} L\right) = 0 \Rightarrow$$

$$\Rightarrow C_2\left[-\left(\frac{\sqrt{\omega}}{c}L\right)^3 \cos \frac{\sqrt{\omega}}{c} L - \left(\frac{kL^3}{EI} - \frac{M}{mL}\left(\frac{\sqrt{\omega}}{c}L\right)^4\right) \sin \frac{\sqrt{\omega}}{c} L\right] +$$

$$+ C_4\left[\left(\frac{\sqrt{\omega}}{c}L\right)^3 \cosh \frac{\sqrt{\omega}}{c} L - \left(\frac{kL^3}{EI} - \frac{M}{mL}\left(\frac{\sqrt{\omega}}{c}L\right)^4\right) \sinh \frac{\sqrt{\omega}}{c} L\right] = 0$$

At this point we introduce the nondimensional quantities,

$$x = \frac{\sqrt{\omega}}{c} L, \quad q = \frac{kL^3}{EI}, \quad \mu = \frac{M}{mL}$$

Normalized frequency

Stiffness ratio

Mass ratio

Then we obtain the following matrix equation:

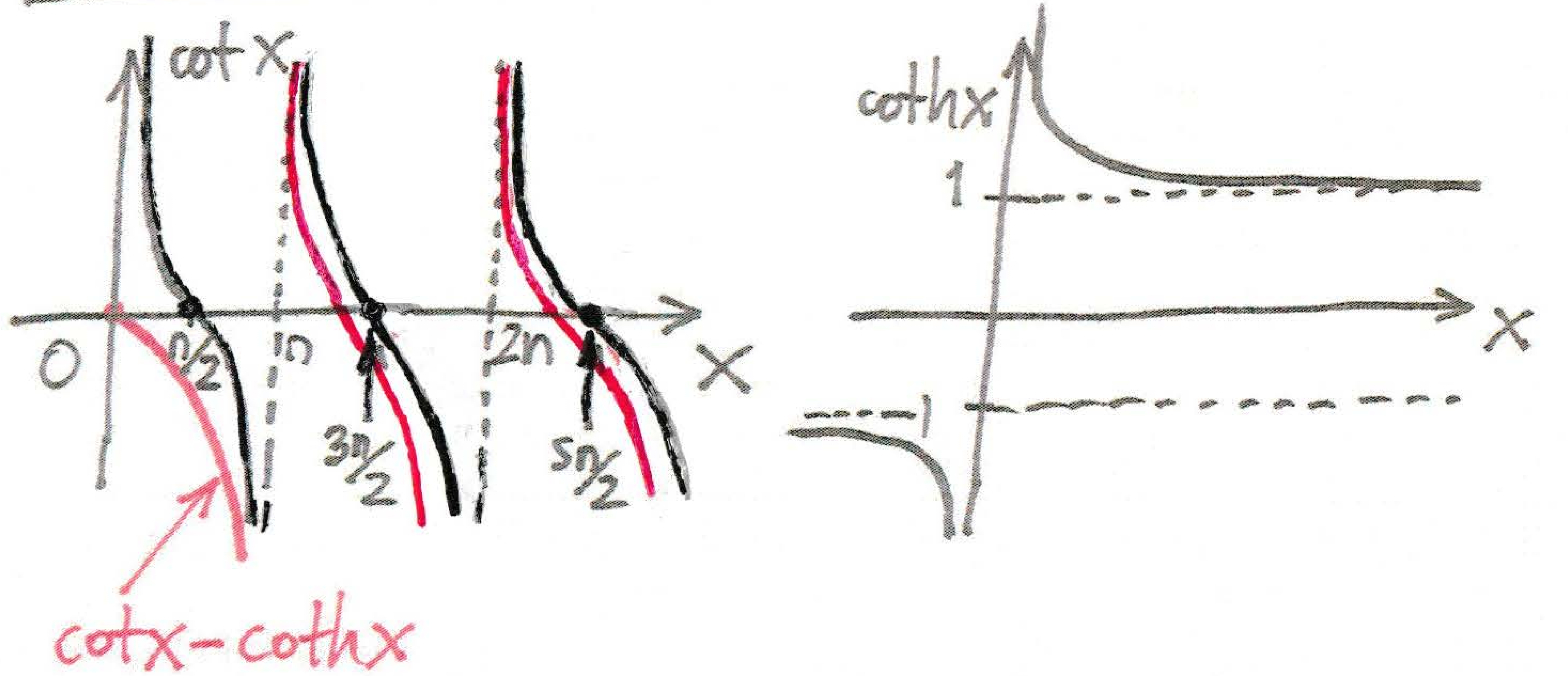
$$\underbrace{\begin{bmatrix} -\sin x & \sinh x \\ -x^3\cos x-(q-\mu x^4)\sin x & x^3\cosh x-(q-\mu x^4)\sinh x \end{bmatrix}}_{[A]}\begin{Bmatrix} C_2 \\ C_4 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For nontrivial solutions we require that $\det[A]=0 \Rightarrow$

