Concepts of Kinetic and Elastic Energy

Consider the unforced generalized wave equation,

No damping effects!

$$\underbrace{B(x)\frac{\partial^2 u}{\partial t^2}}_{\text{Inertia effects}} - \underbrace{\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right]}_{\text{Elastic effects}} = 0\,, \quad \begin{matrix} 0\le x\le L \\ t\ge 0 \end{matrix} \qquad (1)$$

Consider non-simple BCs,

$$-A(0)\frac{\partial u(0,t)}{\partial x} + k_1\, u(0,t) + M_1\frac{\partial^2 u(0,t)}{\partial t^2} = 0 \qquad (1a)$$

$$A(L)\frac{\partial u(L,t)}{\partial x} + k_2\, u(L,t) + M_2\frac{\partial^2 u(L,t)}{\partial t^2} = 0 \qquad (1b)$$

We wish to show energy conservation in the system, in the absence of dissipative forces $\Rightarrow \displaystyle\int_0^L (1)\frac{\partial u}{\partial t}\,dx \Rightarrow$

$$\Rightarrow \int_0^L \underbrace{B(x)\frac{\partial^2 u}{\partial t^2}\frac{\partial u}{\partial t}}_{\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2\right)}dx - \underbrace{\int_0^L \frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right]\frac{\partial u}{\partial t}\,dx}_{A(x)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L - \int_0^L A(x)\frac{\partial u}{\partial x}\underbrace{\frac{\partial^2 u}{\partial x\partial t}}_{\frac{\partial}{\partial t}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)}dx} = 0 \Rightarrow$$

$$A(x)\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L - \int_0^L A(x)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\,dx = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\int_0^L \frac{1}{2}B(x)\left(\frac{\partial u}{\partial t}\right)^2 dx - A(L)\frac{\partial u(L,t)}{\partial x}\frac{\partial u(L,t)}{\partial t} + A(0)\frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t} +$$

$$+\frac{\partial}{\partial t}\int_0^L \frac{1}{2}A(x)\left(\frac{\partial u}{\partial x}\right)^2 dx = 0 \qquad (2)$$

Now $(1a)\times\frac{\partial u(0,t)}{\partial t} \Rightarrow +A(0)\frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t} = \underbrace{k_1 u(0,t)\frac{\partial u(0,t)}{\partial t}}_{\frac{\partial}{\partial t}\left(\frac{1}{2}k_1 u(0,t)^2\right)} +$

$$+\underbrace{M_1\frac{\partial^2 u(0,t)}{\partial t^2}\frac{\partial u(0,t)}{\partial t}}_{\frac{\partial}{\partial t}\left(\frac{1}{2}M_1\left(\frac{\partial u(0,t)}{\partial t}\right)^2\right)} \Rightarrow$$

$$\Rightarrow A(0)\frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}k_1 u^2(0,t)\right] + \frac{\partial}{\partial t}\left[\frac{1}{2}M_1\left(\frac{\partial u(0,t)}{\partial t}\right)^2\right] \qquad (2a)$$

Similarly, $(1b)\times\frac{\partial u(L,t)}{\partial t} \Rightarrow -A(L)\frac{\partial u(L,t)}{\partial x}\frac{\partial u(L,t)}{\partial t} =$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2}k_2 u^2(L,t)\right] + \frac{\partial}{\partial t}\left[\frac{1}{2}M_2\left(\frac{\partial u(L,t)}{\partial t}\right)^2\right] \qquad (2b)$$

Substituting (2a), (2b) into (2) $\Rightarrow$

$$\Rightarrow \frac{\partial}{\partial t}\left\{ \underbrace{\frac{1}{2}\int_0^L B(x)\left(\frac{\partial u}{\partial t}\right)^2 dx}_{\text{Kinetic energy in the elastodynamic system}} + \underbrace{\frac{1}{2}\int_0^L A(x)\left(\frac{\partial u}{\partial x}\right)^2 dx}_{\text{Potential or elastic energy in the system}} + \right.$$

$$\underbrace{+\frac{1}{2}M_1\left(\frac{\partial u(0,t)}{\partial t}\right)^2 + \frac{1}{2}M_2\left(\frac{\partial u(L,t)}{\partial t}\right)^2}_{\text{Kinetic energy stored in the boundaries}} +$$

$$\left. \underbrace{+\frac{1}{2}k_1 u^2(0,t) + \frac{1}{2}k_2 u^2(L,t)}_{\text{Potential energy stored at the boundaries}} \right\} = 0 \Rightarrow$$

Expression that shows conservation of energy in this elastodynamic system.

