

# One-to-one correspondence between deterministic port-based teleportation and unitary estimation

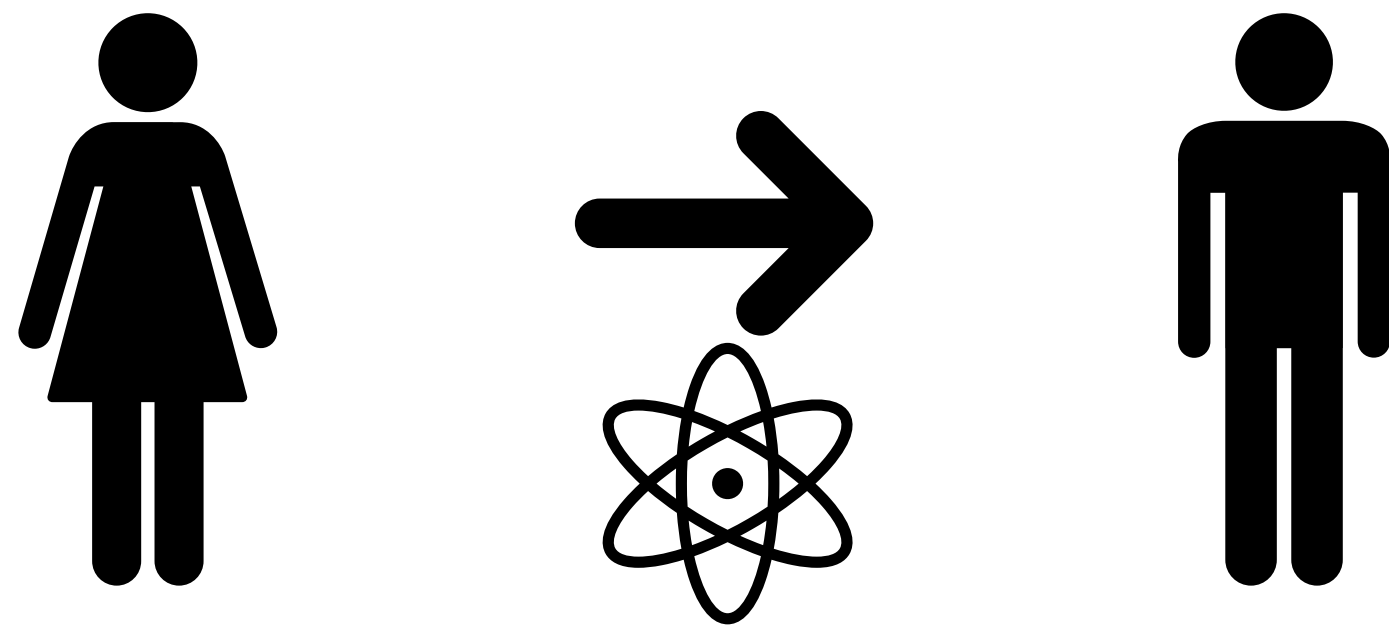
Based on [arXiv:2408.11902](#)

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Marco Túlio Quintino, Mio Murao



# What this talk is about

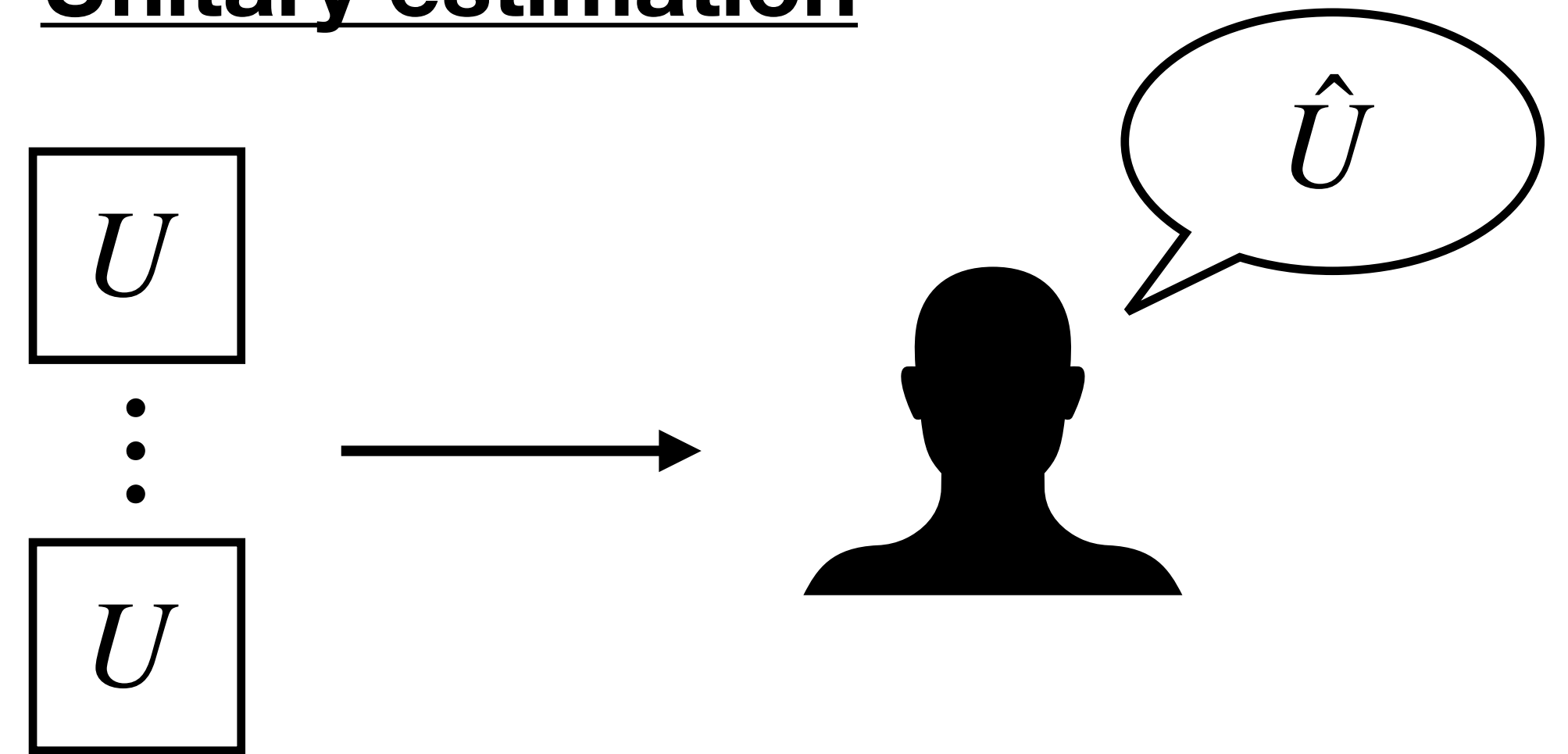
## Port-based teleportation



Teleportation fidelity  $F_{\text{PBT}}$

$$\begin{aligned} F_{\text{PBT}} &\mapsto F'_{\text{est}} \geq F_{\text{PBT}} \\ F_{\text{est}} &\mapsto F'_{\text{PBT}} \geq F_{\text{est}} \end{aligned}$$

## Unitary estimation

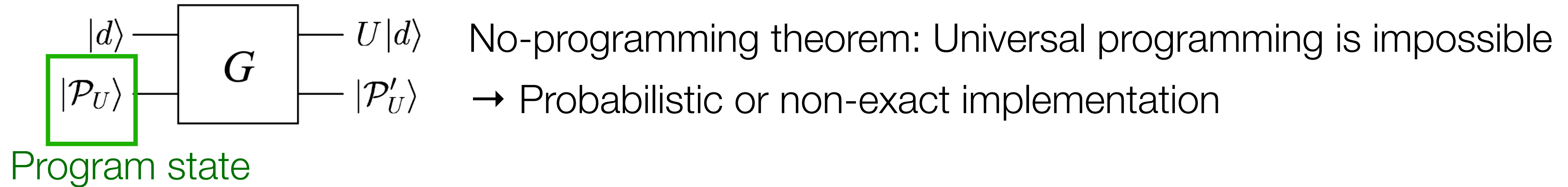


Estimation fidelity  $F_{\text{est}}$

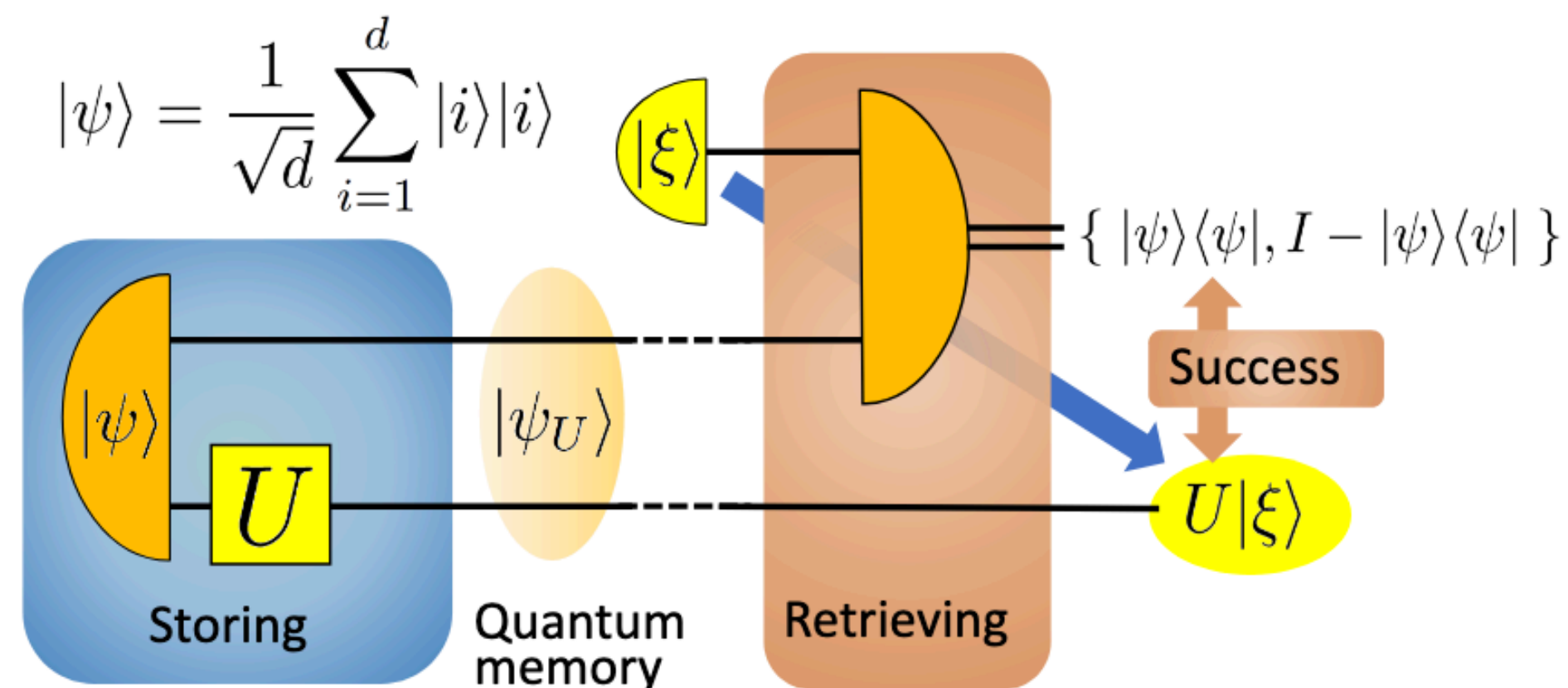
$$\max F_{\text{PBT}} = \max F_{\text{est}}$$