$$\Rightarrow -x^3\sin x\cosh x+(q-\mu x^4)\sin x\sinh x+x^3\cos x\sinh x+(q-\mu x^4)\sin x\sinh x=0$$

$\Rightarrow$ Dividing by $\sin x\sinh x \Rightarrow -x^3\coth x+(q-\mu x^4)+x^3\cot x+(q-\mu x^4)=0 \Rightarrow$

$$\Rightarrow \boxed{\cot x-\coth x=2\mu x-\frac{2q}{x^3}}$$ (Frequency equation)

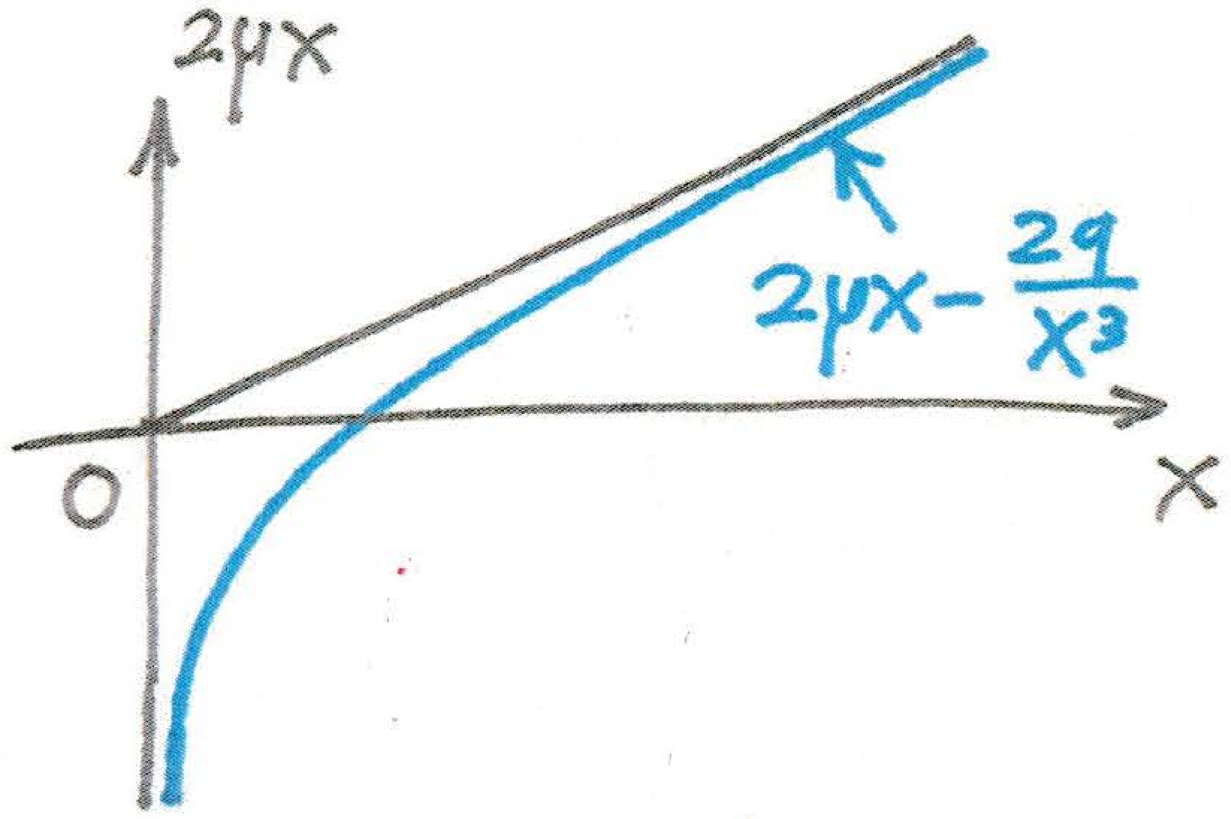


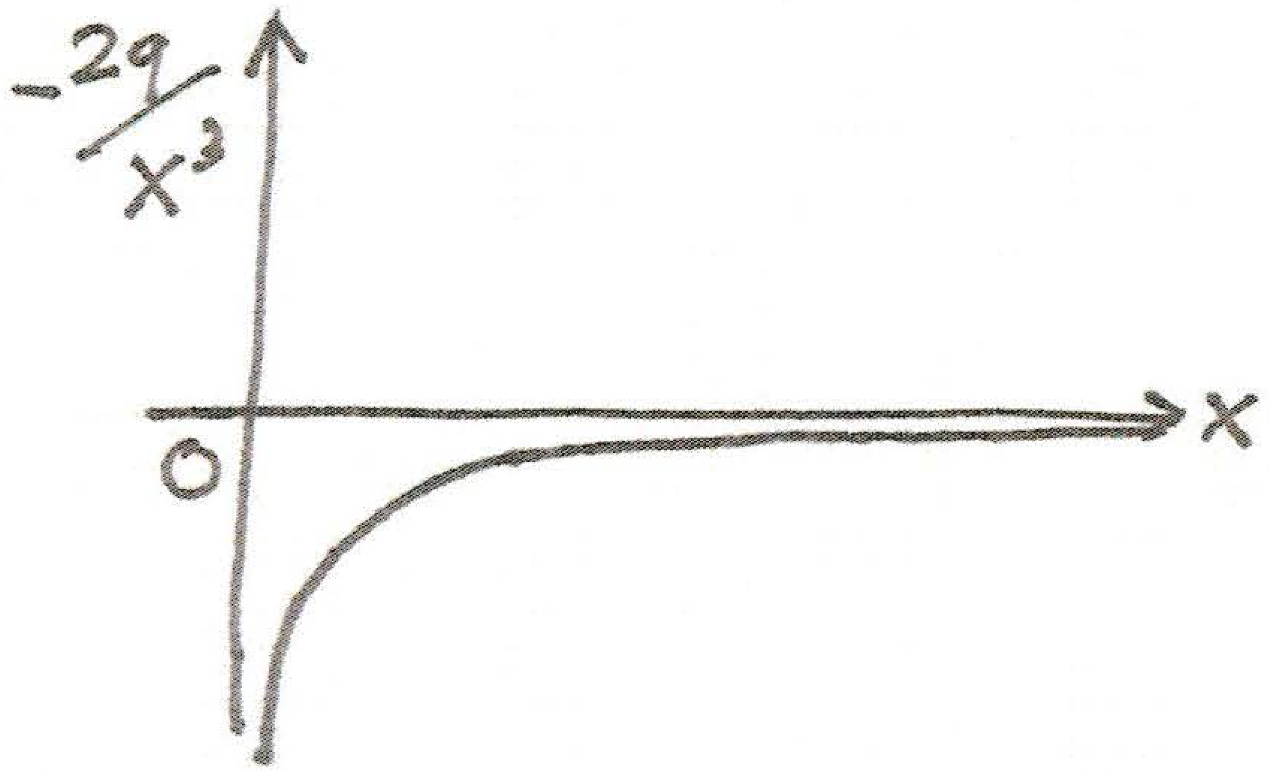
As $x\to 0+ \Rightarrow \cot x \sim \frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}+\dots$

