(7.123) 及びその周辺を確認する。

(7.118) までは、別資料で確認した。

$$\mathcal{L}[q] = \int q(U) \left\{ \int q(X)p(F\ U, X) ln\, p(Y|F) dF dX + ln \frac{p(U)}{q(U)} \right\} dU - D_{KL}[q(X)||p(X)] \tag{1}$$

これの中括弧の中の 1 項目に関して、検討する。

$$\begin{aligned}
\int q(X)p(F|U,X)ln\, p(Y|F)dFdX &= \int q(X) \int ... \int \left\{ \prod_d p(F_d|U,X) \right\} ln \left\{ \prod_d p(Y_d|F) \right\} dF_1 \cdots dF_D dX \\
&= \int q(X) \int ... \int \left\{ \prod_d p(F_{:,d}|U_{:,d},X) \right\} ln \left\{ \prod_d p(Y_{:,d}|F_{:,d}) \right\} dF_{:,1} \cdots dF_{:,D} dX \\
&= \int q(X) \int ... \int \left\{ \prod_d p(F_{:,d}|U_{:,d},X) \right\} \sum_d \{ln\, p(Y_{:,d}|F_{:,d})\} dF_{:,1} \cdots dF_{:,D} dX \\
= \int q(X) \sum_d \left\{ \int p(F_{:,d}|U_{:,d},X)\, ln\, p(Y_{:,d}|F_{:,d}) dF_{:,d} \right\} dX &= \sum_d \int q(X) \int p(F_{:,d}|U_{:,d},X)\, ln\, p(Y_{:,d}|F_{:,d}) dF_{:,d} dX \\
&= \sum_d \int q(X) ln\, G_d(U_{:,d}, Y_{:,d}, X) dX
\end{aligned} \tag{2}$$

2 個目の等号は (7.112),(7.116) にあるようなモデルを仮定する。ここで、$ln\, G_d(U_{:,d}, Y_{:,d}, X) = \int p(F_{:,d}|U_{:,d},X)\, ln\, p(Y_{:,d}|F_{:,d}) dF_{:,d}$。

付録 A.2 の (A.26) と比較して、F_Z と $U_{:,d}$、Y と $Y_{:,d}$、F_X と $F_{:,d}$ が対応し、(A.26) では X は省略されていることを考慮すると、それぞれがガウス過程になっていることも一致するので、計算結果は一致し、(A.30) が成り立つ。$\mu_d = K_{XZ}{K_{ZZ}}^{-1}U_{:,d}, Q = K_{XZ}{K_{ZZ}}^{-1}K_{ZX}$ として、$\sigma^2 I$ と $\beta^{-1}I$ が対応しているので、

$$\begin{aligned}
\int q(X)p(F|U,X)ln\, p(Y|F)dFdX &= \sum_d \int q(X) ln\, G_d(U_{:,d}, Y_{:,d}, X) dX \\
&= \sum_d \int q(X)(ln\, \mathcal{N}(Y_{:,d}|\mu_d, \beta^{-1}I) - \frac{\beta}{2} Tr[K_{XX} - Q])dX \\
&= \sum_d (\mathbb{E}_{q(X)}[ln\, \mathcal{N}(Y_{:,d}|\mu_d, \beta^{-1}I)] - \frac{\beta}{2} Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]])
\end{aligned} \tag{3}$$

(7.119) が求まった。

これを (7.118) に代入する。μ_d が $U_{:,d}$ の関数であることに注意すると (Tr の部分は $U_{:,d}$ に依存しない。)

$$\begin{aligned}
\mathcal{L}[q] &= \int q(U) \{ \int q(X)p(F|U,X) ln\, p(Y|F)dFdX + ln \frac{p(U)}{q(U)} \} dU - D_{KL}[q(X)||p(X)] \\
= \int q(U) \{ \sum_d (\mathbb{E}_{q(X)}[ln\, \mathcal{N}(Y_{:,d}|\mu_d, \beta^{-1}I)] &- \frac{\beta}{2} Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]]) + ln \frac{p(U)}{q(U)} \} dU - D_{KL}[q(X)||p(X)] \\
&= \int \cdots \int \{ \prod_d q(U_{:,d}) \} \{ \sum_d (\mathbb{E}_{q(X)}[ln\, \mathcal{N}(Y_{:,d}|\mu_d, \beta^{-1}I)] - \frac{\beta}{2} Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]]) \\
&+ \sum_d ln \frac{p(U_{:,d})}{q(U_{:,d})} \} dU_{:,1} \cdots dU_{:,D} - D_{KL}[q(X)||p(X)] \\
= \sum_d [\int q(U_{:,d}) (\mathbb{E}_{q(X)}[ln\, \mathcal{N}(Y_{:,d}|\mu_d, \beta^{-1}I)] &- \frac{\beta}{2} Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]] + ln \frac{p(U_{:,d})}{q(U_{:,d})}) dU_{:,d}] - D_{KL}[q(X)||p(X)]
\end{aligned} \tag{4}$$

