Concatenate codes, save qubits

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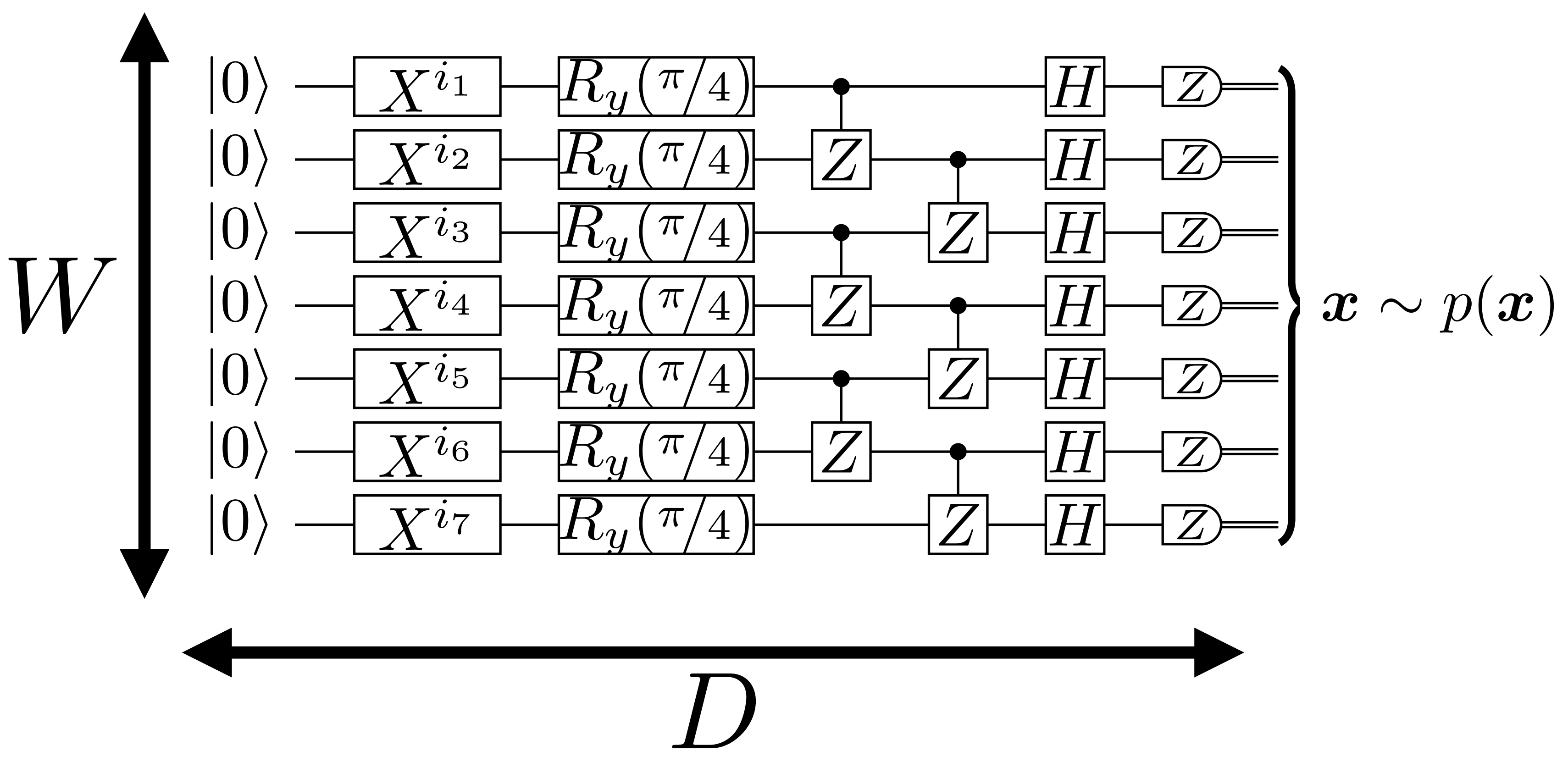
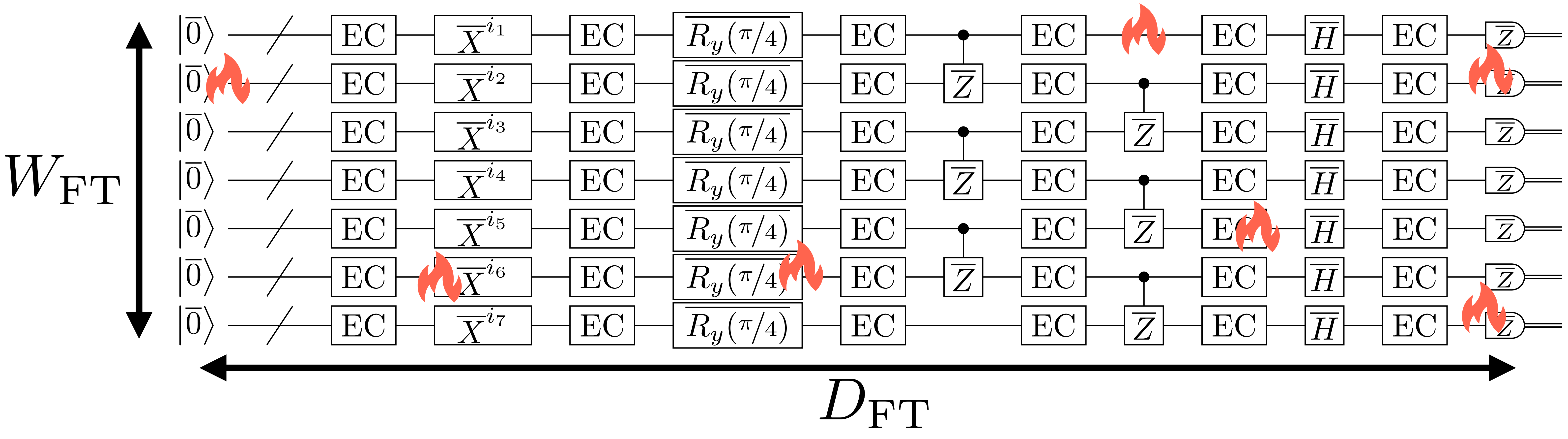
UTokyo

Fault-Tolerant Quantum Computation (FTQC)

Original circuit



Noisy circuit with **physical error rate** $p_0 > 0$



Task: Given ϵ and an original circuit, perform **the noisy circuit** to output $\boldsymbol{x} \sim \tilde{p}(\boldsymbol{x}) \approx_\epsilon p(\boldsymbol{x})$
(the fault-tolerant circuit)

→ Use **quantum error-correcting codes** that can suppress **logical error rate** arbitrarily

$$p_0 < p_{\mathrm{th}} \Rightarrow \boxed{p_L \lessapprox \frac{\epsilon}{WD}}$$ for achieving the overall error $O(\epsilon)$

Below threshold

Width × Depth = Circuit size

Space overhead := $\dfrac{W_{\mathrm{FT}}}{W}$ Time overhead := $\dfrac{D_{\mathrm{FT}}}{D}$

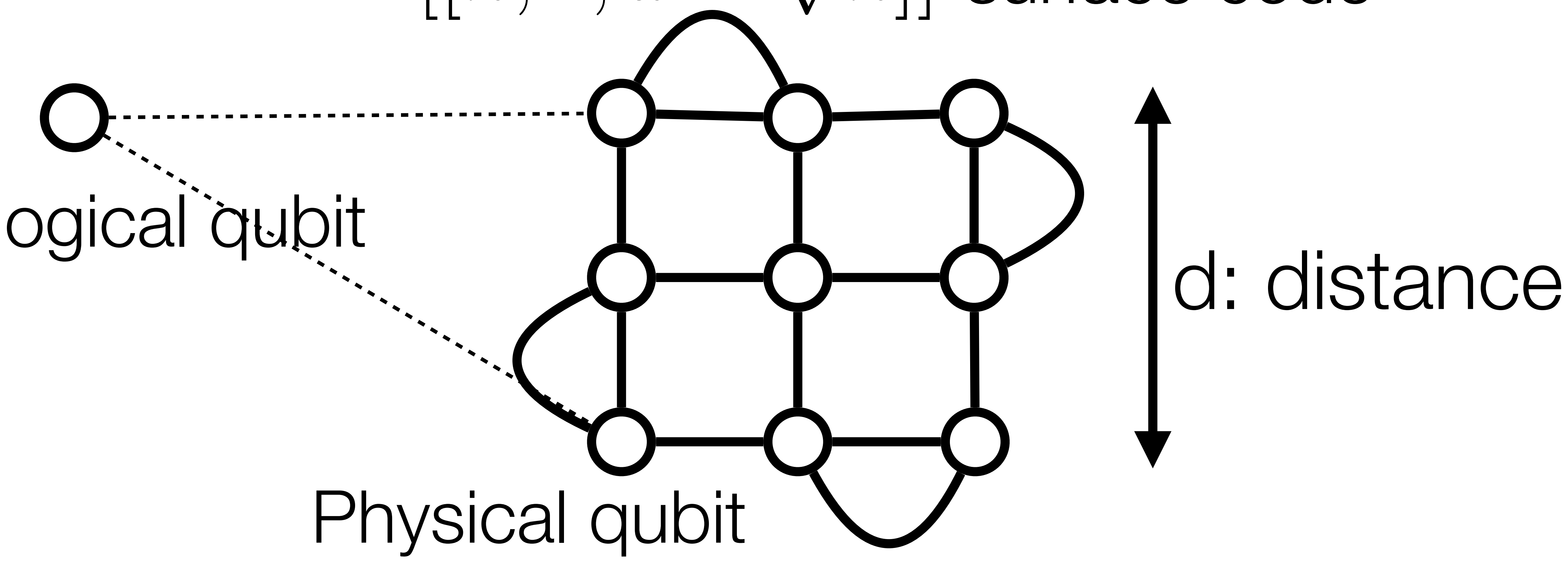
Obstacle: Overheads of FTQC

Two conventional approaches for FTQC to achieve $p_0 < p_{\mathrm{th}} \Rightarrow \boxed{p_L \lessapprox \frac{\epsilon}{WD}}$

