Classification of 2^{nd} order partial differential equations

Complemented by initial and boundary conditions on $\partial\mathcal{D}$

General form:

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0, \quad (x,y) \in \mathcal{D}$$

Domain

where $A, \dots, G$ are functions of x and y, and $u_{xy} \equiv \dfrac{\partial^2 u}{\partial x \partial y}, \dots$

- If $\boxed{B^2 - AC < 0 \ \forall\, (x,y) \in \mathcal{D}}$ then the pde is elliptic. Representative equations of this type is the Laplace equation $\Delta u \equiv u_{xx} + u_{yy} = 0$ and Poisson equation $\Delta u \equiv u_{xx} + u_{yy} = f(x,y)$. Note that we obtain the Laplace equation from the heat equation $u_t = \Delta u$ by setting $u_t = 0 \Rightarrow$
$\Rightarrow$ Laplace equation represents the steady state of the heat equation.
In elliptic equations "information" propagates with infinite speed $\Rightarrow$
$\Rightarrow$ Changes in initial conditions "inform" the domain of the solution instantaneously.

Basic example: Potential flow in fluids (irrotational) $\bar{V} = \nabla \varphi$ satisfying $\left.\begin{matrix}\nabla \times \bar{V} = 0 \\ \nabla \cdot \bar{V} = 0\end{matrix}\right\} \Rightarrow \nabla^2 \varphi = 0$

- If $\boxed{B^2 - AC > 0 \ \forall\, (x,y)} \in \mathcal{D}$ then the pde is hyperbolic. Representative equation of this type is the wave equation $\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0$. Note that as $c \to \infty$ this becomes Laplace equation.

In hyperbolic pdes "information" propagates with finite speed $\Rightarrow$
$\Rightarrow$ A change in initial conditions propagates with finite speed within the domain of the solution, and not instantaneously. In fact, disturbances in the wave equation propagate along characteristic directions ("characteristics") with finite speed.

- If $\boxed{B^2 - AC = 0 \quad \forall (x,y) \in \mathcal{D}}$, then the pde is parabolic. The prototypical example is the heat equation $u_t = \Delta u$.

The heat equation combines properties from both elliptic and hyperbolic equations.

Wave solution of the classical wave equation

Consider again the classical wave equation, defined, however, over an infinite domain:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad -\infty < x < \infty, \; t > 0 \qquad (1)$$

In this case we cannot use a vibrations-based solution (since there are no vibration modes!). Hence we will use a wave-based approach and introduce the new variables $\xi = x - ct$ and $\eta = x + ct$. Then,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \Rightarrow \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \qquad (2)$$

Similarly, $\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta} \Rightarrow \frac{\partial^2}{\partial t^2} = c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\frac{\partial^2}{\partial \xi \partial \eta} + c^2\frac{\partial^2}{\partial \eta^2} \qquad (3)$

Hence, using the variable transformation $(x,t) \mapsto (\xi,\eta)$ the wave equation (1) becomes [where $u(x,t) \mapsto u(\xi,\eta)$]:

$$\cancel{c^2\frac{\partial^2 u}{\partial \xi^2}} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{c^2\frac{\partial^2 u}{\partial \eta^2}} = \cancel{c^2\frac{\partial^2 u}{\partial \xi^2}} + 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{c^2\frac{\partial^2 u}{\partial \eta^2}} \Rightarrow \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

Hence, $\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \xRightarrow{\int d\eta} \frac{\partial u}{\partial \xi} = F(\xi) \xRightarrow{\int d\xi} u(\xi,\eta) = \underbrace{\int F(\xi)d\xi}_{f(\xi)} + g(\eta) \Rightarrow$

$\Rightarrow u(\xi,\eta) = \underbrace{f(\xi) + g(\eta)}_{\text{constants of integration or characteristics}} \Rightarrow$ Reverting back to original coordinates

the solution of the wave equation is, $u(x,t) = \underbrace{f(x-ct)}_{\text{Wave propagating in the positive direction}} + \underbrace{g(x+ct)}_{\text{Wave propagation in the negative direction}}$,

$-\infty < x < \infty, \quad t \geq 0$ (4)

<u>D'Alembert's solution of the wave equation</u>

Note that the "waveforms" $f(\cdot)$ and $g(\cdot)$ maintain their exact shapes during wave propagation $\Rightarrow$ This is because the classical wave equation is a <u>non-dispersive</u> system $\Rightarrow$ Waves propagate with undistorted shapes.

<u>The initial value problem (infinite domain)</u>

We now determine the waveforms $f(\cdot)$ and $g(\cdot)$ for prescribed initial conditions $u(x,0) = U(x)$ and $\frac{\partial u}{\partial t}(x,0) = V(x)$.

Now, $u(x,t) = f(x-ct) + g(x+ct) \Rightarrow$

$$\Rightarrow u(x,0) = V(x) = f(x) + g(x) \text{ at } t=0 \qquad (5)$$

Also, $\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial t} f(x-ct) + \frac{\partial}{\partial t} g(x+ct) = \frac{\partial \xi}{\partial t}\frac{df}{d\xi} + \frac{\partial y}{\partial t}\frac{dg}{dy} \Rightarrow$

$$\Rightarrow \frac{\partial u}{\partial t}(x,t) = -c\frac{df}{d\xi} + c\frac{dg}{dy} \Rightarrow \frac{\partial u}{\partial t}(x,0) = V(x) = -c\frac{df(x)}{dx} + c\frac{dg(x)}{dx} \Rightarrow$$

at $t=0$

$$\Rightarrow f(x) - g(x) = -\frac{1}{c}\int_b^x V(\lambda)\,d\lambda \text{ at } t=0 \qquad (6)$$

Solving (5) and (6) we get,

Computation of the characteristics

$$\text{at } t=0 \rightarrow \left.\begin{aligned} f(x) &= \frac{V(x)}{2} - \frac{1}{2c}\int_b^x V(\lambda)\,d\lambda \\ g(x) &= \frac{V(x)}{2} + \frac{1}{2c}\int_b^x V(\lambda)\,d\lambda \end{aligned}\right\} \qquad (7)$$

These are the initial values of the functions f and g for the given initial conditions at $t=0$.

