Eigenvalue Examples

- Eigenvalues are of central importance in understanding the behavior of time-dependent problems, both from a physics standpoint and from numerical convergence and stability concerns.

- Consider the time-dependent PDE,

$$\frac{\partial \tilde{u}}{\partial t} \;=\; -\mathcal{L}\tilde{u}, \qquad \tilde{u}(\mathbf{x}, t=0) = u^0(\mathbf{x}), \tag{1}$$

  where associated boundary conditions are embedded in the linear operator, $\mathcal{L}$.

- Suppose there exists a set of eigenfunctions $\tilde{s}_k(\mathbf{x})$ and associated eigenvalues such that

$$\mathcal{L}\,\tilde{s}_k \;=\; \tilde{\lambda}\,\tilde{s}_k, \tag{2}$$

  and that, for any given $\tilde{u}$ having sufficient regularity that we can find coefficients $\hat{u}_k$ such that

$$\tilde{u}(\mathbf{x}, t) \;=\; \sum_k \hat{u}_k(t)\,\tilde{s}_k(\mathbf{x}). \tag{3}$$

- Then,

$$\frac{\partial \tilde{u}}{\partial t} \;=\; \sum_k \frac{d\hat{u}_k}{dt}\,\tilde{s}_k(\mathbf{x}) \;=\; -\sum_k \hat{u}_k(t)\,\mathcal{L}\,\tilde{s}_k(\mathbf{x}) \;=\; -\sum_k \tilde{\lambda}_k \hat{u}_k(t)\,\tilde{s}_k(\mathbf{x}). \tag{4}$$

- Since we have diagonalized $\mathcal{L}$, we have a set of decoupled ODEs in time,

$$\frac{d\hat{u}_k}{dt} \;=\; -\tilde{\lambda}_k \hat{u}_k(t), \qquad \hat{u}_k(0) =: \hat{u}_k^0, \tag{5}$$

  for which the solution is

$$\hat{u}_k(t) \;=\; \hat{u}_k^0\, e^{-\tilde{\lambda}_k t}. \tag{6}$$

Interpretation of Analytical Eigenvalues

- We make several remarks regarding (6).
- First, if the eigenvalues $\tilde{\lambda}_k$ are real and positive, then the solution exhibits exponential decay, with high wavenumber (i.e., large eigenvalue) components decaying much more rapidly than the low wavenumber components.
- For second-order operators in space we can anticipate from dimensional analysis that $\tilde{\lambda}_k \sim Ck^2$, which means that the solution will rapidly evolve to a multiple of the eigenmode with the smallest eigenvalue.
- Specifically, assuming $0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots$, we'll find, after a relatively short time,

$$\tilde{u}(\mathbf{x}, t) \;\; \sim \;\; \hat{u}_1^0 \, e^{-\tilde{\lambda}_1 t} \, \tilde{s}_1(\mathbf{x}) \tag{7}$$

- This physical behavior is characteristic of the unsteady heat (or diffusion) equation.

- Conversely, if the eigenvalues are imaginary, the soution will exhibit *no decay.*
- The modulus of the coefficients,

$$|\,\hat{u}_k(t)\,| = |\,\hat{u}_k^0\,|\,|\,e^{-\tilde{\lambda}_k t}\,|, \tag{8}$$

  will be unchanged for all time.
- This physical behavior is characteristic of the advection equation with minimal or no diffusion, modulo bounary effects.
- The fact that the solution does not decay often implies that errors also do not decay. As they are generated in the course of the numerical solution, they tend to stay in the system and continually accumulate until the solution is essentially garbage.
- For this reason, many practioners apply low pass filtering or some sort of high wavenumber damping (or diffusion) to try to suppress spurious modes.
- As long as the effect of these smoothing operators is high order, there is no particular harm in suppressing noise with these approaches.
- It is possible to develop filters that are quite robust and that require minimal or no parametric tuning.

