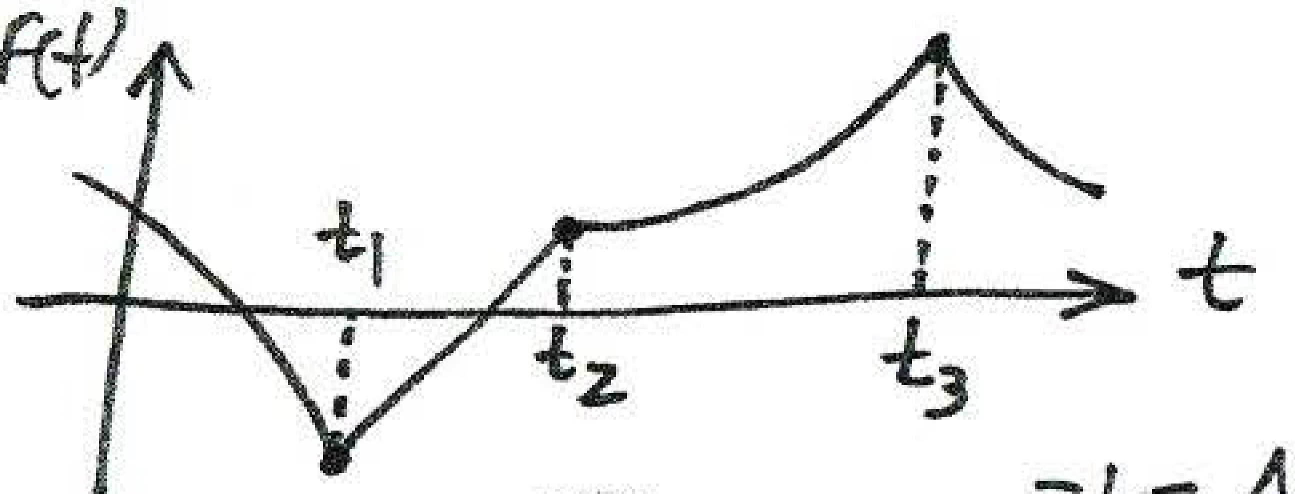
1.2 Forced Response

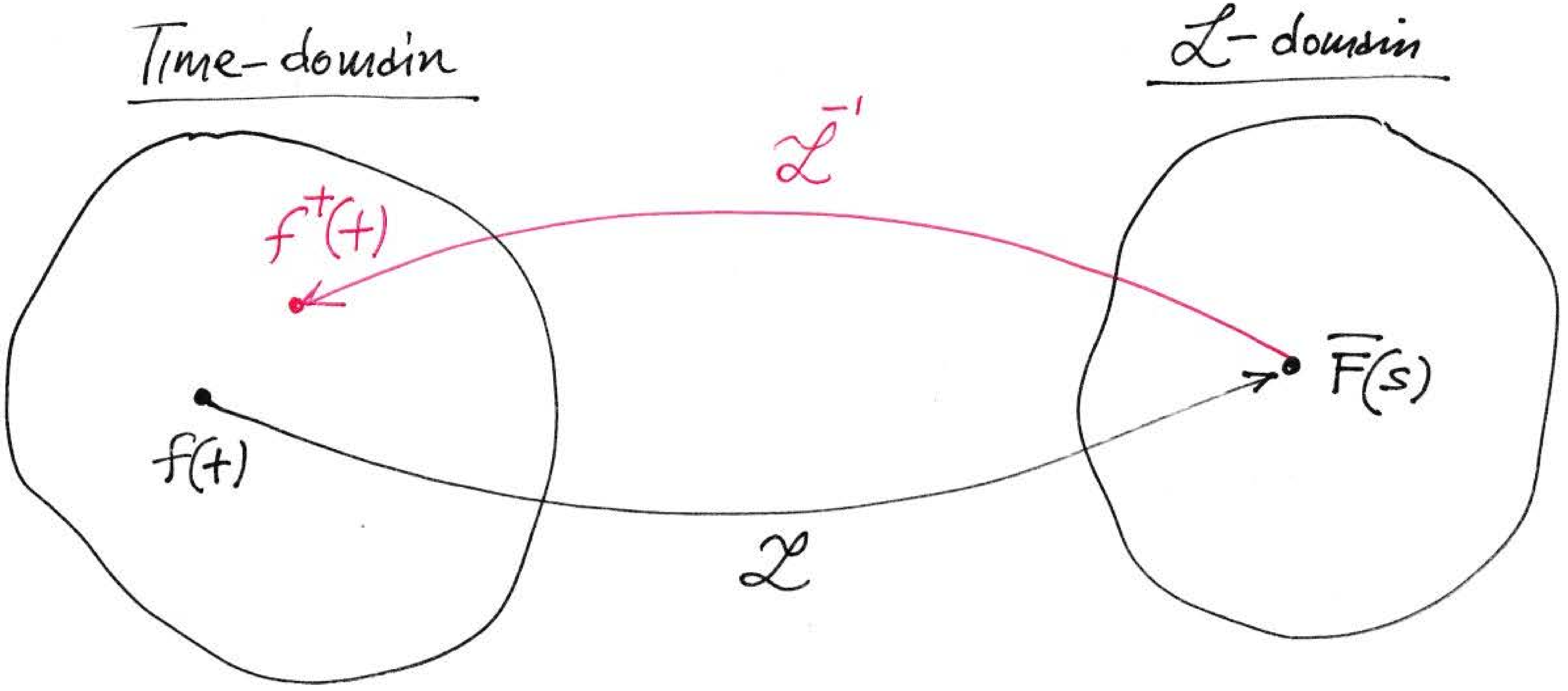
Aside: A function $f(t)$ is Laplace-transformable, if it is of Exponential order (EO) and is almost piecewise continuous (APC). A function $f(t)$ is of EO if $\exists\, a_0 \in R$, such that $\lim_{t\to\infty} f(t)e^{-\alpha t} = 0$ when $\alpha > a_0$. A function $f(t)$ is APC in every finite interval, if there are a finite number of points $t_1, \dots, t_N$ such that at each of these points the behavior of $f(t)$ is described as follows:

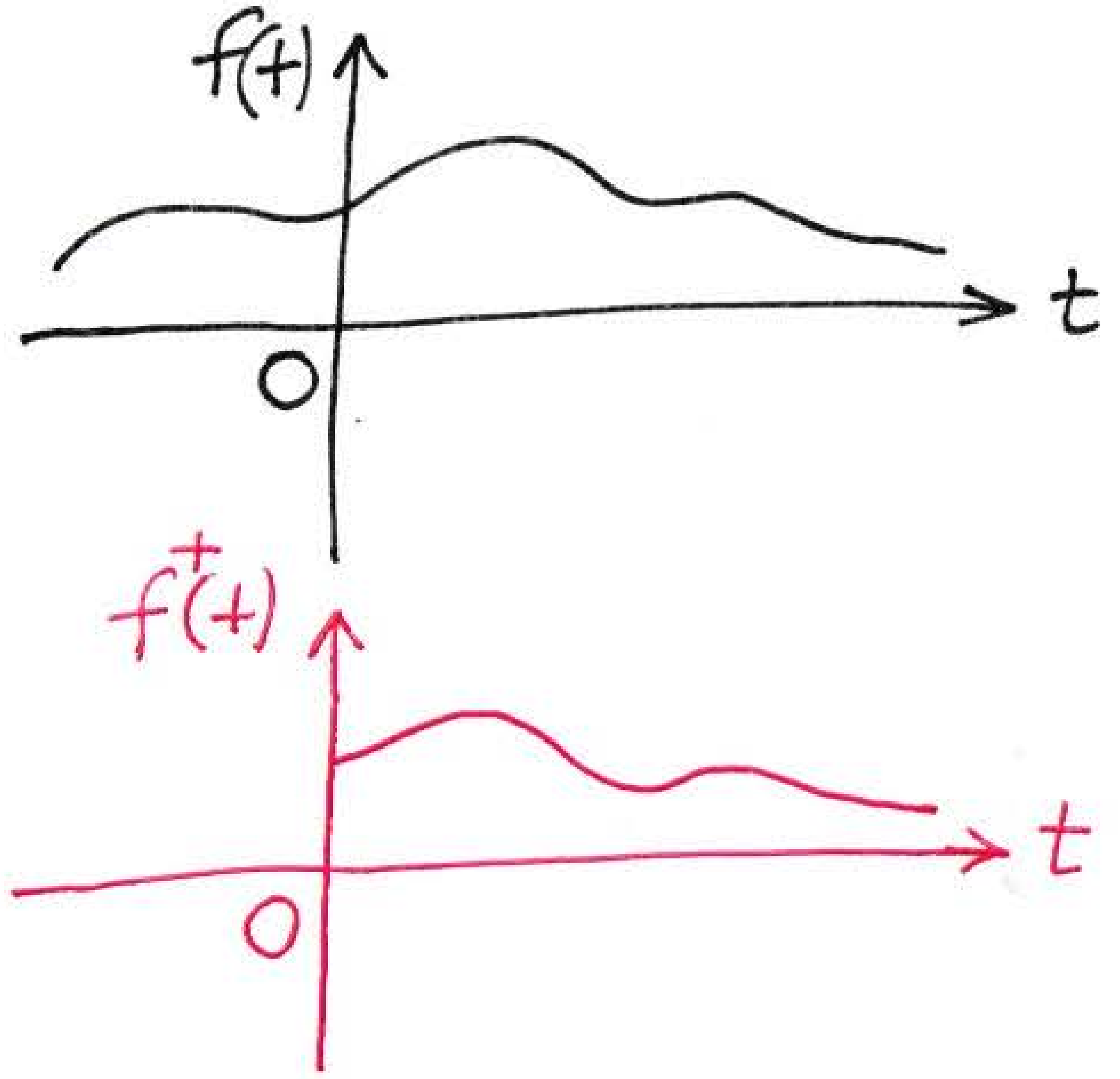
- t_k is a singular point of $f(t)$, but in some neighborhood of t_k, $|t-t_k|<\delta$ the function has the property $|f(t)| < \frac{M}{(t-t_k)^{\mu}}$, $\mu<1$, where M is a constant. Then $|f(t)|$ is integrable and $f(t)$ is absolutely integrable at each interval containing $t_k \Rightarrow$ An APC function $f(t)$ is integrable over any finite interval in t.