となり、(7.120) が求まる。
　これを更に計算すると

$$
\begin{aligned}
\mathcal{L}[q] &= \sum_d [\int q(U_{:,d})(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d,\beta^{-1}I)] - \frac{\beta}{2}Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]] + ln\,\frac{p(U_{:,d})}{q(U_{:,d})})dU_{:,d}] - D_{KL}[q(X)||p(X)] \\
&= \sum_d [\int q(U_{:,d})ln\,\frac{exp(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d,\beta^{-1}I)]p(U_{:,d})}{q(U_{:,d})})dU_{:,d} - \frac{\beta}{2}Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]]] - D_{KL}[q(X)||p(X)] \\
&= \sum_d \{-D_{KL}[q(U_{:,d})||\frac{1}{Z_d}exp(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d,\beta^{-1}I)]p(U_{:,d})] + ln\,Z_d - \frac{\beta}{2}Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]]\} \\
&\quad - D_{KL}[q(X)||p(X)]
\end{aligned} \tag{5}
$$

$\mathcal{L}[q]$ を最大にするには (A.31),(A.32) のように、$D_{KL}[q(U_{:,d})||\frac{1}{Z_d}exp(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d,\beta^{-1}I)]p(U_{:,d})]$ を 0 とするので、(7.121) のように、

$$
q_{opt.}(U_{:,d}) = \frac{1}{Z_d}exp(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d,\beta^{-1}I)]p(U_{:,d}) \tag{6}
$$

Z_d は積分定数なので、

$$
Z_d = \int exp(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d,\beta^{-1}I)]p(U_{:,d})dU_{:,d} \tag{7}
$$

