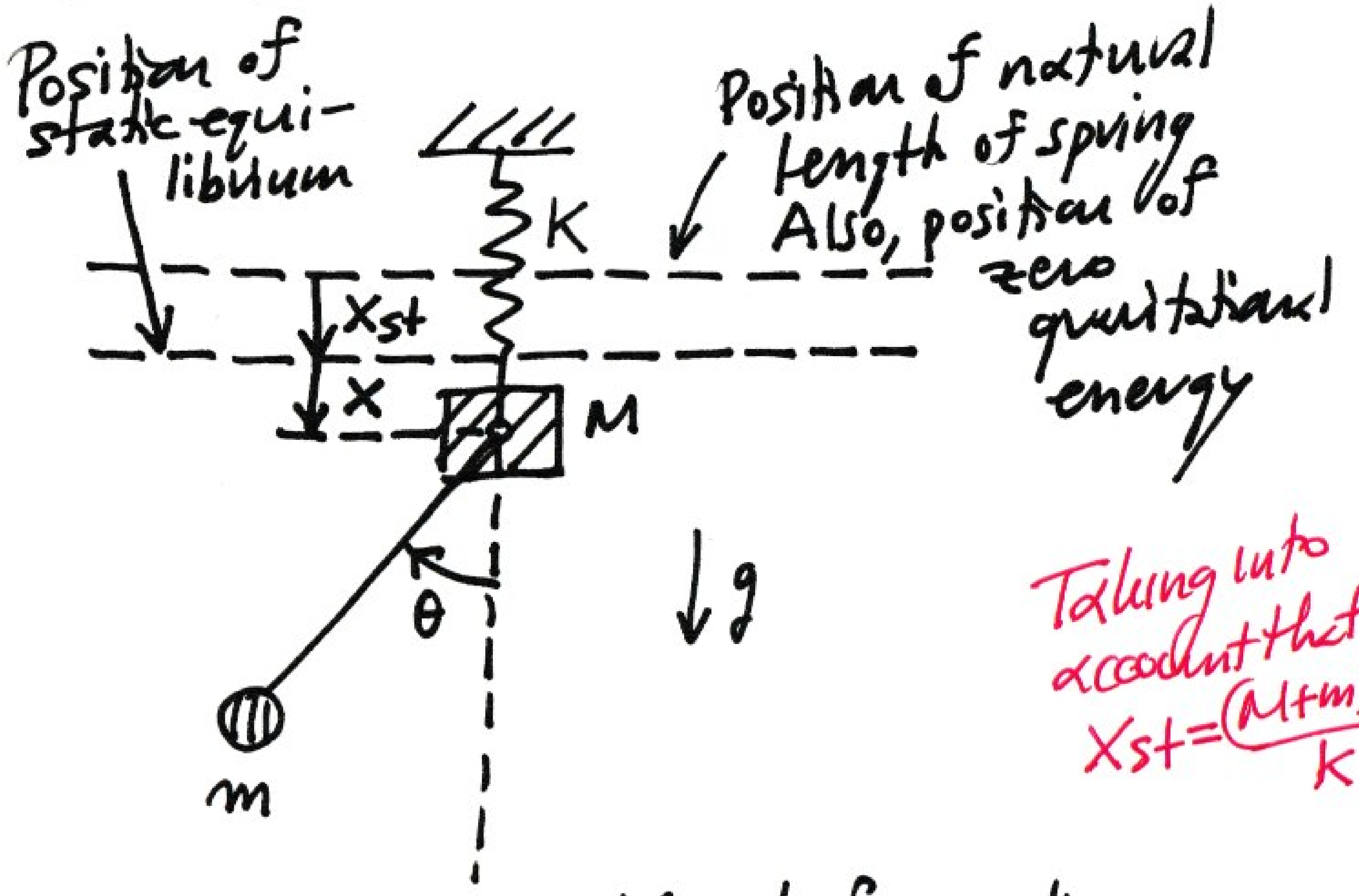