$[[n,k,d]]$ — n #physical qubits, k #logical qubits, d distance

1: Quantum low-density parity-check (LDPC) code

$[[n, 1, d = \sqrt{n}]]$ surface code

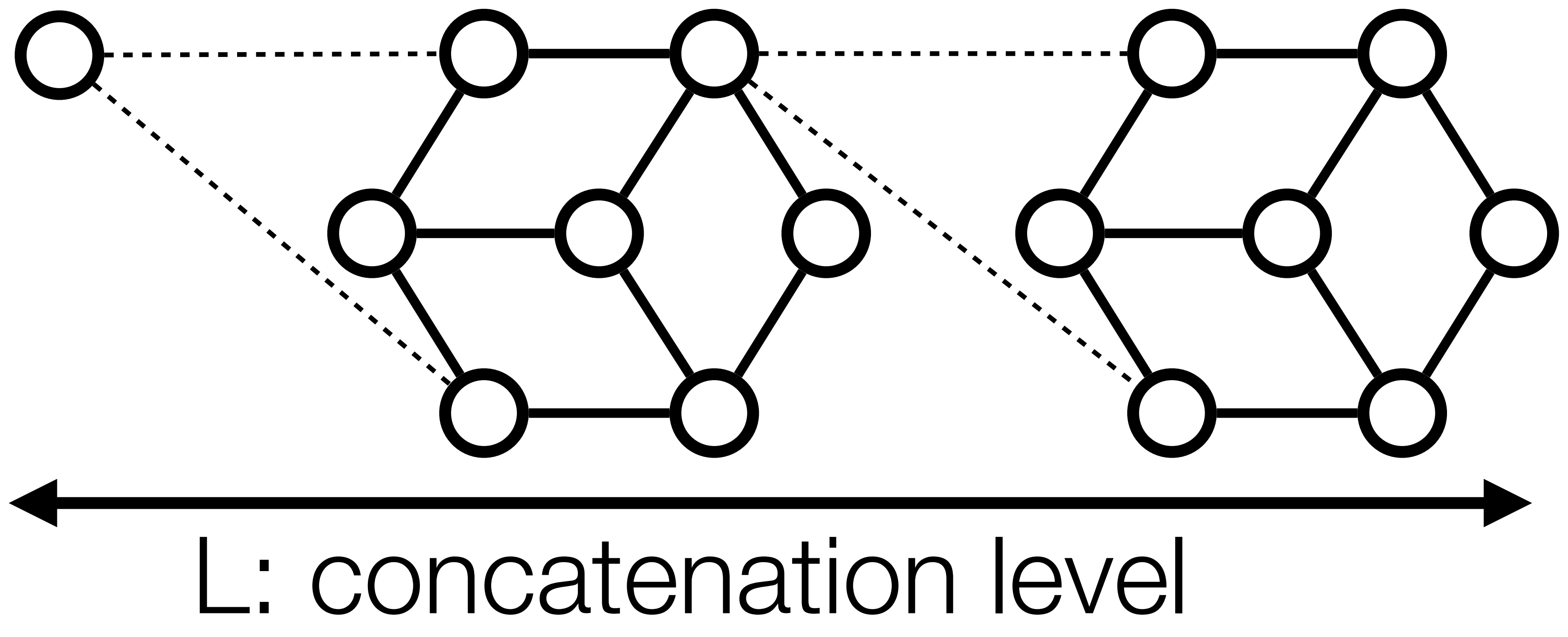


Increase **distance d**

$$n = d^2,\ p_L \lessapprox \left(\frac{p_0}{p_{\mathrm{th}}}\right)^d$$

2: Concatenated code

$[[7,1,3]]$ 7-qubit code



Increase **concatenation level L**

$$n = 7^L,\ p_L \lessapprox \left(\frac{p_0}{p_{\mathrm{th}}}\right)^{2^L}$$

Obstacle in realizing FTQC: Polylog overhead → diverging to infinity on large scales

Space $\frac{W_{\mathrm{FT}}}{W} \approx \mathrm{polylog}\left(\frac{WD}{\epsilon}\right)$ Time $\frac{D_{\mathrm{FT}}}{D} \approx \mathrm{polylog}\left(\frac{WD}{\epsilon}\right)$

Solution: Constant-overhead FTQC

Constant-overhead protocols

1: Quantum LDPC code

$$[[n, k = \Theta(n), d = \Theta(n^\alpha)]]$$

+ efficient, robust decoder

$\longrightarrow$ ✅ Constant space overhead

$$\frac{W_{\mathrm{FT}}}{W} = O(1)$$

❌ Large time overhead

$$\frac{D_{\mathrm{FT}}}{D} = \mathrm{poly}\left(\frac{WD}{\epsilon}\right)$$

$\downarrow$

✅ Short time overhead

$$\frac{D_{\mathrm{FT}}}{D} = \mathrm{polylog}\left(\frac{WD}{\epsilon}\right)$$

Kovalev and Pryadko, PRA 87, 020304 (2013), Gottesman, Quantum Info. Comput. 14, 1338–1372 (2014).
Fawzi, Grospellier, and Leverrier, FOCS2018, Krishna and Poulin, PRX 11, 011023 (2021),
Cohen et al. Sci. Adv. 8, eabn1717 (2022), Tremblay, Delfosse, and Beverland, PRL 129, 050504 (2022).

Tamiya, Yamasaki, Koashi, AQIS 2024 (2024); Tamiya, PhD thesis (2024).

2: Concatenated quantum Hamming code

Concatenation of

$$[[N_r = 2^r - 1, N_r = 2^r - 2r - 1, 3]]$$

$$\frac{N}{K} = \prod_{l=1}^{\infty} \frac{N_{r_l}}{K_{r_l}} \to O(1)$$

$\longrightarrow$ ✅ Constant space overhead

$$\frac{W_{\mathrm{FT}}}{W} = O(1)$$

✅ Short time overhead

$$\frac{D_{\mathrm{FT}}}{D} = \exp(O(\mathrm{polylog}(\log(WD/\epsilon))))$$

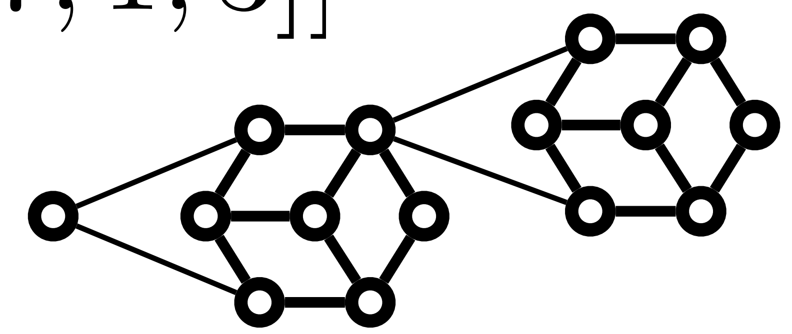
Yamasaki and Koashi, Nature Physics 20, 247 (2024).

