Homework # 1

Problem 4

$$f(\underset{\sim}{x}) = \sigma\left(\log\left(5\left(\max(x_1, x_2)\frac{x_3}{x_4} - (x_5 + x_6)\right)\right) + \frac{1}{2}\right)$$

where $\sigma(x) = \dfrac{1}{1+e^{-x}}$

$\Rightarrow$ Replace $\max\{x_1, x_2\}$ by a smooth maximum function. Here, I use p-norm function:

$$\max(x_1, x_2) = y = \left(x_1^p + x_2^p\right)^{\frac{1}{p}} \quad ; p = 10$$

$$\rightarrow \frac{\partial y}{\partial x_1} = \left(x_1^p + x_2^p\right)^{\frac{1}{p}-1} \cdot x_1^{p-1}$$

$$\frac{\partial y}{\partial x_2} = \left(x_1^p + x_2^p\right)^{\frac{1}{p}-1} \cdot x_2^{p-1}$$

$\Rightarrow$ let: $z = 5\left[y\dfrac{x_3}{x_4} - x_5 - x_6\right]$

$$\frac{\partial z}{\partial x_1} = 5\frac{\partial y}{\partial x_1} \quad ; \quad \frac{\partial z}{\partial x_2} = 5\frac{\partial y}{\partial x_2}$$

$$\frac{\partial z}{\partial x_3} = 5y\frac{1}{x_4} \quad ; \quad \frac{\partial z}{\partial x_4} = -5y\frac{x_3}{x_4^2}$$

$$\frac{\partial z}{\partial x_5} = -5 ; \quad \frac{\partial z}{\partial x_6} = -5$$

+) Let $u = \log(z) + \frac{1}{2}$

+) We have:

$$f(\underset{\sim}{x}) = \sigma(u)$$

+) Gradient $\frac{\partial f}{\partial x_i} = ?$

$$\frac{\partial f}{\partial x_i} = \frac{\partial \sigma}{\partial u} \frac{\partial u}{\partial z} \frac{\partial z}{\partial x_i}$$

$$\frac{\partial \sigma}{\partial u} = \sigma(1-\sigma) \; ; \; \frac{\partial u}{\partial z} = \frac{1}{z}$$

+) We have

$$\nabla f(\underset{\sim}{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \\ \frac{\partial f}{\partial x_4} \\ \frac{\partial f}{\partial x_5} \\ \frac{\partial f}{\partial x_6} \end{bmatrix} = \begin{bmatrix} \sigma(1-\sigma) \cdot \frac{1}{z} \cdot (x_1^p + x_2^p)^{\frac{1}{p}-1} \cdot x_1^{p-1} \\ \sigma(1-\sigma) \cdot \frac{1}{z} \cdot (x_1^p + x_2^p)^{\frac{1}{p}-1} \cdot x_2^{p-1} \\ \sigma(1-\sigma) \cdot \frac{1}{z} \cdot 5y \frac{1}{x_4} \\ \sigma(1-\sigma) \cdot \frac{1}{z} \cdot 5y \frac{-x_3}{x_4^2} \\ \sigma(1-\sigma) \cdot \frac{1}{z} \cdot (-5) \\ \sigma(1-\sigma) \cdot \frac{1}{z} \cdot (-5) \end{bmatrix}$$

$\Rightarrow$ Evaluate at $\underset{\sim}{x} = (5, -1, 6, 12, 7, -5)$

$$f(\underset{\sim}{x}) = 0.8048$$

$$\nabla f(\underset{\sim}{x}) = \begin{bmatrix} 0.0628 \\ 0 \\ 0.1309 \\ -0.0655 \\ -0.3142 \\ -0.3142 \end{bmatrix}$$

Problem 1

$$Ex[payout] = \$1 \times \frac{1}{6} + \frac{-\$1}{4} \times \frac{5}{6} = -\$0.042$$

$\Rightarrow$ We are expected to pay money to Bob!

Problem 2

$$(x-\mu)^2 = x^2 - 2x\mu + \mu^2$$

$$E[(x-\mu)^2] = E[x^2 - 2x\mu + \mu^2]$$
$$= E[x^2] - 2E[x\mu] + \mu^2$$
$$= E[x^2] - 2\mu E[x] + \mu^2$$
$$= E[x^2] - \mu^2 \quad ; (\mu = E[x])$$
$$= E[x^2] - (E[x])^2$$

Problem 3

$$z = Wx + b$$

$$z = \begin{bmatrix} 0.2 & -0.5 & 0.1 \\ 0.7 & 0.3 & -0.8 \end{bmatrix} \begin{bmatrix} 1.0 \\ -2.0 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix} = \begin{bmatrix} 1.35 \\ -0.5 \end{bmatrix}$$