一般的に独立した K 個の分布の足し合わせた値は以下のように表したときを考える。

$$x_{mix} = \sum_{k}^{K} \pi_k x_k \tag{1}$$

また、その分布の確率は以下のようになる。

$$p(x_{mix}) = \int \cdots \int_{x_{mix}=\sum_k^n \pi_k x_k} p_1(x_1) \cdots p_n(x_n) dx_1 \cdots dx_n \tag{2}$$

このとき、x_{mix}の 平均を求めると

$$\begin{aligned}\mathbb{E}[x_{mix}] &= \int x_{mix} p(x_{mix}) dx_{mix} = \int \sum_{k}^{K} \pi_k x_k \int \cdots \int_{x_{mix}=\sum_k^n \pi_k x_k} p_1(x_1) \cdots p_n(x_n) dx_1 \cdots dx_n dx_{mix} \\ &= \sum_{k}^{K} \int \cdots \int (\pi_k x_k p_1(x_1) \cdots p_n(x_n)) dx_1 \cdots dx_n = \sum_{k}^{K} \int \pi_k x_k p_k(x_k) dx_k = \sum_{k}^{K} \pi_k \mu_k\end{aligned} \tag{3}$$

同様に分散を求めると

$$\begin{aligned}&\mathbb{V}_{p(x_{mix})}[x_{mix}] = \mathbb{E}_{p(x_{mix})}[x_{mix}^2] - \mathbb{E}[x_{mix}]^2 \\ &= \int (\sum_{k}^{K} \pi_k x_k)^2 \int \cdots \int_{x_{mix}=\sum_k^n \pi_k x_k} p_1(x_1) \cdots p_n(x_n) dx_1 \cdots dx_n dx_{mix} - \mathbb{E}[x_{mix}]^2 \\ &= \int \cdots \int (\sum_{k}^{K} \pi_k x_k)^2 p_1(x_1) \cdots p_n(x_n) dx_1 \cdots dx_n - \mathbb{E}[x_{mix}]^2 \\ &= \int \cdots \int (\sum_{k}^{K} (\pi_k^2 x_k^2) + \sum_{i \neq j} (2\pi_i \pi_j x_i x_j)) p_1(x_1) \cdots p_n(x_n) dx_1 \cdots dx_n - \mathbb{E}[x_{mix}]^2 \\ &= \int \sum_{k}^{K} (\pi_k^2 x_k^2) dx_k + \sum_{i \neq j} \int \int (2\pi_i \pi_j x_i x_j) p_i(x_i) p_j(x_j) dx_i dx_j - \mathbb{E}[x_{mix}]^2 \\ &= \int \sum_{k}^{K} (\pi_k^2 x_k^2) dx_k + \sum_{i \neq j} 2\pi_i \pi_j \mu_i \mu_j - \mathbb{E}[x_{mix}]^2 = \int \sum_{k}^{K} (\pi_k^2 x_k^2) dx_k + \sum_{i} \pi_i \mu_i \sum_{j \neq i} \pi_j \mu_j - \mathbb{E}[x_{mix}]^2 \\ &= \int \sum_{k}^{K} (\pi_k^2 x_k^2) dx_k + \sum_{i} \pi_i \mu_i (\mathbb{E}(x_{mix}) - \pi_i \mu_i) - \mathbb{E}[x_{mix}]^2 \\ &= \int \sum_{k}^{K} (\pi_k^2 x_k^2) dx_k + \sum_{i} \pi_i \mu_i \mathbb{E}[x_{mix}] - \sum_{i} \pi_i^2 \mu_i^2 - \mathbb{E}[x_{mix}]^2 \\ &= \int \sum_{k}^{K} (\pi_k^2 x_k^2) dx_k + \mathbb{E}[x_{mix}]^2 - \sum_{k}^{K} \pi_k^2 \mu_k^2 - \mathbb{E}[x_{mix}]^2 = \sum_{k}^{K} \pi_k^2 (v_k + \mu_k^2) - \sum_{k}^{K} \pi_k^2 \mu_k^2 = \sum_{k}^{K} \pi_k^2 v_k\end{aligned} \tag{4}$$