where $u_0(x), \ldots, u_n(x)$ are the eigenfunctions corresponding to the degenerate eigenvalues $\lambda = 0$ of multiplicity $n+1$.

Equivalence of Integral and Differential Equations

Using Green's functions we can convert differential equations and boundary value problems to integral equations. Let's consider the nonhomogeneous ordinary differential equation,

$$\boxed{L[u] + \lambda \rho u = \psi(x)} \Rightarrow L[u] = \overbrace{\psi(x) - \lambda \rho u}^{-\varphi(x)} \quad + BCs \qquad (1)$$

where $L[u]$ is a self-adjoint operator, $\psi(x)$ is piecewise continuous, and $\rho(x)$ is positive and continuous, and λ is a positive parameter. Considering homogeneous boundary conditions in the domain $[x_0, x_1]$, say $u(x_0) = u(x_1) = 0$, we can immediately write,

$$\boxed{u(x) = \lambda \int_{x_0}^{x_1} K(x,\xi)\rho(\xi)u(\xi)\,d\xi - \int_{x_0}^{x_1} K(x,\xi)\psi(\xi)\,d\xi} \quad \cancel{+BCs} \qquad (2)$$

$K(x,\xi)$ is the Green's function of $L[u]$

Nonhomogeneous term of this integral equation

Considering λ as the eigenvalue of the resulting boundary value problem, there is an equivalence of the boundary value problem (1) and the integral equation (2).

Moreover, if $L[u]$ is self-adjoint $\Rightarrow$ $K(x,\xi)$ is symmetric in its arguments $\Rightarrow$ The original BVP has been reduced to an integral equation with symmetric kernel $\Rightarrow$ We can apply powerful theorems valid for integral equations with symmetric kernels (directly to (1)!).

Hence consider again,
$$L[u]+\lambda\rho u=\psi(x) \iff u(x)=\lambda\int_{x_0}^{x_1}K(x,\xi)\rho(\xi)u(\xi)d\xi-\int_{x_0}^{x_1}K(x,\xi)\psi(\xi)\,d\xi$$
$u(x_0)=0$
$u(x_1)=0$
BCs

Green's function corresponding to $L[u]$

There exist the following two alternatives.

Either, for fixed λ every solution of the homogeneous equation $L[u]+\lambda\rho u=0$ vanishes identically (λ is not an eigenvalue) $\Rightarrow$ Then the nonhomogeneous equation $L[u]+\lambda\rho u=\psi(x)$ has a unique solution.

Or, for some $\lambda=\lambda_i$ the homogeneous equation has a nontrivial solution $u_i(x)$ (λ_i is the i-th eigenvalue, $u_i(x)$ is the i-th eigenfunction) $\Rightarrow$ Then the solution of the nonhomogeneous differential equation exists iff it holds that
$$\int_{x_0}^{x_1}\rho u_i(x)\psi(x)\,dx=0$$
for all eigenfunctions.

Fredholm's alternative

from the theory of integral equations with symmetric kernels we have the following result regarding the homogeneous operator. $\exists$ a sequence of eigenvalues $\lambda_1, \lambda_2, \dots$ with $\lambda_n \to \infty$, with associated eigenfunctions $u_1(x), u_2(x), \dots$ which form a basis for the infinite-dimensional functional space; moreover, they satisfy the orthogonality conditions,

$$\int_{x_0}^{x_1} \rho u_i u_j \, dx = 0, \quad i \neq j$$

and we can normalize them so that $\int_{x_0}^{x_1} \rho u_i^2 dx = 1, \; i = 1, 2, \dots \Rightarrow$ The basis of eigenfunctions resulting from $L[u] + \lambda \rho u = 0$, $L[u]$ self-adjoint and homogeneous BCs, is orthogonal. Moreover, this basis is also complete.

Any solution of the nonhomogeneous problem $L[u] + \lambda \rho u = \psi(x)$ can be represented using the Green's function formulation,

$L[u] = \psi(x) - \lambda \rho u \equiv -\varphi(x)$

This is based on the operator $L[u]$ only.

$$w(x) = + \int_{x_0}^{x_1} K(x, \xi) \varphi(x) \, dx,$$ where $K(x, \xi)$ is the Green's function of the left-hand side operator $L[u]$.

