Based on the solutions of Yichen Shi

Problem 1

Governing equations and BC: $m\dfrac{\partial^2 u_1}{\partial t^2}=\dfrac{\partial}{\partial x}\left[EA\dfrac{\partial u_1}{\partial x}\right], m\dfrac{\partial^2 u_2}{\partial t^2}=\dfrac{\partial}{\partial x}\left[EA\dfrac{\partial u_2}{\partial x}\right], u_1(0)=u_2(L)=0$

$$\int_{L_-}^{L_+} m\frac{\partial^2 u_1}{\partial t^2}\mathrm{dx}=\int_{L_-}^{L_+}\frac{\partial}{\partial x}\left[EA\frac{\partial u_1}{\partial x}\right]\mathrm{dx}+\int_{L_-}^{L_+}K(u_2(0)-u_1(L))\delta(x-L)\mathrm{dx}$$

$\Longrightarrow EA\dfrac{\partial u_1(L)}{\partial x}=K(u_2(0)-u_1(L))$, similarly, $EA\dfrac{\partial u_2(0,t)}{\partial x}-K(u_2(0)-u_1(L))=0$

Assume $u_1=\varphi_1(x)f(t), u_2=\varphi_2(x)f(t), \ddot{f}=-\omega^2 f$

$-m\omega^2\varphi_1=EA\varphi_1'', -m\omega^2\varphi_2=EA\varphi_2'', EA\varphi_1'(L)+K(\varphi_1(L)-\varphi_2(0))=0$

$EA\varphi_2'(0)-K(\varphi_2(0)-\varphi_1(L))=0$, $\varphi_1(0)=\varphi_2(L)=0$

Take $c^2=EA/m$, assume $\varphi_1=A\cos\dfrac{\omega}{c}x+B\sin\dfrac{\omega}{c}x, \varphi_2=C\cos\dfrac{\omega}{c}(L-x)+D\sin\dfrac{\omega}{c}(L-x)$, then

$\varphi_1(0)=\varphi_2(L)=0\Longrightarrow A=C=0, \varphi_1'=B\dfrac{\omega}{c}\cos\dfrac{\omega}{c}x, \varphi_2'=-D\dfrac{\omega}{c}\cos\dfrac{\omega}{c}(L-x)$

$EA\varphi_1'(L)+K(\varphi_1(L)-\varphi_2(0))=0\Longrightarrow EAB\dfrac{\omega}{c}\cos\dfrac{\omega L}{c}+K\left(B\sin\dfrac{\omega L}{c}-D\sin\dfrac{\omega L}{c}\right)=0$

$EA\varphi_2'(0)-K(\varphi_2(0)-\varphi_1(L))=0\Longrightarrow -EAD\dfrac{\omega}{c}\cos\dfrac{\omega L}{c}-K\left(D\sin\dfrac{\omega L}{c}-B\sin\dfrac{\omega L}{c}\right)=0$

Or equivalently,

$$\begin{bmatrix} EAB\dfrac{\omega}{c} & KB-KD \\ -EAD\dfrac{\omega}{c} & -KD+KB \end{bmatrix}\begin{bmatrix}\cos\dfrac{\omega L}{c} \\ \sin\dfrac{\omega L}{c}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Take the determinant to be 0, $EAB\dfrac{\omega}{c}(-KD+KB)+EAD\dfrac{\omega}{c}(KB-KD)=0\Longrightarrow\dfrac{\omega}{c}(B^2-D^2)=0$

Since the two far ends are fixed, $\omega=0$ is not an eigenvalue.

Case 1, B=D, $EAB\dfrac{\omega}{c}\cos\dfrac{\omega L}{c}+\cancel{(KB-KD)}\sin\dfrac{\omega L}{c}=0\Longrightarrow\dfrac{\omega_i L}{c}=i\pi-\dfrac{\pi}{2}$, eigenfunction

$\varphi_{1i}=H_i\sin\dfrac{\omega_i}{c}x, \varphi_{2i}=H_i\sin\dfrac{\omega_i}{c}(L-x)$, where $H_i$ are multiplicative constants.

Case 2, B=-D, $EAB\frac{\omega}{c}\cos\frac{\omega L}{c}+2KB\sin\frac{\omega L}{c}=0 \Longrightarrow EA\frac{\omega_i L}{c}+2KL\tan\frac{\omega_i L}{c}=0$ ,eigenfunction

$\varphi_{1i}=K_i\sin\frac{\omega_i}{c}x, \varphi_{2i}=-K_i\sin\frac{\omega_i}{c}(L-x)$, where $K_i$ are multiplicative constants.

$$-\omega_i^2\int_0^L \varphi_{1j}m\varphi_{1i}\mathrm{dx}=\int_0^L \varphi_{1j}EA\varphi_{1i}''\mathrm{dx}=[\varphi_{1j}EA\varphi_{1i}']_0^L-\int_0^L \varphi_{1j}'EA\varphi_{1i}'\mathrm{dx}$$

$$=\varphi_{1j}(L)EA\varphi_{1i}'(L)-\cancel{\varphi_{1j}(0)EA\varphi_{1i}'(0)}-\int_0^L \varphi_{1j}'EA\varphi_{1i}'\mathrm{dx}\,(1)$$

Similarly,

$$-\omega_i^2\int_0^L \varphi_{2j}m\varphi_{2i}\mathrm{dx}=[\varphi_{2j}EA\varphi_{2i}']_0^L-\int_0^L \varphi_{2j}'EA\varphi_{2i}'\mathrm{dx}=-\varphi_{2j}(0)EA\varphi_{2i}'(0)-\int_0^L \varphi_{2j}'EA\varphi_{2i}'\mathrm{dx}\,(2)$$

Switch i and j,

$$-\omega_j^2\int_0^L \varphi_{1j}m\varphi_{1i}\mathrm{dx}=\varphi_{1i}(L)EA\varphi_{1j}'(L)-\int_0^L \varphi_{1j}'EA\varphi_{1i}'\mathrm{dx}\ (3)$$

$$-\omega_j^2\int_0^L \varphi_{2j}m\varphi_{2i}\mathrm{dx}=-\varphi_{2i}(0)EA\varphi_{2j}'(0)-\int_0^L \varphi_{2j}'EA\varphi_{2i}'\mathrm{dx}\,(4)$$

(1)-(3), $(\omega_j^2-\omega_i^2)\int_0^L \varphi_{1j}m\varphi_{1i}\mathrm{dx}=\varphi_{1j}(L)EA\varphi_{1i}'(L)-\varphi_{1i}(L)EA\varphi_{1j}'(L)$ (5)

(2)-(4), $(\omega_j^2-\omega_i^2)\int_0^L \varphi_{2j}m\varphi_{2i}\mathrm{dx}=\varphi_{2i}(0)EA\varphi_{2j}'(0)-\varphi_{2j}(0)EA\varphi_{2i}'(0)$ (6)

(5)+(6), Substitute in the boundary conditions

$$(\omega_j^2-\omega_i^2)\int_0^L \varphi_{1j}m\varphi_{1i}\mathrm{dx}+(\omega_j^2-\omega_i^2)\int_0^L \varphi_{2j}m\varphi_{2i}\mathrm{dx}$$

$$=\varphi_{1j}(L)EA\varphi_{1i}'(L)-\varphi_{1i}(L)EA\varphi_{1j}'(L)+\varphi_{2i}(0)EA\varphi_{2j}'(0)-\varphi_{2j}(0)EA\varphi_{2i}'(0)$$

$$=-\varphi_{1j}(L)K(\varphi_{1i}(L)-\varphi_{2i}(0))+\varphi_{1i}(L)K(\varphi_{1j}(L)-\varphi_{2j}(0))+\varphi_{2i}(0)K(\varphi_{2j}(0)-\varphi_{1j}(L))-\varphi_{2j}(0)K(\varphi_{2i}(0)-\varphi_{1i}(L))=0$$

So mass-orthonormality condition:

$$(\omega_j^2-\omega_i^2)\left[\int_0^L \varphi_{1j}m\varphi_{1i}\mathrm{dx}+\int_0^L \varphi_{2j}m\varphi_{2i}\mathrm{dx}\right]=0\Longrightarrow\int_0^L \varphi_{1j}m\varphi_{1i}\mathrm{dx}+\int_0^L \varphi_{2j}m\varphi_{2i}\mathrm{dx}=\delta_{ij}\ (7)$$

Substitute (7) into (1)+(2),

$$\varphi_{1j}(L)EA\varphi_{1i}'(L)-\int_0^L \varphi_{1j}'EA\varphi_{1i}'\mathrm{dx}-\varphi_{2j}(0)EA\varphi_{2i}'(0)-\int_0^L \varphi_{2j}'EA\varphi_{2i}'\mathrm{dx}=-\omega_i^2\left[\int_0^L \varphi_{1j}m\varphi_{1i}\mathrm{dx}+\int_0^L \varphi_{2j}m\varphi_{2i}\mathrm{dx}\right]=-\omega_i^2\delta_{ij}$$