$\mathcal{L}[f(t)] = \bar{F}(s)$, $\mathcal{L}[e^{-\alpha t} f(t)] = \bar{F}(s+\alpha)$, $\mathcal{L}^{-1}\left[\frac{A_k}{s-s_k}\right] = A_k e^{s_k t}$ $(t \geq 0)$,

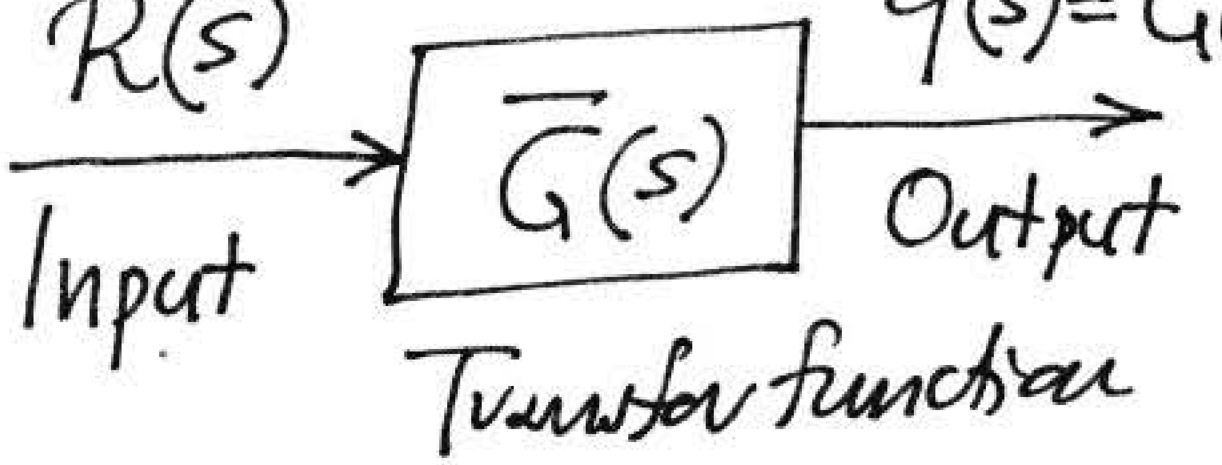
$\mathcal{L}^{-1}\left[\frac{A_k}{(s-s_k)^p}\right] = A_k \frac{t^{p-1}}{(p-1)!} e^{s_k t}$ $(t\geq 0)$, $\mathcal{L}^{-1}\left[\frac{B_1 s + B_2}{(s+\alpha)^2+\omega^2}\right] = B_1 e^{-\alpha t}\cos\omega t + \frac{B_2 - \alpha B_1}{\omega} e^{-\alpha t}\sin\omega t$ $(t\geq 0)$





$\mathcal{L}$-domain:

$\bar{q}(s) = \bar{G}(s)\bar{R}(s) \Rightarrow$



$$\Rightarrow q(t) = \mathcal{L}^{-1}[\bar{q}(s)]$$

Convolution integral in terms of $\mathcal{L}^{-1}[\bar{R}(s)]$ and $\mathcal{L}^{-1}[\bar{G}(s)]$

i) General formulation

$[M]\{\ddot{q}\}+[K]\{q\}=\{Q(t)\}$ n-DOF oscillating system $\Rightarrow$ Suppose that $\{q(t)\}$ and $\{Q(t)\}$ are $\mathcal{L}$-transformable $\Rightarrow$ $\mathcal{L}[q_i(t)]=\int_0^\infty e^{-st}q_i(t)\,dt\equiv$

$$\equiv \overline{q_i}(s),\ i=1,\dots,n$$

$$\mathcal{L}[Q_i(t)]=\int_0^\infty e^{-st}Q_i(t)\,dt\equiv$$

$$\equiv \overline{Q_i}(s),\ i=1,\dots,n$$

Then $\mathcal{L}$-transform the equations of motion $\Rightarrow$

$$[M]\{s^2\bar{q}(s)-sq(0+)-\dot{q}(0+)\}+[K]\{\bar{q}(s)\}=\{\overline{Q}(s)\}\Rightarrow$$

$$\Rightarrow \underbrace{[M]\{s^2\bar{q}(s)\}+[K]\{\bar{q}(s)\}}=\underbrace{\{\overline{Q}(s)\}}_{\text{Effect of external forcing}}+\underbrace{[M]s\{q(0+)\}+[M]\{\dot{q}(0+)\}}_{\text{Effect of initial conditions}}$$

$\{\bar{R}(s)\}$ Generalized transformed input

$(\underbrace{s^2[M]+[K]}_{})\{\bar{q}(s)\}$ $[Z(s)]$ Generalized Impedance Matrix

So, rewrite the equations of motion in the $\mathcal{L}$-domain as,

$$[Z(s)]\{\bar{q}(s)\}=\{\bar{R}(s)\}\Rightarrow\{\bar{q}(s)\}=[Z(s)]^{-1}\{\bar{R}(s)\}$$

Transfer function $[G(s)]$

So, we get the relation,

$$\underset{\text{output}}{\{\bar{q}(s)\}} = \underset{\text{Transfer matrix}}{[G(s)]}\ \underset{\text{input}}{\{\bar{R}(s)\}} \quad (\mathcal{L}\text{-domain})$$

Let us assume that $\{Q(s)\} = \{0\}$.

