Green's Functions and Reduction of Differential Equations to Integral Equations

We will show how we can represent the solutions of boundary value problems in terms of Green's functions. In this way eigenvalue differential equations can be reduced to symmetric integral equations; thereby automatically proving existence, completeness of solutions and the validity of the expansion theorem.

Green's function and relation to bvp of odes

Consider a linear self-adjoint homogeneous differential expression of the form,

self-adjointness

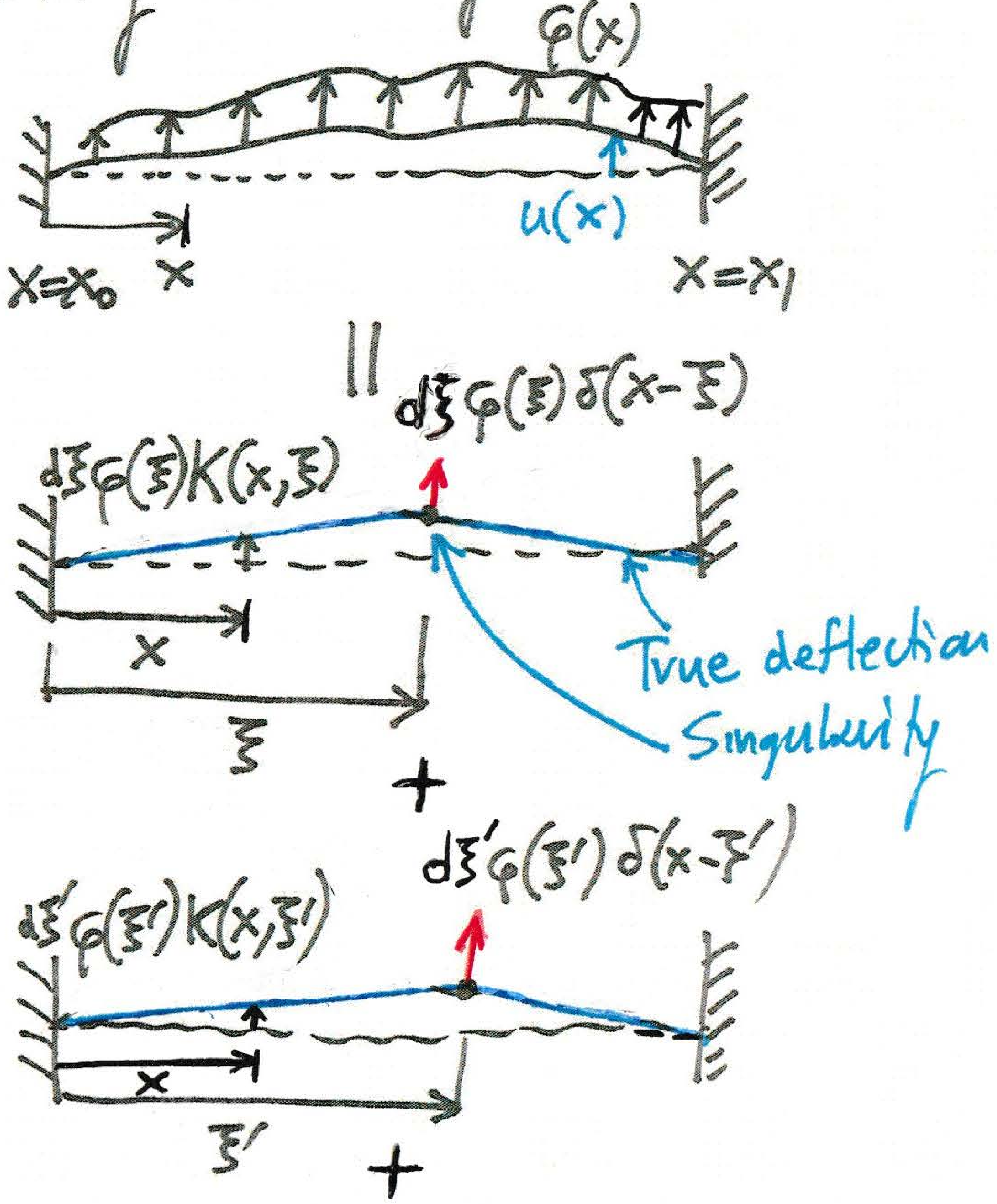
$$L[u] = pu'' + p'u' - qu, \quad G: x_0 \leq x \leq x_1$$

where p, p', q are continuous functions of x, and $p > 0$. The associated nonhomogeneous differential equation is,

$$L[u] = -\varphi(x), \quad x \text{ in } G. \qquad (1)$$

where $\varphi(x)$ is a piecewise continuous function in G. We are interested in the following bvp: find a solution of (1), $u = f(x)$, which satisfies the homogeneous boundary conditions at ∂G, e.g., $u = 0$ at ∂G.

