QED Renormalisation

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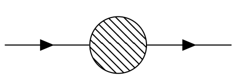
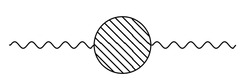
Contents

1 Lagrangian and Feynman Rules of QED

$$\mathcal{L}_{QED} = -\frac{1}{4}(F_{0\mu\nu})^2 + (i\bar{\psi}_0\partial\!\!\!/\psi_0 - m_0\bar{\psi}_0\psi_0) - e_0\bar{\psi}_0\gamma^\mu A_{0\mu}\psi_0 \tag{1}$$

If we find the propagator in interacting vacuum, then we will get some modifications to the propagators which is shown as below.

 $= \dfrac{iZ_2}{p\!\!\!/ - m} + \dots$  $= \dfrac{-iZ_3}{q^2} + \dots$

If we choose similar terminology for A_μ and ψ, and making some other modifications, Then the lagrangian becomes

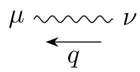
$$e_0 Z_2 Z_3^{1/2} = eZ_1, \quad \psi_0 = Z_2^{1/2}\psi, \quad A_0^\mu = Z_3^{1/2}A^\mu, \quad \delta_m = Z_2 m - m_0, \quad \delta_{1,2,3} = Z_{1,2,3} - 1 \tag{2}$$

We get the Lagrangian

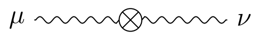
$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\partial\!\!\!/\psi - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi \tag{3}$$

$$- \frac{1}{4}\delta_3(F_{\mu\nu})^2 + \bar{\psi}(i\delta_2\partial\!\!\!/\psi - \delta_m)\psi - e\delta_1\bar{\psi}\gamma^\mu A_\mu \psi \tag{4}$$

Feynman Rules are then given in terms of Renormalised fields

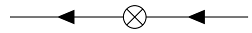


$$= \frac{-ig_{\mu\nu}}{q^2},$$

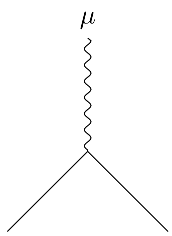


$$= -i(g_{\mu\nu}q^2 - q^\mu q^\nu)\delta_3 \tag{5}$$

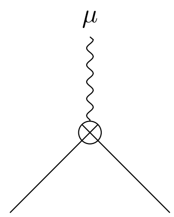
$$= \frac{i}{p\!\!\!/ - m},$$



$$= i(p\!\!\!/\delta_2 - \delta_m) \tag{6}$$



$$= -ie\gamma^\mu,$$



$$= -ie\gamma^\mu\delta_1 \tag{7}$$

2 Divergent Diagrams

There are three divergent diagrams in QED, which are given below and solved.

2.1 Dimensional Analysis

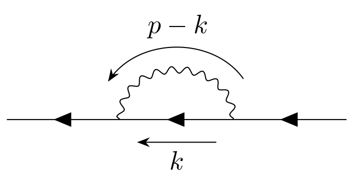
In d dimension using the term $e\bar{\psi}A_\mu\psi$ and $[\psi] = \frac{d-1}{2}, \quad [A_\mu] = \frac{d-2}{2}$, we can get

$$[e] + 2[\psi] + [A_\mu] = d \Rightarrow [e] = 2 - d/2 \tag{8}$$

Then we can find a scale which can make e dimensionless, $e \to e\mu$ where $[\mu] = 2 - d/2$

2.2 Self Interaction Energy of e^-

The Diagram and it's evaluation is given below



$$= -i\Sigma_2(p) = (-ie)^2 \int \frac{d^dk}{(2\pi)^d}\gamma^\mu \frac{i(k\!\!\!/ + m)}{k^2 - m^2}\gamma^\nu \frac{-ig_{\mu\nu}}{(p-k)^2 - \lambda^2} \tag{9}$$

where λ is the fictitious mass of the photon to avoid infrared divergence.

$$\begin{aligned} -i\Sigma_2(p) &= (-ie)^2 \int \frac{d^dk}{(2\pi)^d}\gamma^\mu \frac{i(k\!\!\!/ + m)}{k^2 - m^2}\gamma^\nu \frac{-ig_{\mu\nu}}{(p-k)^2 - \lambda^2} &(10)\\ &= -e^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{(-(d-2)k\!\!\!/ + dm)}{(k^2 - m^2)((p-k)^2 - \lambda^2)}, \quad \text{using} \quad (86), (87) &(11)\end{aligned}$$

