例 2 について、再度、検討してみる。n=2,r=3 のとき、

$$\boldsymbol{x_A} = \begin{pmatrix} x_{A1} \\ x_{A2} \end{pmatrix} \tag{1}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{2}$$

$$\boldsymbol{x_B} = \begin{pmatrix} x_{B1} \\ x_{B2} \end{pmatrix} = A\boldsymbol{x_A} = \begin{pmatrix} a_{11}x_{A1} + a_{12}x_{A2} \\ a_{21}x_{A1} + a_{22}x_{A2} \end{pmatrix} \tag{3}$$

このとき、単項式は $x_{p1} = {x_{A1}}^3, x_{p2} = {x_{A1}}^2 x_{A2}, x_{p3} = x_{A1}{x_{A2}}^2, x_{p4} = {x_{A2}}^3$ があるが (順番は任意)、

$$\boldsymbol{x_p} = \begin{pmatrix} x_{p1} \\ x_{p2} \\ x_{p3} \\ x_{p4} \end{pmatrix} \tag{4}$$

とする。この場合、

$$P_3(A) = \begin{pmatrix} {a_{11}}^3 & 3{a_{11}}^2 a_{12} & 3a_{11}{a_{12}}^2 & {a_{12}}^3 \\ {a_{11}}^2 a_{21} & {a_{11}}^2 a_{22} + 2a_{11}a_{12}a_{21} & {a_{12}}^2 a_{21} + 2a_{11}a_{12}a_{22} & {a_{12}}^2 a_{22} \\ a_{11}{a_{21}}^2 & a_{12}{a_{21}}^2 + 2a_{11}a_{21}a_{22} & a_{11}{a_{22}}^2 + 2a_{12}a_{21}a_{22} & a_{12}{a_{22}}^2 \\ {a_{21}}^3 & 3{a_{21}}^2 a_{22} & 3a_{21}{a_{22}}^2 & {a_{22}}^3 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}P_2(A)_{({x_1}^2)({x_1'}^2)} & \begin{matrix} a_{11}P_2(A)_{({x_1}^2)({x_1}'{x_2}')} \\ +a_{12}P_2(A)_{({x_1}^2)({x_1'}^2)} \end{matrix} & \begin{matrix} a_{11}P_2(A)_{({x_1}^2)({x_2'}^2)} \\ +a_{12}P_2(A)_{({x_1}^2)({x_1}'{x_2}')} \end{matrix} & a_{12}P_2(A)_{({x_1}^2)({x_2'}^2)} \\ a_{11}P_2(A)_{(x_1x_2)({x_1'}^2)} & \begin{matrix} a_{11}P_2(A)_{(x_1x_2)({x_1}'{x_2}')} \\ +a_{12}P_2(A)_{(x_1x_2)({x_1'}^2)} \end{matrix} & \begin{matrix} a_{11}P_2(A)_{(x_1x_2)({x_2'}^2)} \\ +a_{12}P_2(A)_{(x_1x_2)({x_1}'{x_2}')} \end{matrix} & a_{12}P_2(A)_{(x_1x_2)({x_2'}^2)} \\ a_{21}P_2(A)_{(x_1x_2)({x_1'}^2)} & \begin{matrix} a_{21}P_2(A)_{(x_1x_2)({x_1}'{x_2}')} \\ +a_{22}P_2(A)_{(x_1x_2)({x_1'}^2)} \end{matrix} & \begin{matrix} a_{21}P_2(A)_{(x_1x_2)({x_2'}^2)} \\ +a_{22}P_2(A)_{(x_1x_2)({x_1}'{x_2}')} \end{matrix} & a_{22}P_2(A)_{(x_1x_2)({x_2'}^2)} \\ a_{21}P_2(A)_{({x_2}^2)({x_1'}^2)} & \begin{matrix} a_{21}P_2(A)_{({x_2}^2)({x_1}'{x_2}')} \\ +a_{22}P_2(A)_{({x_2}^2)({x_1'}^2)} \end{matrix} & \begin{matrix} a_{21}P_2(A)_{({x_2}^2)({x_2'}^2)} \\ +a_{22}P_2(A)_{({x_2}^2)({x_1}'{x_2}')} \end{matrix} & a_{22}P_2(A)_{({x_2}^2)({x_2'}^2)} \end{pmatrix} \tag{5}$$