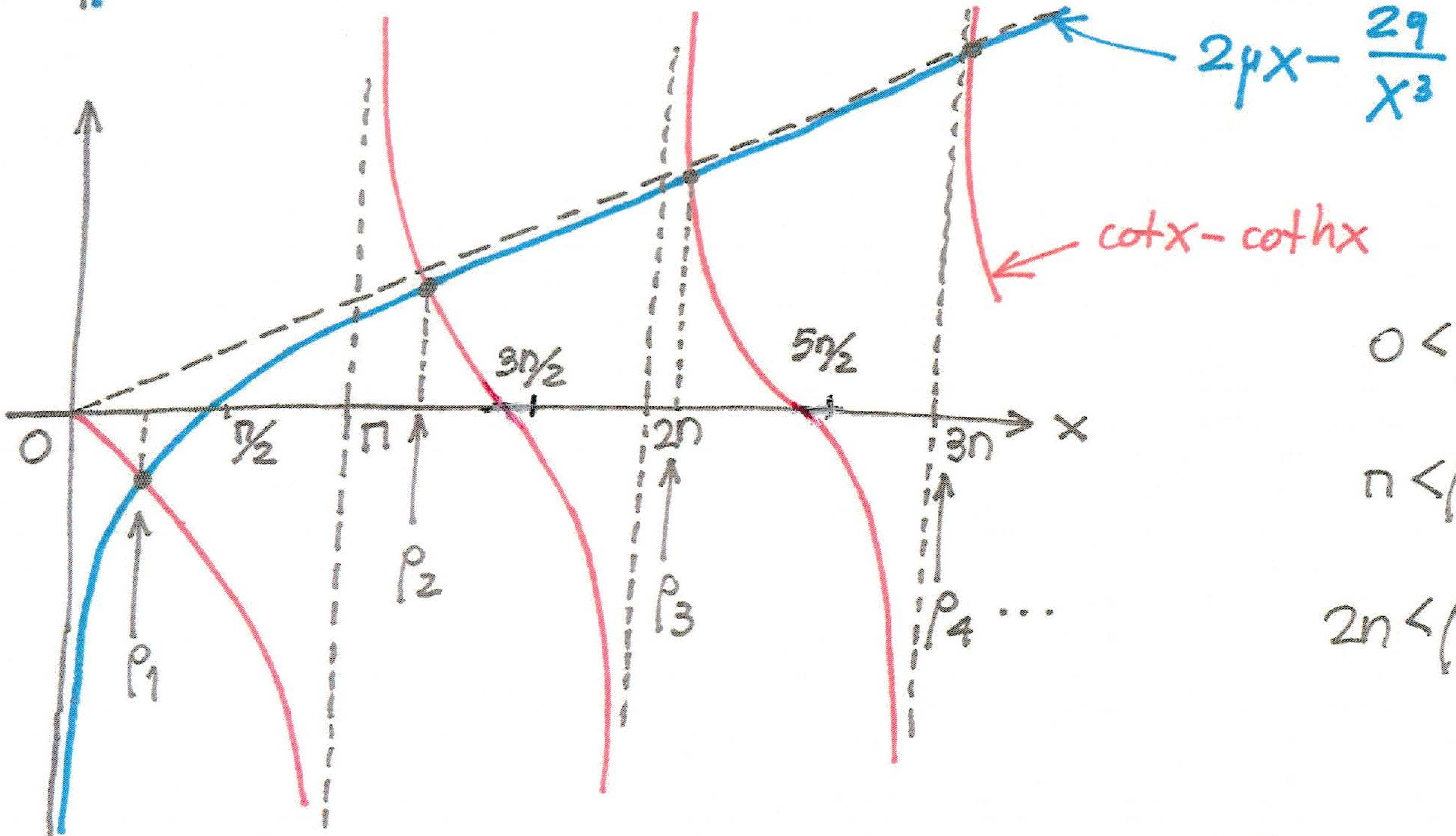
$\coth x \sim \frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\dots$ $\Rightarrow$