Note, that if the boundary conditions are 'simple', i.e., there are no inertia or stiffness elements, then the conservation of energy expression simplifies to,

$$\frac{\partial}{\partial t}\left\{ \frac{1}{2}\int_0^L B(x)\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^L A(x)\left(\frac{\partial u}{\partial x}\right)^2 dx \right\} = 0$$

Rayleigh's Quotient

We now formulate approximate methods that allow us to solve a broad class of problems governed by the generalized wave equation, even when analytical solutions are not available. To this end, consider the following general eigenvalue problem in space:

$$\boxed{\begin{aligned} &\frac{d}{dx}\left[A(x)\,\varphi'(x)\right] + \omega^2 B(x)\,\varphi(x) = 0 && (1)\\ &A(0)\,\varphi'(0) - (k_1 - \omega^2 M_1)\,\varphi(0) = 0 && (1a)\\ &A(L)\,\varphi'(L) + (k_2 - \omega^2 M_2)\,\varphi(L) = 0 && (1b) \end{aligned}}$$

Suppose that $\varphi(x)$ is a trial function $\Rightarrow$

$$\Rightarrow \int_0^L (1)\,\varphi(x)\,dx \Rightarrow \int_0^L \frac{d}{dx}\left[A(x)\varphi'(x)\right]\varphi(x)\,dx + \omega^2 \int_0^L B(x)\varphi^2(x)\,dx \cong 0 \Rightarrow \quad (2)$$

$$\Rightarrow \boxed{\omega^2 \simeq \frac{-\int_0^L \frac{d}{dx}\left[A(x)\varphi'(x)\right]\varphi(x)\,dx}{\int_0^L B(x)\,\varphi^2(x)\,dx} \equiv R[\varphi(x)]} \quad (3a)$$

Functional of $\varphi(x)$

Rayleigh's quotient (does not incorporate the BCs)

Now, let's try to include the boundary conditions into Rayleigh's quotient

⇒ Reconsider (2) ⇒ Perform integration by parts ⇒

$$\Rightarrow \left[A(x)\varphi'(x)\varphi(x)\right]\Big|_0^L - \int_0^L A(x)\varphi'^2(x)\,dx + \omega^2\int_0^L B(x)\varphi^2(x)\,dx = 0 \Rightarrow$$

$$\Rightarrow \underbrace{A(L)\varphi'(L)\varphi(L)}_{-(k_2-\omega^2 M_2)\varphi^2(L)} - \underbrace{A(0)\varphi'(0)\varphi(0)}_{-(k_1-\omega^2 M_1)\varphi^2(0)} - \int_0^L A(x)\varphi'^2(x)\,dx + \omega^2\int_0^L B(x)\varphi^2(x)\,dx = 0$$

$$\Rightarrow \omega^2\left[M_1\varphi^2(0) + M_2\varphi^2(L) + \int_0^L B(x)\varphi^2(x)\,dx\right] =$$

$$= k_1\varphi^2(0) + k_2\varphi^2(L) + \int_0^L A(x)\varphi'^2(x)\,dx \Rightarrow$$

Maximum potential of elastic energy, V_{max}

$$\Rightarrow \boxed{\omega^2 \cong \frac{k_1\varphi^2(0) + k_2\varphi^2(L) + \int_0^L A(x)\varphi'^2(x)\,dx}{M_1\varphi^2(0) + M_2\varphi^2(L) + \int_0^L B(x)\varphi^2(x)\,dx} \equiv R[\varphi(x)]} \quad (3b)$$

Alternative form of Rayleigh's quotient (does incorporate the BCs)

Maximum kinetic energy, T_{max}

Remarks

1) Expressions (3a) and (3b) are of the form $\omega^2 = R[\varphi(x)]$, so these are functionals.

2) If $\varphi(x) = \varphi_r(x)$, $r = 1, 2, \ldots \Rightarrow$ By construction (3a) and (3b) will compute the corresponding exact natural frequencies ω_r^2, $r = 1, 2, \ldots$

3) Since in the RQ (3a) the boundary conditions are not taken into account, the trial functions should satisfy the BCs as well; also the trial functions $\varphi(x)$ should be as differentiable as needed in order to compute the RQ. We will designate such trial functions as <u>comparison functions.</u>

4) However, considering the RQ (3b), the BCs are fully accounted for in the Quotient, so the trial functions do not need to satisfy them. In addition we note that there are less smoothness requirements for the trial functions in this case (just continuously differentiable). We will designate such trial functions as <u>admissible functions</u>.