For $t>0$, replace $x \mapsto \xi$ for f, and $x \mapsto y$ for $g \Rightarrow$

$$\Rightarrow f(\xi) = \frac{V(\xi)}{2} - \frac{1}{2c}\int_b^{\xi} V(\lambda)\,d\lambda \quad \text{and} \quad g(\eta) = \frac{V(\eta)}{2} + \frac{1}{2c}\int_b^{\eta} V(\lambda)\,d\lambda \Bigg\} \Rightarrow$$

But $u(x,t) = f(\xi) + g(\eta)$

$$\Rightarrow u(x,t) = \frac{V(\xi) + V(\eta)}{2} + \frac{1}{2c}\int_{\xi}^{\eta} V(\lambda)\,d\lambda \Rightarrow$$

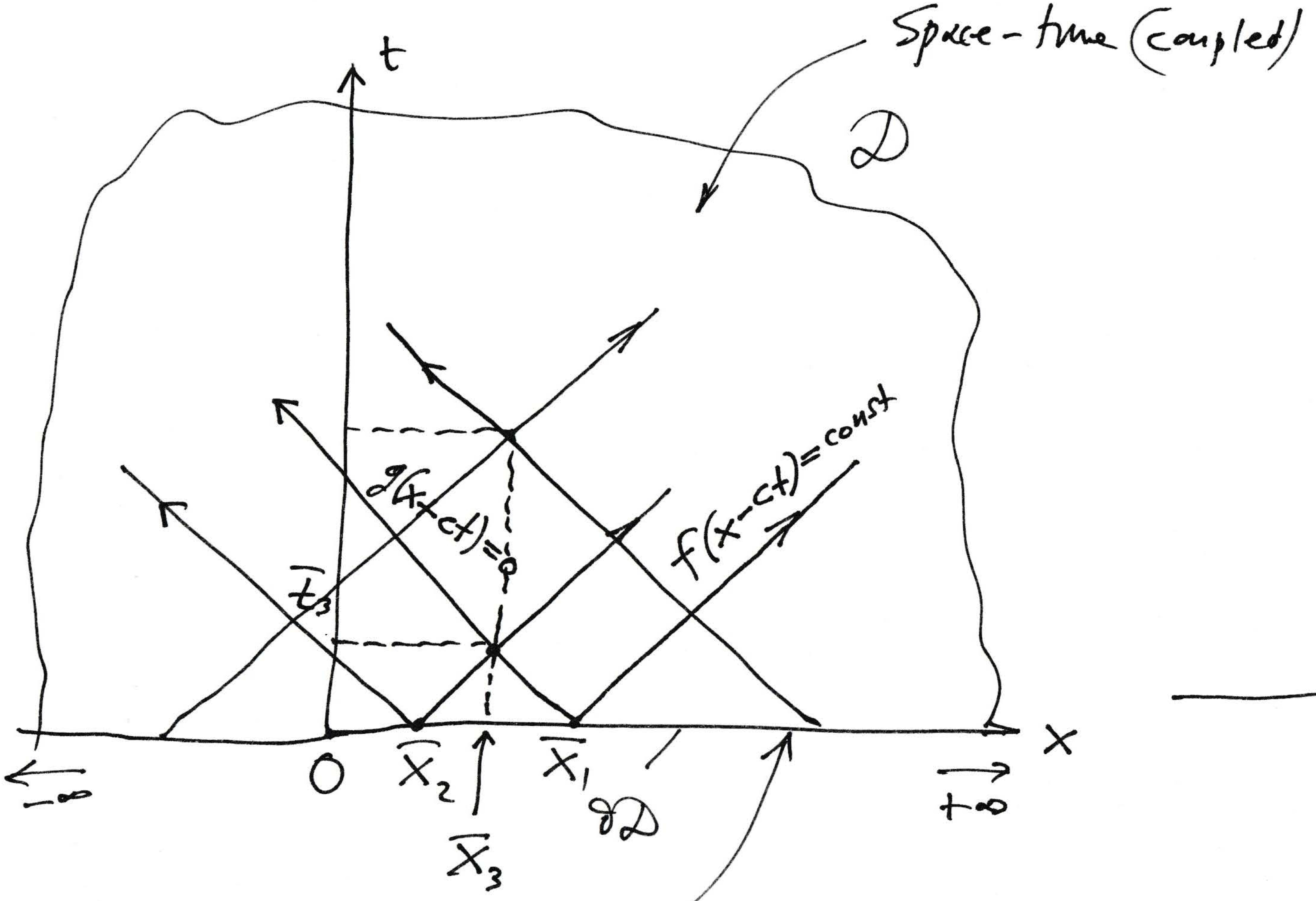
$$\Rightarrow \boxed{u(x,t) = \frac{V(x-ct) + V(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} V(\lambda)\,d\lambda} \qquad (8)$$

This solution holds for $-\infty < x < \infty$ and $t \geq 0$.

