Problem 3

In this problem you want to make a plot of the $log_{10}T - log_{10}\rho$ plane showing which form of pressure (ideal gas, degeneracy, or radiation) dominates in each region. Consider density to range from $10^{-4}\ \mathrm{g\ cm^{-3}} < \rho < 10^{10}\ \mathrm{g\ cm^{-3}}$ and temperature to range from $10\ \mathrm{K}\ < T < 10^{10}\ \mathrm{K}$. We can estimate the boundaries between the different pressure regimes by setting the pressures equal. For instance, the boundary between ideal gas and radiation is given by:

$$\frac{\rho k T}{\mu m_u} \sim \frac{1}{3} a T^4$$

You can choose values of μ and μ_e appropriate for the Sun. Draw the following on your plot:

(a) the boundary between ideal gas and radiation

(b) the boundary between ideal gas and electron degeneracy

(c) the boundary non-relativistic and relativistic electron degeneracy

(d) draw a point indicate approximately the central conditions of the Sun are

Solution

(a) the boundary between ideal gas and radiation

The boundary between ideal gas and radiation is defined by $P_{\mathrm{gas}} = P_{\mathrm{rad}}$:

```
syms P_rad P_gas P_e P_e_r rho a T k_B mu m_H K_1 K_2 mu_e positive
digits(5)
P_rad=a*T^4/3;
P_gas=rho*k_B*T/mu/m_H;
disp(P_gas==P_rad)
```

$$\frac{T\, k_B\, \rho}{m_H\, \mu} = \frac{T^4\, a}{3}$$

Express T as a function of ρ

```
T1=solve(P_gas==P_rad,T)
```

T1 =

$$\left(\frac{3\, k_B\, \rho}{a\, m_H\, \mu}\right)^{1/3}$$

Put in values (in SI units)

```
T1=simplify(subs(T1,[k_B a m_H rho],[const.k_B 8*pi^5*const.k_B^4/const.c^3/15/
const.h^3 const.m_H symunit().kg/symunit().m^3*rho]))
```

T1 =

$$\frac{3.2065\text{e+}6\,\rho^{1/3}}{\mu^{1/3}}\ \mathrm{K}$$

(b) the boundary between ideal gas and degenarate electron

The boundary between ideal gas and (non-relativistic) degenerate electron is defined by $P_{\mathrm{gas}} = P_{\mathrm{e}}$

```
P_e=K_1*(rho/mu_e)^(5/3);
disp(P_gas==P_e)
```

$$\frac{T\,k_B\,\rho}{m_H\,\mu} = K_1\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

Express T as a function of ρ

```
T2=simplify(solve(P_gas==P_e,T))
```

T2 =

$$\frac{K_1\,m_H\,\mu\,\rho^{2/3}}{k_B\,{\mu_e}^{5/3}}$$

Put in values (in SI units)

```
T2=simplify(subs(T2,[k_B K_1 m_H rho],[const.k_B
1e7*symunit().m^4*symunit().kg^(-2/3)/symunit().s^2 const.m_H symunit().kg/
symunit().m^3*rho]))
```

T2 =

$$\frac{1202.7\,\mu\,\rho^{2/3}}{{\mu_e}^{5/3}}\ \mathrm{K}$$

Similarly, we can find the boundary between ideal gas and relativistic degenerate electron as

```
P_e_r=K_2*(rho/mu_e)^(4/3);
T3=simplify(solve(P_gas==P_e_r,T));
T3=simplify(subs(T3,[k_B K_2 m_H rho],[const.k_B
1.24e10*symunit().m^3*symunit().kg^(-1/3)/symunit().s^2 const.m_H symunit().kg/
symunit().m^3*rho]))
```

T3 =

$$\frac{1.4914\text{e+}6\,\mu\,\rho^{1/3}}{{\mu_e}^{4/3}}\ \mathrm{K}$$

(c) the boundary non-relativistic and relativistic degenararte electron

The boundary non-relativistic and relativistic electron degeneracy is given by $P_{\mathrm{e}} = P_{\mathrm{e,r}}$

```
disp(P_e==P_e_r)
```

$$K_1\left(\frac{\rho}{\mu_e}\right)^{5/3} = K_2\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

