AE 502: Homework Project 2

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1 Problem 1

Problem Statement: (25 points) What orbital elements a, e, i would you choose for your Molniya orbit if the perigee altitude cannot be lower than 600km to keep drag at bay, and you want your satellite to orbit the Earth three times per day? What is your lowest $\dot{\Omega}$ drift rate?

Given that the orbital period is given by the problem, we can determine the semi-major axis of the given orbit. As the satellite must orbit the Earth three times a day, the orbital period is fixed at 28800 seconds.

$$T_p = 2\pi\sqrt{a^3/\mu}$$
$$a = (\mu(\frac{T_p}{2\pi})^2)^{1/3}$$
$$a \approx 20307.39319km$$

Additionally, as $\dot{\omega}$ is frozen under Earth's J_2 perturbation, we are left with two options for inclination.

$$\dot{\omega} = 3/4 \cdot n \cdot J_2 \cdot (\frac{R}{a})^2 \frac{5cos^2(i) - 1}{(1-e^2)^2} = 0$$
$$cos(i) = \pm\sqrt{\frac{1}{5}}$$
$$i \approx 1.107149rad, 2.034444rad$$

Next, as our desire is to maximize dwell time over the northern latitudes, we will set our perigee altitude to be 600 km as this is the minimum given by the problem.

$$r_p \geq R + 600$$
$$a(1-e) \geq R + 600$$
$$e \leq 1 - \frac{R+600}{a}$$
$$e \leq 1 - \frac{6970km}{20307.39319km}$$

Again, as we want our perigee altitude to be as low as possible, we will maximize our eccentricity. As such,

$$e = 1 - \frac{6970km}{20307.39319km}$$
$$e \approx 0.656775$$

Now that we have our a, e, and two options for i selected, we can determine which generates the lowest $\dot{\Omega}$ drift rate.

$$\dot{\Omega} = -3/2 \cdot n \cdot J_2 \cdot (\frac{R}{a})^2 \frac{cos(i)}{(1-e^2)^2} = 0$$