Concatenated Hamming Codes

H. Yamasaki and M. Koashi, Nature Physics 20, 247 (2024).

Steane's 7-qubit code

$[[7,1,3]]$

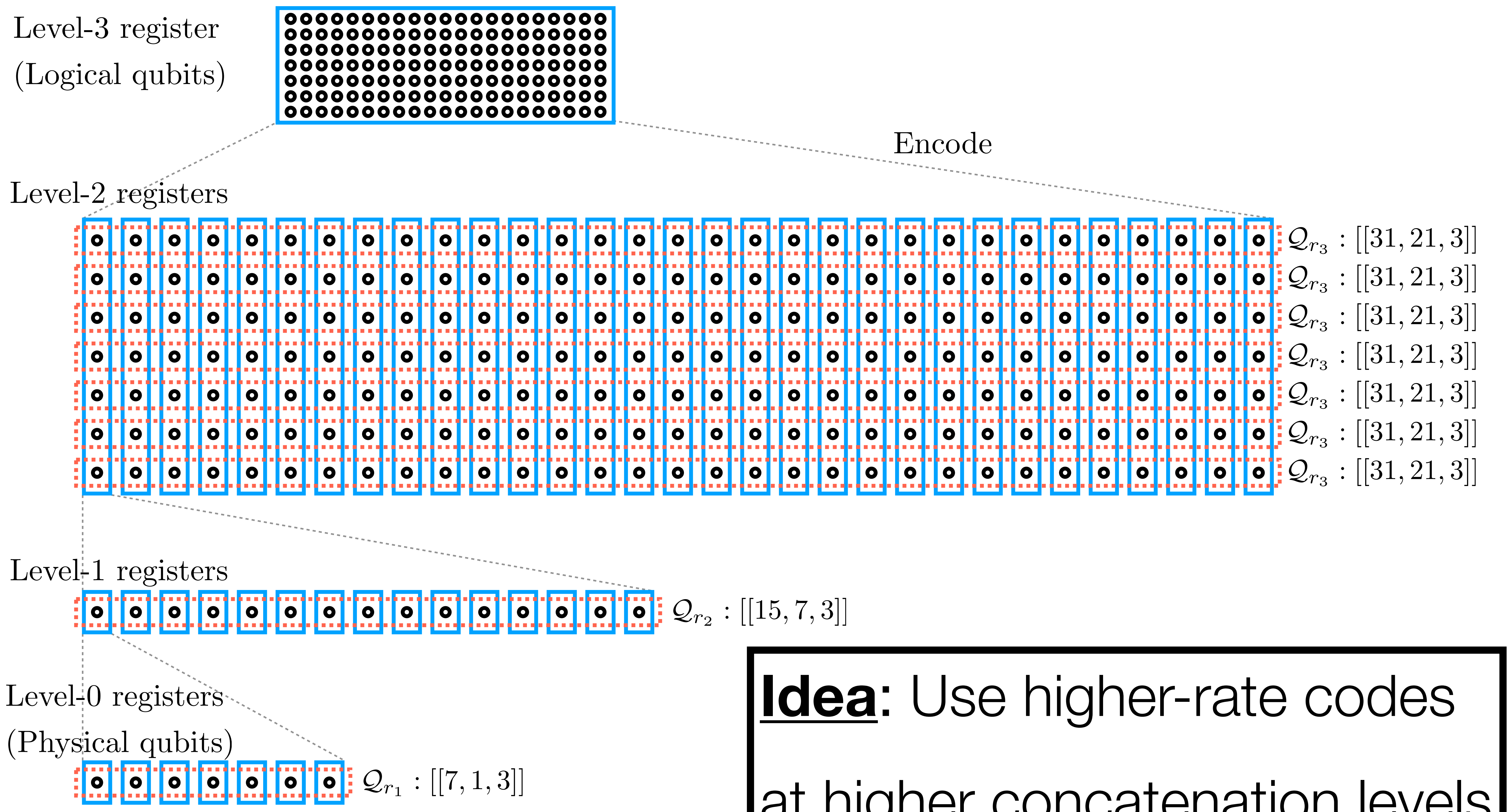


Generalization

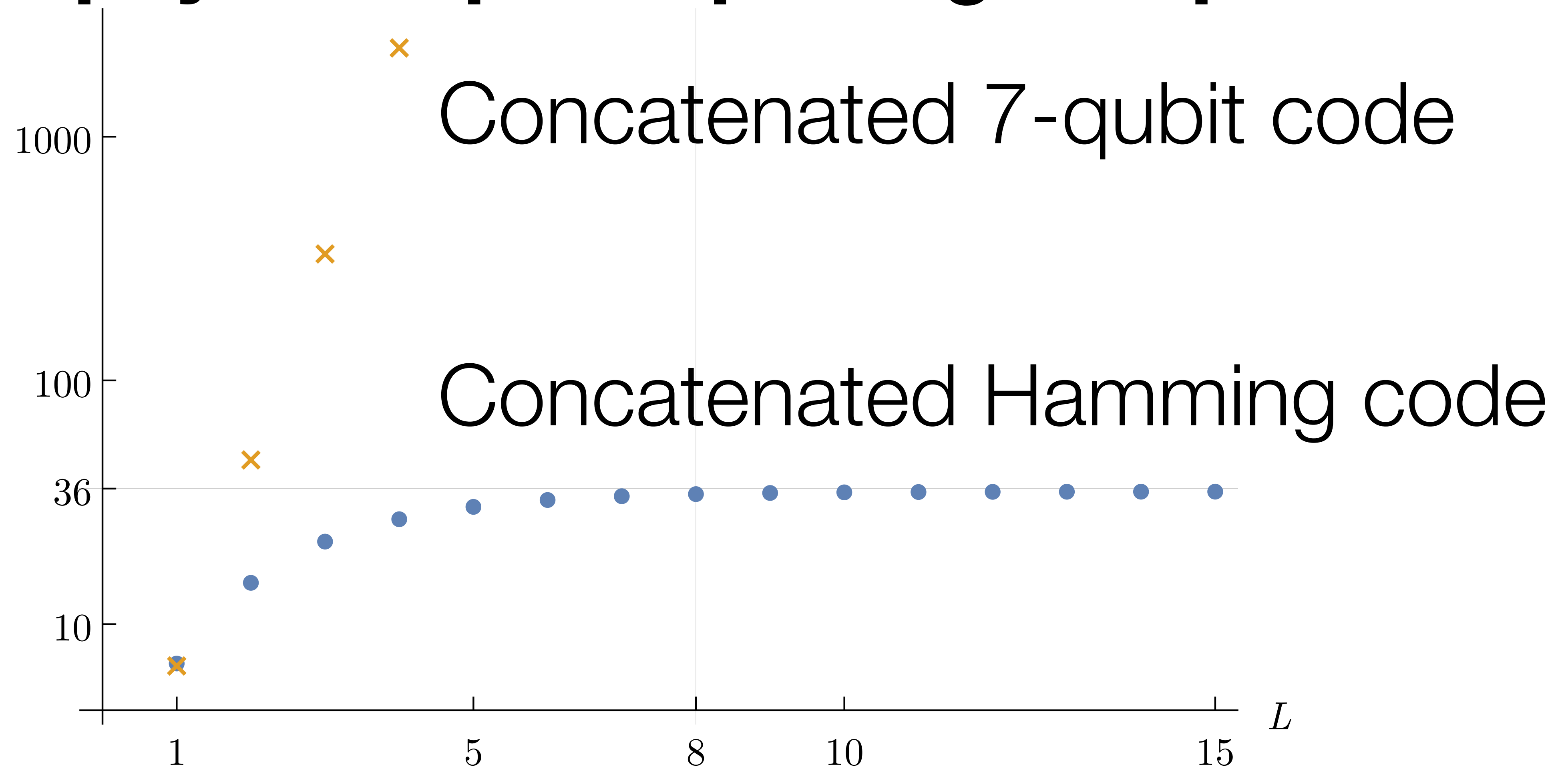
Quantum Hamming codes

$[[2^r-1, 2^r-2r-1, 3]]$ Many logical qubits

$[[7,1,3]], [[15,7,3]], [[31,21,3]], [[63,51,3]], \ldots$



#physical qubits per logical qubit



Idea: Use higher-rate codes at higher concatenation levels

Concatenating quantum Hamming codes at growing rate yields a **non-vanishing overall rate**

Practical requirements for FTQC

	Concatenated Q Hamming code	**This work**
- Low space overhead	✅	✅
- High threshold	❌	✅
- Modularity	✅	✅
- Quantitative comparison	❌	✅

This work

- **High threshold & low space overhead** protocol
- Quantitative comparison with **surface code**
  → **Reduction of space overhead** at the practical regime

Outline of this talk

- **Backgrounds**
- Main results
- Discussions

Underlying code to improve threshold

Concatenated Hamming code

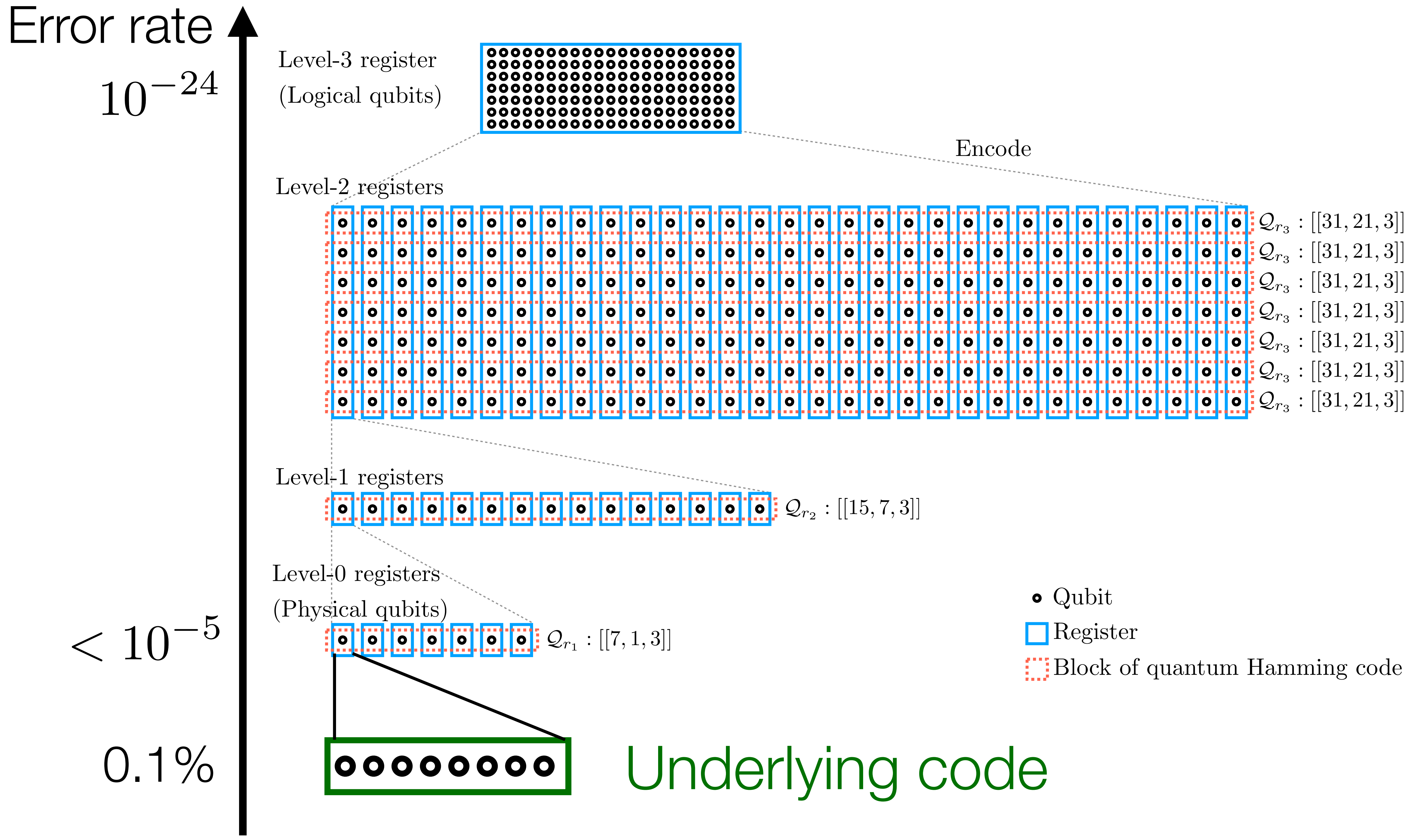
✅ Small space overhead: $\frac{W_{\mathrm{FT}}}{W} = O(1)$

❌ Low threshold: $\sim 10^{-5}$

Surface code

❌ Large space overhead: $\frac{W_{\mathrm{FT}}}{W} \approx \mathrm{polylog}\left(\frac{WD}{\epsilon}\right)$

✅ High threshold: $\sim 10^{-2}$



Error suppression

Underlying code:

0.1% → 10^{-5}

Concatenated Hamming code:

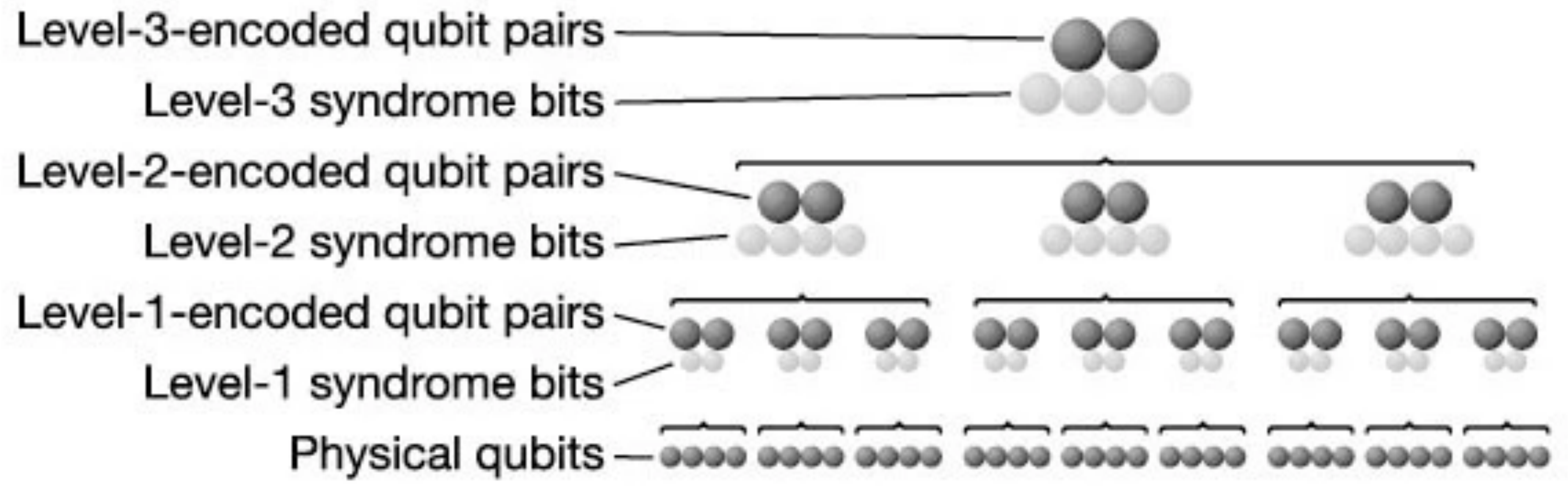
10^{-5} → 10^{-24}

Put high threshold code on the bottom layer **to improve the threshold**

High threshold code: C4/C6 code

E. Knill, Nature 434, 39-44 (2005).

<u>Merit</u>: Threshold for protocol based on C4/C6 code : 2.4% vs Threshold for surface code: 0.3%



