Analytical lower bound on query complexity for universal transformations of unitary operations

Based on arXiv:2405.07625

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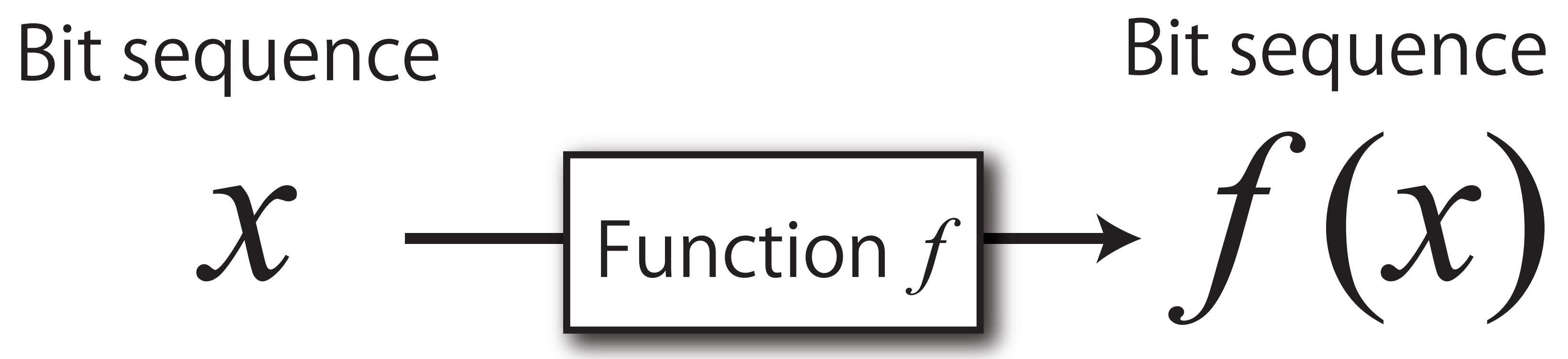
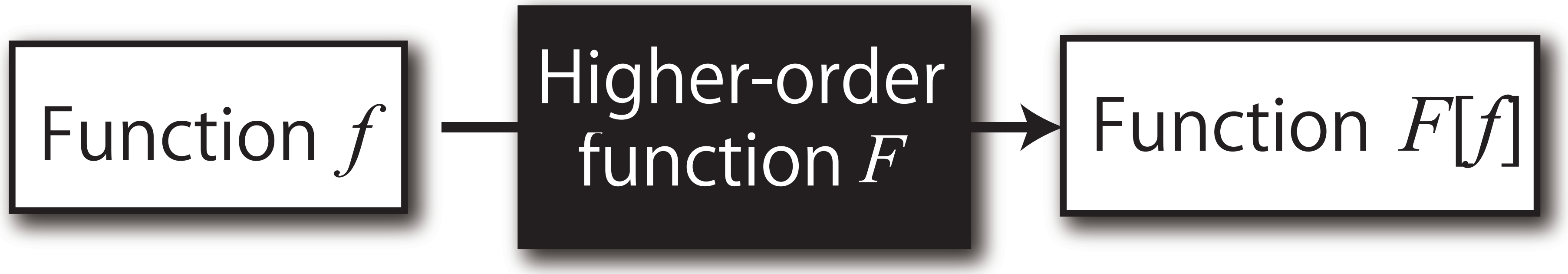
QIP2025