We start with the heuristic consideration that we have a uniform string forced by a force distribution $\varphi(x)$. We visualized a limiting process



$$u(x) = \int_{x_0}^{x_1} \varphi(\xi) K(x,\xi)\, d\xi$$

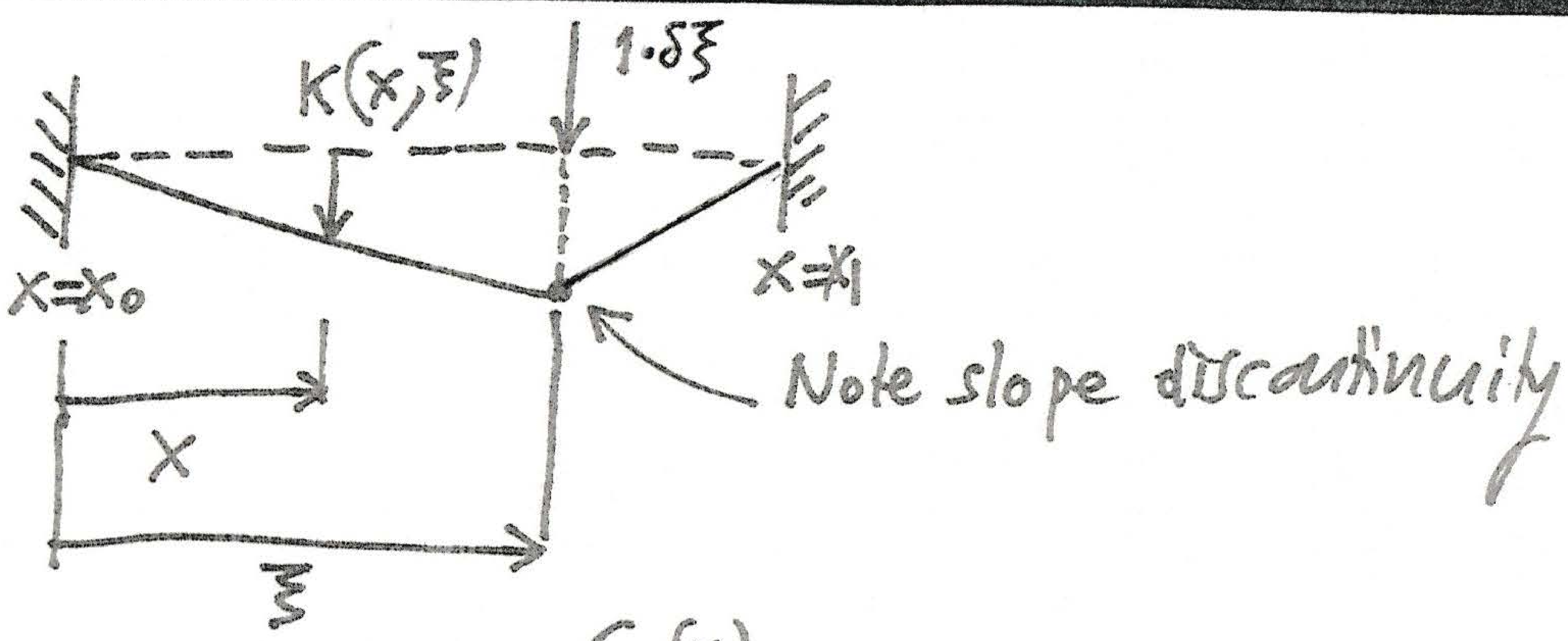
whereby the continuous force distribution $\varphi(x)$ is a superposition of an infinite number of concentrated (point) forces, each applied individually. We can do this since this is a linear system $\Rightarrow$ superposition holds. The principle of linear superposition is tied to the notion of Green's function. Considering a point force acting at position $x=\xi$ of the string with the given intensity, we denote by $K(x,\xi)$ the deflection of the string at a point x as a result of the action of the point force of unit intensity acting at point ξ. Then, the effect at position x of the continuously distributed force $\varphi(x)$ can be considered as superposition of the effects of continuously distributed point forces whose local intensity at $x=\xi$ is $\varphi(\xi) \Rightarrow$