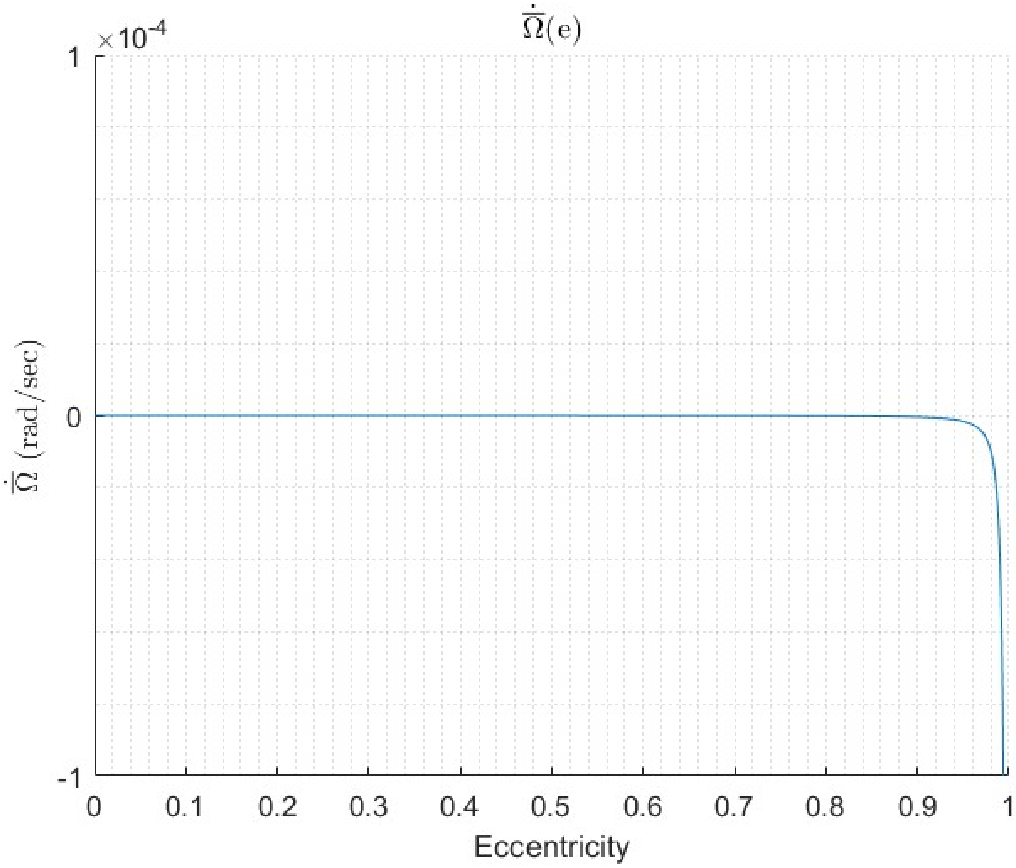

Figure 1: Problem 1 $\dot{\overline{\Omega}}$ as a function of Eccentricity

As our two options of i have no impact on the magnitude of the $\dot{\overline{\Omega}}$ drift rate. We will select $i \approx 1.107149 rad$. And given these choices the $\dot{\overline{\Omega}}$ drift rate is fully constrained, giving us the final answer of:

$$\dot{\overline{\Omega}} \approx -4.8095 \cdot 10^{-08} rad/sec$$

$$\dot{\overline{\Omega}} \approx -1.5178 rad/year \approx -86.9621 deg/year$$

Although this drift is quite significant, utilizing a constellation of satellites could compensate for the drift and result in long dwell times as well. That being said, the minimum $\dot{\overline{\Omega}}$ if the Eccentricity was not set can be found by setting the Eccentricity to 0. Although this would no longer be a Molniya orbit and no longer have those desired long dwell times, the resulting $\dot{\overline{\Omega}}$ would be $\approx -1.5552 \cdot 10^{-08} rad/sec$.

1.1 Final Results

$$\dot{\overline{\Omega}} \approx -4.8095 \cdot 10^{-08} rad/sec$$

$$a \approx 20307.39319 km$$

$$e \approx 0.656775$$

$$i \approx 1.107149 rad$$

2 Problem 2

Problem Statement: (25 points) What orbital elements a, e, i would you choose to optimize your Molniya orbit around Mars, if the perigee altitude cannot be lower than 400km (J2 = 0.00196, R = 3390 km, GM = $4.282 \cdot 10^4 \frac{km^3}{s^2}$)? Assume the orbital period of your satellite is one Martian Day (24 hours, 39 minutes and 35 seconds). What is your lowest $\dot{\Omega}$ drift rate?

Given that the orbital period is given by the problem, we can determine the semi-major axis of the given orbit. From the problem, the orbital period is fixed at 88775 seconds.

$$T_p = 2\pi\sqrt{a^3/\mu}$$
$$a = (\mu(\frac{T_p}{2\pi})^2)^{1/3}$$
$$a \approx 20446.6782km$$

Additionally, as $\dot{\omega}$ is frozen under Mar's J_2 perturbation, we are left with two options for inclination.

$$\dot{\omega} = 3/4 \cdot n \cdot J_2 \cdot (\frac{R}{a})^2 \frac{5cos^2(i) - 1}{(1-e^2)^2} = 0$$
$$cos(i) = \pm\sqrt{\frac{1}{5}}$$
$$i \approx 1.107149rad, 2.034444rad$$

Next, as our desire is to maximize dwell time over the northern latitudes, we will set our perigee altitude to be 400 km as this is the minimum given by the problem.

$$r_p \geq R + 400$$
$$a(1-e) \geq R + 400$$
$$e \leq 1 - \frac{R+400}{a}$$
$$e \leq 1 - \frac{3790km}{20446.67824km}$$

Again, as we want our perigee altitude to be as low as possible, we will maximize our eccentricity. As such,

$$e = 1 - \frac{3790km}{20446.67824km}$$
$$e \approx 0.8146398$$

Now that we have our a, e, and two options for i selected, we can determine which generates the lowest $\dot{\Omega}$ drift rate.

$$\dot{\Omega} = -3/2 \cdot n \cdot J_2 \cdot (\frac{R}{a})^2 \frac{cos(i)}{(1-e^2)^2} = 0$$

As our two options of i have no impact on the magnitude of the $\dot{\Omega}$ drift rate. We will select $i \approx 1.107149rad$. And given these choices the $\dot{\Omega}$ drift rate is fully constrained, giving us the final answer of:

$$\dot{\Omega} \approx -2.2610 \cdot 10^{-08} rad/sec$$
$$\dot{\Omega} \approx -0.7135rad/year \approx -40.8807deg/year$$

Although this drift is quite significant, utilizing a constellation of satellites could compensate for the drift and result in long dwell times as well. That being said, the minimum $\dot{\Omega}$ if the Eccentricity was not set can be found by setting the Eccentricity to 0. Although this would no longer be a Molniya orbit and no longer have those desired long dwell times, the resulting $\dot{\Omega}$ would be $\approx -2.55803 \cdot 10^{-09} rad/sec$.