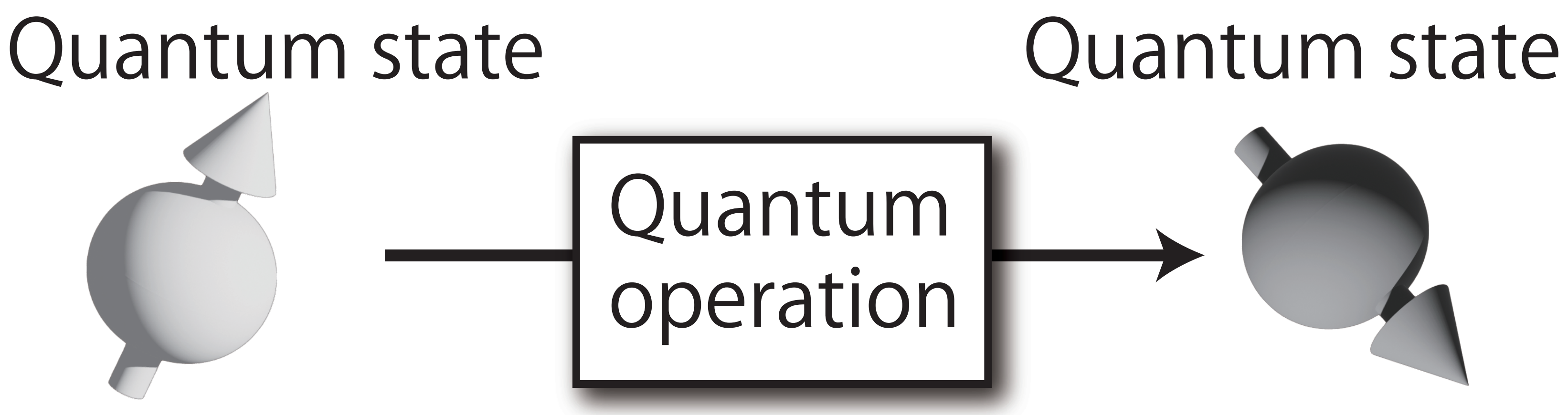
Higher-order quantum operation

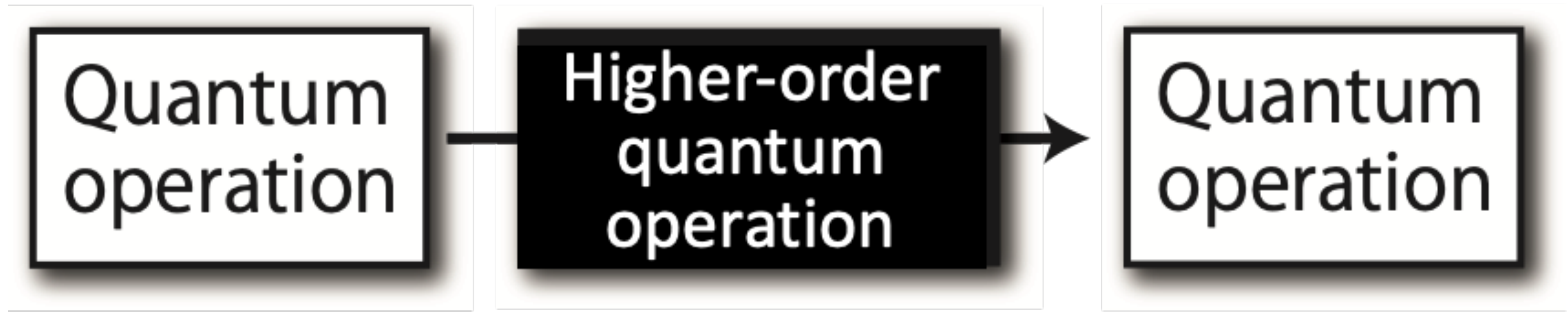
Higher-order function in classical information processing





Higher-order quantum operation in quantum information processing



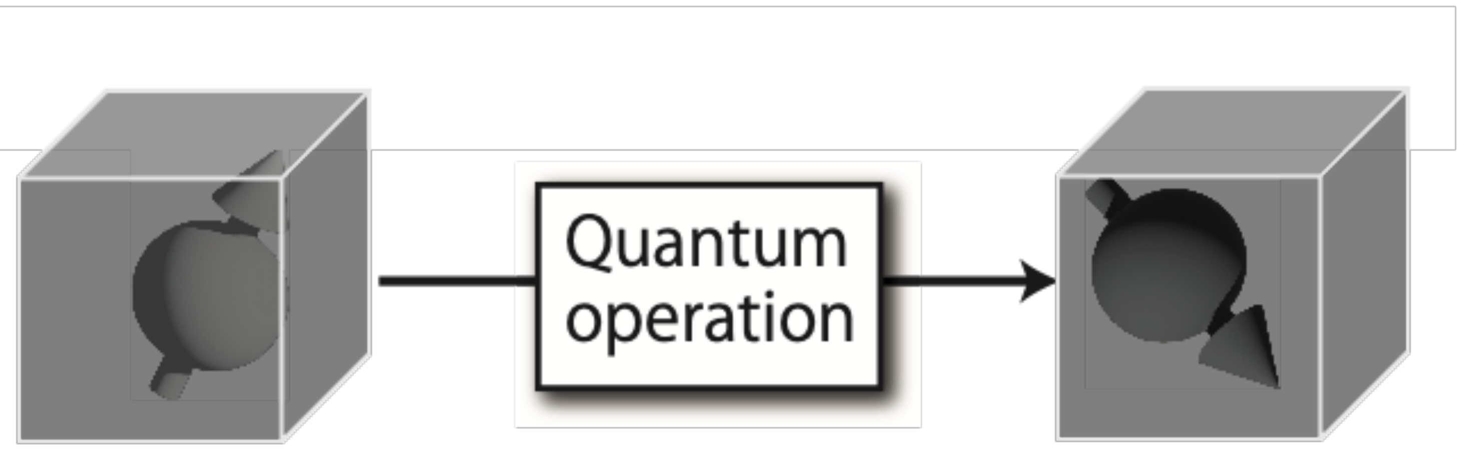


Higher-order quantum operation = Transformation of quantum operation

Universal transformation of q. states and operations

Universal transformation of quantum states

Given ρ



$\sigma = f(\rho)$

Unknown state

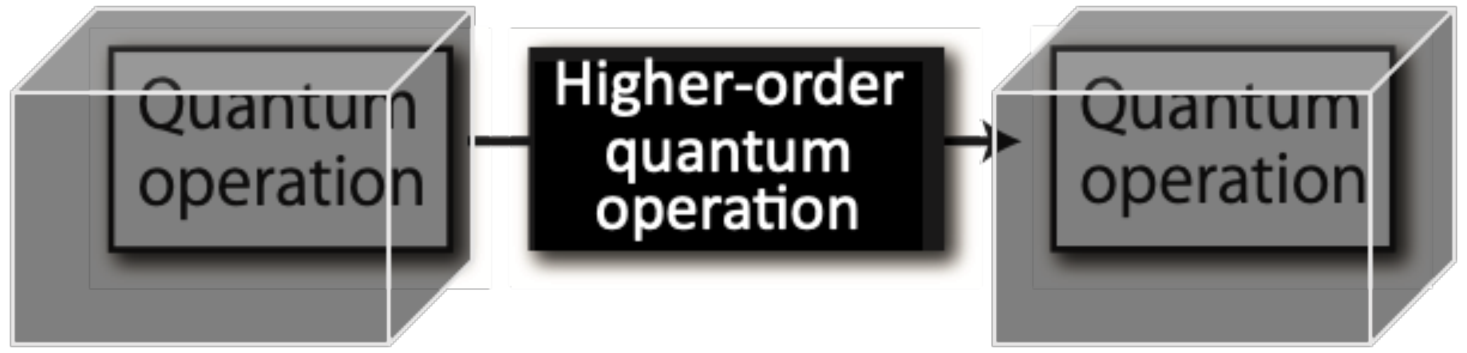
This is NOT $|0\rangle \mapsto |\psi\rangle$