<u>Solution of the wave equation over a bounded domain</u>

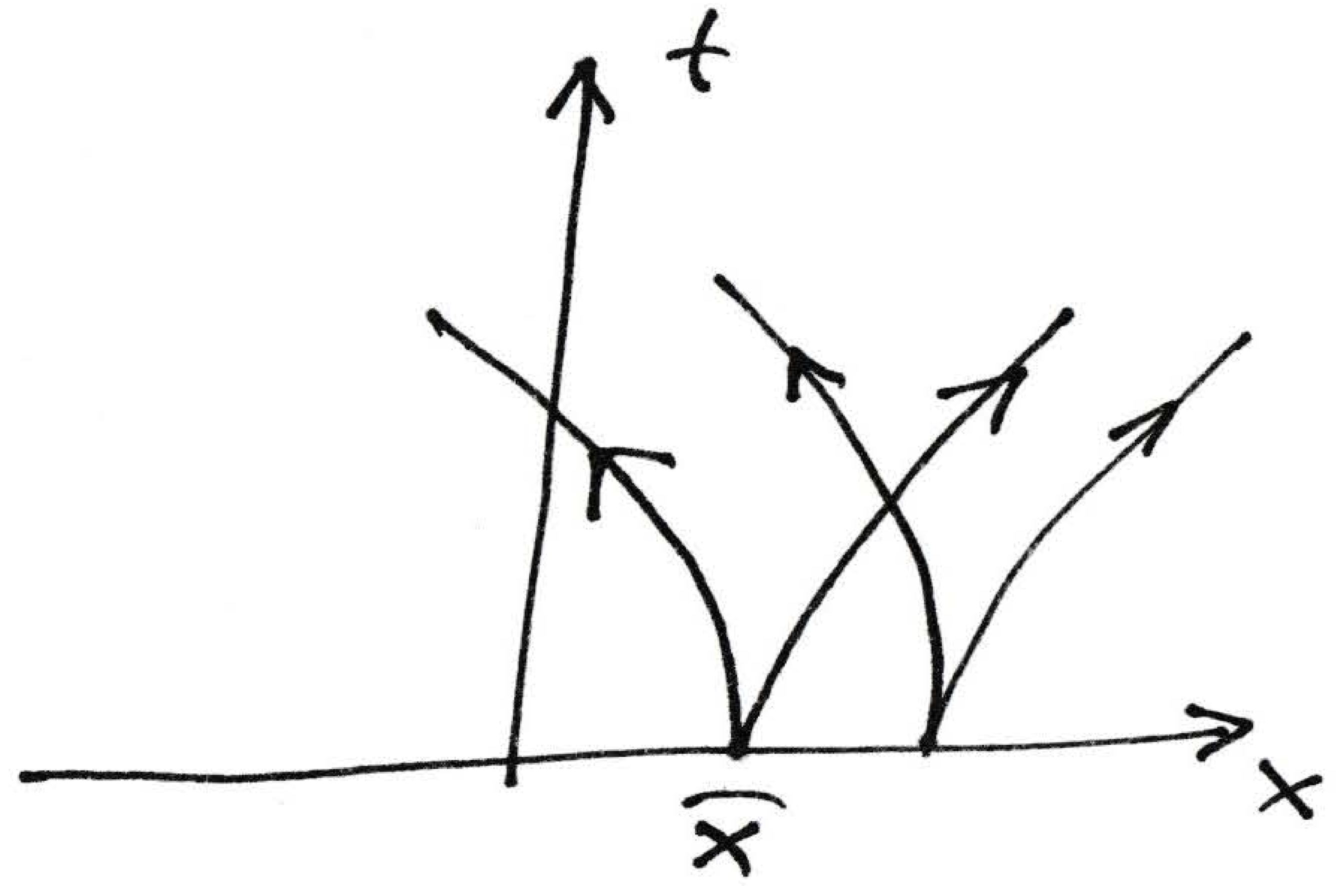
We now solve the wave equation over a finite domain subject to prescribed initial conditions (speed of sound has been normalized to 1):

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 \leq x \leq L, \quad t \geq 0 \\ &u(0,t) = 0, \quad u(L,t) = 0 \\ &u(x,0) = U(x), \quad \frac{\partial u}{\partial t}(x,0) = V(x) \end{aligned}\right\} \quad (9)$$

