Plane Poiseuille Flow and the Orr-Sommerfeld Problem

- Here, we consider viscous 2D flow between a pair of flat plates separated by a distance $2H$.
- Let the domain be $\Omega = [0 : L] \times [-H, H]$, with $H = 1$.
- On the top and bottom, we have $\mathbf{u} = \mathbf{0}$.
- On the left and right we use periodic boundary conditions and force the flow with a body force, $\mathbf{f} = [f_x, 0]$, where f_x corresponds to the mean pressure drop required to sustain the flow at a steady-state condition.
- The steady-state base flow is $\mathbf{u} = (U_0, 0) := \mathbf{U}$, with

$$U_0(y) \;\; = \;\; 1 - y^2, \tag{1}$$

which is illustrated by the parabolic velocity profile in the figure below.

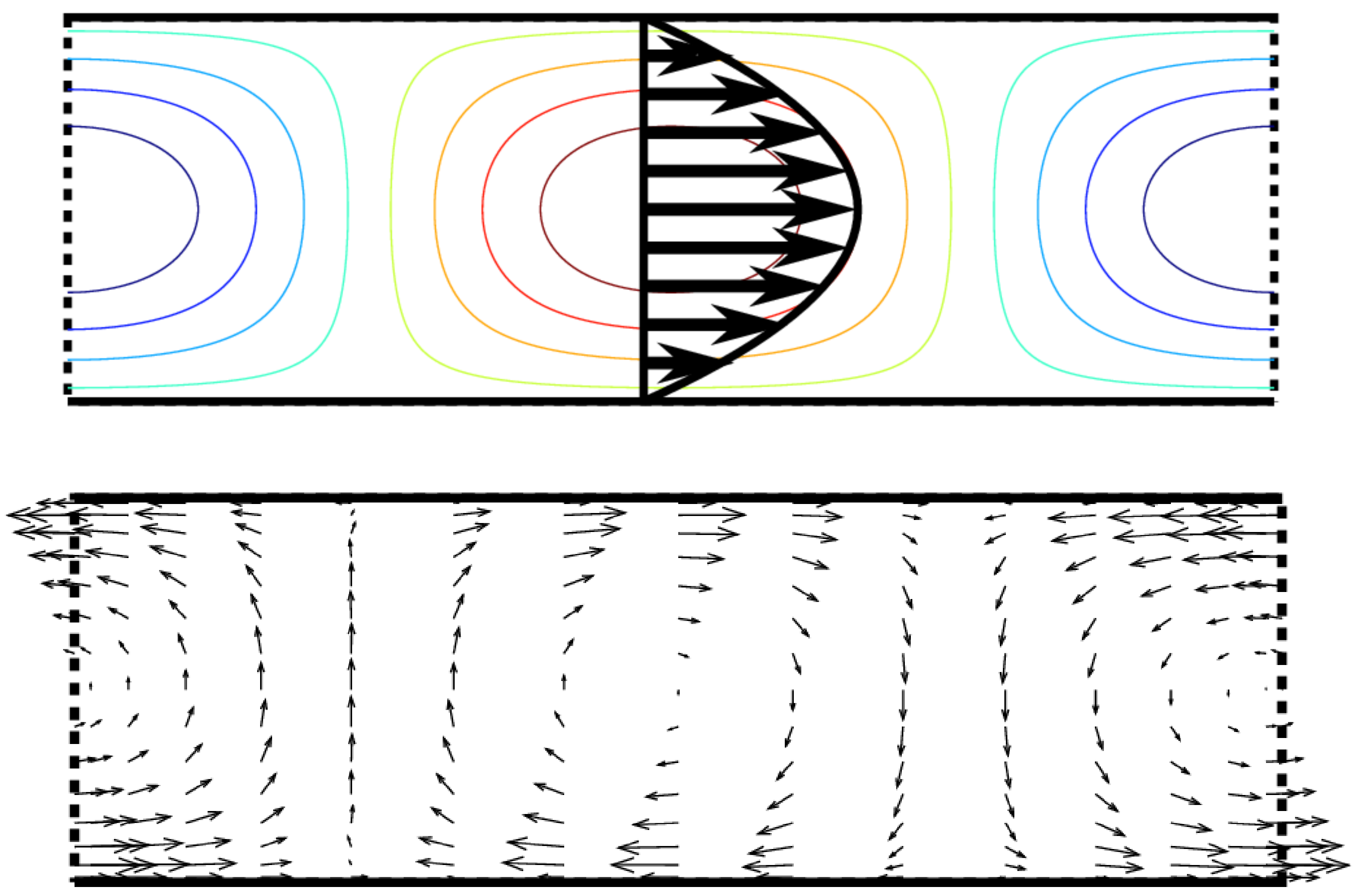


Figure 1: Top: parabolic base flow, with streamfunction contour lines indicating the addition of small amplitude TS waves. Bottom: perturbation velocity field for TS waves.

