(1) Review of theory of MDOF Vibrating Systems

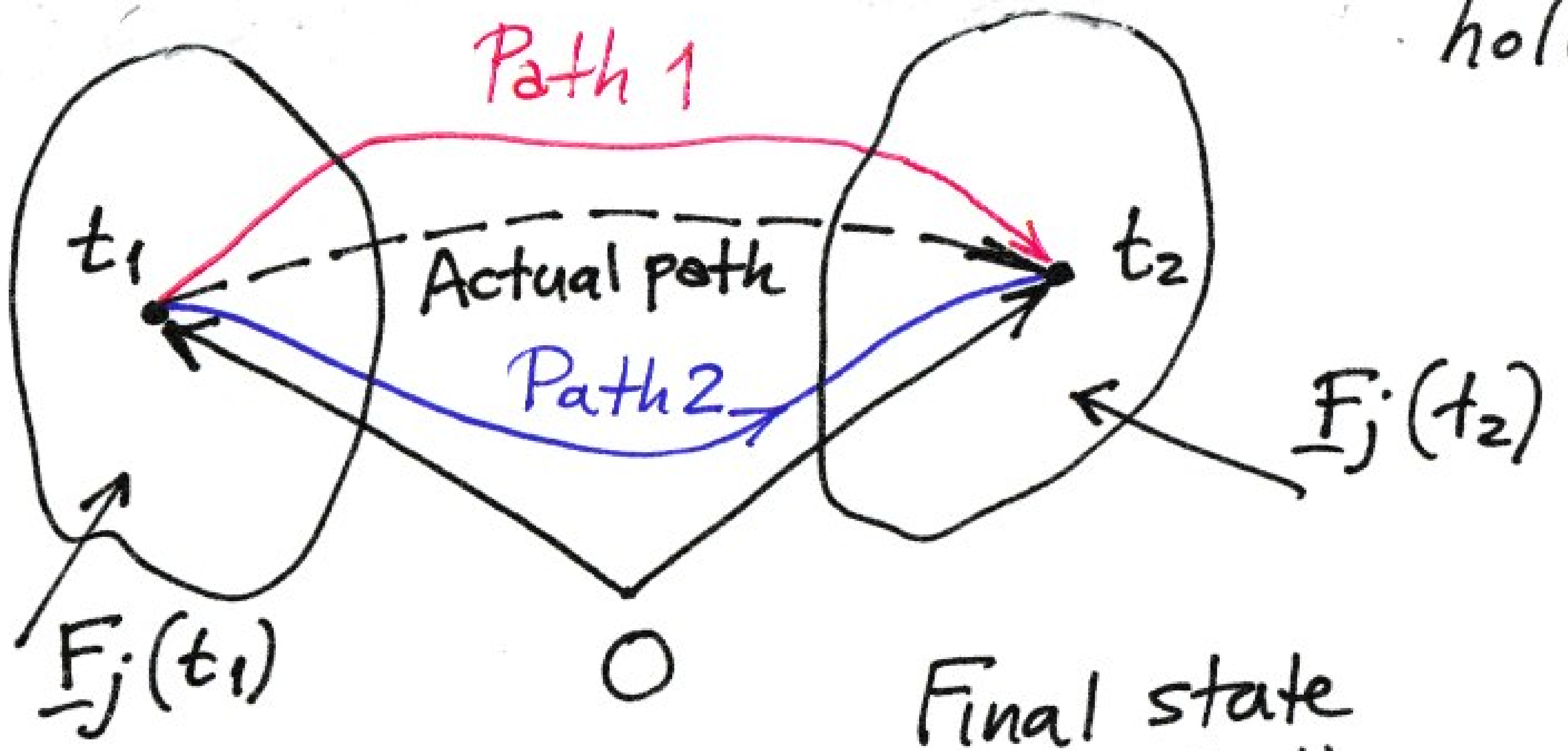
Generalized coordinates of a MDOF system is the minimum number of coordinates $\{q_1(t), \dots, q_n(t)\}$ independent with respect to each other, that are necessary to describe the configuration of the system completely. State coordinates $\{q_1, \dot{q}_1, \dots, q_n, \dot{q}_n\}$ describe the state of the system completely, where $n = \text{DOF}$. Note that $q_i(t), \dot{q}_i(t)$ are assumed to be finite, single-valued, C^1.