Numerical Eigenproblem

- Because of their impact on time-dependent problems, we wish to understand how the eigenvalues of our discrete system relate to their continuous counterparts.
- In the following, we consider a domain $\Omega = [0, L]$.
- To make things precise, consider

$$\mathcal{L}\tilde{u} \;=\; -\nu\frac{d^2\tilde{u}}{dx^2} \;=\; \tilde{\lambda}\,\tilde{u}, \quad \tilde{u}(0) = \tilde{u}(L) = 1. \tag{9}$$

- The eigenfunctions in this case are

$$\tilde{s}_k \;=\; \sin\, k\pi x/L, \tag{10}$$

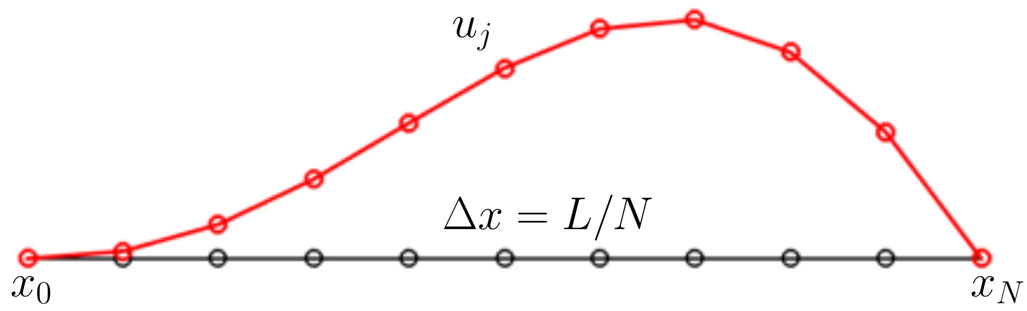
  with corresponding eigenvalues,

$$\tilde{\lambda}_k \;=\; \nu\frac{k^2\pi^2}{L^2}, \quad k = 1,\, 2\,\ldots \tag{11}$$

- **Q:** How much larger/smaller is $\tilde{\lambda}_k$ if we double the domain size?
- **Q:** Suppose it takes 1 second for a parabolic initial condition in a 1D thermal problem to decay to 10% of its original amplitude for a domain of size L. What would be the relative amplitude after 1 second for a parabolic initial condition on a domain comprising the same material but of size $L/2$? (Hint—it's not 0.1/2 and it's not 0.1/4.)

- In the following pages, we'll look at the eigenvalues and eigenvectors for finite differences, Fourier methods, and Legendre nodal-spectral methods.

Finite Differences



- We begin with finite differences.
- Here, we have $\underline{u} = [u_1 \ \ldots \ u_{N-1}]^T$, satisfying the system of ordinary differential equations (ODEs),

$$\frac{d\underline{u}}{dt} \; = \; -L\underline{u} \; = \; -\frac{\nu}{\Delta x^2} \begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & 1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \underline{u} \; =: -\, A\underline{u}. \tag{12}$$

Note that we introduce ν into A for dimensional consistency.

- For uniform grid spacing, $\Delta x := L/N$, the eigenvectors of the second-order centered difference scheme given by (12) are the discrete equivalents to their analytical counterparts, (10).

- That is,

$$[\underline{s}_k]_j \;=\; \sin k\pi x_j/L \;\text{ – } k\text{th eigenvector.} \tag{13}$$

- We find the corresponding eigenvalue by setting $\underline{u} = \underline{s}_k$ in (12).

- For the jth equation,

$$\begin{aligned} -L\underline{s}_k\big|_j &= -\frac{\nu}{\Delta x^2}\left[\sin(\pi k x_{j-1}/L) \;-\; 2\,\sin(\pi k x_j/L) \;+\; \sin(\pi k x_{j+1}/L)\right] & (14)\\ &= \frac{2\nu}{\Delta x^2}\left[1 \;-\; \cos(\pi k \Delta x/L)\right]\,\sin(\pi k x_j/L), & (15)\end{aligned}$$

  where we have used the identity $\sin(a+b) \;-\; \sin(a-b) \;=\; 2\sin a\,\cos b$ to make the substitution,

$$\sin(\pi k x_{j-1}/L) \;+\; \sin(\pi k x_{j-1}/L) \;=\; \sin(\pi k x_j/L)\;\; cos(\pi k \Delta x/L). \tag{16}$$

- The eigenvalues for the discrete Poisson operator based on 2nd-order center differences are thus,

$$\boxed{\lambda_k \;=\; \frac{2\nu}{\Delta x^2}\left[1 \;-\; \cos(\pi k \Delta x/L)\right]}. \tag{17}$$

