P.88 にある式

$$|A\boldsymbol{x_1}, \cdots, A\boldsymbol{x_r}| = C_r(A)|\boldsymbol{x_1}, \cdots, \boldsymbol{x_r}|$$

を確認する。検討しやすい n = 3, r = 2 の外積を検討する。このときの列の選択は (2, 3), (3, 1), (1, 2) となる。そうすると、

$$\boldsymbol{x_1} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}, \boldsymbol{x_2} = \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|\boldsymbol{x_1}, \boldsymbol{x_2}| = \begin{pmatrix} \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} \\ \begin{vmatrix} x_{31} & x_{32} \\ x_{11} & x_{12} \end{vmatrix} \\ \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \end{pmatrix}, \boldsymbol{A}\boldsymbol{x_1} = \begin{pmatrix} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} \\ a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} \end{pmatrix}, \boldsymbol{A}\boldsymbol{x_2} = \begin{pmatrix} a_{11}x_{12} + a_{12}x_{22} + a_{13}x_{32} \\ a_{21}x_{12} + a_{22}x_{22} + a_{23}x_{32} \\ a_{31}x_{12} + a_{32}x_{22} + a_{33}x_{32} \end{pmatrix},$$

$$C_r(A) = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} & \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

$$|A\boldsymbol{x_1}, A\boldsymbol{x_2}| = \begin{pmatrix} \begin{vmatrix} a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} & a_{21}x_{12} + a_{22}x_{22} + a_{23}x_{32} \\ a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} & a_{31}x_{12} + a_{32}x_{22} + a_{33}x_{32} \end{vmatrix} \\ \begin{vmatrix} a_{31}x_{11} + a_{32}x_{21} + a_{33}x_{31} & a_{31}x_{12} + a_{32}x_{22} + a_{33}x_{32} \\ a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} & a_{11}x_{12} + a_{12}x_{22} + a_{13}x_{32} \end{vmatrix} \\ \begin{vmatrix} a_{11}x_{11} + a_{12}x_{21} + a_{13}x_{31} & a_{11}x_{12} + a_{12}x_{22} + a_{13}x_{32} \\ a_{21}x_{11} + a_{22}x_{21} + a_{23}x_{31} & a_{21}x_{12} + a_{22}x_{22} + a_{23}x_{32} \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} a_{21}a_{32}x_{11}x_{22} + a_{21}a_{33}x_{11}x_{32} + a_{22}a_{31}x_{12}x_{21} + a_{22}a_{33}x_{21}x_{32} + a_{23}a_{31}x_{12}x_{31} + a_{23}a_{32}x_{22}x_{31} \\ -a_{22}a_{31}x_{11}x_{22} - a_{23}a_{31}x_{11}x_{32} - a_{21}a_{32}x_{12}x_{21} - a_{23}a_{32}x_{21}x_{32} - a_{21}a_{33}x_{12}x_{31} - a_{22}a_{33}x_{22}x_{31} \\ \\ a_{31}a_{12}x_{11}x_{22} + a_{31}a_{13}x_{11}x_{32} + a_{32}a_{11}x_{12}x_{21} + a_{32}a_{13}x_{21}x_{32} + a_{33}a_{11}x_{12}x_{31} + a_{33}a_{12}x_{22}x_{31} \\ -a_{32}a_{11}x_{11}x_{22} - a_{33}a_{11}x_{11}x_{32} - a_{31}a_{12}x_{12}x_{21} - a_{33}a_{12}x_{21}x_{32} - a_{31}a_{13}x_{12}x_{31} - a_{32}a_{13}x_{22}x_{31} \\ \\ a_{11}a_{22}x_{11}x_{22} + a_{11}a_{23}x_{11}x_{32} + a_{12}a_{21}x_{12}x_{21} + a_{12}a_{23}x_{21}x_{32} + a_{13}a_{21}x_{12}x_{31} + a_{13}a_{22}x_{22}x_{31} \\ -a_{12}a_{21}x_{11}x_{22} - a_{13}a_{21}x_{11}x_{32} - a_{11}a_{22}x_{12}x_{21} - a_{13}a_{22}x_{21}x_{32} - a_{11}a_{23}x_{12}x_{31} - a_{12}a_{23}x_{22}x_{31} \end{pmatrix}$$

$$
= \begin{pmatrix}
(a_{21}a_{32}x_{11}x_{22} - a_{21}a_{32}x_{12}x_{21} - a_{22}a_{31}x_{11}x_{22} + a_{22}a_{31}x_{12}x_{21}) + \\
(a_{21}a_{33}x_{11}x_{32} - a_{21}a_{33}x_{12}x_{31} - a_{23}a_{31}x_{11}x_{32} + a_{23}a_{31}x_{12}x_{31}) + \\
(a_{22}a_{33}x_{21}x_{32} - a_{23}a_{32}x_{21}x_{32} - a_{22}a_{33}x_{22}x_{31} + a_{23}a_{32}x_{22}x_{31}) \\
\\
(a_{31}a_{12}x_{11}x_{22} - a_{31}a_{12}x_{12}x_{21} - a_{32}a_{11}x_{11}x_{22} + a_{32}a_{11}x_{12}x_{21}) + \\
(a_{31}a_{13}x_{11}x_{32} - a_{31}a_{13}x_{12}x_{31} - a_{33}a_{11}x_{11}x_{32} + a_{33}a_{11}x_{12}x_{31}) + \\
(a_{32}a_{13}x_{21}x_{32} - a_{33}a_{12}x_{21}x_{32} - a_{32}a_{13}x_{22}x_{31} + a_{33}a_{12}x_{22}x_{31}) \\
\\
(a_{11}a_{22}x_{11}x_{22} - a_{11}a_{22}x_{12}x_{21} - a_{12}a_{21}x_{11}x_{22} + a_{12}a_{21}x_{12}x_{21}) + \\
(a_{11}a_{23}x_{11}x_{32} - a_{11}a_{23}x_{12}x_{31} - a_{13}a_{21}x_{11}x_{32} + a_{13}a_{21}x_{12}x_{31}) + \\
(a_{12}a_{23}x_{21}x_{32} - a_{13}a_{22}x_{21}x_{32} - a_{12}a_{23}x_{22}x_{31} + a_{13}a_{22}x_{22}x_{31})
\end{pmatrix}
$$

$$
= C_r(A)|\boldsymbol{x_1}, \boldsymbol{x_2}|
$$

以上で、外積の場合の具体的な計算が終わったわけだが、一般にどうなるかを上記の例を踏まえて、記載する。例 1 では要素を比較することで、

$$
C_r(C) = C_r(A)C_r(B) \tag{1}
$$

を示した。外積の場合も同様に要素を比較すると、$\boldsymbol{x_1}, \boldsymbol{x_2}$ に適当な $\boldsymbol{x_3}$ を加えて、$X = (\boldsymbol{x_1}\boldsymbol{x_2}\boldsymbol{x_2}), B = X$ とすることにより、(1) の 3 列目が、求めたい関係、$|A\boldsymbol{x_1}, A\boldsymbol{x_2}, A\boldsymbol{x_3}| = C_r(A)|\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}|$ となっている。($|\boldsymbol{x_1}, \boldsymbol{x_2}|$ は 1の証明の B'に相当する。) よって、一般に、

$$
|A\boldsymbol{x_1}, \cdots, A\boldsymbol{x_r}| = C_r(A)|\boldsymbol{x_1}, \cdots, \boldsymbol{x_r}| \tag{2}
$$

が成り立つ。