C=AB なので、

$$C = P_1(C) = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} = P_1(A)P_1(B) = AB \tag{6}$$

$$
P_3(C)=\begin{pmatrix}
c_{11}P_2(C)_{(x_1{}^2)(x_1{}'^2)} & \begin{array}{l}c_{11}P_2(C)_{(x_1{}^2)(x_1{}'x_2{}')}\\+c_{12}P_2(C)_{(x_1{}^2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{11}P_2(C)_{(x_1{}^2)(x_2{}'^2)}\\+c_{12}P_2(C)_{(x_1{}^2)(x_1{}'x_2{}')}\end{array} & c_{12}P_2(C)_{(x_1{}^2)(x_2{}'^2)}\\
c_{11}P_2(C)_{(x_1x_2)(x_1{}'^2)} & \begin{array}{l}c_{11}P_2(C)_{(x_1x_2)(x_1{}'x_2{}')}\\+c_{12}P_2(C)_{(x_1x_2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{11}P_2(C)_{(x_1x_2)(x_2{}'^2)}\\+c_{12}P_2(C)_{(x_1x_2)(x_1{}'x_2{}')}\end{array} & c_{12}P_2(C)_{(x_1x_2)(x_2{}'^2)}\\
c_{21}P_2(C)_{(x_1x_2)(x_1{}'^2)} & \begin{array}{l}c_{21}P_2(C)_{(x_1x_2)(x_1{}'x_2{}')}\\+c_{22}P_2(C)_{(x_1x_2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{21}P_2(C)_{(x_1x_2)(x_2{}'^2)}\\+c_{22}P_2(C)_{(x_1x_2)(x_1{}'x_2{}')}\end{array} & c_{22}P_2(C)_{(x_1x_2)(x_2{}'^2)}\\
c_{21}P_2(C)_{(x_2{}^2)(x_1{}'^2)} & \begin{array}{l}c_{21}P_2(C)_{(x_2{}^2)(x_1{}'x_2{}')}\\+c_{22}P_2(C)_{(x_2{}^2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{21}P_2(C)_{(x_2{}^2)(x_2{}'^2)}\\+c_{22}P_2(C)_{(x_2{}^2)(x_1{}'x_2{}')}\end{array} & c_{22}P_2(C)_{(x_2{}^2)(x_2{}'^2)}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
c_{11}P_2(C)_{(x_1{}^2)(x_1{}'^2)} & \begin{array}{l}c_{11}P_2(C)_{(x_1{}^2)(x_1{}'x_2{}')}\\+c_{12}P_2(C)_{(x_1{}^2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{11}P_2(C)_{(x_1{}^2)(x_2{}'^2)}\\+c_{12}P_2(C)_{(x_1{}^2)(x_1{}'x_2{}')}\end{array} & c_{12}P_2(C)_{(x_1{}^2)(x_2{}'^2)}\\
c_{21}P_2(C)_{(x_1{}^2)(x_1{}'^2)} & \begin{array}{l}c_{21}P_2(C)_{(x_1{}^2)(x_1{}'x_2{}')}\\+c_{22}P_2(C)_{(x_1{}^2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{21}P_2(C)_{(x_1{}^2)(x_2{}'^2)}\\+c_{22}P_2(C)_{(x_1{}^2)(x_1{}'x_2{}')}\end{array} & c_{22}P_2(C)_{(x_1{}^2)(x_2{}'^2)}\\
c_{11}P_2(C)_{(x_2{}^2)(x_1{}'^2)} & \begin{array}{l}c_{11}P_2(C)_{(x_2{}^2)(x_1{}'x_2{}')}\\+c_{12}P_2(C)_{(x_2{}^2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{11}P_2(C)_{(x_2{}^2)(x_2{}'^2)}\\+c_{12}P_2(C)_{(x_2{}^2)(x_1{}'x_2{}')}\end{array} & c_{12}P_2(C)_{(x_2{}^2)(x_2{}'^2)}\\
c_{21}P_2(C)_{(x_2{}^2)(x_1{}'^2)} & \begin{array}{l}c_{21}P_2(C)_{(x_2{}^2)(x_1{}'x_2{}')}\\+c_{22}P_2(C)_{(x_2{}^2)(x_1{}'^2)}\end{array} & \begin{array}{l}c_{21}P_2(C)_{(x_2{}^2)(x_2{}'^2)}\\+c_{22}P_2(C)_{(x_2{}^2)(x_1{}'x_2{}')}\end{array} & c_{22}P_2(C)_{(x_2{}^2)(x_2{}'^2)}
\end{pmatrix} \tag{7}
$$