これは更に計算できる。

$$Z_d = \int exp(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d, \beta^{-1}I)]p(U_{:,d})dU_{:,d} = \int exp(\mathbb{E}_{q(X)}[ln\,\mathcal{N}(Y_{:,d}|\mu_d, \beta^{-1}I)]\mathcal{N}(U_{:,d}|0, K_{ZZ})dU_{:,d}$$
$$= \int \left[\frac{\beta^{N/2}}{(2\pi)^{N/2}} exp(\mathbb{E}_{q(X)}[-\frac{\beta}{2}(Y_{:,d} - K_{XZ}K_{ZZ}{}^{-1}U_{:,d})^T(Y_{:,d} - K_{XZ}K_{ZZ}{}^{-1}U_{:,d})]\right]$$
$$\left[\frac{1}{(2\pi)^{M/2}|K_{ZZ}|^{1/2}} exp(U_{:,d}{}^T K_{ZZ}{}^{-1}U_{:,d})\right] dU_{:,d}$$
$$= \frac{\beta^{N/2}}{(2\pi)^{(N+M)/2}|K_{ZZ}|^{1/2}} \int exp(-\frac{1}{2}[U_{:,d}^T(K_{ZZ}{}^{-1} + \beta K_{ZZ}{}^{-1}\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}]K_{ZZ}{}^{-1})U_{:,d}$$
$$-2\beta Y_{:,d}^T \mathbb{E}_{q(X)}[K_{XZ}]K_{ZZ}{}^{-1}U_{:,d} + \beta Y_{:,d}^T Y_{:,d}])dU_{:,d}$$
$$= \frac{\beta^{N/2}}{(2\pi)^{(N+M)/2}|K_{ZZ}|^{1/2}} \int exp(-\frac{1}{2}[U_{:,d}^T K_{ZZ}{}^{-1}(K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])K_{ZZ}{}^{-1}U_{:,d}$$
$$-2\beta Y_{:,d}^T \mathbb{E}_{q(X)}[K_{XZ}]K_{ZZ}{}^{-1}U_{:,d} + \beta Y_{:,d}^T Y_{:,d}])dU_{:,d}$$
$$= \frac{\beta^{N/2}}{(2\pi)^{(N+M)/2}|K_{ZZ}|^{1/2}} \int exp(-\frac{1}{2}[(U_{:,d} - \beta K_{ZZ}(K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}]K_{ZZ})^{-1}\mathbb{E}_{q(X)}[K_{ZX}]Y_{:,d})^T$$
$$K_{ZZ}{}^{-1}(K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])K_{ZZ}{}^{-1}(U_{:,d} - \beta K_{ZZ}(K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])^{-1}\mathbb{E}_{q(X)}[K_{ZX}]Y_{:,d})$$
$$-\beta^2 Y_{:,d}^T \mathbb{E}[K_{XZ}](K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])^{-1}\mathbb{E}[K_{ZX}]Y_{:,d} + \beta Y_{:,d}^T Y_{:,d}])dU_{:,d}$$
$$= \frac{\beta^{N/2}}{(2\pi)^{(N+M)/2}|K_{ZZ}|^{1/2}} \int [(2\pi)^{M/2}|K_{ZZ}(K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])^{-1}K_{ZZ}|^{1/2}$$
$$\mathcal{N}(U_{:,d}|\beta K_{ZZ}(K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}]K_{ZZ})^{-1}\mathbb{E}_{q(X)}[K_{ZX}]Y, K_{ZZ}(K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])^{-1}K_{ZZ})]$$
$$exp(-\frac{1}{2}Y_{:,d}^T[\beta I - \beta^2\mathbb{E}[K_{XZ}](K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])^{-1}\mathbb{E}[K_{ZX}]]Y_{:,d})dU_{:,d}$$
$$= \frac{\beta^{N/2}|K_{ZZ}|^{1/2}}{(2\pi)^{N/2}|K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}]|^{1/2}} exp(-\frac{1}{2}Y^T(\beta I - \beta^2\mathbb{E}[K_{XZ}](K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])^{-1}\mathbb{E}[K_{ZX}])Y)$$
$$= \frac{\beta^{N/2}|K_{ZZ}|^{1/2}}{(2\pi)^{N/2}|K_{ZZ} + \beta\Psi_2|^{1/2}} exp(-\frac{1}{2}Y^T WY) \tag{8}$$

最後の等号は $W = \beta I - \beta^2\mathbb{E}[K_{XZ}](K_{ZZ} + \beta\mathbb{E}_{q(X)}[K_{ZX}K_{XZ}])^{-1}\mathbb{E}[K_{ZX}] = \beta I - \beta^2\Psi_1(K_{ZZ} + \beta\Psi_2)^{-1}\Psi_1^T, \Psi_1 = \mathbb{E}_{q(X)}[K_{XZ}], \Psi_2 = \mathbb{E}_{q(X)}[K_{ZX}K_{XZ}]$ とする。

$\Psi_0 = Tr(\mathbb{E}_{q(X)}[K_{XX}])$ とすると、(5) を最小化する場合を考える。$q(U_{:,d})$ に関して、KL ダイバージェンスを 0 になるようにする。

$$\mathcal{L}[q] = \sum_d \{ln\,Z_d - \frac{\beta}{2}Tr[\mathbb{E}_{q(X)}[K_{XX}] - \mathbb{E}_{q(X)}[Q]]\} - D_{KL}[q(X)||p(X)]$$
$$= \sum_d \{ln\,(\frac{\beta^{N/2}|K_{ZZ}|^{1/2}}{(2\pi)^{N/2}|K_{ZZ} + \beta\Psi_2|^{1/2}} exp(-\frac{1}{2}Y^T WY)) - \frac{\beta}{2}Tr[\mathbb{E}_{q(X)}[K_{XX}]]$$
$$+\frac{\beta}{2}Tr[\mathbb{E}_{q(X)}[K_{XZ}(K_{ZZ})^{-1}K_{ZX}]]\} - D_{KL}[q(X)||p(X)]$$
$$= \sum_d \{ln\,(\frac{\beta^{N/2}|K_{ZZ}|^{1/2}}{(2\pi)^{N/2}|K_{ZZ} + \beta\Psi_2|^{1/2}} exp(-\frac{1}{2}Y^T WY)) - \frac{\beta\Psi_0}{2} + \frac{\beta}{2}Tr[\mathbb{E}_{q(X)}[(K_{ZZ})^{-1}K_{ZX}K_{XZ}]]\} - D_{KL}[q(X)||p(X)]$$
$$= \sum_d \{ln\,(\frac{\beta^{N/2}|K_{ZZ}|^{1/2}}{(2\pi)^{N/2}|K_{ZZ} + \beta\Psi_2|^{1/2}} exp(-\frac{1}{2}Y^T WY)) - \frac{\beta\Psi_0}{2} + \frac{\beta}{2}Tr[(K_{ZZ})^{-1}\Psi_2]\} - D_{KL}[q(X)||p(X)] \tag{9}$$