Hence, the transformed output is $\{\bar{q}(s)\} = \dfrac{[A(s)]}{|Z(s)|}[M]\{s\,\underset{\{q(0+)\}}{q_0} + \underset{\{\dot{q}(0+)\}}{\dot{q}_0}\}$

Adjoint matrix of $[Z(s)]$

Determinant of $[Z(s)]$

$[Z(s)]^{-1}$

But note that the denominator $|Z(s)|$ can be expressed in terms of the natural frequencies of the unforced problem,

$$|Z(s)| = \overbrace{(s-j\omega_1)(s+j\omega_1)}^{(s^2+\omega_1^2)} \dots \overbrace{(s-j\omega_n)(s+j\omega_n)}^{(s^2+\omega_n^2)},$$

$j = (-1)^{1/2}$, and we assume that the unforced system has distinct eigenfrequencies $\Rightarrow |Z(s)| = \prod_{r=1}^{n}(s^2+\omega_r^2) \Rightarrow \{\bar{q}(s)\} = \dfrac{[A(s)]}{\prod_{r=1}^{n}(s^2+\omega_r^2)}[M]\{sq_0+\dot{q}_0\} \Rightarrow$

$\Rightarrow \{q(t)\} = \mathcal{L}^{-1}\{\bar{q}(s)\} \Rightarrow$ We use the method of partial fraction expansion $\Rightarrow$

Then, $\{q(t)\} = \sum_{k=1}^{n} \left\{ (s-j\omega_k)\{\bar{q}(s)\} e^{st} \Big|_{s=j\omega_k} + (s+j\omega_k)\{\bar{q}(s)\} e^{st} \Big|_{s=-j\omega_k} \right\} =$

$$= \sum_{k=1}^{n} \left\{ \frac{[A(s)][M]\{s q_0 + \dot{q}_0\} e^{st}}{(s+j\omega_k) \prod_{\substack{r=1 \\ r\neq k}}^{n} (s^2 + j\omega_r^2)} \Bigg|_{s=j\omega_k} + \frac{[A(s)][M]\{s q_0 + \dot{q}_0\} e^{st}}{(s-j\omega_k) \prod_{\substack{r=1 \\ r\neq k}}^{n} (s^2 + j\omega_r^2)} \Bigg|_{s=-j\omega_k} \right\}$$

Note that the response is expressed as superposition of the n modes!

But $[A(s=j\omega_k)] = [A(s=-j\omega_k)] \Rightarrow$

$$\Rightarrow \boxed{\{q(t)\} = \sum_{k=1}^{n} \frac{[A(j\omega_k)][M]}{\prod_{\substack{r=1 \\ r\neq k}}^{n} (\omega_r^2 - \omega_k^2)} \left\{ \{q_0\} \cos\omega_k t + \{\dot{q}_0\} \frac{\sin\omega_k t}{\omega_k} \right\}}$$

Response due to initial conditions, under the assumption of distinct natural frequencies and no damping

$$\bar{q}(s) = \frac{[A(s)]}{(s^2+\omega_1^2)(s^2+\omega_2^2)\dots(s^2-\omega_n^2)} =$$

$$= \frac{[A(s)]}{(s-j\omega_1)(s+j\omega_1)(s-j\omega_2)(s+j\omega_2)\dots(s-j\omega_n)(s+j\omega_n)}$$

$$\{q(t)\} = \sum_{k=1}^{n}\left[\left\{\cancel{(s-j\omega_k)}\frac{[A(s)]\,e^{st}}{(s-j\omega_1)\dots\cancel{(s-j\omega_k)}(s+j\omega_k)\dots(s+j\omega_n)}\right\}_{s=j\omega_k} +\right.$$

$$\left. + \left\{\cancel{(s+j\omega_k)}\frac{[A(s)]\,e^{st}}{\dots\,(s-j\omega_k)\cancel{(s+j\omega_k)}\dots}\right\}_{s=-j\omega_k}\right]$$

ii) Response to harmonic excitation

Suppose now that $\{Q(t)\} = [Q_1(t) \ldots Q_n(t)]^T = [Q_{01} e^{j\Omega_1 t} \ldots Q_{0n} e^{j\Omega_n t}]^T \Rightarrow$

Complex amplitudes

$\Rightarrow$ $\mathcal{L}$-transform the forcing vector $\{\bar{Q}(s)\} = \left[\frac{Q_{01}}{s-j\Omega_1} \ \cdots \ \frac{Q_{0n}}{s-j\Omega_n}\right]^T \equiv \left\{\frac{Q_0}{s-j\Omega}\right\}$

Assuming zero initial conditions $\Rightarrow \{q_0\} = \{\dot{q}_0\} = \{0\} \Rightarrow$

$[G(s)] = [Z(s)]^{-1}$

$$\Rightarrow \{\bar{q}(s)\} = \frac{[A(s)]}{|Z(s)|}\left[\frac{Q_{01}}{s-j\Omega_1} \cdots \frac{Q_{0n}}{s-j\Omega_n}\right]^T$$

Now, let's express the adjoint matrix $[A(s)]$ in terms of its columns $\Rightarrow$

$$\Rightarrow [A(s)] = [\{A_1(s)\} \ldots \{A_n(s)\}]$$

$\Rightarrow$

$$\Rightarrow \{\bar{q}(s)\} = \sum_{k=1}^{n} \frac{\{A_k(s)\}}{|Z(s)|}\frac{Q_{0k}}{s-j\Omega_k}$$

Has as poles the imaginary eigenvalues $\pm j\omega_k$

Additional poles introduced by the external harmonic forces!

The roots of $|Z(s)|=0$ are the poles $s=\pm j\omega_r$, $r=1,\dots,n$, which contribute to the transient response of the system which is zero for zero initial conditions; however, the contributions of the poles $s=j\Omega_k$ resulting from the external harmonic excitations contribute to the steady state response (which does not depend on the initial conditions of the problem) $\Rightarrow$ Hence, we compute the steady state response as,

$$\boxed{\{q(t)\}=\sum_{k=1}^{n}\frac{\{A_k(j\Omega_k)\}}{|Z(j\Omega_k)|}Q_{0k}\,e^{j\Omega_k t}}$$ (steady-state response)

provided that $\Omega_k \neq \omega_m \;\forall\, k,m\in[1,\dots,n]$ (lack of resonance). We note that when $\Omega_k \to \omega_m$ for some $k,m\in[1,\dots,n] \Rightarrow |Z(j\Omega_k)|\to 0$ and $\|\{q(t)\}\|\to\infty \Rightarrow$ In such a case the previous inversion cannot be performed and an alternative inversion must be followed that takes into account the resonance condition, i.e., $\Omega_k=\omega_m$ (linear resonance).

In the case of resonance, we get $\longrightarrow$ When $\omega_m = \Omega_k$

$$\{q(t)\} = \frac{d}{ds}\left[(s - j\omega_m)^2 \frac{\{A_k(s)\}}{|Z(s)|} \frac{Q_{ok}}{s - j\Omega_k} e^{st} \right]\Bigg|_{s = j\omega_m} +$$

$$+ \sum_{\substack{p=1 \\ p \neq k}}^{n} \frac{\{A_p(j\Omega_p)\}}{|Z(j\Omega_p)|} Q_{op}\, e^{j\Omega_p t} =$$

Note that this term explodes as $t \to \infty$

$$= \frac{Q_{ok}\, e^{j\Omega_k t}}{2j\Omega_k \sum\limits_{\substack{r=1 \\ r \neq m}}^{n} (\omega_r^2 - \Omega_k^2)} \Bigg\{ t A_k(j\Omega_k) + A_k'(j\Omega_k) -$$

$$- \left(\frac{1}{2j\Omega_k} + \sum_{\substack{r=1 \\ r \neq m}}^{n} \frac{2j\Omega_k}{\omega_r^2 - \Omega_k^2} \right) A_k(j\Omega_k) \Bigg\} +$$

$$+ \sum_{\substack{p=1 \\ p \neq k}}^{n} \frac{\{A_p(j\Omega_p)\}}{|Z(j\Omega_p)|} Q_{op}\, e^{j\Omega_p t}$$

Frequency response – resonanse

$$\boxed{\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 = P\cos\omega t} \rightarrow \text{SS solution (not interested in ICs)}$$