Fundamental question: What is the actual path followed by a dynamical system during the dynamics? Assuming holonomic systems, the following Extended Hamilton's principle holds:

$$\int_{t_1}^{t_2} \left(\delta T + \overline{dW}\right) dt = 0$$

δT: Variation of kinetic energy

$\overline{dW}$: Infinitesimal work performed by applied forces (not necessarely a perfect differential)



Initial state (prescribed) at time t_1

Final state (prescribed) at time t_2

If, however the virtual work can be expressed in terms of a potential, $\overline{dW} = -\delta V \leftarrow$ Perfect differential $\Rightarrow$

Hamilton's principle $\rightarrow$ $\Rightarrow \int_{t_1}^{t_2} (\delta T - \delta V)\, dt = 0 \Rightarrow \boxed{\int_{t_1}^{t_2} \delta L \, dt = 0, \quad L = T - V}$

Remarks

1) We define a system as <u>holonomic</u> if all of its constraints are holonomic. For a constraint to be holonomic, it must be expressible as a function,

$$f(\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_N, t) = 0$$

If we have N coordinates and m constraints $\Rightarrow$ The DOF are $n = N - m$

i.e., a holonomic constraint depends only on the coordinates $q_i(t)$ and possibly time. But it does not depend on the velocities. Alternatively, we may imagine the holonomic constraint as an $N-1$-dimensional hypersurface in the N-dimensional configuration space of the system.

2) If a system is holonomic we can express its motion by a set of n linearly independent generalized coordinates. Then, Hamilton's principle can be expressed as, $\int_{t_1}^{t_2} \delta L\, dt = 0 \Rightarrow \delta \int_{t_1}^{t_2} L\, dt = 0$

3) Hamilton's extended principle is derived directly from the <u>principle of virtual work</u> in conjuction with <u>D'Alembert's principle</u>.

If a system is in equilibrium, then the work performed by all external forces/moments during virtual displacements/rotations is zero. (Static equilibrium)

In dynamic equilibrium the inertia forces counter-balance the external forces (so dynamic equilibrium can be 'transformed' to static one)

Hence $\delta \int_{t_1}^{t_2} L\, dt = 0$, $L = T - V$. Hence, the actual path taken by the system renders the value of the integral $\int_{t_1}^{t_2} L\, dt$ stationary with respect to all possible neighbouring paths that the system could virtually take between two instants of time provided that the initial and final states are prescribed. The stationary value is a minimum.

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \xrightarrow{\text{Variational calculus}} \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = Q_r, \quad r = 1, \dots, n}$$

Lagrange's equations

A frequent case is when the kinetic energy depends only on the generalized velocities and not on the coordinates $\Rightarrow T = T(\underline{\dot{q}}) \Rightarrow$

$\Rightarrow$ If we <u>linearize</u> for small motions close to a stable equilibrium then we can express $T = \frac{1}{2}\{\dot{q}\}^T [M] \{\dot{q}\}$ (quadratic form) $\Rightarrow$ $T > 0$ for $\{\dot{q}\} \neq 0$

Mass matrix $\rightarrow$ Positive definite matrix

Similarly for the linearized system, $V = \frac{1}{2}\{q\}^T [K] \{q\} \Rightarrow$

Stiffness matrix $\rightarrow$ Positive semi-definite matrix

$\Rightarrow V \geq 0$ for $\{q\} \neq 0$

Suppose that $V = \{x\}^T[A]\{x\}$, $[A]$ is a square matrix, is a quadratic form $\Rightarrow$ Sylvester's theorem states that the necessary and sufficient conditions for V to be a positive definite quadratic form is that all the principal minor determinants of $[A]$ be positive. V is a positive semi-definite quadratic form is if all these principal minor determinants are non-negative.