**Constructive proof for the equivalence!**

## Universal programming of unitary operations



## Storage and retrieval (SAR) of quantum program via teleportation



If the Bell measurement outcome is  $(i, j)$ , we get

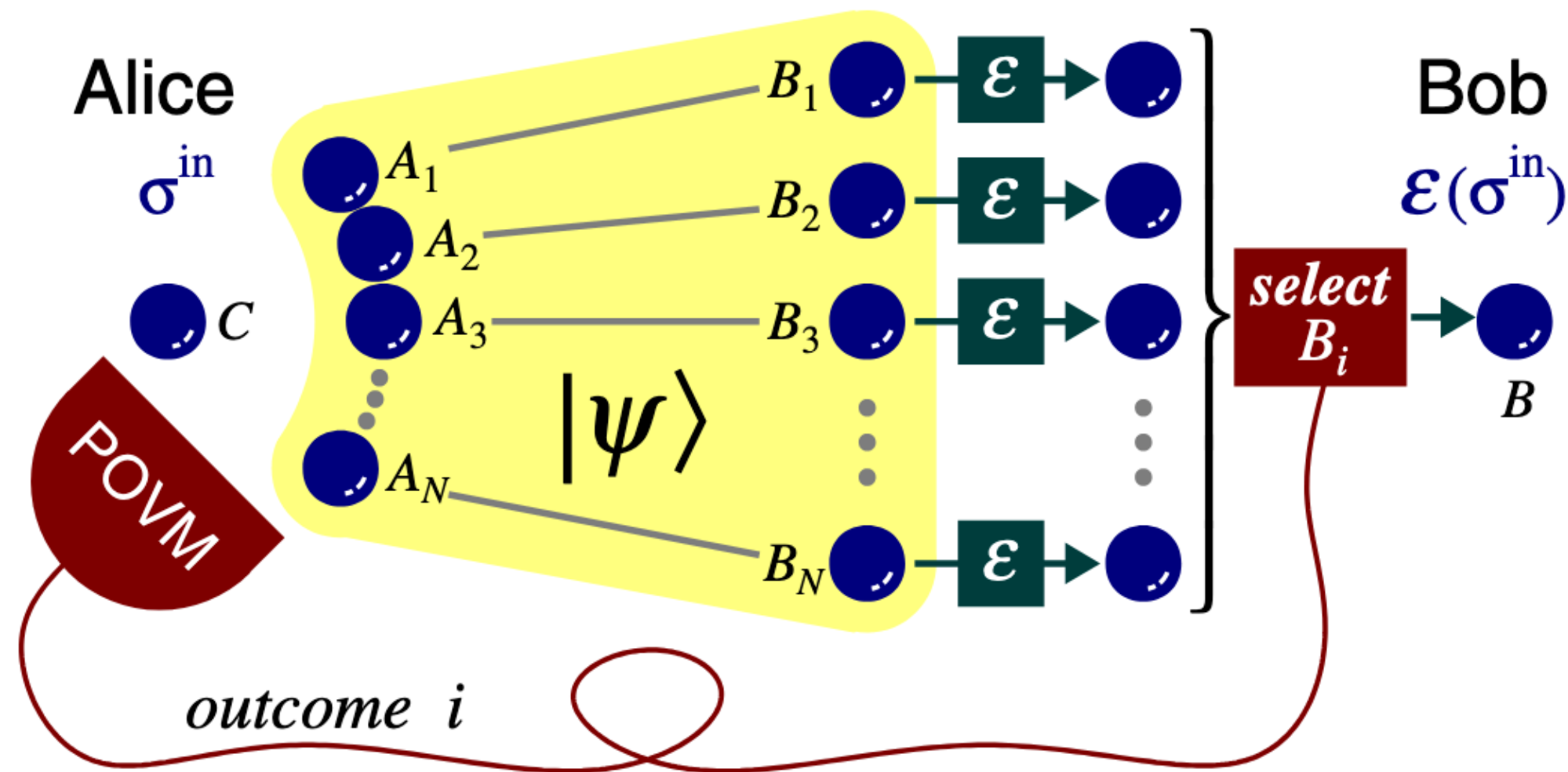
$$UX^iZ^j|\xi\rangle$$

→ Retrieval succeeds when  $(i, j) = (0, 0)$

Success probability =  $1/d^2$

**Retrieval succeeds when Pauli correction is not needed**

## SAR of quantum program via port-based teleportation (PBT)



1. Alice & Bob share  $2N$ -qudit entangled state  $|\psi\rangle$
2. Alice measures input state  $\sigma^{\text{in}}$  with her share of  $|\psi\rangle$
3. Alice sends the measurement outcome  $i$  to Bob
4. Bob selects  $i$ -th port **(no Pauli correction)**

Quantum state  $(I^{\otimes N} \otimes \mathcal{E}^{\otimes N})(|\psi\rangle\langle\psi|)$  is the program state of  $\mathcal{E}$

**Port-based teleportation is quantum teleportation without Pauli correction**



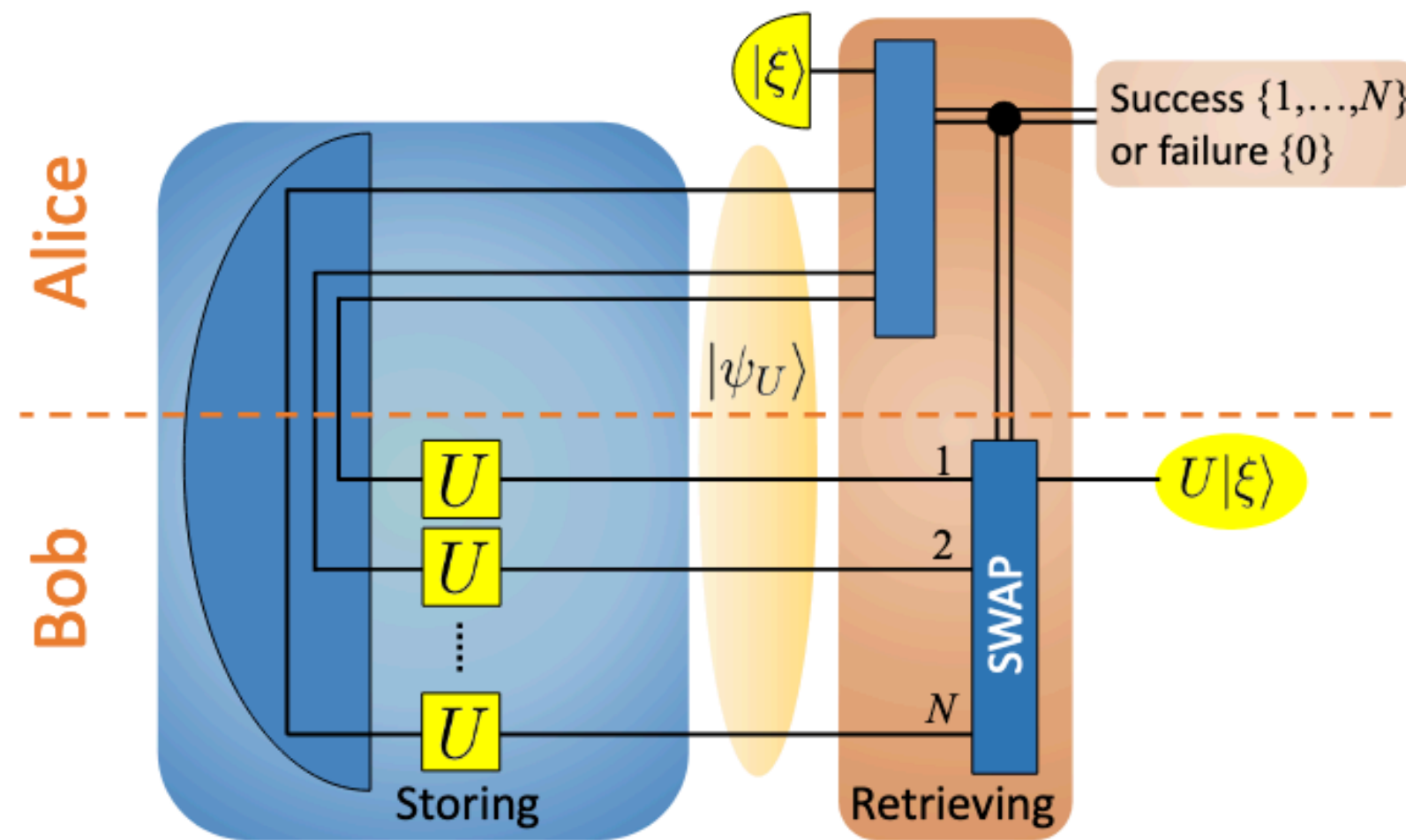
# Port-based teleportation (PBT)

