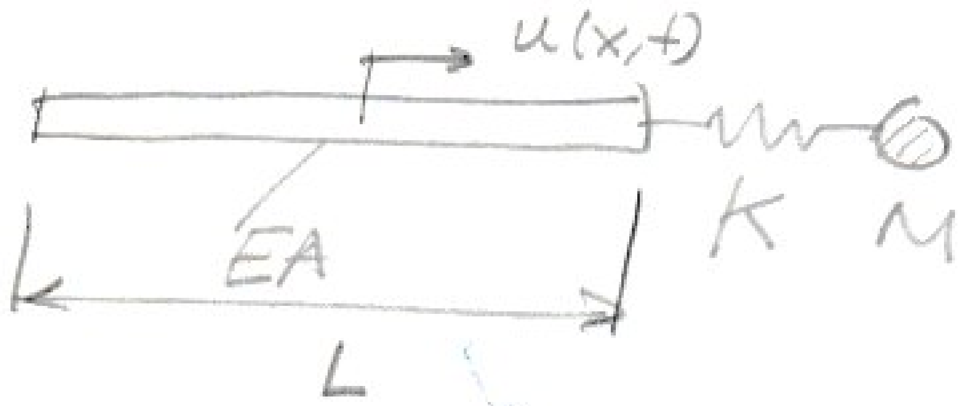

$$m\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right]$$

$$EA\frac{\partial u(0,t)}{\partial x}=0$$

$$-EA\frac{\partial u(L-,t)}{\partial x}+K\left[y(t)-u(L-,t)\right]=0$$

$$M\ddot{y}(t)=-K\left[y(t)-u(L-,t)\right]$$

Seek normal mode solution $\Rightarrow$ $\left.\begin{array}{l} u(x,t)=\varphi(x)f(t) \\ y(t)=\psi f(t)\end{array}\right\} \Rightarrow \ddot{f}(t)+\omega^2 f(t)=0$

$$\varphi''(x)+\left(\frac{\omega}{c}\right)^2\varphi(x)=0\Rightarrow\varphi(x)=C_1\cos\left(\frac{\omega x}{c}\right)+C_2\sin\left(\frac{\omega x}{c}\right)$$

But $M\psi\left(-\omega^2\cancel{f(t)}\right)=-K\left[\psi\cancel{f(t)}-\varphi(L)\cancel{f(t)}\right]\Rightarrow$

$$\Rightarrow (K-\omega^2M)\psi=K\varphi(L)\Rightarrow\psi=\left(\frac{K}{K-\omega^2M}\right)\varphi(L)$$

$$-EA\,\varphi'(L)+K\left[\psi-\varphi(L)\right]=0\Rightarrow$$

$$\Rightarrow -\frac{EA}{K}\varphi'(L)+\left(\frac{K}{K-\omega^2M}-1\right)\varphi(L)=0\Rightarrow\boxed{-\frac{EA}{K}\varphi'(L)+\left(\frac{1}{1-\left(\frac{\omega}{\Omega}\right)^2}-1\right)\varphi(L)=0}$$

where $\Omega^2=\frac{K}{M}$

Also, $\boxed{\varphi'(0)=0}\Rightarrow C_2=0\Rightarrow\varphi(x)=C_1\cos\left(\frac{\omega x}{c}\right)$

Hence

$$\left[-\left(\frac{EA}{K}\right)\left(\frac{\omega}{c}\right)\sin\frac{\omega L}{c}+\left(\frac{1}{1-\left(\frac{\omega}{\Omega}\right)^2}-1\right)\cos\frac{\omega L}{c}\right]C_1=0\Rightarrow$$

Note that this admits the solution $\omega=0$ rigid body mode.

$$\Rightarrow\tan\frac{\omega L}{c}=\left(1-\frac{1}{1-\left(\frac{\omega}{\Omega}\right)^2}\right)\Big/-\left(\frac{EA}{K}\frac{\omega}{c}\right)\Rightarrow\boxed{\tan\frac{\omega L}{c}=\frac{\left(\frac{1}{1-\left(\frac{\omega}{\Omega}\right)^2}-1\right)}{\frac{1}{L}\frac{EA}{K}\left(\frac{\omega L}{c}\right)}=\frac{\left(\frac{LK}{EA}\right)\left(\frac{\omega L}{c}\right)}{\left(\frac{\Omega L}{c}\right)^2-\left(\frac{\omega L}{c}\right)^2}}$$

Derive orthogonality conditions.

Consider the eigenfunction $\varphi_r(x) = C_r \cos\left(\frac{\omega_r x}{c}\right) \Rightarrow$

$$-\omega_r^2 m \varphi_r(x) = EA\, \varphi_r''(x) \Rightarrow -\omega_r^2 \int_0^L m \varphi_r(x) \varphi_k(x)\, dx = \int_0^L EA\, \varphi_r''(x) \varphi_k(x)\, dx =$$

$$= \underbrace{EA\, \varphi_r'(x) \varphi_k(x) \Big|_0^L}_{} - \int_0^L EA\, \varphi_r'(x) \varphi_k'(x)\, dx$$

$$-EA\, \varphi_r'(0) \cancel{\varphi_k(0)} + EA\, \varphi_r'(L) \varphi_k(L) = \cancel{EA} \frac{K}{\cancel{EA}} \varphi_r(L) \varphi_k(L) \left(\frac{\omega_r^2}{\Omega^2 - \omega_r^2}\right)$$