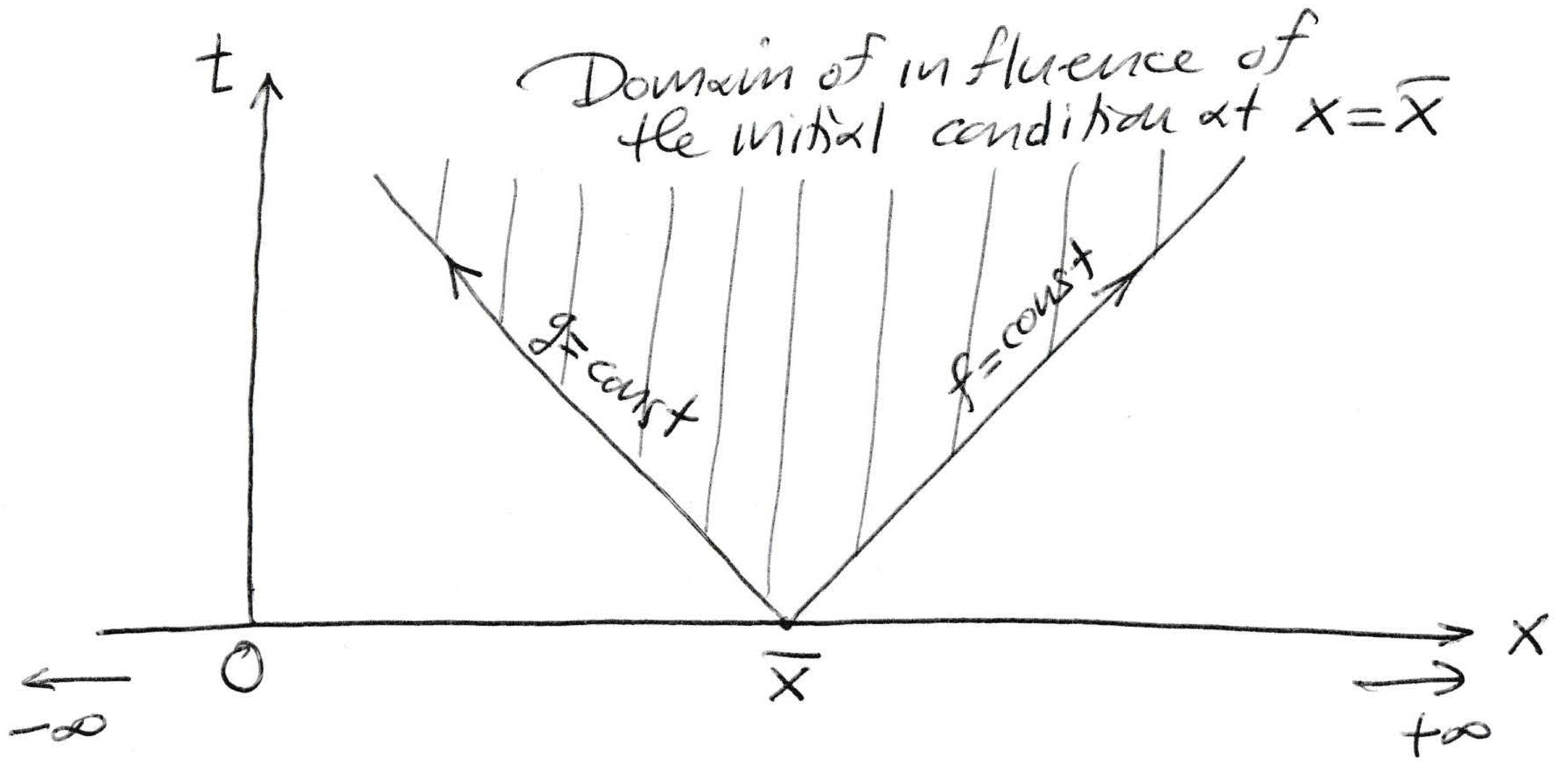


"characteristic" "Line" of the initial conditions

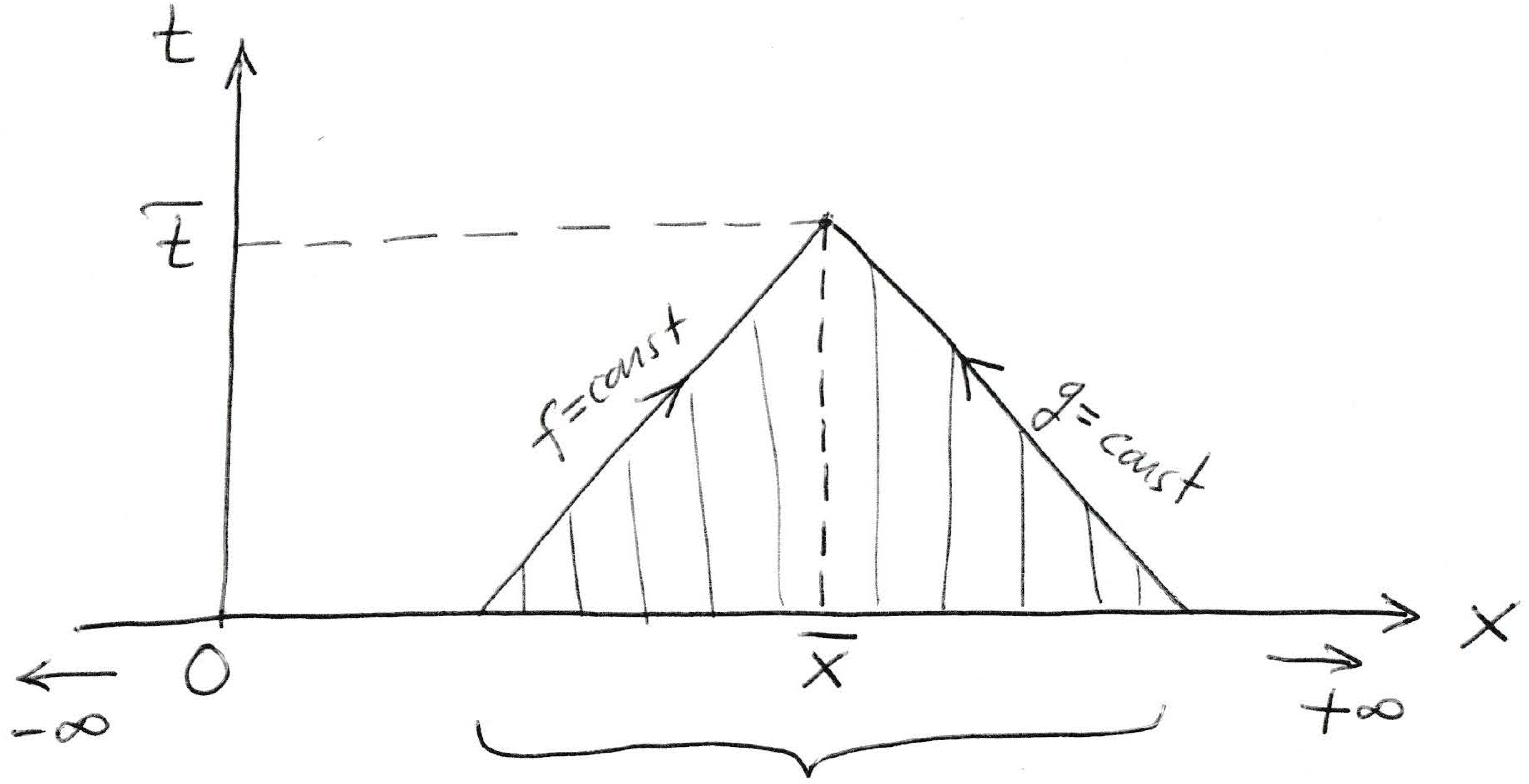
Straight lines since we assumed homogeneous wave equation



