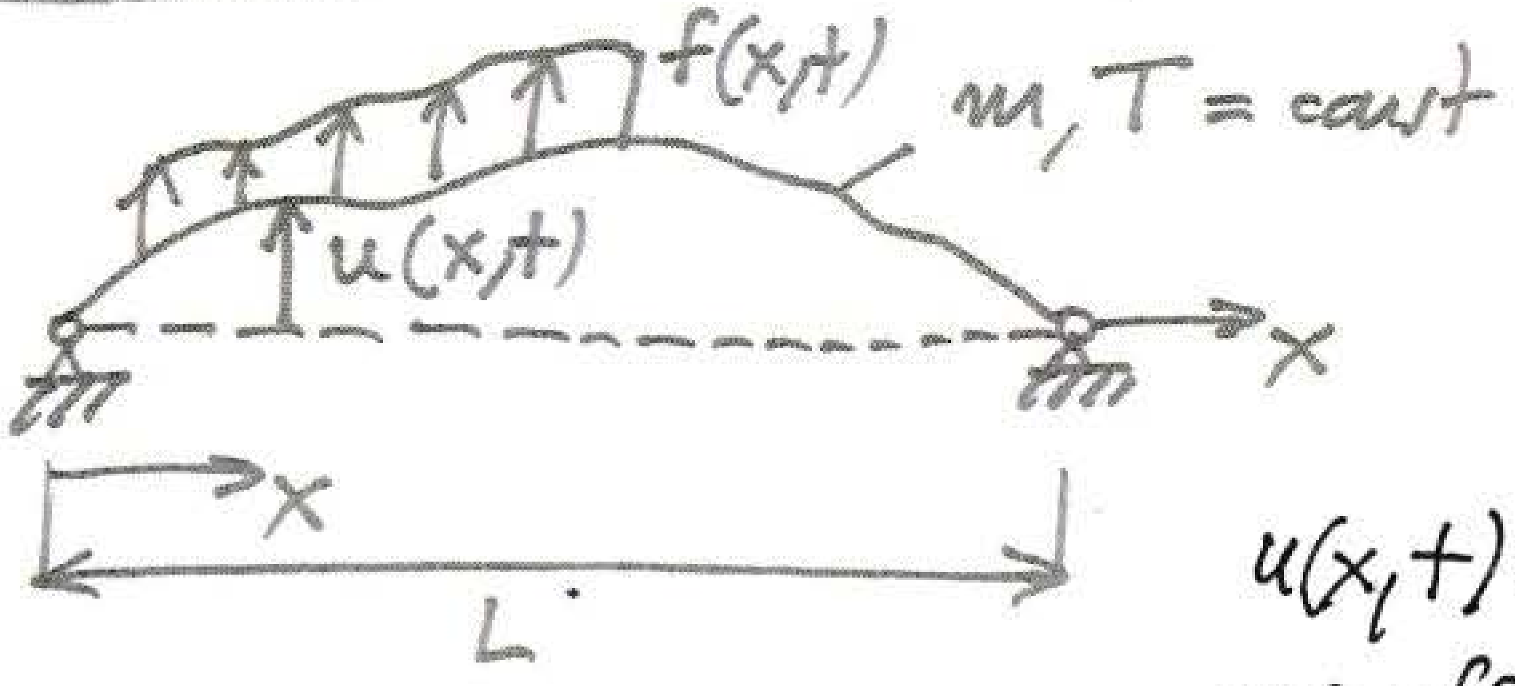
Example 1: fixed-fixed elastic string



The equation of motion in this case is,

$$u(x,t) = \varphi(x) f(t) \quad \begin{cases} T\dfrac{\partial^2 u}{\partial x^2} + F(x,t) = m\dfrac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le L, \; t \ge 0 & (1) \\[2ex] \text{BCs: } u(0,t) = u(L,t) = 0 & (1a) \\[1ex] \text{ICs: } u(x,0) = g(x), \; \dfrac{\partial u}{\partial t}(x,0) = h(x) & (1b) \end{cases}$$

