Torsional spring at end:

$$GJ_p(0)\frac{\partial \varphi(0,t)}{\partial x} - k_T\varphi(0,t)=0 \quad \text{or} \quad GJ_p(L)\frac{\partial \varphi(L,t)}{\partial x} + k_T\varphi(L,t)=0$$

Disk at the end

$$GJ_p(0)\frac{\partial \varphi(0,t)}{\partial x} - I\frac{\partial^2 \varphi(0,t)}{\partial t^2}=0 \quad \text{or} \quad GJ_p(L)\frac{\partial \varphi(L,t)}{\partial x} + I\frac{\partial^2 \varphi(L,t)}{\partial t^2}=0$$

moment of inertia of disk

Summarizing, for all three systems considered thus far, i.e., elastic string performing transverse vibrations, the elastic rod performing axial vibrations, and the circular shaft performing torsional vibrations, the governing partial differential equations can be expressed in the form,

$$\boxed{f(x,t)+\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right]= B(x)\frac{\partial^2 u}{\partial t^2}\,,\quad 0\le x\le L}$$

We'll start our analysis of the generalized wave equation by assuming that there is no forcing, so we set $f(x,t)=0$ and study the free vibrations of these elastic systems.

Study of free vibrations with 'simple' boundary conditions

Consider again the unforced generalized wave equation,

$$\frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right] = B(x)\frac{\partial^2 u}{\partial t^2}, \qquad 0 \le x \le L \qquad (1)$$

with 'simple' boundary conditions,

$$\left.\begin{array}{ll} (a) & u(0,t)=0,\ u(L,t)=0 \\ (b) & \dfrac{\partial u(0,t)}{\partial x}=0,\ \dfrac{\partial u(L,t)}{\partial x}=0 \\ (c) & \dfrac{\partial u(0,t)}{\partial x}=0,\ u(L,t)=0 \\ (d) & u(0,t)=0,\ \dfrac{\partial u(L,t)}{\partial x}=0 \end{array}\right\} \quad (1a)$$

and initial conditions $u(x,0)=g(x)$ and $\dfrac{\partial u(x,0)}{\partial t}=h(x)$ (1b)

It is natural to seek normal modes for system (1), that is, synchronous vibrations of the form,

$$\underbrace{u(x,t)=\varphi(x)\,f(t)}_{\text{space-time separation!}} \qquad (2)$$

← We adopt a dynamics-based approach

Are such motions possible? We need to perform analysis! Substitute (2) into the governing partial differential equation (1) ⇒

$$\frac{\partial}{\partial x}\left[A(x)\frac{d\varphi(x)}{dx}f(t)\right] = B(x)\,\varphi(x)\,\frac{d^2 f(t)}{dt^2} \Rightarrow$$

$$\Rightarrow f(t)\,\frac{d}{dx}\left[A(x)\frac{d\varphi(x)}{dx}\right] = B(x)\,\varphi(x)\,\frac{d^2 f(t)}{dt^2} \Rightarrow \quad \frac{\ddot{f}(t)}{f(t)} = \frac{\frac{d}{dx}\left[A(x)\frac{d\varphi(x)}{dx}\right]}{B(x)\,\varphi(x)}, \quad \forall t \geq 0,\ \forall x \in [0, L]$$

where $(\dot{\ }) \equiv \frac{d}{dt}$ and we assume that the denominators in this relation $\neq 0$.

Hence we conclude that it must be satisfied that,

$$(3) \begin{cases} \dfrac{\ddot{f}(t)}{f(t)} = -\omega^2 < 0 & \text{(otherwise } f(t) \text{ would be unbounded at } t\to\infty) \\[2ex] \dfrac{\frac{d}{dx}\left[A(x)\frac{d\varphi(x)}{dx}\right]}{B(x)\,\varphi(x)} = -\omega^2 & \end{cases}$$

Hence, we split the problem into two separate subproblems:

In time: $\boxed{\ddot{f}(t) + \omega^2 f(t) = 0\,, \quad t \geq 0}$ (4a)

In space: $\boxed{\frac{d}{dx}\left[A(x)\frac{d\varphi(x)}{dx}\right] + \omega^2 B(x)\varphi(x) = 0\,, \quad 0 \leq x \leq L}$ (4b)

Considering the boundary conditions, we can also split them in space and time,

$$\left.\begin{aligned} (a) &\rightarrow \quad \varphi(0)=0,\ \varphi(L)=0 \\ (b) &\rightarrow \quad \varphi'(0)=0,\ \varphi'(L)=0 \\ (c) &\rightarrow \quad \varphi'(0)=0,\ \varphi(L)=0 \\ (d) &\rightarrow \quad \varphi(0)=0,\ \varphi'(L)=0 \end{aligned}\right\} \quad (4c)$$

These relations affect only the subproblem in space, i.e., (4b) and not the subproblem in time (since there are BCs).