M. Sedlák, A. Bisio, M. Ziman, PRL 122, 170502 (2019)  
A. Bisio et al. PRA 81, 032324 (2010)  
Y. Yang, R. Renner, G. Chiribella, PRL 125, 210501 (2020)

## Optimal protocols for SAR and PBT

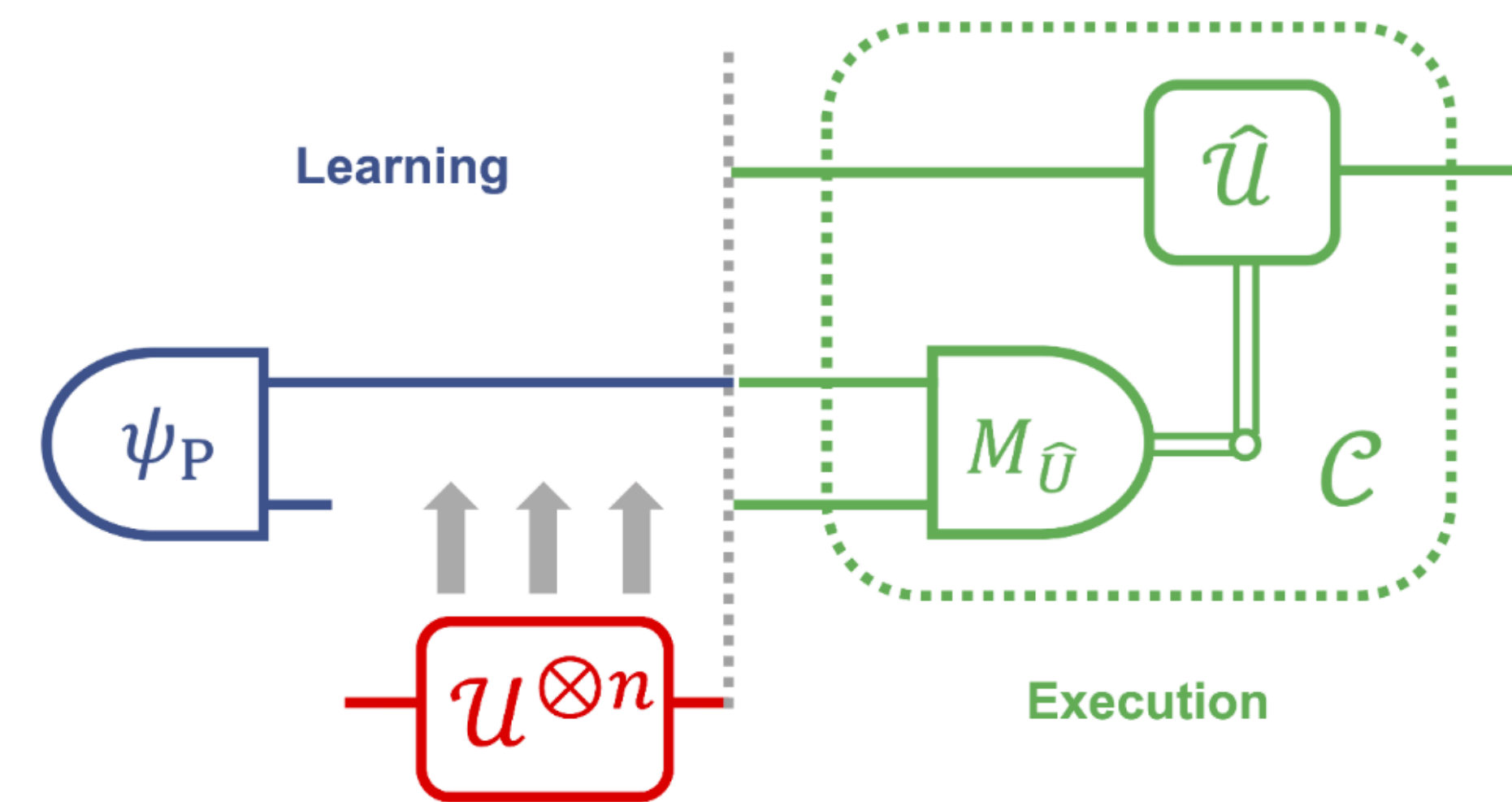
### Probabilistic exact (pPBT)

Optimal pSAR = pPBT



### Deterministic non-exact (dPBT)

Optimal dSAR = unitary estimation  $\neq$  dPBT



Optimal probability  $p_{\text{PBT}}(N, d) = N/(N - 1 + d^2)$

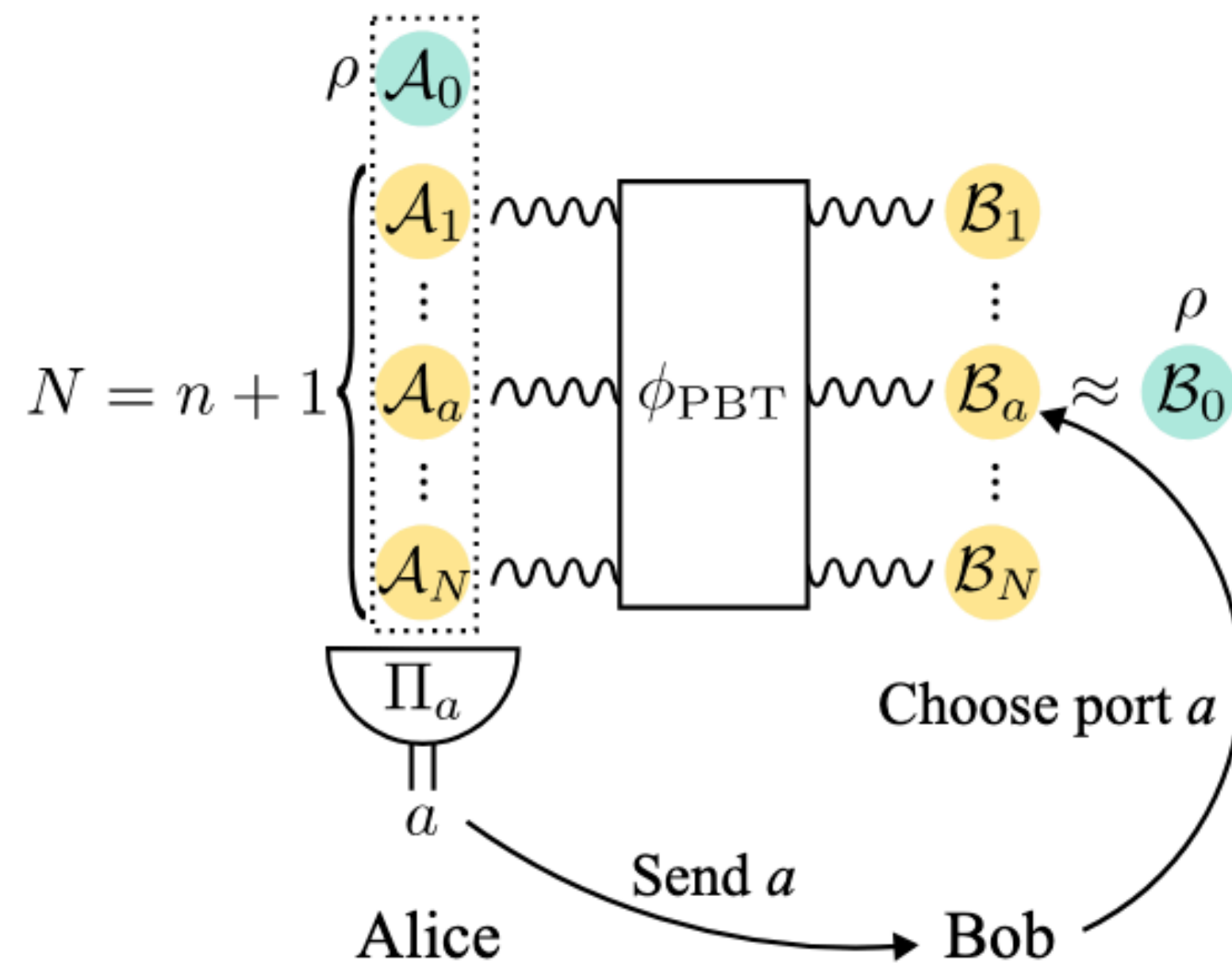
Optimal fidelity  $F_{\text{PBT}}(N, d) = ???$