So the stiffness orthogonality condition is:

$$\varphi_{1j}(L)EA\varphi_{1i}'(L)-\int_0^L \varphi_{1j}'EA\varphi_{1i}'\mathrm{dx}-\varphi_{2j}(0)EA\varphi_{2i}'(0)-\int_0^L \varphi_{2j}'EA\varphi_{2i}'\mathrm{dx}=-\omega_i^2\delta_{ij}$$

$\frac{\omega_i L}{c}=i\pi-\frac{\pi}{2}$ don't satisfy $EA\frac{\omega_i L}{c}+2KL\tan\frac{\omega_i L}{c}=0$, no repeating eigenvalues with two eigenfunctions of both first and second type. We first mass-orthogonalize the first type of eigenfunctions.

$$1=\int_0^L \varphi_{1i}m\varphi_{1i}\mathrm{dx}+\int_0^L \varphi_{2i}m\varphi_{2i}\mathrm{dx}=mH_i^2\int_0^L\left[\sin^2\frac{\omega_i}{c}x+\sin^2\frac{\omega_i}{c}(L-x)\right]\mathrm{dx}=mH_i^2\left[L-\frac{c\sin(2\omega_i L/c)}{2\omega_i}\right]=mLH_i^2$$

So $H_i=1/\sqrt{mL}$, the first type of eigenfunctions are: $\varphi_{1i}=1/\sqrt{mL}\sin\frac{\omega_i}{c}x,\varphi_{2i}=1/\sqrt{mL}\sin\frac{\omega_i}{c}(L-x)$, where $\frac{\omega_i L}{c}=i\pi-\frac{\pi}{2}$.

Mass-orthogonalize the second type of eigenfunctions.

$$1=\int_0^L \varphi_{1i}m\varphi_{1i}\mathrm{dx}+\int_0^L \varphi_{2i}m\varphi_{2i}\mathrm{dx}=mK_i^2\left[L-\frac{c\sin(2\omega_i L/c)}{2\omega_i}\right]\Longrightarrow K_i=\left[m\left(L-\frac{c\sin(2\omega_i L/c)}{2\omega_i}\right)\right]^{-1/2}$$

The second type of eigenfunctions are:

$\varphi_{1i}=\left[m\left(L-\frac{c\sin(2\omega_i L/c)}{2\omega_i}\right)\right]^{-1/2}\sin\frac{\omega_i}{c}x,\varphi_{2i}=-\left[m\left(L-\frac{c\sin(2\omega_i L/c)}{2\omega_i}\right)\right]^{-1/2}\sin\frac{\omega_i}{c}(L-x)$, where $EA\frac{\omega_i L}{c}+2KL\tan\frac{\omega_i L}{c}=0$.

Choose $E=10^9 Pa, A=10^{-4}m^2, L=1m, K=10^5 N/m, m=0.1kg/m$. $c=\sqrt{EA/m}=10^3$, $1/\sqrt{mL}=\sqrt{10}$, the first type of eigenfunctions are:

$$\varphi_{1i}=1/\sqrt{mL}\sin\left[\left(i\pi-\frac{\pi}{2}\right)\frac{x}{L}\right]=\sqrt{10}\sin\left[\left(i\pi-\frac{\pi}{2}\right)x\right]$$

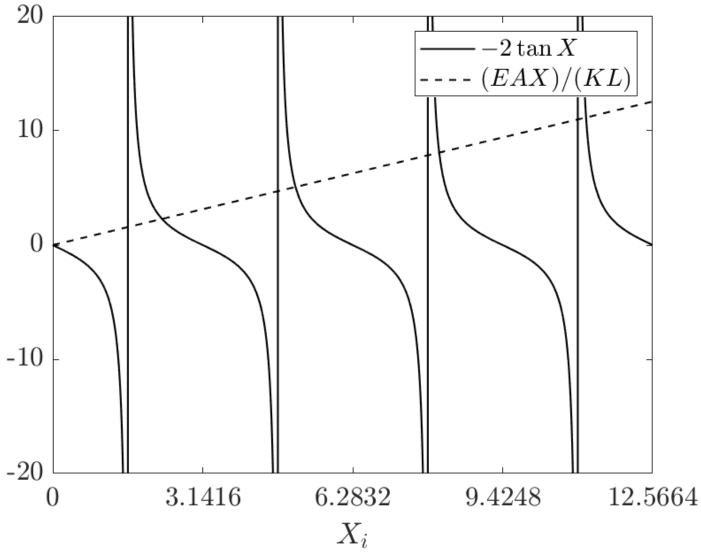
$$\varphi_{2i}=1/\sqrt{mL}\sin\left[\left(i\pi-\frac{\pi}{2}\right)\frac{L-x}{L}\right]=\sqrt{10}\sin\left[\left(i\pi-\frac{\pi}{2}\right)(1-x)\right]$$

Take $X_i=\omega_i L/c$, the second type of eigenfunctions are:

$$\varphi_{1i}=\left[m\left(L-\frac{c\sin(2X_i)}{2X_i c/L}\right)\right]^{-1/2}\sin\left[X_i\frac{x}{L}\right]=\left[0.1\left(1-\frac{\sin(2X_i)}{2X_i}\right)\right]^{-1/2}\sin\left[X_i\frac{x}{L}\right],$$

$\varphi_{2i}=-\left[0.1\left(1-\frac{\sin(2X_i)}{2X_i}\right)\right]^{-1/2}\sin\left[X_i\frac{L-x}{L}\right]$, where

$EAX_i+2KL\tan X_i=0\Longrightarrow 10^5X_i+2\times10^5\tan X_i=0$,shown below as intersections of curves.




Code:

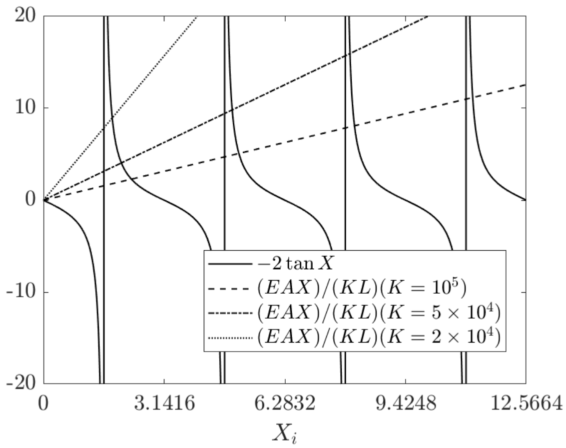
```
K=1e5;

L=1;

E=1e9;

A=1e-4;

s=(E*A)/(K*L);

xis=linspace(0,4*pi,4001);

y1=-2*tan(xis);

plot(xis,y1,'k-',xis,s*xis,'k--','LineWidth',1)

xlim([0 4*pi])

ylim([-20 20])

xticks((0:4)*pi);

xlabel('$X_i$','Interpreter','latex')

legend({'$-2 \tan X$','$(EAX)/(KL)$'},'Interpreter','latex')

set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

the change of K has no impact on the first type of modes. For the second type, when $K \to 0$, $X_i \to \pi i - \pi/2$, shown below. And the eigenfunctions approach to

$$\varphi_{1i} = \left[ mL\left(1 - \frac{\sin(2(\pi i - \pi/2))}{2X_i}\right)\right]^{-1/2} \sin\left[(\pi i - \pi/2)\frac{x}{L}\right] = \left[0.1\left(1 - \frac{\sin(2(\pi i - \pi/2))}{2X_i}\right)\right]^{-1/2} \sin\left[(\pi i - \pi/2)\frac{x}{L}\right]$$

$$\varphi_{2i} = -\left[0.1\left(1 - \frac{\sin(2X_i)}{2X_i}\right)\right]^{-1/2} \sin\left[X_i \frac{L-x}{L}\right] = -\left[0.1\left(1 - \frac{\sin(2(\pi i - \pi/2))}{2(\pi i - \pi/2)}\right)\right]^{-1/2} \sin\left[(\pi i - \pi/2)\frac{L-x}{L}\right]$$

$\varphi_{1i}$ become the mode shape of fix-free rod with length L, $\varphi_{2i}$ become the mode shape of free-fix rod with length L.



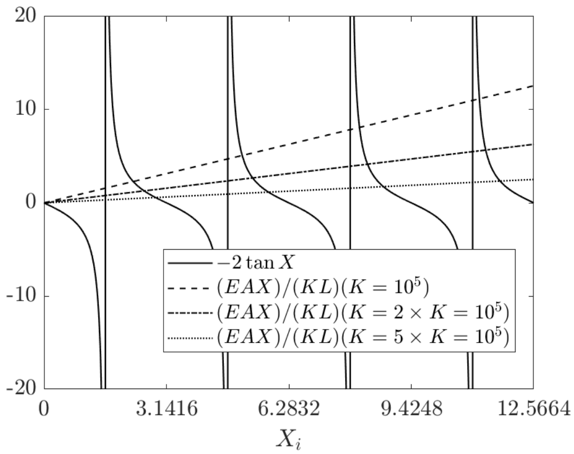

Code:

```
K=1e5;
L=1;
E=1e9;
A=1e-4;
s=(E*A)/(K*L);
xis=linspace(0,4*pi,4001);
y1=-2*tan(xis);
plot(xis,y1,'k-',xis,s*xis,'k--',xis,2*s*xis,'k-.',xis,5*s*xis,'k:','LineWidth',1)
xlim([0 4*pi])
ylim([-20 20])
xticks((0:4)*pi);
xlabel('$X_i$','Interpreter','latex')
legend({'$-2 \tan X$','$(EAX)/(KL) (K=10^5)$','$(EAX)/(KL) (K=5\times 10^4)$','$(EAX)/(KL) (K=2 \times 10^4)$'},'Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

when $K\to\infty$, $X_i\to i\pi$, shown below. And the eigenfunctions approach to

$$\varphi_{1i}=[mL]^{-1/2}\sin\left[(\pi i)\frac{x}{L}\right]=[0.1]^{-1/2}\sin\left[(\pi i)\frac{x}{L}\right]\ \varphi_{2i}=-[0.1]^{-1/2}\sin\left[(\pi i)\frac{L-x}{L}\right]$$

$\varphi_{1i}$ become the mode shape of fix-fix rod with length L, $\varphi_{2i}$ become the mode shape of fix-fix rod with length L.