First we consider the eigenvalue problem:

$$\left.\begin{aligned} &\frac{d^2\varphi(x)}{dx^2} + \left(\frac{\omega}{c}\right)^2 \varphi(x) = 0, \quad c = \sqrt{\frac{T}{m}} \\ &\varphi(0) = \varphi(L) = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \varphi(x) = c_1 \underbrace{\cos\frac{\omega}{c}x}_{\varphi_1(x,\omega)} + c_2 \underbrace{\sin\frac{\omega}{c}x}_{\varphi_2(x,\omega)}$$

$$\left.\begin{aligned} &\varphi(0) = 0 \Rightarrow c_1 = 0 \\ &\varphi(L) = 0 \Rightarrow c_1 \cos\frac{\omega L}{c} + c_2 \sin\frac{\omega L}{c} = 0 \end{aligned}\right\} \to \begin{bmatrix} 1 & 0 \\ \cos\frac{\omega L}{c} & \sin\frac{\omega L}{c} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\Rightarrow$ Setting the determinant of coefficients equal to zero $\Rightarrow \sin\frac{\omega L}{c} = 0 \Rightarrow$

$$\Rightarrow \frac{\omega L}{c} = k\pi, \; k = 1,2,\ldots \Rightarrow \boxed{\omega_k = \frac{k\pi c}{L}, \; k = 1,2\ldots}, \quad \omega_1 = \frac{\pi c}{L} < \omega_2 = \frac{2\pi c}{L} < \omega_3 < \ldots$$

Then, the corresponding eigenfunctions are $\varphi_r(x) = C_r \sin\omega_r x \Rightarrow$

$\Rightarrow$ Mass-orthonormalizing $\Rightarrow \displaystyle\int_0^L \underbrace{B(x)}_{m}\varphi_r^2(x)\,dx = 1 \Rightarrow \int_0^L C_r^2 \sin^2\omega_r x\,dx = \frac{1}{m} \Rightarrow$

$\Rightarrow C_r = \sqrt{\dfrac{2}{mL}} \Rightarrow \boxed{\varphi_r(x) = \sqrt{\dfrac{2}{mL}}\,\sin\dfrac{r\pi x}{L}, \quad r=1,2,\ldots}$ ← Mass-orthonormalized eigenfunction

Then, we express the solution of problem (1) as,

$$u(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\,\varphi_r(x) = \sum_{i=1}^{\infty} \eta_i(t)\sqrt{\frac{2}{mL}}\,\sin\frac{r\pi x}{L}$$

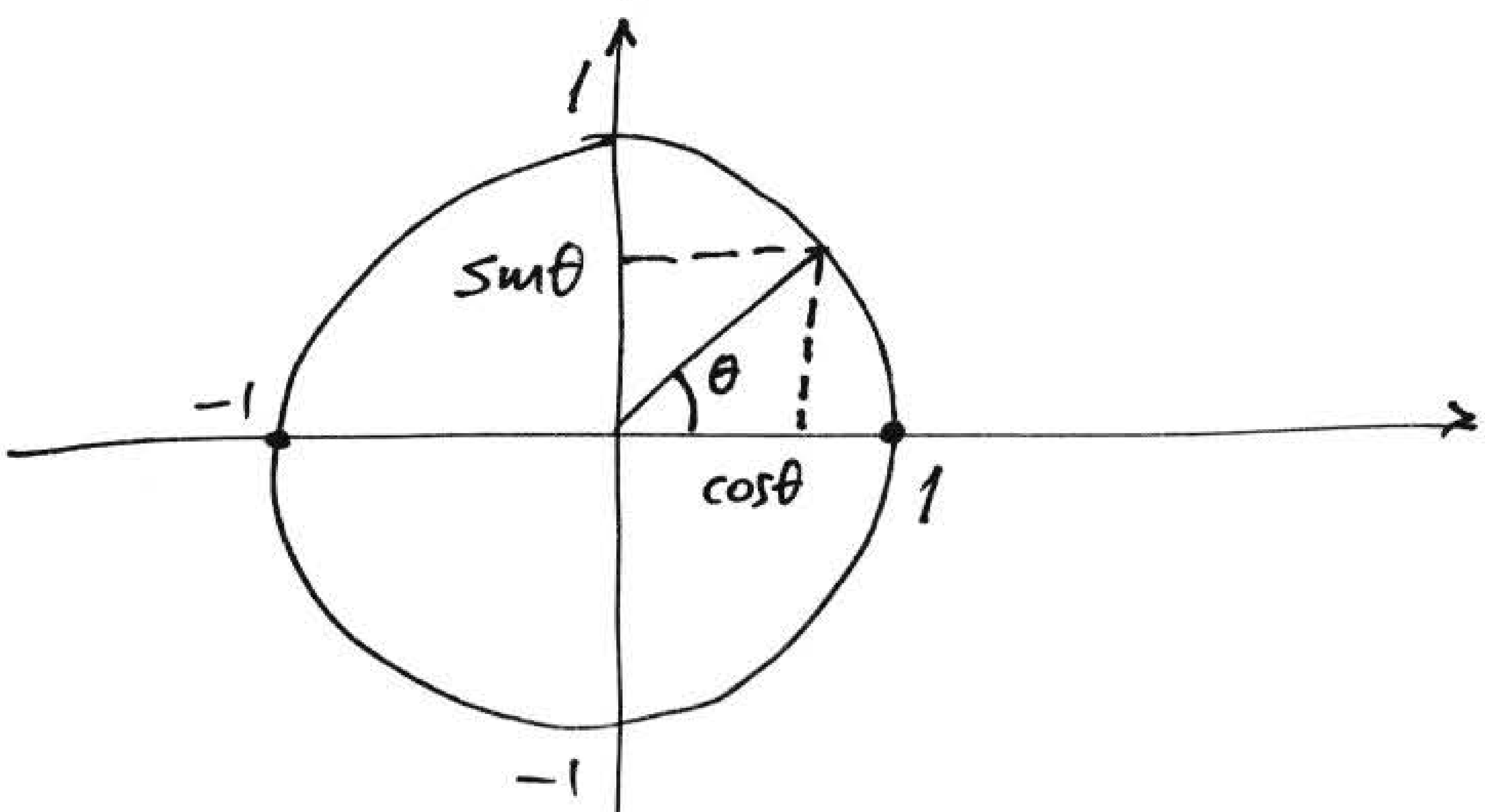
$$\text{where } \ddot{\eta}_i(t) + \omega_i^2\,\eta_i(t) = N_i(t) = \int_0^L m\,F(x,t)\,\varphi_i(x)\,dx$$