$\Rightarrow$ As $x\to 0+ \Rightarrow$

$\Rightarrow \cot x-\coth x \sim -\frac{2x}{3}+\dots$







$$0 < \rho_1 = \frac{\sqrt{\omega_1}\,L}{c} < \pi$$

$$\pi < \rho_2 = \frac{\sqrt{\omega_2}}{c}L < 2\pi$$

$$2\pi < \rho_3 = \frac{\sqrt{\omega_3}\,L}{c} < 3\pi$$

$$\vdots$$

Then, the corresponding mode shapes are computed by,

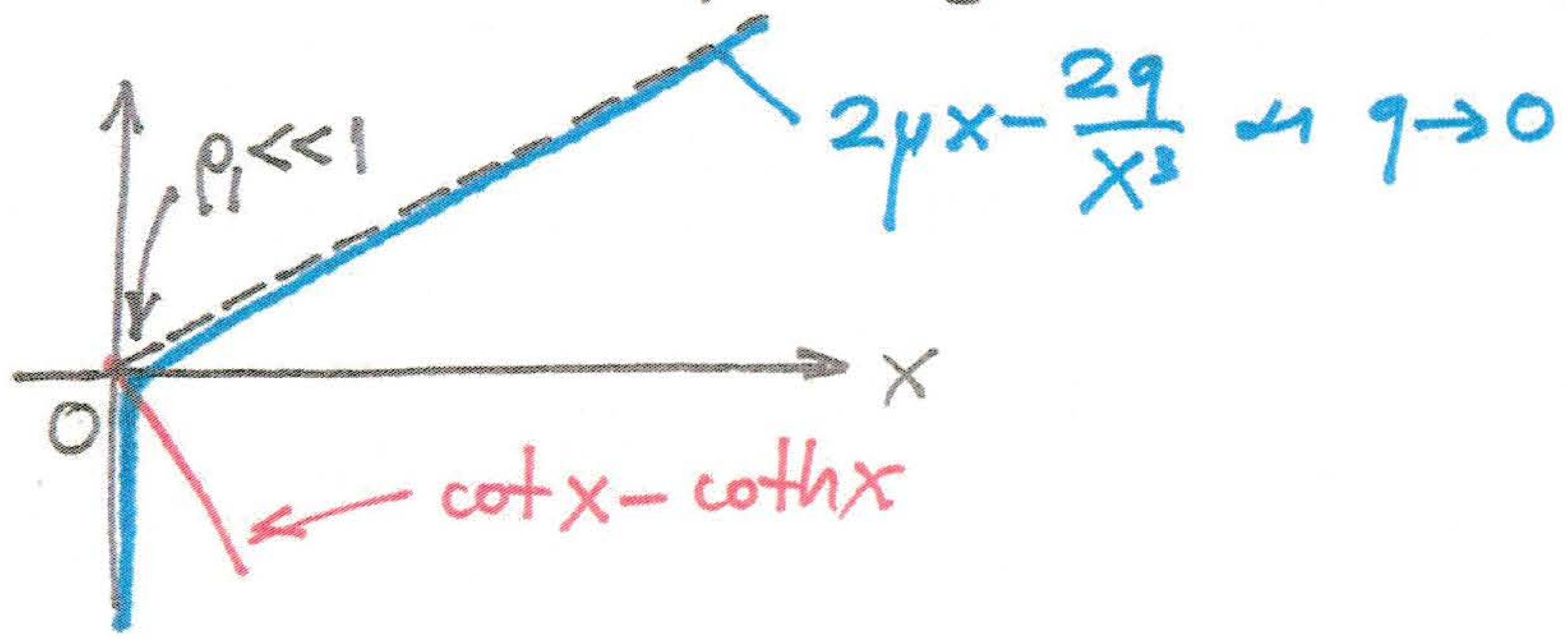
$$C_2^{(i)} = C_4^{(i)} \frac{\sinh\frac{\sqrt{\omega_i}}{c}L}{\sin\frac{\sqrt{\omega_i}}{c}L} \Rightarrow \varphi_i(x) = C_4^{(i)}\left[\frac{\sinh\rho_i}{\sin\rho_i}\sin\rho_i\frac{x}{L} + \sinh\frac{\rho_i x}{L}\right],$$