Code:

```
K=1e5;

L=1;

E=1e9;

A=1e-4;

s=(E*A)/(K*L);

xis=linspace(0,4*pi,4001);

y1=-2*tan(xis);

plot(xis,y1,'k-',xis,s*xis,'k--',xis,1/2*s*xis,'k-.',xis,1/5*s*xis,'k:','LineWidth',1)

xlim([0 4*pi])

ylim([-20 20])

xticks((0:4)*pi);

xlabel('$X_i$','Interpreter','latex')

legend({'$-2 \tan X$','$(EAX)/(KL) (K=10^5)$','$(EAX)/(KL) (K=2\times K=10^5)$','$(EAX)/(KL) (K=5 \times K=10^5)$'},'Interpreter','latex')

set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

Problem 2

(i)

The Rayleigh quotient is: $R\{\tilde{\varphi}\} = \dfrac{\displaystyle\int_0^L T\tilde{\varphi}'^2 \mathrm{dx} + k_1\tilde{\varphi}^2(a) + k_2\tilde{\varphi}^2(c) + k_0\tilde{\varphi}^2(L)}{\displaystyle\int_0^L m\tilde{\varphi}^2 \mathrm{dx} + M_2\tilde{\varphi}^2(d) + M_1\tilde{\varphi}^2(L-b) + M_0\tilde{\varphi}^2(L)}$

Take $L = 1m, a = 0.2m, d = 0.4m, c = 0.6m, b = 0.2m, k_1 = k_2 = k_0 = 1N/m, m = 1kg/m, T = 1N$
$M_0 = M_1 = M_2 = 1kg$

So $R\{\tilde{\varphi}\}=\dfrac{\displaystyle\int_0^1 \tilde{\varphi}'^2 \mathrm{dx} + \tilde{\varphi}^2(1/5)+\tilde{\varphi}^2(3/5)+\tilde{\varphi}^2(1)}{\displaystyle\int_0^1 \tilde{\varphi}^2 \mathrm{dx} + \tilde{\varphi}^2(2/5)+\tilde{\varphi}^2(4/5)+\tilde{\varphi}^2(1)}$

For trial function $\tilde{\varphi}_1=\sin x$, RQ gives

$$R\{\tilde{\varphi}_1\}=\frac{\displaystyle\int_0^1 \cos^2 x \mathrm{dx} + \sin^2(1/5)+\sin^2(3/5)+\sin^2(1)}{\displaystyle\int_0^1 \sin^2 x \mathrm{dx} + \sin^2(2/5)+\sin^2(4/5)+\sin^2(1)}=1.08907$$

For trial function $\tilde{\varphi}_2=\sin 1.5x$, $R\{\tilde{\varphi}_2\}=1.0808$

For trial function $\tilde{\varphi}_3=\sin 2x$, $R\{\tilde{\varphi}_2\}=1.18179$

So we estimate the first natural frequency as $\tilde{\omega}=\sqrt{1.0808}=1.03962$.

Mathematica code:

```
ϕ=Sin[2*x];
ϕp=D[ϕ,x];
n=Integrate[ϕp^2,{x,0,1}]+(ϕ^2/.{x->1/5})+(ϕ^2/.{x->3/5})+(ϕ^2/.{x->1});
d=Integrate[ϕ^2,{x,0,1}]+(ϕ^2/.{x->2/5})+(ϕ^2/.{x->4/5})+(ϕ^2/.{x->1});
R=N[n/d]
```

(ii)

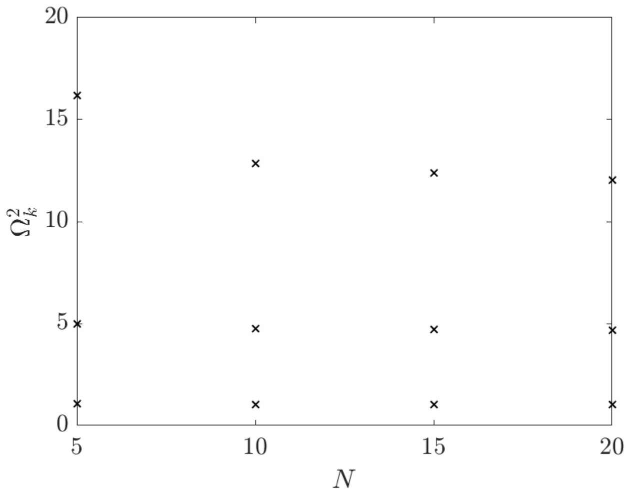
Assume trial functions $\tilde{\varphi}=\sum_{i=1}^{N}\alpha_i\psi_i$, where $\psi_i=\sin((i-1/2)\pi x)$.

So $K_{ij}=\psi_i(1/5)\psi_j(1/5)+\psi_i(3/5)\psi_j(3/5)+\psi_i(1)\psi_j(1)+\int_0^1 \varphi_i{}'\varphi_j{}'\mathrm{dx}$

$$M_{ij}=\psi_i(2/5)\psi_j(2/5)+\psi_i(4/5)\psi_j(4/5)+\psi_i(1)\psi_j(1)+\int_0^1 \varphi_i\varphi_j\mathrm{dx}$$

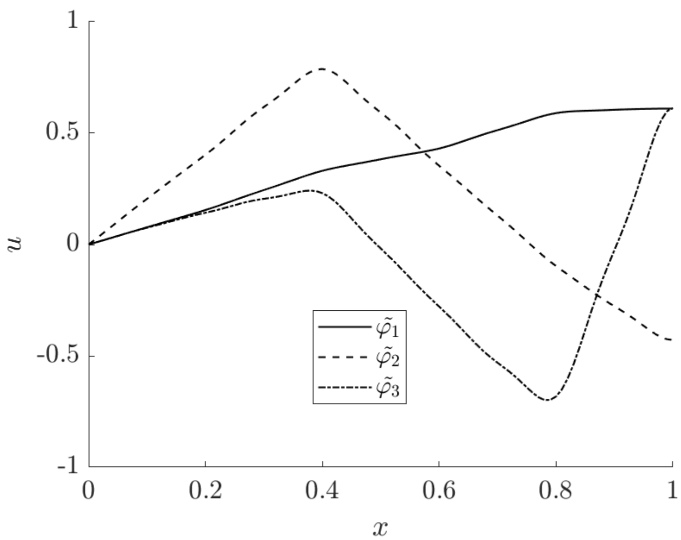
We solve the discretized problem $(\boldsymbol{K}-\Omega^2\boldsymbol{M})\boldsymbol{\alpha}=\mathbf{0}$ and get $\Omega_1^2<\Omega_2^2<\cdots\Omega_N^2$, the first 3 are eigenvalue estimates of the original eigenvalue problem. And $\tilde{\varphi}_i=\sum_{j=1}^{N}\psi(x)q_j, \boldsymbol{q}=\boldsymbol{\alpha}_i, i=1,2,3$ are the estimates of the first 3 eigenfunctions.

The convergence plot is as follows.




For N=20, the computed first three eigenvalues are: $\Omega_1^2 = 1.0406, \Omega_2^2 = 4.6632, \Omega_3^2 = 12.0384$.

The first three eigenfunctions are:




Code:

```
%% Rayleigh-Ritz

phi=@(n,x) sin((n-1/2)*pi*x);

phip=@(n,x) (n-1/2)*pi*cos((n-1/2)*pi*x);

for N=5:5:20

    G=zeros(N,N);

    H=G;

    for m=1:N

        for n=1:N

            G(m,n)=phi(m,1/5)*phi(n,1/5)+phi(m,3/5)*phi(n,3/5)+phi(m,1)*phi(n,1)+integral(@(x)
phip(m,x).*phip(n,x),0,1);
```