[[6, 2, 2]] code

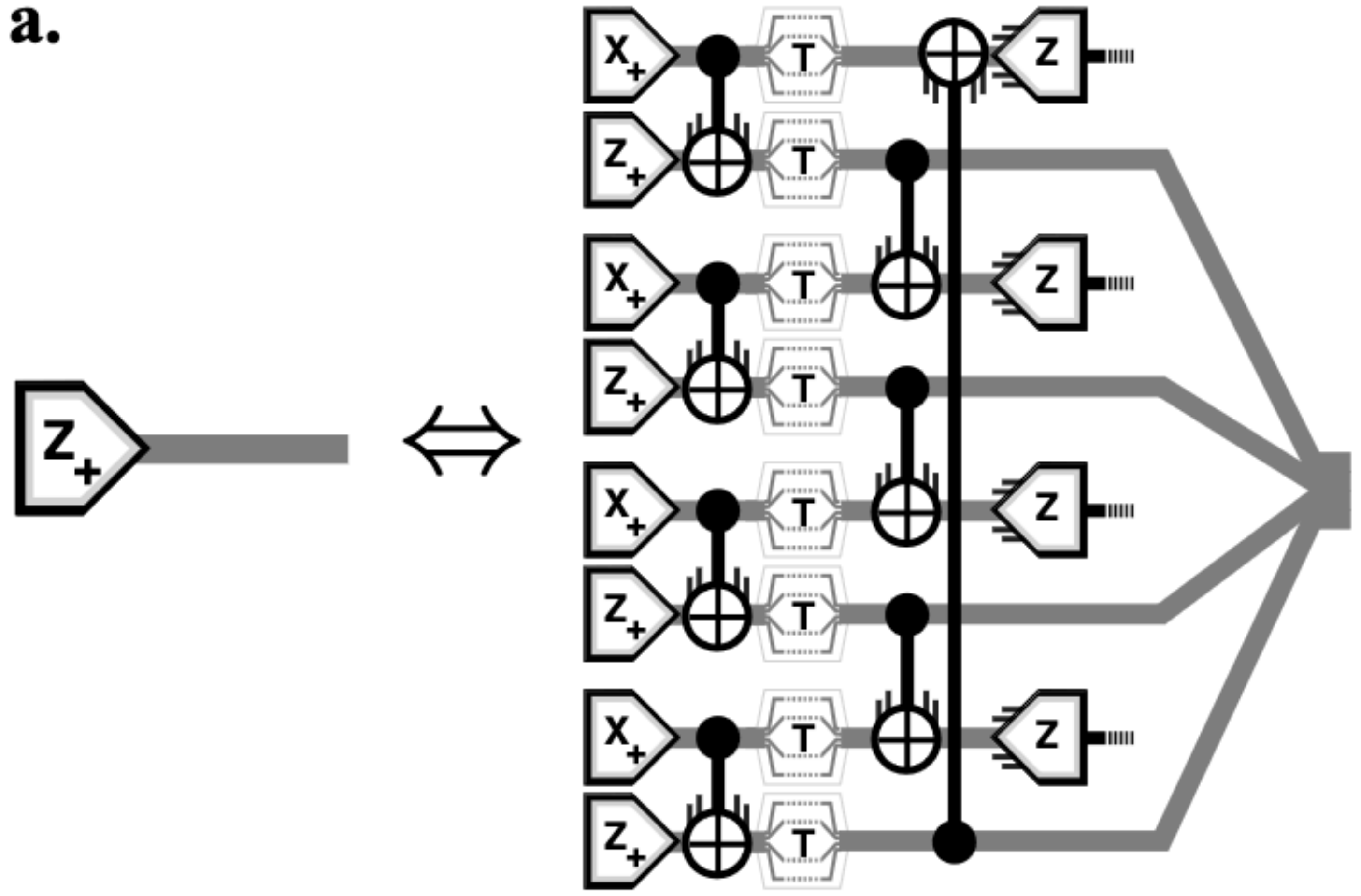
[[6, 2, 2]] code   Distance 2 for qubit pairs

[[4, 2, 2]] code

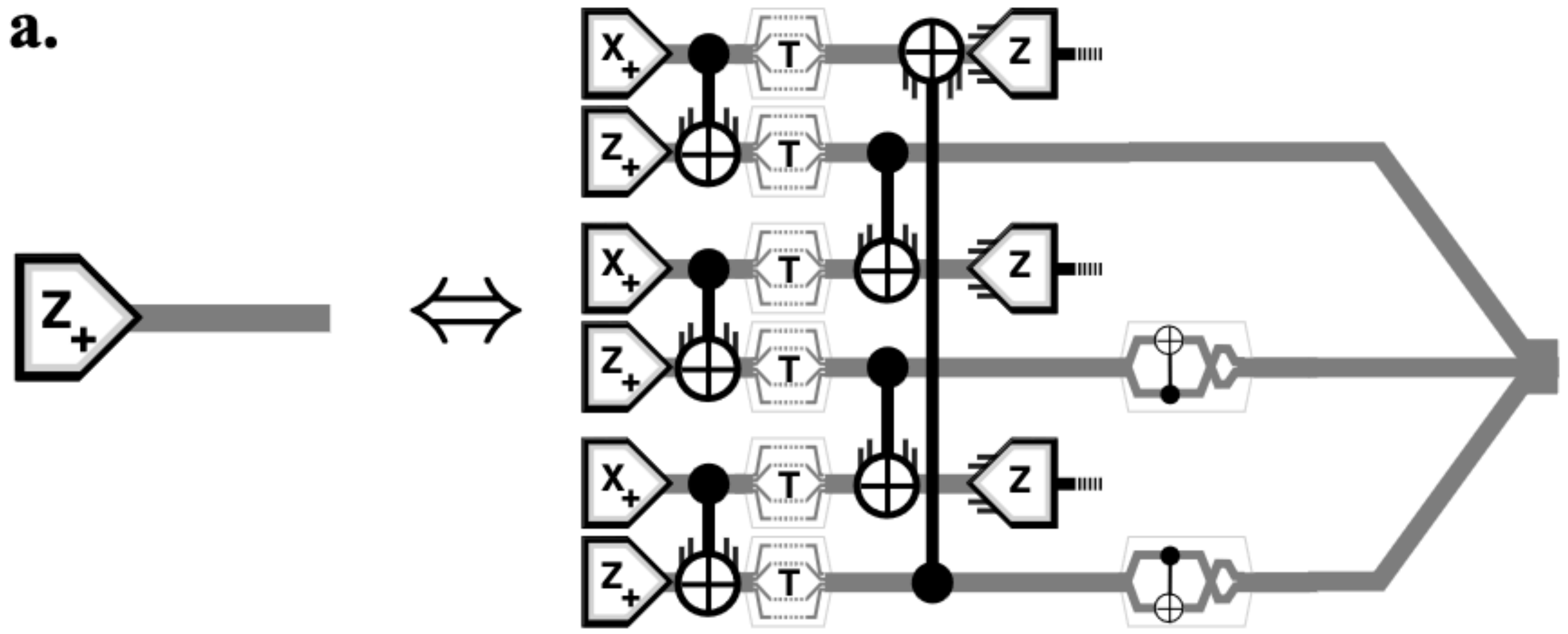
<u>Protocol</u>

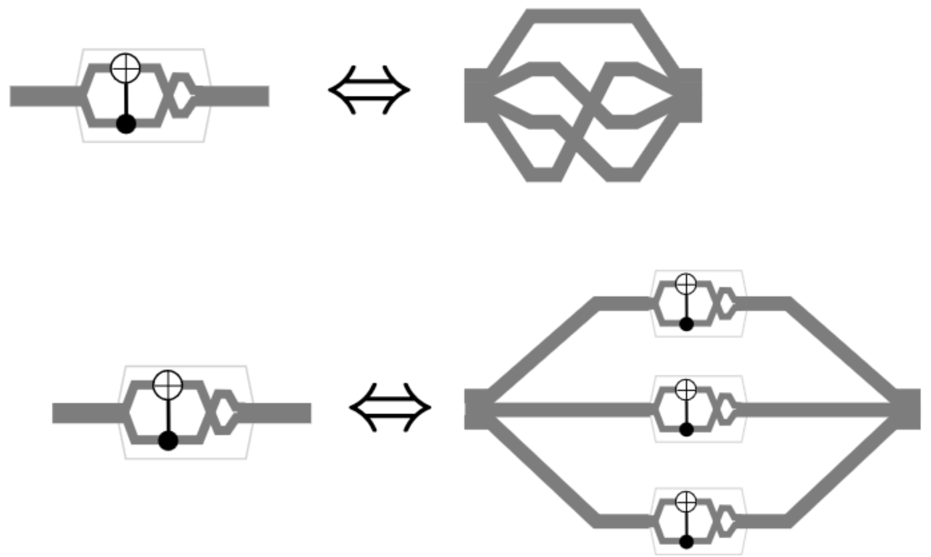
8 qubits in 1D periodic arrangement

6 level-1 registers in 1D periodic arrangement



Preparation of [[4, 2, 2]] code





Preparation of [[6, 2, 2]] code

Outline of this talk

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Code construction

Guiding principle

Underlying code: High threshold (but many qubits)

Higher-level code: Few qubits (but low threshold)

Optimization

- Underlying code → C4/C6 code
- Higher-level code → Concatenation of Q Hamming codes

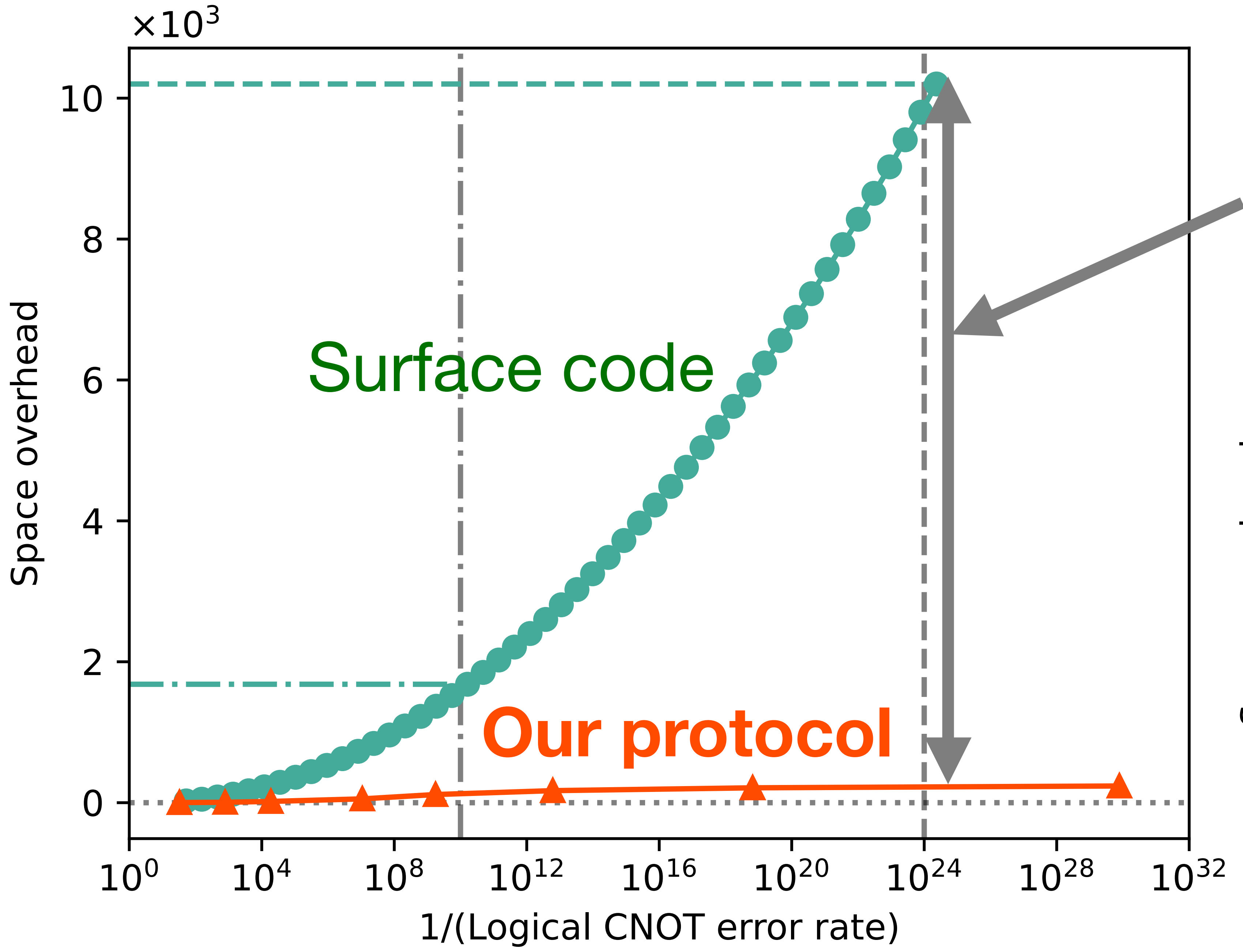
	Quantum code	N	K	N/K
level-1	$C_4 (= [[4,2,2]])$	4	2	2
level-2	$C_6 (= [[6,2,2]])$	12	2	6
level-3	$C_6 (= [[6,2,2]])$	36	2	18
level-4	$C_6 (= [[6,2,2]])$	108	2	54
level-5	$\mathcal{Q}_4 (= [[15,7,3]])$	1.6×10^3	14	1.2×10^2
level-6	$\mathcal{Q}_5 (= [[31,21,3]])$	5.0×10^4	2.9×10^2	1.7×10^2
level-7	$\mathcal{Q}_6 (= [[63,51,3]])$	3.2×10^6	1.5×10^4	2.1×10^2
level-8	$\mathcal{Q}_7 (= [[127,113,3]])$	4.0×10^8	1.7×10^6	2.4×10^2