$$
\begin{gathered}
P_3(C)_{(x_1{}^3)(x_1{}'^3)}=c_{11}P_2(C)_{(x_1{}^2)(x_1{}'^2)}=(a_{11}b_{11}+a_{12}b_{21})\\
(P_2(A)_{(x_1{}^2)(x_{1\prime}{}^2)}P_2(B)_{(x_1{}^2)(x_1{}'^2)}+P_2(A)_{(x_1{}^2)(x_{1\prime}x_{2\prime})}P_2(B)_{(x_1x_2)(x_1{}'^2)}+P_2(A)_{(x_1{}^2)(x_{2\prime}{}^2)}P_2(B)_{(x_2{}^2)(x_1{}'^2)})
\end{gathered} \tag{8}
$$

$$
\begin{aligned}
(P_3(A)P_3(B))_{(x_1{}^3)(x_1{}'^3)}&=a_{11}b_{11}P_2(A)_{(x_1{}^2)(x_{1\prime}{}^2)}P_2(B)_{(x_1{}^2)(x_1{}'^2)}+a_{11}b_{11}P_2(A)_{(x_1{}^2)(x_{1\prime}x_{2\prime})}P_2(B)_{(x_1x_2)(x_1{}'^2)}\\
&+a_{12}b_{21}P_2(A)_{(x_1{}^2)(x_{1\prime}{}^2)}P_2(B)_{(x_1{}^2)(x_1{}'^2)}+a_{11}b_{11}P_2(A)_{(x_1{}^2)(x_{2\prime}{}^2)}P_2(B)_{(x_2{}^2)(x_1{}'^2)}\\
&+a_{12}b_{21}P_2(A)_{(x_1{}^2)(x_{1\prime}x_{2\prime})}P_2(B)_{(x_1x_2)(x_1{}'^2)}+a_{12}b_{21}P_2(A)_{(x_1{}^2)(x_{2\prime}{}^2)}P_2(B)_{(x_2{}^2)(x_1{}'^2)}
\end{aligned} \tag{9}
$$

