7.1. Basic Properties of CI

- Example 7.1

Sample $n=31$. $\bar{x}=80$. Normal Distribution.
with $\sigma=2$. Q: CI of μ?

Since $n=31>30$, sample mean. $\bar{x}$ is normally distributed
with $E=\mu$ and $SD=\frac{\sigma}{\sqrt{n}}$

Thus $Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$.

$P(-1.96<Z<1.96)=0.95$.

Thus, the interval $\bar{x}\pm1.96\cdot\frac{\sigma}{\sqrt{n}}$ or $(\bar{x}-1.96\frac{\sigma}{\sqrt{n}}, \bar{x}+1.96\frac{\sigma}{\sqrt{n}})$

- Define: 95% of CI for μ.

$$(\bar{x}-1.96\cdot\frac{\sigma}{\sqrt{n}}, \bar{x}+1.96\cdot\frac{\sigma}{\sqrt{n}})$$

observed sample mean — Population SD.

- Other levels of Confidence Interval.

A $100(1-\alpha)\%$ confidence Interval:

$$\bar{x}\pm Z_{\alpha/2}\cdot\sigma/\sqrt{n}.$$

Example 7.3: 90% of CI

$\Rightarrow \alpha=0.1$, $Z_{\alpha/2}=Z_{0.05}$.

$\Rightarrow$ Find 0.95% on z-table

$\Rightarrow 1.645$.

- Confidence level, Precision and Sample Size.
  - The price for higher CI is wider Interval.
  - Sample 7.4.

Normally distributed. $\sigma=25$,

Q: What sample size n must satisfy 95% of CI has a width of at most 10.

$10 = 2 \cdot (1.96)\left(\frac{25}{\sqrt{n}}\right)$ $\qquad z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

$$W = 2 \times z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

· Define:

The sample size necessary for the CI is

$$n = \left(2 \cdot z_{\alpha/2} \cdot \frac{\sigma}{W}\right)^2.$$

7.2 Large sample Confidence interval for a population mean and Proportion.

– Proposition: $\rightarrow n > 40.$

If n is sufficiently large, by CLT, $\bar{x}$ has approximately normal distribution. Therefore, large-sample CI for μ with CI approximately $100(1-\alpha)\%$ is $\bar{x} \pm z_{\alpha/2}\frac{\boxed{s}}{\sqrt{n}}$ $\rightarrow$ replace population σ with s.

• A confidence interval for a population Proportion.

– Proposition:

Let $\tilde{p} = \dfrac{\hat{p} + z_{\alpha/2}^2/2n}{1 + z_{\alpha/2}^2/n}$

Then a confidence interval for a population proportion p with Confidence level approximately $100(1-\alpha)\%$ is

$$\tilde{p} \pm z_{\alpha/2}\frac{\sqrt{\hat{p}\hat{q}/n + z_{\alpha/2}^2/4n^2}}{1 + z_{\alpha/2}^2/n} \qquad \hat{q} = 1 - \hat{p}$$

- One-sided confidence intervals.
  - Proposition:
    - A large-sample upper confidence bound for μ is.

$$\mu < \bar{x} + z_{\alpha} \cdot \frac{s}{\sqrt{n}}.$$ Not $z_{\alpha/2}$.

    - . . . . lower . . . . . .

$$\mu > x - z_{\alpha} \cdot \frac{s}{\sqrt{n}}.$$

- General large-sample CI.

  θ from sample, $\hat{\theta}$ estimator.

$$P\left(-z_{\alpha/2} < \frac{\hat{\theta}-\theta}{\sigma_{\theta}} < z_{\alpha/2}\right) \approx 1-\alpha.$$

7.3. Intervals based on a Normal Population Distribution

- When n is small, CLT cannot be invoked. so T-distribution is introduced.

Theorm:

When x is the mean of a random sample of size n from a normal distribution with mean μ.

$$T = \frac{\bar{x}-\mu}{S/\sqrt{n}} \quad (T\text{-distribution})$$

with $n-1$ degree of freedom. (ν).

- Properties of T-distribution
  - Each t-distribution is more spread out than standard normal curve.
  - As ν increase, the spread increase.

- One-sample t CI
  - $P(-t_{\alpha/2, n-1} < T < t_{\alpha/2, n-1}) = 1-\alpha$
  - $100(1-\alpha)\%$ CI for μ:

    $$\left(\bar{x} - t_{\alpha/2, n-1} \cdot \frac{s}{\sqrt{n}},\ \bar{x} + t_{\alpha/2, n-1} \cdot \frac{s}{\sqrt{n}}\right).$$
  - One-sided CI

    $$\bar{x} \pm t_{\alpha, n-1} \cdot \frac{s}{\sqrt{n}}.$$

- Prediction Interval for a single future value.
  - Define: A Prediction Interval (PI) for a single observation to be selected from future interval:

    $$\bar{x} \pm t_{\alpha/2, n-1} \cdot S\sqrt{1+\frac{1}{n}}.$$

- Tolerance Interval.
  - Let k be a number between 0 and 100. A tolerance interval for capturing at least $k\%$ of the value in a normal distribution with confidence level of $\underline{95\%}$

    $$\bar{x} \pm \text{tolerance value} \cdot S.$$