例 2 について、検討してみる。まず、具体例を考えてみる。n=2,r=2 のとき、

$$\boldsymbol{x_A} = \begin{pmatrix} x_{A1} \\ x_{A2} \end{pmatrix} \tag{1}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{2}$$

$$\boldsymbol{x_B} = \begin{pmatrix} x_{B1} \\ x_{B2} \end{pmatrix} = A\boldsymbol{x_A} = \begin{pmatrix} a_{11}x_{A1} + a_{12}x_{A2} \\ a_{21}x_{A1} + a_{22}x_{A2} \end{pmatrix} \tag{3}$$

このとき、単項式は $x_{p1} = {x_{A1}}^2, x_{p2} = x_{A1}x_{A2}, x_{p3} = {x_{A2}}^2$ があるが、

$$\boldsymbol{x_p} = \begin{pmatrix} x_{p1} \\ x_{p2} \\ x_{p3} \end{pmatrix} \tag{4}$$

この場合、

$$P_r(A) = \begin{pmatrix} {a_{11}}^2 & 2a_{11}a_{12} & {a_{12}}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ {a_{21}}^2 & 2a_{21}a_{22} & {a_{22}}^2 \end{pmatrix} \tag{5}$$

C=AB なので、

$$C = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \tag{6}$$

(6) から $P_r(C)$ を構成する。

$$P_r(C) = \begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21})^2 & 2(a_{11}b_{11} + a_{12}b_{21})(a_{11}b_{12} + a_{12}b_{22}) & (a_{11}b_{12} + a_{12}b_{22})^2 \\ \begin{matrix} (a_{11}b_{11} + a_{12}b_{21}) \\ (a_{21}b_{11} + a_{22}b_{21}) \end{matrix} & \begin{matrix} (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) \\ +(a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \end{matrix} & \begin{matrix} (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{12} + a_{22}b_{22}) \end{matrix} \\ (a_{21}b_{11} + a_{22}b_{21})^2 & 2(a_{21}b_{11} + a_{22}b_{21})(a_{21}b_{12} + a_{22}b_{22}) & (a_{21}b_{12} + a_{22}b_{22})^2 \end{pmatrix} =$$

$$\begin{pmatrix} \begin{matrix} {a_{11}}^2{b_{11}}^2 + 2a_{11}b_{11}a_{12}b_{21} \\ +{a_{12}}^2{b_{21}}^2 \end{matrix} & \begin{matrix} 2({a_{11}}^2b_{11}b_{12} + a_{11}a_{12}b_{11}b_{22} \\ +a_{11}a_{12}b_{12}b_{21} + {a_{12}}^2b_{21}b_{22}) \end{matrix} & \begin{matrix} {a_{11}}^2{b_{12}}^2 + 2a_{11}a_{12}b_{12}b_{22} \\ +{a_{12}}^2{b_{22}}^2 \end{matrix} \\ \begin{matrix} a_{11}a_{21}{b_{11}}^2 + a_{12}a_{21}b_{11}b_{21} \\ +a_{11}a_{22}b_{11}b_{21} + a_{12}a_{22}{b_{21}}^2 \end{matrix} & \begin{matrix} 2a_{11}a_{21}b_{11}b_{12} + a_{12}a_{21}b_{12}b_{21} \\ +2a_{12}a_{22}b_{21}b_{22} + a_{11}a_{22}b_{11}b_{11} \\ +a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{12}b_{21} \end{matrix} & \begin{matrix} a_{11}a_{21}{b_{12}}^2 + a_{12}a_{21}b_{12}b_{22} \\ +a_{11}a_{22}b_{12}b_{22} + a_{12}a_{22}{b_{22}}^2 \end{matrix} \\ \begin{matrix} {a_{21}}^2{b_{11}}^2 + 2a_{21}b_{11}a_{22}b_{21} \\ +{a_{22}}^2{b_{21}}^2 \end{matrix} & \begin{matrix} 2({a_{21}}^2b_{11}b_{12} + a_{21}a_{22}b_{11}b_{22} \\ +a_{21}a_{22}b_{12}b_{21} + {a_{22}}^2b_{21}b_{22}) \end{matrix} & \begin{matrix} {a_{21}}^2{b_{12}}^2 + 2a_{21}a_{22}b_{12}b_{22} \\ +{a_{22}}^2{b_{22}}^2 \end{matrix} \end{pmatrix} \tag{7}$$