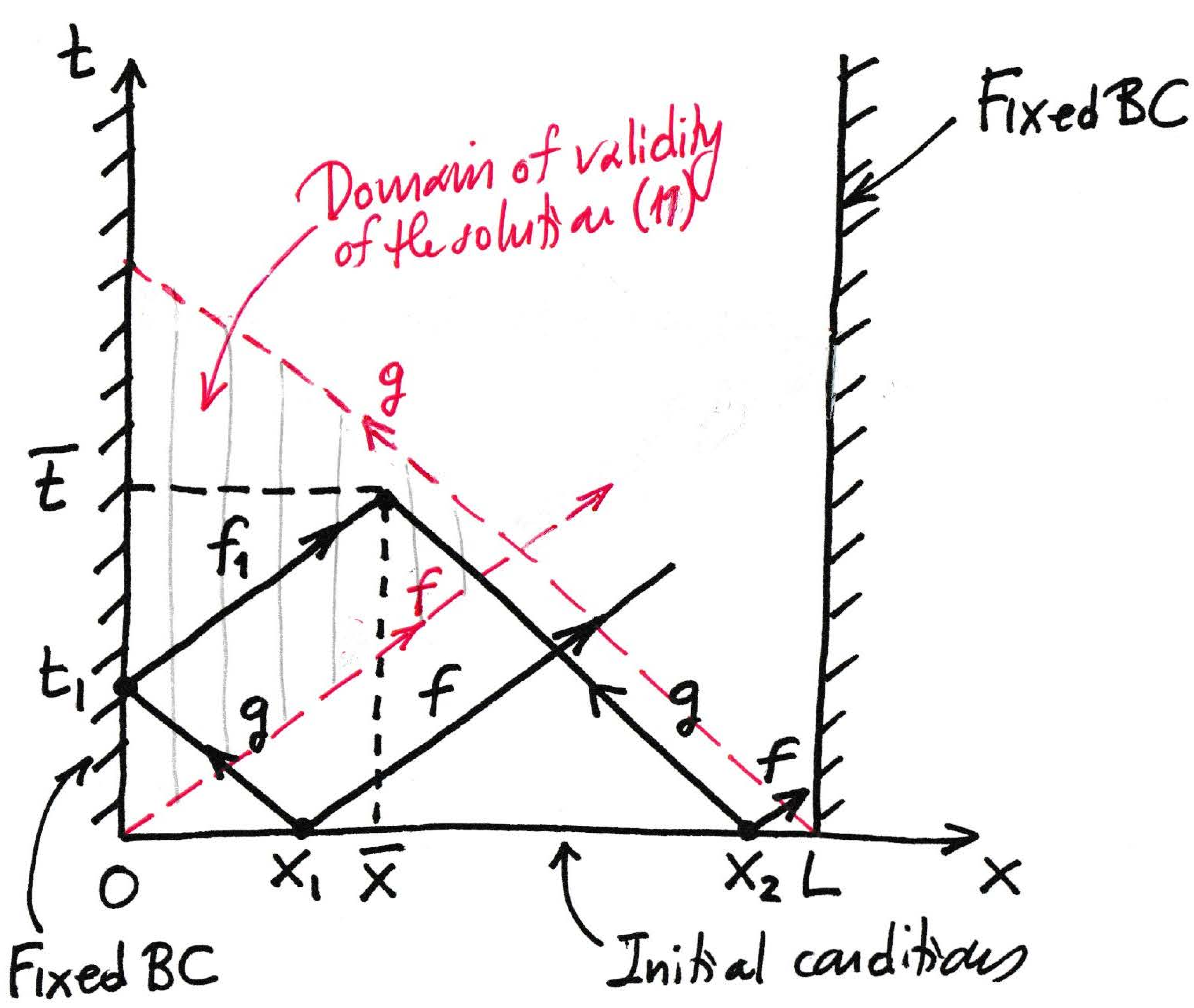
Nonhomogeneous wave equation



Since disturbances propagate with finite speed in the medium, we can identify domains of influence of the initial conditions.



The subset of initial conditions that influence the solution at $(\bar{x},\bar{t})$



Suppose that we want to compute the solution at point $(\bar{x},\bar{t})$. First, we consider the line of initial conditions

$\Rightarrow \forall\, x\in[0,L]$ at $t=0 \Rightarrow$

$$\Rightarrow u(x,t)\Big|_{t=0} = \left\{f(x-t)+g(x+t)\right\}_{t=0}$$

$$\frac{\partial u}{\partial t}(x,t)\Big|_{t=0} = \left\{-f'(x-t)+g'(x+t)\right\}_{t=0}$$

Hence, $\left.\begin{aligned} U(x) &= f(x)+g(x) \\ V(x) &= -f'(x)+g'(x) \end{aligned}\right\} \Rightarrow$

$$\Rightarrow \left.\begin{aligned} U'(x) &= f'(x)+g'(x) \\ V(x) &= -f'(x)+g'(x) \end{aligned}\right\} \overset{+}{\Rightarrow}$$

$$\Rightarrow g(x) = \int^x \frac{U'(\lambda)+V(\lambda)}{2}\,d\lambda \Rightarrow f(x) = U(x) - \int^x \frac{U'(\lambda)+V(\lambda)}{2}\,d\lambda \Rightarrow$$

$\Rightarrow$ A $t=0$ we compute
$$\left.\begin{aligned} f(x) &= \frac{U(x)}{2} - \int^x \frac{V(\lambda)}{2}\,d\lambda \\ g(x) &= \frac{U(x)}{2} + \int^x \frac{V(\lambda)}{2}\,d\lambda \end{aligned}\right\} \quad (10)$$

This "information" will propagate in the spatio-temporal domain.