**Is there any connection between unitary estimation and dPBT?**

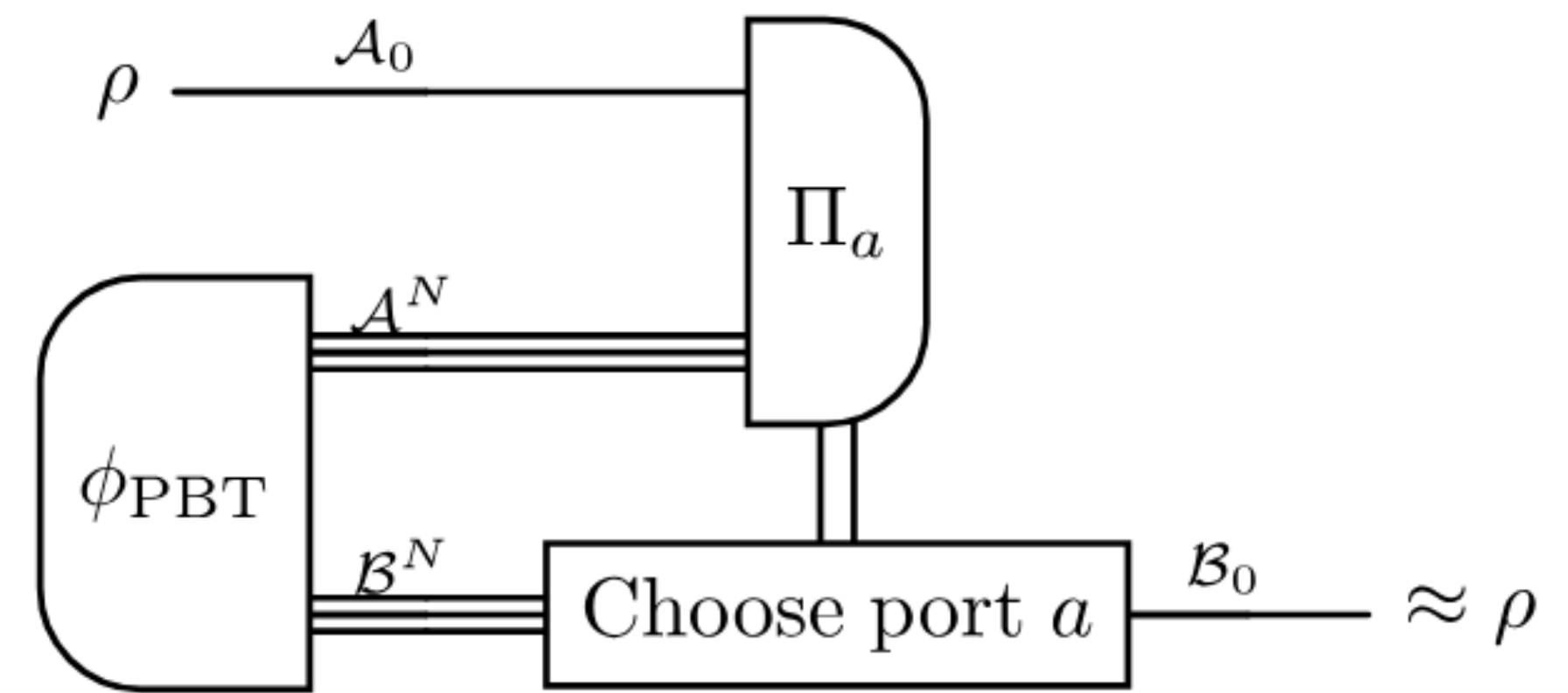
# Outline of this talk

- **Definition of the tasks**
- Main result
- Applications
- Proof techniques
- Conclusion

## Deterministic port-based teleportation (dPBT)



Quantum circuit

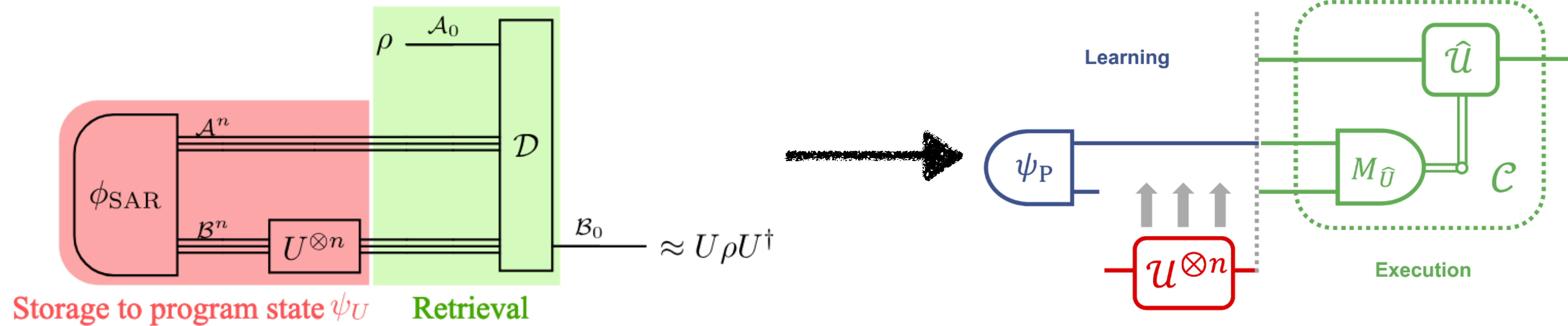


$$\text{Teleportation channel } \Lambda(\rho) = \sum_{a=1}^N \text{Tr}_{\mathcal{A}_0 \mathcal{A}^N \overline{\mathcal{B}}_a} [(\Pi_a \otimes I_{\mathcal{B}^N})(\rho \otimes |\phi_{\text{PBT}}\rangle\langle\phi_{\text{PBT}}|)]$$

$$\text{Figure of merit: } F_{\text{PBT}} = f(I_d, \Lambda)$$

Channel fidelity

## Deterministic storage and retrieval (dSAR)



$$\text{Retrieved channel } \Lambda_U(\rho) = \mathcal{D}[\rho \otimes (I_{\mathcal{A}^n} \otimes U_{\mathcal{B}^n}^{\otimes n}) | \phi_{\text{SAR}} \rangle \langle \phi_{\text{SAR}} | (I_{\mathcal{A}^n} \otimes U_{\mathcal{B}^n}^{\otimes n})^\dagger]$$

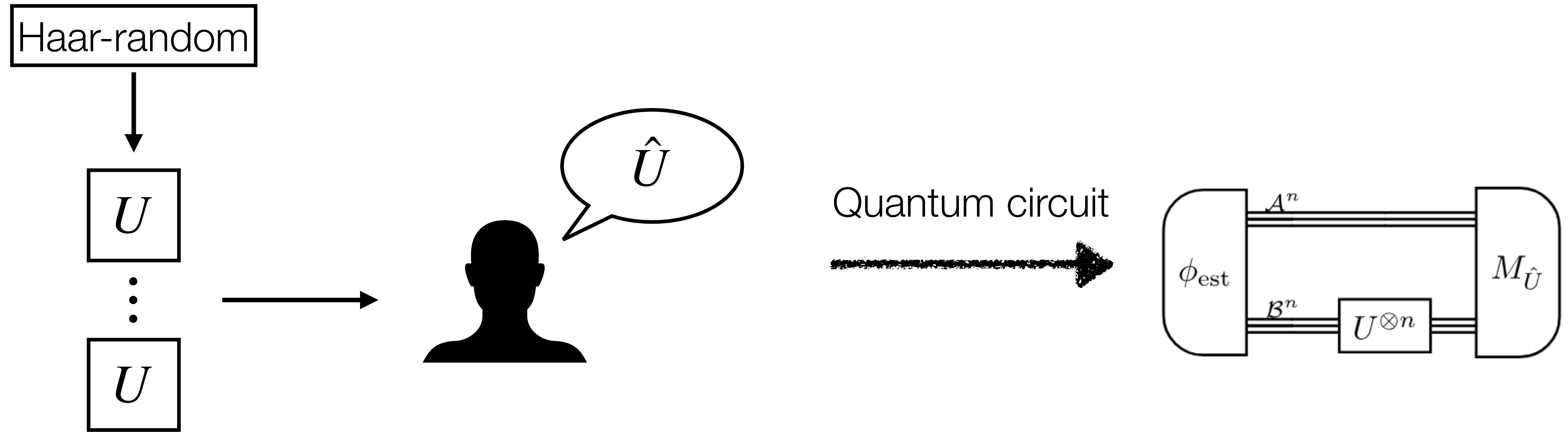
$$\text{Figure of merit: } F_{\text{SAR}} = \int dU f(U, \Lambda_U)$$

**dSAR = unitary estimation**



# Definition of the tasks

## Unitary estimation



Guess probability  $p(\hat{U} | U) = \text{Tr}[M_{\hat{U}}(I_{\mathcal{A}^n} \otimes U_{\mathcal{B}^n}^{\otimes n}) |\phi_{\text{est}}\rangle\langle\phi_{\text{est}}| (I_{\mathcal{A}^n} \otimes U_{\mathcal{B}^n}^{\otimes n})^\dagger]$

Figure of merit:  $F_{\text{est}} = \int dU d\hat{U} p(\hat{U} | U) f(U, \hat{U})$