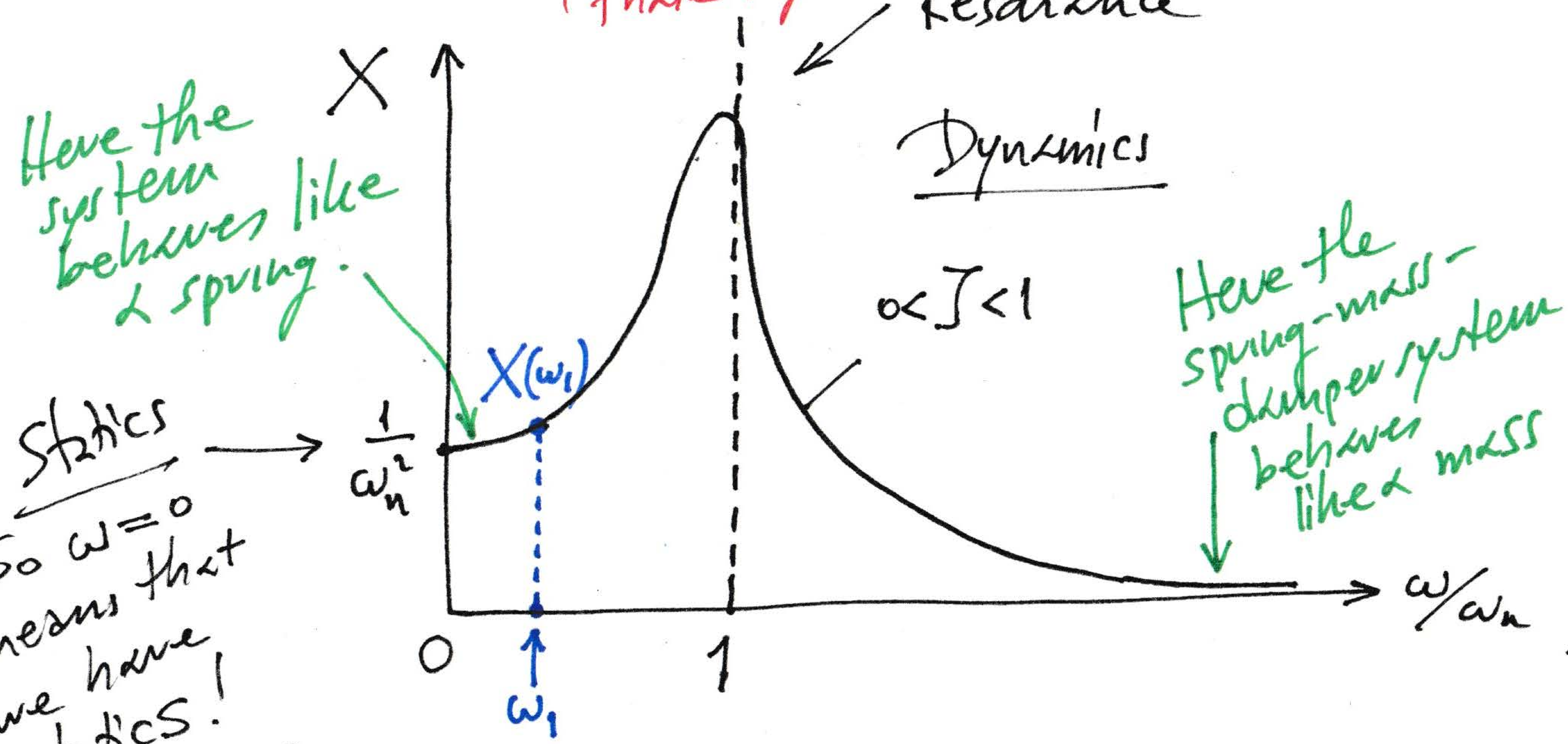
$$\downarrow$$

$$\boxed{x_{ss}(t) = X\cos(\omega t - \varphi)} \Rightarrow \text{Note that, e.g., if } \varphi = 0 \Rightarrow x_{ss}(t) = X\cos\omega t$$

$$\text{if } \varphi = \pi \Rightarrow x_{ss}(t) = -X\cos\omega t$$

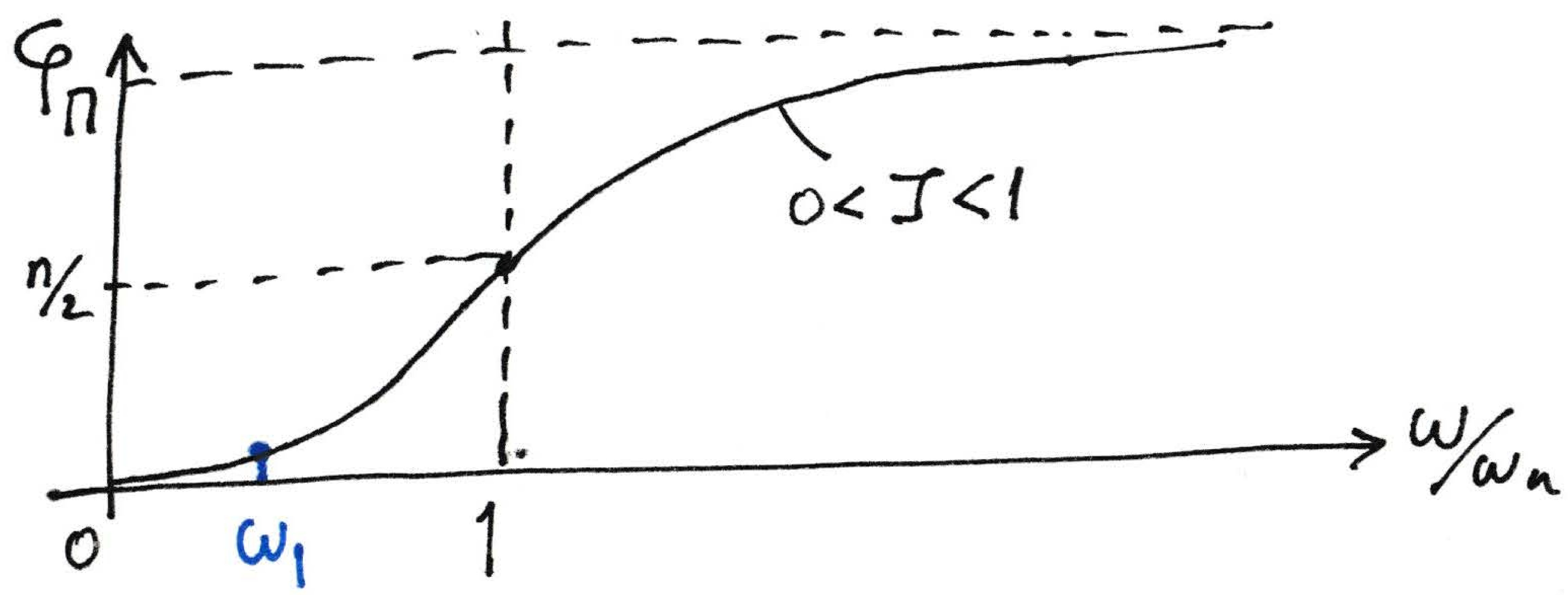
X: Always positive (modulus)