次に $P_r(A)P_r(B)$ を検討する。

$$P_r(A)P_r(B) = \begin{pmatrix} {a_{11}}^2 & 2a_{11}a_{12} & {a_{12}}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ {a_{21}}^2 & 2a_{21}a_{22} & {a_{22}}^2 \end{pmatrix} \begin{pmatrix} {b_{11}}^2 & 2b_{11}b_{12} & {b_{12}}^2 \\ b_{11}b_{21} & b_{11}b_{22} + b_{12}b_{21} & b_{12}b_{22} \\ {b_{21}}^2 & 2b_{21}b_{22} & {b_{22}}^2 \end{pmatrix} =$$

$$
\begin{pmatrix}
\begin{array}{c} {a_{11}}^2{b_{11}}^2+2a_{11}b_{11}a_{12}b_{21} \\ +{a_{12}}^2{b_{21}}^2 \end{array} & \begin{array}{c} 2({a_{11}}^2b_{11}b_{12}+a_{11}a_{12}b_{11}b_{22} \\ +a_{11}a_{12}b_{12}b_{21}+{a_{12}}^2b_{21}b_{22}) \end{array} & \begin{array}{c} {a_{11}}^2{b_{12}}^2+2a_{11}a_{12}b_{12}b_{22} \\ +{a_{12}}^2{b_{22}}^2 \end{array} \\
\begin{array}{c} a_{11}a_{21}{b_{11}}^2+a_{12}a_{21}b_{11}b_{21} \\ +a_{11}a_{22}b_{11}b_{21}+a_{12}a_{22}{b_{21}}^2 \end{array} & \begin{array}{c} 2a_{11}a_{21}b_{11}b_{12}+a_{12}a_{21}b_{12}b_{21} \\ +2a_{12}a_{22}b_{21}b_{22}+a_{11}a_{22}b_{11}b_{11} \\ +a_{12}a_{21}b_{11}b_{22}+a_{11}a_{22}b_{12}b_{21} \end{array} & \begin{array}{c} a_{11}a_{21}{b_{12}}^2+a_{12}a_{21}b_{12}b_{22} \\ +a_{11}a_{22}b_{12}b_{22}+a_{12}a_{22}{b_{22}}^2 \end{array} \\
\begin{array}{c} {a_{21}}^2{b_{11}}^2+2a_{21}b_{11}a_{22}b_{21} \\ +{a_{22}}^2{b_{21}}^2 \end{array} & \begin{array}{c} 2({a_{21}}^2b_{11}b_{12}+a_{21}a_{22}b_{11}b_{22} \\ +a_{21}a_{22}b_{12}b_{21}+{a_{22}}^2b_{21}b_{22}) \end{array} & \begin{array}{c} {a_{21}}^2{b_{12}}^2+2a_{21}a_{22}b_{12}b_{22} \\ +{a_{22}}^2{b_{22}}^2 \end{array}
\end{pmatrix} \tag{8}
$$

よって、上記の場合、$P_r(C) = P_r(A)P_r(B)$。

以下、一般的に考える。まず、単項式の個数が

$$\binom{n+r-1}{r}$$

あることを確認する。これは、r 個 + 仕切り (n-1) 個の中から、仕切りを選ぶことになる。上記の具体例を利用すると、

$$●○○->|, x_2, x_2->{x_2}^2$$

$$○●○->x_1, |, x_2->x_1x_2$$

$$○○●->x_1, x_1, |->{x_1}^2$$

よって、個数は、

$$\binom{n+r-1}{n-1} = \binom{n+r-1}{(n+r-1)-(n-1)} = \binom{n+r-1}{r}$$

$P_r(C) = P_r(A)P_r(B)$ が成立するとすると、$|P_r(C)| = |P_r(A)||P_r(B)|$ が言えて、$|P_r(A)| = |A|^k$ が言える。そこで、$|P_r(A)|, |A|^k$ の次数を比較する。$|A|$ は A が n 次の正方行列なので、a_{ij} について、n 次になる。

$|P_r(A)|$ は ${}_{n+r-1}C_r$ 次の正方行列で、要素は a_{ij} に関して r 次になっている。そのため、

$$r\,{}_{n+r-1}C_r = r\frac{(n+r-1)!}{(n-1)!r!} = \frac{(n+r-1)!}{(n-1)!(r-1)!} = nk \tag{9}$$

よって、

$$k = \frac{1}{n}\frac{(n+r-1)!}{(n-1)!(r-1)!} = \frac{(n+r-1)!}{n!(r-1)!} = {}_{n+r-1}C_{r-1} \tag{10}$$

そのため、

$$|P_r(A)| = |A|^{{}_{n+r-1}C_{r-1}} \tag{11}$$