At $t=t_1$ and $x=0 \Rightarrow u(0,t_1)=0 \Rightarrow g(0+t_1)+f_1(0-t_1)=0 \Rightarrow$

$\Rightarrow f_1(-t_1)=-g(t_1) \Rightarrow f_1(z)=-g(-z)$

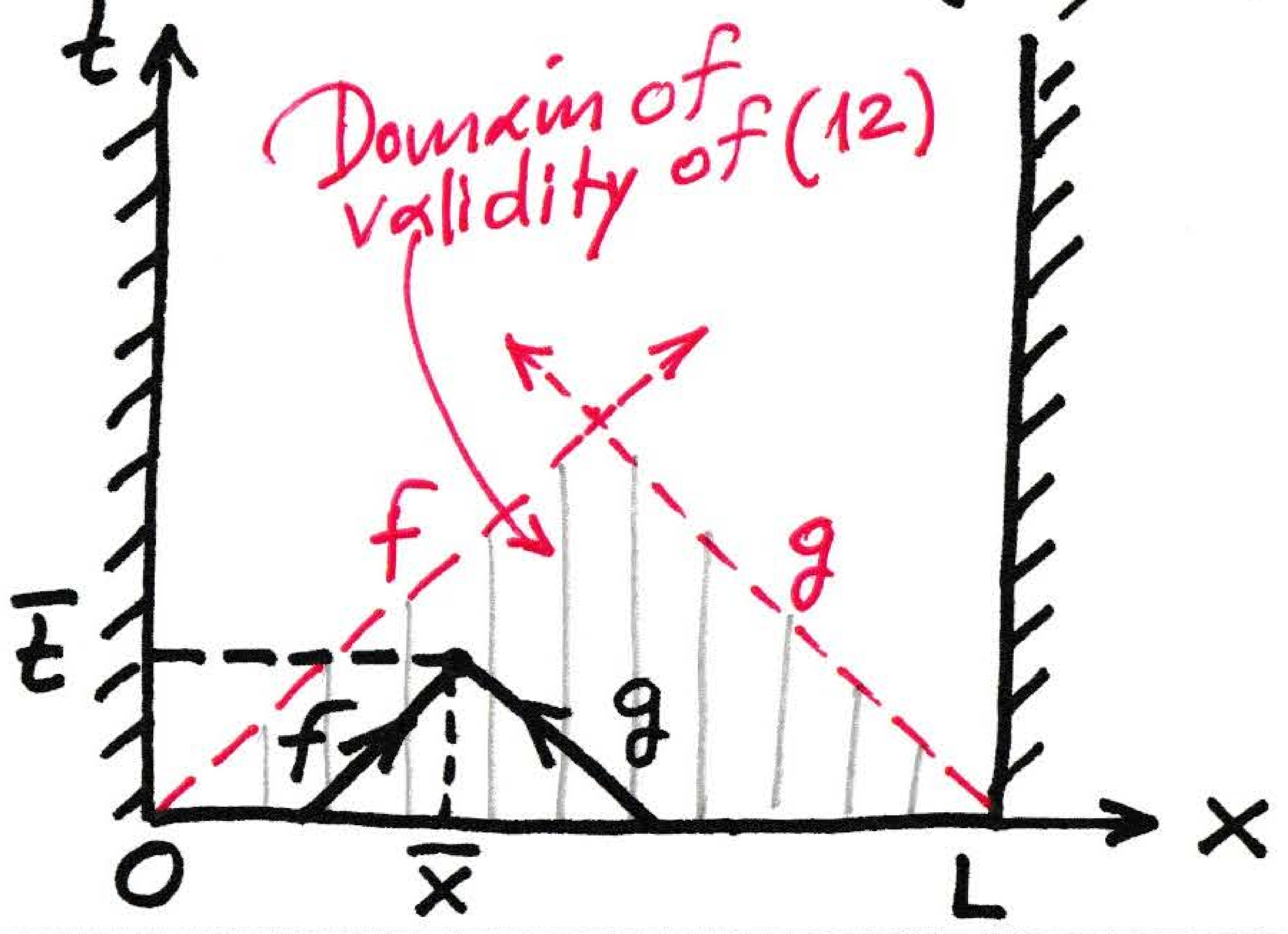
Hence, $u(\bar{x},\bar{t})=f_1(\bar{x}-\bar{t})+g(\bar{x}+\bar{t})=-g(\bar{t}-\bar{x})+g(\bar{x}+\bar{t}) \Rightarrow$

$\Rightarrow$ Taking into account the expressions (10) we find that

$$u(\bar{x},\bar{t})=\frac{V(\bar{t}+\bar{x})-V(\bar{t}-\bar{x})}{2}+\frac{1}{2}\int_{\bar{t}-\bar{x}}^{\bar{t}+\bar{x}} V(\lambda)\,d\lambda \qquad (11)$$