$$i = 1, 2, \dots$$

Now we may study certain limiting cases. Recall frequency equation,

$$\boxed{\cot X - \coth X = 2\mu X - \frac{2q}{X^3}}$$

Case 1: Let $q = \frac{kL^3}{EI} \to 0 \Rightarrow$ This can be interpreted as a very 'rigid beam'. In this case $\rho_1 \to 0 \Rightarrow$ Then the frequency equation can be expanded for $X \to 0$ as follows:

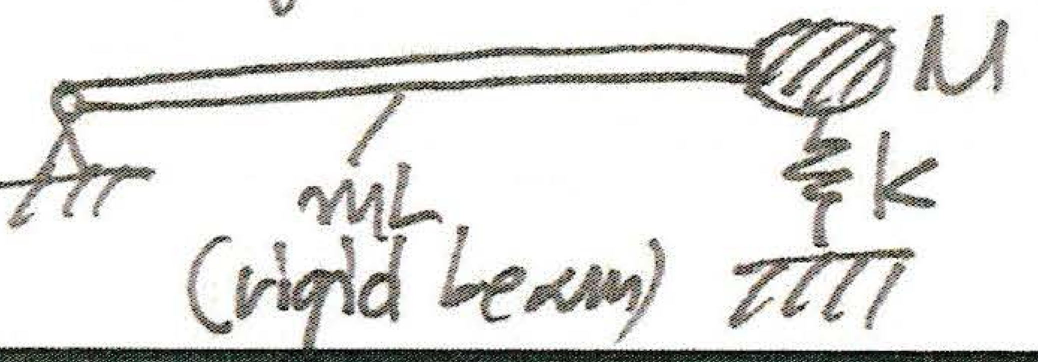


$$-\frac{2X}{3} + \ldots \sim 2\mu X - \frac{2q}{X^3} \sim -\frac{2q}{X^3} \Rightarrow$$

$$\Rightarrow X^4 \sim \frac{3q}{3\mu+1} \Rightarrow \frac{\omega_1^2}{C^4} L^4 \sim \frac{\frac{3kL^3}{EI}}{3\frac{M}{mL}+1} \Rightarrow$$

$$\Rightarrow \omega_1 \sim \sqrt{\frac{3k}{3M+mL}} = \sqrt{\frac{k}{M+\frac{mL}{3}}}$$

