4. Some Results on Vibration and Eigenvalue Problems

'Methods of Mathematical Physics', R. Courant + D. Hilbert.

A linear homogeneous differential operator (LHD) for a function $u(\cancel{x})$ is in the form
$$\mathcal{L}[u] = Au + Bu_x + Cu_{xy} + \dots \quad \text{on } \mathcal{D} \qquad (1)$$
where $x, y, \dots$ are independent variables and $u = u(x, y, \dots)$, $x, y, \dots \in \mathcal{D}$

If the derivatives of $\mathcal{L}[u]$ end up at a given order, then this is the order of $\mathcal{L}[u]$. A LHD operator satisfies the principle of superposition,
$$\mathcal{L}[c_1 u_1 + \dots + c_n u_n] = c_1 \mathcal{L}[u_1] + \dots + c_n \mathcal{L}[u_n], \quad c_i \in R, \text{ constants} \qquad (2)$$

A linear nonhomogeneous differential operator has the form,
$$\mathcal{L}[u] = f(x, y, \dots), \quad x, y, \dots \in \mathcal{D} \qquad (3)$$

'discrete' superposition

Suppose now that $f = 0 \Rightarrow$ If $u_1, \dots, u_n, \dots$ are linearly independent solutions of $\mathcal{L}[u] = 0$, then the convergent series $\sum_{n=1}^{\infty} c_n u_n$ represents the general solution of $\mathcal{L}[u] = 0$. Also, if a solution $u(x, y, \dots; \alpha)$ of $\mathcal{L}[u] = 0$ with parameter α is known, then new solutions of the following form can be constructed:

can be regarded as weighting function

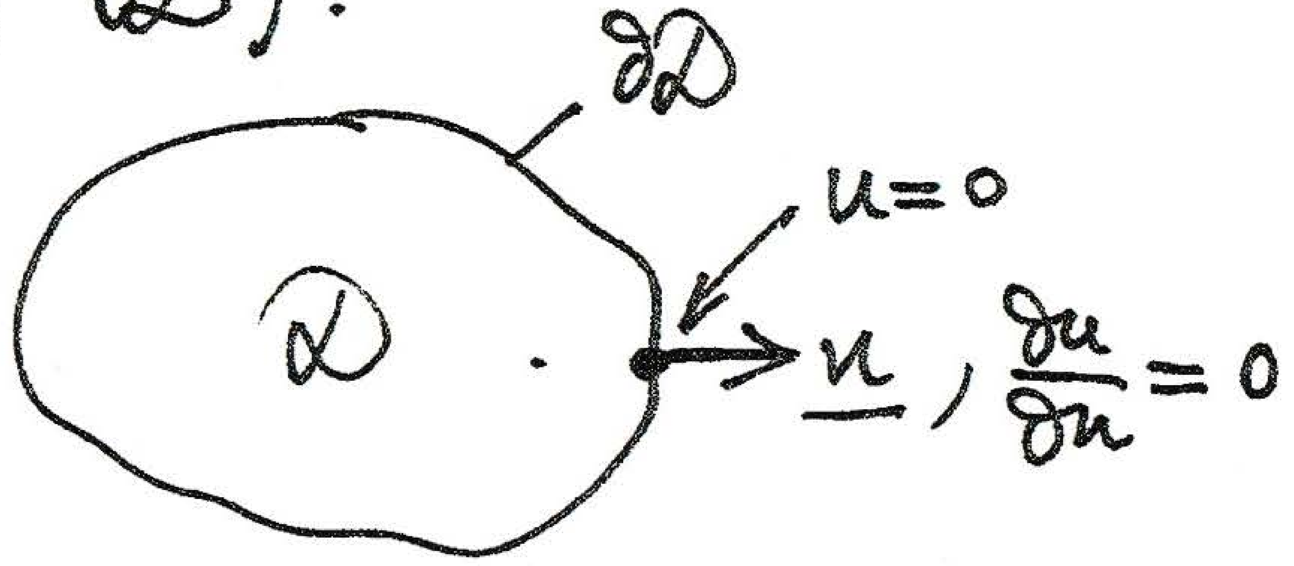
$$v = \int_A w(\alpha)\, u(x, y, \dots; \alpha)\, d\alpha, \quad \alpha \in A \qquad (4)$$

Note 'continuous' superposition

where $w(\alpha)$ is an arbitrary function defined in the domain $A \subset R$. The only restriction on $w(\alpha)$ is that the integral (4) must exist and that the operator $\mathcal{L}[\cdot]$ can have meaning under the integral sign (these conditions are satisfied if $w(\alpha)$ is piecewise continuous and the domain of integration A is finite).