5) Hence, $\{\text{normal modes}\} \subset \underset{(3a)}{\{\text{comparison functions}\}} \subset \underset{(3b)}{\{\text{admissible functions}\}}$

6) Rayleigh's principle: It can be shown that if a trial function is $O(\varepsilon)$ close to the eigenfunction $\varphi_r(x)$, i.e., $\varphi(x) = \varphi_r(x) + \underbrace{\sum_{\substack{i=1 \\ i \neq r}}^{\infty} \varepsilon_i \varphi_i(x)}_{\text{Error}}$,

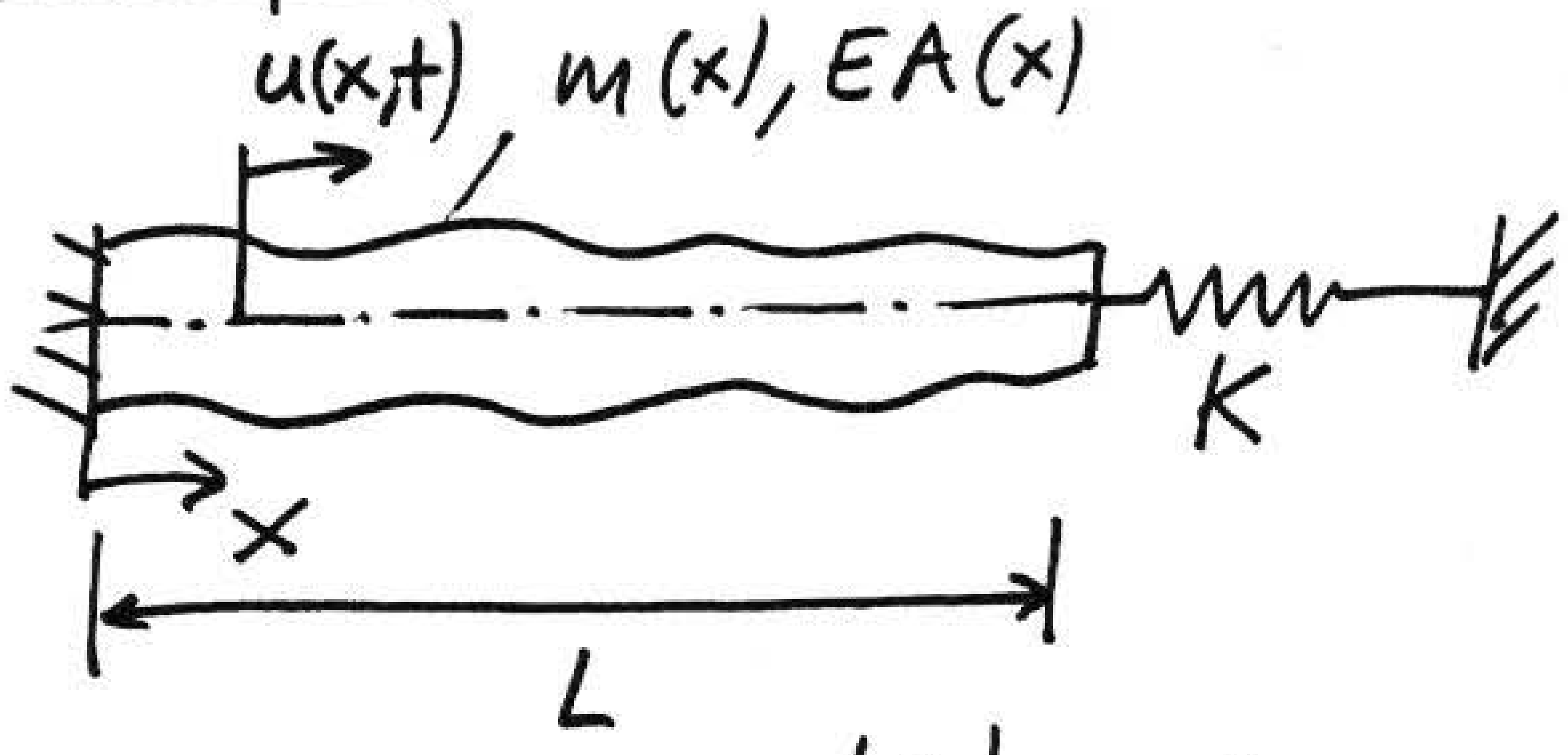
$\varepsilon_i = O(\varepsilon)$, $0 < \varepsilon \ll 1$, then the estimate for the r-th natural frequency that we get using RQ can be expressed as,

$$\omega^2 = R[\varphi(x)] = \omega_r^2 + \sum_{i=1}^{\infty} (\omega_i^2 - \omega_r^2)\varepsilon_i^2 \qquad (4)$$