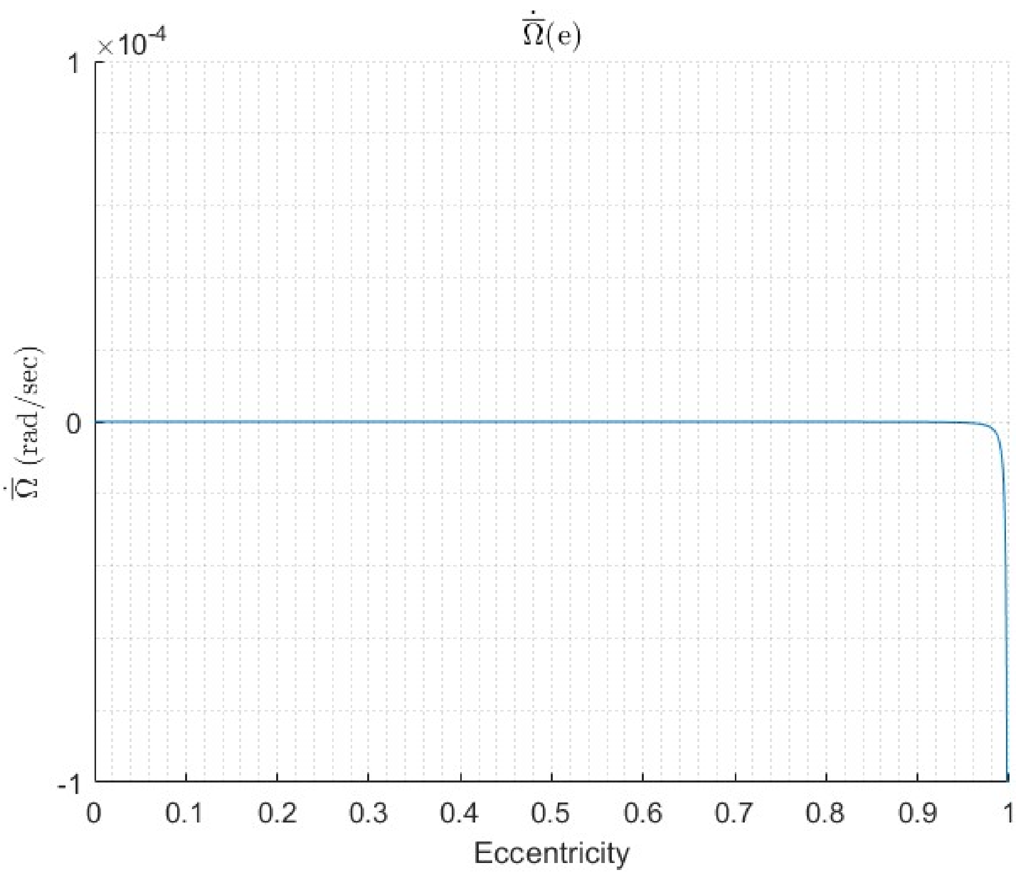

Figure 2: Problem 2 $\dot{\bar{\Omega}}$ as a function of Eccentricity

2.1 Final Results

$$\dot{\bar{\Omega}} \approx -2.2610 \cdot 10^{-08} rad/sec$$
$$a \approx 20446.6782 km$$
$$e \approx 0.8146398$$
$$i \approx 1.107149 rad$$

3 Problem 3

Problem Statement: (50 points) Code up your favorite (non-averaged) perturbed equations of motion [e.g. Curtis, 2014, Ch.12], either Planetary Equations or special perturbations equations, and solve them numerically to study the actual effect of Earth's J2 perturbation on the following Molniya orbit: a = 26600 km, i = 1.10654 rad, e = 0.74, ω = 5 deg, Ω = 90 deg, M0 = 10 deg. Plot the time evolution of all orbital elements except the mean anomaly (M) for 100 days. Describe your results!

From propagating these elements for 100 days, several things can be clearly seen. The first is that most of the elements, despite fluctuations over short time-scales show no secular change over longer time-scales. Indeed, the Semimajor Axis (a), Inclination (i), Eccentricity (e) do not exhibit any significant secular drift. The two items that do drift are the Argument of Periapsis (w) and the Right Ascension (Ω). The other intriguing item is that the Right Ascension despite containing fluctuations about the "secular line" does not deviate significantly from that line.