Eg. State cloning $\rho \mapsto \rho \otimes \rho$

Universal NOT $\rho \mapsto \rho^{\perp}$

Universal transformation of quantum operations

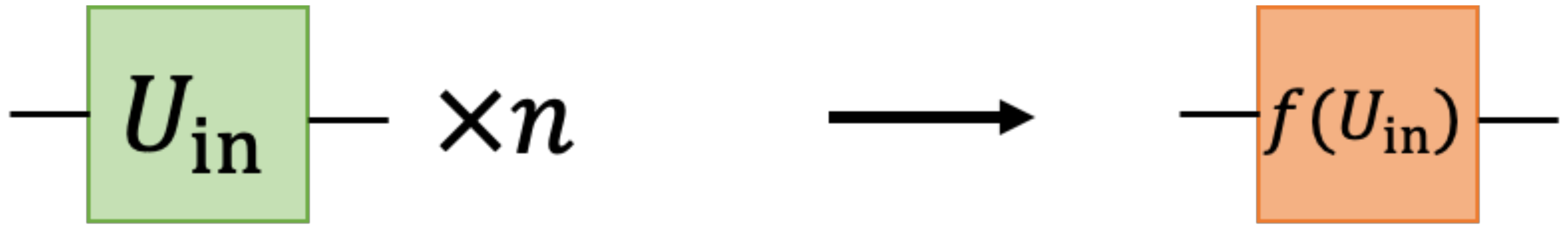
Given Φ



$f(\Phi)$

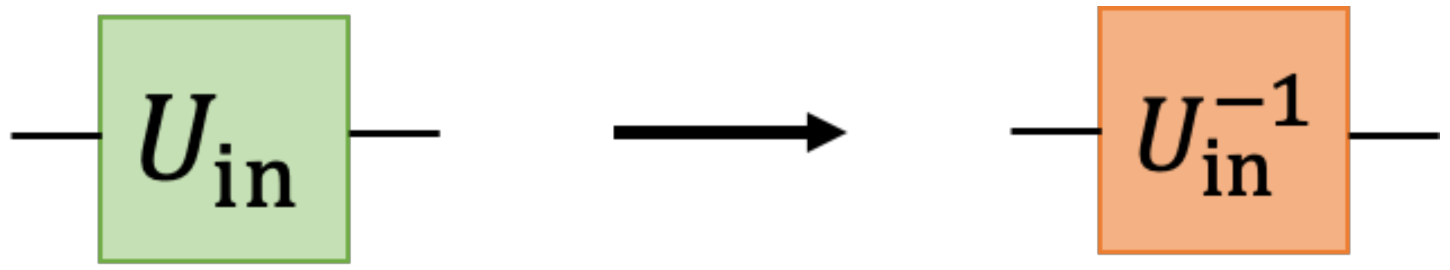
Unknown operation

In particular, we consider



Unknown unitary

Example: Unitary inversion



Simulation of "time inversion":

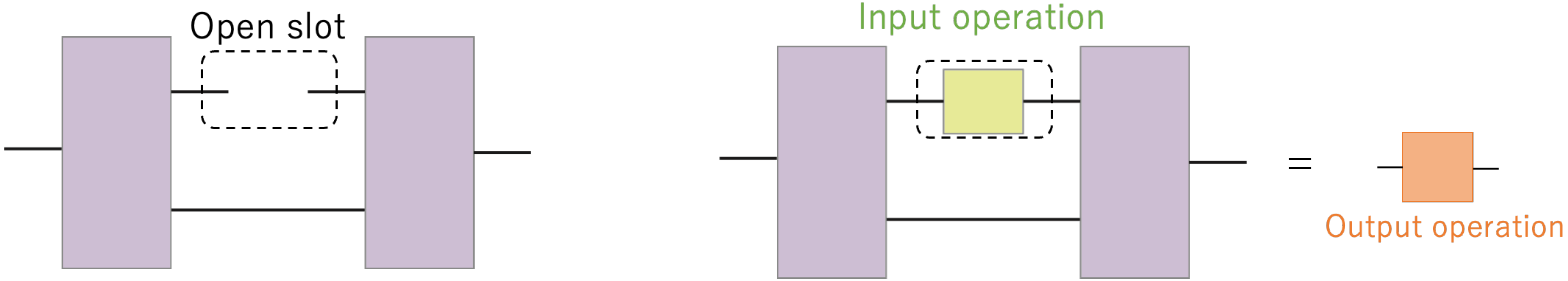
$$U_{\mathrm{in}} = e^{-iHt} \mapsto U_{\mathrm{in}}^{-1} = e^{iHt}$$

Applications
- Quantum control
- Learning (e.g. OTOC)