So the estimate for the natural frequency is $O(\varepsilon^2)$ close to the true value! Note that if we order $\omega_1 < \omega_2 < \omega_3 < \ldots < \omega_r < \ldots \Rightarrow$

$$\Rightarrow \boxed{\omega^2 = R[\varphi(x)] \geq \omega_1^2} \qquad (5)$$

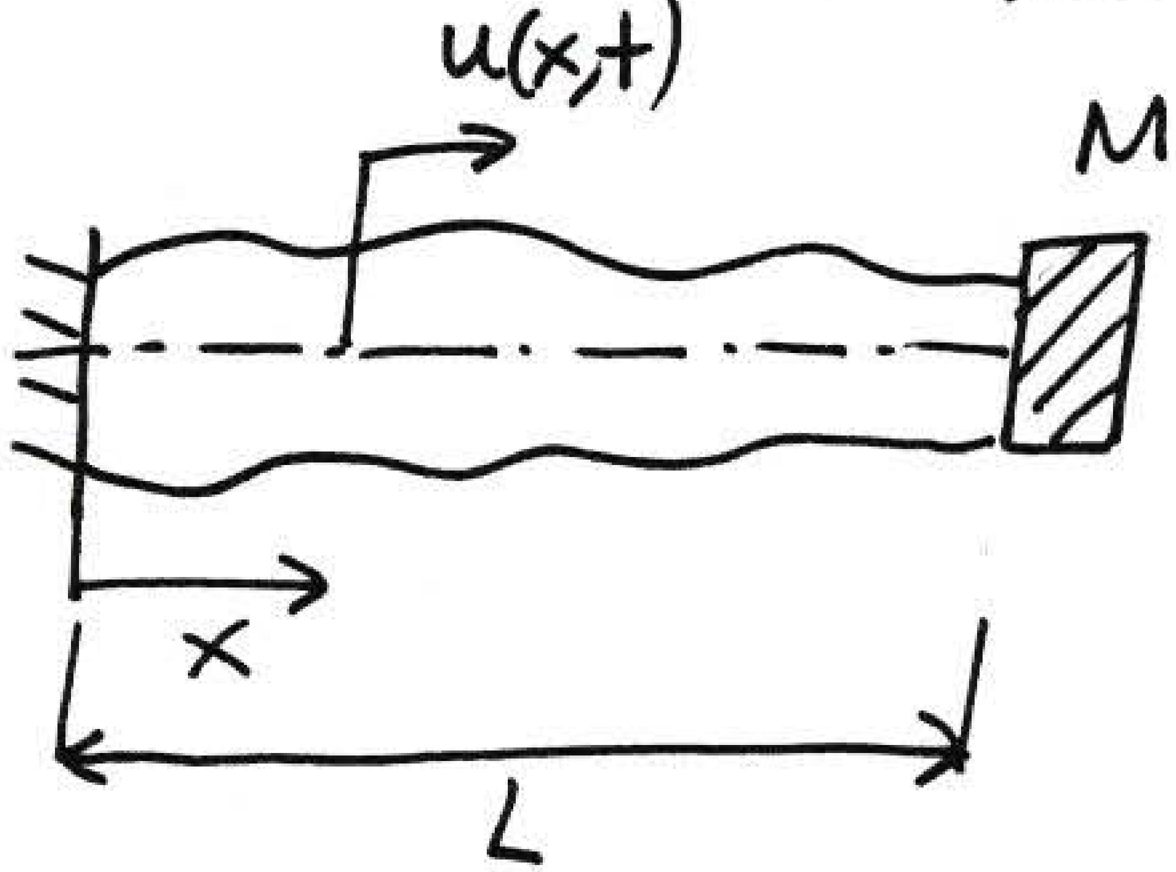
Examples



$$R[\varphi(x)]=\frac{\int_0^L EA(x)\left(\frac{d\varphi(x)}{dx}\right)^2 dx+K\varphi^2(L)}{\int_0^L m(x)\,\varphi^2(x)\,dx}$$

$\varphi(0)=0,\ EA(x)\left.\frac{d\varphi}{dx}\right|_{x=L}=-K\varphi(L)$

$\varphi(x)$ is an admissible function



Term from integration by parts

$$\omega^2=R[\varphi(x)]=\frac{-\omega^2 M\varphi^2(L)+\int_0^L EA(x)\left(\frac{d\varphi(x)}{dx}\right)^2 dx}{\int_0^L m(x)\,\varphi^2(x)\,dx}\Rightarrow$$

$\varphi(0)=0$

$EA\left.\frac{d\varphi}{dx}\right|_{x=L}=\omega^2 M\varphi(L)$

$$\Rightarrow\ \omega^2=\frac{\int_0^L EA(x)\left(\frac{d\varphi(x)}{dx}\right)^2 dx}{\int_0^L m(x)\,\varphi^2(x)\,dx+M\varphi^2(L)}$$

$\varphi(x)$ is an admissible function