4.2.4.4 を考慮する。(4.58) を踏まえると、

$$Z_{i+1} = Z(a_i, b_i) = \int f_{i+1}(\theta)\, Gam(\theta|a_i, b_i)\, d\theta \tag{1}$$

そうすると、(4.58), (3.22),(3.23),(3.24) より、

$$\begin{aligned}
\mathbb{E}_{q_{i+1}}[\theta] &= \int \theta q_{i+1}(\theta) d\theta = \int \theta \frac{1}{Z_{i+1}} f_{i+1}(\theta)\, Gam(\theta|a_i, b_i)\, d\theta = \frac{1}{Z_{i+1}} \int \theta f_{i+1}(\theta)\, C_G(a_i, b_i) \theta^{a_i - 1} e^{-b_i \theta}\, d\theta \\
&= \frac{1}{Z_{i+1}} \int f_{i+1}(\theta)\, (\frac{{b_i}^{a_i}}{\Gamma(a_i)}) \theta^{(a_i+1)-1} e^{-b_i\theta}\, d\theta = \frac{1}{Z_{i+1}} \int f_{i+1}(\theta)\, (\frac{{b_i}^{a_i+1} a_i}{\Gamma(a_i+1) b_i}) \theta^{(a_i+1)-1} e^{-b_i\theta}\, d\theta \\
&= \frac{a_i}{Z_{i+1} b_i} \int f_{i+1}(\theta)\, (\frac{{b_i}^{a_i+1}}{\Gamma(a_i+1)}) \theta^{(a_i+1)-1} e^{-b_i\theta}\, d\theta = \frac{a_i}{Z_{i+1} b_i} \int f_{i+1}(\theta)\, C_G(a_i+1, b_i) \theta^{(a_i+1)-1} e^{-b_i\theta}\, d\theta \\
&= \frac{a_i}{Z(a_i, b_i) b_i} \int f_{i+1}(\theta)\, Gam(a_i+1, b_i)\, d\theta = \frac{a_i}{Z(a_i, b_i) b_i} Z(a_i+1, b_i) = \frac{Z(a_i+1, b_i) a_i}{Z(a_i, b_i) b_i}
\end{aligned} \tag{2}$$

また、2 次モーメントは

$$\begin{aligned}
\mathbb{E}_{q_{i+1}}[\theta^2] &= \int \theta^2 q_{i+1}(\theta) d\theta = \int \theta^2 \frac{1}{(a_i, b_i)} f_{i+1}(\theta)\, Gam(\theta|a_i, b_i)\, d\theta = \frac{1}{Z(a_i, b_i)} \int \theta^2 f_{i+1}(\theta)\, C_G(a_i, b_i) \theta^{a_i - 1} e^{-b_i \theta}\, d\theta \\
&= \frac{1}{Z(a_i, b_i)} \int f_{i+1}(\theta)\, (\frac{{b_i}^{a_i}}{\Gamma(a_i)}) \theta^{(a_i+2)-1} e^{-b_i\theta}\, d\theta = \frac{1}{Z(a_i, b_i)} \int f_{i+1}(\theta)\, (\frac{{b_i}^{a_i+2} a_i(a_i+1)}{\Gamma(a_i+2) {b_i}^2}) \theta^{(a_i+2)-1} e^{-b_i\theta}\, d\theta \\
&= \frac{a_i(a_i+1)}{Z(a_i, b_i) {b_i}^2} \int f_{i+1}(\theta)\, (\frac{{b_i}^{a_i+2}}{\Gamma(a_i+2)}) \theta^{(a_i+2)-1} e^{-b_i\theta}\, d\theta = \frac{a_i(a_i+1)}{Z(a_i, b_i) {b_i}^2} \int f_{i+1}(\theta)\, C_G(a_i+2, b_i) \theta^{(a_i+2)-1} e^{-b_i\theta}\, d\theta \\
&= \frac{a_i(a_i+1)}{Z(a_i, b_i) {b_i}^2} \int f_{i+1}(\theta)\, Gam(a_i+2, b_i)\, d\theta = \frac{a_i(a_i+1)}{Z(a_i, b_i) {b_i}^2} Z(a_i+2, b_i) = \frac{Z(a_i+2, b_i) a_i(a_i+1)}{Z(a_i, b_i) {b_i}^2}
\end{aligned} \tag{3}$$

本当はガンマ分布の十分統計量を一致させる必要があるが、例えば、文献 [47] の (10)(11) 式の間にあるように解析的に求まらないので、1,2 次のモーメントを揃える。

すると、ガンマ分布の平均、分散を考慮して、

$$\mathbb{E}_{q_{i+1}}[\theta] = \frac{Z(a_i+1, b_i) a_i}{Z(a_i, b_i) b_i} = \frac{a_{i+1}}{b_{i+1}} \tag{4}$$

$$\mathbb{E}_{q_{i+1}}[(\theta - \mathbb{E}_{q_{i+1}}[\theta])^2] = \mathbb{E}_{q_{i+1}}[\theta^2] - \mathbb{E}_{q_{i+1}}[\theta]^2 = \frac{Z(a_i+2, b_i) a_i(a_i+1)}{Z(a_i, b_i) {b_i}^2} - \left(\frac{Z(a_i+1, b_i) a_i}{Z(a_i, b_i) b_i} \right)^2 = \frac{a_{i+1}}{{b_{i+1}}^2} \tag{5}$$

この 2 式を用いて、a_{i+1}, b_{i+1} を求める。(4) を (5) で割って、

$$b_{i+1} = \left(\frac{Z(a_i+2, b_i)(a_i+1)}{Z(a_i+1, b_i) b_i} - \left(\frac{Z(a_i+1, b_i) a_i}{Z(a_i, b_i) b_i} \right) \right)^{-1} \tag{6}$$

(4) に (6) をかけて、

$$a_{i+1} = \left(\frac{Z(a_i+2, b_i) Z(a_i, b_i)(a_i+1)}{{Z(a_i+1, b_i)}^2 a_i} - 1 \right)^{-1} \tag{7}$$