<u>If the homogeneous equation can be solved, then just one (any) solution of the nonhomogeneous equation yields all its solutions.</u>

Up to now we considered the linear operators, now we focus on initial and boundary conditions. If the BCs of a LHD operator are homogeneous (i.e., they do not depend on the independent variables $x, y, \ldots$), then the resulting BVP is homogeneous. In this case, if u is a solution of a homogeneous BVP, then so is cu, $c \in R$. Typical homogeneous BCs are relations between the values of u and its derivatives on ∂D (for example, $u=0$ or $\frac{\partial u}{\partial n}=0$ on ∂D).



Now, given linear nonhomogeneous BCs, for example $u = f(x, y, \ldots)$ on ∂D, we can obtain an equivalent problem with homogeneous BCs $\Rightarrow$ Assume that $\mathcal{L}[u]=0$ is a

linear homogeneous equation, and that the boundary values f can be extended continuously in $\mathcal{D}$. Then we can define the new function $v = f - u \Rightarrow$
$\Rightarrow$ We obtain the non-homogeneous problem $\mathcal{L}[v] = q$ on $\mathcal{D}$, but with homogeneous boundary condition $v = 0$ on $\partial\mathcal{D}$. Hence, homogeneous differential equations with nonhomogeneous BCs are equivalent to nonhomogeneous differential equations with homogeneous BCs.

4.1. Adjoint Differential Expressions, Green's Formulas

Differential expressions that arise from a variational problem with a homogeneous quadratic integrand are called self-adjoint.

(a) One independent variable, $x \in R$

Consider the quadratic expression $Q[u,u] = a u'^2 + 2b u'u + d u^2$, $u = u(x)$, $a = a(x)$, $b = b(x)$, $d = d(x)$. Then, the corresponding symmetric billinear expression is $Q[u,v] = a u'v' + b(u'v + v'u) + d u v$, $x \in \mathcal{D} = [x_0, x_1]$.

Then, $$\int_{x_0}^{x_1} Q[u,v]\, dx = -\int_{x_0}^{x_1} v\, \mathcal{L}[u]\, dx + (au' + bu)v \Big|_{x_0}^{x_1} \quad (5)$$

Operator on u arising from the integration of the symmetric billinear expression $Q[u,v]$, $Q[u,v] = Q[v,u]$

where $\boxed{\mathcal{L}[u] = (au')' + (b' - d)u}$, $(\cdot)' \equiv \frac{d}{dx} \cdot$ (6)

Similarly we can perform the integration by parts in a different way,

$$\int_{x_0}^{x_1} Q[u,v]\,dx = -\int_{x_0}^{x_1} u\,\mathcal{L}[v]\,dx + (av'+bv)u\Big|_{x_0}^{x_1} \qquad (7)$$

same operator in v now appears

Now substructing (5)−(7) ⇒ we obtain the symmetric Green's formula:

$$\underbrace{\int_{x_0}^{x_1}\Big(v\mathcal{L}[u]-u\mathcal{L}[v]\Big)dx}_{\text{Depends on the values of } u \text{ and } v \text{ inside } \mathcal{D}=[x_0,x_1]} = \underbrace{a(u'v-v'u)\Big|_{x_0}^{x_1}}_{\text{Depends on the values of } u,v \text{ on } \partial\mathcal{D}} \qquad (8)$$

