Nodal Fourier Bases

- We consider the use of Fourier bases to solve boundary value problems in d space dimensions.
- Because of their limited applicability (i.e., limited rate of convergence) in general domains, these will generally be combined with polynomial bases in one or more space dimensions, depending on the boundary conditions.
- Our interest in Fourier bases will thus be confined to problems that are *periodic* in one or more dimensions.
- We further assume that the domain is either translationally invariant or smoothly varying in the periodic direction.
- Despite their limitations, Fourier bases offer some significant advantages over other spectral bases, including:
  - No numerical dispersion—all resolved waves are propagated at the correct speed.
  - Uniform point distribution and CFL $\sim O(N^{-1})$ instead of $O(N^{-2})$.
  - $O(N \log N)$ operator application instead of $O(N^2)$.
- For these reasons, they are often used for targeted applications.

Why Nodal?

- If we use complex exponentials for the underlying basis, the discrete differential operators are (complex) diagonal matrices, which makes them fast to invert.
- We can achieve the same result with nodal bases, as one can convert from modal to nodal bases in $O(\log_2 N)$ time using the FFT.
- More precisely, for $d = 2$ the complexity is $O(N^2 \log_2 N)$.
- Unfortunately, there is no standardization of the FFT interface.
- Moreover it requires working in complex arithmetic, which is not as highly optimized as real floating-point arithmetic.
- If we work strictly in real arithmetic with nodal bases, we can evaluate all 2D operators in $O(N^2)$ storage and $O(N^3)$ work using *matrix-matrix* products (i.e., BLAS3), which are extremely fast.
- In fact, one needs $N \gtrsim 200$ before the FFT-based implementation outperforms the matrix-matrix based one.
- For these reasons, and for portability, it easier to simply treat the Fourier bases just as we do the Legendre bases, save that we have different $\hat{D}$, $\hat{B}$, and $\hat{J}$ (differentiation, mass, and interpolation) matrices.
- With these established, we can also use the same prolongation matrix, $R^T = Q$, that we introduced earlier for the nodal Legendre method with periodic BCs.

Real Fourier Bases

- Consider an expansion of the form

$$u(x,t) \;=\; \sum_{k=1}^{N} \phi_k(x)\hat{u}_k(t), \tag{1}$$

with (real-valued) bases,

$$\begin{aligned} X_f^N &= \text{span}\{\phi_1,\, \phi_2,\, \dots\, ,\phi_N\} \\ &= \text{span}\{1,\, \cos x,\, \sin x,\, \cos 2x,\, \sin 2x,\, \dots\, ,\cos \tfrac{N-1}{2}x,\, \sin \tfrac{N-1}{2}x\}. \end{aligned} \tag{2}$$

- In this case, if N is even, we drop the last sine function in the basis set and replace the last cosine function by $\cos \frac{N}{2}x$.
- Somewhat curiously, if N=4, we have a set of four basis functions,

$$X_f^N \;=\; \text{span}\{1,\, \cos x,\, \sin x,\, \cos 2x\}, \tag{3}$$

which exhibits a preponderence of cosines corresponding to $k = 0$, 1, and 2, and only one sine function.

- This apparent asymmetry stems from the facts that $\sin kx \equiv 0$ for $k = 0$ and that $\sin \frac{N}{2}x$ vanishes on the uniform point distribution $x_j = jh$ with $h = 2\pi/N$, $j = 0, \dots, N-1$.
- The basis set (2) is *modal.*
- To be consistent with our Legendre implementations we'd like to have a *nodal* representation.
- That is, find $l_j(x) \in X_f^N$ such that

$$l_j(x_i) \;=\; \delta_{ij}, \quad i,\, j\; \in \{1, \dots, N\}. \tag{4}$$

- For the moment, we'll consider nodal points $x_j = 2\pi j/N$, $j = 1, \dots, N$, but we will later expand this set to include $x_0 = 0$.