Periodic Boundary Conditions

- The Navier-Stokes equations with boundary and initial conditions read,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} \;=\; -\nabla\tilde{p} + \frac{1}{Re}\nabla^2\mathbf{u} + \mathbf{f}, \tag{2}$$

$$\nabla\cdot\mathbf{u} \;=\; 0, \tag{3}$$

$$\mathbf{u}(x,\pm H,t) = 0, \quad \mathbf{u}(0,y,t) = \mathbf{u}(L,y,t), \quad \mathbf{u}(x,y,0) = \mathbf{u}^0(x,y). \tag{4}$$

- Here, $\tilde{p}$ is the periodic *perturbation* pressure.
- Let $p(x,y,t)$ be the actual pressure (normalized by density in this nondimensional formulation).
- Unlike velocity, p is *not* L-periodic. There is a mean pressure drop, $< \frac{\partial p}{\partial x} >$, that is a constant.
- Consequently, we break the pressure into two parts,

$$p(x,y,t) \;=\; \tilde{p}(x,y,t) + < \frac{\partial p}{\partial x} > x, \tag{5}$$

where $\tilde{p}$ is the unknown pressure field which is L-periodic,

$$p(0,y,t) \;=\; p(L,y,t). \tag{6}$$

- Notice that

$$-\nabla p \;=\; -\nabla\tilde{p} - < \frac{\partial p}{\partial x} > \hat{\imath} \tag{7}$$

$$= \;-\nabla\tilde{p} + f_x\,\hat{\imath} \tag{8}$$

- The required body force to move the viscous flow through the channel is thus

$$f_x \;=\; - < \frac{\partial p}{\partial x} > . \tag{9}$$

- It is straightforward to show that

$$f_x \;=\; \frac{2}{Re} \tag{10}$$

will maintain the parabolic profile with centerline velocity $U_0(y=0) = 1.$.

- **Ex 1:** Check this assertion both numerically and analytically.
- **Ex 2:** If you start with $\mathbf{u}^0 = 0$ and the correct body force, roughly how long will it take for the peak velocity to reach 0.99?

The Orr-Sommerfeld Problem

- Although plane Poiseuille flow solves the incompressible Navier-Stokes equations, it is not the only possible solution.
- At sufficiently high Reynolds numbers, the flow can of course be turbulent.
- There is also a family of solutions known as Tollmein-Schlicting (TS) waves that are conditionally stable/unstable, depending on the domain length (L/H) and half-height Reynolds number, $Re = UH/\nu$, where U is (in this case) the centerline velocity of the base parabolic profile.
- The TS waves are solutions of the Orr-Sommerfeld (OS) problem, which is a semi-analytical approach to probe the stability of plane-Poiseuille and other boundary-layer flows.
- Let $\alpha = \frac{1}{2\pi}\frac{L}{H}$ be a nondimensional measure of the domain length.
- For $\alpha = 1.02056$ (corresponding to a domain that is $\approx 2\%$ longer than $2\pi H$), it is found that the TS waves will grow for any $Re > Re_c$, where $Re_c \approx 5772.6$ is the lowest realizable critical Reynolds number.
- If $\alpha \neq 1.02056$, the critical Reynolds number is larger, which means that the system is more stable.

- The figure below (from Wikipedia) shows the real component of the eigenvalue for the OS problem that has the *least negative* real part.
- Any point in this (α, Re) parameter space for which $\lambda > 0$ indicates that the TS waves will grow in time.
- To better understand this, we take a quick look at the linearized Navier-Stokes equations.