C4/C6 code (levels 1–4)

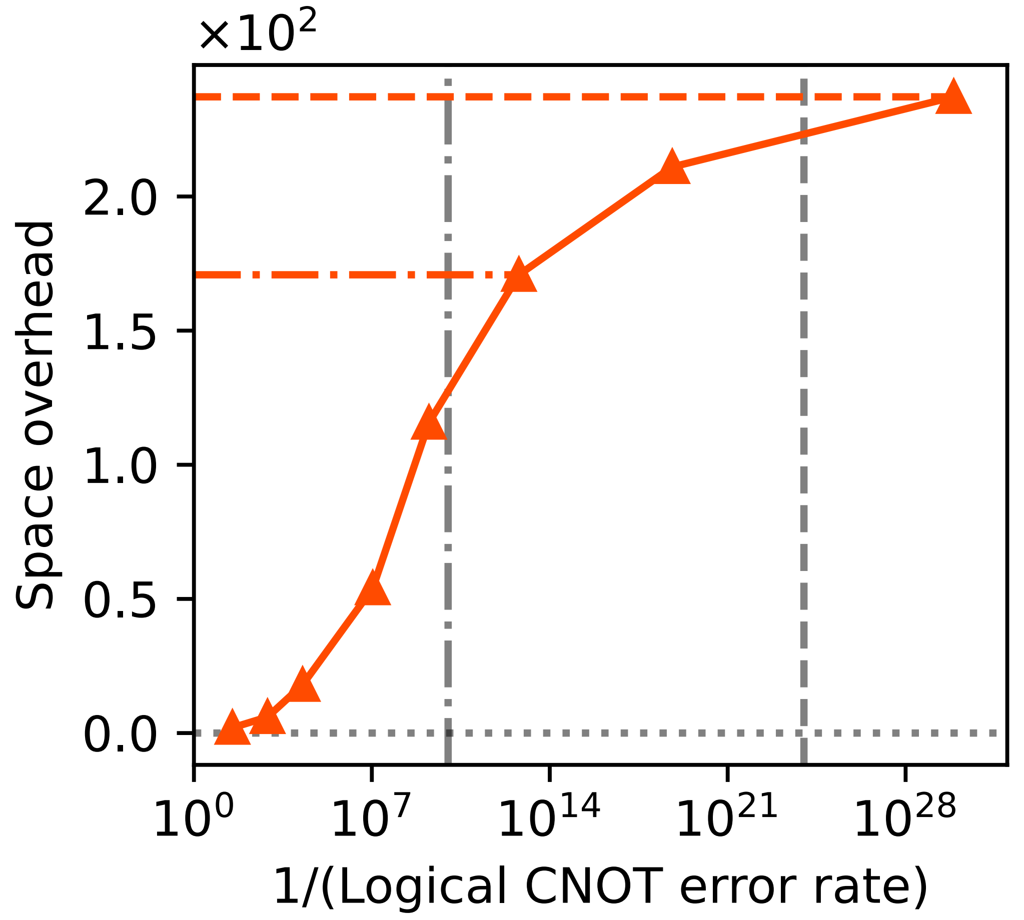
Q Hamming codes (levels 5–8)

Main result: Comparison of our protocol with surface code

Space overhead: physical error rate = 0.1%



Saving of space overhead



Space overhead = #(Logical qubits)/#(Physical qubits) per code block

Threshold:

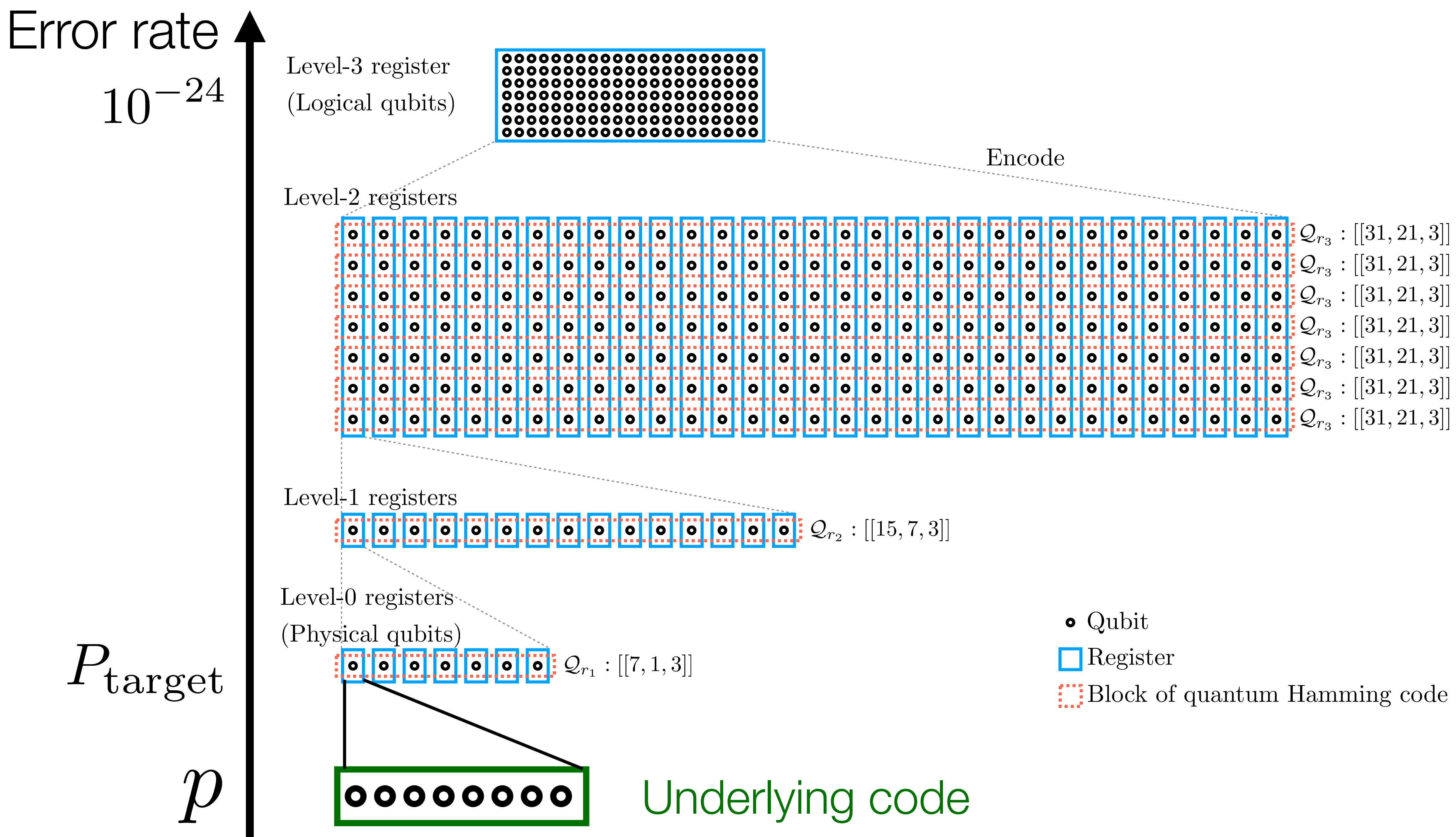
	Threshold
C_4/C_6 code	2.4%
Surface code	0.31%
Steane code	0.030%
C_4/Steane code	0.15%

- Circuit-level error model
- Logical CNOT error rate

Higher threshold, lower space overhead than surface code

Remark: Requires all-to-all connectivity → Suitable for neutral atoms, ion traps, and optics

Optimization over underlying code



H. Yamasaki and M. Koashi, Nature Physics 20, 247 (2024).

Space overhead to achieve P_{target}

	Threshold	Space overhead		
		$p = 0.01\%$	$p = 0.1\%$	$p = 1\%$
C_4/C_6 code	2.4%	18	**54**	**1458**
Surface code	0.31%	121	841	-
Steane code	0.030%	343	-	-
C_4/Steane code	0.15%	**14**	4802	-