Using Feynman parametrisation and shifting to $l = k - yp$ and ignoring the odd powers of $l\!\!\!/$, we get

$$-i\Sigma_2(p\!\!\!/) = -e^2\mu^{4-d}\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{-(d-2)xp\!\!\!/ + dm}{[l^2 - \Delta]^2}, \quad \Delta = -x(1-x)p^2 + x\lambda^2 + (1-x)m^2 \tag{12}$$

Doing Wick Rotation we get

$$-i\Sigma_2(\not{p}) = -ie^2\mu^{4-d}\int_0^1 dx \int \frac{d^d l_E}{(2\pi)^d}\frac{-(d-2)x\not{p} + dm}{[l_E^2+\Delta]^2} \tag{13}$$

$$= -i\frac{e^2}{16\pi^2}\int_0^1 dx[-2x\not{p} + 4m + \epsilon(x\not{p} - m)]\left(1+\frac{\epsilon}{2}\log\mu^2\right)\left(\frac{2}{\epsilon}+\log\left(\frac{4\pi e^{-\gamma}}{\Delta}\right)\right) \text{ using(89)} \tag{14}$$

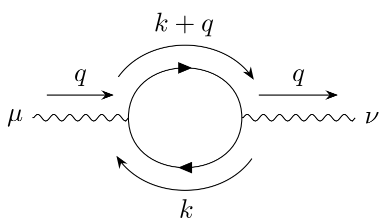
$$= -i\frac{e^2}{16\pi^2}\int_0^1 dx[-2x\not{p} + 4m + \epsilon(x\not{p} - m)]\left(\frac{2}{\epsilon}+\log\left(\frac{4\pi e^{-\gamma}\mu^2}{\Delta}\right)\right) \tag{15}$$

$$= -i\frac{e^2}{16\pi^2}\left[\frac{2}{\epsilon}(-\not{p}+4m)+\not{p}-2m+2\int_0^1 dx(-x\not{p}+2m)\log\left(\frac{\bar{\mu}^2}{\Delta}\right)\right], \quad \bar{\mu}^2 = 4\pi e^{-\gamma}\mu^2 \tag{16}$$

$$\Sigma_2(\not{p}) = \frac{1}{\epsilon}\frac{\alpha}{2\pi}(-\not{p}+4m)+\frac{\alpha}{4\pi}\left[\not{p}-2m+2\int_0^1 dx(-x\not{p}+2m)\log\left(\frac{\bar{\mu}^2}{\Delta}\right)\right], \tag{17}$$

$$\Delta = -x(1-x)p^2 + x\lambda^2 + (1-x)m^2$$

2.3 Vacuum Polarisation



$$= i\Pi_2^{\mu\nu}(q) = (-1)(-ie)^2\int\frac{d^dk}{(2\pi)^d}Tr\left[\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right] \tag{18}$$

After applying (90) and (91) in the numerator and doing feynman parametrisation I got,

$$i\Pi_2^{\mu\nu}(q) = -4e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k.(k+q)-m^2)}{[k^2+2xk.q+xq^2-m^2]^2} \tag{19}$$

$$= -4e^2\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{2l^\mu l^\nu - g^{\mu\nu}l^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}x(1-x)q^2 + g^{\mu\nu}m^2}{[l^2-\Delta]^2} \tag{20}$$

$$= -4e^2\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{g^{\mu\nu}(\frac{2}{d}-1)l^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}x(1-x)q^2 + g^{\mu\nu}m^2}{[l^2-\Delta]^2} \tag{21}$$

$$= -4ie^2\int_0^1 dx\left[g^{\mu\nu}(1-\frac{2}{d})\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2+\Delta]^2} + B\int\frac{d^dl}{(2\pi)^d}\frac{1}{[l^2+\Delta]^2}\right] \text{ Using Wick's Rotation} \tag{22}$$

where $B = g^{\mu\nu}(x(1-x)q^2+m^2) - 2x(1-x)q^\mu q^\nu$ and $\Delta = m^2 - x(1-x)q^2$