0

But $\varphi_r'(L) = \left(\frac{1}{1 - \left(\frac{\omega_r}{\Omega^2}\right)^2} - 1\right) \frac{K}{EA} \varphi_r(L) =$

$$= \left(\frac{\Omega^2}{\Omega^2 - \omega_r^2} - 1\right) \frac{K}{EA} \varphi_r(L) = \left(\frac{\omega_r^2}{\Omega^2 - \omega_r^2}\right) \frac{K}{EA} \varphi_r(L)$$

$\Rightarrow$

$K = \Omega^2 M$

$$\Rightarrow -\omega_r^2 \int_0^L m \varphi_r(x) \varphi_k(x)\, dx = K \varphi_r(L) \varphi_k(L) \left(\frac{\omega_r^2}{\Omega^2 - \omega_r^2}\right) - \int_0^L EA\, \varphi_r'(x) \varphi_k'(x)\, dx$$

Similarly, $-\omega_k^2 \int_0^L m \varphi_r(x) \varphi_k(x)\, dx = \Omega^2 M \varphi_r(L) \varphi_k(L) \left(\frac{\omega_k^2}{\Omega^2 - \omega_k^2}\right) - \int_0^L EA\, \varphi_r'(x) \varphi_k'(x)\, dx$

$\Rightarrow$

$$\Rightarrow (\omega_k^2 - \omega_r^2) \int_0^L m \varphi_r(x) \varphi_k(x)\, dx = \Omega^2 M \varphi_r(L) \varphi_k(L) \underbrace{\left(\frac{\omega_r^2}{\Omega^2 - \omega_r^2} - \frac{\omega_k^2}{\Omega^2 - \omega_k^2}\right)}_{\frac{\omega_r^2 \Omega^2 - \cancel{\omega_k^2 \omega_r^2} - \omega_k^2 \Omega^2 + \cancel{\omega_k^2 \omega_r^2}}{(\Omega^2 - \omega_r^2)(\Omega^2 - \omega_k^2)}} \Rightarrow$$

$$\Rightarrow (\omega_k^2 - \omega_r^2) \int_0^L m \varphi_r(x) \varphi_k(x)\, dx =$$

$$= \Omega^2 M \varphi_r(L) \varphi_k(L) \frac{\Omega^2 (\omega_r^2 - \omega_k^2)}{(\Omega^2 - \omega_r^2)(\Omega^2 - \omega_k^2)} \Rightarrow \boxed{\int_0^L m \varphi_r(x) \varphi_k(x)\, dx + \frac{\Omega^4 M \varphi_r(L) \varphi_k(L)}{(\Omega^2 - \omega_r^2)(\Omega^2 - \omega_k^2)} = \delta_{rs} \quad \omega_k \neq \omega_r \text{ (Mass orthonormality)}}$$

Also, $\int_0^L EA\varphi_r''(x)\varphi_k(x)\,dx = -\omega_r^2\delta_{rs} + \omega_r^2 \frac{\Omega^4 M\varphi_r(L)\varphi_k(L)}{(\Omega^2-\omega_r^2)(\Omega^2-\omega_k^2)} \Rightarrow$

$$\Rightarrow K\varphi_r(L)\varphi_k(L)\left(\frac{\omega_r^2}{\Omega^2-\omega_r^2}\right) - \int_0^L EA\varphi_r'(x)\varphi_k'(x)\,dx = -\omega_r^2\delta_{rs} + \omega_r^2\frac{\Omega^4 M\varphi_r(L)\varphi_k(L)}{(\Omega^2-\omega_r^2)(\Omega^2-\omega_k^2)} \Rightarrow$$

$K\Omega^2$

$$\Rightarrow \int_0^L EA\,\varphi_r'(x)\varphi_k'(x)\,dx = K\varphi_r(L)\varphi_k(L)\frac{\omega_r^2}{(\Omega^2-\omega_r^2)} -$$

$$-\omega_r^2\frac{K\varphi_r(L)\varphi_k(L)\Omega^2}{(\Omega^2-\omega_r^2)(\Omega^2-\omega_k^2)} + \omega_r^2\delta_{rs} =$$

$$= \omega_r^2\delta_{rs} + \frac{K\omega_r^2}{\Omega^2-\omega_r^2}\varphi_r(L)\varphi_k(L)\underbrace{\left[1-\frac{\Omega^2}{\Omega^2-\omega_k^2}\right]}$$

$$\frac{\cancel{\Omega^2}-\omega_k^2-\cancel{\Omega^2}}{\Omega^2-\omega_k^2} = -\frac{\omega_k^2}{\Omega^2-\omega_k^2}$$

$$\Rightarrow \int_0^L EA\,\varphi_r'(x)\varphi_k'(x)\,dx = \omega_r^2\delta_{rs} - \frac{K\omega_r^2\omega_k^2}{(\Omega^2-\omega_r^2)(\Omega^2-\omega_k^2)}\varphi_r(L)\varphi_k(L) \Rightarrow$$

$$\Rightarrow \boxed{\int_0^L EA\,\varphi_r'(x)\varphi_k'(x)\,dx + \frac{K\omega_r^2\omega_k^2}{(\Omega^2-\omega_r^2)(\Omega^2-\omega_k^2)}\varphi_r(L)\varphi_k(L) = \omega_r^2\delta_{rs}}$$

(stiffness orthonormality)