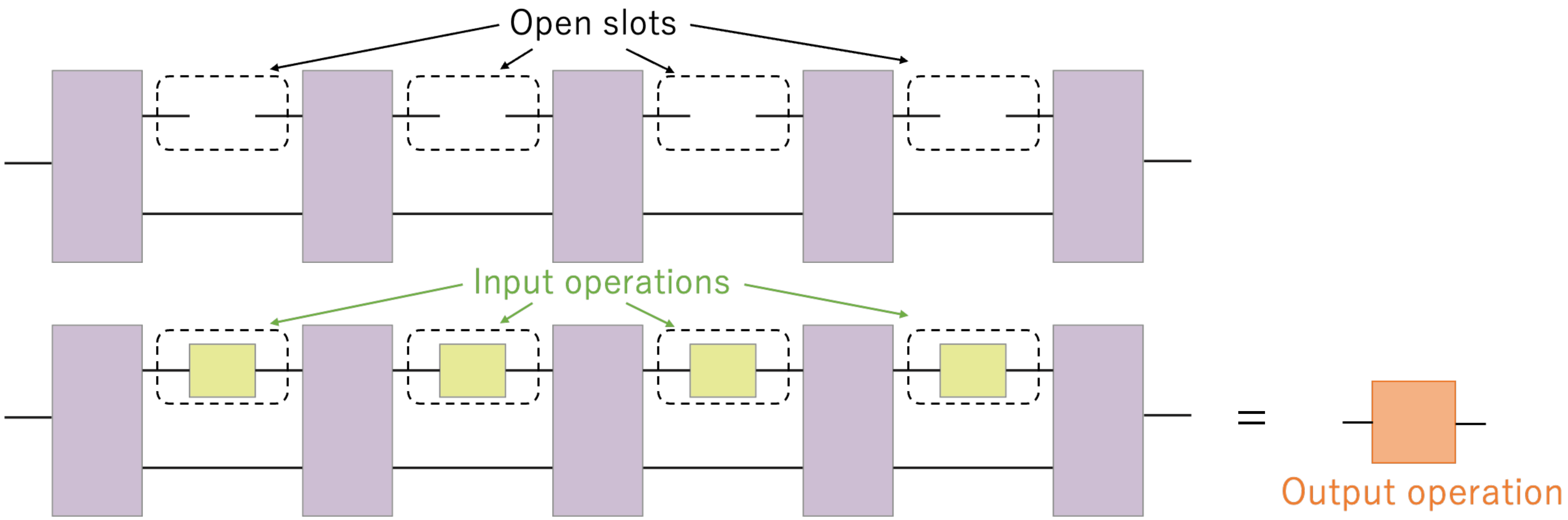
Quantum comb

Circuit implementation of higher-order quantum operation

Single slot:



Multi slot:



Deterministic, exact and universal transformations

No-go theorems on universal transformations

States: **Impossible**, e.g., no-cloning theorem, no-universal-NOT theorem → Only prob./approx.

How about universal transformations of unitary operations?

$$-\boxed{U_{\mathrm{in}}}- \longrightarrow -\boxed{U_{\mathrm{in}}^{-1}}-$$ is impossible, but

G. Chiribella and D. Ebler, NJP 18, 093053 (2016).

SY, A. Soeda, M. Murao, PRL 131, 120602 (2023).

Y.-A. Chen et al. arXiv:2403.04704.

$$\overbrace{-\boxed{U_{\mathrm{in}}}- \cdots -\boxed{U_{\mathrm{in}}}-}^{n \text{ calls}} \longrightarrow -\boxed{U_{\mathrm{in}}^{-1}}-$$ is **POSSIBLE** for $n = O(d^2)$!

Other possible transformations

- Complex conjugation: $U_{\mathrm{in}}^{\otimes n} \mapsto U_{\mathrm{in}}^{*}$ for $n = d - 1$
- Transposition: $U_{\mathrm{in}}^{\otimes n} \mapsto U_{\mathrm{in}}^{T}$ for $n = O(d^2)$

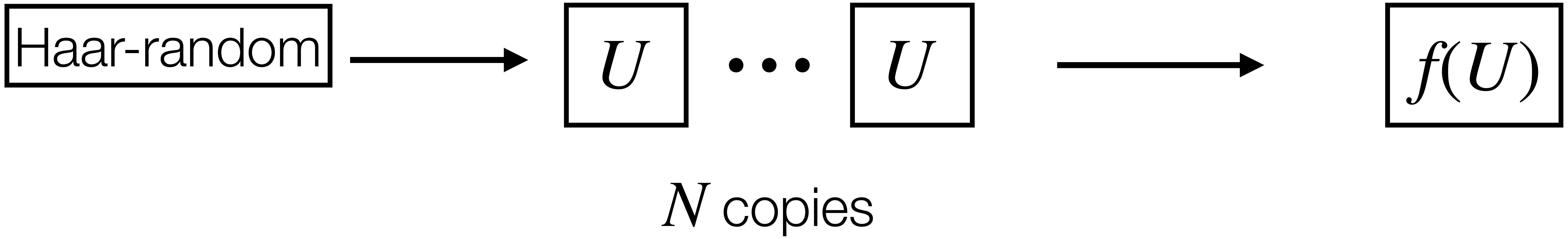
Research question:

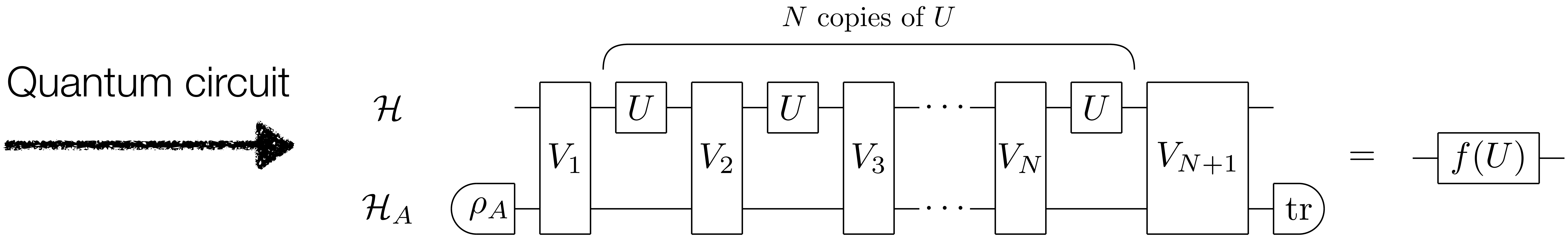
What is the fundamental limit of universal transformations of unitary operations?