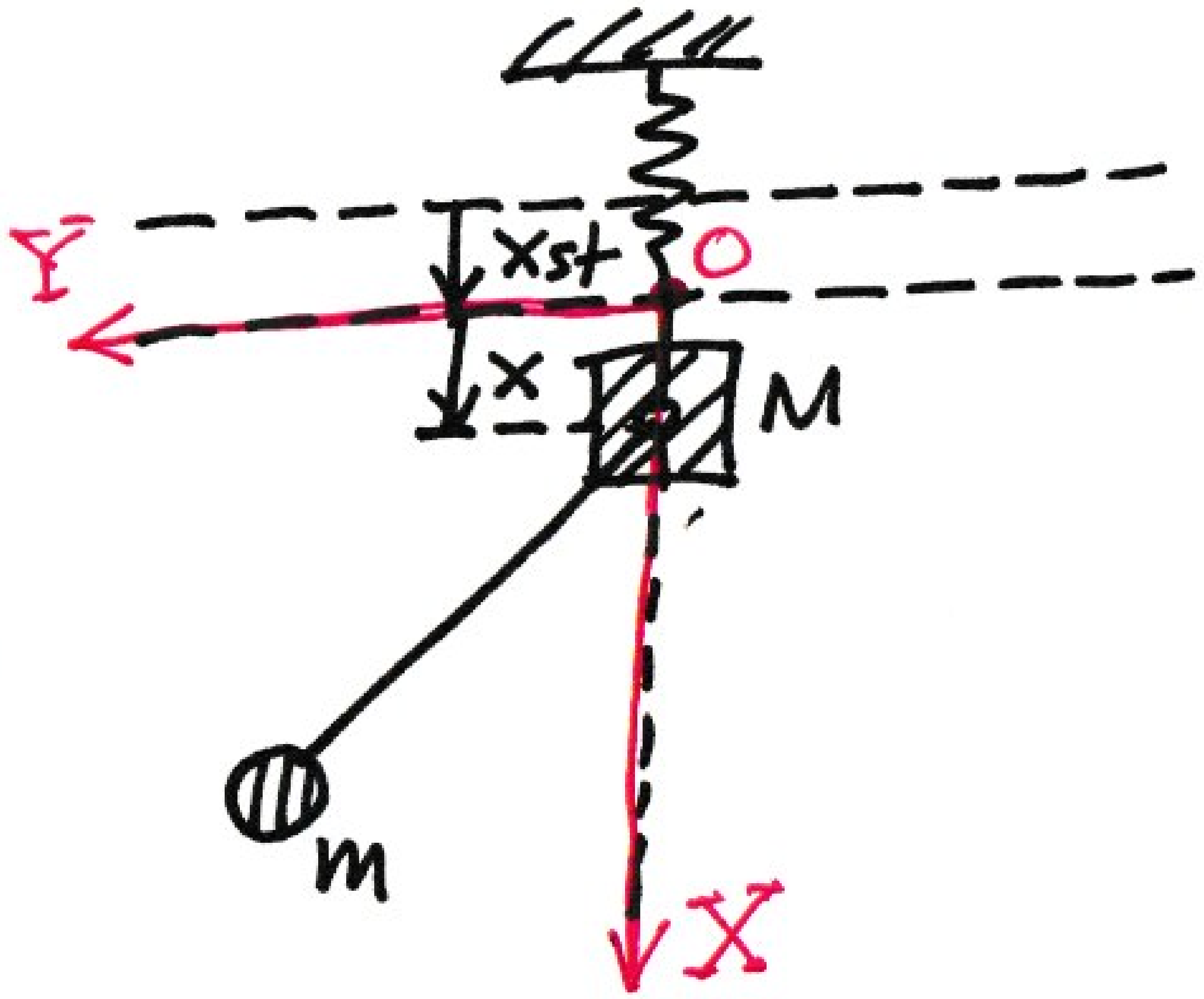
Example