- Note that, with $\theta := \pi k \Delta x / L$, we have $\cos\theta = (1 - \theta^2/2 + \theta^4/4! - h.o.t.)$, so,

$$\lambda_k = \frac{2\nu}{\Delta x^2}\frac{\theta^2}{2}\left[1 - \frac{\theta^2}{12} + h.o.t.\right] \tag{18}$$

$$= \nu\left(\frac{\pi k}{L}\right)^2\left[1 - \frac{\theta^2}{12} + h.o.t.\right]. \tag{19}$$

$$= \tilde{\lambda}_k\left[1 - \frac{\theta^2}{12} + h.o.t.\right]. \tag{20}$$

- In particular, for $k \ll N$,

$$\tilde{\lambda}_k - \lambda_k \sim \nu(\pi k/N)^2/12 = O(k\Delta x)^2, \tag{21}$$

and for $k = N - 1$,

$$\tilde{\lambda}_{N-1} = \nu\frac{\pi^2(N-1)^2}{L^2} \tag{22}$$

$$\lambda_{N-1} \sim \nu\frac{4N^2}{L^2}. \tag{23}$$

- Asymptotically, as $k \longrightarrow N$,

$$\frac{\tilde{\lambda}_k}{\lambda_k} \sim \frac{\pi^2}{4}. \tag{24}$$

- This *spectral equivalence* result provided major insight into the development of fast sparse preconditioners for spectral methods, as introduced in Orszag's seminal 1980 JCP paper on spectral methods for complex geometries.

Fourier Spectral Methods

- Here, for the time-dependent PDE we have the expansion,

$$u(x,t) \;=\; \sum_{k=1}^{N-1} \hat{u}_k(t)\, \underbrace{\sin(k\pi x/L)}_{\phi_k(x)}, \tag{25}$$

  and

$$\frac{\partial u}{\partial t} \;=\; -\mathcal{L}u(x,t) \;=\; -\nu \sum_k \frac{k^2\pi^2}{L^2}\phi_k(x)\hat{u}_k(t). \tag{26}$$

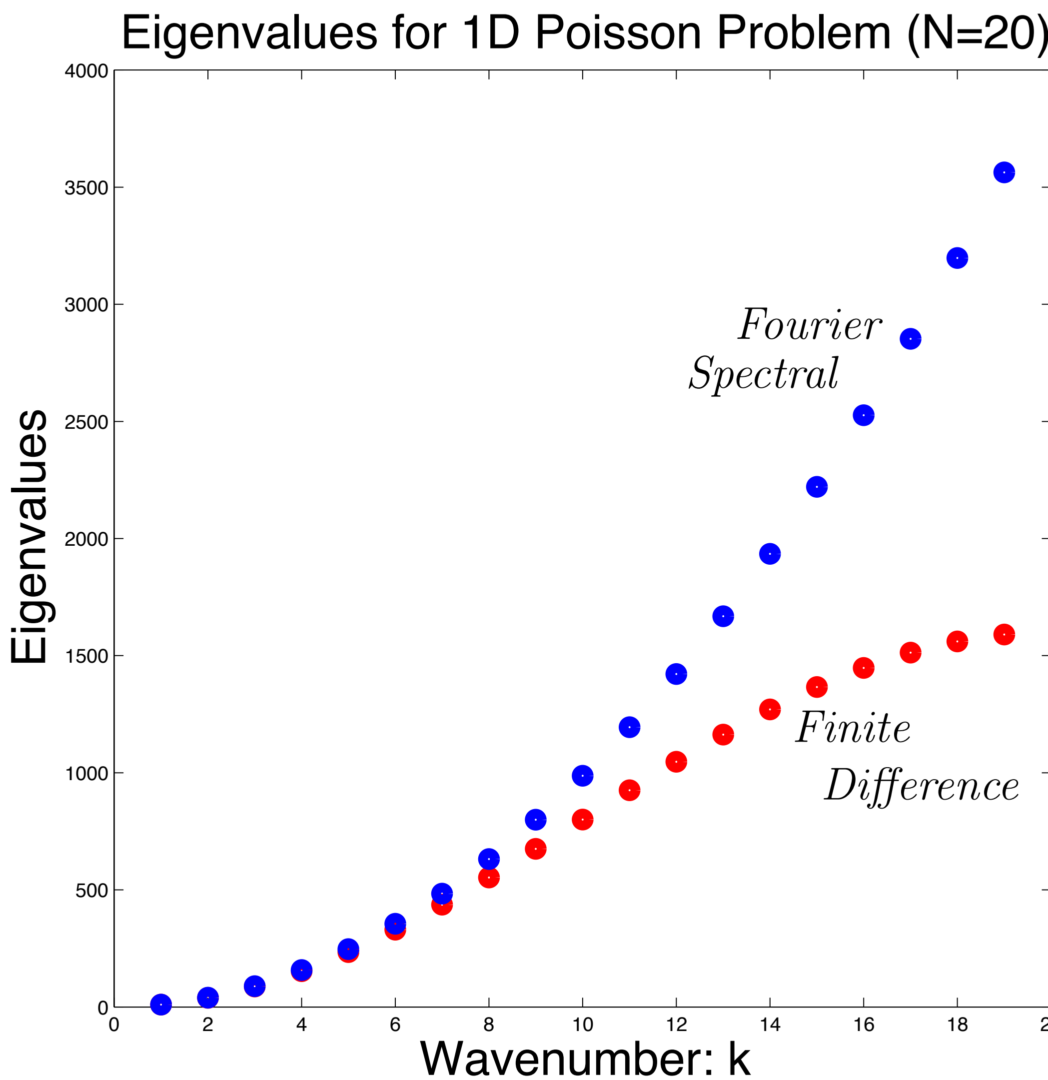
- The ODE governing the evolution of the basis coefficients is thus,