Outline of this talk

- **Problem setting:**
  Query complexity for universal transformation of unitary operations
- Main result:
  Lower bound of query complexity based on SDP
- Proof techniques
- Extension to relaxed situations:
  Subgroup and probabilistic settings
- Conclusion & outlook

Task: Universal transformation of unitary operations





Def: [Query complexity of $f : \mathrm{SU}(d) \to \mathrm{SU}(d)$] = $\min N$

Problem setting

Question

What is the fundamental limit of universal transformation of unitary operations?

= Lower bound on the query complexity of $f : \mathrm{SU}(d) \to \mathrm{SU}(d)$

Assumption

Suppose f is differentiable, i.e., $f(e^{i\epsilon H}) = [I + i\epsilon g(H) + O(\epsilon^2)]f(I)$ holds for

- $H \in \mathfrak{su}(d)$: Hermitian (traceless) matrix
- $g : \mathfrak{su}(d) \to \mathfrak{su}(d)$: linear map

Eg. $f(U) = U^{-1}, U^*, U^T, U^n$

Previous works on query complexity

Polynomial method

$F(|U\rangle\rangle\langle\langle U|) = |f(U)\rangle\rangle\langle\langle f(U)|$ should have polynomial degree $\leq n$

e.g. $f(U) = U^\dagger$ → Polynomial degree $d-1$ → $n \geq d-1$

Non-tight

J. Miyazaki et al. PRR 1, 013007 (2019).

Topological method

Property on continuous function $\phi : \mathrm{SU}(d) \to S^1$ to show no-go for controllization

Z. Gavorová et al. PRA 109, 032625 (2024).

Non-universal

Numerical method

Solve the optimization problem of fidelity for a given d, n

M. Quintino and D. Ebler, Quantum 6, 679 (2022).

Non-scalable

We need tight, universal, and scalable lower bound

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Main result

<u>Theorem 1</u>

Query complexity of $f \geq \quad \min \operatorname{tr}\beta$

s.t. $\beta \in \mathscr{L}(\mathbb{C}^d), J_g + \beta \otimes I \geq 0$

where $J_g := \sum_{i,j \neq (0,0)} X^i Z^j \otimes g(X^i Z^j)$: Choi matrix of g cf. $f(e^{i\epsilon H}) = [I + i\epsilon g(H) + O(\epsilon^2)]f(I)$

X, Z: Generalized Pauli matrix

<u>Remark</u>

- This result can be applied to **<u>any</u>** differentiable function $f : \mathrm{SU}(d) \to \mathrm{SU}(d)$
- RHS: semidefinite programming (SDP) which does not depend on n → **<u>Scalable</u>**
- This lower bound is **<u>tight</u>** for some cases (in next slide)

Main result

Theorem 2

Inversion: [Query complexity of $f(U) = U^{-1}$] $\geq d^2$

Transposition: [Query complexity of $f(U) = U^T$] $\geq 4(d=2), d+3(d \geq 3)$

Complex conjugation: [Query complexity of $f(U) = U^*$] $\geq d-1$

Thm. 1 $+\alpha$

Tightness of the lower bound

Function	Lower bound	Minimum known	
$f(U) = U^{-1}$	$n \geq d^2$	$n \leq 4\ (d=2), \lesssim \frac{\pi}{2} d^2\ (d \geq 3)$	✔ Matching lower bound
$f(U) = U^T$	$n \geq 4\ (d=2), d+3\ (d \geq 3)$	$n \leq 4\ (d=2), \lesssim \frac{\pi}{2} d^2\ (d \geq 3)$	△ Only tight for $d=2$
$f(U) = U^*$	$n \geq d-1$	$n \leq d-1$	✔✔ Tight

SY, A. Soeda, M. Murao, PRL 131, 120602 (2023).

Y.-A. Chen et al. arXiv:2403.04704.

J. Miyazaki et al. PRR 1, 013007 (2019).

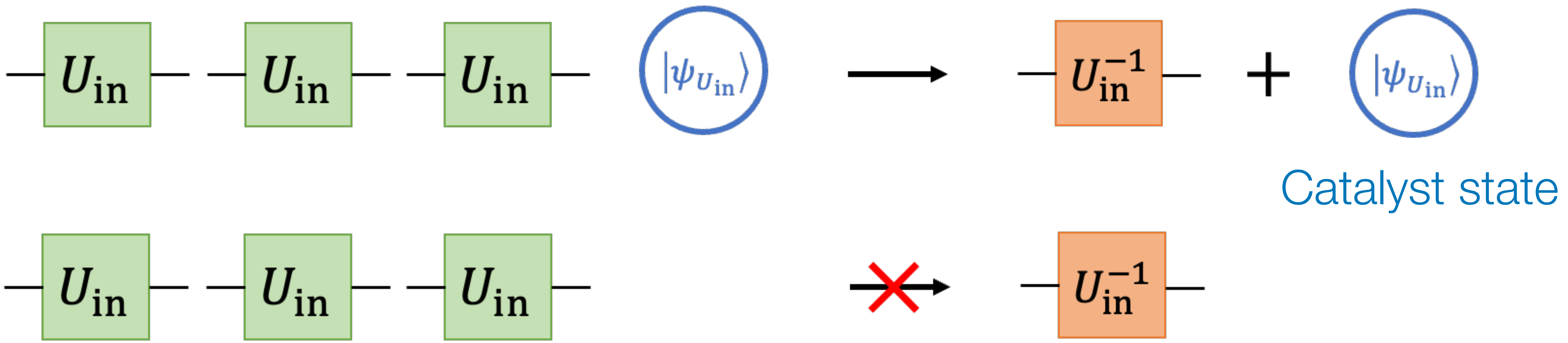
Bonus result: No-go on catalytic transformation

