Consider the limiting process where $n \to \infty$, $h \to 0$ but $L = \text{constant}$. Then we can pass to a continuum approximation whereby we can transform the index i to a continuous variable. This is done as follows:

$$v_i(t) \mapsto v(i,t)$$

$$v_{i+1}(t) \mapsto v(i+1,t) \sim v(i,t) + \frac{\partial v(i,t)}{\partial i} + \frac{1}{2!}\frac{\partial^2 v(i,t)}{\partial i^2} + \cdots$$

$$v_{i-1}(t) \mapsto v(i-1,t) \sim v(i,t) - \frac{\partial v(i,t)}{\partial i} + \frac{1}{2!}\frac{\partial v(i,t)}{\partial i^2} + \cdots$$

$$\ddot{v}_i(t) \mapsto \frac{\partial^2 v(i,t)}{\partial t^2}$$

$$T_i \mapsto T(i)$$

$$T_{i-1} \mapsto T(i-1) = T(i) - \frac{dT(i)}{di} + \cdots$$

$$F_i(t) \mapsto F(i,t)$$

$$m_i \mapsto m(i)$$

$\Rightarrow$ We can rewrite the i-th ode of motion as:

$$m(i)\frac{\partial^2 v(i,t)}{\partial t^2} = F(i,t) - \left[T(i) - \frac{dT(i)}{di}\right]\frac{1}{h}\left[\cancel{v_i(t)} - \cancel{v(i,t)} + \frac{\partial v(i,t)}{\partial i} - \frac{1}{2}\frac{\partial^2 v(i,t)}{\partial i^2}\right] -$$

$$- T(i)\frac{1}{h}\left[\cancel{v(i,t)} - \cancel{v(i,t)} - \frac{\partial v(i,t)}{\partial i} - \frac{1}{2}\frac{\partial^2 v(i,t)}{\partial i^2}\right] + \cdots \Rightarrow$$

$$\Rightarrow m(i)\,\frac{\partial^2 v(i,t)}{\partial t^2} = F(i,t) + T(i)\,\frac{\partial^2 v(i,t)}{\partial i^2}\,\frac{1}{h} + $$
$$+ \frac{\partial T(i)}{\partial i}\,\frac{1}{h}\left[\frac{\partial v(i,t)}{\partial i} - \frac{1}{2}\,\frac{\partial^2 v(i,t)}{\partial i^2}\right] + \ldots \Rightarrow$$

$$\Rightarrow \frac{m(i)}{h}\,\frac{\partial^2 v(i,t)}{\partial t^2} = \frac{F(i,t)}{h} + T(i)\,\frac{\partial^2 v(i,t)}{\partial (hi)^2} + $$
$$+ \frac{\partial T(i)}{\partial (ih)}\left[\frac{\partial v(i,t)}{\partial (ih)} - \frac{1}{2}\,\frac{\partial^2 v(i,t)}{h\,\partial i^2}\right] + \ldots \rightarrow \cancel{\frac{1}{2}h\,\frac{\partial^2 v(i,t)}{\partial (ih)^2}} \text{ as } h \to 0$$

Let us introduce a new spatial variable $x = ih$, $0 \leq x \leq L \Rightarrow$

$\Rightarrow$ As $i \to \infty$, $h \to 0$, $\dfrac{m(i)}{h} \to m(x)$, $\dfrac{F(i,t)}{h} \to F(x,t) \Rightarrow$

$$\Rightarrow m(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} = F(x,t) + T(x)\,\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial T(x)}{\partial x}\left[\frac{\partial v(x,t)}{\partial x} - \cancel{\frac{h}{2}\,\frac{\partial^2 v(x,t)}{\partial x^2}}\right] + \ldots$$

0 as $h \to 0$

$$\Rightarrow \boxed{m(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} = F(x,t) + \frac{\partial}{\partial x}\left[T(x)\,\frac{\partial v(x,t)}{\partial x}\right] + \ldots}$$

Generalized wave equation!

But how do the BCs transform?

$v_0(t) = 0 \Rightarrow v(0,t) = 0$ $\qquad$ $v_{n+1}(t) = 0 \longmapsto v(L,t) = 0$

Discrete system $\qquad$ continuous system

And how do ICs transform?

$$v_i(0) = c_i \longmapsto v(x,0) = c(x)$$

$$\dot{v}_i(0) = g_i \longmapsto \frac{\partial v(x,0)}{\partial t} = g(x)$$

Now we get a well-posed problem to solve!