$$\eta_i(0) = \sqrt{\frac{2}{mL}}\int_0^L m\,g(x)\sin\left(\frac{i\pi x}{L}\right)dx$$

$$\dot{\eta}_i(0) = \sqrt{\frac{2}{mL}}\int_0^L m\,h(x)\sin\left(\frac{i\pi x}{L}\right)dx$$

$\Rightarrow$

$$\boxed{u(x,t) = \sum_{i=1}^{\infty}\left[\eta_i(0)\cos\left(\frac{i\pi c t}{L}\right) + \frac{\dot{\eta}_i(0)}{\frac{i\pi c}{L}}\sin\left(\frac{i\pi c}{L}t\right) + \frac{1}{\left(\frac{i\pi c}{L}\right)}\int_0^L N_i(\tau)\sin\left[\frac{i\pi c}{L}(t-\tau)\right]d\tau\right]\cdot\sqrt{\frac{2}{mL}}\,\sin(i\pi x/L)}$$



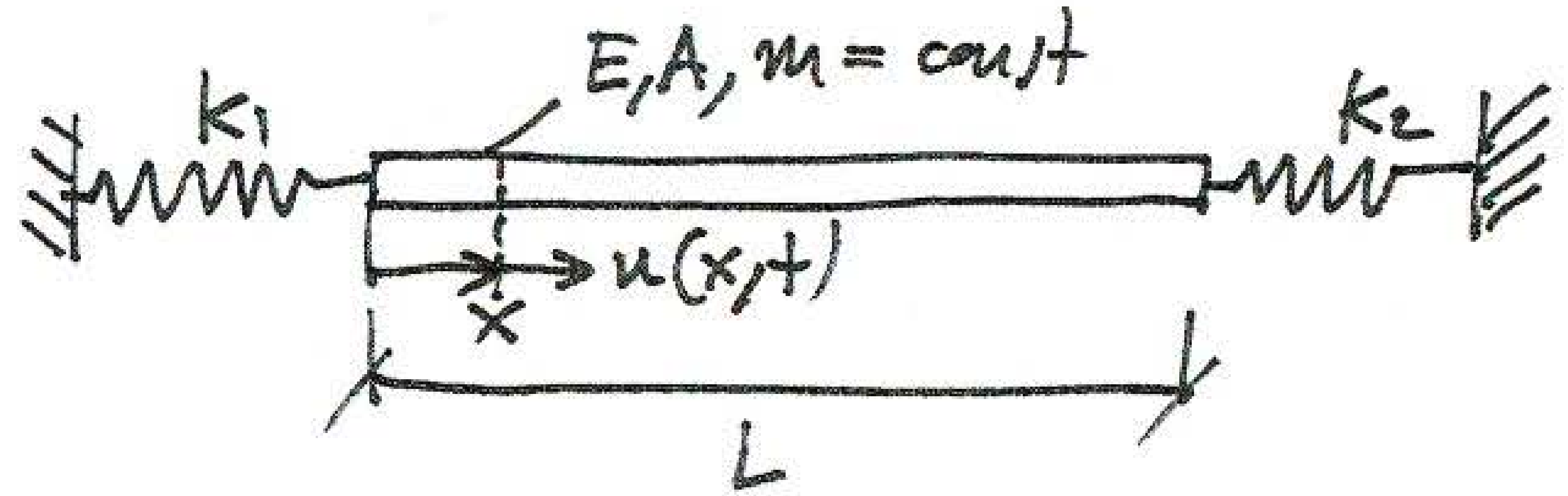
$\sin\theta = 0 \Rightarrow \theta = k\pi, \quad k = 0, 1, 2, \cdots$