Suppose now that we start with a general (non-symmetric) quadratic expression $B[u,v] = au'v' + bu'v + cuv' + duv \Rightarrow B[u,v] \neq B[v,u] \Rightarrow$

$$\left.\begin{aligned} \Rightarrow \int_{x_0}^{x_1} B[u,v]\,dx &= -\int_{x_0}^{x_1} v\,\mathcal{L}[u]\,dx + (au'+cu)v\Big|_{x_0}^{x_1} = \\ &= -\int_{x_0}^{x_1} u\,\mathcal{M}[v]\,dx + (av'+bv)u\Big|_{x_0}^{x_1} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \int_{x_0}^{x_1} \left(v\mathcal{L}[u] - uM[v]\right) dx = \left[a(u'v - v'u) + (c-b)uv \right]\Big|_{x_0}^{x_1}$$

Nonsymmetric Green's formula

where $\mathcal{L}[u] = (au')' - bu' + (cu)' - du$,

$M[v] = (av')' + (bv)' - cv' - dv \Rightarrow$ In this general case, $\mathcal{L}[u] \neq M[u]$

so $\mathcal{L}[u]$ is not self-adjoint; $M[u]$ is called the adjoint operator of $\mathcal{L}[u]$.

So $\mathcal{L}[\cdot]$ and $M[\cdot]$ are said to be adjoint to each other. Now, if

$\mathcal{L}[u] = M[u]$ holds identically $\forall\, x \in \mathcal{D} \Rightarrow$ Then, $\mathcal{L}[u] = M[u]$ is self-adjoint operator $\Rightarrow$ Then it may be derived from a quadratic expression $Q[u,u]$.

The adjoint of the differential expression $[pu'' + ru' + qu]$ is $[(pv)'' - (rv)' + qv]$, and the requirement that $\boxed{p' = r}$ is a necessary and sufficient condition for this differential expression to be self-adjoint.

Proof: $(pv)'' - (rv)' + qv = (p'v + v'p)' - r'v - rv' + qv =$

$= \cancel{p''v} + v''p + \cancel{2}p'v' - \cancel{r'v} - \cancel{rv'} + qv = pv'' + p'v' + qv =$

Assume that $p' = r \Rightarrow p'' = r'$

$= pv'' + rv' + qv$ ✓

Remark 1

Note that the eigenvalue problem for a nonuniform rod is

$$\frac{d}{dx}\left[EA(x)\frac{d\varphi(x)}{dx}\right] + m(x)\omega^2\varphi(x)=0 \Rightarrow$$

$$\Rightarrow \underbrace{\frac{d}{dx}[EA(x)]}_{r(x)}\frac{d\varphi(x)}{dx} + \underbrace{EA(x)}_{p(x)}\frac{d^2\varphi(x)}{dx^2} + \underbrace{m(x)\omega^2}_{q(x)}\varphi(x)=0 \Rightarrow$$

$$\Rightarrow \quad p(x)\varphi''(x)+r(x)\varphi'(x)+q(x)\varphi(x)=0, \quad \text{with} \quad p'(x)=r(x)\Rightarrow$$

$\Rightarrow$ Hence, the eigenvalue problem is <u>self-adjoint!</u>

Remark 2

For an operator $F[y(x), y'(x), x]$, Euler's differential is defined as

$$[F]_y = -\left\{\frac{d}{dx}F_{y'} - F_y\right\}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]-\frac{\partial \mathcal{L}}{\partial q}=0$$

Note, Lagrange's equations!

Note quadratic expression

Hence, if $Q[u,u] = F(u,u') = au'^2 + 2bu'u + du^2 \Rightarrow$

$$\Rightarrow [F]_u = -\left\{\frac{d}{dx}\left[2au'+2bu\right] - 2bu' + 2du\right\} =$$

$$= -\left\{2(au')' + \cancel{2b'u} + 2bu' - \cancel{2bu'} + 2du\right\} = -2(au')' - 2b'u + 2du \Rightarrow$$

Self-adjoint operator!

$$\Rightarrow \boxed{\mathcal{L}[u] = -\frac{1}{2}\left[Q[u,u]\right]_u}$$

So, a self-adjoint differential expression can be derived by means of the Euler differential of an expression associated with the quadratic integrand $Q[u,u]$.