(7.123) が成立する。(7.125)-(7.127) が成立することは定義より明らか。

(7.128) に関して確認する。

$$\begin{aligned}\Psi_0 = \sum_{n=1}^{N}\int k(x_n,x_n)q(x_n|\mu_n,diagm(v_n))dx_n &= \sum_{n=1}^{N}\int\cdots\int {\sigma_f}^2(\prod_{i=1}^{H_0}\mathcal{N}(x_{n,i}|\mu_{n,i},v_{n,i}))dx_{n,1}\cdots dx_{H_0} \\ &= \sum_{n=1}^{N}{\sigma_f}^2\prod_{i=1}^{H_0}(\int\mathcal{N}(x_{n,i}|\mu_{n,i},v_{n,i})dx_{n,i}) = \sum_{n=1}^{N}{\sigma_f}^2 = N\sigma_f\end{aligned} \tag{10}$$

(7.129) に関して確認する。

$$\begin{aligned}[\Psi_1]_{n,m} = \int k(x_n,z_m)q(x_n)dx_n &= \int\cdots\int {\sigma_f}^2 exp(-\frac{1}{2}\sum_{i=1}^{H_0}w_i(x_{n,i}-z_{m,i})^2)(\prod_{i=1}^{H_0}\mathcal{N}(x_{n,i}|\mu_{n,i},v_{n,i}))dx_{n,1}\cdots dx_{H_0} \\ &= {\sigma_f}^2\int\cdots\int(\prod_{i=1}^{H_0}\mathcal{N}(x_{n,i}|\mu_{n,i},v_{n,i})exp(-\frac{1}{2}w_i(x_{n,i}-z_{m,i})^2))dx_{n,1}\cdots dx_{H_0} \\ &= {\sigma_f}^2\prod_{i=1}^{H_0}(\int\frac{1}{\sqrt{2\pi v_i}}exp(-\frac{(x_{n,i}-\mu_{n,i})^2}{2v_i})exp(-\frac{1}{2}w_i(x_{n,i}-z_{m,i})^2)dx_{n,i}) \\ &= {\sigma_f}^2\prod_{i=1}^{H_0}(\int\frac{1}{\sqrt{2\pi v_i}}exp(-\frac{1}{2}(\frac{(x_{n,i}-\mu_{n,i})^2}{v_i}+w_i(x_{n,i}-z_{m,i})^2))dx_{n,i}) \\ &= {\sigma_f}^2\prod_{i=1}^{H_0}(\int\frac{1}{\sqrt{2\pi v_i}}exp(-\frac{1}{2}((w_i+\frac{1}{v_i})(x_{n,i}-\frac{w_iv_iz_{n,i}+\mu_{n,i}}{w_iv_i+1})^2+\frac{w_i(\mu_{n,i}-z_{n,i})^2}{w_iv_i+1}))dx_{n,i}) \\ &= {\sigma_f}^2\prod_{i=1}^{H_0}exp(-\frac{w_i(\mu_{n,i}-z_{n,i})^2}{2(w_iv_i+1)})(\frac{1}{\sqrt{v_i(w_i+\frac{1}{v_i})}}\int\frac{\sqrt{w_i+\frac{1}{v_i}}}{\sqrt{2\pi}}exp(-\frac{1}{2}((w_i+\frac{1}{v_i})(x_{n,i}-\frac{w_iv_iz_{n,i}+\mu_{n,i}}{w_iv_i+1})^2))dx_{n,i}) \\ &= {\sigma_f}^2\prod_{i=1}^{H_0}\frac{1}{\sqrt{w_iv_i+1}}exp(-\frac{w_i(\mu_{n,i}-z_{n,i})^2}{2(w_iv_i+1)})(\int\mathcal{N}(x_{n,i}|\frac{w_iv_iz_{n,i}+\mu_{n,i}}{w_iv_i+1},(w_i+\frac{1}{v_i})^{-1})dx_{n,i}) \\ &= {\sigma_f}^2\prod_{i=1}^{H_0}\frac{1}{\sqrt{w_iv_i+1}}exp(-\frac{w_i(\mu_{n,i}-z_{n,i})^2}{2(w_iv_i+1)})\end{aligned} \tag{11}$$