- Notice that we can construct a square invertible matrix

$$J_{jk} \;=\; \phi_k(x_j) \tag{5}$$

- This is the *synthesis* matrix that combines sines and cosines ($\phi_k(x)$), multiplied by modal coefficients, $\hat{u}_k$, to construct $u_j = u(x_j)$,

$$\underline{u} \;=\; J\underline{\hat{u}}. \tag{6}$$

- Here, $\underline{\hat{u}} = [\hat{u}_1\ \hat{u}_2\ \ldots \hat{u}_N]^T$ are the modal coefficients and $\underline{u} = [u_1\ u_2\ \ldots u_N]^T$ are the nodal coefficients.
- J^{-1} is the matrix that takes us from nodal to modal, which is what we will want in order to evaluate, in a preprocessing step, interpolants and derivatives.
- So, given a vector of nodal values $\underline{u} = [u_1 \ldots u_N]^T$, we find the modal coefficients as

$$\underline{\hat{u}} \;=\; J^{-1}\underline{u}. \tag{7}$$

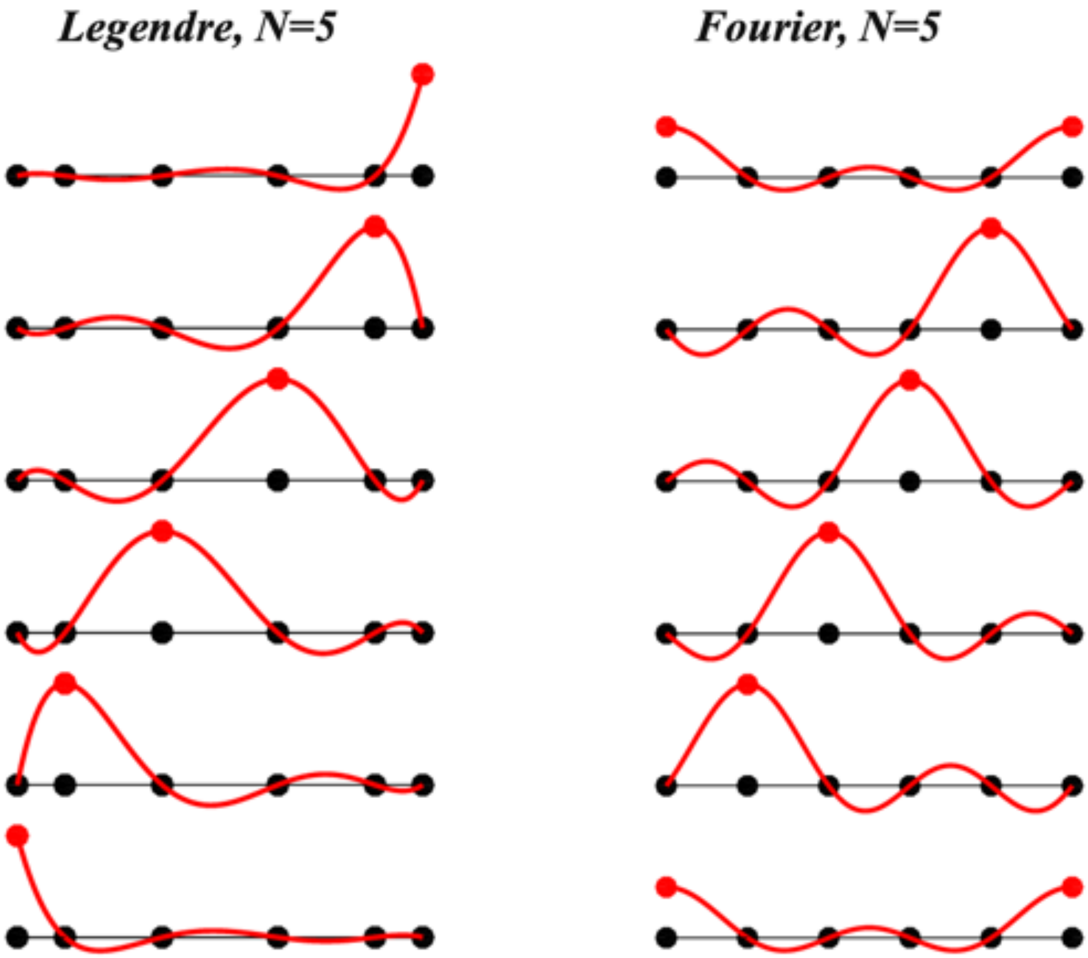
- Let $\tilde{J}$ be the matrix with entries