$$i\Pi_2^{\mu\nu}(q) = -4ie^2\int_0^1\frac{dx}{(4\pi)^{d/2}}\left[g^{\mu\nu}\frac{d}{2}(1-\frac{2}{d})\frac{\Gamma(1-d/2)}{\Delta^{(1-d/2)}} + B\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\right] \tag{23}$$

$$= -4ie^2\int_0^1\frac{dx}{(4\pi)^{d/2}}\left[-g^{\mu\nu}\Gamma(2-d/2)\frac{\Delta}{\Delta^{(2-d/2)}} + B\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\right] \tag{24}$$

$$= -4ie^2\int_0^1\frac{dx}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}[-g^{\mu\nu}\Delta + B] \tag{25}$$

$$= -4ie^2\int_0^1\frac{dx}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}[g^{\mu\nu}q^2 - q^\mu q^\nu]2x(1-x) \tag{26}$$

$$i\Pi_2^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)i\Pi_2(q^2) \tag{27}$$

where $\Pi_2(q^2) = -8e^2\int_0^1\frac{dx}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}x(1-x)$
We could have seen this from Ward Identity $q_\mu\Pi_2^{\mu\nu}(q) = 0$. Then the tensor can only depend on two tensors $q^\mu q^\nu$ and $g^{\mu\nu}$. Choosing the appropriate coefficients for these and we get $(g^{\mu\nu}q^2 - q^\mu q^\nu)$ which when taken product with q_μ becomes 0.

The common coefficient can only depend on q^2 which we called $\Pi_2(q^2)$.
Now calculating the common coefficient, with scale μ

$$\Pi_2(q^2) = -8e^2 \int_0^1 \frac{dx}{(4\pi)^{d/2}} \mu^{4-d} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} x(1-x) \tag{28}$$

$$= -8\frac{e^2}{16\pi^2} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma}\mu^2}{\Delta} \right) \right) \tag{29}$$

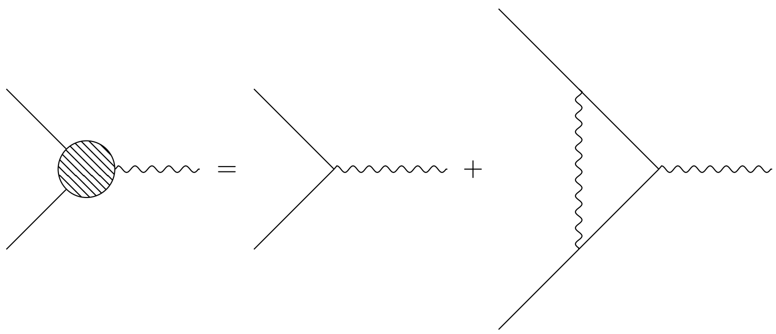
$$= -\frac{2\alpha}{\pi} \left(\frac{1}{3\epsilon} + \int_0^1 dx\, x(1-x) \log\left(\frac{\bar{\mu}^2}{\Delta} \right) \right) \tag{30}$$

$$\Pi_2(q^2) = -\frac{2\alpha}{3\pi\epsilon} - \frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \log\left(\frac{\bar{\mu}^2}{\Delta} \right), \quad \Delta = m^2 - x(1-x)q^2 \tag{31}$$

We can see that $\Pi(q^2 = 0)$ is regular.

2.4 Vertex Function

Vertex function is given as



(32)

$$-ie\Gamma^\mu = -ie\gamma^\mu - ie\delta\Gamma^\mu \tag{33}$$

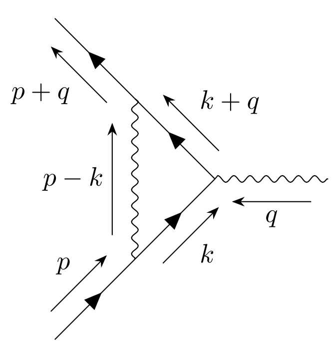
As there are only three tensors possible to have this form, we can say that

$$\Gamma^\mu = A\gamma^\mu + B(p' + p)^\mu + C(p' - p)^\mu \tag{34}$$

where $p', p, q = p' - p$ are the final momentum, inital momentum, and photon momentum. By Ward Identity, $q_\mu \Gamma^\mu = 0$ we can see that third term doesn't necessarily vanish when squeezed between $\bar{u}(p'), u(p)$, but we want the whole to be 0. Therefore, $C = 0$.
Then using Gordon's Identity (93), we get the form