Then, if V is positive definite $\Rightarrow$

$V > 0$ for $\{x\} \neq 0$, and $V = 0$ iff $\{x\} = \{0\}$

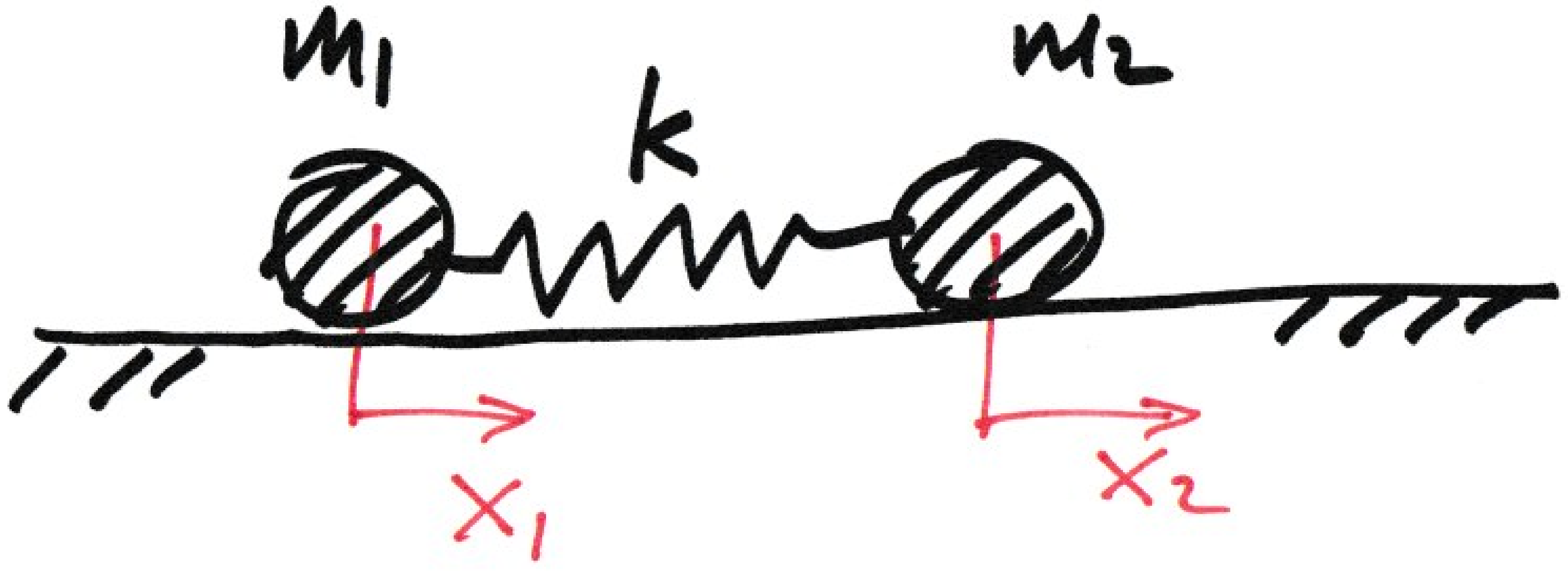
All eigenvalues of V are positive

Also, if V is positive semi-definite $\Rightarrow$