Catalytic transformation

SY, A. Soeda, M. Murao, PRL 131, 120602 (2023).

M. Quintino and D. Ebler, Quantum 6, 679 (2022)

Unitary inversion for $d = 2$



Theorem 3

If the SDP lower bound is achievable, there is no catalytic transformation

→ No catalytic transformation for $f(U) = U^*$

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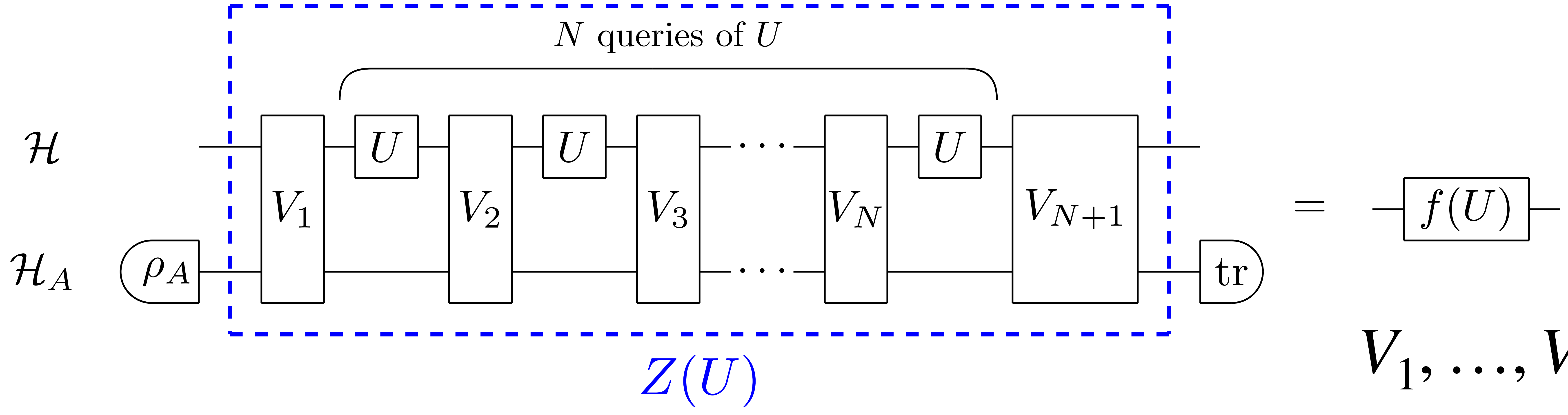
Proof of Theorem 1

Idea: Differentiation



$V_1, \ldots, V_{N+1}$: Unitary

Differentiating $Z(e^{i\epsilon H})$ with respect to ϵ, we obtain

$$\mathscr{E}(H) = g(H) + \alpha(H)I, \qquad \alpha(H) \in \mathbb{R} \text{ :Global phase}$$

where $\mathscr{E}$ is a CP map defined by

$$\mathscr{E}(H) = \sum_{s=1}^{N} \sum_{jk} M_{jk}^{(s)\dagger} H M_{jk}^{(s)}, \qquad \sum_{jk} M_{jk}^{(s)} \otimes |j\rangle\langle k| := V_s \cdots V_1$$

Proof of Theorem 1

$$\mathscr{E}(H) = g(H) + \alpha(H)I,$$
$$\mathscr{E}(H) = \sum_{s=1}^{N} \sum_{jk} M_{jk}^{(s)\dagger} H M_{jk}^{(s)}$$
$$\sum_{jk} M_{jk}^{(s)} \otimes |j\rangle\langle k| := V_s \cdots V_1$$

$\Rightarrow$

$$\mathscr{E}(H) = g(H) + \alpha(H)I$$
$\mathscr{E}$ is CP
$$\mathscr{E}(I) = NI$$

Choi matrix $\longrightarrow$

$$J_{\mathscr{E}} = J_g + \beta \otimes I \geq 0$$
$$N = \frac{1}{d}\mathrm{tr}\mathscr{E}(I) = \frac{1}{d}\mathrm{tr}J_{\mathscr{E}}$$

SDP constraints!