4 Problem 4

Problem Statement: (BONUS QUESTION - no partial credit - 100 points) One of the undesirable properties of the Planetary Equations in Keplerian orbital elements that were presented in the lecture are their singularities. Several of Lagrange's Planetary equations, for instance, become singular for co-planar orbits (i = 0 deg). This issue can be solved by using an alternative set of orbital elements, for instance, equinoctial elements. a = a h = e $\cdot$ sin ($\omega + \Omega$) k = e $\cdot$ cos ($\omega + \Omega$) p = tan (i/2) sin Ω q = tan (i/2) cos Ω λ0 = M0 + ω + Ω Use the method of Lagrange/Poisson brackets to construct Gauss Planetary Equations in those elements! Equations (13) - (19) in Broucke and Cefola [1972] could be helpful. Present the resulting equations! Use the resulting Gauss planetary equations to solve the following J2 perturbed two body problem around the Earth: a = 7000km, e = 0.05, i = 0 rad, h = p = q = 0, k = e, λ0 = 0.

Was unable to complete fully in the alotted time.

5 Acknowledgements

The idea to use Matlab's ODE solver rather than a iterative for-loop was from Curtis's 10.6 example problem.

6 Raw Code

`Github Link: https://github.com/syalla2/AE502_HW2`

6.1 Problem 1

```
%%HW 2 Problem 1
clear
J_2 = 0.00108; R = 6370; mu = 3.986e05;
```

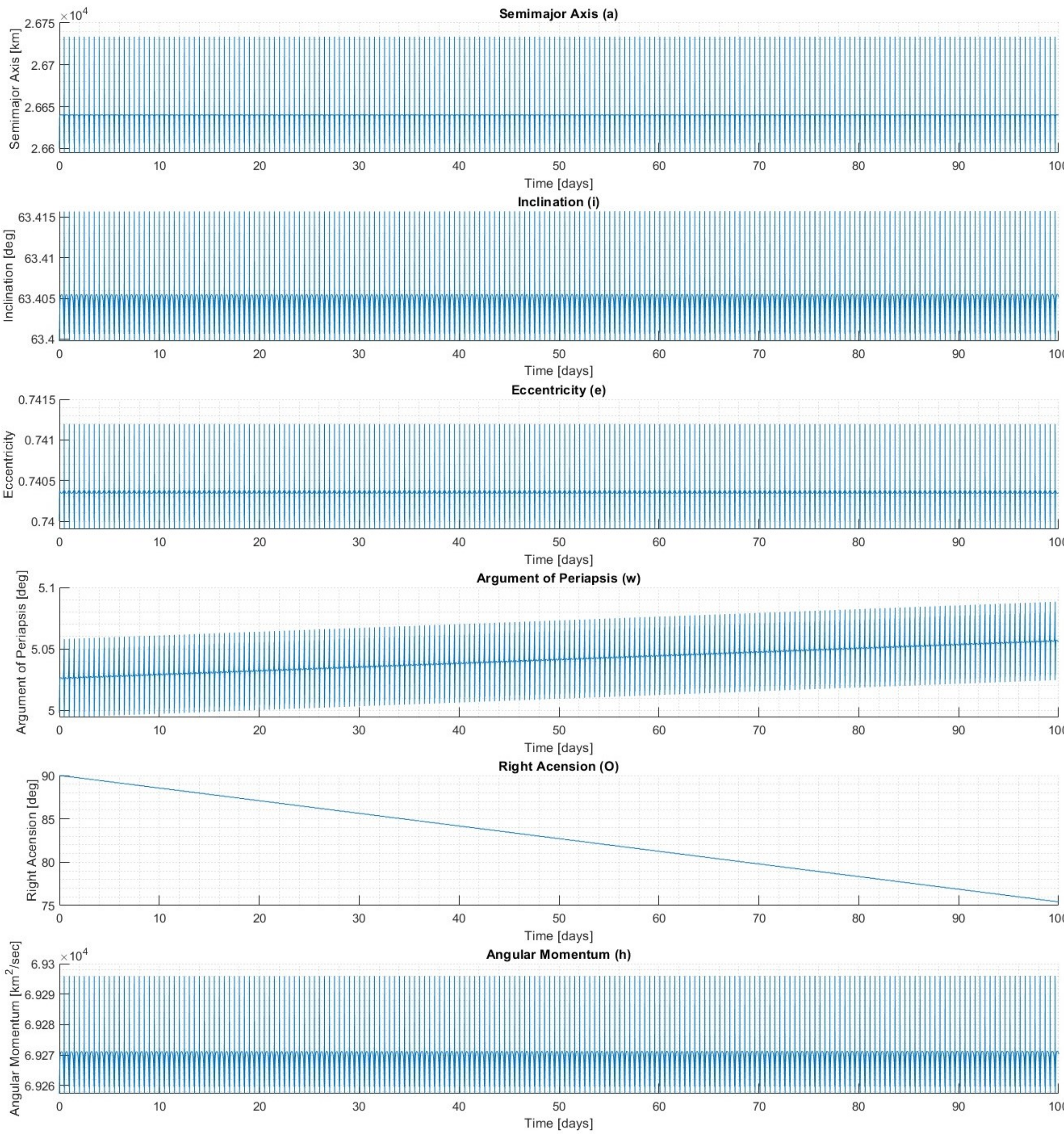

Figure 3: Problem 3 Orbital Elements propagated for 100 days