```
            H(m,n)=phi(m,2/5)*phi(n,2/5)+phi(m,4/5)*phi(n,4/5)+phi(m,1)*phi(n,1)+integral(@(x)
phi(m,x).*phi(n,x),0,1);
        end
    end
    [V,D]=eig(G,H);
    plot(N,diag(D),'kx','LineWidth',1);
    hold on;
end
xticks(5:5:20);
xlabel('$N$','Interpreter','latex')
ylabel('$\Omega_k^2$','Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
ylim([0 20])
%%
[E,I]=sort(diag(D));
%%
hold on;
xs=linspace(0,1,101);
k=1;
ys1=0*xs;
for l=1:20
    ys1=ys1+phi(l,xs)*V(l,I(k));
end
plot(xs,ys1,'k-','LineWidth',1);
k=2;
ys1=0*xs;
for l=1:20
    ys1=ys1+phi(l,xs)*V(l,I(k));
end
plot(xs,ys1,'k--','LineWidth',1);
```

```
k=3;
ys1=0*xs;
for l=1:20
    ys1=ys1+phi(l,xs)*V(l,I(k));
end
plot(xs,ys1,'k-.','LineWidth',1);
legend({'$\tilde{\varphi_1}$','$\tilde{\varphi_2}$','$\tilde{\varphi_3}$'},'Interpreter','latex')
xlabel('$x$','Interpreter','latex')
ylabel('$u$','Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

Problem 3

The Rayleigh quotient is: $R\{\varphi\}=\dfrac{\displaystyle\int_0^L EA\varphi'^2\mathrm{dx}}{\displaystyle\int_0^L m\varphi^2\mathrm{dx}}$

We first find the normal modes for the uniform rod.

$$m\frac{\partial^2 u}{\partial t^2}=EA\frac{\partial^2 u}{\partial x^2}, u=\varphi f, \ddot{f}=-\omega^2 f \Longrightarrow -m\omega^2\varphi=EA\varphi'' \Longrightarrow \varphi=C_1\cos\frac{\omega}{c}x+C_2\sin\frac{\omega}{c}x, c^2=EA/m$$

boundary conditions $u(0,t)=0, EA\dfrac{\partial u(L,t)}{\partial x}=0 \Longrightarrow \varphi(0)=0, \varphi'(L)=0$

$$\Longrightarrow C_1=0, C_2\frac{\omega}{c}\cos\frac{\omega}{c}L=0, \omega L/c=\pi(i-1/2), i=1,2,\cdots$$

so the natural frequencies of uniform rod are: $\omega_i=(i-1/2)\pi c/L=\dfrac{(i-1/2)\pi\sqrt{EA/m}}{L}, i=1,2,\cdots$, mode shapes are $\varphi_i(x)=A_i\sin\dfrac{\omega_i x}{c}=A_i\sin\dfrac{(i-1/2)\pi x}{L}, i=1,2,\cdots$, where $A_i$ are multiplicative constants.

now use those for the Rayleigh-Ritz method for the nonuniform rod. Assume $\psi_i=\sin\dfrac{(i-1/2)\pi x}{L}=\sin(i-1/2)\pi x, i=1,2,\cdots$.

$$K_{ij}=\int_0^L EA(x)\psi_i'\psi_j'\mathrm{dx}=(i-1/2)(j-1/2)\pi^2 EA\int_0^1[1-\sin^{3/2}(x/2)]\cos(i-1/2)\pi x\cos(j-1/2)\pi x\,\mathrm{dx}=\pi^2 EAG_{ij}$$

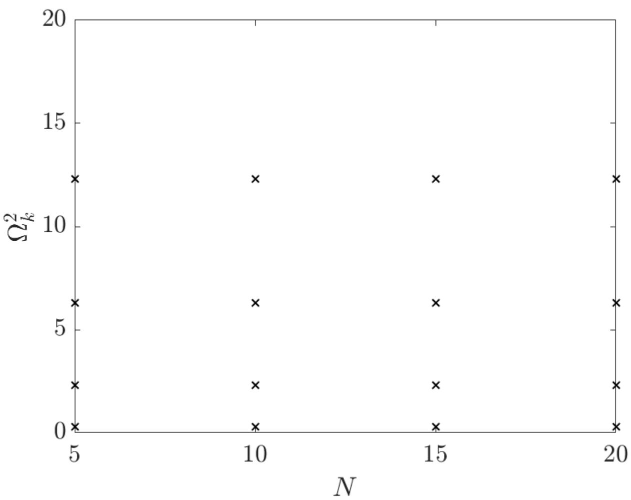
$$M_{ij}=\int_0^L m(x)\psi_i\psi_j\mathrm{dx}=m\int_0^1[1-\sin^{3/2}(x/2)]\sin(i-1/2)\pi x\sin(j-1/2)\pi x\,\mathrm{dx}=mH_{ij}$$

Where $G_{ij} = (i-1/2)\,(j-1/2)\int_0^1 [1-\sin^{3/2}(x/2)]\cos(i-1/2)\pi x\cos(j-1/2)\pi x\,\mathrm{dx}$,

$H_{ij} = \int_0^1 [1-\sin^{3/2}(x/2)]\sin(i-1/2)\pi x\sin(j-1/2)\pi x\,\mathrm{dx}$ are found by numerical integration using matlab subroutine integral().

The eigenvalues $\tilde{\omega}_k^2$ and eigenfunctions $\tilde{\varphi}_k$ are found by solving $(\boldsymbol{K}-\tilde{\omega}_k^2\boldsymbol{M})\boldsymbol{a}_k = \boldsymbol{0}, \tilde{\varphi}_k = \sum_{\ell=1}^{N}\psi_\ell b_\ell$, where $\boldsymbol{b} = \boldsymbol{a}_k$. $(\boldsymbol{K}-\tilde{\omega}_k^2\boldsymbol{M})\boldsymbol{a}_k = \boldsymbol{0} \Longrightarrow (\pi^2 EA\boldsymbol{G}-\tilde{\omega}_k^2 m\boldsymbol{H})\boldsymbol{a}_k = \boldsymbol{0} \Longrightarrow (\boldsymbol{G}-\Omega_k^2\boldsymbol{H})\boldsymbol{a}_k = \boldsymbol{0}$, where $\Omega_k^2 = \tilde{\omega}_k^2\dfrac{m}{\pi^2 EA}$.

The convergence plot for values of $\Omega_k^2$ are as follows:




We see the convergence of $\Omega_k^2$.

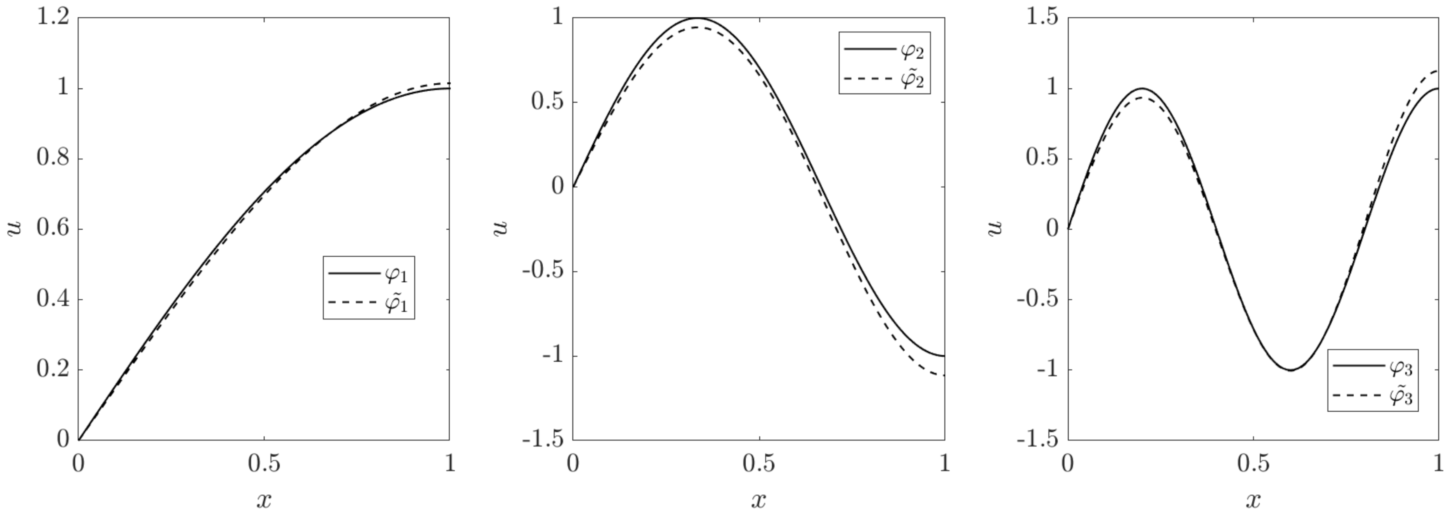
Use the data of N=20, the first three $\Omega_k^2$ are: 0.2938,2.2942,6.2934, so the first three eigenvalues $\tilde{\omega}_k^2 = \dfrac{\pi^2 EA}{m}\Omega_k^2$ are: $\dfrac{\pi^2 EA}{m}\times 0.2938, \dfrac{\pi^2 EA}{m}\times 2.2942, \dfrac{\pi^2 EA}{m}\times 6.2934$.