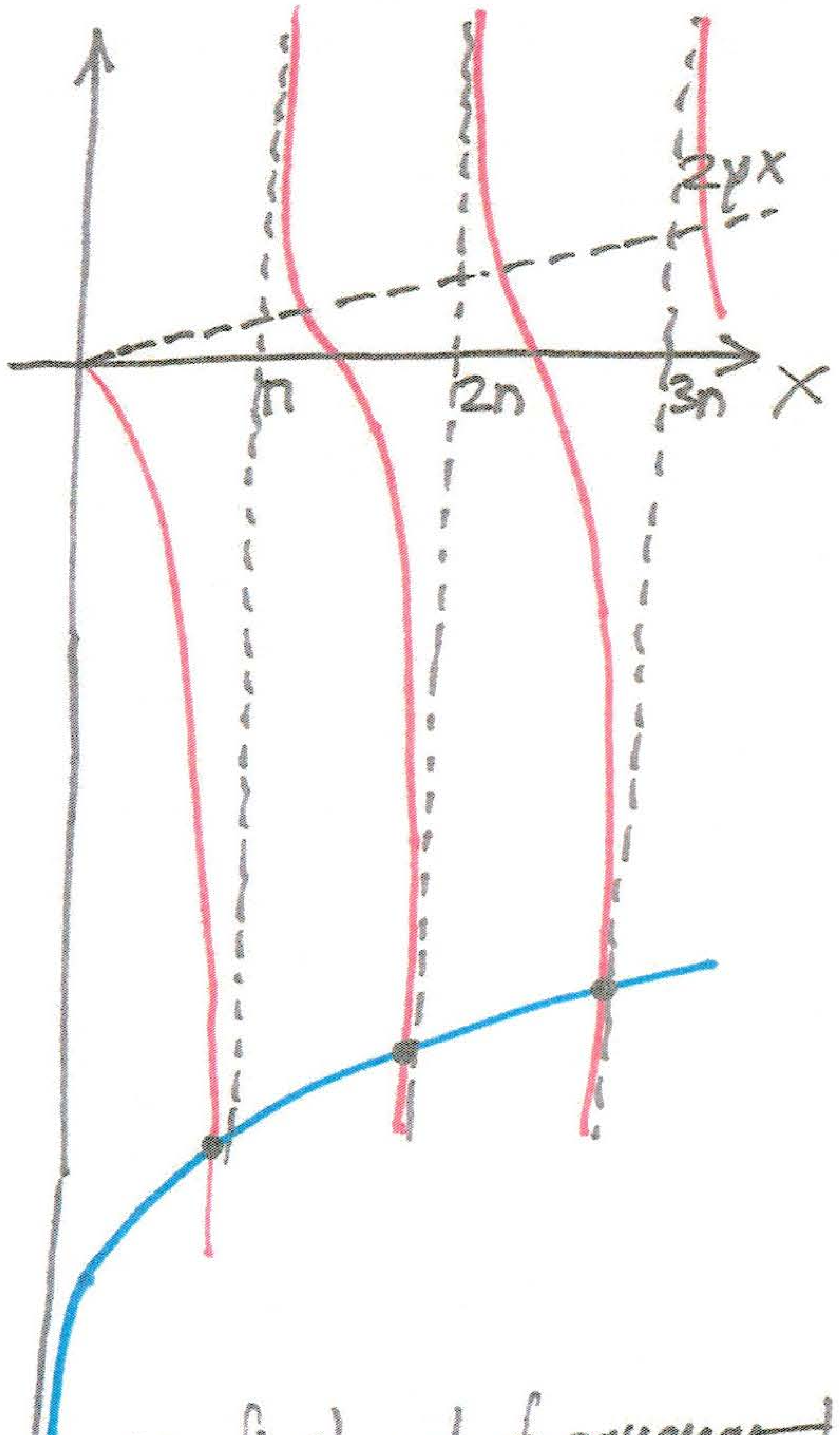
This is the natural frequency of a rigid uniform beam of mass distribution m with a spring mass at its right end



Remark
The rest of the roots are computed by
$\cot X - \coth X \sim 2\mu X \Rightarrow$

$\Rightarrow \rho_2 \gg \rho_1, \rho_3 \gg \rho_1, \ldots$

Case 2: Let $q = \frac{kL^3}{EI} \to \infty \Rightarrow$ This is the case of a very stiff end spring $\Rightarrow$ In this case $\rho_1 \sim \pi$, $\rho_2 \sim 2\pi$, $\rho_3 \sim 3\pi, \ldots \Rightarrow$ We approach the limit of a simply supported beam.



Case 3: When $\mu = \frac{M}{mL} \to 0 \Rightarrow$ We get the limit of a simply supported beam at its left boundary and a spring at its right boundary.

Case 4: When $\mu = \frac{M}{mL} \to \infty \Rightarrow$ The mass at the right boundary dominates the dynamics $\Rightarrow$

$\Rightarrow$ As $x \to 0+ \Rightarrow -\frac{2x}{3} + \ldots = 2\mu x - \frac{2q}{x^3} \Rightarrow$

$\Rightarrow \mu + \cancel{\frac{1}{3}} \sim \frac{q}{x^4} \Rightarrow \mu \sim \frac{q}{x^4} \Rightarrow x^4 \sim \frac{q}{\mu} \Rightarrow \omega_1 = \sqrt{\frac{K}{M}}$

for the first nat. frequency $\Rightarrow$ Limit of spring-mass oscillator!

for higher modes, $\rho_2, \rho_3, \rho_4, \ldots \gg \rho_1$