$$\frac{d\underline{\hat{u}}}{dt} \;=\; -\frac{\nu\pi^2}{L^2}\begin{bmatrix} 1 & & \\ & \ddots & \\ & & (N-1)^2 \end{bmatrix}\begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{pmatrix}. \tag{27}$$

- L is diagonal in this case, which implies,

$$\lambda_k \;=\; L_{kk} \;=\; \tilde{\lambda}_k, \quad k = 1, \ldots, N-1. \tag{28}$$

- The Fourier spectral method eigenvalues are *exact*, up to $k = N-1$.
- The eigenvalues for $\nu = L = 1$ and $N = 20$ are shown below.
- **Q:** *How do the maximum eigenvalues scale with N?*



Eigenvalues for 1D Poisson Problem (N=20)
Eigenvalues
Wavenumber: k
Fourier Spectral
Finite Difference
4000
3500
3000
2500
2000
1500
1000
500
0
0
2
4
6
8
10
12
14
16
18
20

Legendre Nodal Spectral Methods

- Here, as discussed in the previous lecture, we expand $u(x)$ in terms of our Lagrange cardinal polynomials based on the (translated) GLL points.

- Let $\tilde{u}(x) \approx u(x) \in X_0^N$ be given by the expansion

$$u(x) \;=\; \sum_{j=0}^{N} u_j l_j(x), \tag{29}$$

  with $u_0 = u_N = 0$.

- We have $N-1$ uknowns and need $N-1$ equations, which we can obtain through our standard variational projection technique.

- We seek $u \in X_0^N$ such that for all $v \in X_0^N$,

$$a(v,u) \;=\; \lambda\,(v,u). \tag{30}$$

- The discrete form is thus

$$A\underline{u} \;=\; \lambda B\underline{u}, \tag{31}$$

  which is a *generalized eigenvalue problem.*

- In matlab, the solution is obtained as

```
A = (2/L)*R*Ah*R'; B = (L/2)*R*Bh*R';
[S,D] = eig(A,B); lam=diag(D);
[lam,ind] = sort(lam);   %% Sort eigenvalues
S=S(:,ind);              %% Sort eigenvectors
```

- Here, $S = [\underline{s}_1 \; \underline{s}_2 \; \ldots \; \underline{s}_{N-1}]$ is the matrix of eigenvectors, sorted such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$.

- Because the low-wavenumber eigenvectors are *smooth*, we can anticipate that they will be well represented in X_0^N for $k < \frac{2}{\pi}N$.
- We would thus also expect that $\tilde{\lambda}_k \sim \lambda_k$ for $k < \frac{2}{\pi}N$.
- In particular, with