Comparing to the eigenvalues of uniform rod:

$$\omega_1^2 = \frac{\pi^2 EA}{m}\times 0.25, \omega_2^2 = \frac{\pi^2 EA}{m}\times 2.25, \omega_3^2 = \frac{\pi^2 EA}{m}\times 6.25$$

We see that the natural frequencies of the nonuniform rod are larger than the uniform rod.

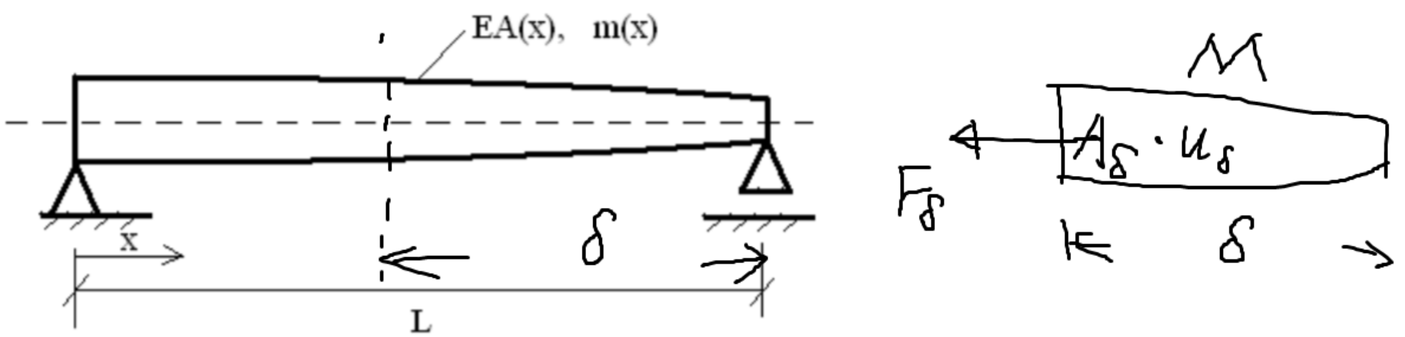
We also plot the eigenfunctions of Rayleigh-Ritz method against the eigenfunctions of uniform rod:




$\varphi_1, \varphi_2, \varphi_3$ are the first three eigenfunctions of the uniform rod; $\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3$ are the first eigenfunctions of the nonuniform rod.

We see the eigenfunctions of the nonuniform rod have larger displacement near the free end comparing to the uniform rod.

To see why these differences make sense, we partition the rod into 2 parts and investigate the right part, shown as below:




We consider the right part as a single oscillator, as the cross section changes, the effect of stiffness drops from EA to $EA[1-\sin^{3/2}(L-\delta)/(2L)]$. The effect of inertia drops from $m\delta$ to

$\int_{L-\delta}^{L}[1-\sin^{3/2}x/(2L)]\mathrm{dx} < m\delta[1-\sin^{3/2}(L-\delta)/(2L)]$, which means both effects of stiffness and inertia drops, but the effect of inertia drops more so that the frequency will need to be higher for inertia effects to balance stiffness effects, natural frequency increases. As $\delta$ increases, the effect of inertia get weakened more, s the part of the rod becomes more compliant, and resulting the eigenfunction to have higher amplitude than uniform beam near the free end.

Code:

```
%% Rayleigh-Ritz

g=@(m,n,x) (m-1/2)*(n-1/2)*(1-sin(x/2).^(3/2)).*cos((m-1/2)*pi*x).*cos((n-1/2)*pi*x);
```

```
h=@(m,n,x) (1-sin(x/2).^(3/2)).*sin((m-1/2)*pi*x).*sin((n-1/2)*pi*x);
for N=5:5:20
    G=zeros(N,N);
    H=G;
    for m=1:N
        for n=1:N
            G(m,n)=integral(@(x) g(m,n,x),0,1);
            H(m,n)=integral(@(x) h(m,n,x),0,1);
        end
    end
    [V,D]=eig(G,H);
    plot(N,diag(D),'kx','LineWidth',1);
    hold on;
end
xticks(5:5:20);
xlabel('$N$','Interpreter','latex')
ylabel('$\Omega_k^2$','Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
ylim([0 20])
%% sort eigenvalues&eigenvectors
[E,I]=sort(diag(D));
%%
k=3;
nexttile;
phi=@(j,x) sin((j-1/2)*pi*x);
xs=linspace(0,1,101);
ys1=phi(k,xs);
ys2=0*xs;
for l=1:20
    ys2=ys2+phi(l,xs)*V(l,I(k));
```

```
end
plot(xs,ys1,'k-',xs,-ys2,'k--','LineWidth',1);
legend({'$\varphi_'+string(k)+'$','$\tilde{\varphi_'+string(k)+'}$'},'Interpreter','latex')
xlabel('$x$','Interpreter','latex')
ylabel('$u$','Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

Problem 4

Suppose $u = u_p + u_f$, where $u_p(0,t) = 2(1-\cos\omega t), u_p(1,t) = \cos 2\omega t - 1$, $\dfrac{\partial^2 u_p}{\partial x^2} = 0$. Substitute into the original equation, $u_f(0,t) = 0, u_f(1,t) = 0$, $\dfrac{\partial^2 u_f}{\partial t^2} = \dfrac{\partial^2 u_f}{\partial x^2} - \dfrac{\partial^2 u_p}{\partial t^2}$.

$\dfrac{\partial^2 u_p}{\partial x^2} = 0 \Longrightarrow u_p = C_1(t)x + C_2$, take in BC, $u_p = 2(1-\cos\omega t) + [\cos 2\omega t + 2\cos\omega t - 3]x$

$\omega^2[2\cos\omega t - x(2\cos\omega t + 4\cos 2\omega t)]$, $\dfrac{\partial^2 u_f}{\partial t^2} = \dfrac{\partial^2 u_f}{\partial x^2} - \dfrac{\partial^2 u_p}{\partial t^2} = \dfrac{\partial^2 u_f}{\partial x^2} - F$

Assume $u_f = \sum_{n=1}^{\infty} \eta_n(t)\varphi_n(x)$, where $-\omega^2\varphi_n = \varphi_n'' \Longrightarrow \varphi_n = C_1\cos\omega_n x + C_2\sin\omega_n x$, take in boundary conditions, $C_1 = 0, \sin\omega_n = 0 \Longrightarrow \omega_n = n\pi, n = 1, 2, \cdots$. Mass-orthogonalize,

$\int_0^1 A_n^2 \sin^2 n\pi x = 1 \Longrightarrow A_n = \sqrt{2}$, so $\varphi_n = \sqrt{2}\sin n\pi x, n = 1, 2, \cdots$.

Substitute back into $u_f = \sum_{n=1}^{\infty} \eta_n(t)\varphi_n(x)$, left multiply by $\varphi_m$ and integrate from 0 to 1,

$$\int_0^1 \sum_{n=1}^{\infty} \ddot{\eta}_n(t)\varphi_m\varphi_n \,\mathrm{dx} = \int_0^1 \sum_{n=1}^{\infty} -\omega_n^2\eta_n(t)\varphi_m\varphi_n \,\mathrm{dx} + \int_0^1 -\varphi_m F \,\mathrm{dx} \Longrightarrow \ddot{\eta}_m(t) + \omega_m^2\eta_m(t) = -\int_0^1 \varphi_m F \,\mathrm{dx}$$

$$N_m(t) = -\int_0^1 \varphi_m F \,\mathrm{dx} = -\omega^2 \int_0^1 \sqrt{2}\sin m\pi x[2\cos\omega t - x(2\cos\omega t + 4\cos 2\omega t)]\,\mathrm{dx}$$

$$= -\frac{\sqrt{2}\,\omega^2}{m\pi}[4(-1)^m \cos 2\omega t + 2\cos\omega t]$$

With zero initial condition, for each modal oscillator, $\eta_m(t) = \dfrac{1}{\omega_m}\int_0^t N_m(\tau)\sin[\omega_m(t-\tau)]\,d\tau$

$$= -\frac{\sqrt{2}\,\omega^2}{m^2\pi^2}\int_0^t [4(-1)^m\cos 2\omega\tau + 2\cos\omega\tau]\sin[m\pi(t-\tau)]\,d\tau$$

So the response is $u = u_p + u_f = 2(1-\cos\omega t) + [\cos 2\omega t + 2\cos\omega t - 3]x$

$$+\sum_{m=1}^{\infty}\left\{-\frac{\sqrt{2}\,\omega^2}{m^2\pi^2}\int_0^t [4(-1)^m \cos 2\omega\tau + 2\cos\omega\tau]\sin[m\pi(t-\tau)]\,d\tau\right\}\sin m\pi x$$

When $\omega = m\pi$ or $2\omega = m\pi$, $m = 1, 2, \cdots$, one or two of the modal oscillators become unbounded in time. So resonance occur when $\omega = m\pi/2, m = 1, 2, \cdots$.

Problem 5.

(i)

Choose $L = 1m, E = 10^7 Pa, A = 10^{-4} m^2, m = 0.1 kg/m, M = 1kg, K = 10^2 N/m$.

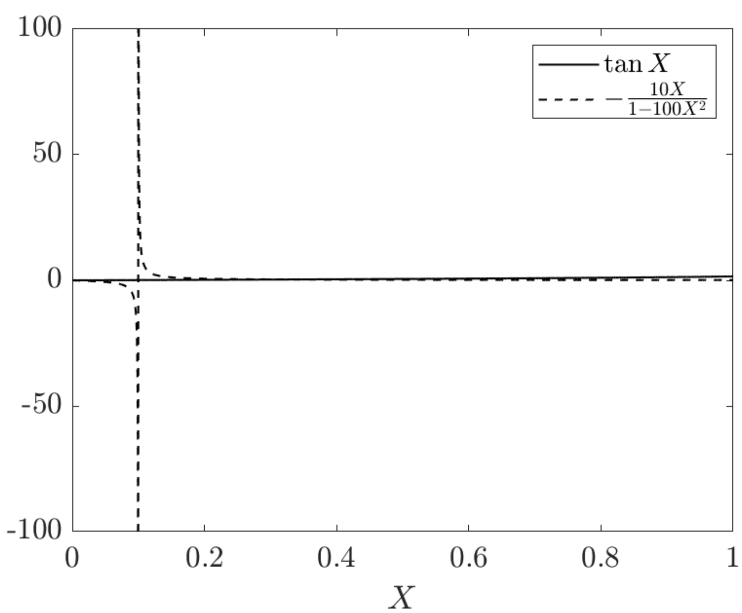
Rigid body mode has frequency $\omega_0 = 0$, with eigenfunction $\varphi_0(x) = 1, \psi = 1$.

The first resonance frequency is the first solution of $X_i = \omega_i L/c$, $\tan(X) = -\dfrac{Kc^2 ML}{EA}\dfrac{X}{KL^2 - c^2 MX^2}$,

where $c^2 = EA/m$. Substitute in the numbers, $c = \sqrt{EA/m} = 100 m/s$,

$$\tan(X) = -\frac{Kc^2ML}{EA}\frac{X}{KL^2 - c^2MX^2} = -\frac{10^2 \cdot 10^4 \cdot 1 \cdot 1}{10^7 \cdot 10^{-4}}\frac{X}{10^2 \cdot 1^2 - 10^4 X^2} = -\frac{10^3 X}{10^2 - 10^4 X^2} = -\frac{10X}{1 - 100X^2}$$

Plot and find the first root is between 0.2 to 0.8.




Use matlab subroutine fzero() to find $X_1 = 0.3262$, so the first natural frequency is $\omega_1 = cX_1/L = 32.62 rad/s$. The first eigenfunction is

$$\varphi_1(x) = A_1(K - \omega_1^2 M)\cos\left(\frac{\omega_1}{c}x\right) = A_1[100 - 32.62^2]\cos(0.3262x) = -964.27A_1\cos(0.3262x)$$

$$\psi_1 = A_1 K\cos\left(\frac{\omega_1}{c}L\right) = 100\cos(0.3262) = 94.7267$$

Code:

```
x1=linspace(0,1,5001);
y1=tan(x1);
y2=-10*x1./(1-100*x1.^2);
plot(x1,y1,'k-',x1,y2,'k--','LineWidth',1)
xlabel('$X$','Interpreter','latex')
legend({'$\tan X$','$-\frac{10X}{1-100X^2}$'},'Interpreter','latex')
set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
ylim([-100 100])
%%
format long
fzero(@(x) tan(x)+10*x/(1-100*x^2),[0.2 0.8])
```

(ii)

The Rayleigh quotient is:

$$\tilde{\omega}^2 = R\left\{\tilde{\varphi},\tilde{\psi}\right\} = \frac{K\left(\tilde{\psi}-\tilde{\varphi}(L)\right)^2 + \int_0^L EA\tilde{\varphi}'^2\mathrm{dx}}{M\tilde{\psi}^2 + \int_0^L m\tilde{\varphi}^2\mathrm{dx}} = \frac{10^2\left(\tilde{\psi}-\tilde{\varphi}(1)\right)^2 + 10^3\int_0^1 \tilde{\varphi}'^2\mathrm{dx}}{\tilde{\psi}^2 + \int_0^1 0.1\cdot\tilde{\varphi}^2\mathrm{dx}}$$

Case 1, test function $\tilde{\varphi} = x, \tilde{\psi} = -0.1$

$$\tilde{\omega}^2 = R\left\{\tilde{\varphi},\tilde{\psi}\right\} = \frac{10^2(-0.1-1)^2 + 10^3\int_0^1 \mathrm{dx}}{0.1^2 + \int_0^1 0.1\cdot x^2\mathrm{dx}} = 25869.2$$

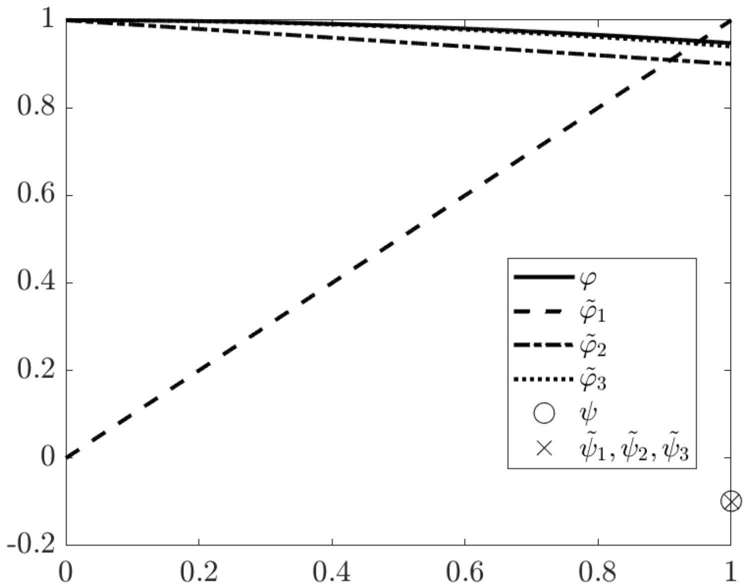
Case 2, test function $\tilde{\varphi} = 1-0.1x, \tilde{\psi} = -0.1$

$$\tilde{\omega}^2 = R\left\{\tilde{\varphi},\tilde{\psi}\right\} = \frac{10^2(-0.1-0.9)^2 + 10^3\int_0^1 0.01\,\mathrm{dx}}{0.1^2 + \int_0^1 0.1\cdot(1-0.1x)^2\mathrm{dx}} = 1096.35$$

Case 3, test function $\tilde{\varphi} = 1-0.06x^2, \tilde{\psi} = -0.1$ , $\tilde{\omega}^2 = R\left\{\tilde{\varphi},\tilde{\psi}\right\} = 1064.94$

RQ of case 1 is very far from the exact value, RQ of case 2 is close to the exact value but case 3 is even closer.

We compare the test functions to the true eigenfunction.




Where $\varphi = \cos(0.3262x)$, $\psi = -94.7267/964.27$ is the exact eigenfunction, $\tilde{\varphi}_1, \tilde{\psi}_1, \tilde{\varphi}_2, \tilde{\psi}_2, \tilde{\varphi}_3, \tilde{\psi}_3$ are trial functions. We see the eigenfunction of case 3 is closest to the exact eigenfunction, case 2 is a little farther, case 1 eigenfunction is very far from the exact eigenfunction.

Mathematica code:

```
ϕ=1-0.06*x^2;

ψ=-0.1;

ϕp=D[ϕ,x];

R=(100*(ψ-ϕ/.{x->1})^2+1000*Integrate[ϕp^2,{x,0,1}])/((ψ)^2+Integrate[0.1*ϕ^2,{x,0,1}])
```

Matlab code:

```
x1=linspace(0,1,5001);

y1=cos(0.3252*x1);

z1=-94.7267/964.27;

y2=x1;

z2=-0.1;

y3=1-0.1*x1;

z3=-0.1;

y4=1-0.06*x1.^2;

z4=-0.1;

plot(x1,y1,'k-',x1,y2,'k--',x1,y3,'k-.',x1,y4,'k:','LineWidth',2)

hold on;

plot(1,z1,'ko',1,z2,'kx','MarkerSize',10)
```

```
legend({'$\varphi$','$\tilde{\varphi}_1$','$\tilde{\varphi}_2$','$\tilde{\varphi}_3$','$\psi$','$\tilde{\psi}_1,\tilde{\psi}_2,\tilde{\psi}_3$'},'Interpreter','latex')

set(gca,'FontSize',15,'BoxStyle','full','TickLabelInterpreter','latex');
```

Problem 6.

We want to show that if a trial function can be expressed as $\varphi(x)=\varphi_r(x)+\sum_{i=1,i\neq r}^{\infty}\varepsilon_i\varphi_i(x),\varepsilon_i=O(\varepsilon)$, then

$$\omega^2=R\{\varphi(x)\}=\omega_r^2+\sum_{i=1,i\neq r}^{\infty}(\omega_i^2-\omega_r^2)\varepsilon_i^2$$

The Rayleigh quotient is given by: $R\{\varphi\}=\dfrac{K_1\varphi^2(0)+K_2\varphi^2(L)+\int_0^L A(x)\varphi'^2(x)\mathrm{dx}}{M_1\varphi^2(0)+M_2\varphi^2(L)+\int_0^L B(x)\varphi^2(x)\mathrm{dx}}$

We define two bilinear forms $P(a,b)=K_1a(0)b(0)+K_2a(L)b(L)+\int_0^L a'(x)A(x)b'(x)\mathrm{dx}$

$$Q(a,b)=b(0)M_1a(0)+b(L)M_2a(L)+\int_0^L b(x)B(x)a(x)\mathrm{dx}$$

The stiffness-orthogonality and mass-orthogonality conditions for the eigenfunctions $\varphi_i(x)$ are given by:

$$\int_0^L \varphi_r{}'(x)A(x)\varphi_s{}'(x)\mathrm{dx}\ +K_1\varphi_r(0)\varphi_s(0)+K_2\varphi_r(L)\varphi_s(L)=\omega_r^2\delta_{rs}$$

$$\int_0^L B(x)\varphi_r(x)\varphi_r(x)\mathrm{dx}\ +M_1\varphi_r(0)\varphi_s(0)+M_2\varphi_r(L)\varphi_s(L)=\delta_{rs}$$

Rewrite the conditions in bilinear forms: $P(\varphi_r,\varphi_s)=\omega^2\delta_{rs},Q(\varphi_r,\varphi_s)=\delta_{rs}$,

Due to the bi-linearity of P, $P(\varphi,\varphi)=P(\varphi,\varphi_r)+\sum_{j=1,j\neq r}^{\infty}P(\varphi,\varepsilon_j\varphi_j)$

$$=P(\varphi_r,\varphi_r)+\cancel{\sum_{i=1,i\neq r}^{\infty}\varepsilon_iP(\varphi_i,\varphi_r)}\ +\sum_{j=1,j\neq r}^{\infty}\left[\varepsilon_jP(\varphi_r,\varphi_j)+\sum_{i=1,i\neq r}^{\infty}\varepsilon_i\varepsilon_jP(\varphi_i,\varphi_j)\right]$$

$$=\omega_r^2+\cancel{\sum_{j=1,j\neq r}^{\infty}\varepsilon_jP(\varphi_r,\varphi_j)}\ +\sum_{j=1,j\neq r}^{\infty}\sum_{i=1,i\neq r}^{\infty}\varepsilon_i\varepsilon_jP(\varphi_i,\varphi_j)$$

$$=\omega_r^2+\sum_{j=1,j\neq r}^{\infty}\sum_{i=1,i\neq r}^{\infty}\varepsilon_i\varepsilon_j\omega_i^2\delta_{ij}=\omega_r^2+\sum_{j=1,j\neq r}^{\infty}\varepsilon_i^2\omega_j^2$$

Similarly, due to bi-linearity of Q, $Q(\varphi,\varphi)=Q(\varphi,\varphi_r)+\sum_{j=1,j\neq r}^{\infty}Q(\varphi,\varepsilon_j\varphi_j)$

$$=Q(\varphi_r,\varphi_r)+\cancel{\sum_{i=1,i\neq r}^{\infty}\varepsilon_i Q(\varphi_i,\varphi_r)}+\sum_{j=1,j\neq r}^{\infty}\left[\varepsilon_j Q(\varphi_r,\varphi_j)+\sum_{i=1,i\neq r}^{\infty}\varepsilon_i\varepsilon_j Q(\varphi_i,\varphi_j)\right]$$

$$=1+\cancel{\sum_{j=1,j\neq r}^{\infty}\varepsilon_j Q(\varphi_r,\varphi_j)}+\sum_{j=1,j\neq r}^{\infty}\sum_{i=1,i\neq r}^{\infty}\varepsilon_i\varepsilon_j Q(\varphi_i,\varphi_j)$$

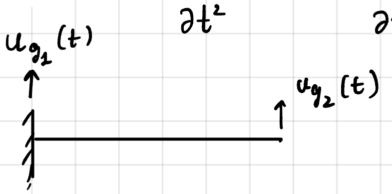
$$=1+\sum_{j=1,j\neq r}^{\infty}\sum_{i=1,i\neq r}^{\infty}\varepsilon_i\varepsilon_j\delta_{ij}=1+\sum_{j=1,j\neq r}^{\infty}\varepsilon_j^2$$

And Rayleigh quotient is:

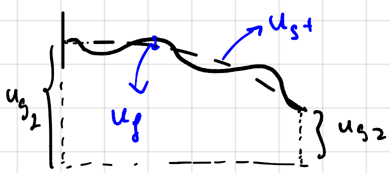
$$R\{\varphi\}=\frac{P(\varphi,\varphi)}{Q(\varphi,\varphi)}=\frac{\omega_r^2+\sum_{j=1,j\neq r}^{\infty}\varepsilon_j^2\omega_j^2}{1+\sum_{j=1,j\neq r}^{\infty}\varepsilon_j^2}=\omega_r^2+\frac{\sum_{j=1,j\neq r}^{\infty}\varepsilon_j^2(\omega_j^2-\omega_r^2)}{1+\sum_{j=1,j\neq r}^{\infty}\varepsilon_j^2}\approx\omega_r^2+\sum_{j=1,j\neq r}^{\infty}\varepsilon_j^2(\omega_j^2-\omega_r^2)$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \le x \le 1, \quad t \ge 0$$

$\rightarrow T/m = 1$




Decompose the motion into free and static: $u = u_{st} + u_f$




where
$$\begin{cases} u_{st}(0,t) = u_{g_1}(t) = 2(1-\cos\omega t) \\ u_{st}(1,t) = u_{g_2}(t) = \cos 2\omega t - 1 \\ u_f(0,t) = 0 \; ; \; T u_f'(1,t) = 0 \rightarrow u_f'(1,t) = 0 \end{cases}$$

Solve for $u_{st}(x,t)$: Treating the beam as non-flexible gives/ neglecting inertia gives:

$$\frac{\partial^2 u_{st}}{\partial x^2} = 0 \Rightarrow u_{st} = C_1(t)x + C_2(t)$$

Apply BCs:
$$\begin{cases} u_{st}(0,t) = C_2(t) = 2 - 2\cos(\omega t) \\ u_{st}(1,t) = C_1(t) + C_2(t) = \cos(2\omega t) - 1 \rightarrow C_1(t) = -3 + \cos(2\omega t) + 2\cos(\omega t) \end{cases}$$

$$\Rightarrow \boxed{u_{st} = (2 - 2\cos\omega t)(1-x) + (\cos 2\omega t - 1)x}$$

Solve for $u_f(x,t)$: GDE becomes:
$$\cancelto{0}{\frac{\partial^2 u_{st}}{\partial x^2}} + \frac{\partial^2 u_f}{\partial x^2} = \frac{\partial^2 u_{st}}{\partial t^2} + \frac{\partial^2 u_f}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u_f}{\partial x^2} - \frac{\partial^2 u_f}{\partial t^2} = \frac{\partial^2 u_{st}}{\partial t^2} = \underbrace{2\omega^2\cos(\omega t)(1-x) - 4\omega^2\cos(2\omega t)x}_{F(x,t)} \quad (*)$$

BCs: $u_f(0,t) = 0 \; ; \; u_f'(1,t) = 0$

ICs: $u_{st}(x,0) + u_f(x,0) = 0 \Rightarrow u_f(x,0) = 0$ ; similarly $u_{f_t}(x,0) = 0$

* Free response: Let $u_f = \varphi(x) f(t) \Rightarrow \begin{cases} \ddot f(t) + \omega^2 f(t) = 0 \\ \varphi''(x) + \omega^2 \varphi(x) = 0 \end{cases}$

$$\Rightarrow \begin{cases} f(t) = A_1\cos\omega t + A_2 \sin\omega t \\ \varphi(x) = B_1\cos\omega x + B_2\sin\omega x \end{cases} ; \text{ BCs} \longrightarrow \begin{cases} B_1 = 0 \\ B_2\,\omega\cos(\omega) = 0 \longrightarrow \cos(\omega) = 0 \end{cases}$$

(for non-trivial solution)

$$\Rightarrow \omega_k = \left(\frac{2k+1}{2}\right)\pi \quad (k = 0, 1, 2, \dots) \Rightarrow \varphi_k = B_k \sin\left(\frac{(2k+1)\pi x}{2}\right)$$

Mass - orthonormalization: $\int_0^1 \varphi_k^2\,dx = 1 \Rightarrow B_k \int_0^1 \sin^2\left[\frac{(2k+1)\pi x}{2}\right] = 1 \Rightarrow B_k = 2$