Then, $w(x)$ can be expanded in terms of the eigenfunctions (since the eigenfunctions form a complete basis), as follows,

$$w(x) = \sum_{n=1}^{\infty} c_n u_n(x), \quad c_n = \int_{x_0}^{x_1} w \rho u_n \, dx \qquad \text{(Expansion theorem)}$$

Then, invoking Mercer's theorem, since all eigenvalues of the self-adjoint operator are positive, we obtain the following relation,

$$\boxed{K(x,\xi) = \sqrt{\rho(x)\rho(\xi)} \sum_{n=1}^{\infty} \frac{u_n(x)\, u_n(\xi)}{\lambda_n}} \qquad \text{(Bilinear expression)} \quad (3)$$

Provides a relation between the Green's function and the eigenfunctions.

where the series converges absolutely and uniformly. This is called spectral decomposition of the Green's function. A sufficient condition for the validity of the expansion theorem is that the first and second derivatives of the continuous function $w(x)$ should be piecewise continuous.

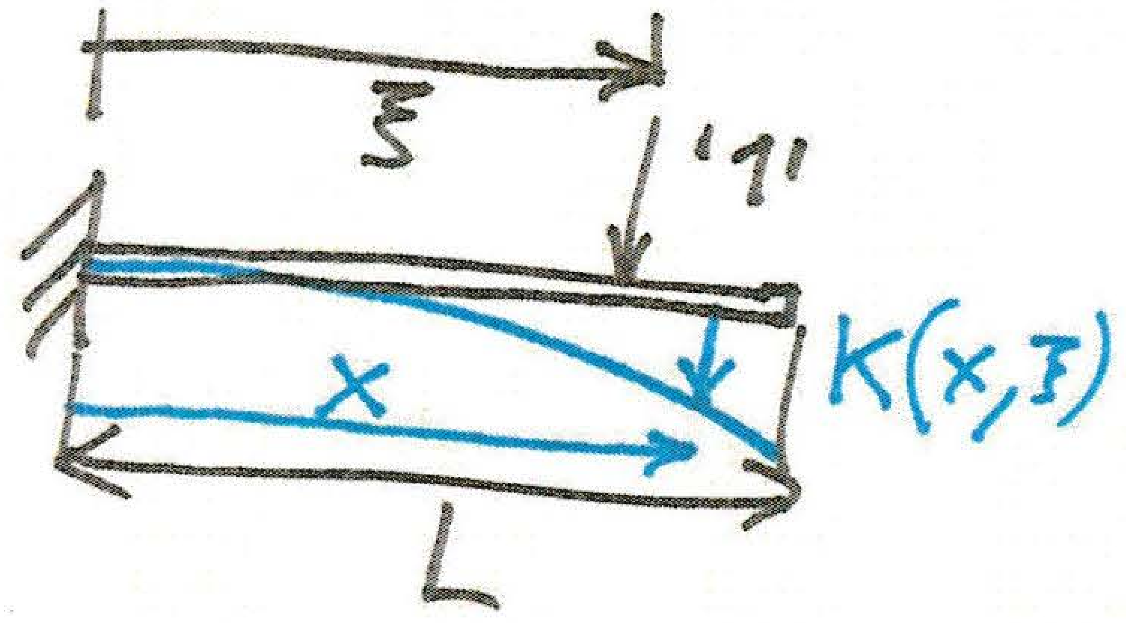
Alternatively we can introduce a different definition for the operator,

$$\underbrace{L[u] + \lambda \rho u = \psi(x)}_{\text{Operator with Green's function } \tilde{K}(x,\xi)} \Rightarrow \text{We can similarly write the solution as,} \quad w(x) = \sum_{n=1}^{\infty} \gamma_n u_n(x), \quad \gamma_n = \frac{\int_{x_0}^{x_1} u_n(x) \psi(x)\, dx}{\lambda - \lambda_n}.$$

from this last expression we note that if $\lambda = \lambda_n$ resonance occurs if $\int_{x_0}^{x_1} u_n(x)\psi(x)dx \neq 0 \Rightarrow$ To avoid resonance (and blow-up of the series) it needs to be satisfied that $\int_{x_0}^{x_1} u_n(x)\psi(x)dx = 0,\ n = 1, 2, \dots$ ← Fredholm's alternative!