Note that x is the deformation of the spring from its static equilibrium.



Remark:
To compute T, refer the motions of the mass to an inertial coordinate frame, e.g., (OXY).
Then, $T=\frac{1}{2}mV_{m\,abs}^2+\frac{1}{2}MV_{M\,abs}^2$

This system has 2DOF with its static equilibrium at $X=X_{st}$, $\theta=0$.
Gravity is taken into account.

Potential energy is:

$$V=\frac{1}{2}K(X_{st}+X)^2-Mg(X_{st}+X)-$$
$$-mg(X_{st}+X+l\cos\theta)=$$

Taking into account that $X_{st}=\frac{(M+m)g}{k}$

$$=\frac{1}{2}kx^2+mgl(1-\cos\theta)+\text{Const}$$

Not needed

Kinetic energy is:

$$T=\frac{1}{2}(M+m)\dot{x}^2+\frac{1}{2}m(\dot{l\sin\theta})^2+$$
$$+\frac{1}{2}m(x(\dot{l-l\cos\theta}))^2$$

So, $V=V(x,\theta)$ and $T=T(\theta,\dot{x},\dot{\theta})$ and this is a <u>nonlinear system</u>.

But, what if we assume small vibrations and <u>linearize</u> close to the position of static equilibrium?

Then, $V \sim \frac{1}{2}kx^2 + \frac{1}{2}mg\ell\theta^2$, since $1-\cos\theta \sim \frac{\theta^2}{2} + \cdots$

$T \sim \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2$, since $\sin\theta \sim \theta + \cdots$

So, both V and T become quadratic forms,

$$V \sim \frac{1}{2}\begin{Bmatrix} x \\ \theta \end{Bmatrix}^T \begin{bmatrix} K & 0 \\ 0 & mg\ell \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix}, \quad T \sim \frac{1}{2}\begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix}^T \begin{bmatrix} M+m & 0 \\ 0 & m\ell^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix}$$

Stiffness matrix [K] is positive definite

Mass matrix [M] is positive definite

Then, the linearized system of equations becomes <u>uncoupled</u> (i.e., both the stiffness and mass matrices are diagonal). Note that the original nonlinear equations of motion are <u>coupled</u> since both x and θ appear in both equations of motion.

For linear vibrating systems, the stiffness matrix $[K]$ is <u>positive semi-definite</u>, whereas $[M]$ is <u>positive definite</u> $\Rightarrow$ We can express the n equations of motion governing the dynamics as (assuming n-DOF):

$$[M]\{\ddot{q}\}+[K]\{q\}=\{Q\}$$

These equations will be <u>coupled</u>, in general!

Now, we may introduce the change of coordinates, $\{q\}_{n\times1}=[B]_{n\times n}\{y\}_{n\times1}$ where $[B]$ is a constant non-singular matrix $\Rightarrow$

$$\Rightarrow [M][B]\{\ddot{y}\}+[K][B]\{y\}=\{Q\} \Rightarrow \text{Pre-multiply by } [B]^{-1} \Rightarrow$$

$$\Rightarrow \underbrace{[B]^{-1}[M][B]}_{[M_B]}\{\ddot{y}\}+\underbrace{[B]^{-1}[K][B]}_{[K_B]}\{y\}=\underbrace{[B]^{-1}\{Q\}}_{\{F\}}$$

<u>Then, we'll denote $[B]\equiv[\Psi]$</u>

Now, if $[B]$ is such that <u>both</u> $[M_B]$ and $[K_B]$ become diagonal matrices $\Rightarrow$ The transformed equations of motion become <u>uncoupled</u>:

$$\lceil m_b \rfloor\{\ddot{y}\}+\lceil k_b \rfloor\{y\}=\{F\} \Rightarrow$$

$$\Rightarrow \lceil I \rfloor\{\ddot{y}\}+\underbrace{\lceil m_b^{-1} \rfloor\lceil k_b \rfloor}_{\lceil k \rfloor}\{y\}=\underbrace{\lceil m_b^{-1} \rfloor\{F\}}_{\{N\}} \Rightarrow \lceil I \rfloor\{\ddot{y}\}+\lceil k \rfloor\{y\}=\{N\}$$

Then, we have n odes of the form,

$$\ddot{\eta}_r + k_r \eta_r = N_r(t), \quad r=1,\dots,n$$

and assuming initial conditions $\eta_r(0) = D_r$, $\dot{\eta}_r(0) = V_r$, we may Duhamel's integral (or convolution integral) to write the solution of each of these equations as,

$$\eta_r(t) = D_r \cos\omega_r t + \frac{V_r}{\omega_r}\sin\omega_r t + \frac{1}{\omega_r}\int_0^t N_r(\tau)\sin\omega_r(t-\tau)\,d\tau$$

where $k_r = \omega_r^2$, $r=1,\dots,n$.

Remark: The initial conditions for $\{y\}$ are expressed in terms of the initial conditions of the problem through the relations

$$\{y(0)\} = [\Psi]^{-1}\{q(0)\}, \quad \{\dot{y}(0)\} = [\Psi]^{-1}\{\dot{q}(0)\}$$

The procedure just outlined that enables us to decouple the equations of motion of a linear vibrating system is termed modal decomposition.