```
%Problem 1
%Want to orbit the Earth three times a day
T_p = 8*60*60;
n   = 2*pi/T_p;
a   = ((T_p/(2*pi))^2*mu)^(1/3);
i   = acos(sqrt(.2));
e_  = [0:.001:1];
e   = 1 - (R+600)/a;

O_dot_ = -3/2 .* n .* J_2 .* (R/a)^2 .* cos(acos(1/sqrt(5))) ./ ((1 - e_.^2) .^ 2);

figure
hold on
plot(e_, O_dot_); xlabel('Eccentricity'); ylabel('$\dot{\overline{\Omega}}$ (rad/sec)', ...
    'Interpreter','Latex');
title('$\dot{\overline{\Omega}}$(e)', 'Interpreter','Latex'); grid minor; ylim([-.0001 .0001])
hold off

%Solution of O_dot1 given the choice of a, e, i
O_dot1 = -3/2 .* n .* J_2 .* (R/a)^2 .* 1/sqrt(5) ./ ((1 - e.^2) .^ 2);
```

6.2 Problem 2

```
%%HW 2 Problem 2
clear
J_2 = 0.00196; R = 3390; mu = 4.282e04;

%Problem 1
%Want to orbit the Earth three times a day
T_p = 24*60*60 + 39*60 + 35;
n   = 2*pi/T_p;
a   = ((T_p/(2*pi))^2*mu)^(1/3);
i   = acos(sqrt(.2));
e_  = [0:.001:1];
e   = 1 - (R+400)/a;

%O_dot_ as a function of e_
O_dot_ = -3/2 .* n .* J_2 .* (R/a)^2 .* cos(acos(1/sqrt(5))) ./ ((1 - e_.^2) .^ 2);

figure
hold on
plot(e_, O_dot_); xlabel('Eccentricity'); ylabel('$\dot{\overline{\Omega}}$ (rad/sec)', ...
    'Interpreter','Latex');
title('$\dot{\overline{\Omega}}$(e)', 'Interpreter','Latex'); grid minor; ylim([-.0001 .0001])
hold off

%Solution of O_dot1 given the choice of a, e, i
O_dot1 = -3/2 .* n .* J_2 .* (R/a)^2 .* 1/sqrt(5) ./ ((1 - e.^2) .^ 2);
```