φ: Phase lag with respect to the excitation

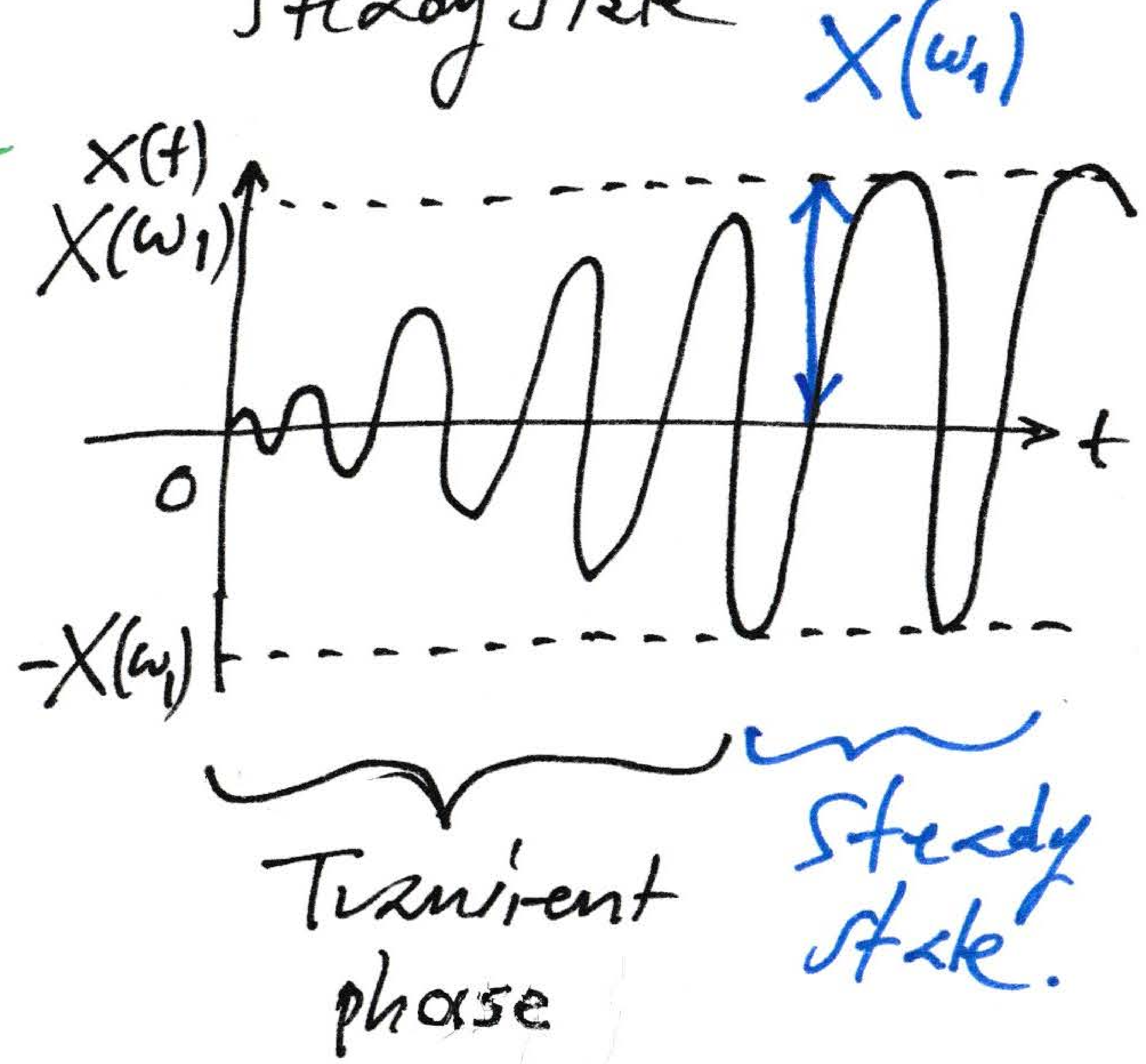


Statics →

So $\omega = 0$ means that we have statics! Whereas $\omega > 0$ means dynamics.

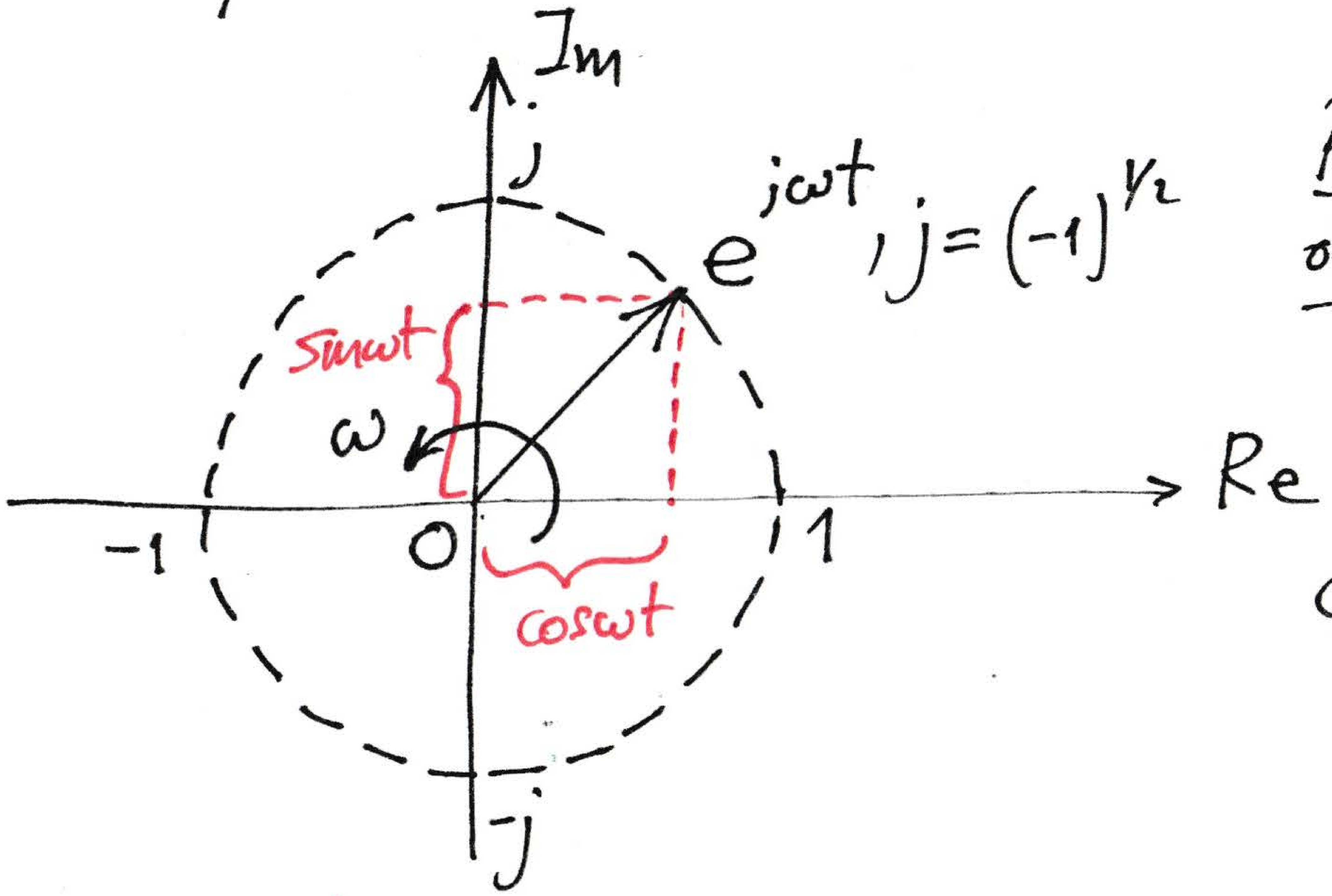


Remark: Transition to steady state



So, the resonance plot depicts only the steady state response (and ignores the transient phase)

We now introduce a complex vector representation to simplify the analysis.



$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$
$$\cos\omega t = Re\left[e^{j\omega t}\right]$$
$$\sin\omega t = Im\left[e^{j\omega t}\right]$$

So, a harmonic motion is represented by a rotating vector (of constant magnitude) in the complex domain.

So, reconsider,

Real domain: $\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \begin{Bmatrix} P\sin\omega t \\ P\cos\omega t \end{Bmatrix}}$

$\downarrow$ Replace by an equivalent problem

Complex domain: $\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = Pe^{j\omega t}}$

Solve this problem and then "go back" to the real domain.

Suppose that the solution in the complex domain is $x_{ss}^{c}(t)$.

Recognizing that $P\sin\omega t = Im\left[Pe^{j\omega t}\right]$

$$x_{ss}^{R}(t) = Im\left[x_{ss}^{c}(t)\right] \leftarrow \boxed{x_{ss}^{c}(t)}$$

If, $P\cos\omega t = Re\left[Pe^{j\omega t}\right]$

$$x_{ss}^{R}(t) = Re\left[x_{ss}^{c}(t)\right] \leftarrow \boxed{x_{ss}^{c}(t)}$$

<u>Digression</u>: Suppose that we have a complex function,

$\frac{a+jb}{c+jd} \Rightarrow$ How do we compute its modulus and phase?

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2} =$$

$$= \underbrace{\left(\frac{ac+bd}{c^2+d^2}\right)}_{\text{Real part } (Re)} + j \underbrace{\left(\frac{bc-ad}{c^2+d^2}\right)}_{\text{Imaginary part } (Im)} \Rightarrow$$