Odes of higher order

Similar formulations hold for operators of higher order. For example, considering $u'''' - \lambda u = 0$, this corresponds to the eigenvalue problem of the beam. As before we can introduce the Green's function $K(x,\xi)$ as the displacement of the beam at position x due to a unit force at position ξ, satisfying the boundary conditions.



$K(x,\xi)$ satisfies the following conditions:

1) $\forall \xi$ the function $K(x,\xi)$ together with its first and second derivatives is continuous and satisfies the boundary conditions.
2) At $\forall\, x \neq \xi$, $x \in [x_0, x_1] = \text{Domain}$, the third and fourth derivatives of $K(x,\xi)$ with respect to x are also continuous. However at $x = \xi$ the following discontinuity is imposed, $K'''(x,\xi)\Big|_{\xi-}^{\xi+} = -1$.

3) Except at $x=\xi$, $K(x,\xi)$ satisfies the differential equation $K''''(x,\xi)=0$, where $()' \equiv d/dx$.

Note here that we chose $L[u]=u''''$

The foundamental property of Green's functions can be stated as follows: Let $u(x)$ be a continuous function satisfying the BCs, and with continuous first, second and third derivatives, and piecewise continuous fourth derivatives; let $\varphi(x)$ be a piecewise continuous function $\Rightarrow$

$\Rightarrow$ If it holds that $\underset{+\text{BCs}}{L[u] \equiv u''''} = -\varphi(x) \Rightarrow u(x) = \int_{x_0}^{x_1} K(x,\xi)\,\varphi(\xi)\,d\xi$

and conversely.

The eigenvalue problem for this operator is formulated as,

$$u'''' - \lambda \rho u = 0$$

with the corresponding nonhomogeneous problem $u'''' - \lambda\rho u = -\psi(x)$, and can be treated as previously. There exists an countable infinity of eigenvalues $\lambda_1, \lambda_2, \ldots$ and associated eigenvectors $u_1(x), u_2(x), \ldots$, with the properties that the scaled modes $\sqrt{\rho}\,u_i$ form a complete, orthogonal system, so that $\forall\, w(x)$ satisfying the BCs and with continuous derivatives up to the 3^{rd} order and piecewise fourth derivatives may be expanded in terms of the eigenfunctions in an absolutely and uniformly convergent series. Moreover, Mercer's theorem implies the bilinear relation,

$$K(x,\xi) = \sum_{n=1}^{\infty} \frac{u_n(x)\,u_n(\xi)}{\lambda_n} = K(\xi,x) \quad \text{in } G$$

This is the Green's function of $L[u] \equiv u''''$

Note symmetry of Green's function

and the corresponding expansion theorem.

Examples of Green's functions

1) $L[u] = u''$, $[x_0, x_1] = [0,1]$, $u(0) = u(1) = 0 \Rightarrow K(x,\xi) = \begin{cases} (1-\xi)x, & x \le \xi \\ (1-x)\xi, & x > \xi \end{cases}$

Remark: $\left.\begin{array}{l} u'' = 0 \Rightarrow u = c_1 + c_2 x \\ u(0) = 0 \end{array}\right\} \Rightarrow c_1 = 0 \Rightarrow u = c_2 x, \ x \le \xi$