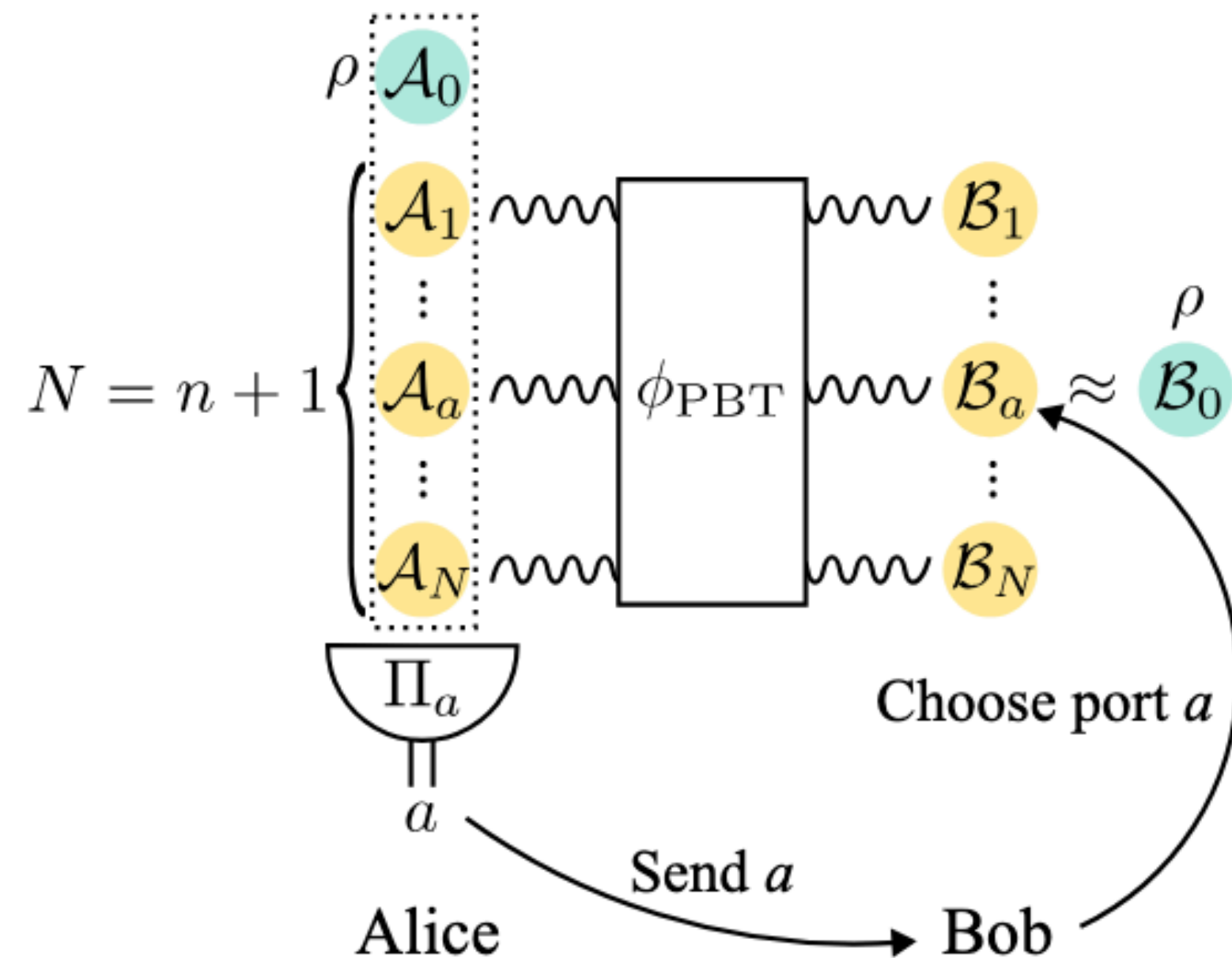
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# Main result

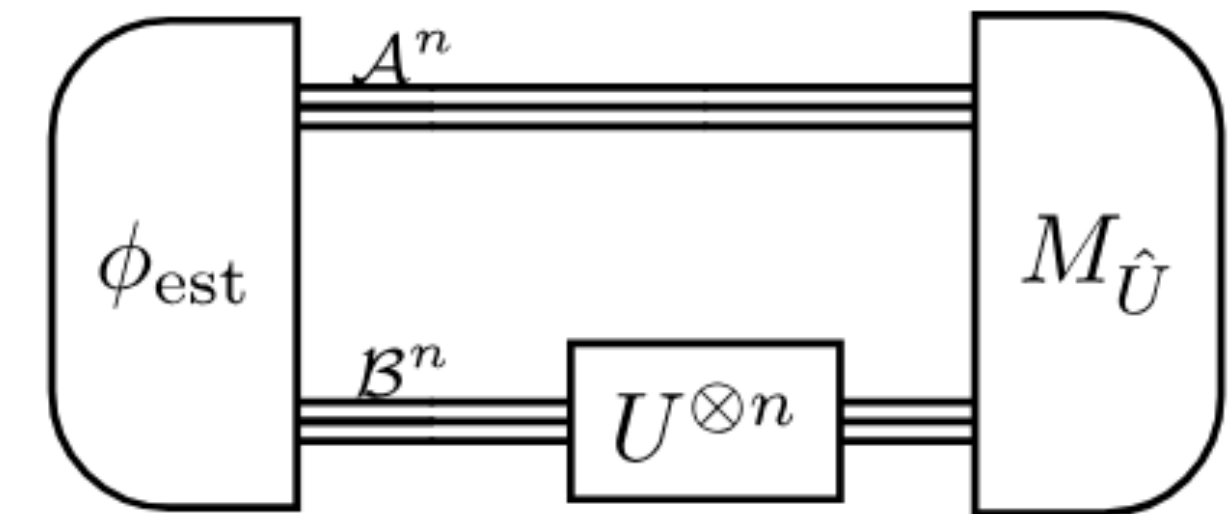
$d$ -dim. dPBT with  $N = n + 1$  ports

$d$ -dim. unitary estimation with  $n$  queries



$$F_{\text{PBT}}(n + 1, d) = F_{\text{est}}(n, d)$$

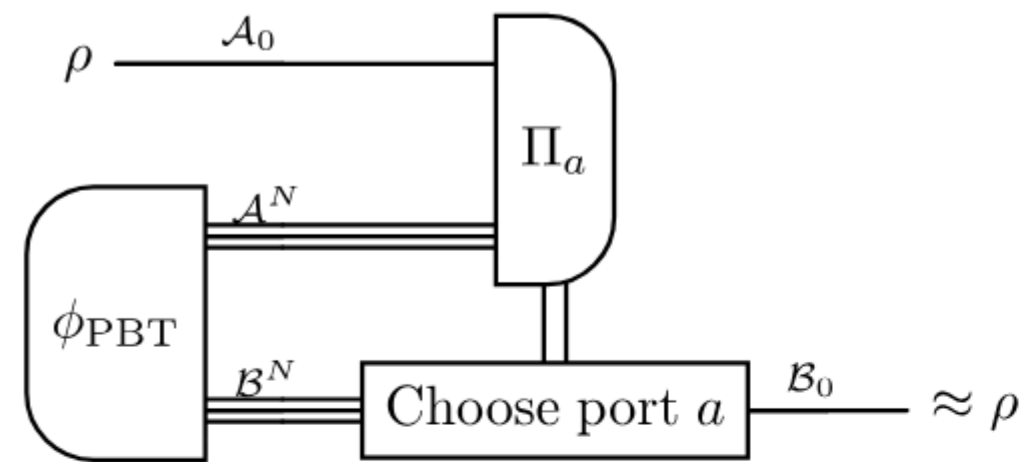
✓ Explicit construction from one protocol to the other



**One-to-one correspondence between dPBT and unitary estimation**

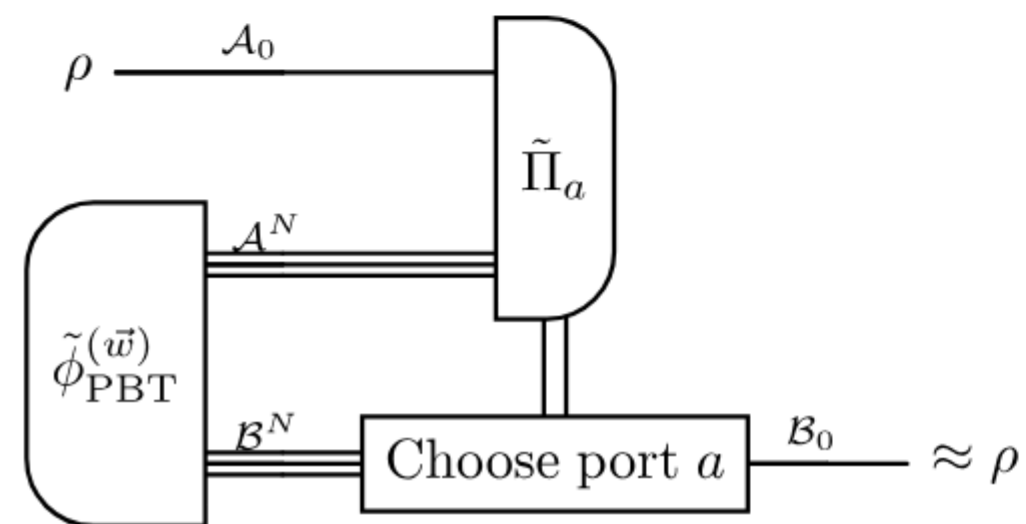
# Main result

(a) General protocol for dPBT



Eqs. (S50), (S52) and (S54)

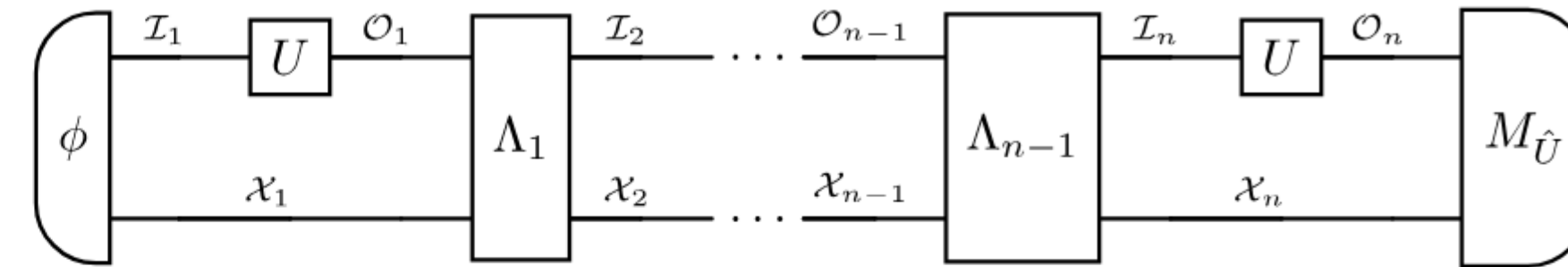
(b) Covariant protocol for dPBT



Eq. (S115)

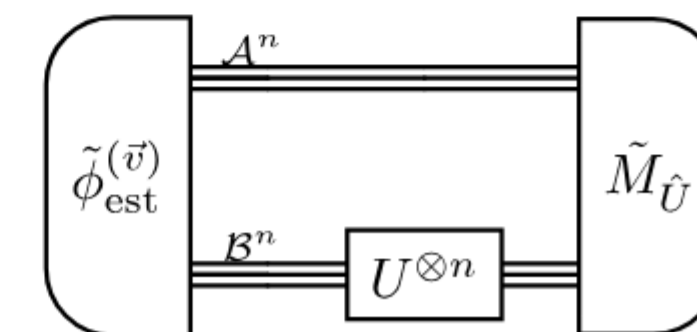
Eq. (S117)