Change underlying code to optimize space overhead for each physical error rate

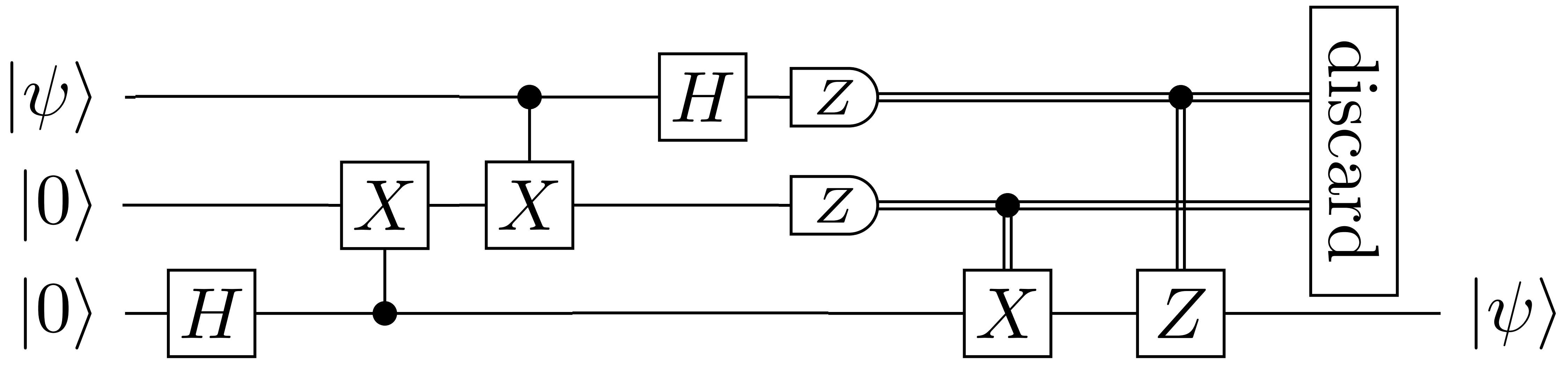
Details of the protocol

H. Yamasaki and M. Koashi, Nature Physics 20, 247 (2024).

Fault-tolerant gate operation = gate gadget + error correction gadget

Knill's EC gadget: based on teleportation

Conventional: Parallel use of auxiliary qubit

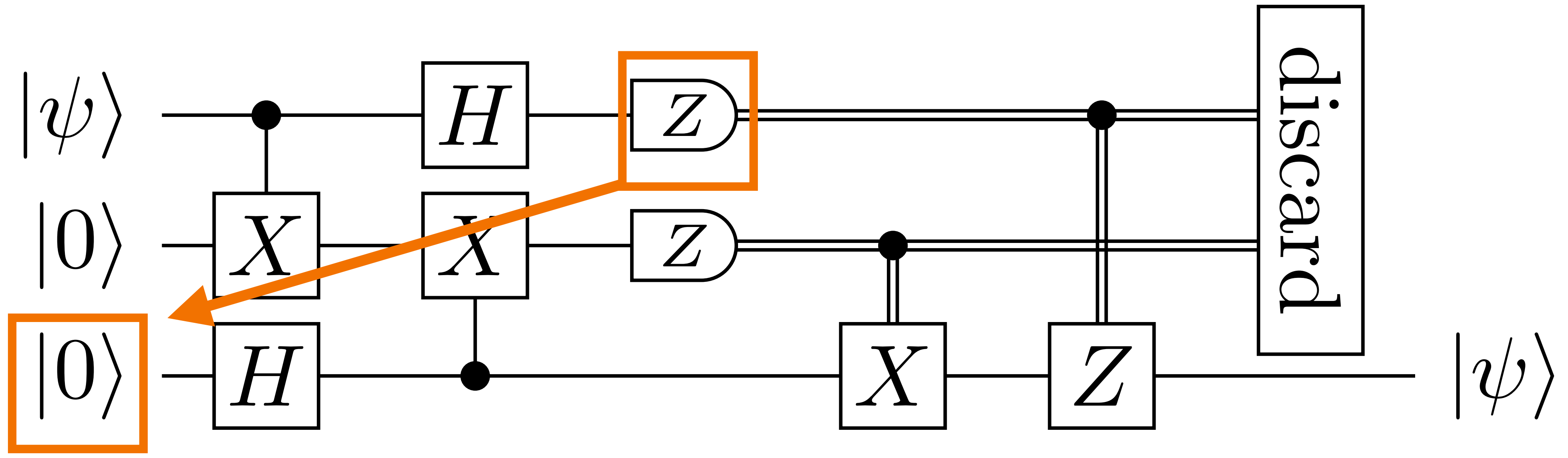


❌ Large space overhead

✅ Less errors

→ Underlying codes

Modified: Sequential use of auxiliary qubit



✅ Small space overhead

❌ More errors

→ Q Hamming codes

Parallel use of auxiliary qubit at underlying code → **High threshold**

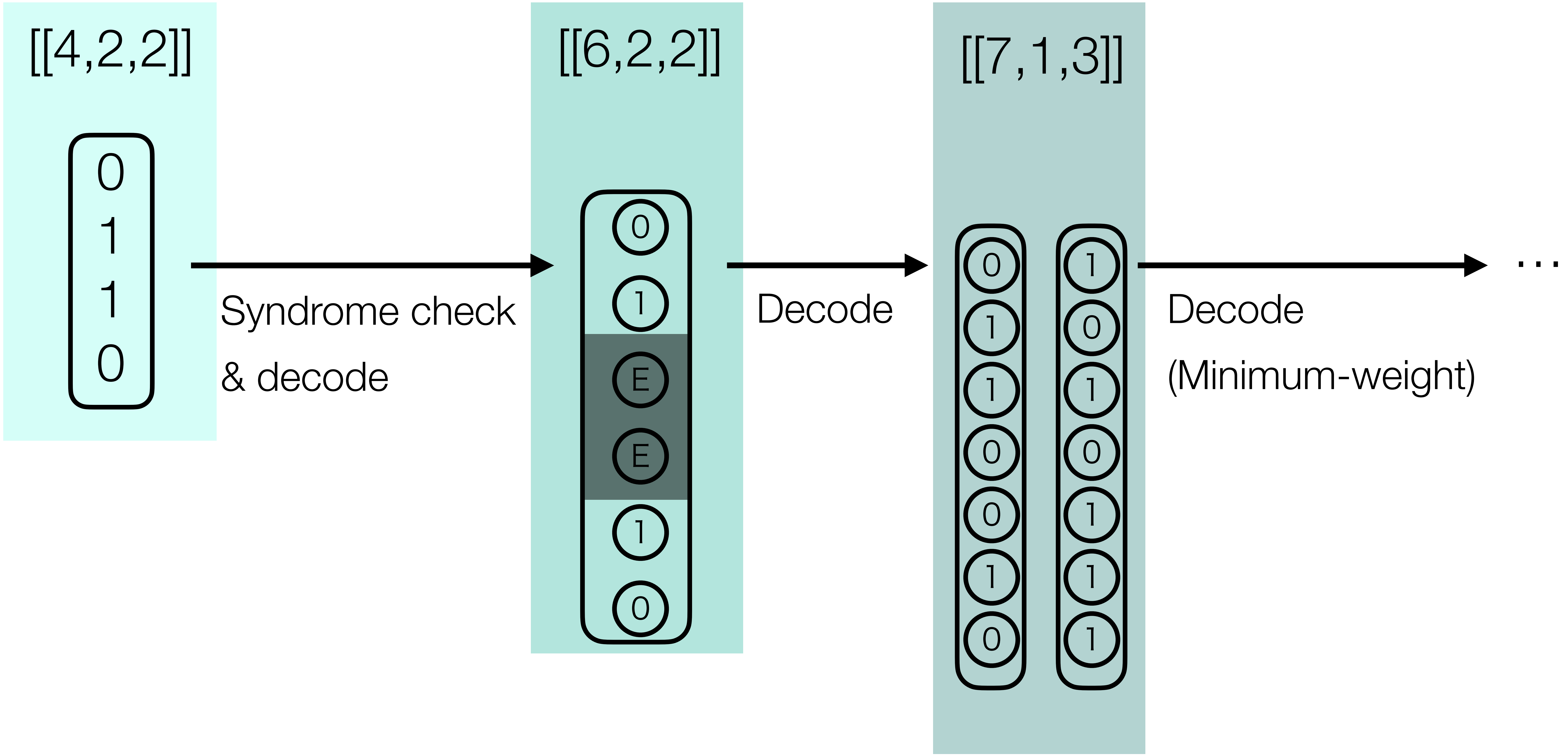
Sequential use at higher-level codes → **Small space overhead**

Similar strategy for initial state preparation

Details of the protocol

H. Goto and H. Uchikawa, Sci. Rep. 3, 2044 (2013).
H. Yamasaki and M. Koashi, Nature Physics 20, 247 (2024).

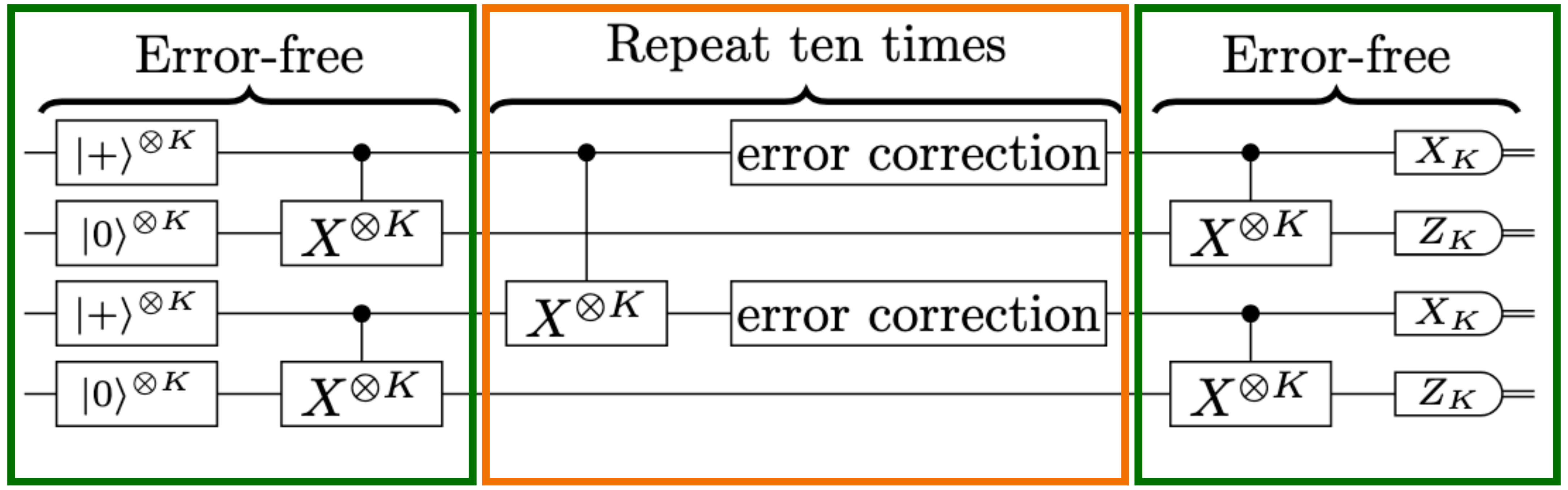
Hard-decision decoder



Details of the protocol

H. Goto, Sci. Rep. 6, 19578 (2016).
D. Horsman et al. NJP 14, 123011 (2012).
C. Vuillot et al. NJP 21, 033028 (2019).
C. Gidney, Quantum 7. 786 (2022).