$\cos\theta = 0 \Rightarrow \theta = \dfrac{(2k+1)\pi}{2}, \quad k = 0, 1, 2, \cdots$

$\tan\theta = 0 \Rightarrow \cdots$

$\cot\theta = 0 \Rightarrow \cdots$

Example 2



the governing pde is,

$$EA\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

BCs are:

$$\left.\begin{aligned} EA\frac{\partial u(0,t)}{\partial x} - k_1 u(0,t) = 0 \\ EA\frac{\partial u(L,t)}{\partial x} + k_2 u(L,t) = 0 \end{aligned}\right\} \quad (1a)$$

ICs are: $u(x,0) = g(x)$, $\frac{\partial u(x,0)}{\partial t} = h(x)$ (1c)

Working similarly we formulate a linear eigenvalue problem:

$$\varphi''(x) + \left(\frac{\omega}{c}\right)^2 \varphi(x) = 0 \quad , \quad c = \sqrt{\frac{E}{\rho}} \qquad (2)$$

$$\left.\begin{aligned} EA\,\varphi'(0) - k_1\varphi(0) = 0 \\ EA\,\varphi'(L) + k_2\varphi(L) = 0 \end{aligned}\right\} \quad (2a)$$

$$(2) \Rightarrow \varphi(x) = C_1\cos\frac{\omega}{c}x + C_2\sin\frac{\omega}{c}x \Rightarrow$$

$$\Rightarrow \varphi'(x) = -\frac{\omega}{c}C_1\sin\frac{\omega x}{c} + \frac{\omega}{c}C_2\cos\frac{\omega x}{c}$$

$$\Rightarrow \quad EA\frac{\omega}{c}C_2\underbrace{\cos\frac{\omega 0}{c}}_{1} - k_1 C_1 \cdot 1 = 0$$

$$EA\left[-\frac{\omega}{c}C_1\sin\frac{\omega L}{c} + \frac{\omega}{c}C_2\cos\frac{\omega L}{c}\right] + k_2\left[C_1\cos\frac{\omega L}{c} + C_2\sin\frac{\omega L}{c}\right] = 0$$