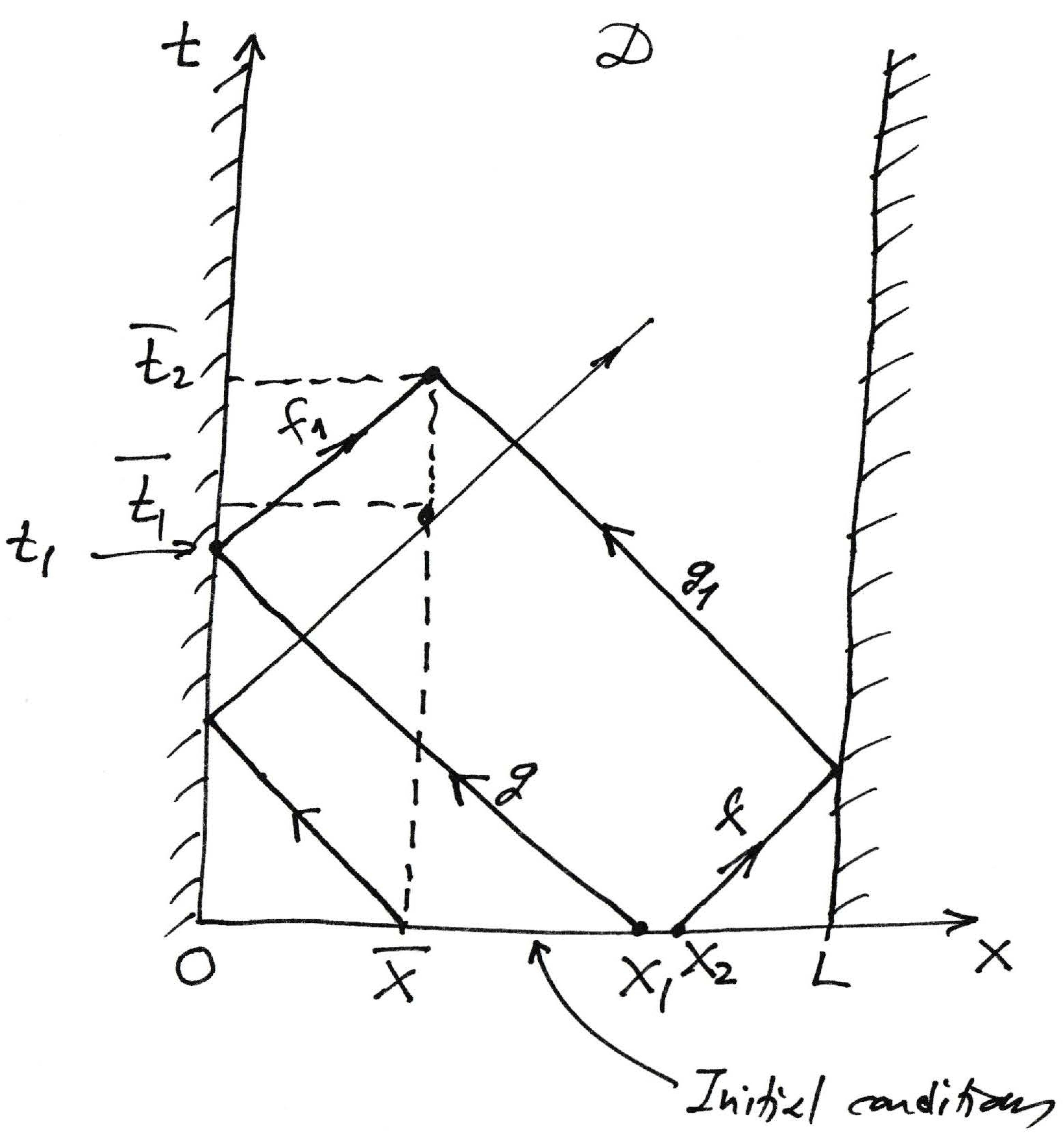
This form of the solution is valid for $\bar{t}+\bar{x}<L$ and $\bar{t}-\bar{x}>0$ since it involves only one "left" reflection.

Now suppose that we want the solution in the domain $t-x<0$ and $t+x<L \Rightarrow u(\bar{x},\bar{t})=f(\bar{x}-\bar{t})+g(\bar{x}+\bar{t})=$

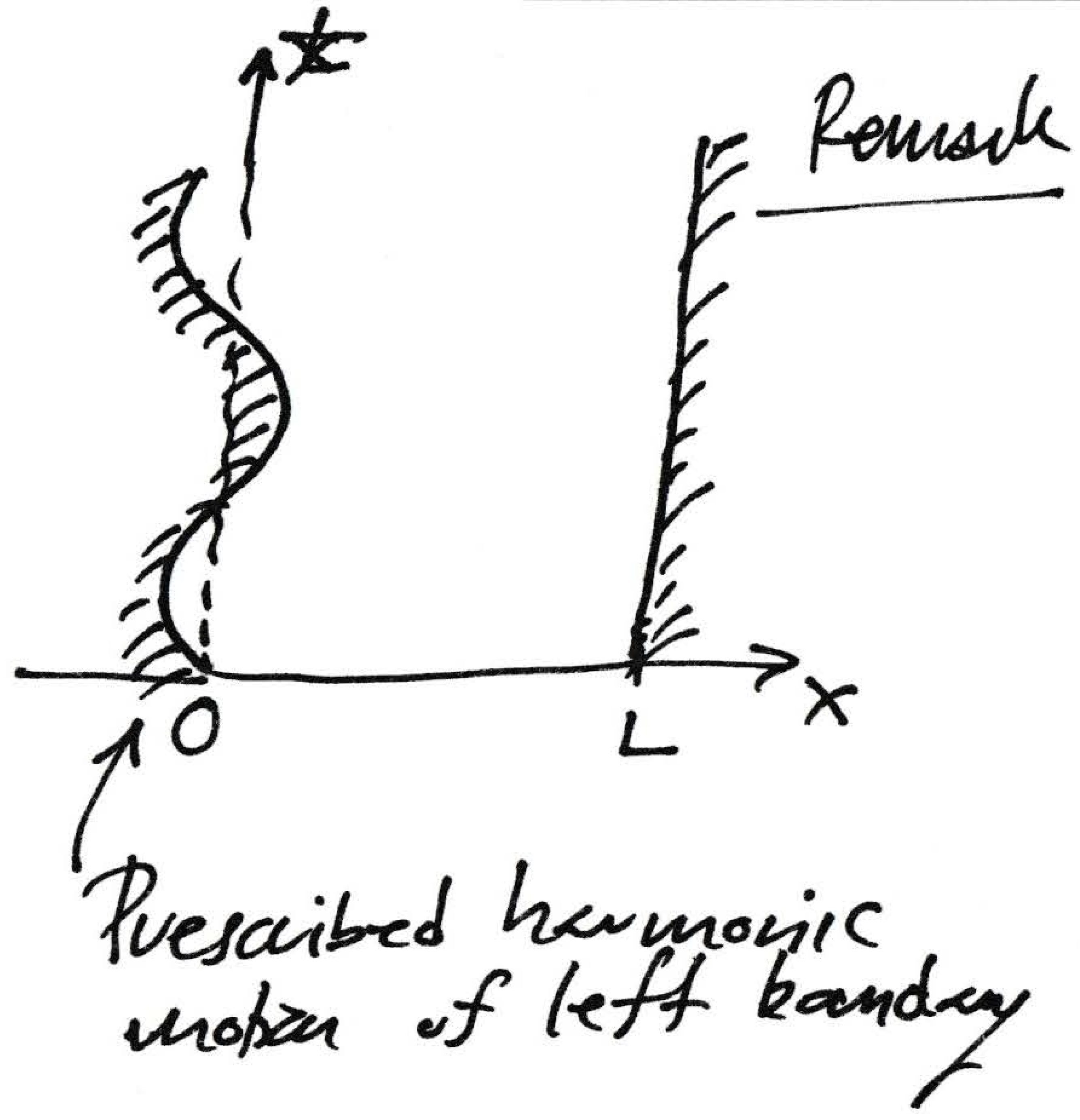


$$=\frac{V(\bar{x}-\bar{t})+V(\bar{x}+\bar{t})}{2}+\int_{\bar{x}-\bar{t}}^{\bar{x}+\bar{t}} V(\lambda)\,d\lambda \qquad (12)$$