$\Rightarrow$ Modulus $= \sqrt{Re^2 + Im^2} = \dots$

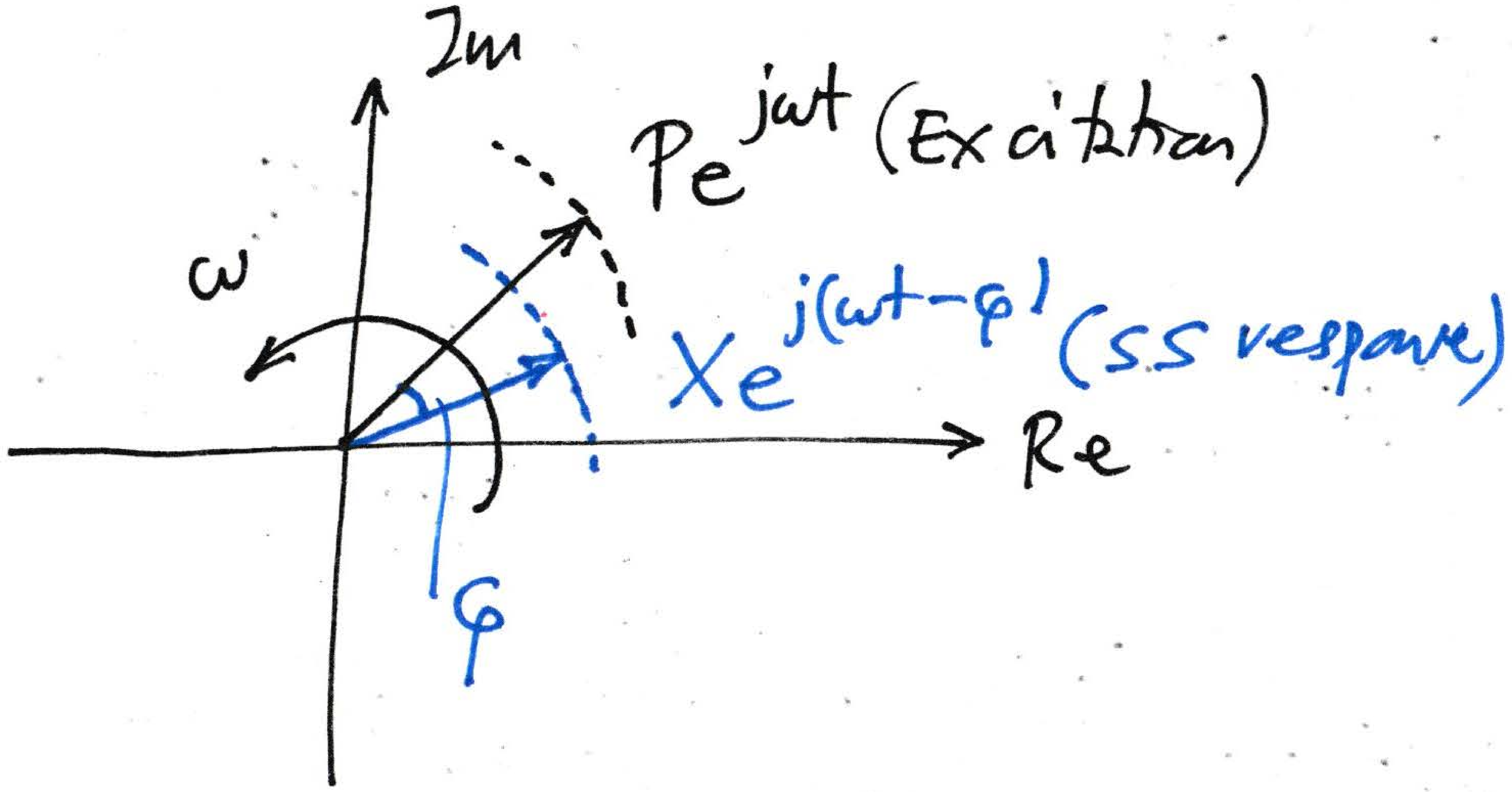
Phase is $\tan^{-1}\left(\frac{Im}{Re}\right) = \dots$

To find the SS solution in the complex domain, look for a solution of the form, $X_{ss}(t) = Xe^{j(\omega t-\varphi)} = Xe^{j\omega t}e^{-j\varphi} =$

Real magnitude > 0

$= X^{*} e^{j\omega t}$ (*)

where $X^{*} \doteq Xe^{-j\varphi}$, X^{*} is complex amplitude



Substituting (*) into the complex problem,

$$-\omega^2 X^{*} \cancel{e^{j\omega t}} + 2\zeta\omega_n j\omega X^{*} \cancel{e^{j\omega t}} + \omega_n^2 X^{*} \cancel{e^{j\omega t}} = P \cancel{e^{j\omega t}} \Rightarrow$$

$\Rightarrow$ Solve directly for $X^{*} = \dfrac{P}{(\omega_n^2-\omega^2)+j2\zeta\omega_n\omega} \Rightarrow$

$$\Rightarrow \boxed{\frac{X^{*}}{P} = \frac{1}{\omega_n^2-\omega^2+j2\zeta\omega_n\omega}}$$

Steady state response (**)

Applying this methodology to (**) $\Rightarrow$

$$\left.\begin{aligned} \frac{X}{P} &= \frac{|X^*|}{P} = \left(\frac{1}{\omega_n^2}\right)\frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2\right\}^{1/2}} \\ \varphi &= \tan^{-1}\left\{\frac{2\zeta\,\omega/\omega_n}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\} \end{aligned}\right\}$$

Recap: The solution in the complex domain is

Solution in the complex plane (also denoted as $X_{ss}^C(t)$) $\longrightarrow$ $X_{ss}(t) = X^* e^{j\omega t} = X e^{j(\omega t-\varphi)}$

$\Big\downarrow$ To the real domain

Solution in the real plane (also denoted as X_{ss}^R)

If excitation is $P\cos\omega t \Rightarrow X_{ss}(t) = \mathrm{Re}\left[X e^{j(\omega t-\varphi)}\right] = X\cos(\omega t-\varphi)$

If excitation is $P\sin\omega t \rightarrow X_{ss}(t) = \mathrm{Im}\left[X e^{j(\omega t-\varphi)}\right] = X\sin(\omega t-\varphi)$

At steady state we have the following dynamic equilibrium:

$$\ddot{x}_{ss} + 2\zeta\omega_n \dot{x}_{ss} + \omega_n^2 x_{ss} = Pe^{j\omega t}$$

Inertia force Damping force Stiffness force Excitation force

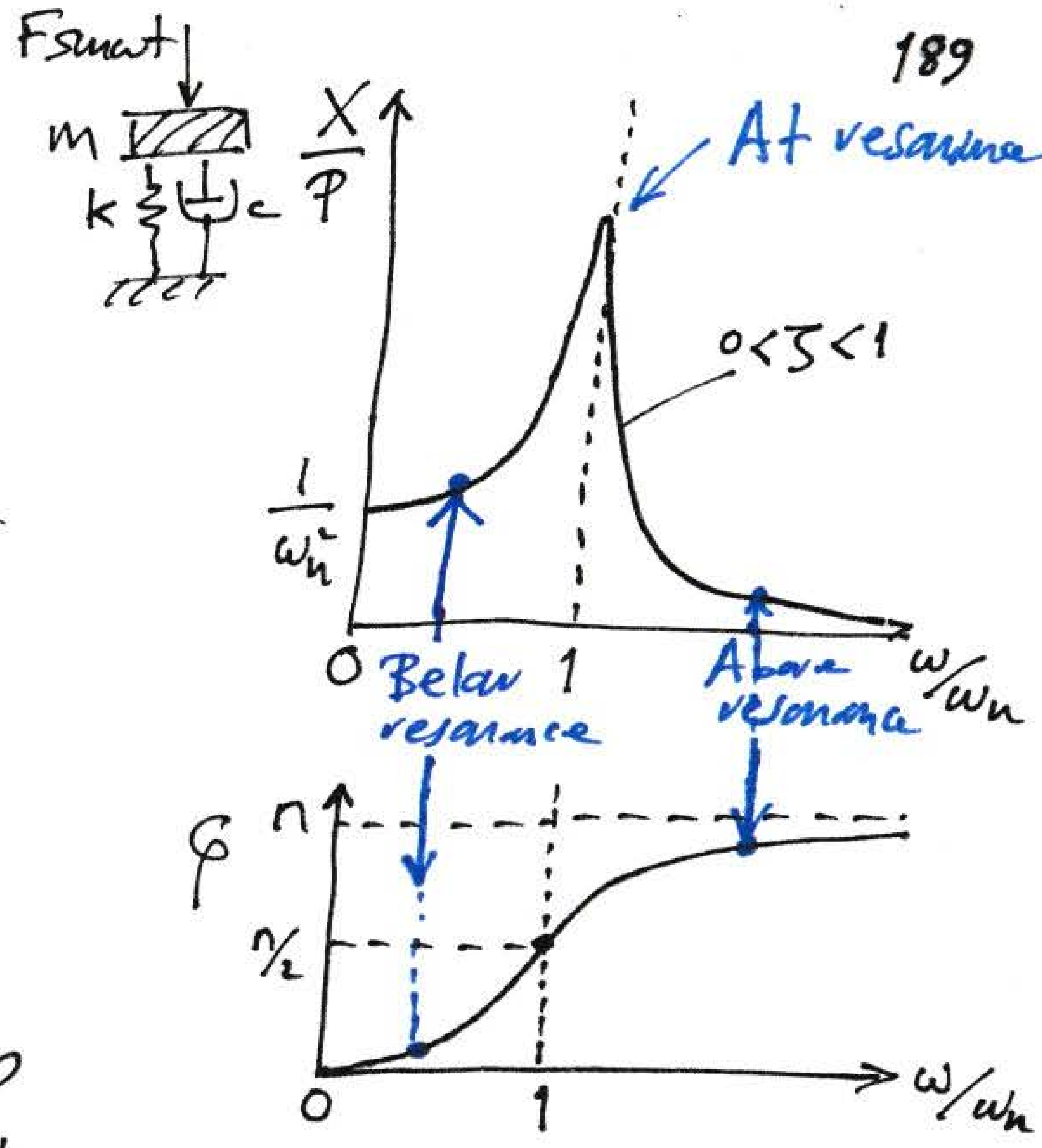
Given that $x_{ss}(t) = Xe^{j(\omega t-\varphi)} = \underbrace{Xe^{-j\varphi}}_{X^*} e^{j\omega t}$

$$\omega_n^2 x_{ss}(t) = \omega_n^2 X^* e^{j\omega t}$$

$$\ddot{x}_{ss} = -\omega^2 X^* e^{j\omega t}$$

$$2\zeta\omega_n \dot{x}_{ss} = 2j\zeta\omega_n\omega X^* e^{j\omega t}$$

These are rotating vectors in the complex plane!