Numerical simulation



Logical Bell pair

Noisy CNOT + EC

Logical Bell measurement

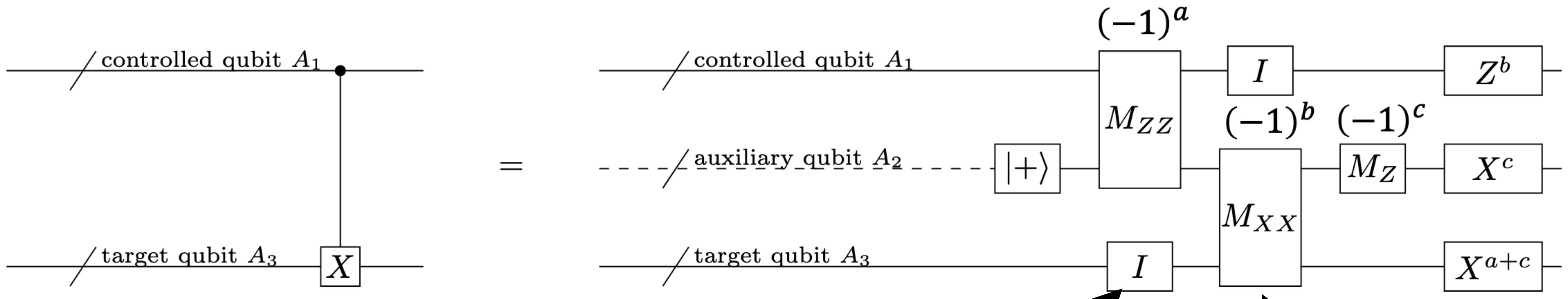
Transversal CNOT + Knill's EC gadget

C4/C6, Hamming

Lattice surgery

Surface code

Logical CNOT for surface code



Memory experiment

Stability experiment

Decoder = PyMatching (MWPM)

Quantitative comparison with the surface code at **the same condition**

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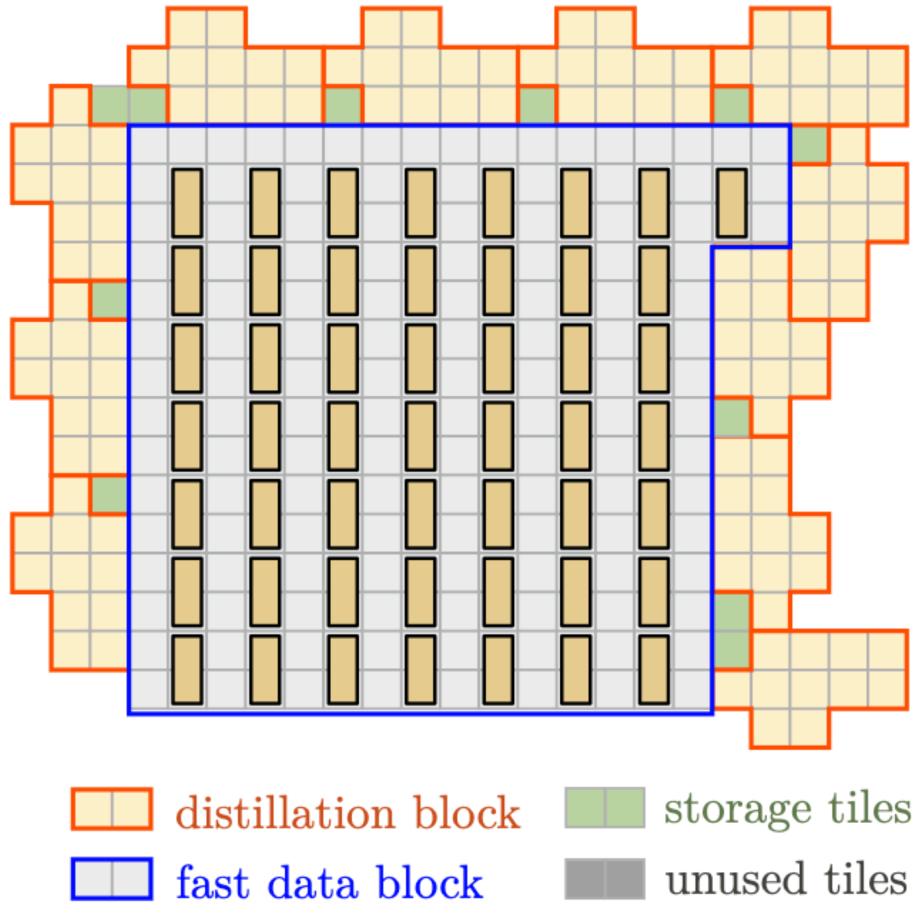
Comparison with QLDPC codes