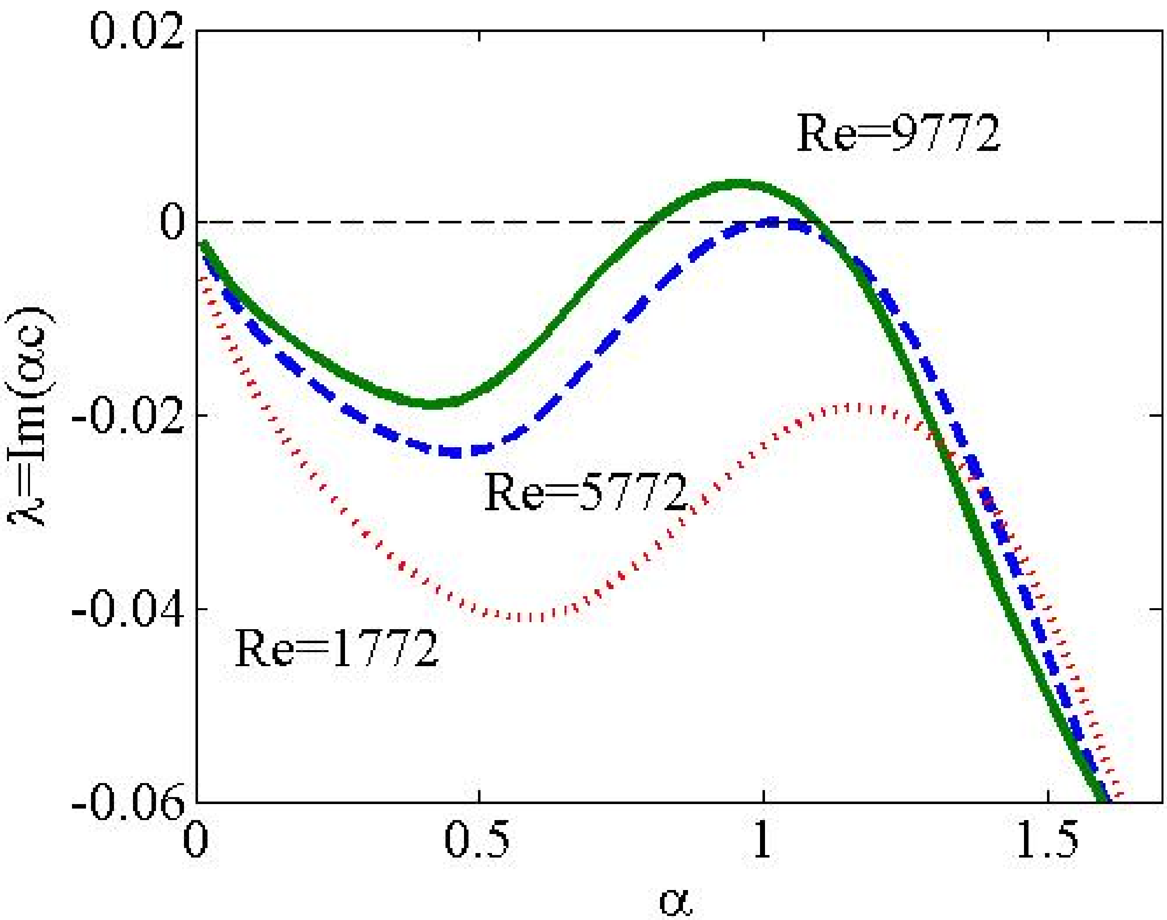


Figure 2: OS eigenvalues as a function of α and Re.

Linearized Navier-Stokes

- Consider solutions to the Navier-Stokes equations of the form

$$\mathbf{u}(x,y,t) \;=\; \mathbf{U} \,+\, \epsilon\,\mathbf{u}', \tag{11}$$

$$p(x,y,t) \;=\; P \,+\, \epsilon\,p', \tag{12}$$

  where $\mathbf{U} = [U_0(y),\ 0]$ is the base flow, $\mathbf{u}'(\mathbf{x},t)$ is an order-unity perturbation velocity, and $\epsilon \ll 1$ is the perturbation amplitude.

- Here, $P = 0$ is the base pressure for the steady state Poiseuille flow and $\epsilon p'$ is the perturbation pressure.

- Since $\mathbf{u}$ and $\mathbf{U}$ are both solutions to the NS equations, we can derive an equation for the evolution of the perturbation field, $\epsilon\mathbf{u}' = \mathbf{u} - \mathbf{U}$,

$$\begin{array}{rcl}
\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\nabla^2\mathbf{u} + \nabla p &=& -\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{f}, \qquad \nabla\cdot\mathbf{u} \;=\; 0,\\
-\;\; \frac{\partial \mathbf{U}}{\partial t} - \frac{1}{Re}\nabla^2\mathbf{U} + \nabla P &=& -\mathbf{U}\cdot\nabla\mathbf{U} + \mathbf{f}, \qquad \nabla\cdot\mathbf{U} \;=\; 0,\\
\hline
\epsilon\frac{\partial \mathbf{u}'}{\partial t} - \frac{\epsilon}{Re}\nabla^2\mathbf{u}' + \epsilon\nabla p' &=& -NL, \qquad\qquad\quad \epsilon\nabla\cdot\mathbf{u}' \;=\; 0.
\end{array}$$

- Here, the nonlinear term is

$$NL \;:=\; (\mathbf{U}+\epsilon\mathbf{u}')\cdot\nabla(\mathbf{U}+\epsilon\mathbf{u}') \;-\; \mathbf{U}\cdot\nabla\mathbf{U} \tag{13}$$

$$\;=\; \epsilon\left(\mathbf{U}\cdot\nabla\mathbf{u}' \,+\, \mathbf{u}'\cdot\nabla\mathbf{U}\right) \,+\, \epsilon^2\,\mathbf{u}'\cdot\nabla\mathbf{u}'. \tag{14}$$

- The idea behind the *linearized* NS equations is to drop terms governing the evolution of $\mathbf{u}'$ that are $O(\epsilon^2)$ in size and to retain only those that are $O(\epsilon)$.