HW: How is the self-adjoint operator, $pu'' + p'u' + qu = \mathcal{L}[u]$ derived from a quadratic potential $Q[u,u]$ and what is the meaning of this potential?

An arbitrary linear differential expression $[pu'' + ru' + qu]$ can be transformed into one which is self-adjoint $\Rightarrow$ multiply by a suitable non-vanishing factor $\rho(x) = e^{\int \frac{r-p'}{p} dx}$, and use the new independent variable,

$x' = \int e^{-\int \frac{r-p'}{p} dx} dx$ in place of x; alternatively introduce a new dependent variable $v = u\, e^{\int \frac{r-p'}{p} dx}$ instead of u.

(b) Several independent variables

Analogous relations hold for several variables. As an example, consider linear, second order partial differential equations $\Rightarrow$ Consider the quadratic integrand

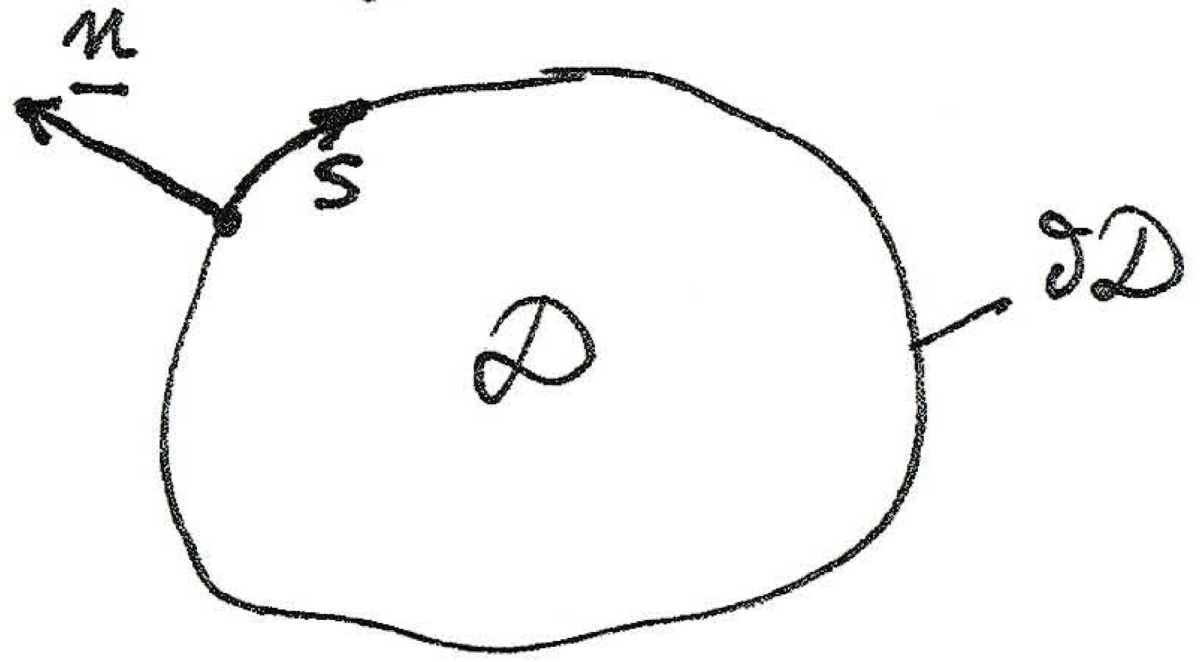
$$Q[u,u] = p\,(u_x^2 + u_y^2) + q u^2 \qquad (9)$$

with the polar form,

$$Q[u,v] = p\,(u_x v_x + u_y v_y) + q u v \qquad (10)$$

We integrate $Q[u,v]$ over the domain $(x,y) \in \mathcal{D}$ with piecewise smooth boundary $\partial\mathcal{D} \Rightarrow$ Integration by parts leads to the following Green's formula:

$$\int_{\mathcal{D}} Q[u,v]\,dx\,dy = -\int_{\mathcal{D}} v\,\mathcal{L}[u]\,dx\,dy + \int_{\partial\mathcal{D}} p v \underbrace{\frac{\partial u}{\partial n}}_{\nabla u \cdot \underline{n}} ds \qquad (11)$$



where $\mathcal{L}[u] = (p u_x)_x + (p u_y)_y - q u$ is a self-adjoint operator. We assume that v is continuous with piecewise continuous first and second derivatives.