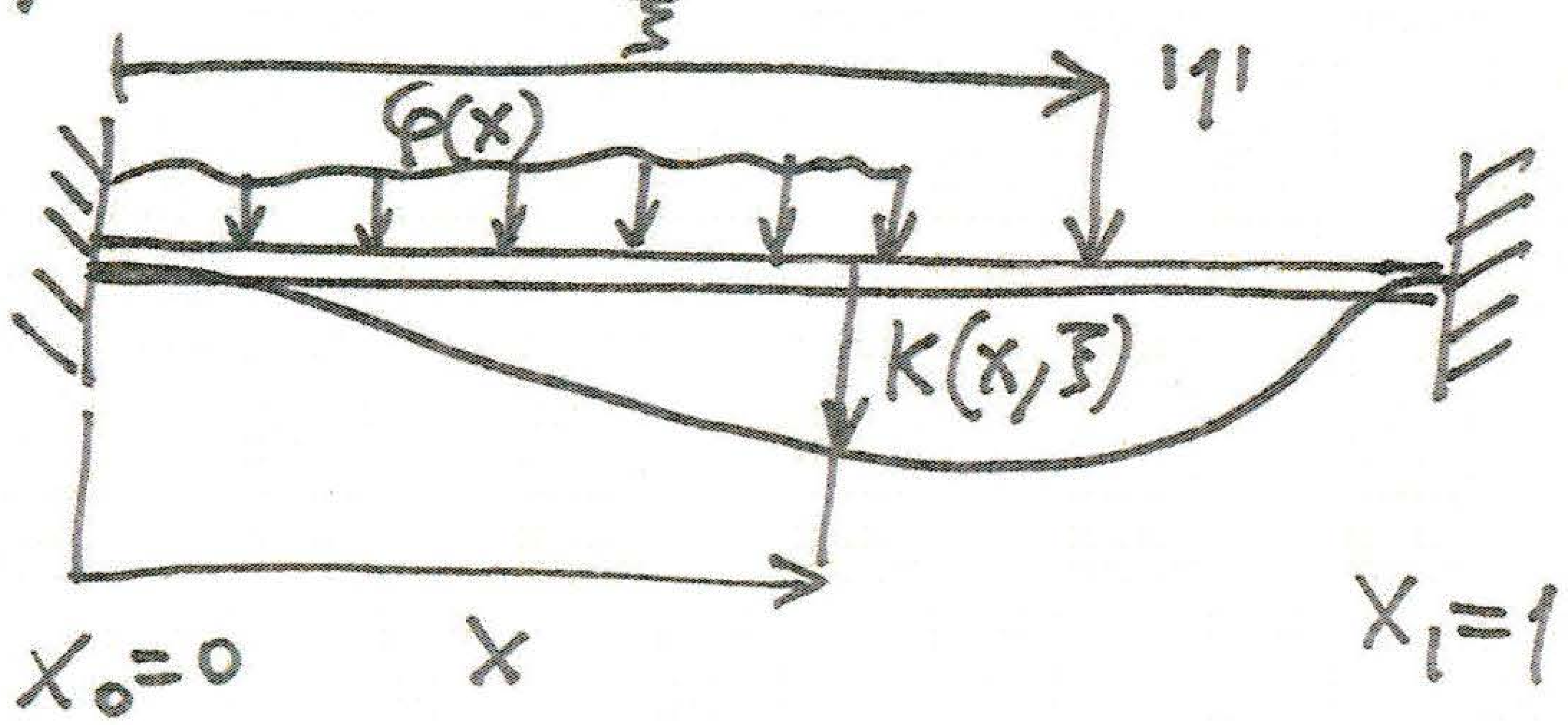
$\left.\begin{array}{l} u'' = 0 \Rightarrow \tilde{u} = d_1 + d_2 (1-x) \\ \tilde{u}(1) = 0 \end{array}\right\} \Rightarrow d_1 = 0 \Rightarrow \tilde{u} = d_2(1-x), \ x > \xi$

$$\left.\begin{array}{l} \text{At } x = \xi \Rightarrow \left.\dfrac{du}{dx}\right|_{\xi-}^{\xi+} = -1 \Rightarrow \left.\dfrac{d\tilde{u}}{d\xi}\right|_{\xi+} - \left.\dfrac{du}{d\xi}\right|_{\xi-} = -1 \Rightarrow -d_2 - c_2 = -1 \\ \text{At } x = \xi \Rightarrow \tilde{u}(\xi+) = u(\xi-) \Rightarrow d_2(1-\xi) = c_2\xi \end{array}\right\}$$

2) $L[u]=u''$, $[x_0, x_1]=[0,1]$, $u(0)=0$, $u'(1)=0 \Rightarrow K(x,\xi)=\begin{cases} x, & x\leq\xi \\ \xi, & x\geq\xi \end{cases}$

Note symmetry

3) Clamped-clamped linear beam



$u''''-\lambda\rho u=\varphi(x)$, $u(0)=u'(0)=u(1)=$
$=u'(1)=0$

$L[u]=u'''' \Rightarrow L[u]=\lambda\rho u+\varphi(x) \Rightarrow$
$\Rightarrow$ Derive $K(x,\xi)$