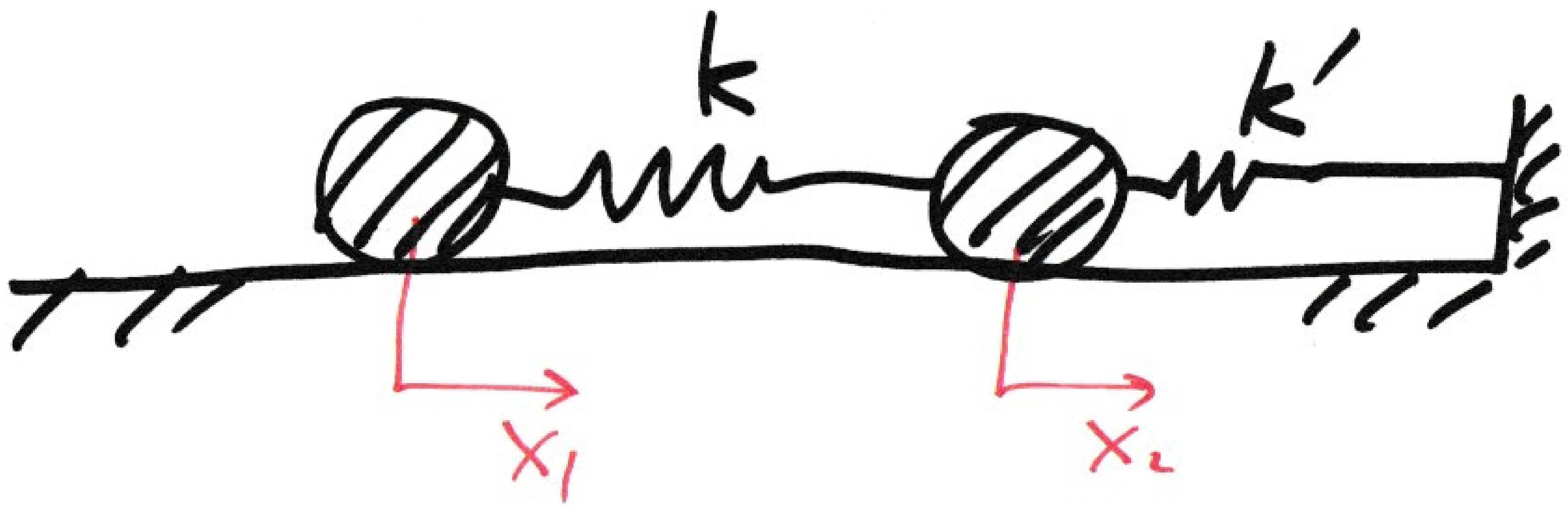
All eigenvalues of V are non-negative

$\Rightarrow$ $V \geq 0$ for $\{x\} \neq 0$, and $V = 0$ does not necessarily imply that $\{x\} = \{0\}$.

Example of positive-semidefinite [K]