$$\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2) \tag{35}$$

Now, we can calculate only the $\delta\Gamma$ part



$$= \bar{u}(p')\delta\Gamma^\mu u(p) = \int \frac{d^d k}{(2\pi)^d} \frac{-ig_{\nu\rho}}{(k-p)^2} \bar{u}(p')(-ie\gamma^\nu) \frac{i(\not{k}' + m)}{k'^2 - m^2} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2} (-ie\gamma^\rho)$$

In 4 dimension only $F_1(q^2)$ will be divergent and $F_2(q^2)$ won't be divergent at all.
So, I calculated $\delta F_1(q^2) = F_1(q^2) - 1$ in d dimensions and $F_2(q^2)$ in 4 dimensions given below. These will be the

First order contribution to the vertex function.

$$\delta F_1(q^2) = \frac{e^2}{(4\pi)^{d/2}} \int dxdydz\delta(x+y+z-1)\Big[\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\frac{(2-\epsilon)^2}{2} \tag{36}$$

$$+ \frac{\Gamma(3-d/2)}{\Delta^{3-d/2}}(q^2[2(1-x)(1-y)-\epsilon xy] + m^2[2(1-4z+z^2)-\epsilon(1-z)^2])\Big] \tag{37}$$

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m^2z(1-z)}{m^2(1-z)^2} \tag{38}$$

$$= \frac{\alpha}{\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{z}{(1-z)} \tag{39}$$

$$F_2(q^2) = \frac{\alpha}{2\pi} \tag{40}$$

For $\delta F_1(q^2)$, I took $d = 4-\epsilon$, neglecting positive powers of ϵ, I got,

$$\delta F_1(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\left[\frac{2}{\epsilon} - 1 + \log\left(\frac{\bar{\mu}^2}{\Delta}\right) + \frac{q^2(1-x)(1-y)+m^2(1-4z+z^2)}{\Delta}\right] \tag{41}$$

$$= \frac{\alpha}{2\pi\epsilon} - \frac{\alpha}{4\pi} + \dots \tag{42}$$

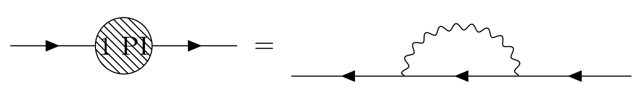
where $\Delta = m^2(1-z)^2 - q^2xy + \lambda^2 z$, λ is the fictitious mass of photon to avoid infrared divergence.

3 Renormalisation conditions

3.1 For self interaction diagram

Let $-i\Sigma(p)$ be the propagators for 1 PI diagrams

$-i\Sigma(p)$ =

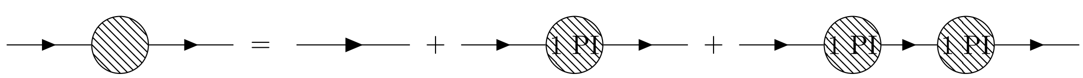


$+ \dots$

For a good enough approximation we can say that it is equal to the first order self interaction energy

$$-i\Sigma(p) = -i\Sigma_2(p)$$

Then the propagator in the new renormalised field will have following sum in terms of bare couplings and fields with additional loops



(43)

which should give the propagator in the renormalised fields and renormalised mass in terms of bare coupling, as mentioned in the first section. Thus, above equation can be interpreted as

$$\frac{iZ_2}{\not{p}-m} = \frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}[-i\Sigma_2(p)]\frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}[-i\Sigma_2(p)]\frac{i}{\not{p}-m_0}[-i\Sigma_2(p)]\frac{i}{\not{p}-m_0} \tag{44}$$

$$= \frac{i}{\not{p}-m_0}\left[1 + \frac{\Sigma}{\not{p}-m_0} + \left(\frac{\Sigma}{\not{p}-m_0}\right)^2 + \dots\right] \tag{45}$$