$$\tilde{J}_{jk} \;=\; \phi_k(x_j^M), \tag{8}$$

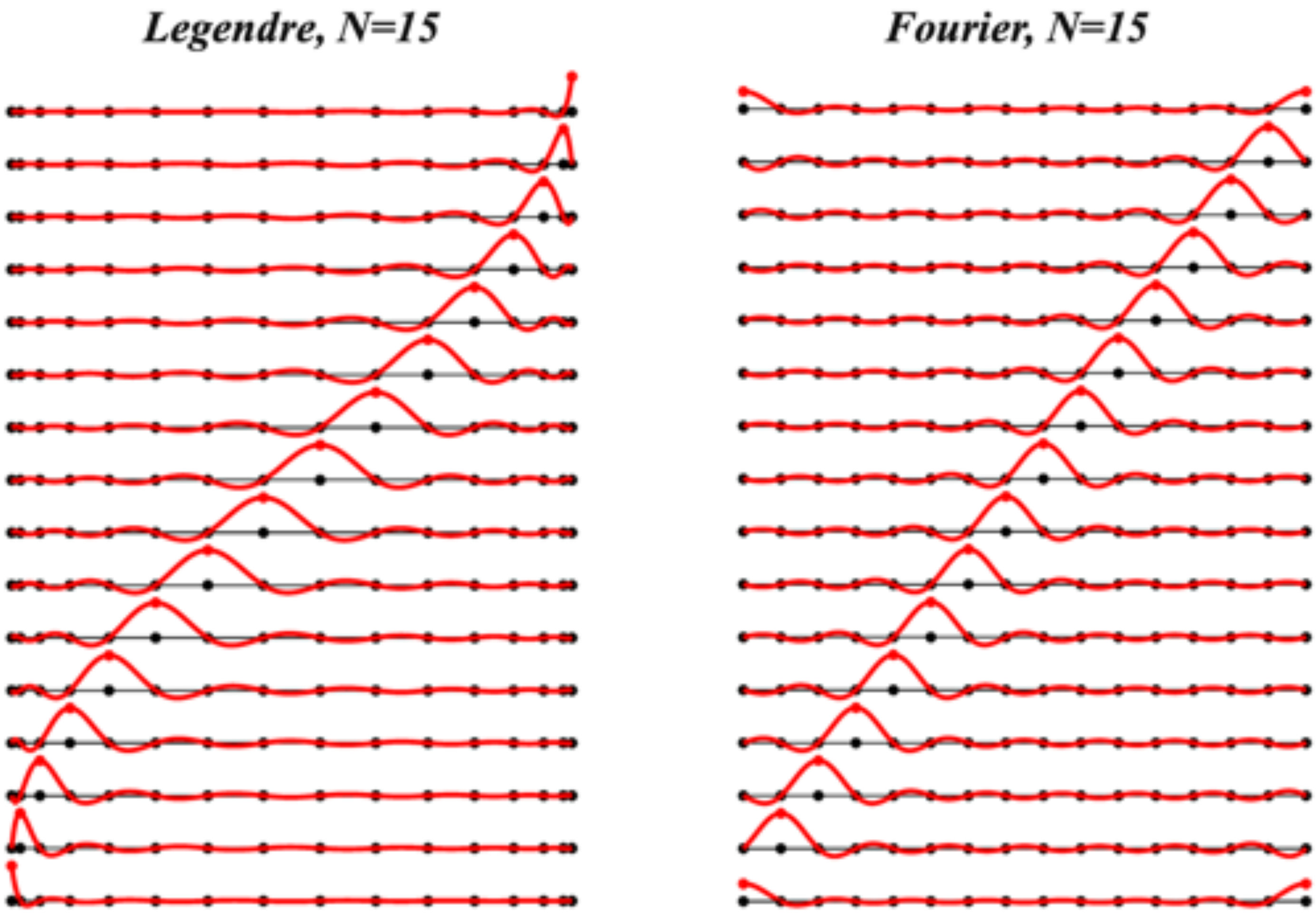
where the x_j^M are the desired output points on a fine mesh.
- Then we can define our *standard interpolation matrix* (nodal-to-nodal) as

$$J^M \;=\; \tilde{J}\,J^{-1}. \tag{9}$$

- J^{-1} takes us from N nodes to N modal coefficients, where we evaluate on the fine (M) mesh, and $\tilde{J}$ maps these functions back to physical space (nodal) values.
- The following figures show the nodal Fourier bases for $N = 5$ and 15.



Legendre, N=5
Fourier, N=5



- Here, we have included x_0 and multiplied the endpoint basis functions by $1/2$ so that we end up with the average of f_0 and f_N in order to treat interpolation of a nonperiodic function $f(x)$.