$$\Rightarrow \boxed{\varphi_k(x) = 2\sin\left[\frac{(2k+1)\pi x}{2}\right]}$$

; Apply modal superposition: $u(x,t) = \sum_{i=1}^{\infty} \eta_i(t)\,\varphi_i(x)$

$$(*) \rightarrow \sum_{i=1}^{\infty} \ddot\eta_i(t)\int_0^1 \varphi_i(x)\varphi_j(x)\,dx = \sum_{i=1}^{\infty} \eta_i(t)\int_0^1 \varphi_i(x)\varphi_j(x)\,dx - \int_0^1 F(x,t)\varphi_j(x)\,dx$$

$$\Rightarrow \sum_{i=1}^{\infty} \ddot{\eta}_i(t)\,\delta_{ik} = -\sum_{i=1}^{\infty} \eta_i(t)\,w^2 \delta_{ik} - \int_0^1 \left[2w^2\cos(wt)(1-x) - 4w^2\cos(2wt)\,x\right] 2\sin(w_k x)\,dx$$

$$\Rightarrow \ddot{\eta}_k(t) + w_k^2\,\eta_k(t) = -\left[\frac{4w^2\left(2\pi k - 2\cos(\pi k) + \pi\right)\cos(wt) - 8w^2\cos(2wt)\,2\cos(\pi k)}{(2\pi k + \pi)^2}\right]$$

$$\Rightarrow \ddot{\eta}_k(t) + w_k^2\,\eta_k(t) = \frac{-4w^2\left(\pi(2k+1) - 2(-1)^k\right)\cos(wt) + 8w^2(-1)^k\cos(2wt)}{\pi^2(2k+1)^2} = N_k(t)$$

ICs $\rightarrow$ $\eta_k(0) = 0$ and $\dot{\eta}_k(0) = 0$

The Solution for the $k^{th}$ modal:

$$\eta_k(t) = \frac{1}{w_k}\int_0^t N_k(\tau)\sin(w_k)(t-\tau)\,d\tau = \frac{1}{w_k}\int_0^t \left(\xi_k\cos(w\tau) + \Psi_k\cos(2w\tau)\right)\sin[w_k(t-\tau)]\,d\tau$$

where $\xi_k = \dfrac{-4w^2\left[\pi(2k+1) - 2(-1)^k\right]}{\pi^2(2k+1)^2}$ and $\Psi_k = \dfrac{8w^2(-1)^k}{\pi^2(2k+1)^2}$

Then $$\eta_k(t) = \frac{\xi_k\left[\cos(wt) - \cos(w_k t)\right]}{w^2 - w_k^2} + \frac{\Psi_k\left[\cos(2wt) - \cos(w_k t)\right]}{w_k^2 - 4w^2}$$

Putting everything together: $u(x,t) = \sum_{k=1}^{\infty} \varphi_k(x)\,\eta_k(t) + u_{st}$

$$\boxed{\begin{aligned}&\Rightarrow u(x,t) = \sum_{k=1}^{\infty}\left(\frac{\xi_k\left[\cos(wt) - \cos(w_k t)\right]}{w^2 - w_k^2} + \frac{\Psi_k\left[\cos(2wt) - \cos(w_k t)\right]}{w_k^2 - 4w^2}\right) 2\sin(w_k x) \\ &\qquad + (2 - 2\cos wt)(1-x) + (\cos 2wt - 1)\,x \\ &\text{where: } \xi_k = \frac{-4w^2\left[\pi(2k+1) - 2(-1)^k\right]}{\pi^2(2k+1)^2} \text{ and } \Psi_k = \frac{8w^2(-1)^k}{\pi^2(2k+1)^2};\quad w_k = \frac{(2k+1)\pi}{2}\end{aligned}}$$

Resonance can occur in this system for when $\boxed{w = w_k \text{ and } w = \frac{w_k}{2}}$

for some k integer

6) Rayleigh's principle: It can be shown that if a trial function is $O(\varepsilon)$ close to the eigenfunction $\varphi_r(x)$, i.e., $\varphi(x) = \varphi_r(x) + \underbrace{\sum_{\substack{i=1 \\ i\neq r}}^{\infty} \varepsilon_i \varphi_i(x)}_{\text{Error}}$,

$\varepsilon_i = O(\varepsilon)$, $0<\varepsilon \ll 1$, then the estimate for the r-th natural frequency that we get using RQ can be expressed as,

$$\omega^2 = R[\varphi(x)] = \omega_r^2 + \sum_{i=1}^{\infty} (\omega_i^2 - \omega_r^2)\varepsilon_i^2 \qquad (4)$$

So the estimate for the natural frequency is $O(\varepsilon^2)$ close to the true value! Note that if we order $\omega_1 < \omega_2 < \omega_3 < \ldots < \omega_r < \ldots \Rightarrow$

$$\Rightarrow \boxed{\omega^2 = R[\varphi(x)] \geq \omega_1^2} \qquad (5)$$

Proof

Consider the RQ in the form (with comparison functions):

$$\omega^2 \simeq \frac{-\int_0^L \frac{d}{dx}\left[A(x)\varphi'(x)\right]\varphi(x)\,dx}{\int_0^L B(x)\varphi^2(x)\,dx} \equiv R[\varphi(x)] \qquad (3a)$$

Now $\varphi(x)$ is a comparison function, so using the expansion theorem,

eigenfunctions (orthonormalized)

$$\varphi(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x) \Rightarrow$$

$$\Rightarrow R[\varphi(x)] = \frac{-\int_0^L \frac{d}{dx}\left[A(x)\frac{d}{dx}\left(\sum_{i=1}^{\infty} a_i\varphi_i(x)\right)\right]\sum_{j=1}^{\infty} a_j\varphi_j(x)\,dx}{\int_0^L B(x)\left(\sum_{i=1}^{\infty} a_i\varphi_i(x)\right)\left(\sum_{j=1}^{\infty} a_j\varphi_j(x)\right)dx} =$$

$$= \frac{-\sum_{j=1}^{\infty} a_j \int_0^L \varphi_j(x) \sum_{i=1}^{\infty} \frac{d}{dx}\left[A(x)\alpha_i\varphi_i'(x)\right]dx}{\int_0^L B(x)\left(\sum_{i=1}^{\infty} a_i\varphi_i(x)\right)\left(\sum_{j=1}^{\infty} a_j\varphi_j(x)\right)dx} \Rightarrow$$

$$\Rightarrow R[\varphi(x)] = \frac{-\sum_{j=1}^{\infty} a_j \sum_{i=1}^{\infty} a_i \int_0^L \overbrace{\varphi_j(x) \frac{d}{dx}\left[A(x)\varphi_i'(x)\right]}^{-\omega_j^2 \delta_{ij}} dx}{\int_0^L B(x)\left(\sum_{i=1}^{\infty} a_i \varphi_i(x)\right)\left(\sum_{j=1}^{\infty} a_j \varphi_j(x)\right) dx} =$$

$$= \frac{-\sum_{j=1}^{\infty} a_j \left(\sum_{i=1}^{\infty} (-a_i)\,\omega_j^2 \delta_{ij}\right)}{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_i a_j \underbrace{\int_0^L B(x)\,\varphi_i(x)\,\varphi_j(x)\,dx}_{\delta_{ij}}} = \frac{\sum_{j=1}^{\infty} a_j^2 \omega_j^2}{\sum_{i=1}^{\infty} a_i \sum_{j=1}^{\infty} a_j \delta_{ij}} \Rightarrow$$

$$\Rightarrow R[\varphi(x)] = \frac{\sum_{j=1}^{\infty} a_j^2 \omega_j^2}{\sum_{j=1}^{\infty} a_j^2}$$

Now, recall that $\varphi(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x)$; suppose that $a_i = \varepsilon_i\, a_k$, ($k$ fixed)
$|\varepsilon_i| \ll 1$, $i = 1, 2, \ldots,\ i \neq k$

So assume that the comparison function $\varphi(x)$ is "close" to the $k$-th orthonormalized eigenfunction $\varphi_k(x)$ $\Rightarrow$

Use bi-nomial expansion $(a+b)^n = a^n + n a^{n-1} b + \dots$

$$\Rightarrow R[\varphi(x)] = \frac{\omega_k^2 + \sum_{j=1}^{\infty}(1-\delta_{jk})\,\omega_j^2 \varepsilon_j^2}{1 + \sum_{j=1}^{\infty}(1-\delta_{jk})\,\varepsilon_j^2} \approx \omega_k^2 + \sum_{j=1}^{\infty}(\omega_j^2 - \omega_k^2)\,\varepsilon_j^2$$

So, if the trial function $\varphi(x)$ differs from the k-th eigenfunction $\varphi_k(x)$ by a small quantity (in functional space) of first order, then the Rayleigh's quotient differs from the eigenfrequency squared $\omega_k^2$ by an even smaller quantity of second order.

Moreover, if we select $k=1 \Rightarrow R[\varphi(x)] \approx \omega_1^2 + \sum_{j=1}^{\infty}\underbrace{(\omega_j^2 - \omega_1^2)}_{\geq 0}\varepsilon_j^2$,

So, $R[\varphi(x)] \geq \omega_1^2$, which represents a <u>global minimum</u> of the Rayleigh Quotient!