  (Think $\epsilon \approx 10^{-5}$ or smaller...)

- Doing so leads to the *linear PDE* for the perturbation field

$$\frac{\partial \mathbf{u}'}{\partial t} - \frac{1}{Re}\nabla^2 \mathbf{u}' + \nabla p' \;=\; -\big(\mathbf{U}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\mathbf{U}\big), \qquad \nabla\cdot\mathbf{u}' \;=\; 0, \tag{15}$$

  where ϵ has been factored out of this homogeneous system.

- The Orr-Sommerfeld analysis proceeds by seeking solutions of the form

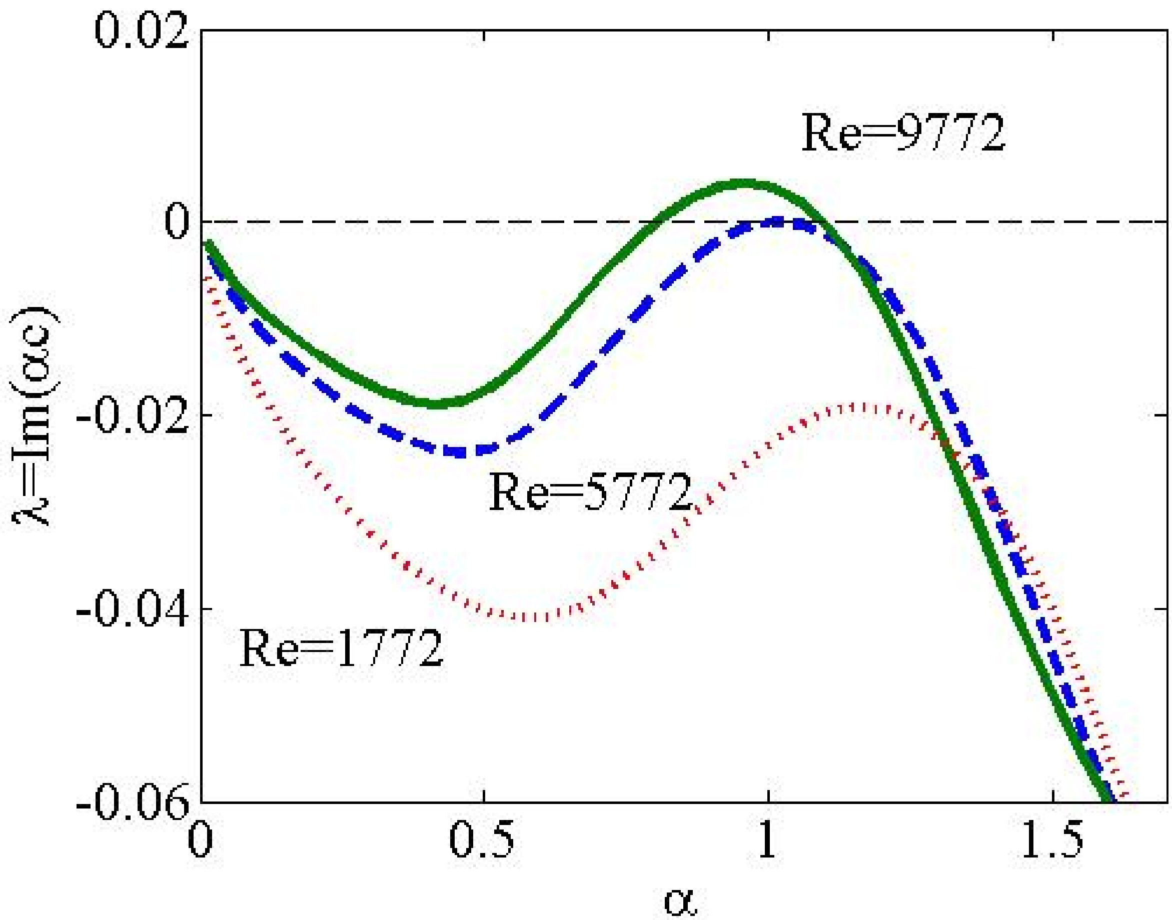
$$\mathbf{u}'(x,y,t) \;=\; \hat{\mathbf{u}}(y)\, e^{i(\alpha x - \omega t)} \tag{16}$$

$$p'(x,y,t) \;=\; \hat{p}(y)\, e^{i(\alpha x - \omega t)}. \tag{17}$$

- Notice that, unlike the case of the Taylor-Green vortices, where the wavenumber for the pressure is twice that of the velocity, the wavenumber for the pressure in the case of this *linear* PDE is exactly the same as that of the velocity.