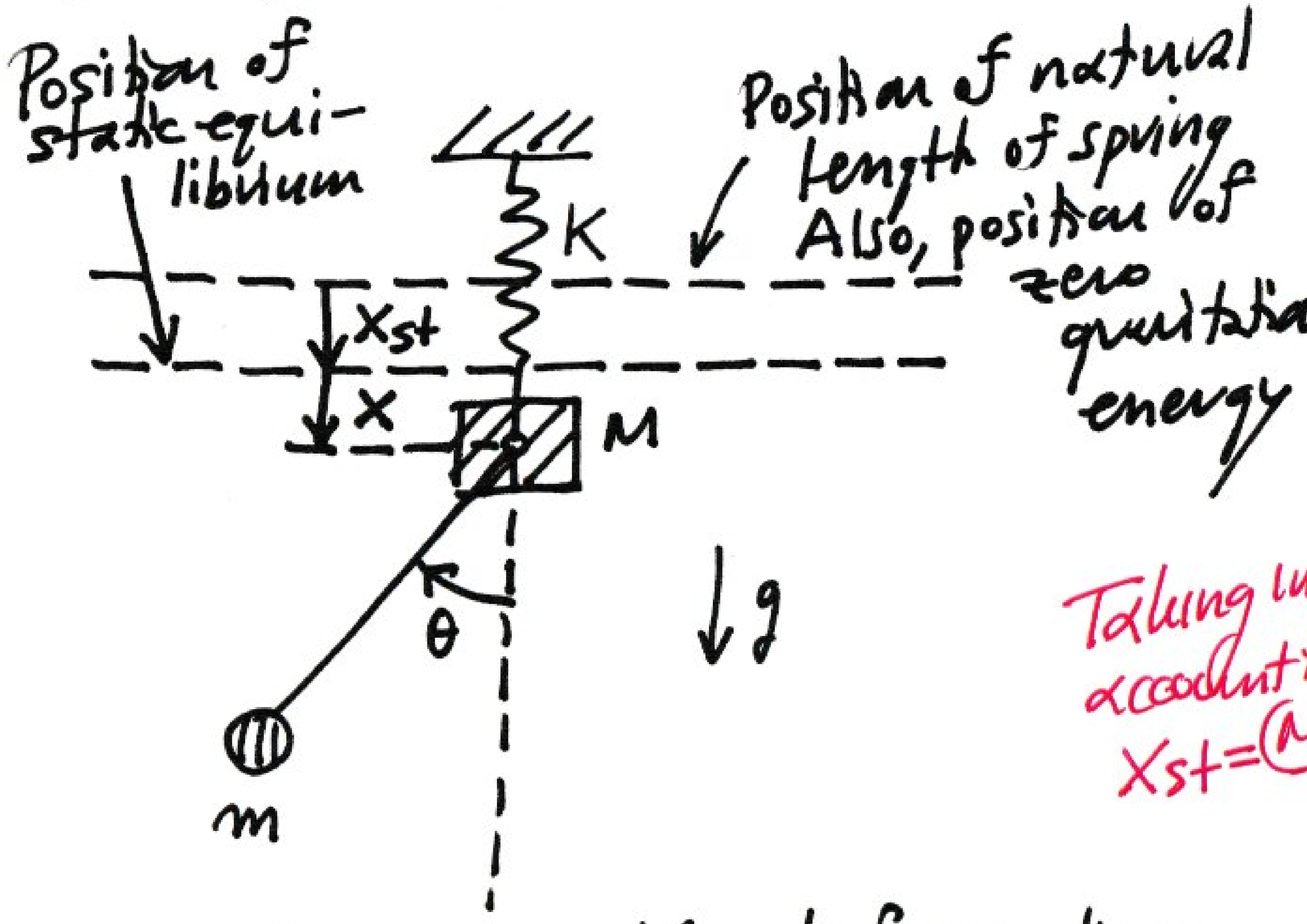
If $x_1 = x_2 \neq 0 \Rightarrow V = 0$!



Now, this is an example of positive-definite [K] !

If $x_1 = x_2 = 0 \Rightarrow V = 0$, but if $x_1, x_2 \neq 0 \Rightarrow V > 0$

Example



This system has 2DOF with its static equilibrium at $X=X_{st}, \theta=0$.
Gravity is taken into account.

Potential energy is:

$$V=\frac{1}{2}K(X_{st}+X)^2-Mg(X_{st}+X)-$$
$$-mg(X_{st}+X+l\cos\theta)=$$
$$=\frac{1}{2}kx^2+mgl(1-\cos\theta)+\text{Const}$$