$$\frac{iZ_2}{\not{p}-m} = \frac{i}{\not{p}-m_0-\Sigma_2} \tag{46}$$

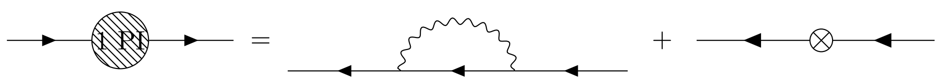
Close to the pole we will have the denominator as

$$\not{p} - m = 0 + (\not{p}-m)\frac{d}{d\not{p}}(\not{p}-m_0-\Sigma_2(\not{p}))\Big|_{\not{p}=m} = (\not{p}-m)\left(1 - \frac{d}{d\not{p}}\Sigma_2(\not{p})\Big|_{\not{p}=m}\right)$$

When this denominator goes into the numerator we get the corrections

$$Z_2 = 1 + \frac{d}{d\not{p}}\Sigma_2(\not{p})\Big|_{\not{p}=m}$$

But these are **divergent terms**.
Now if we add the counterterms too in the sum for propagator, then it should become finite, i.e.



(47)

$$-i\Sigma_f(\not{p}) = -i\Sigma_2(\not{p}) + i(\not{p}\delta_2 - \delta_m) \tag{48}$$

Adding these counter terms is equivalent to renormalising the theory, so the new 1 PI diagrams will be finite only and adding all the contributions will give a geometric series in terms of σ_f and m.
But We know that there should be a pole at $\not{p} = m$(physical mass observed experimentally), as shown below,

$$\frac{iZ_2}{\not{p} - m} = \frac{i}{\not{p} - m - \Sigma_f(\not{p})} \xrightarrow{\not{p}\to m} \frac{i}{\not{p} - m} \tag{49}$$

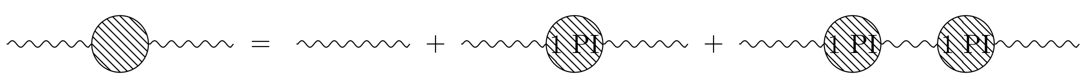
which gives us the Renormalisation conditions namely,

$$\Sigma_f(\not{p} = m) = 0 \tag{50}$$

$$\frac{d}{d\not{p}}\Sigma_f(\not{p})\Big|_{\not{p}=m} = 0, \quad \text{as } Z_2 \xrightarrow{\not{p}\to m} 1 \tag{51}$$

3.2 Vacuum Polarisation

Similar to above if we say that the renormalised propagator for photon can be given as sum of all the inner polarisations. Then

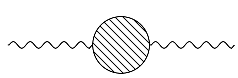


(52)

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}[i(q^2g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)]\frac{-ig_{\sigma\nu}}{q^2} + \ldots \tag{53}$$

$$= \frac{-i}{q^2(1-\Pi(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right) \tag{54}$$

Now, when added into a Feynman diagram and connected to fermion, there should be only $g^{\mu\nu}$ in the propagator, thus taking only this we get

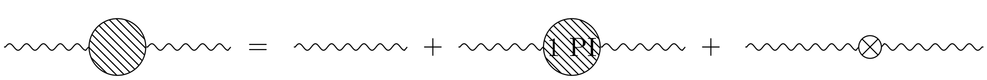


$$= \frac{-iZ_3}{q^2} = \frac{-ig_{\mu\nu}}{q^2(1-\Pi(q^2))}$$

Thus,

$$Z_3 = \frac{1}{1-\Pi(q^2)} \tag{55}$$

Similar to the Self interaction energy of e^-, now if we do the whole procedure with counter terms feynman diagrams too, then the summation will have the form



(56)

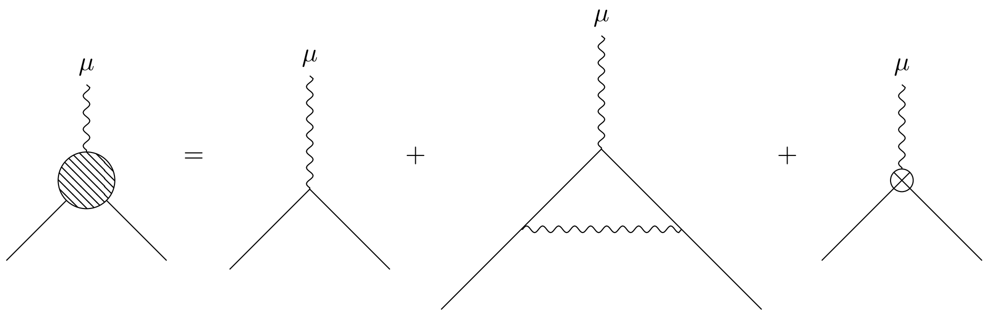
$$= \frac{-ig^{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}[i(q^2g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)]\frac{-ig_{\sigma\nu}}{q^2}\frac{-ig_{\mu\rho}}{q^2}[-i(q^2g^{\rho\sigma} - q^\rho q^\sigma)\delta_3]\frac{-ig_{\sigma\nu}}{q^2} \tag{57}$$