Plug in numbers and solve for ρ:

```
rho1=solve(P_e==P_e_r,rho)
```

rho1 =

$$\frac{{K_2}^3\,\mu_e}{{K_1}^3}$$

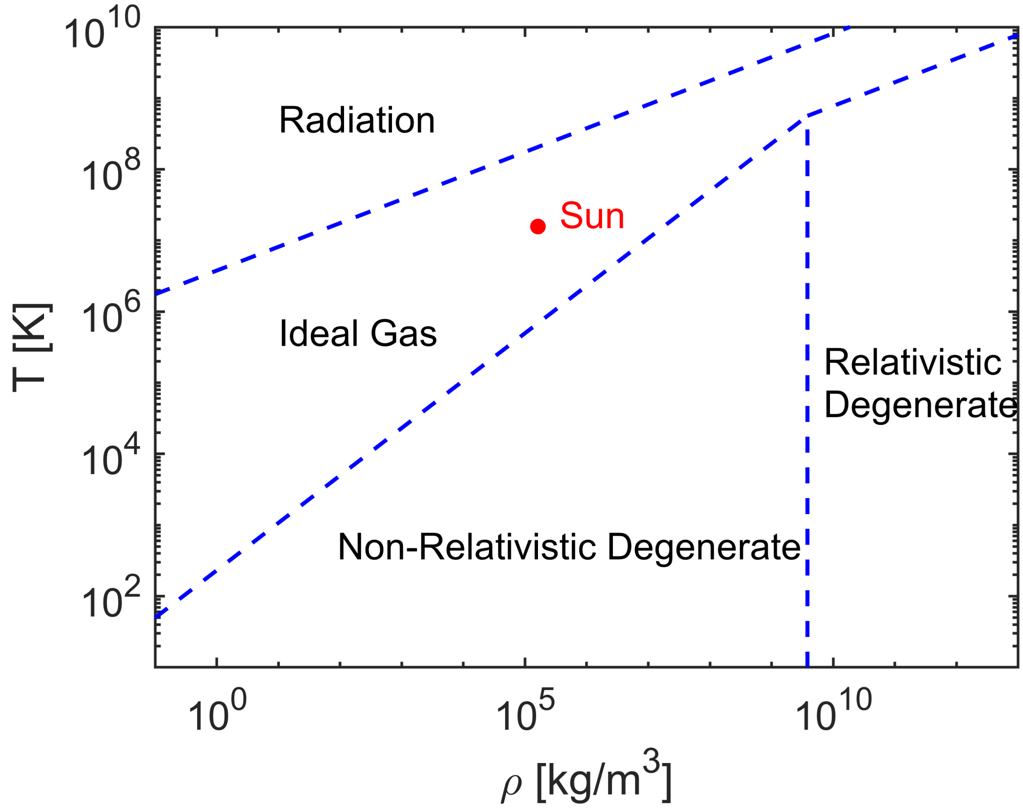
```
rho1=vpa(subs(rho1,[K_1 K_2],[1e7*symunit().m^4*symunit().kg^(-2/3)/symunit().s^2
1.24e10*symunit().m^3*symunit().kg^(-1/3)/symunit().s^2]))
```

rho1 =

$$1.9066e+9\,\mu_e\,\frac{\mathrm{kg}}{\mathrm{m}^3}$$

(d) Plot the diagram and draw the point indicates Sun

```
%Take mu = 0.61 and mu_e = 2.
T1=separateUnits(subs(T1,[mu mu_e],[0.61 2]));
T2=separateUnits(subs(T2,[mu mu_e],[0.61 2]));
T3=separateUnits(subs(T3,[mu mu_e],[0.61 2]));
rho1=double(separateUnits(subs(rho1,[mu mu_e],[0.61 2])));
%Plot the boundary between ideal gas and radiation
fplot(T1,"LineWidth",1.5,"LineStyle","--","Color","b")
hold on
%Plot the boundary between ideal gas and non-relativistic degenerate electron
fplot(T2,[0.1 rho1],"LineWidth",1.5,"LineStyle","--","Color","b")
%Plot the boundary between ideal gas and relativistic degenerate electron
fplot(T3,[rho1 1e13],"LineWidth",1.5,"LineStyle","--","Color","b")
%Plot the boundary between non-relativistic degenerate and relativistic degenerate
plot([rho1 rho1],[10
subs(T2,rho,rho1)],"LineWidth",1.5,"LineStyle","--","Color","b")
%Plot the position of the Sun. The Sun's data is from Nasa
plot(1.622e5,1.571e7,".","MarkerSize",20,"Color","r")
%Set logarithmic scale and axis range
set(gca(),'XScale','log','YScale','log')
xlim([0.1 1e13]);ylim([10 1e10]);
set(gca(),"FontSize",15,"LineWidth",1)
%Set lable
xlabel("\rho [kg/m^3]");ylabel("T [K]");
%Set Text
text(1.622e5+2e5,1.571e7+7e6,"Sun","FontSize",14,"Color","r")
text(9e1,5e2,"Non-Relativistic Degenerate","FontSize",14,"Color","k")
text(7e9,1e5,["Relativistic" "Degenerate"],"FontSize",14,"Color","k")
text(1e1,5e5,"Ideal Gas","FontSize",14,"Color","k")
text(1e1,5e8,"Radiation","FontSize",14,"Color","k")
```



This shows the Sun's model is well within the ideal-gas region of the equation of state.