$$\Rightarrow \quad u(x)=\int_{x_0}^{x_1} K(x,\xi)\underbrace{\varphi(\xi)\,d\xi}_{\text{Differential force}}, \quad \text{where } K(x,\xi) \text{ is the Green's function.} \quad (2)$$

The Green's function $K(x,\xi)$ for the differential expression $L[u]$ satisfies the boundary conditions and represents the response at point x due to a unit impulse at position $\xi \Rightarrow$ Hence the expression for $u(x)$ in (2) automatically satisfies the boundary conditions at ∂G, since $K(x,\xi)$ and $\varphi(\xi)$ satisfy the boundary conditions. This means that the Green's function $K(x,\xi)$ satisfies the equation $L[K]=0$ everywhere expect at $x=\xi \Rightarrow$

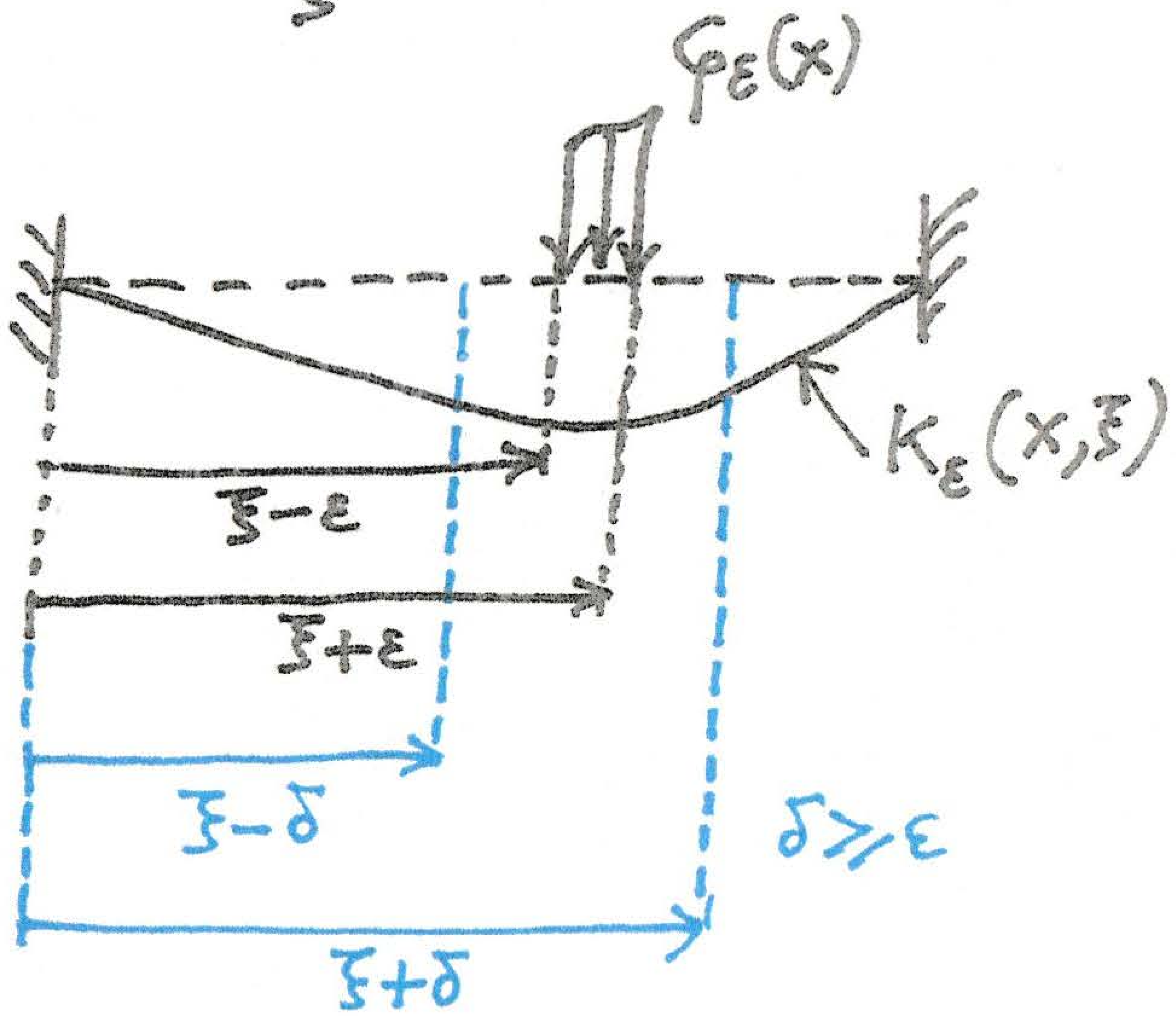
$$L[K]=0, \quad x \in G-\{x=\xi\}$$

At the point $x=\xi$ where the impulse of unit magnitude is applied $K(x,\xi)$ must have a singularity which is computed heuristically based on the following argument.



We consider the point force at $x=\xi$ as the limiting case of a $\varphi_\varepsilon(x)$ which vanishes in G for $|x-\xi|>\varepsilon$, and its total intensity is

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon}\varphi_\varepsilon(x)\,dx=1$$



We denote the corresponding deflection of the string by $K_\varepsilon(x,\xi)\Rightarrow$

$$\Rightarrow L[K_\varepsilon]=-\varphi_\varepsilon(x)\Rightarrow(pK_\varepsilon')'-qK_\varepsilon=-\varphi_\varepsilon(x)$$

$$\Rightarrow\int_{\xi-\delta}^{\xi+\delta}(\cdot),\quad\delta\geq\varepsilon\Rightarrow\int_{\xi-\delta}^{\xi+\delta}\left\{\frac{d}{dx}\left(p\frac{dK_\varepsilon}{dx}\right)-qK_\varepsilon\right\}dx=$$