(7.130) に関して確認する。

$$[\Psi_2]_{m,m'} = \sum_{n=1}^{N}\int k(x_n,z_m)k(x_n,z_{m'})q(x_n)dx_n \equiv \sum_{n=1}^{N}[{\Psi_2}^n]_{m,m'} \tag{12}$$

つまり、$[\Psi_2{}^n]_{m,m'} = \int k(x_n, z_m)k(x_n, z_{m'})q(x_n)dx_n$。これは $[\Psi_1]_{n,m'}$ と同様に計算できる。

$$
\begin{aligned}
&[\Psi_2{}^n]_{m,m'} = \int k(x_n, z_m)k(x_n, z_{m'})q(x_n)dx_n \\
&= \int \cdots \int [{\sigma_f}^2 exp(-\frac{1}{2}\sum_{i=1}^{H_0} w_i(x_{n,i} - z_{m,i})^2)][{\sigma_f}^2 exp(-\frac{1}{2}\sum_{i=1}^{H_0} w_i(x_{n,i} - z_{m',i})^2)](\prod_{i=1}^{H_0}\mathcal{N}(x_{n,i}|\mu_{n,i}, v_{n,i}))dx_{n,1}\cdots dx_{H_0} \\
&= {\sigma_f}^4 \int \cdots \int (\prod_{i=1}^{H_0}\mathcal{N}(x_{n,i}|\mu_{n,i}, v_{n,i})exp(-\frac{1}{2}w_i[(x_{n,i} - z_{m,i})^2 + (x_{n,i} - z_{m',i})^2]))dx_{n,1}\cdots dx_{H_0} \\
&= {\sigma_f}^4 \prod_{i=1}^{H_0}(\int \frac{1}{\sqrt{2\pi v_i}} exp(-\frac{1}{2}(\frac{(x_{n,i} - \mu_{n,i})^2}{v_i} + w_i[(x_{n,i} - z_{m,i})^2 + (x_{n,i} - z_{m',i})^2]))dx_{n,i}) \\
&= {\sigma_f}^4 \prod_{i=1}^{H_0}(\int \frac{1}{\sqrt{2\pi v_i}} exp(-\frac{1}{2}((2w_i + \frac{1}{v_i})(x_{n,i} - \frac{w_i v_i z_{m,i} + w_i v_i z_{m',i} + \mu_{n,i}}{2w_i v_i + 1})^2 \\
&+ \frac{2w_i {\mu_{n,i}}^2 + w_i {z_{m,i}}^2 + w_i {z_{m',i}}^2 - 2w_i z_{m,i}\mu_{n,i} - 2w_i z_{m',i}\mu_{n,i} + {w_i}^2 v_i {z_{m,i}}^2 + {w_i}^2 v_i {z_{m',i}}^2 - 2{w_i}^2 v_i z_{m,i} z_{m',i}}{2w_i v_i + 1}))dx_{n,i}) \\
&= {\sigma_f}^4 \prod_{i=1}^{H_0} \frac{exp(-\frac{{w_i}^2 {z_{m,i}}^2 - 2{w_i}^2 z_{m,i} z_{m',i} + {w_i}^2 {z_{m',i}}^2 + \frac{w_i {z_{m,i}}^2}{v_i} + \frac{w_i {z_{m',i}}^2}{v_i} + \frac{2w_i {\mu_{n,i}}^2}{v_i} - \frac{2w_i z_{m,i}\mu_{n,i}}{v_i} - \frac{2w_i z_{m',i}\mu_{n,i}}{v_i}}{2(2w_i + \frac{1}{v_i})})}{\sqrt{v_i(2w_i + \frac{1}{v_i})}} \\
&(\int \mathcal{N}(x_{n,i}|\frac{w_i v_i z_{n,i} + w_i v_i z_{m',i} + \mu_{n,i}}{2w_i v_i + 1}, (2w_i + \frac{1}{v_i})^{-1})dx_{n,i}) \\
&= {\sigma_f}^4 \prod_{i=1}^{H_0} \frac{1}{\sqrt{2w_i v_i + 1}} \\
&exp(-\frac{{w_i}^2 {z_{m,i}}^2 - 2{w_i}^2 z_{m,i} z_{m',i} + {w_i}^2 {z_{m',i}}^2 + \frac{w_i {z_{m,i}}^2}{v_i} + \frac{w_i {z_{m',i}}^2}{v_i} + \frac{2w_i {\mu_{n,i}}^2}{v_i} - \frac{2w_i z_{m,i}\mu_{n,i}}{v_i} - \frac{2w_i z_{m',i}\mu_{n,i}}{v_i}}{2(2w_i + \frac{1}{v_i})})
\end{aligned} \tag{13}
$$