よって、$P_3(C)_{(x_1{}^3)(x_1{}'^3)}=(P_3(A)P_3(B))_{(x_1{}^3)(x_1{}'^3)}$．また、

$$
\begin{aligned}
P_3(C)_{(x_1{}^2x_2)(x_1{}'x_2{}'^2)}&=c_{11}P_2(C)_{(x_1x_2)(x_2{}'^2)}+c_{12}P_2(C)_{(x_1x_2)(x_1{}'x_2{}')}\\
&=(a_{11}b_{11}+a_{12}b_{21})(P_2(A)_{(x_1x_2)(x_{1\prime}{}^2)}P_2(B)_{(x_1{}^2)(x_2{}'^2)}\\
+P_2(A)_{(x_1x_2)(x_{1\prime}x_{2\prime})}P_2(B)_{(x_1x_2)(x_2{}'^2)}&+P_2(A)_{(x_1x_2)(x_{2\prime}{}^2)}P_2(B)_{(x_2{}^2)(x_1{}'^2)})\\
&+(a_{11}b_{12}+a_{12}b_{22})(P_2(A)_{(x_1x_2)(x_{1\prime}{}^2)}P_2(B)_{(x_1{}^2)(x_1{}'x_2{}')}\\
+P_2(A)_{(x_1x_2)(x_{1\prime}x_{2\prime})}P_2(B)_{(x_1x_2)(x_1{}'x_2{}')}&+P_2(A)_{(x_1x_2)(x_{2\prime}{}^2)}P_2(B)_{(x_2{}^2)(x_1{}'x_2{}')})
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
(P_3(A)P_3(B))_{(x_1{}^2x_2)(x_1{}'x_2{}'^2)} = {} & a_{11}b_{11}P_2(A)_{(x_1x_2)(x_1{}'^2)}P_2(B)_{(x_1{}^2)(x_2{}'^2)} + a_{11}b_{12}P_2(A)_{(x_1x_2)(x_1{}'^2)}P_2(B)_{(x_1{}^2)(x_1{}'x_2{}')} \\
& +a_{11}b_{11}P_2(A)_{(x_1x_2)(x_1{}'x_2{}')}P_2(B)_{(x_1x_2)(x_2{}'^2)} + a_{11}b_{12}P_2(A)_{(x_1x_2)(x_1{}'x_2{}')}P_2(B)_{(x_1x_2)(x_1{}'x_2{}')} \\
& +a_{12}b_{21}P_2(A)_{(x_1x_2)(x_1{}'^2)}P_2(B)_{(x_1{}^2)(x_2{}'^2)} + a_{12}b_{22}P_2(A)_{(x_1x_2)(x_1{}'^2)}P_2(B)_{(x_1{}^2)(x_1{}'x_2{}')} \\
& +a_{11}b_{11}P_2(A)_{(x_1x_2)(x_{2'}{}^2)}P_2(B)_{(x_2{}^2)(x_2{}'^2)} + a_{11}b_{12}P_2(A)_{(x_1x_2)(x_{2'}{}^2)}P_2(B)_{(x_2{}^2)(x_1{}'x_2{}')} \\
& +a_{12}b_{21}P_2(A)_{(x_1x_2)(x_{1'}x_{2'})}P_2(B)_{(x_1x_2)(x_2{}'^2)} + a_{12}b_{22}P_2(A)_{(x_1x_2)(x_{1'}x_{2'})}P_2(B)_{(x_1x_2)(x_1{}'x_2{}')} \\
& +a_{12}b_{21}P_2(A)_{(x_1x_2)(x_{2'}{}^2)}P_2(B)_{(x_2{}^2)(x_2{}'^2)} + a_{12}b_{22}P_2(A)_{(x_1x_2)(x_{2'})}P_2(B)_{(x_2{}^2)(x_1{}'x_2{}')}
\end{aligned}
\tag{11}
$$

よって、$P_3(C)_{(x_1{}^2x_2)(x_1{}'x_2{}'^2)} = (P_3(A)P_3(B))_{(x_1{}^2x_2)(x_1{}'x_2{}'^2)}$.

つまり、全ての $p_k \geq 0$ となる k に対して、以下が成り立つ。

$$
P_{(p_1,\cdots,p_n)(p'_1,\cdots,p'_n)}{}^{(r')}(C) = \sum_{m=1,p_m\geq 1}^{n} c_{km}P_{(p_1,\cdots,(p_k-1),\cdots,p_n)(p'_1,\cdots,(p'_m-1),\cdots,p'_n)}{}^{(r'-1)}(C) \tag{12}
$$

この式は C だけでなく、A,B に対しても成立する。

$$
c_{km} = \sum_{l}^{n} a_{kl}b_{lm} \tag{13}
$$

$$
\begin{aligned}
& P_{(p_1,\cdots,(p_k-1),\cdots,p_n)(p'_1,\cdots,(p'_m-1),\cdots,p'_n)}{}^{(r'-1)}(C) = \\
& \sum_{Pset_{all}} P_{(p_1,\cdots,(p_k-1),\cdots,p_n)(Pset_{all})}{}^{(r'-1)}(A)P_{(Pset_{all})(p'_1,\cdots,(p'_m-1),\cdots,p'_n)}{}^{(r'-1)}(B)
\end{aligned}
\tag{14}
$$