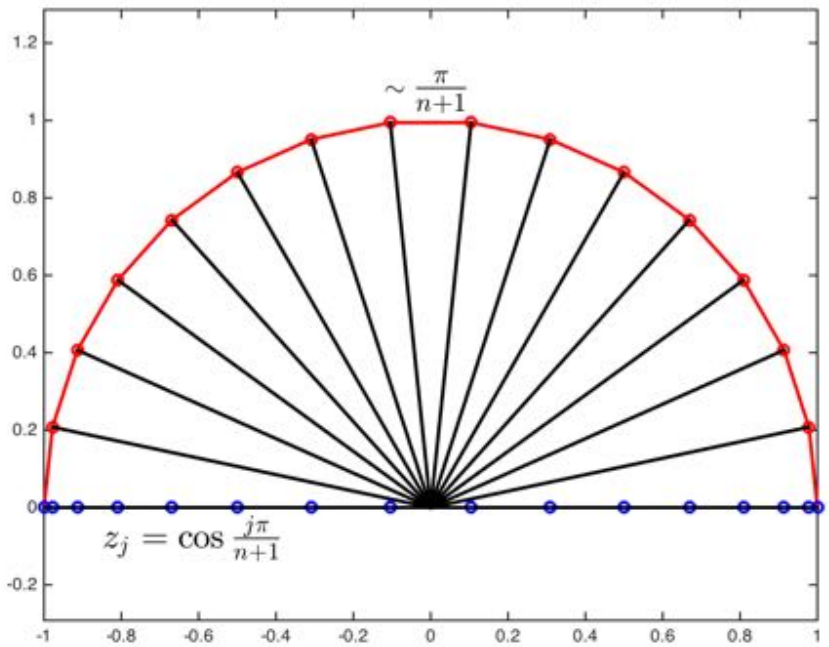
$$s_k(x) \;:=\; \sum_{j=1}^{N-1} l_j(x)[\underline{s}_k]_j, \tag{32}$$

we have the following relationships for all $v \in X_0^N$ and $k < \frac{2}{\pi}N$,

$$\lambda_k\,(v, s_k) \;\equiv\; a(v, s_k) \;\approx\; a(v, \tilde{s}_k) \;\equiv\; \tilde{\lambda}_k(v, \tilde{s}_k), \;\approx\; \tilde{\lambda}_k(v, s_k), \tag{33}$$

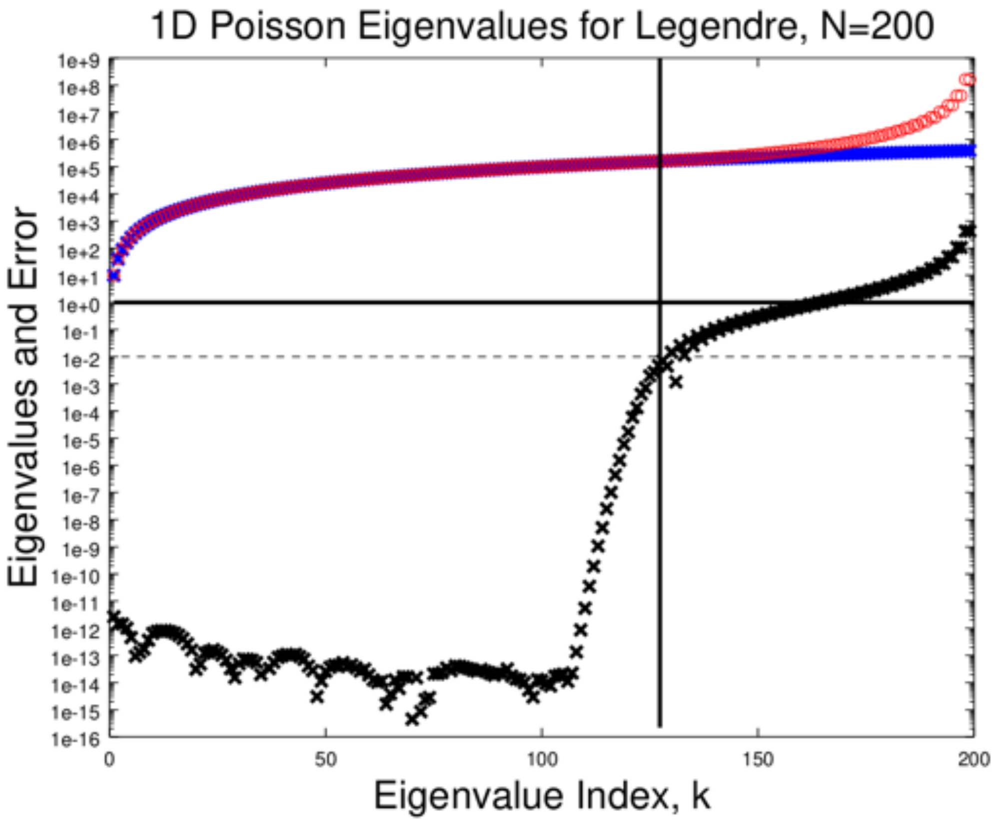
$$\Longrightarrow\; \lambda_k \;\approx\; \tilde{\lambda}_k. \tag{34}$$