The original response of the system is computed simply as,

$$\{q\} = [\Psi]\{y\}$$

1.1 Normal Modes of Vibration

Let's reconsider the unforced n-DOF linear vibrating system,

$$[M]\{\ddot{q}\}+[K]\{q\}=\{0\}$$

Seek synchronous motions of this system, i.e., oscillations where all DOFs move in-unison, they pass through zero at the same instant of time and they reach their extremum values at the same instant of time (oscillations with trivial phase differences between DOFs, 0 or π) $\Rightarrow$ In that case the motions will be separable in time, $\boxed{\{q(t)\}=\{C\}f(t)}$

Constant vector ($\{C\}$); scalar function of time ($f(t)$)

Example: $\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}=\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} f_1(t) \Rightarrow$

$\Rightarrow q_1=f_1(t)=q_2$

or

$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}=\begin{Bmatrix} 0.5 \\ -1 \end{Bmatrix} f_2(t) \Rightarrow$

$\Rightarrow q_1=0.5\ f_2(t)$

$q_2=-f_2(t)$

Remark:

Can such motions exist?

Remark: Note that we have omitted damping from this discussion.

Then, we get:

$$[M]\{C\}\ddot{f}(t)+[K]\{C\}f(t)=\{0\} \Rightarrow$$ Obtain equations of the form,

(i,j)-th element of [M]

$$\sum_{j=1}^{n}(M_{ij}c_j)\ddot{f}(t)+\sum_{j=1}^{n}(K_{ij}c_j)f(t)=0,\quad i=1,\dots,n \Rightarrow$$

$$\Rightarrow -\frac{\ddot{f}(t)}{f(t)}=\frac{\sum_{j=1}^{n}K_{ij}c_j}{\sum_{j=1}^{n}M_{ij}c_j},\quad i=1,\dots,n,\quad \forall t\geq 0 \Rightarrow$$ Necessarily it must hold that

$$-\frac{\ddot{f}(t)}{f(t)}\equiv\omega^2>0$$

Depends on t

Does not depend on t

We require this to be a positive number (we exclude at this point the case $\omega=0$), since otherwise $f(t)$ would not be an oscillating function (actually it would be unbounded).

? ? ?

Then, $\ddot{f}(t)+\omega^2 f(t)=0 \Rightarrow f(t)=A\cos(\omega t-\varphi)$

But then, $\sum_{j=1}^{n}(K_{ij}-\omega^2 M_{ij})\,C_j=0,\ i=1,\dots,n \Rightarrow$

$$\Rightarrow \boxed{\underbrace{([K]-\omega^2[M])}_{[A(\omega^2)]}\{C\}=\{0\}}$$

Either $[A(\omega^2)]$ is nonsingular $\Rightarrow \{C\}=\{0\}$

or $[A(\omega^2)]$ is singular $\Rightarrow$ Problem may have an infinity of solutions!

This outlines the solution of a linear eigenvalue problem with ω^2 being the eigenvalue. For nontrivial solutions it must hold that $\det[A(\omega^2)]=0 \Rightarrow$ $\Rightarrow$ We can compute ω^2! Once ω^2 is computed, the vector $\{C\}$ can be computed as well. In general, we obtain n roots $\omega_1^2 \le \omega_2^2 \le \dots \le \omega_n^2$ for the eigenvalue, $\omega_i^2>0,\ i=1,\dots,n$ for $[K]$ positive definite. Moreover, all roots are real, but it is possible that two or more roots are equal, e.g., $\omega_1^2=\omega_2^2$. We will provide a detailed example of the solution of the eigenvalue problem in the next lecture!

Notes

Newton's law — D'Alembert — Dynamical equilibrium

$$m\underline{\ddot{x}} = \underline{F} \longrightarrow m\underline{\ddot{x}} - \underline{F} = \underline{0}$$

$$\underline{A}\,\underline{x} = 0$$

Either $\underline{A}$ singular $\Rightarrow$ $\det \underline{A} = 0$

∞ solutions

Or $\underline{A}$ nonsingular $\Rightarrow$ $\underline{x} = \underline{A}^{-1}\underline{0} = 0$

$$\underline{A}\,\underline{x} = \underline{b}$$

Either $\underline{A}$ nonsingular $\Rightarrow$ $\underline{x} = \underline{A}^{-1}\underline{b}$

Or $\underline{A}$ singular

- If $\underline{b}$ is in the range of $\underline{A}$ $\Rightarrow$ ∞ solutions

  Fredholm's alternative!

- If $\underline{b}$ is not in the range of $\underline{A}$ $\Rightarrow$ No solution.