に注意して、

$$
\begin{aligned}
& P_{(p_1,\cdots,p_n)(p'_1,\cdots,p'_n)}{}^{(r')}(C) = \sum_{m=1,p_m\geq 1}^{n} c_{km}P_{(p_1,\cdots,(p_k-1),\cdots,p_n)(p'_1,\cdots,(p'_m-1),\cdots,p'_n)}{}^{(r'-1)}(C) = \\
& \sum_{m=1,p_m\geq 1}^{n}(\sum_{l}^{n} a_{kl}b_{lm})(\sum_{Pset_{all}{}^{(r-1)}} P_{(p_1,\cdots,(p_k-1),\cdots,p_n)(Pset_{all}{}^{(r-1)})}{}^{(r'-1)}(A)P_{(Pset_{all}{}^{(r-1)})(p'_1,\cdots,(p'_m-1),\cdots,p'_n)}{}^{(r'-1)}(B))
\end{aligned}
\tag{15}
$$

一方、$P_{r'}(A)P_{r'}(B)$ の $(p_1,\cdots,p_n)(p'_1,\cdots,p'_n)$ 成分を考慮する。

$$
\begin{aligned}
& \sum_{Pset_{all}{}^{(r)}}(\sum_{m'=1,p_{m'}\geq 1}^{n} a_{k'm'}P_{(p_1,\cdots,(p_{k'}-1),\cdots,p_n)(Pset_{all}{}^{(r)}(p'_{m'}-1))}{}^{(r'-1)}(A)) \\
& (\sum_{m''=1,p_{m''}\geq 1}^{n} b_{k''m''}P_{(Pset_{all}{}^{(r)}(p_{k''}-1))(p'_1,\cdots,(p'_{m''}-1),\cdots,p'_n)}{}^{(r'-1)}(B))
\end{aligned}
\tag{16}
$$

ここで、$\sum_{Pset_{all}{}^{(r)}}(\sum_{m'=1,p_{m'}\geq 1}^{n} a_{k'm'}P_{(p_1,\cdots,(p_{k'}-1),\cdots,p_n)(Pset_{all}{}^{(r)}(p'_{m'}-1))}{}^{(r'-1)}(A))$ を考慮すると、$\sum_{Pset_{all}{}^{(r-1)}}(\sum_{l}^{n} a_{k'l}P_{(p_1,\cdots,(p_{k'}-1),\cdots,p_n)(Pset_{all}{}^{(r-1)})}{}^{(r'-1)}(A))$ となる。(一つ引いて、r で全体を走らせるのは、r-1 で全体を走らせる場合に相当。)

また、k', k'' は任意の数字にできるが、$k' = k, k'' = m' = l$ として、列が $(p'_1,\cdots,p'_n)$ にて、定義されるので、$m'' = m$ となる。

よって、$P_{r'}(A)P_{r'}(B)$ の $(p_1,\cdots,p_n)(p'_1,\cdots,p'_n)$ 成分は、

$$
\sum_{Pset_{all}{}^{(r-1)}}\sum_{l}^{n}(a_{kl}P_{(p_1,\cdots,(p_k-1),\cdots,p_n)(Pset_{all}{}^{(r-1)})}{}^{(r'-1)}(A))(\sum_{m=1,p_m\geq 1}^{n} b_{lm}P_{(Pset_{all}{}^{(r-1)})(p'_1,\cdots,(p'_m-1),\cdots,p'_n)}{}^{(r'-1)}(B))
\tag{17}
$$

ここで、a,b は $Pset_{all}{}^{(}r-1)$ に依存しないため、また、m は b にしか影響しないため、和の順番を入れ替えて、

$$\sum_{m=1,p_m\geq 1}^{n}\sum_{l}^{n}(a_{kl}b_{lm})\sum_{Pset_{all}{}^{(r-1)}}(P_{(p_1,\cdots,(p_k-1),\cdots,p_n)(Pset_{all}{}^{(r-1)})}{}^{(r'-1)}(A)P_{(Pset_{all}{}^{(r-1)})(p'_1,\cdots,(p'_m-1),\cdots,p'_n)}{}^{(r'-1)}(B)) \tag{18}$$

となり、上記の $P_{(p_1,\cdots,p_n)(p'_1,\cdots,p'_n)}{}^{(r')}(C)$ と一致する。