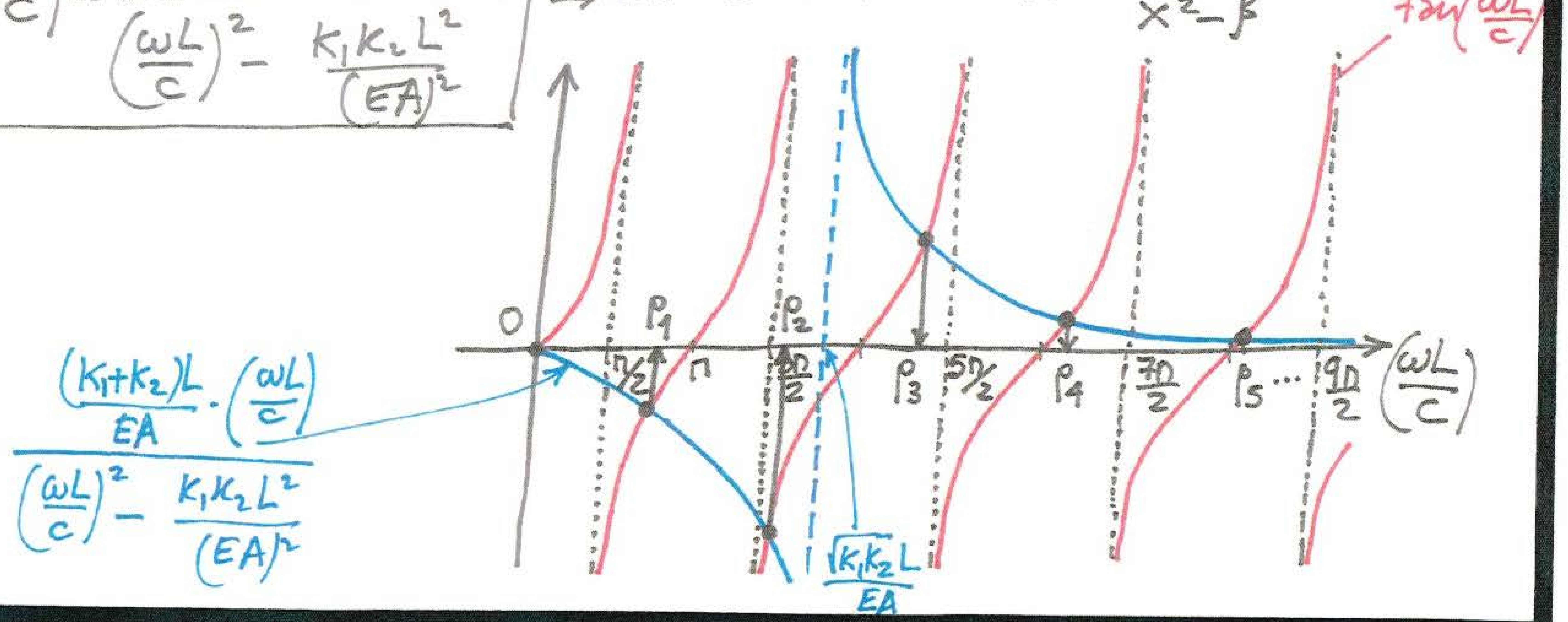
Hence, we get:

$$\begin{bmatrix} -k_1 & \frac{EA\omega}{c} \\ -EA\frac{\omega}{c}\sin\frac{\omega L}{c}+k_2\cos\frac{\omega L}{c} & EA\frac{\omega}{c}\cos\frac{\omega L}{c}+k_2\sin\frac{\omega L}{c} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow$$

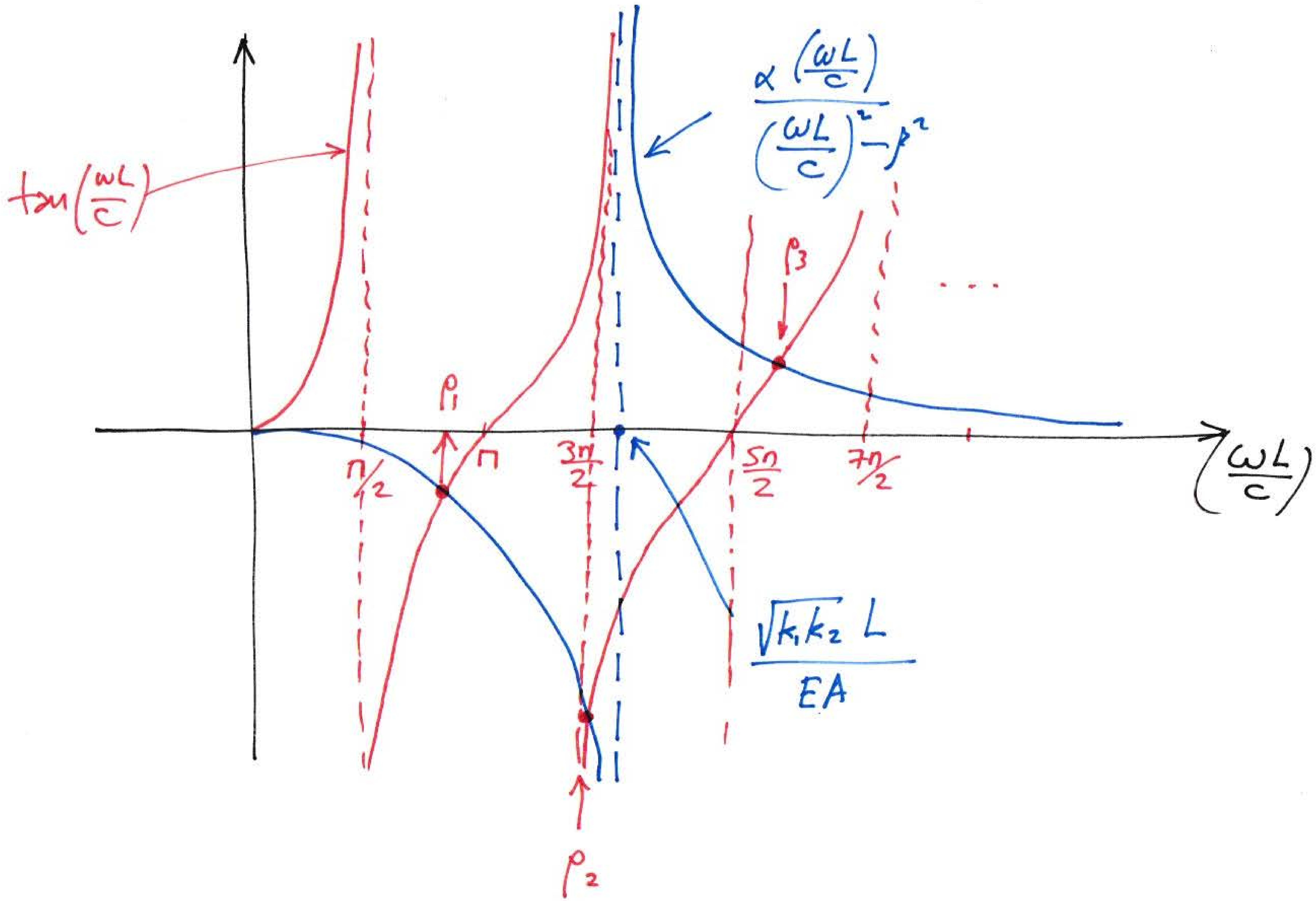
$\Rightarrow$ for nontrivial solutions we require that $\det[\ ]=0 \Rightarrow$