Remark. Only considers necessary conditions → Our argument does not imply achievability

Similar technique is used in our concurrent work (H. Kristjánsson et al. arXiv:2409.18420. Poster No. 77 (Thursday))

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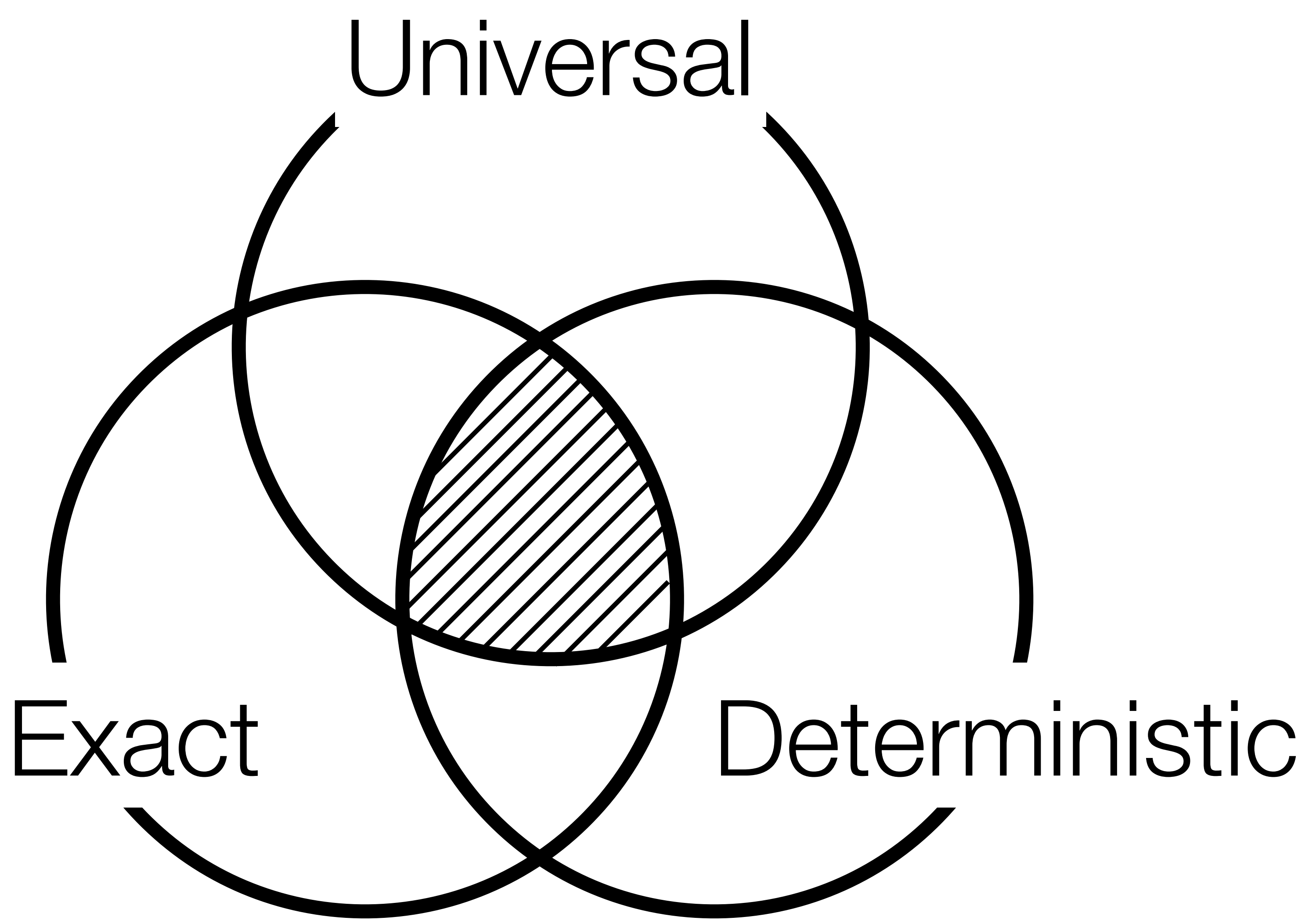
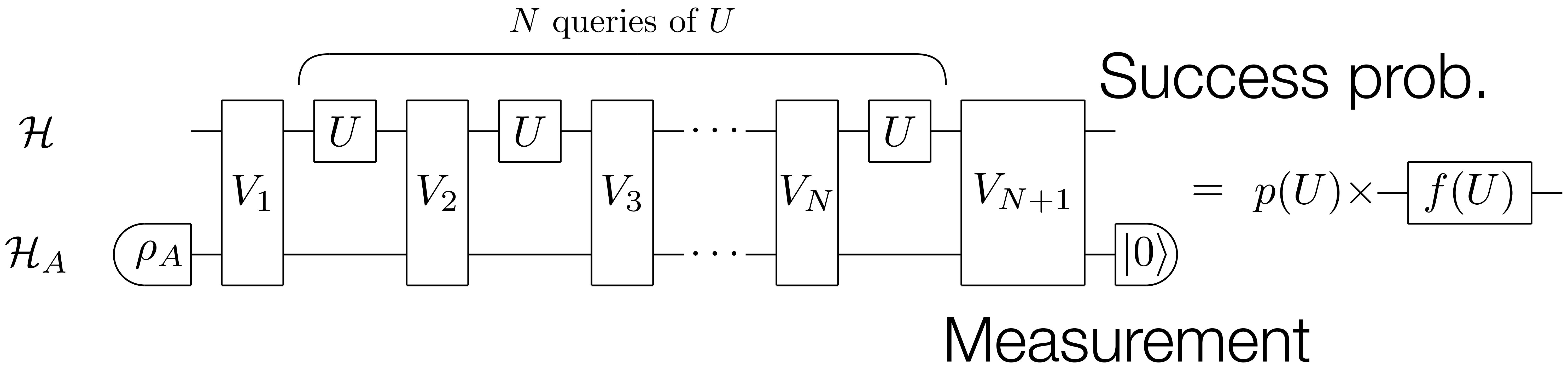
Extension to relaxed settings