  This result is also conditioned on U being constant in the x direction.

- In (16)–(17), α is governed by our periodic domain length and $i\,\omega$ is an unknown complex eigenvalue associated with (15).



- The real part of $i\,\omega$ is plotted above.
- If the real part is positive, the solution will grow for *any* nontrivial perturbation amplitude, ϵ.
- Numerically, this means that even round-off error is sufficient to trigger this exponential growth.
- We refer to such a system as being *linearly unstable*, because a small-amplitude perturbation (linear in ϵ) is sufficient to trigger the instability.
- By contrast, something that is linearly stable, requires a large-amplitude perturbation, which effectively modifies the base profile, to trigger an instability.

Finding Solutions to the OS Problem

- The semi-analytical approach to "solving" the OS problem is to plug the expansions (16)–(17) into the linearized system and to then formally eliminate p' from the equations, which leads to a 4th-order PDE governing $\hat{\mathbf{u}}(y)$.
- From there, one can solve the associated eigenproblem for each point in the (α, Re) parameter space, which is how Fig. 2 was obtained.
- The eigenproblem is quite ill-conditioned because the governing system is *non-normal*—the eigenfunctions are nearly parallel.
- We will take a different approach to investigating this problem.
- With our highly accurate NS solver, however, we can perturb the base-flow solution by adding a mode that "looks like" the TS-wave perturbation.

- We consider ICs of the form

$$u^0(x,y) \;=\; U \,+\, \epsilon \cos(2\pi x/L)\,\sin(\pi y/H) \tag{18}$$

$$v^0(x,y) \;=\; V \,-\, \epsilon \sin(2\pi x/L)\,\cos(\pi y/H), \tag{19}$$

  with $U = 1 - y^2$, $V = 0$, and $\epsilon \approx 10^{-5}$.

- We then solve the fully-nonlinear Navier-Stokes equations, recognizing that these differ from the OS equations by a term that is $O(\epsilon)$.

- For unstable cases, we should see growth (or decay) in the perturbation energy,

$$E(t) \;:=\; \int_\Omega (\mathbf{u}-\mathbf{U})\cdot(\mathbf{u}-\mathbf{U})\,dV, \tag{20}$$

  that has the following form

$$E(t) \;=\; E(0)\,e^{2\,\omega_i t}. \tag{21}$$

- Here, ω_i is the imaginary part of $\omega \,:=\, \omega_r \,+\, i\,\omega_i$.

- The “2” in the exponential comes from energy being the square of the velocity.

- An additional diagnostic is simply to check, under the given (α, Re) parameters, whether the initial perturbation grows or decays as $t \longrightarrow \infty$. If $\alpha = 1.02056$ and $Re > Re_c \approx 5772.22$ (a result found by Orszag using spectral methods in 1971), then we should have growth for sufficiently large t.

- For shorter times, one may observe decay because our IC contains multiple eigenmodes, all of which will decay save for the critical one.

- If $\alpha = 1$ (i.e., $L = 2\pi H$), the critical Reynolds number is larger than 5772.22.

- There are known eigenvalues (easily computed with an OS solver) for many Reynolds numbers.

- From the Nek5000 `os7000` example, we have the following for $\alpha = 1$.

	ω_i (growth)	ω_r (frequency)
Re=7000	0.001715391802	0.2529293654
Re=7500	0.002234975649	0.2498915356

- Notice that the growth rate is lower for $Re = 7000$ than for 7500, as might be expected.

Summary

- Here we have considered plane Poiseuille flow in a periodic domain, which raises two questions:
  - What is f_x if $U = 1 - y^2$?
  - How long does it take to reach the steady state?

- We have also considered the Orr-Sommerfeld question, which is a challenging problem in linear stability theory.
- This problem raises several questions, including
  - For what (α, Re) parameters do we see instabilities?
  - What is the growth/decay rate for a given unstable/stable configuration?

- We have shown how we might answer these questions by running some NS solutions in a 2D rectangular domain with homogeneous Dirichlet conditions.