(c) General adaptive protocol for unitary estimation



Eqs. (S87), (S91) and (S92)

(d) Parallel covariant protocol for unitary estimation



**Conversion of the protocols between dPBT and unitary estimation**

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## Asymptotically optimal dPBT

$$\textbf{Corollary. } F_{\text{PBT}}(N, d) = 1 - \Theta(d^4 N^{-2})$$

Optimal unitary estimation obeys Heisenberg limit:  $F_{\text{est}}(n, d) = 1 - \Theta(d^4 n^{-2})$

→ Optimal dPBT also satisfies  $F_{\text{PBT}}(N, d) = 1 - \Theta(d^4 N^{-2})$

✓ Improved over the previous result  $1 - O(d^5 N^{-2}) \leq F_{\text{PBT}}(N, d) \leq 1 - \Omega(d^2 N^{-2})$

✓ Partially solves an open problem to determine  $h(d) = \lim_{N \rightarrow \infty} [1 - F_{\text{PBT}}(N, d)]N^2 = \Theta(d^4)$

## Optimal unitary estimation for $n \leq d - 1$

$$\textbf{Corollary. } F_{\text{est}}(n, d) = \frac{n + 1}{d^2} \quad \text{for } n \leq d - 1$$

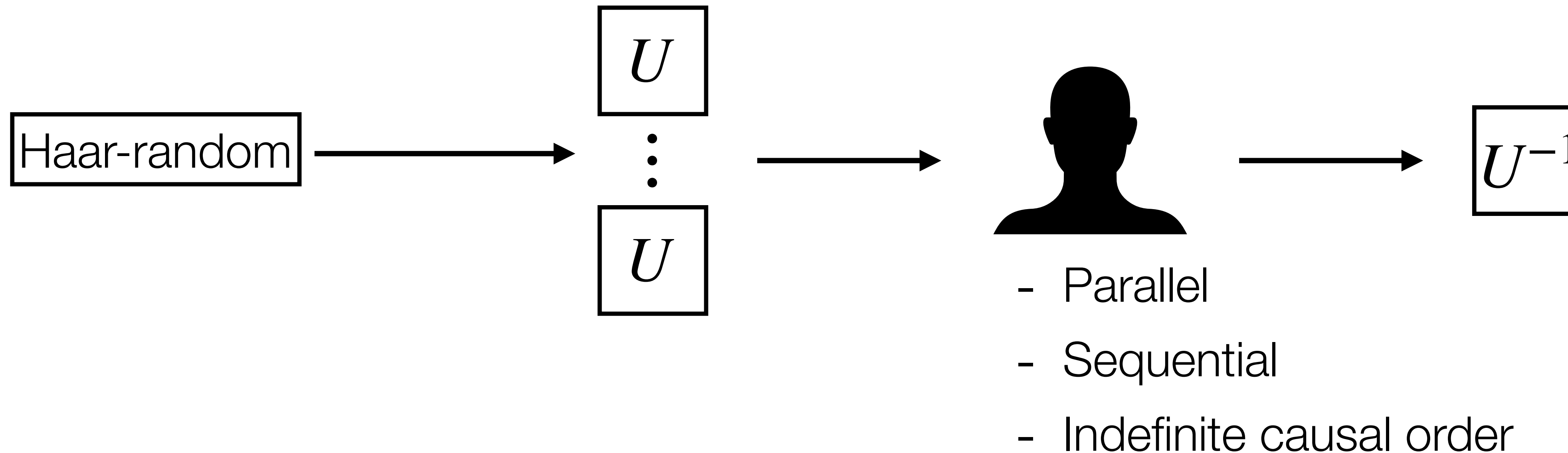
Optimal PBT for  $N \leq d$ :  $F_{\text{PBT}}(N, d) = \frac{N}{d^2}$

→ Optimal unitary estimation for  $n \leq d - 1$ :  $F_{\text{est}}(n, d) = \frac{n + 1}{d^2}$

✓ Optimal resource state for PBT = Optimal resource state for unitary estimation

## Optimal unitary inversion for $n \leq d - 1$

### Unitary inversion



$$\textbf{Theorem. } F_{\text{inv}}^{(\text{PAR})}(n, d) = F_{\text{inv}}^{(\text{SEQ})}(n, d) = F_{\text{inv}}^{(\text{GEN})}(n, d) = \frac{n + 1}{d^2} \quad \text{for } n \leq d - 1$$

✓ Measure-and-prepare is optimal for  $n \leq d - 1$  even allowing indefinite causal order

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# Proof techniques

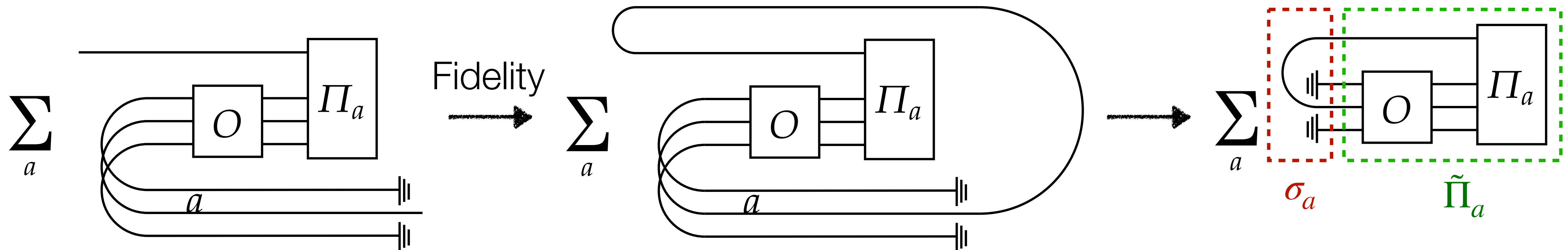
1. **Symmetry of the problem** → **Covariant protocol**
2. Conversion between covariant protocols



## Optimal fidelity of dPBT given by semidefinite programming (SDP)

W.l.o.g. we can consider  $|\phi_{\text{PBT}}\rangle = (O_{\mathcal{A}^N} \otimes I_{\mathcal{B}^N}) |\phi^+\rangle_{\mathcal{A}^N \mathcal{B}^N}$ ,

where  $\text{Tr}[O^\dagger O] = d^N$  and  $|\phi^+\rangle$  is the maximally entangled state



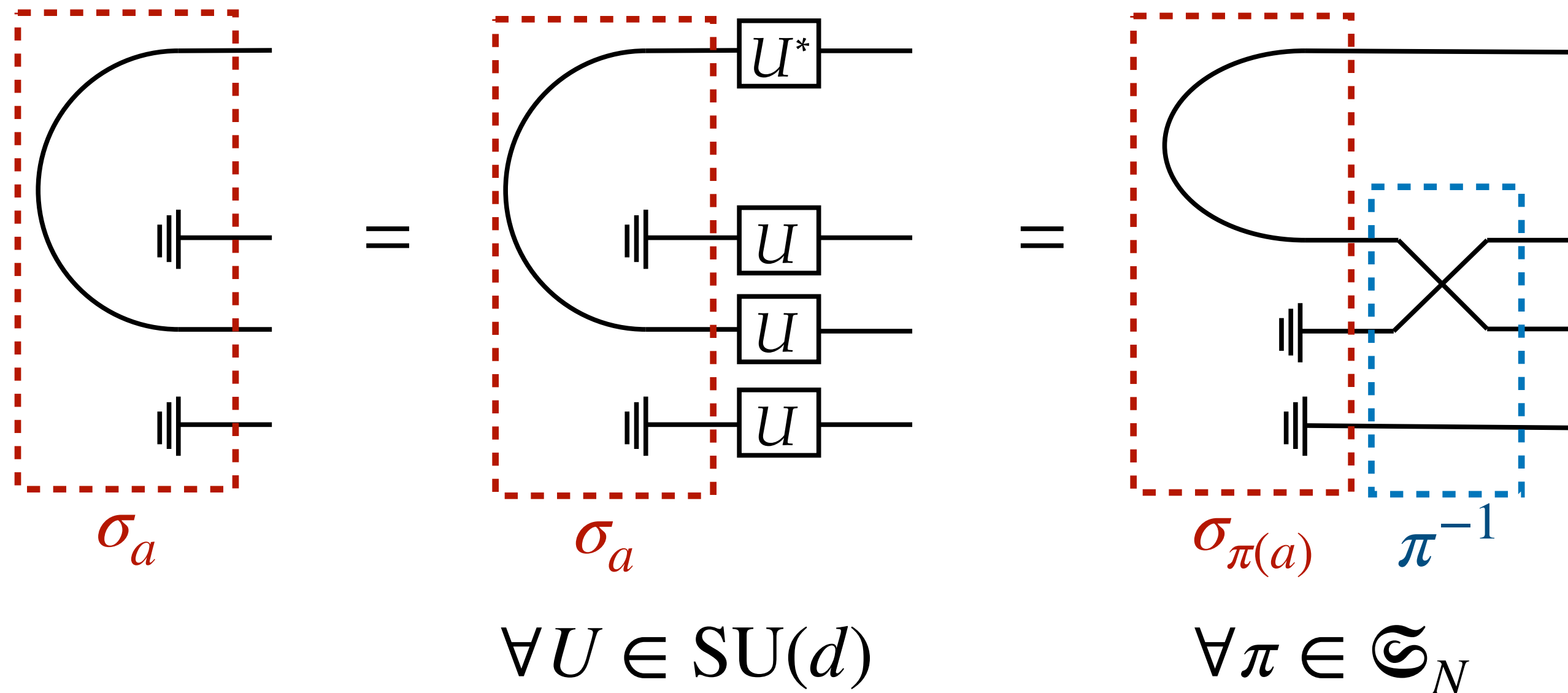
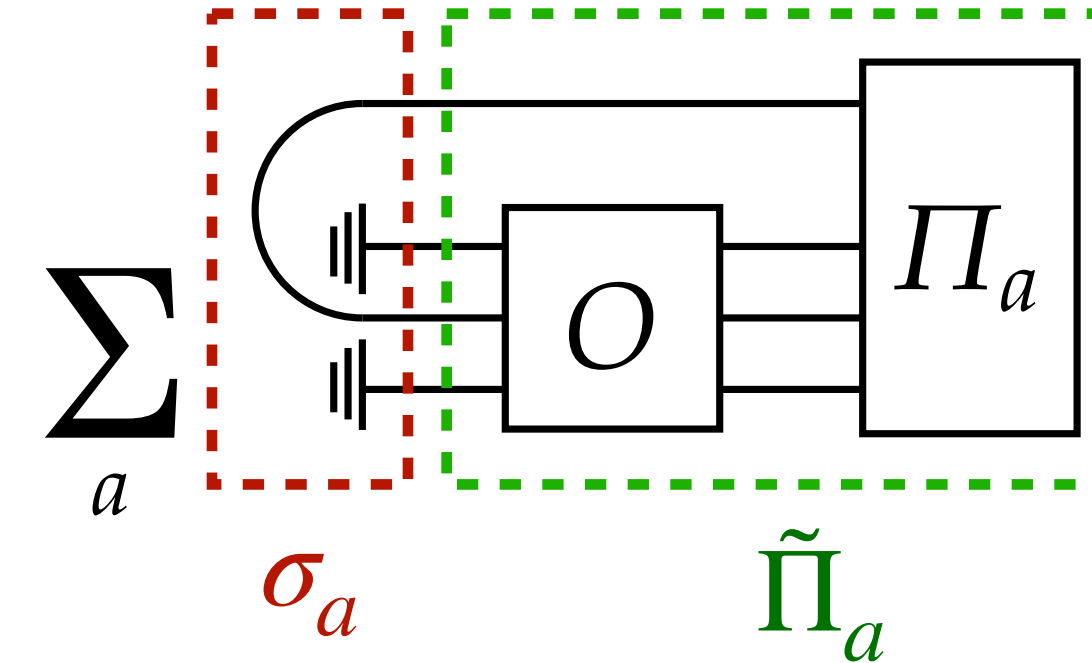
$$F_{\text{PBT}} = \sum_a \text{Tr}[\tilde{\Pi}_a \sigma_a], \quad \sigma_a := |\phi^+\rangle_{\mathcal{A}_0 \mathcal{A}_a} \otimes I_{\overline{\mathcal{A}_a}} / d^{N-2}$$