$$= \frac{-ig^{\mu\nu}}{q^2(1-\Pi_f(q^2))} \tag{58}$$

We know that the renormalised photon propagator should have the pole at $q^2 = 0$. Therefore, we should get that $\Pi_f(q^2 = 0) = 0$, where $\Pi_f(q^2) = \Pi(q^2) - \delta_3$, by the addition of counter term contribution.

3.3 Vertex Correction

If we add the counter term to the diagrams we get



(59)

$$
\begin{aligned}
-ie\Gamma^{\mu}_{f}(p',p) &= -ie(\gamma^{\mu}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2)) - ie\delta_1 &(60)\\
&= -ie(\gamma^{\mu}(1+\delta F_1(q^2)+\delta_1) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2m}F_2(q^2)) &(61)\\
&\xrightarrow{p',p\,\text{on-shell},q\to 0} -ie\gamma^{\mu}(1+\delta F_1(0)+\delta_1) &(62)
\end{aligned}
$$

On shell condition says that after renormalisation and adding counter terms we should get at just a vertex which is $-ie\gamma^{\mu}$,i.e.

$$-ie\Gamma^{\mu}_{f}(p'-p=0) = -ie\gamma^{\mu} \tag{63}$$

4 Finding the counter terms coefficients

The Renormalisation conditions for on-shell renormalisation was found as

$$
\begin{aligned}
\Sigma_2(\not{p}=m) - (\not{p}\delta_2 - \delta_m)|_{\not{p}=m} &= & \Sigma_f(\not{p}=m) && = 0 &&(64)\\
\frac{d}{d\not{p}}\Sigma_2(\not{p})\Big|_{\not{p}=m} - \delta_2 &= & \tfrac{d}{d\not{p}}\Sigma_f(\not{p})\Big|_{\not{p}=m} && = 0 &&(65)\\
\Pi_2(q^2=0) - \delta_3 &= & \Pi_f(q^2=0) && = 0 &&(66)\\
-ie\gamma^{\mu}(1+\delta F_1(0)+\delta_1) &= & -ie\Gamma^{\mu}_{f}(p'-p=0) && = -ie\gamma^{\mu} &&(67)
\end{aligned}
$$

These conditions helps us find the δ's as given below

$$
\begin{aligned}
m\delta_2 - \delta_m &= \Sigma_2(\not{p}=m) &(68)\\
\delta_2 &= \frac{d}{d\not{p}}\Sigma_2(\not{p})\Big|_{\not{p}=m} &(69)\\
\delta_3 &= \Pi_2(q^2=0) &(70)\\
\delta_1 &= -\delta F_1(0) &(71)
\end{aligned}
$$

Now, we can find the corrections upto only $1/\epsilon$ order

$$
\begin{aligned}
\delta_1 &= -\frac{\alpha}{2\pi\epsilon} &(72)\\
\delta_2 &= -\frac{\alpha}{2\pi\epsilon} &(73)\\
\delta_3 &= -\frac{2\alpha}{3\pi\epsilon} &(74)\\
\delta_m &= -\frac{\alpha(3+m)}{2\pi\epsilon} &(75)
\end{aligned}
$$