$\tilde{L}[u]=u''''-\lambda\rho u \Rightarrow \tilde{L}[u]=\varphi(x) \Rightarrow$
$\Rightarrow$ Derive $\tilde{K}(x,\xi)$

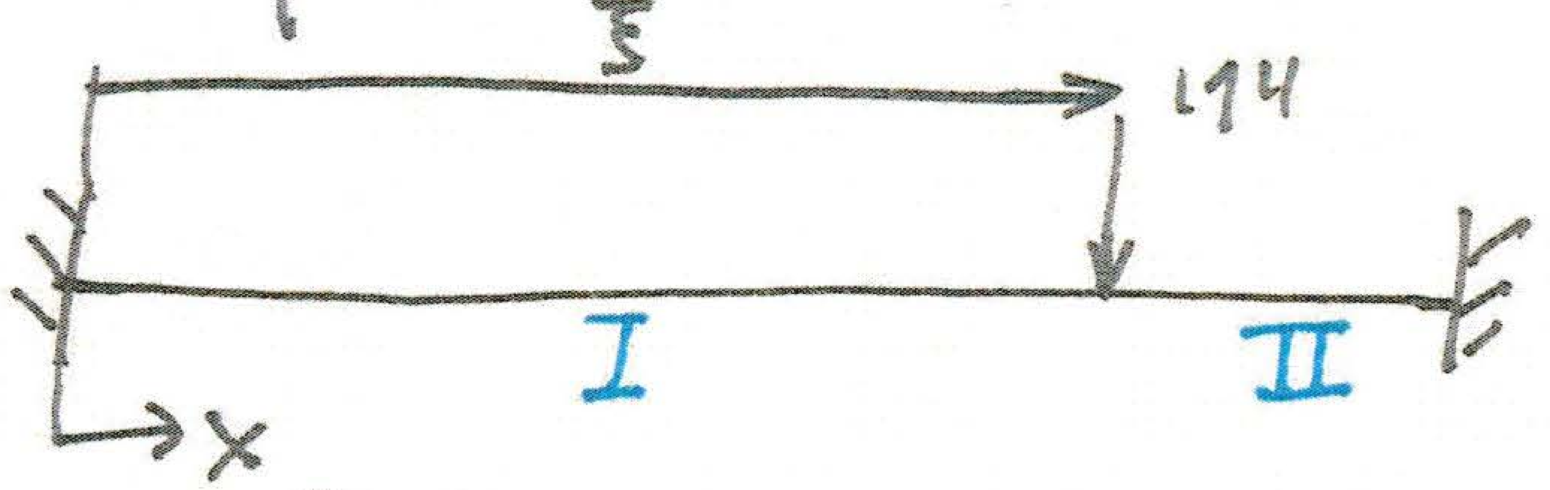
Then the response would be,

$$u(x)=\int_0^1 -(\lambda\rho u+\varphi)\,K(x,\xi)\,dx$$

$$u(x)=\int_0^1 -\varphi(x)\,\tilde{K}(x,\xi)\,dx$$

Remark

To compute $\tilde{K}(x,\xi) \Rightarrow$ It satisfies
$u''''-\lambda\rho u=0 \quad \forall\, x\in[0,1]-\{x=\xi\}$.



Part I: $\tilde{K}^{I}(x,\xi)=c_1\cos\gamma x+c_2\cosh\gamma x+c_3\sin\gamma x+c_4\sinh\gamma x$ $\Big\}$ $\tilde{K}^{I}(0,\xi)=0$, $\tilde{K}^{I\prime}(0,\xi)=0$

Part II: $\tilde{K}^{II}(x,\xi)$ $\Big\}$ $\tilde{K}^{II}(1,\xi)=0$, $\tilde{K}^{II\prime}(1,\xi)=0$

$\tilde{K}^{II'''}(\xi,\xi)-\tilde{K}^{I'''}(\xi,\xi)=-1$

$\tilde{K}^{II''}(\xi,\xi)=\tilde{K}^{I''}(\xi,\xi)$

Continuity in displacement at $x=\xi \Rightarrow K^{II}(\xi,\xi) = K^{I}(\xi,\xi)$

Continuity in slope at $x=\xi \Rightarrow K^{II'}(\xi,\xi) = K^{I'}(\xi,\xi)$

Then, obtain 4 non-homogeneous algebraic equations for the remaining four unknown coefficients.