Note that this is identical to the solution of the infinite system (i.e., $L\to\infty$), before any reflections of waves from the boundaries occur.

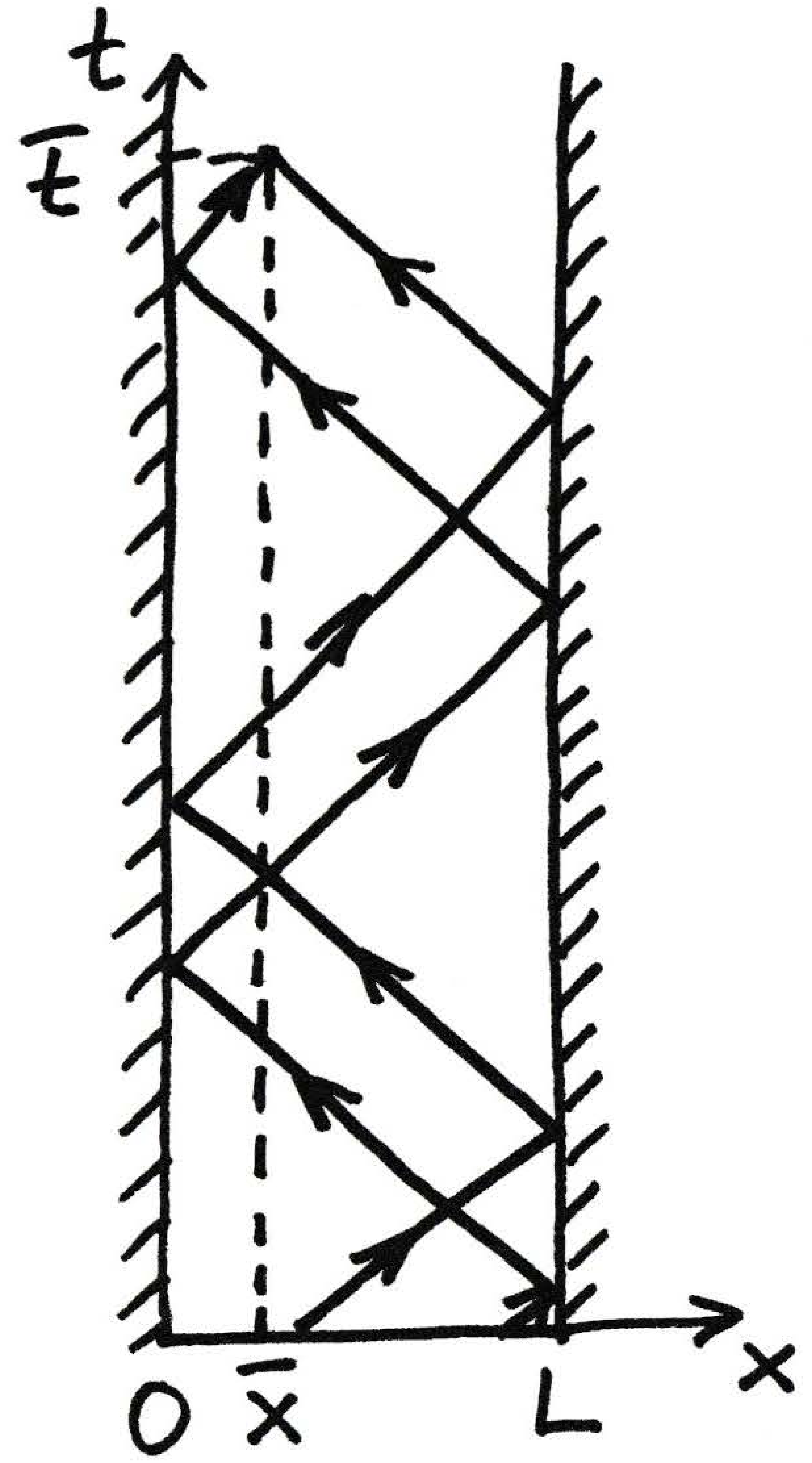


Remark



$$u(\bar x, \bar t_2) = f_1(\bar x - c\bar t_2) + g_1(\bar x + c\bar t_2)$$

Similarly we walk in domains where there are multiple "right" and "left" reflections from the boundaries. After a "large" number of



reflections at the boundaries a steady state is reached $\Rightarrow$ Vibrations (or standing waves) are generated!

Relation between vibrations and waves

Consider again the initial value problem over a finite domain,

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L \equiv \pi \\ &u(0,t) = u(\pi,t) = 0 \\ &u(x,0) = U(x), \ \frac{\partial u}{\partial t}(x,0) = V(x) \end{aligned}\right\} \quad (13)$$

Computing the normal modes of this problem and using the expansion theorem we express the solution as,

Vibrations-based solution

$$\boxed{\begin{aligned} &u(x,t) = \sum_{k=1}^{\infty} (a_k \cos kct + b_k \sin kct) \sin kx \\ &a_k = \frac{2}{\pi}\int_0^{\pi} U(x) \sin kx \, dx, \quad b_k = \frac{2}{kc\pi}\int_0^{\pi} V(x) \sin kx \, dx \end{aligned}} \quad (14)$$