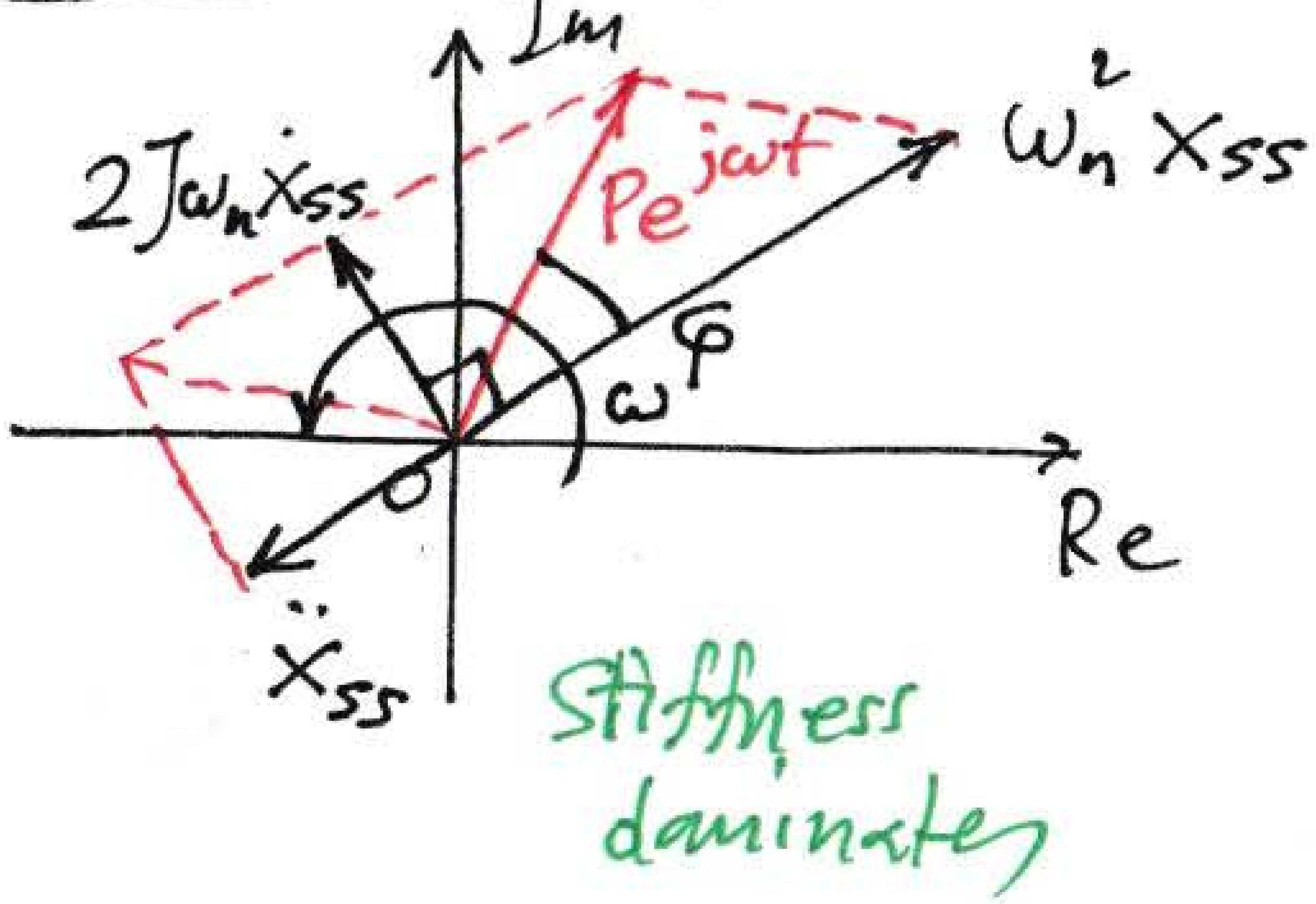


Reason for resonance

$$\ddot{x}_{ss} + 2\zeta\omega_n \dot{x}_{ss} + \omega_n^2 x_{ss} = Pe^{j\omega t}$$

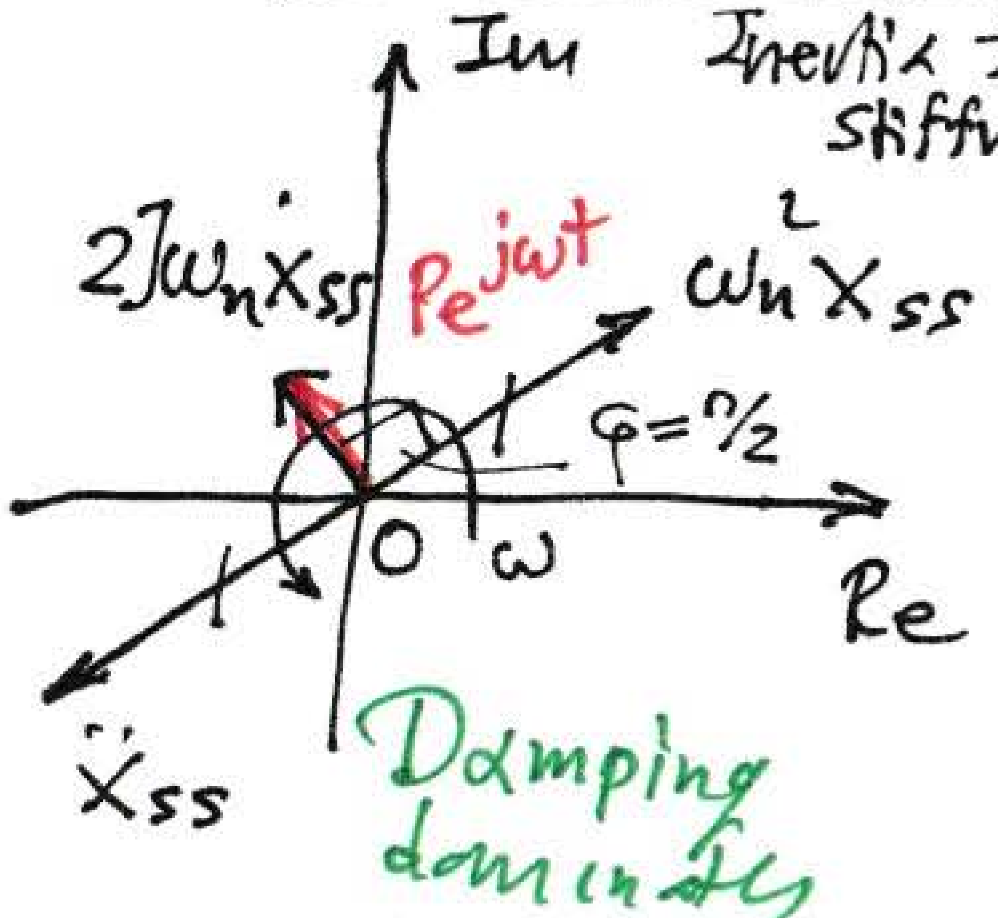
$$x_{ss}(t) = Xe^{j(\omega t-\varphi)}$$