Modularity

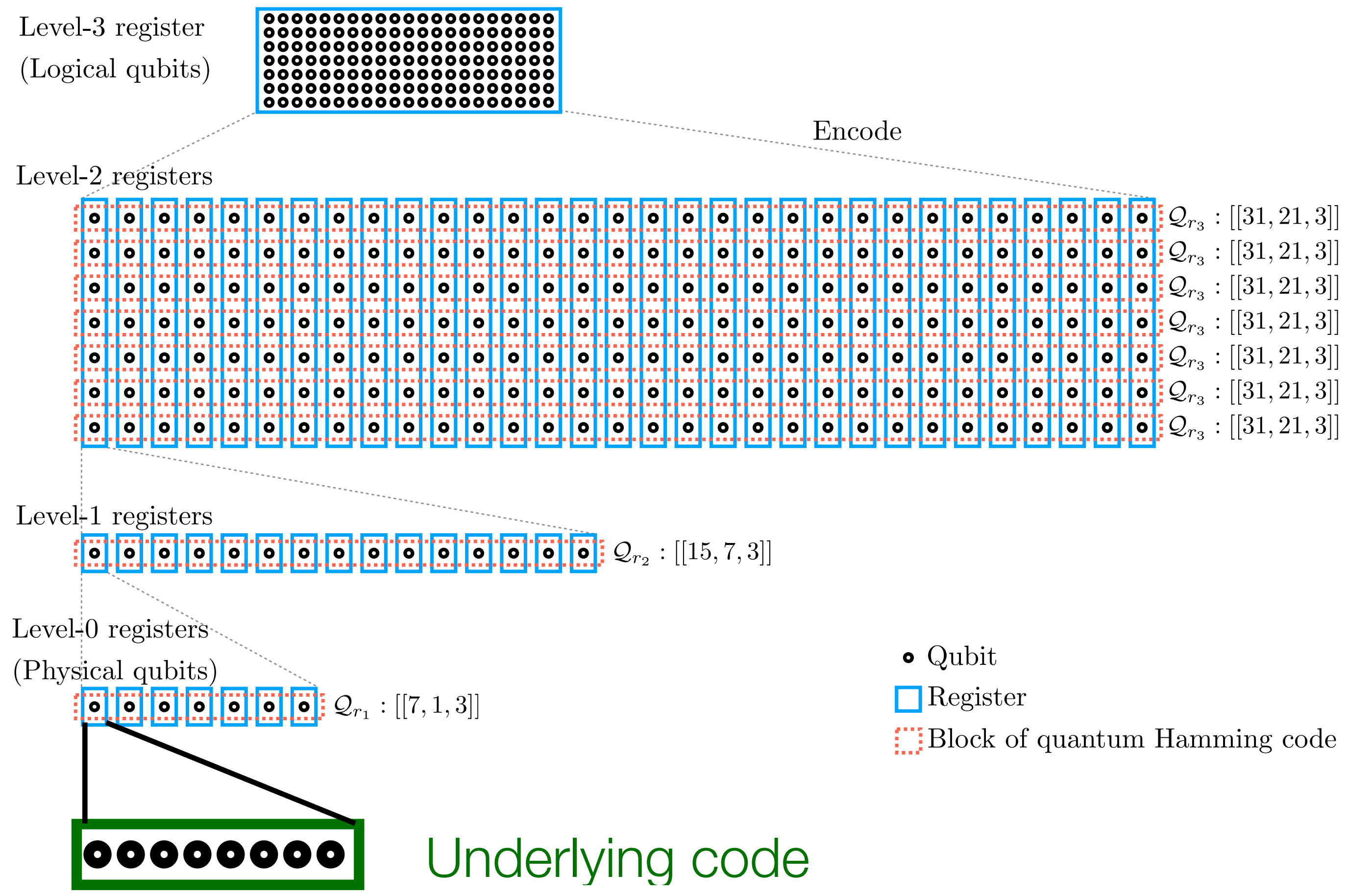
QLDPC



Litinski, arXiv:1808.02892

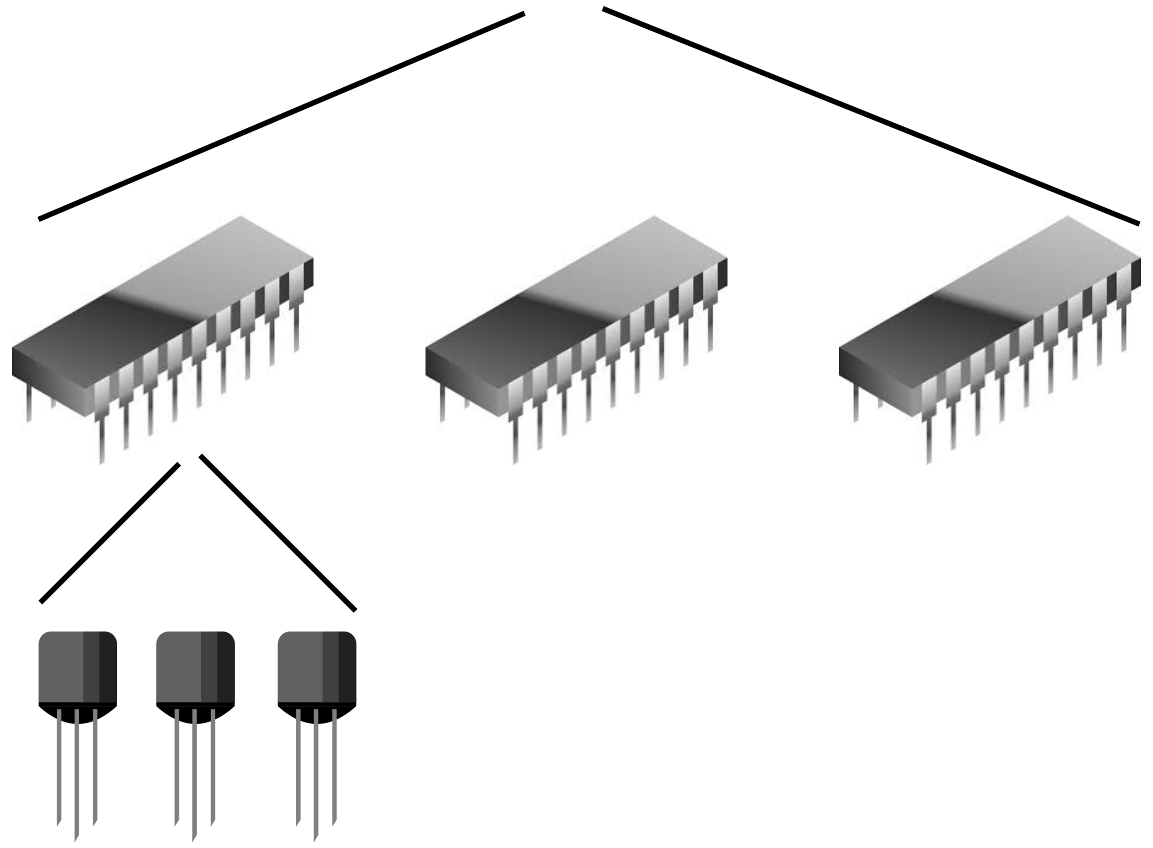
❌ Large code at once

Concatenated code



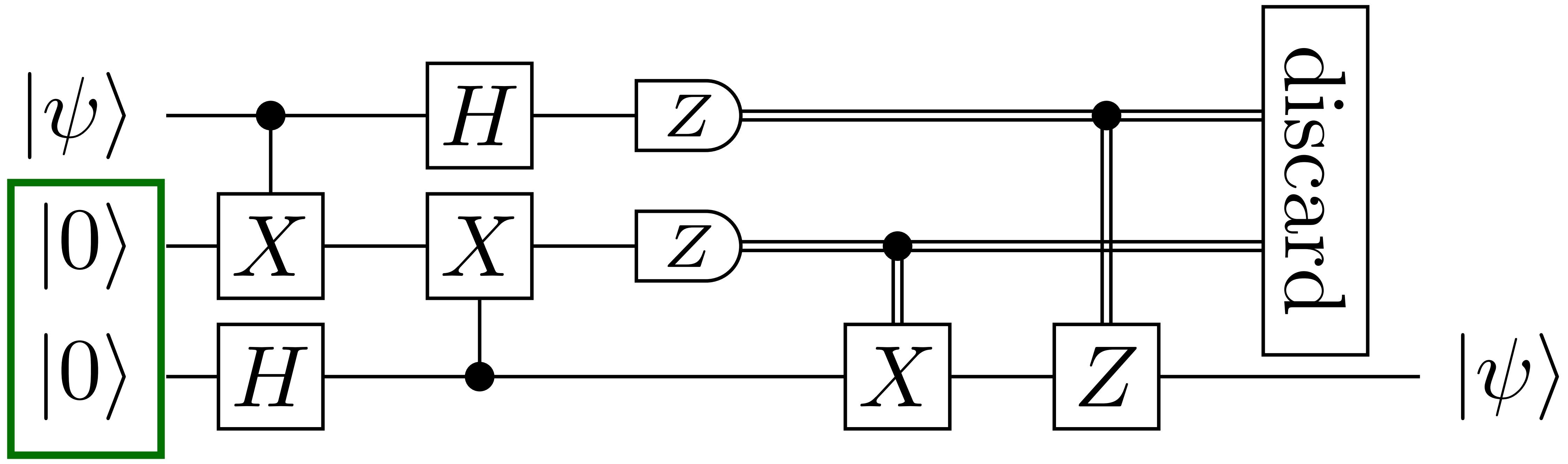
✅ Abstraction layers

https://commons.wikimedia.org/wiki/File:Summit_Supercomputer_2018.jpg



Future works

- Optimization of space overhead including auxiliary qubits



Auxiliary qubits

Space overhead $3 \times \frac{N}{K}$

On underlying codes

- Different error model other than depolarizing error model

Abbreviation	SD6	SI1000	EM3
Name	Standard Depolarizing	Superconducting Inspired	Entangling Measurements
Noisy Gateset	CX(p) AnyClifford$_1$(p) Init$_Z$(p) M_Z(p) Idle(p)	CZ(p) AnyClifford$_1$($p/10$) Init$_Z$($2p$) M_Z($5p$) Idle($p/10$) ResonatorIdle($2p$)	M$_{PP}$(p) M$_{PI}$(p) Init$_Z$($p/2$) M$_Z$($p/2$) Idle(p)
Measurement Ancillae	Yes	Yes	No
Honeycomb Cycle Length	6 time steps	9 time steps ($\approx$ 1000ns)	3 time steps

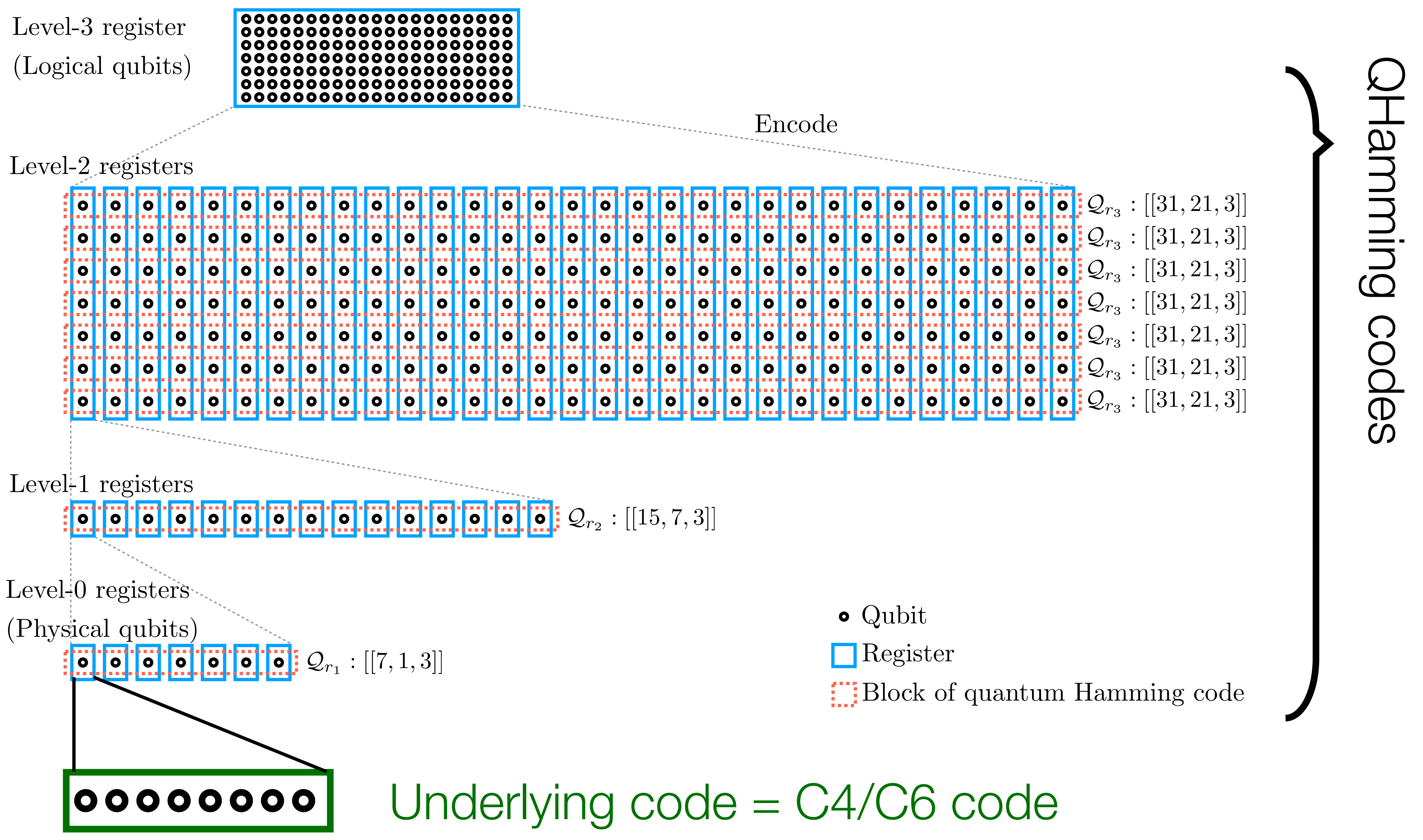
C. Gidney, M. Newman, M. McEwen, Quantum 6, 813 (2022).

Do we still have an advantage over surface code including auxiliary qubits/on other error models?

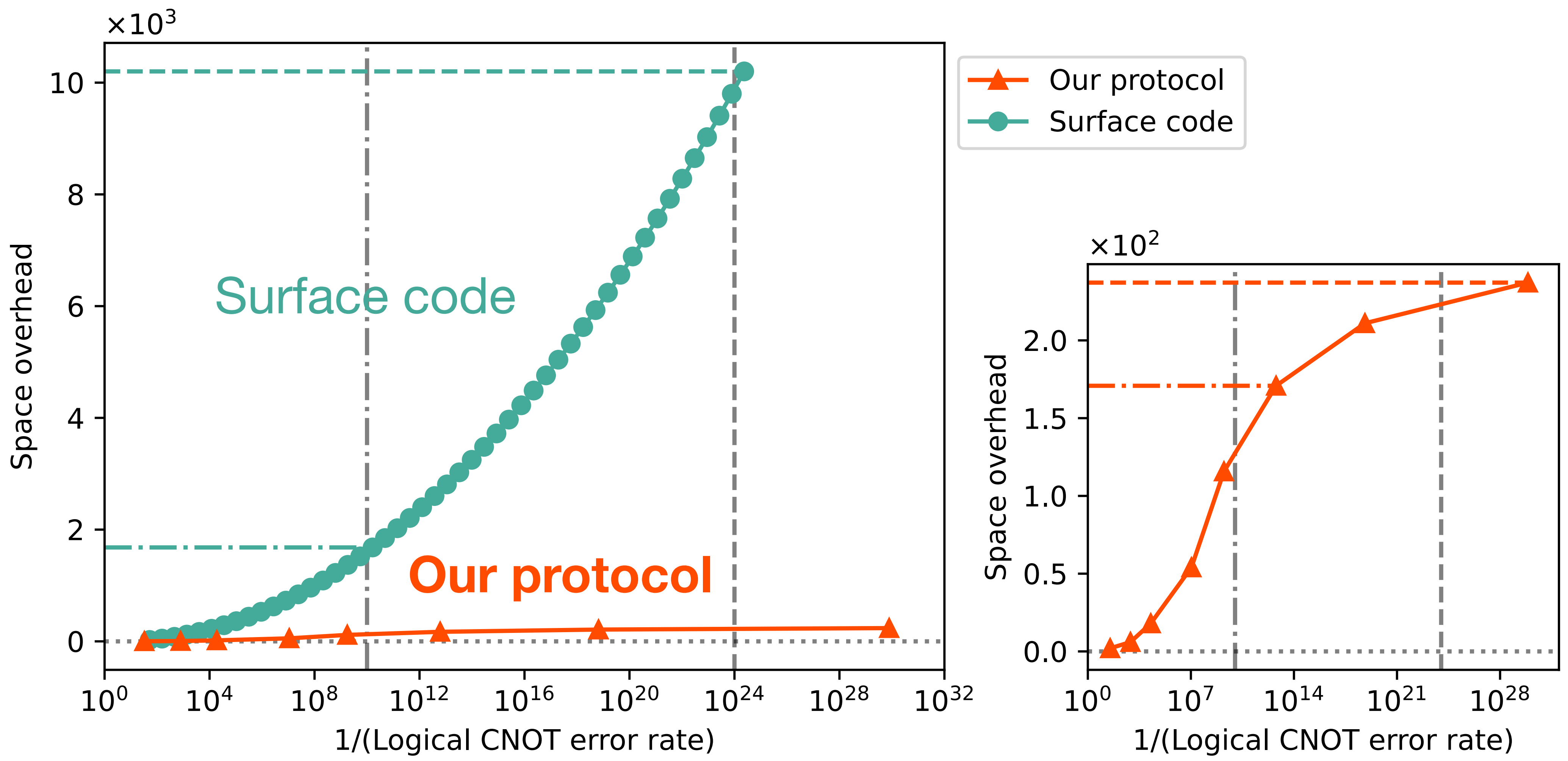
→ Ongoing

Conclusion

Code construction



Space overhead



Threshold

Our protocol: 2.4% > Surface: 0.3%

- **Lower space overhead, higher threshold** than the surface code
- Underlying code can be optimized for each physical error rates

Concatenate codes, save qubits!

Thank you!