6.3 Problem 3

```
%% HW 2 Problem 3
clear all; clc
%% Parameters from problem statement
a_0 = 26600;
i_0 = 1.10654;
e_0 = .74;
```

```
w_0 = 5*pi/180;
O_0 = 90*pi/180;
M_0 = 10*pi/180;

% J2 of Earth
J_2 = 0.00108;
% Equatorial radius of the Earth
R = 6370;
% Gravitational Parameter
mu = 3.986e05;

%time of propogation
t_end = 100*24*60*60;

%% Given above -> Initial Calculations
h_0 = sqrt(mu*a_0*(1 - e_0^2));

%% Convert Mean Anomaly to True Anamoly
%% Implementation of Laguerre-Conway: My own code from a previous class
n = 5;
E = M_0;
M = M_0;
e = e_0;

del = 1000000000;
count = 0;
while(del > .000001)
    F = E - e*sin(E) - M;
    F_1 = 1 - e*cos(E);
    F_2 = e*sin(E);
    E_old = E;
    %Laguerre-Conway
    E = E - (n * F)/ ...
        (F_1+sign(F_1)*abs((n-1)^2*(F_1)^2 - n*(n-1)*F*F_2)^.5);
    del = (E - E_old);
    count  = count + 1;
end

% True Anamoly from eccentric anamoly
temp = tan(E/2) * sqrt( (1 + e) / (1 - e) );
TA = 2 * atan(temp);
TA_0 = TA;

%% Perturbed Equations
a = a_0;
i = i_0;
w = w_0;
h = h_0;
u_0 = w_0 + TA_0;
O = O_0;

options = odeset(...
 'reltol', 1.e-10, ...
 'abstol', 1.e-10, ...
'initialstep', 10);
```

```
%% This method is similar to a method in Curtis Matlab Example 10_6.
%  Integrating the rates given the intial conditions of the problem
elements_0 = [h_0, e_0, TA_0, O_0, i_0, w_0];
y0 = elements_0';
t_array = 0:10:t_end;
[t, y] = ode89(@rates, t_array, y0, options);

h  = y(:, 1);
e  = y(:, 2);
TA = y(:, 3);
O  = y(:, 4);
i  = y(:, 5);
w  = y(:, 6);

a = h.^2./mu .* 1 ./ (1 - e.^2);

%% Plotting Orbital Elements
figure
subplot(6, 1, 1)
hold on
plot(t_array/(3600*24), a); title('Semimajor Axis (a)')
ylabel('Semimajor Axis [km]'); xlabel(['Time [days]']); grid minor;
hold off

subplot(6, 1, 2)
hold on
plot(t_array/(3600*24), i*180/pi); title('Inclination (i)')
ylabel('Inclination [deg]'); xlabel(['Time [days]']); grid minor;
hold off

subplot(6, 1, 3)
hold on
plot(t_array/(3600*24), e); title('Eccentricity (e)')
ylabel('Eccentricity'); xlabel(['Time [days]']); grid minor;
hold off

subplot(6, 1, 4)
hold on
plot(t_array/(3600*24), w*180/pi); title('Argument of Periapsis (w)')
ylabel('Argument of Periapsis [deg]'); xlabel(['Time [days]']); grid minor;
hold off

subplot(6, 1, 5)
hold on
plot(t_array/(3600*24), O*180/pi); title('Right Acension (O)')
ylabel('Right Acension [deg]'); xlabel(['Time [days]']); grid minor;
hold off

subplot(6, 1, 6)
hold on
plot(t_array/(3600*24), h); title('Angular Momentum (h)')
ylabel('Angular Momentum [km^2/sec]'); xlabel(['Time [days]']); grid minor;
hold off

%% Subfunction that generates rates
function dfdt = rates(t, f)
```

```
    % Earth Constants
    mu = 3.986e05;
    J_2 = 0.00108;
    R = 6370;

    h  = f(1);
    e  = f(2);
    TA = f(3);
    O  = f(4);
    i  = f(5);
    w  = f(6);

    % Curtis pg 511, Get r, u from parameters
    r = h^2/(mu * (1 + e*cos(TA)));
    u = w + TA;

    % Gravitational Perturbation in the RSW frame, Curtis 10.88
    p_r = (-3/2) * (J_2*mu*R^2) / (r^4) * (1 - 3*sin(i)^2 * sin(u)^2);
    p_s = (-3/2) * (J_2*mu*R^2) / (r^4) * sin(i)^2 * sin(2*u);
    p_w = (-3/2) * (J_2*mu*R^2) / (r^4) * sin(2*i) * sin(u);

    % Gauss Planetary Equations, Curtis 10.84
    hdot = r*p_s;
    edot = h/mu * sin(TA) * p_r + 1/(mu*h) * ((h^2 + mu*r) * cos(TA) + mu*e*r) * p_s;
    TAdot = h/(r^2) + 1/(e*h) * (h^2/mu * cos(TA) * p_r - (r + h^2/mu) * sin(TA) * p_s);
    Odot = r/(h*sin(i)) * sin(u) *p_w;
    idot = r/h * cos(u) * p_w;
    wdot = -1/(e*h) * (h^2/mu * cos(TA) *p_r - (r + h^2/mu) * sin(TA) * p_s) - ...
        ( r*sin(u) ) / ( h * tan(i) ) * p_w;


    dfdt = [hdot, edot, TAdot, Odot, idot, wdot]';
end
```