Below resonance, $\omega < \omega_n$

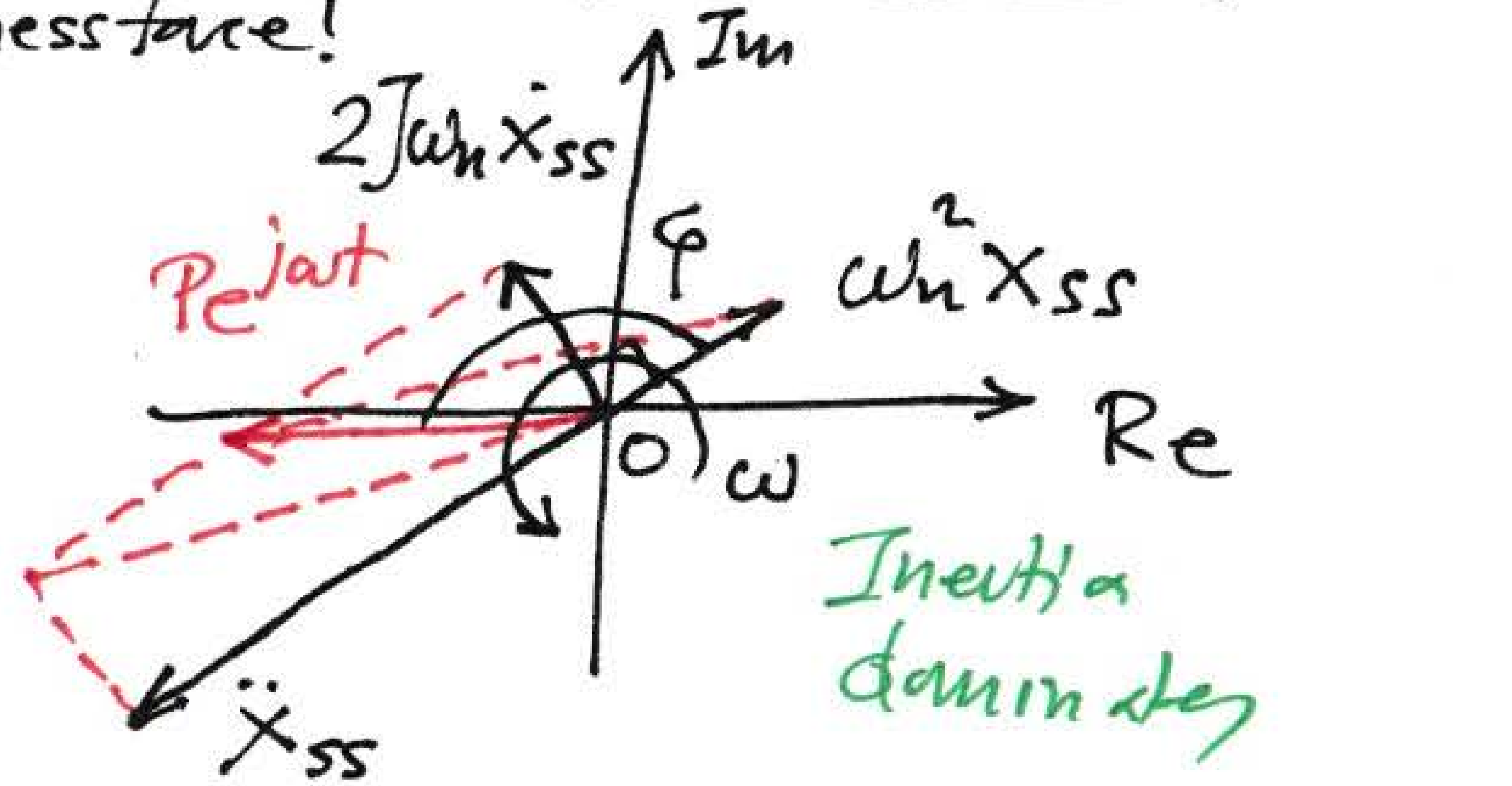


At resonance, $\omega = \omega_n$

Inertia force cancels stiffness force!



Above resonance, $\omega > \omega_n$



Remarks

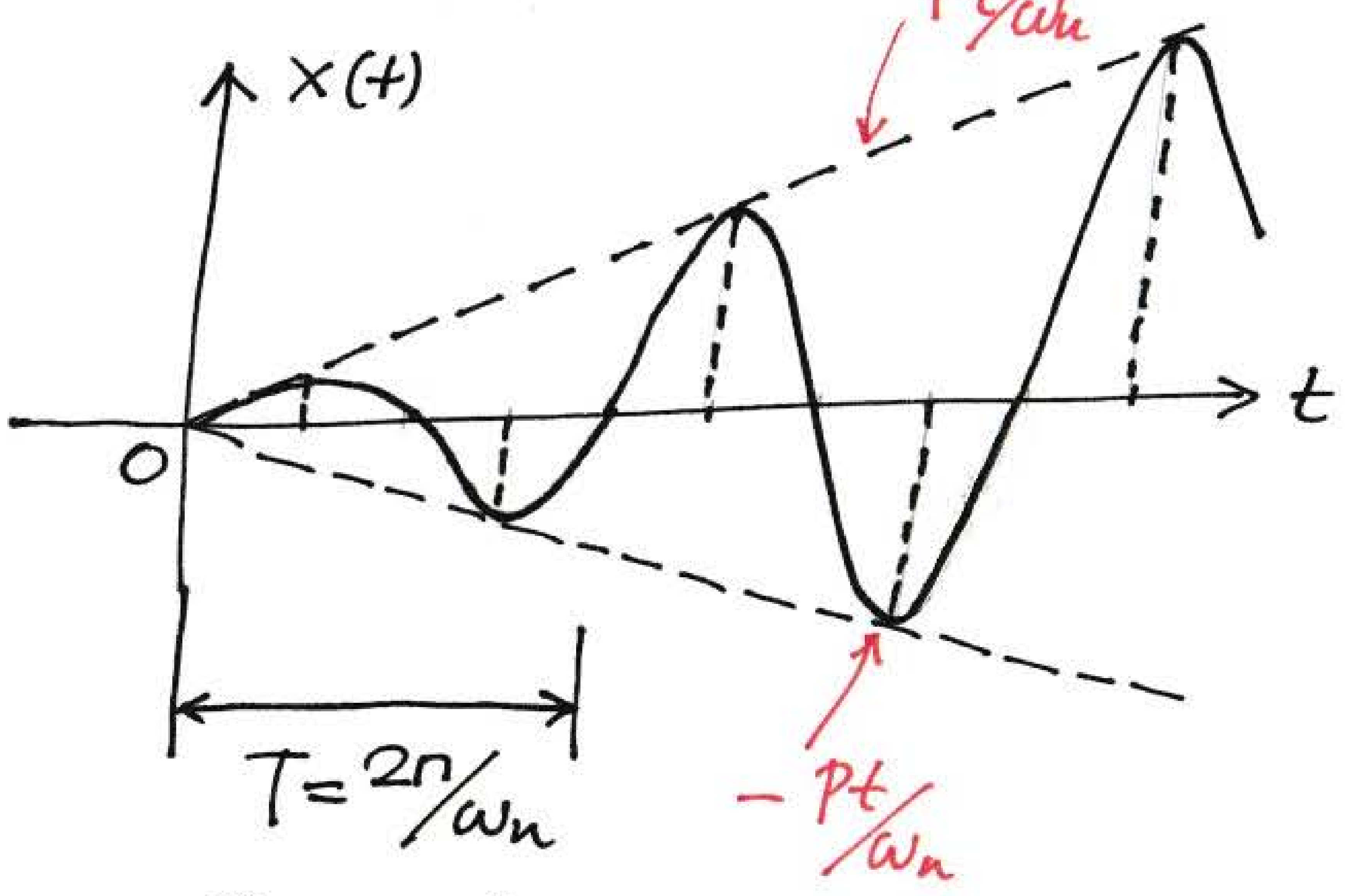
1) In the <u>undamped</u> system at resonance the external harmonic excitation cannot be balanced by the inertia or stiffness forces which cancel each other $\Rightarrow$ We get infinite steady state response at resonance!

This can be concluded by solving the initial value (transient) problem as follows:

Consider, $\left.\begin{aligned} &\ddot{x} + \omega_n^2 x = P\cos\omega_n t \\ &x(0) = \dot{x}(0) = 0 \end{aligned}\right\} \Rightarrow$ The solution is the particular solution,

$$x(t) = \frac{Pt}{\omega_n}\sin\omega_n t$$

So, by solving the transient problem at resonance we recover the uncontrolable growth of the response to infinity!



Period of the oscillation