Maybe overkilling for practical applications

→ Relaxed setting

- Probabilistic setting



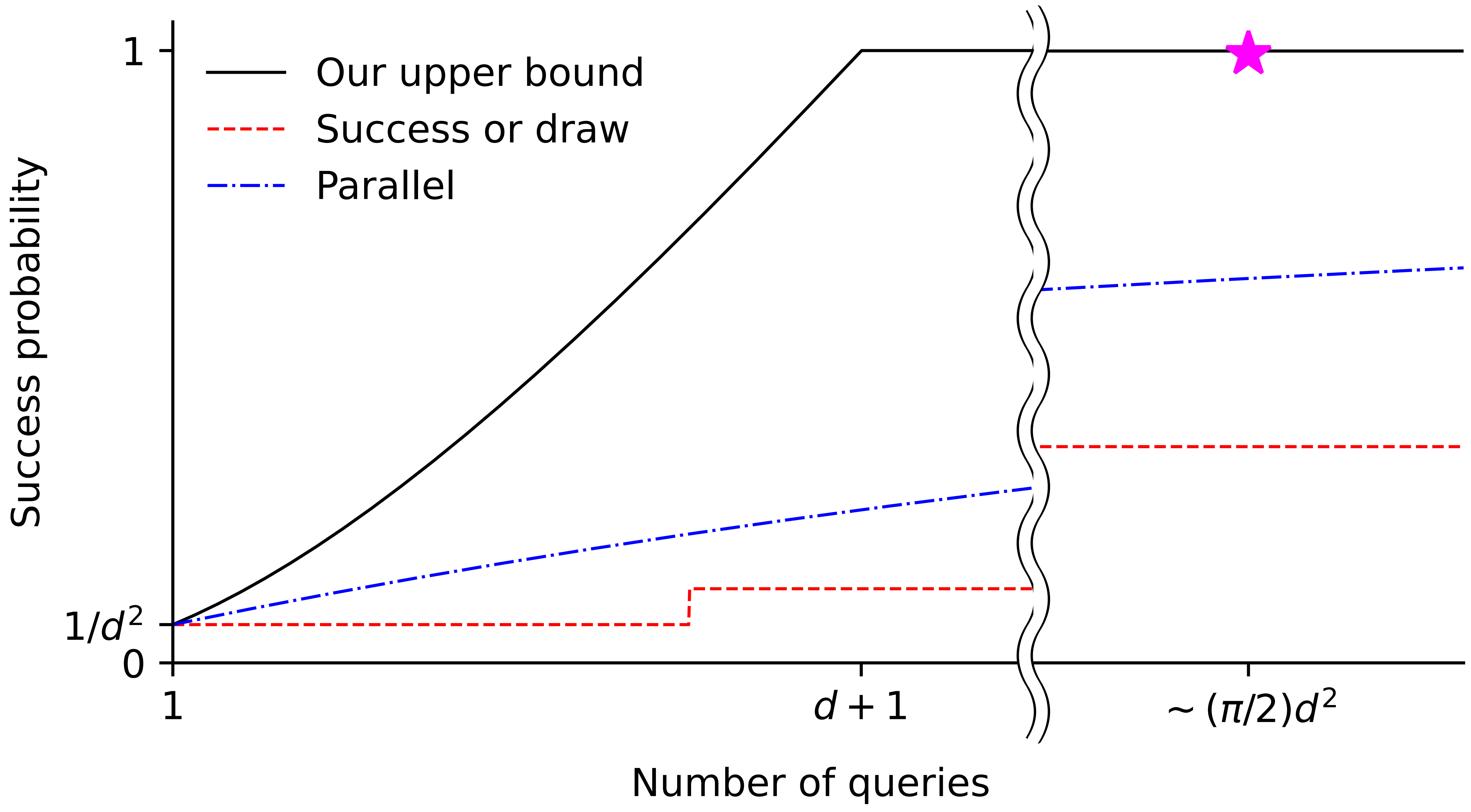
- Partial information of input unitary is given

$$U \in G \subset \mathrm{SU}(d) \qquad G\text{: subgroup}$$

We also obtain SDP lower bound for the query complexity in these relaxed settings

Example: Probabilistic unitary transposition

$$f(U) = U^T$$



Upper bound on the success probability

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Conclusion

- Non-trivial universal transformation of unitary operation is possible

$$\underbrace{\boxed{U}\cdots\boxed{U}}_{N \text{ copies}} \longrightarrow \boxed{f(U)}$$

- We provide tight, universal, and scalable SDP to bound N

Function	Lower bound	Minimum known
$f(U) = U^{-1}$	$n \geq d^2$	$n \leq 4\ (d=2), \lesssim \frac{\pi}{2} d^2\ (d \geq 3)$
$f(U) = U^T$	$n \geq 4\ (d=2), d+3\ (d \geq 3)$	$n \leq 4\ (d=2), \lesssim \frac{\pi}{2} d^2\ (d \geq 3)$
$f(U) = U^*$	$n \geq d-1$	$n \leq d-1$

- Proof technique: differentiation
- Extend to probabilistic/partially-known settings

Outlook

- We derive the SDP lower bound based on the differentiation

→ Only considers local properties

$\boxed{\textbf{Q1.}\text{ Tighter lower bound by higher-order derivatives?}}$

- Our lower bound for unitary transposition does not match the minimum known number

Lower bound: $N = \Omega(d)$

Minimum known: $N = O(d^2)$

Q2. Unitary transposition protocol with $N = O(d)$?

or

Q2'. Tighter lower bound for unitary transposition to provide $N = \Omega(d^2)$?

Thank you!