- To see where the $\frac{2}{\pi}$ ratio comes in, recall that the spacing of the Legendre nodes is similar to that of Chebyshev.
- In particular, for $\Omega = [-1, 1]$, the widest gap is $\sim \frac{\pi}{N}$, which corresponds to an equipartion of the unit circle, as illustrated below for the case of $n = N - 1$ points.



- Note that *uniformly* spaced points on this interval have a maximum gap of only $2/N$, which is a factor of $2/\pi$ smaller than that of the maximum Chebyshev/Legendre node spacing.
- Thus, while Fourier methods can resolve up to $k_{\max} \sim N/2$ modes, the polynomial-based methods are able to resolve only to $k_{\max} \sim N/\pi$.
- Note that the above argument is a heuristic, but the conclusions in fact carry over to *projection-based* approximations where one expands in terms of Legendre polynomials, rather than Lagrange cardinal functions on the GLL nodes.

- A demonstration of the eigenvalue behavior for the Legendre spectral method is shown in the accompanying figure.



- One can see that, for $N = 200$, as used here, there is a rapid drop in the error (indicated by the black symbols) for $k < \frac{2}{\pi}N$, which is delineated by the vertical line on the plot.

- For those values of k, the relative error is less than 0.01, a typical engineering tolerance, and it decreases exponentially fast with $N - k$ as $k \longrightarrow 1$.

- The blue symbols indicate the analytical eigenvalues, while the red ones show the numerical behavior.

- The significant growth arises because the largest eigenvalues scale as $\Delta x_{\min}^{-2}$, and $\Delta x_{\min}\ CN^{-2}$, which implies,

$$\lambda_{\max}(B^{-1}A) \ \sim \ CN^4, \tag{35}$$

  where C is a constant that is independent of N.

Methods/Eigenvalues for Advection

- Here, we are concerned with the 1D *periodic problem*,

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= -c\frac{\partial \tilde{u}}{\partial x}, \quad \tilde{u}(0) \;=\; \tilde{u}(L) & (36)\\ &= -\mathcal{L}\tilde{u}, & (37) \end{aligned}$$

  with c a constant.

- If $c > 0$, this represents advection of a signal to the left at speed c.

- Because of periodicity, this signal will wrap and repeat ad infinitum.
  - *demo 1:* `single_fd4.m`
  - *demo 2:* `single_sem_demo.m`

- Here, periodicity simply serves as a mechanism to investigate *long time-integration*, without having a huge domain.

- It also admits straightforward analysis (known as von Neumann analysis) of our PDE.

- That analysis comes down to understanding the eigenvalues of our system.

Analytical Eigenvalues for Advection

- We start with

$$\mathcal{L}\tilde{u} \;=\; c\frac{\partial \tilde{u}}{\partial x} \;=\; \tilde{\lambda}\tilde{u}, \quad \tilde{u}(0) = \tilde{u}(L). \tag{38}$$

- For the constant-coefficient periodic case, the eigenfunctions are the Fourier modes,

$$\boxed{\tilde{s}_k(x) \;=\; e^{2\pi i\, kx/L},} \tag{39}$$

for which

$$\mathcal{L}\tilde{s}_k \;=\; 2\pi i\frac{ck}{L}e^{2\pi i\, kx/L}. \tag{40}$$

- The analytical eigenvalues are thus

$$\boxed{\tilde{\lambda}_k \;=\; 2\pi i\frac{ck}{L}}, \tag{41}$$

for integer $k \in [-\infty, \infty]$.