5 β function and Coupling running

Relation between bare coupling and renormalised coupling is given as

$$\mu^{\epsilon/2} e = Z_3 e_0 \Rightarrow \alpha_0 = \frac{\mu^\epsilon \alpha}{Z_3} \tag{76}$$

where $Z_3 = 1 + \delta_3 = 1 - \frac{2\alpha}{3\pi\epsilon}$, and choosing $\beta = \mu \frac{d\alpha}{d\mu}$
Now, bare coupling α_0 doesn't depend on the scale we have used, so,

$$\mu \frac{d\alpha_0}{d\mu} = \mu \frac{d}{d\mu} \frac{\mu^\epsilon \alpha}{Z_3} = 0 \tag{77}$$

$$\Rightarrow \epsilon \frac{\mu^\epsilon \alpha}{Z_3} + \frac{\mu^\epsilon}{Z_3}\beta - \frac{\mu^\epsilon \alpha}{Z_3^2} \mu \frac{dZ_3}{d\mu} = 0 \tag{78}$$

$$\Rightarrow \epsilon \frac{\mu^\epsilon \alpha}{Z_3} + \frac{\mu^\epsilon}{Z_3}\beta + \frac{\mu^\epsilon \alpha}{Z_3^2} \frac{2\beta}{3\pi\epsilon} = 0 \tag{79}$$

$$\Rightarrow \epsilon \alpha Z_3 + \beta \left(Z_3 + \frac{2\alpha}{3\pi\epsilon} \right) = 0 \tag{80}$$

$$\Rightarrow \epsilon \alpha \left(1 - \frac{2\alpha}{3\pi\epsilon} \right) + \beta \left(1 - \frac{2\alpha}{3\pi\epsilon} + \frac{2\alpha}{3\pi\epsilon} \right) = 0 \tag{81}$$

$$\Rightarrow \epsilon \alpha - \frac{2\alpha^2}{3\pi} + \beta = 0 \tag{82}$$

$$\boxed{\beta = \frac{2\alpha^2}{3\pi}} \tag{83}$$

Solving the beta function we get the running of the coupling constant α

$$\boxed{\alpha = \frac{\alpha_0}{1 - \frac{2\alpha_0}{3\pi} \log\left(\frac{\mu}{\mu_0}\right)}} \tag{84}$$

6 Pressing questions

6.1 Why Photon is massless?

As $\Pi(q^2)$ is regular at $q^2 = 0$, the renormalized exact propagator always has the pole at only $q^2 = 0$. That means the photon always remains massless in Renormalised Theory. This remains true because we used the Ward identity where we only had the tensor $[g^{\mu\nu} q^2 - q^\mu q^\nu]$. If we had a term such as $M^2 g^{\mu\nu}$, then we would get a pole at $q^2 = M^2$, which will be the photon mass.

6.2 Why is $\delta_1 = \delta_2$?

Under the given transformation of fields and charges,

$$e_0 Z_2 Z_3^{1/2} = e Z_1, \quad A_0^\mu = Z_3^{1/2} A^\mu$$

$$D_\mu = \partial_\mu - i e_0 A_{0\mu} \rightarrow \partial_\mu - i \left(\frac{e Z_1}{Z_2 Z_3^{1/2}} \right) Z_3^{1/2} A^\mu \rightarrow \partial_\mu - i e \left(\frac{Z_1}{Z_2} \right) A^\mu \tag{85}$$

Thus, for keeping gauge invariance we would expect that $Z_1 = Z_2$ is same for all orders, which we showed for $O(\alpha)$.

7 Appendix

Identities used are given here

$$I_1 = \gamma^\mu \gamma_\mu = d,\ \gamma^\mu \tag{86}$$
$$I_2 = \gamma^\nu \gamma_\mu = -(d-2)\gamma^\mu \tag{87}$$
$$I_3 = \int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2+\Delta]^n} = \frac{\Gamma(n-d/2)}{(4\pi)^{d/2}\Gamma(n)} \frac{1}{\Delta^{n-d/2}} \tag{88}$$
$$I_4 = \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} = \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} + \log\left(\frac{4\pi e^{-\gamma}}{\Delta}\right)\right), \text{in the limit}\, d = 4-\epsilon \tag{89}$$
$$I_5 = Tr[\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma] = 4(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{90}$$
$$I_6 = Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \tag{91}$$
$$I_7 = \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2+\Delta]^n} = \frac{d\Gamma(n-d/2-1)}{(4\pi)^{d/2}2\Gamma(n)} \frac{1}{\Delta^{n-d/2-1}} \tag{92}$$
$$I_8 = \bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right] u(p) \tag{93}$$