Taking into account that $X_{st}=\frac{(M+m)g}{k}$

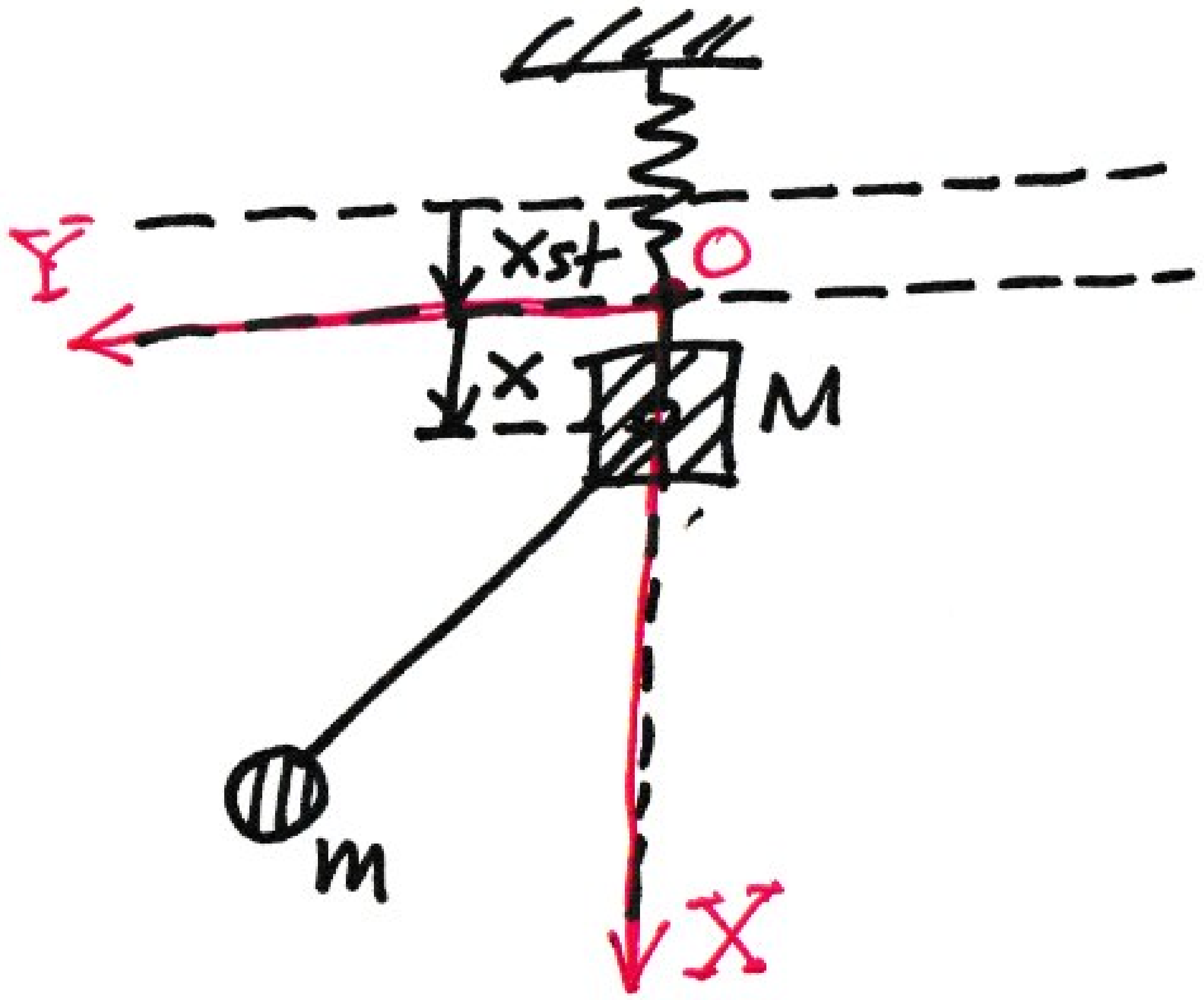
Not needed

Kinetic energy is:

$$T=\frac{1}{2}(M+m)\dot{x}^2+\frac{1}{2}m(l\sin\theta\,\dot{\theta})^2+$$
$$+\frac{1}{2}m(\dot{x}(l-l\cos\theta))^2$$

So, $V=V(x,\theta)$ and $T=T(\theta,\dot{x},\dot{\theta})$ and this is a <u>nonlinear system</u>.

Note that x is the deformation of the spring from its static equilibrium.



Remark:
To compute T, refer the motions of the mass to an inertial coordinate frame, e.g., (OXY).
Then, $T=\frac{1}{2}mV_{m\,abs}^2+\frac{1}{2}MV_{M\,abs}^2$

But, what if we assume small vibrations and <u>linearize</u> close to the position of static equilibrium?