Prove $H(n,2,2) = 4$, for all $n \in \mathbb{Z}^+$

Hypotheses:

1. $H(0,a,b) = b+1$
2. $H(1,a,0) = a$
3. $H(2,a,0) = 0$
4. $H(n,a,0) = 1$ if $n \geq 3$
5. $H(n,a,b) = H(n-1, a, H(n,a,b-1))$ if $n \geq 1$ & $b \geq 1$

the hypothesis will be referred to as $h_1, h_2, \dots, h_5$ in this proof.

$$H(n,2,2) = H(n-1, 2, H(n,2,1)) \quad \dots h_5$$

start by looking at third parameter $H(n,2,1)$

$$H(n,2,1) = H(n-1, 2, H(n,2,0)) \text{ if } n=1$$

i) if $n=1$:

$$\begin{aligned} H(1,2,1) &= H(0,2,H(1,2,0)) && \dots h_5 \dots (1) \\ &= H(0,2,2) && \dots h_2 \\ &= 3 && \dots h_1 \end{aligned}$$

ii) if $n=2$:

$$\begin{aligned} H(2,2,1) &= H(1,2,H(2,2,0)) && \dots h_5 \\ &= H(1,2,0) && \dots h_3 \\ &= 2 && \dots h_2 \end{aligned}$$

iii) if $n \geq 3$

$H(n,2,1) = H(n-1, 2, H(n,2,0))$ ... h5 ... (2)
$= H(n-1, 2, 1)$ ... h4

For base case when $n=2$, $H(2,2,1) = 2$. $H_1(x) = H(x,2,1)$
assume $H_1(n)$ is true, then we must show $H_1(n+1)$ is true for $n \geq 2$

$H_1(n) = H(n,2,1) = H_1(n-1,2,1)$ ... (2)
$H_1(n+1) = H(n+1,2,1) = H(n,2,1)$ ... (2)
Now, we know that $H_1(n) = H_1(n+1)$ for $n \geq 2$
Therefore, for all $n \geq 2$, $H_1(n,2,1) = 2$ ... (a)
$n=1$, $H_1(n,2,1) = 3$

Now we will prove $H(n,2,2) = 4$. Define $H_2(x) = H(x,2,2)$.
$H(n,2,2) = H(n-1, 2, H(n,2,1))$ ... h5
i) $n=1$:
$H_2(1) = H(1,2,2) = H(0,2, H(1,2,1))$ ... h5
$= H(0,2,3)$ ... (a)
$= 4$ ... h1

ii) $n=2$:
$H_2(2) = H(2,2,2) = H(1,2,H(2,2,1))$ ... h5
$= H(1,2,2)$ ... (a)
$= H_2(1)$
$= 4$

We assume $H_2(n)$ is true for all $n \geq 2$
From this we will show that $H_2(n) = H_2(n+1)$ for all $n \geq 2$

$$H_2(n+1) = H(n+1, 2, 2) = H(n, 2, H(n+1, 2, 1)) \quad \ldots h.5$$
$$= H(n, 2, 2) \quad \ldots (9)$$
$$= H_2(n)$$

We proved that $H_2(n+1) = H_2(n)$ for all $n \geq 2$

Also, $H_2(1) = H_2(2) = 4$

Therefore, $H_2(n) = H(n, 2, 2) = 4$ for all $n \geq 1$