Finite-Difference Eigenvalues for Advection

- Here we have the advection matrix, C,

$$L\underline{u} \;=\; C\underline{u}, \tag{42}$$

- For uniformly-spaced grid with $\Delta x = L/N$, the standard central differerence operator is the circulant matrix,

$$C \;=\; \frac{c}{2\Delta x}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & -1 & 0 \end{bmatrix}.$$

- For any circulant matrix (i.e., a matrix that is *constant* on the "wrapped" diagonals), the eigenvectors are

$$\left(\underline{s}_k\right)_j \;=\; e^{i2\pi k x_j/L} \;=\; e^{i2\pi kj/N}, \quad k = -\frac{N}{2},\ldots,\frac{N}{2}-1. \tag{43}$$

Letting $\underline{u} = \underline{s}_k$ and inserting this expression into $L\underline{u} := C\underline{u} = \lambda\underline{u}$, we have

$$\begin{aligned} C\underline{s}_k\big|_j \;&=\; \frac{c}{2\Delta x}\left[e^{i2\pi k(j+1)/N} - e^{-i2\pi k(j+1)/N}\right] \\ &=\; \frac{c}{2\Delta x}\left[e^{i2\pi k/N} - e^{-i2\pi k/N}\right] e^{i2\pi kj)/N} \\ &=\; \underbrace{\frac{ic}{\Delta x}\sin\frac{2\pi k}{N}}_{\lambda_k}\,\underline{s}_k. \end{aligned} \tag{44}$$

These eigenvalues are purely imaginary and have maximum modulus of $\frac{|c|}{\Delta x}$.

- Let $k_{\max} := N/2$ denote the upper bound on the wavenumber and

$$\theta \;=\; \theta_k \;:=\; 2\pi\frac{k\Delta x}{L} \;=\; 2\pi\frac{k}{N} \;=\; \pi\frac{k}{k_{\max}}. \tag{45}$$

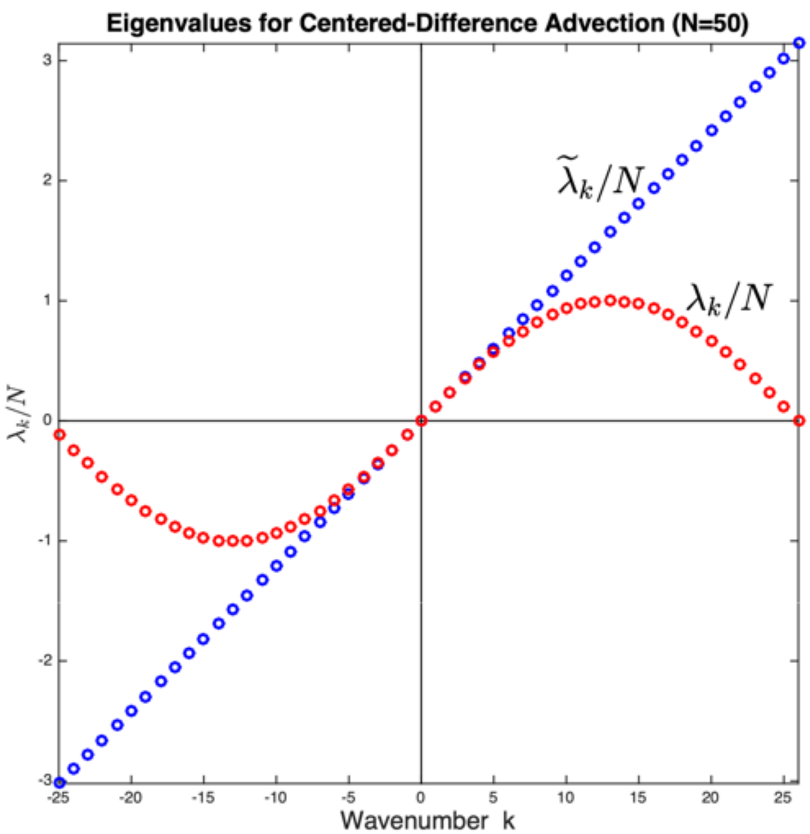
- Then,

$$\lambda_k \;=\; \frac{ic}{\Delta x}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \tag{46}$$

$$=\; 2\pi i\frac{ck}{L}\left(1 - \frac{1}{6}\theta^2 + h.o.t.\right) \tag{47}$$

$$=\; \tilde{\lambda}_k\left(1 - \frac{1}{6}\theta^2 + h.o.t.\right). \tag{48}$$

- Thus, the error in $\tilde{\lambda}_k$ is $O(\Delta x^2)$ or $O(k\Delta x)^2$ as $k\Delta x \longrightarrow 0$.
- The figure below shows the eigenvalue distributions for $c = L = 1$, $N = 50$, which implies that $k_{\max} = 25$.