$$\tilde{\Pi}_a \geq 0, \quad \sum_a \tilde{\Pi}_a = O^\dagger O \otimes I =: X \otimes I$$

$$\begin{aligned} & \max \sum_a \text{Tr}[\tilde{\Pi}_a \sigma_a] \\ & \text{s.t. } \tilde{\Pi}_a \geq 0, \quad \sum_a \tilde{\Pi}_a = X \otimes I, \quad \text{Tr}[X] = d^N \end{aligned}$$

## Unitary and symmetric group symmetry of the SDP for dPBT

$$\begin{aligned} & \max \sum_a \text{Tr}[\tilde{\Pi}_a \sigma_a] \\ \text{s.t. } & \tilde{\Pi}_a \geq 0, \sum_a \tilde{\Pi}_a = X \otimes I, \text{Tr}[X] = d^N \end{aligned}$$



$$\begin{aligned} & \rightarrow [X, U^{\otimes N}] = 0 \quad \forall U \in \text{SU}(d) \\ & [X, \pi] = 0 \quad \forall \pi \in \mathfrak{S}_N \end{aligned}$$

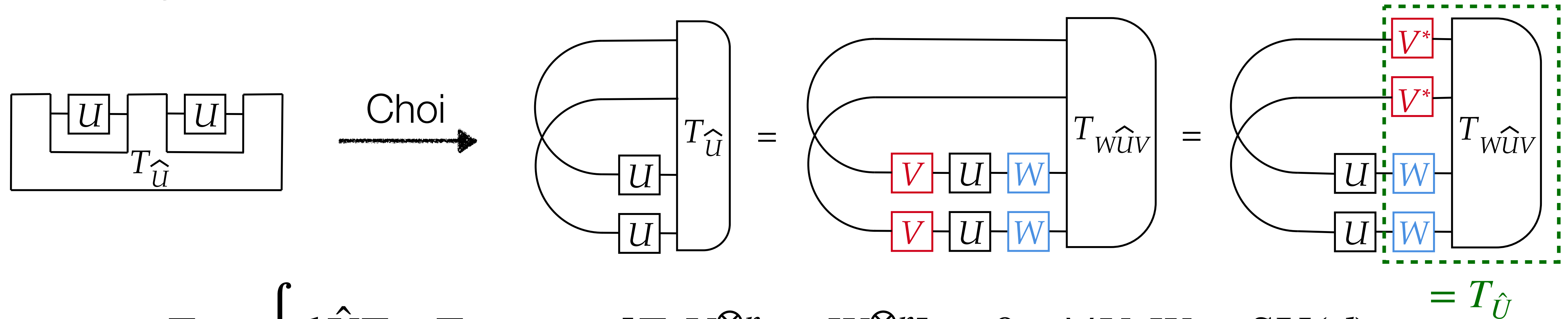
$$\Rightarrow X = \sum_{\mu} w_{\mu} P_{\mu} \quad P_{\mu}: \text{Young projector}$$

$$(\mathbb{C}^d)^{\otimes N} = \bigoplus_{\mu} \mathcal{H}_{\mu} \otimes \mathbb{C}^{m_{\mu}}$$

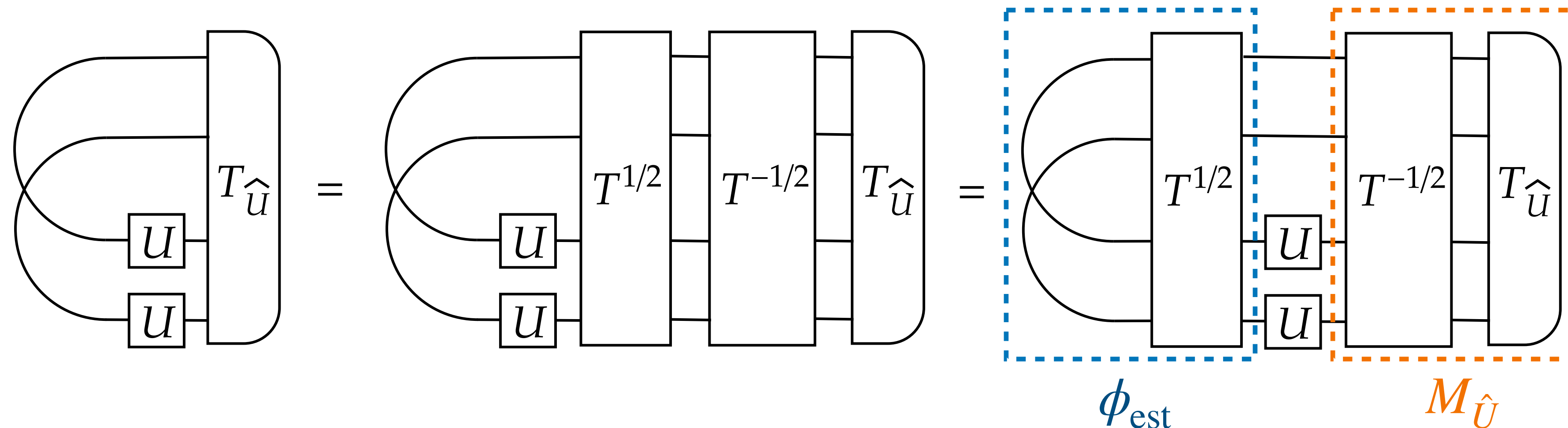
$\mu$ : Young diagram

$$F_{\text{PBT}} = \vec{w}^T M_{\text{PBT}} \vec{w}$$

## Unitary group symmetry for unitary estimation



Defining  $T := \int d\hat{U} T_{\hat{U}}$ ,  $T$  satisfies  $[T, V^{\otimes n} \otimes W^{\otimes n}] = 0 \quad \forall V, W \in \text{SU}(d)$



$$\sqrt{T} = \sum_{\alpha} v_{\alpha} \Pi_{\alpha} \otimes \Pi_{\alpha}$$

$\alpha$ : Young diagram

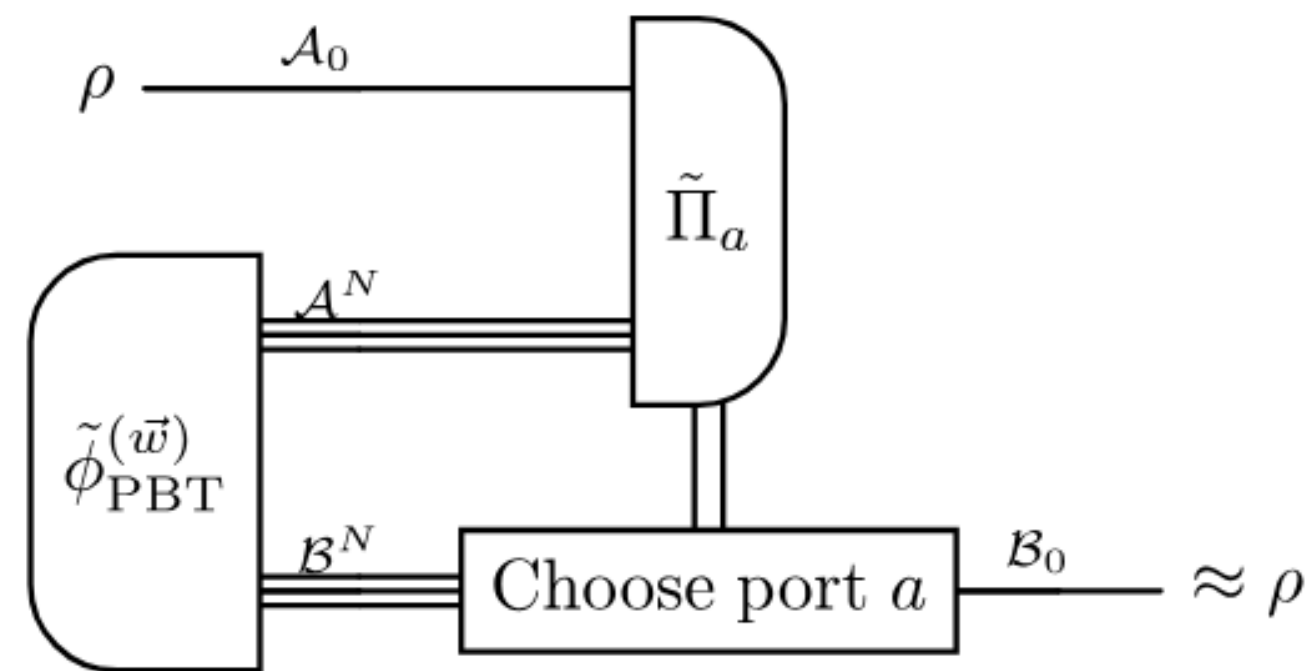
$$F_{\text{est}} = \vec{v}^{\top} M_{\text{est}} \vec{v}$$