$$=-\int_{\xi-\delta}^{\xi+\delta}\varphi_\varepsilon(x)\,dx=-1\Rightarrow$$

$$\Rightarrow\int_{\xi-\delta}^{\xi+\delta}\left\{\frac{d}{dx}\left(p\frac{dK_\varepsilon}{dx}\right)-qK_\varepsilon\right\}dx=-1\Rightarrow$$

$\Rightarrow$ first let $\varepsilon\to0$, and assume $\lim_{\varepsilon\to0}K_\varepsilon(x,\xi)=K(x,\xi)$ <u>continuously dif $\Rightarrow$ except at $x=\xi$</u>

$$\underset{\varepsilon\to 0}{\Rightarrow} \int_{\xi-\delta}^{\xi+\delta} \left\{ \frac{d}{dx}\left(p\frac{dK}{dx}\right) - \underbrace{qK}_{\text{continuous, } x\in G} \right\} dx = -1 \Rightarrow \text{As } \delta\to 0 \text{ we get,}$$

$$\lim_{\delta\to 0} \int_{\xi-\delta}^{\xi+\delta} \frac{d}{dx}\left(p\frac{dK}{dx}\right) dx = -1 \Rightarrow \lim_{\delta\to 0} \left.\frac{dK(x;\xi)}{dx}\right|_{\xi-\delta}^{\xi+\delta} = -\frac{1}{p(\xi)} \Rightarrow$$

This "jump" condition distinguishes $K(x,\xi)$ from any other homogeneous solution

$$\Rightarrow \left[\frac{dK(x,\xi)}{dx}\right]_{\xi-}^{\xi+} = -\frac{1}{p(\xi)}$$

This heuristic argument computes the jump in the slope of $K(x,\xi)$ at the singularity $x=\xi$.

This holds only since $L[u]$ was of second order in x

Based on the above discussion we can define the Green's function $K(x,\xi)$ as the solution of $L[u]=\delta(x-\xi)$, so that it satisfies the following two conditions:

i) It solves the homogeneous equation $L[K]=0$, $x\in G-\{\xi\}$ satisfying the boundary conditions.

ii) At the point of singularity $x=\xi$ it satisfies $\left[K'(x,\xi)\right]_{\xi-}^{\xi+} = -\frac{1}{p(\xi)}$