Consider the limiting process where $n \to \infty$, $h \to 0$ but $L =$ constant. Then we can pass to a continuum approximation whereby we can transform the index i to a continuous variable. This is done as follows:

$$\left.\begin{aligned}
v_i(t) &\mapsto v(i,t) \\
v_{i+1}(t) &\mapsto v(i+1,t) \sim v(i,t) + \frac{\partial v(i,t)}{\partial i} + \frac{1}{2!}\frac{\partial^2 v(i,t)}{\partial i^2} + \dots \\
v_{i-1}(t) &\mapsto v(i-1,t) \sim v(i,t) - \frac{\partial v(i,t)}{\partial i} + \frac{1}{2!}\frac{\partial v(i,t)}{\partial i^2} + \dots \\
\ddot{v}_i(t) &\mapsto \frac{\partial^2 v(i,t)}{\partial t^2} \\
T_i &\mapsto T(i) \\
T_{i-1} &\mapsto T(i-1) = T(i) - \frac{dT(i)}{di} + \dots \\
F_i(t) &\mapsto F(i,t) \\
m_i &\mapsto m(i)
\end{aligned}\right\} \Rightarrow$$

We can rewrite the i-th ode of motion as:

$$m(i)\frac{\partial^2 v(i,t)}{\partial t^2} = F(i,t) - \left[T(i) - \frac{dT(i)}{di}\right]\frac{1}{h}\left[\cancel{v_i(t)} - \cancel{v(i,t)} + \frac{\partial v(i,t)}{\partial i} - \frac{1}{2}\frac{\partial^2 v(i,t)}{\partial i^2}\right] -$$

$$- T(i)\frac{1}{h}\left[\cancel{v(i,t)} - \cancel{v(i,t)} - \frac{\partial v(i,t)}{\partial i} - \frac{1}{2}\frac{\partial^2 v(i,t)}{\partial i^2}\right] + \dots \Rightarrow$$

$$\Rightarrow m(i)\frac{\partial^2 v(i,t)}{\partial t^2} = F(i,t) + T(i)\frac{\partial^2 v(i,t)}{\partial i^2}\frac{1}{h} +$$

$$+ \frac{\partial T(i)}{\partial i}\frac{1}{h}\left[\frac{\partial v(i,t)}{\partial i} - \frac{1}{2}\frac{\partial^2 v(i,t)}{\partial i^2}\right] + \ldots \Rightarrow$$

$$\Rightarrow \frac{m(i)}{h}\frac{\partial^2 v(i,t)}{\partial t^2} = \frac{F(i,t)}{h} + T(i)\frac{\partial^2 v(i,t)}{\partial (hi)^2} +$$

$$+ \frac{\partial T(i)}{\partial (ih)}\left[\frac{\partial v(i,t)}{\partial (ih)} - \frac{1}{2}\frac{\partial^2 v(i,t)}{h\,\partial i^2}\right] + \ldots$$

$$\cancel{\frac{1}{2}h\frac{\partial^2 v(i,t)}{\partial (ih)^2}}$$
as $h \to 0$

Let us introduce a new spatial variable $x = ih$, $0 \le x \le L \Rightarrow$

$\Rightarrow$ As $i \to \infty$, $h \to 0$, $\frac{m(i)}{h} \to m(x)$, $\frac{F(i,t)}{h} \to F(x,t) \Rightarrow$

$$\Rightarrow m(x)\frac{\partial^2 v(x,t)}{\partial t^2} = F(x,t) + T(x)\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial T(x)}{\partial x}\left[\frac{\partial v(x,t)}{\partial x} - \cancel{\frac{h}{2}\frac{\partial^2 v(x,t)}{\partial x^2}}\right] + \ldots$$

0 as $h \to 0$

$$\Rightarrow \boxed{m(x)\frac{\partial^2 v(x,t)}{\partial t^2} = F(x,t) + \frac{\partial}{\partial x}\left[T(x)\frac{\partial v(x,t)}{\partial x}\right] + \ldots}$$

Generalized wave equation!

But how do the BCs transform?

$v_0(t)=0 \Rightarrow v(0,t)=0$ $\qquad$ $v_{n+1}(t)=0 \longmapsto v(L,t)=0$

Discrete system $\qquad$ continuous system

And how do ICs transform?

$$v_i(0) = c_i \longmapsto v(x,0) = c(x)$$

$$\dot{v}_i(0) = g_i \longmapsto \frac{\partial v(x,0)}{\partial t} = g(x)$$

Now we get a well-posed problem to solve!