$$\Rightarrow -k_1 EA\frac{\omega}{c}\cos\frac{\omega L}{c}-k_1k_2\sin\frac{\omega L}{c}+\left(EA\frac{\omega}{c}\right)^2\sin\frac{\omega L}{c}-k_2EA\frac{\omega}{c}\cos\frac{\omega L}{c}=0 \Rightarrow$$

$$\Rightarrow \boxed{\tan\left(\frac{\omega L}{c}\right)=\frac{\frac{(k_1+k_2)L}{EA}\cdot\left(\frac{\omega L}{c}\right)}{\left(\frac{\omega L}{c}\right)^2-\frac{k_1k_2L^2}{(EA)^2}}} \Rightarrow \text{In the form } \tan x=\frac{\alpha x}{x^2-\beta}$$



$$\tan\left(\frac{\omega L}{c}\right) = \frac{\alpha\left(\frac{\omega L}{c}\right)}{\left(\frac{\omega L}{c}\right)^2 - \beta^2}, \quad \alpha = \frac{(k_1+k_2)L}{EA}, \quad \beta = \frac{k_1 k_2 L^2}{(EA)^2}$$



Then we see that ρ_i stays within certain limits:

$$\frac{\pi}{2} < \rho_1 < \pi, \quad \frac{3\pi}{2} < \rho_2 < 2\pi, \quad \cancel{2\pi}\ \frac{5\pi}{2} < \rho_3 < \cancel{\frac{5\pi}{2}}\ 3\pi, \ \ldots$$

Once we have computed $\rho_i \Rightarrow \rho_i = \frac{\omega_i L}{c} \Rightarrow \omega_i = \frac{c\rho_i}{L}, \ i=1,2,\ldots$

Then we can show that the corresponding eigenfunction is,

$$\boxed{\varphi_i(x) = C_i \frac{\sin\left(\rho_i \frac{x}{L} + \psi_i\right)}{\cos\psi_i}, \quad i=1,2,\ldots}, \quad \tan\psi_i = \frac{EA\rho_i}{K_1 L} \Rightarrow$$

$$\Rightarrow 0 < \psi_i < \frac{\pi}{2}, \ i=1,2,\ldots$$

Then we can mass-orthonormalize, $\int_0^L m\varphi_i^2(x)\,dx = 1 \Rightarrow$

$\Rightarrow m\frac{C_i^2}{\cos^2\psi_i}\int_0^L \sin^2\left(\rho_i\frac{x}{L} + \psi_i\right)dx = 1 \Rightarrow$ from this we may compute the constant C_i, $i=1,2,\ldots \Rightarrow$

$$\boxed{C_i = 2\cos\psi_i\sqrt{\frac{\rho_i}{mL\left(2\rho_i - \sin 2(\rho_i + \psi_i) + 2\sin 2\psi_i\right)}}}$$