- Note that, for $k = k_{\max} = N/2$,

$$\tilde{\lambda}_k/N \;=\; 2\pi\, i\,\frac{k_{\max}}{N} \;=\; \pi\, i, \tag{49}$$

whereas, the maximum modulus of λ_k/N is

$$\max\frac{|\tilde{\lambda}_k|}{N} \;=\; \frac{|c|}{N\Delta x} \;=\; \frac{|c|}{L} \;=\; 1. \tag{50}$$

Fourier Spectral Eigenvalues

- Again, we have the Fourier expansion,

$$u(x,t) \;=\; \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k(t)\, \underbrace{e^{2\pi i\, kx/L}}_{\phi_k}. \tag{51}$$

- As before,

$$\mathcal{L}u \;=\; c\frac{\partial u}{\partial x} \;=\; \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{\lambda}_k\, \phi_k(x)\, \hat{u}_k(t), \tag{52}$$

and the discrete system is

$$L\underline{\hat{u}} \;=\; 2\pi i \frac{c}{L} \begin{bmatrix} -\frac{N}{2} & & & & & & \\ & \ddots & & & & & \\ & & -1 & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & \frac{N}{2}-1 \end{bmatrix} \underline{\hat{u}}, \tag{53}$$

which implies that the eigenvalues are $\lambda_k = \tilde{\lambda}_k$, as indicated in (52).

- Here, we can see the significance of the range

$$k \;\in\; \left[-\frac{N}{2} \;:\; \frac{N}{2} - 1 \right]. \tag{54}$$

All eigenvalues are correct up to $k = k_{\max} := N/2$.

Legendre Spectral Eigenvalues

- As before, we derive the discrete operator L by consider the *unsteady* problem,

$$\frac{\partial \tilde{u}}{\partial t} \;=\; -c\frac{\partial \tilde{u}}{\partial x}, \quad \tilde{u}(0,t) = \tilde{u}(L,t), \quad \text{plus ICs.} \tag{55}$$

- Our variational formulation is, for each time instance t, *Find* $u(x,t) \in X_p^N$ *such that, for all* $v(x) \in X_p^N$,

$$\frac{d}{dt}\,(v,u)_N \;=\; -c(v,u), \tag{56}$$

  which leads to the system

$$B\frac{d\underline{u}}{dt} \;=\; -C\underline{u}, \tag{57}$$

  where we devine the bilinear form $c(v,u)$ momentarily.

- To put this in terms of L, factor B from both sides to yield

$$\frac{d\underline{u}}{dt} \;=\; -B^{-1}C\underline{u} \;=:\; -L\underline{u}. \tag{58}$$

- Thus,

$$L \;:=\; B^{-1}C. \tag{59}$$

- Let's look at the bilinear form.
- Starting with (55), bilinear for in the rhs of (56) is

$$c(v,u) \;:=\; \int_0^L v\,c\,\frac{\partial u}{\partial x}\,dx. \tag{60}$$

- Using our usual expansions for v and u on $\hat{\Omega}$, we have, for all v, $u \in X^N$,

$$c(v,u) \;=\; \underline{\bar{v}}^T \bar{C} \underline{\bar{u}}, \tag{61}$$

  with, $C_{ij} := c(\phi_i, \phi_j)$, this yields

$$\bar{C} \;=\; \bar{B}\bar{D} \;=\; \left(\frac{L}{2}\hat{B}\right)\left(\frac{2}{L}\hat{D}\right) \;=\; \hat{B}\hat{D}. \tag{62}$$

- Recall, the "bar" notation implies that we are considering all v, u in X^N.
- However, we need to restrict our attention to v, u in X_p^N, which implies that there exists a Q that will map $\underline{u}$ to $\underline{\bar{u}}$ such that $u \in X_p^N$.
- Applying this to the inner product in (65) yields

$$\underline{\bar{v}}^T \bar{C} \underline{\bar{u}} \;=\; (Q\underline{v})^T\,\bar{C}\,(Q\underline{u}) \;=\; \underline{v}^T Q^T\,\bar{C}\,Q\underline{u}, \tag{63}$$

  or

$$C \;=\; Q^T\,\bar{C}\,Q. \tag{64}$$

- Finally, with $B = Q^T\bar{B}Q$, we have for L:

$$L \;=\; B^{-1}\,C. \tag{65}$$

- The figure below shows the eigenvalue distributions for N=1024.