Starting from the problem in time; (4a) $\Rightarrow$

$$\left.\begin{aligned} f(t) &= A_1 \cos\omega t + A_2 \sin\omega t = \\ &= C \cos(\omega t - \varphi) \end{aligned}\right\} (5)$$

The coefficients A_1, A_2, C and φ cannot yet be determined, we must wait first for the solution of the subproblem in space.

Now consider the subproblem in space, (4b) and (4c). This is a Sturm-Liouville eigenvalue problem whose general analytical solution we can't write. However, since this is a linear problem, if we find two linearly independent solutions say $\varphi_1(x,\omega)$ and $\varphi_2(x,\omega)$, then the general solution can be expressed as a superposition of these solutions, $\varphi(x,\omega) = C_1 \varphi_1(x,\omega) + C_2 \varphi_2(x,\omega)$, $C_1, C_2 \in R \Rightarrow$

Substituting into the BCs, say conditions (a),

$$(6a)\ \left\{\begin{array}{l} C_1\varphi_1(0,\omega)+C_2\varphi_2(0,\omega)=0 \\ C_1\varphi_1(L,\omega)+C_2\varphi_2(L,\omega)=0 \end{array}\right\} \Rightarrow$$ For nontrivial solutions for C_1 and C_2, we must require that

$$\begin{vmatrix} \varphi_1(0,\omega) & \varphi_2(0,\omega) \\ \varphi_1(L,\omega) & \varphi_2(L,\omega) \end{vmatrix}=0 \Rightarrow$$

$\Rightarrow$ This computes the eigenvalues of the problem. It turns out that the problems that we'll be concerned with (i.e., finite elastic structures with simple or complex boundary conditions) admit a countable infinity of eigenvalues (natural frequencies),

$$0<\omega_1<\omega_2<\ldots<\omega_n<\ldots$$

Fundamental natural frequency

For each natural frequency ω_r, $r=1,\ldots,n,\ldots$ we may replace it into (6a) and get: $C_1^{(r)}\varphi_1(0,\omega_r)+C_2^{(r)}\varphi_2(0,\omega_r)=0 \Rightarrow \dfrac{C_1^{(r)}}{C_2^{(r)}}=\underbrace{-\dfrac{\varphi_2(0,\omega_r)}{\varphi_1(0,\omega_r)}}_{K_r} \Rightarrow \left\{\dfrac{C_1}{C_2}\right\}^{(r)}=-\dfrac{\varphi_2(0,\omega_r)}{\varphi_1(0,\omega_r)}$

Then, the corresponding mode shape is

$$\varphi_r(x) = C_1^{(r)}\varphi_1(x,\omega_r) + C_2^{(r)}\varphi_2(x,\omega_r) =$$
$$= \left[K_r\,\varphi_1(x,\omega_r) + \varphi_2(x,\omega_r)\right] C_2^{(r)} \Rightarrow$$

$$\Rightarrow \boxed{\varphi_r(x) = A\left[K_r\,\varphi_1(x,\omega_r) + \varphi_2(x,\omega_r)\right]}$$

Cannot be determined by the analysis thus far.

We note that the r-th mode shape corresponding to the r-th natural frequency is determined up to a multiplicative constant

So, we may normalize the eigenfunction $\varphi_r(x)$ such that,

$$\int_0^L B(x)\,\varphi_r^2(x)\,dx = 1\ ,\ r = 1, 2, \ldots \qquad (7)$$

This condition determines the multiplicative constant uniquely

Then, the pairs $\{\omega_r, \varphi_r(x)\}$, $r=1,2,\ldots$ determine the normal modes of vibration of the free system (1).

Summarizing the general solution of problem (1), (1a) and (1b) can be expressed as follows:

$$u(x,t) = \sum_{r=1}^{\infty} \varphi_r(x) f_r(t) =$$

Only unknown constants

$$= \sum_{r=1}^{\infty} \varphi_r(x) \left[A_{1r} \cos\omega_r t + A_{2r} \sin\omega_r t \right] \quad (8)$$

with $\int_0^L B(x)\,\varphi_r^2(x)\,dx = 1$

This normalization condition enables us to compute uniquely the unknown multiplicative constant in $\varphi_r(x)$.

To compute the double infinity of unknown constants in (8) using only the two available ICs, (1b), we need to study the orthogonality properties of the normal modes!