ここで、(7.130) の一部を検討する。$\overline{z_i} = \frac{z_{m,i} - z_{m',i}}{2}$ として、

$$
\begin{aligned}
&\frac{w_i(z_{m,i} - z_{m',i})^2}{4} + \frac{w_i(\mu_{n,i} - \overline{z_i})^2}{2w_i v_i + 1} = \frac{w_i(z_{m,i} - z_{m',i})^2}{4} + \frac{\frac{w_i}{v_i}(\mu_{n,i} - \frac{z_{m,i} - z_{m',i}}{2})^2}{2w_i + \frac{1}{v_i}} \\
&= \frac{w_i(z_{m,i} - z_{m',i})^2(2w_i + \frac{1}{v_i}) + \frac{4w_i}{v_i}(\mu_{n,i} - \frac{z_{m,i} + z_{m',i}}{2})^2}{4(2w_i + \frac{1}{v_i})} \\
&= \frac{w_i(z_{m,i} - z_{m',i})^2(2w_i + \frac{1}{v_i}) + \frac{w_i}{v_i}(2\mu_{n,i} - z_{m,i} - z_{m',i})^2}{4(2w_i + \frac{1}{v_i})} \\
&= \frac{2{w_i}^2 {z_{m,i}}^2 - 4{w_i}^2 z_{m,i} z_{m',i} + 2{w_i}^2 {z_{m',i}}^2 + \frac{2w_i {z_{m,i}}^2}{v_i} + \frac{2w_i {z_{m',i}}^2}{v_i} + \frac{4w_i {\mu_{n,i}}^2}{v_i} - \frac{4w_i z_{m,i}\mu_{n,i}}{v_i} - \frac{4w_i z_{m',i}\mu_{n,i}}{v_i}}{4(2w_i + \frac{1}{v_i})} \\
&= \frac{{w_i}^2 {z_{m,i}}^2 - 2{w_i}^2 z_{m,i} z_{m',i} + {w_i}^2 {z_{m',i}}^2 + \frac{w_i {z_{m,i}}^2}{v_i} + \frac{w_i {z_{m',i}}^2}{v_i} + \frac{2w_i {\mu_{n,i}}^2}{v_i} - \frac{2w_i z_{m,i}\mu_{n,i}}{v_i} - \frac{2w_i z_{m',i}\mu_{n,i}}{v_i}}{2(2w_i + \frac{1}{v_i})}
\end{aligned} \tag{14}
$$

よって、

$$
\begin{aligned}
[\Psi_2{}^n]_{m,m'} &= \sigma_f{}^4 \prod_{i=1}^{H_0} \frac{1}{\sqrt{2w_i v_i + 1}} \\
exp(-\frac{w_i{}^2 z_{m,i}{}^2 - 2w_i{}^2 z_{m,i} z_{m',i} + w_i{}^2 z_{m',i}{}^2 + \frac{w_i z_{m,i}{}^2}{v_i} + \frac{w_i z_{m',i}{}^2}{v_i} + \frac{2w_i \mu_{n,i}{}^2}{v_i} - \frac{2w_i z_{m,i}\mu_{n,i}}{v_i} - \frac{2w_i z_{m',i}\mu_{n,i}}{v_i}}{2(2w_i + \frac{1}{v_i})}) \\
&= \sigma_f{}^4 \prod_{i=1}^{H_0} \frac{1}{\sqrt{2w_i v_i + 1}} exp(-\frac{w_i(z_{m,i} - z_{m',i})^2}{4} - \frac{w_i(\mu_{n,i} - \overline{z_i})^2}{2w_i v_i + 1})
\end{aligned}
\tag{15}
$$