# Proof techniques

1. Symmetry of the problem  $\rightarrow$  Covariant protocol
- 2. Conversion between covariant protocols**

## Conversion between the covariant protocols

Covariant dPBT protocol



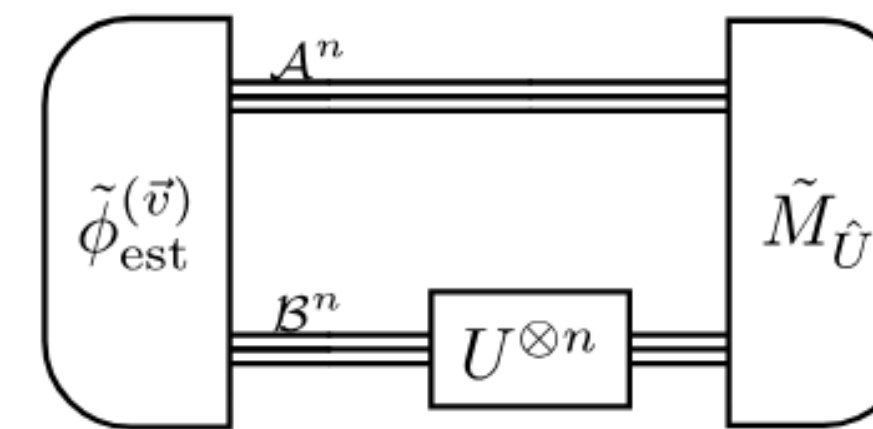
$$F_{\text{PBT}} = \vec{w}^T M_{\text{PBT}} \vec{w}$$

$$(M_{\text{PBT}})_{\mu\nu} = \frac{R(N-1, d)^T R(N-1, d)}{d^2}$$

$$\max F_{\text{PBT}} = \max \text{eig}(M_{\text{PBT}}) = \max \text{sing}(R)^2 / d^2$$

$$\Rightarrow F_{\text{PBT}}(N, d) = F_{\text{est}}(n, d) \quad \text{for } N = n + 1$$

Covariant unitary estimation protocol



$$F_{\text{est}} = \vec{v}^T M_{\text{est}} \vec{v}$$

$$(M_{\text{est}})_{\alpha\beta} = \frac{R(n, d) R(n, d)^T}{d^2} \quad [R(n, d)]_{\alpha\mu} = \delta_{\mu \in \alpha + \square}$$

$$\max F_{\text{est}} = \max \text{eig}(M_{\text{est}}) = \max \text{sing}(R)^2 / d^2$$

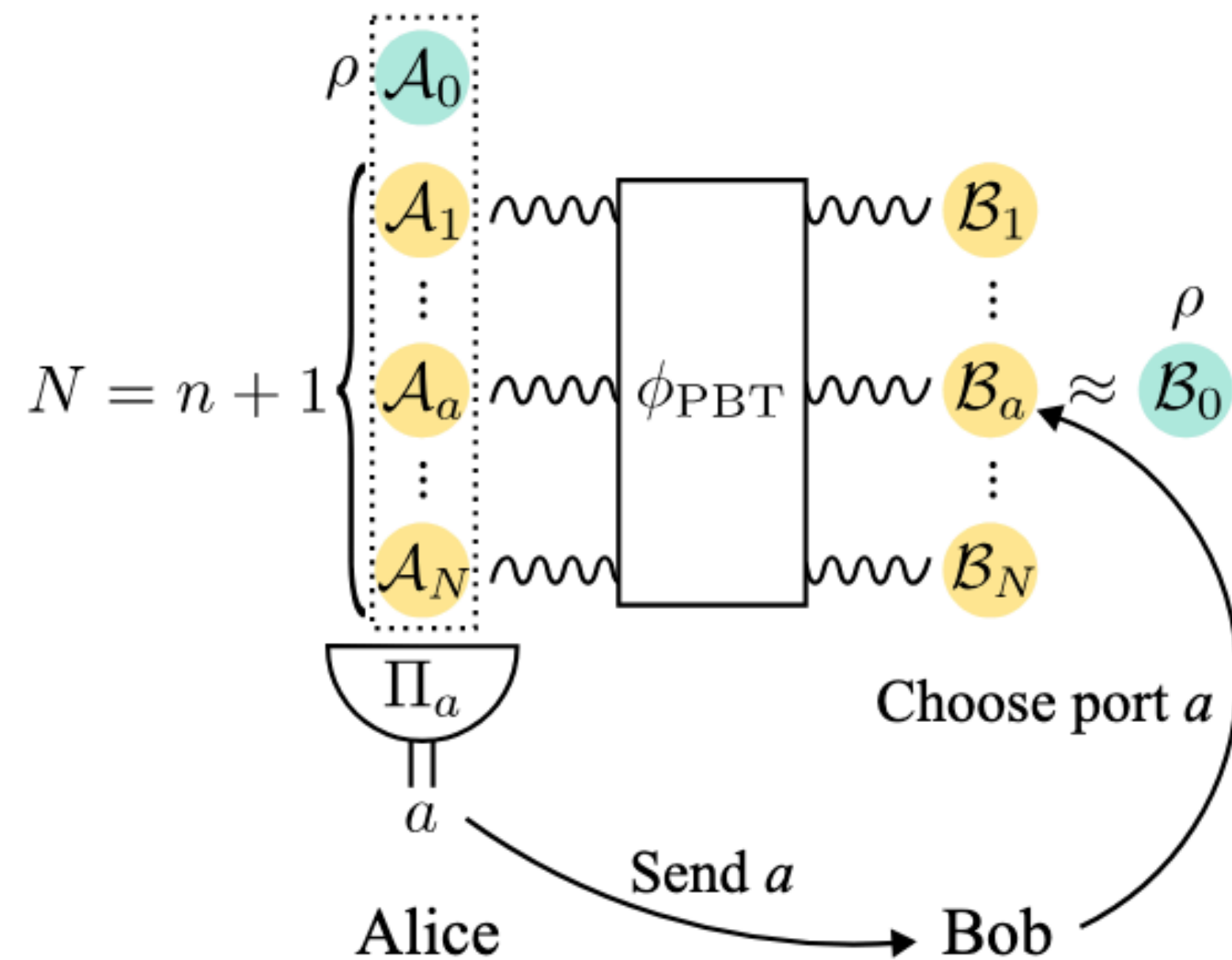


# Outline of this talk

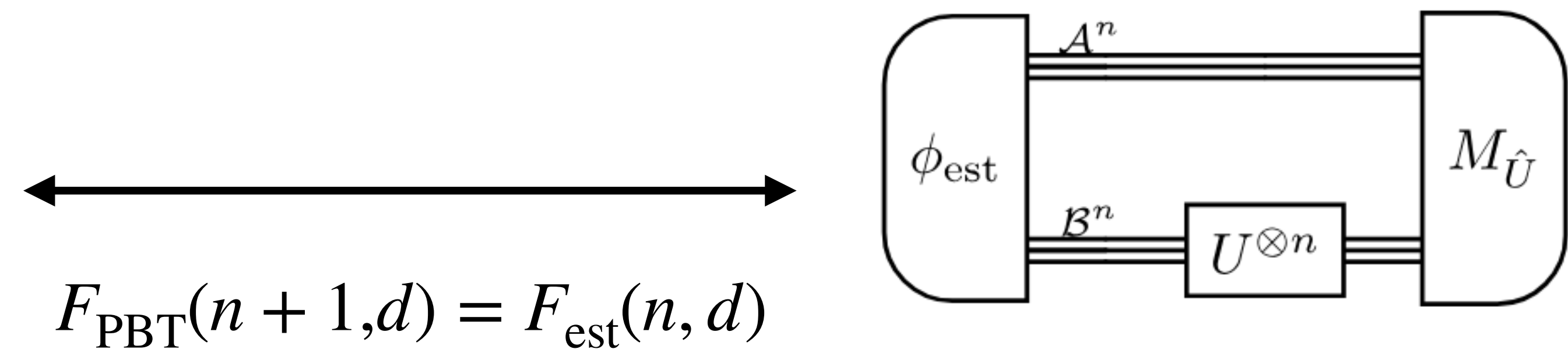
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# Conclusion

$d$ -dim. dPBT with  $N = n + 1$  ports



$d$ -dim. unitary estimation with  $n$  queries



**Take-home: dPBT = unitary estimation!**

- ✓ Explicit construction from one protocol to the other protocol via covariant protocol
  - ✗ Covariant protocol may require more resources than non-covariant one (e.g. #qubits)
- E.g. Adaptive unitary estimation using no auxiliary qubits in [J. Haah et al. FOCS (2023)]

**Future work:** Efficient conversion between unitary estimation and dPBT?

**Thank you!**