If v is as smooth as u, we may obtain the symmetric Green's formula:

$$\int_D \left(v\,\mathcal{L}[u] - u\,\mathcal{L}[v] \right) dx\,dy = \int_{\partial D} p \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) ds \qquad (12)$$

$\int u\,dv = uv - \int v\,du$

Integration by path

1) $$\int_{x_0}^{x_1} Q[u,v]\,dx = \int_{x_0}^{x_1} \left[a u' v' + b\,(u'v + v'u) + d u v\right] dx =$$

$$= \int_{x_0}^{x_1} (au' + bu)\,v'\,dx + \int_{x_0}^{x_1} \left[b u' v + d u v\right] dx =$$

$$= (au' + bu)\,v \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} (au' + bu)'\,v\,dx + \int_{x_0}^{x_1} (b u' v + d u v)\,dx =$$

$$= (au' + bu)\,v \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \left[(au')'v + (b'u + b\cancel{u'})v - b\cancel{u'}v - d u v\right] dx \Rightarrow$$

$$\Rightarrow \int_{x_0}^{x_1} Q[u,v]\,dx = (au' + bu)\,v \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \underbrace{\left[(au')' + (b' - d)u\right]}_{\mathcal{L}[u]} v\,dx \Rightarrow$$

$$\Rightarrow \int_{x_0}^{x_1} Q[u,v]\,dx = (au' + bu)\,v \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} v\,\mathcal{L}[u]\,dx \qquad (5)$$

2) Green's formula:

$$\int_{x_0}^{x_1} Q[u,v]\,dx = -\int_{x_0}^{x_1} v\,\mathcal{L}[u]\,dx + (au'+bu)v\Big|_{x_0}^{x_1} \qquad (5)$$

$$\int_{x_0}^{x_1} Q[u,v]\,dx = -\int_{x_0}^{x_1} u\,\mathcal{L}[v]\,dx + (av'+bv)u\Big|_{x_0}^{x_1} \qquad (7)$$

$\Rightarrow$

$$\Rightarrow -\int_{x_0}^{x_1}\left(v\,\mathcal{L}[u]-u\,\mathcal{L}[v]\right)dx + \left(au'v+\cancel{buv}-av'u\mp\cancel{bvu}\right)\Big|_{x_0}^{x_1} = 0 \Rightarrow$$

$$\Rightarrow \int_{x_0}^{x_1}\left(v\,\mathcal{L}[u]-u\,\mathcal{L}[v]\right)dx = a\,(u'v-v'u)\Big|_{x_0}^{x_1} \qquad (8)$$

3) So, the self-adjoint operator is

$$\mathcal{L}[u] = (au')' - bu' + (cu)' - du =$$

$$= M[u] = (au')' + (bu)' - cu' - du$$ Adjoint operator

But $\mathcal{L}[u] = a'u' + au'' - bu' + c'u + cu' - du =$

$$= \underbrace{a}_{p}u'' + \underbrace{(a' - b + c)}_{r}u' + (\underbrace{c' - d}_{q})u \equiv pu'' + ru' + qu$$

$$M[v] = a'v' + av'' + b'v + bv' - cv' - dv =$$

$$= av'' + (a' + b - c)v' + (b' - d)v =$$

$$= (\underbrace{a}_{p}v)'' - [\underbrace{(a' - b + c)}_{r}v]' + (\underbrace{c' - d}_{q})v \equiv (pv)'' - (rv)' + qv$$

So, in order that $\mathcal{L}[u] = M[u] \Rightarrow p' = r$

Operator = to its adjoint operator