NOTES ON ZAREV'S ALGORITHM FOR COMPUTING *HFK* USING BORDERED-SUTURED HEEGAARD FLOER HOMOLOGY

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Abstract. These are notes for Adam Knapp's REU project at Columbia University in the Summer of 2012. The goal is to explain succinctly Rumen Zarev's algorithm for computing the Heegaard Floer knot invariant $\widehat{HFK}(K)$ for a knot $K \subset S^3$. (Much of what is in these notes was explained to me by Zarev.)

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1. Overview of the algorithm

Heegaard Floer homology, and Zarev's extension of it, bordered-sutured Floer homology, does not assign invariants directly to topological objects but rather to combinatorial representations of those objects. The representations for surfaces are called *arc diagrams*. We give some notation for arc diagrams in Section 2; for now, it sufffices to know that there is a surface with boundary $F(\mathcal{Z})$ associated to an arc diagram $\mathcal{Z}$, with certain distinguished intervals (arcs) in $\partial F(\mathcal{Z})$.

We view tangles as subsets of $[0,1]\times\mathbb{D}^2$. Given a tangle T, let $X(T) = [0,1]\times\mathbb{D}^2 \setminus \mathrm{nbd}(T)$ denote its exterior. Divide $X(T)$ into two parts: the *horizontal boundary* consisting of $(\partial X(T)) \setminus (\{0,1\}\times\mathbb{D}^2)$, and the *vertical boundary*, which is the rest of $\partial X(T)$. A *bordered-sutured structure* for $X(T)$ consists of:

- A diffeomorphism from $-F(\mathcal{Z}_1) \amalg F(\mathcal{Z}_2)$ to the horizontal boundary of $X(T)$, for some arc diagrams $\mathcal{Z}_1$ and $\mathcal{Z}_2$.

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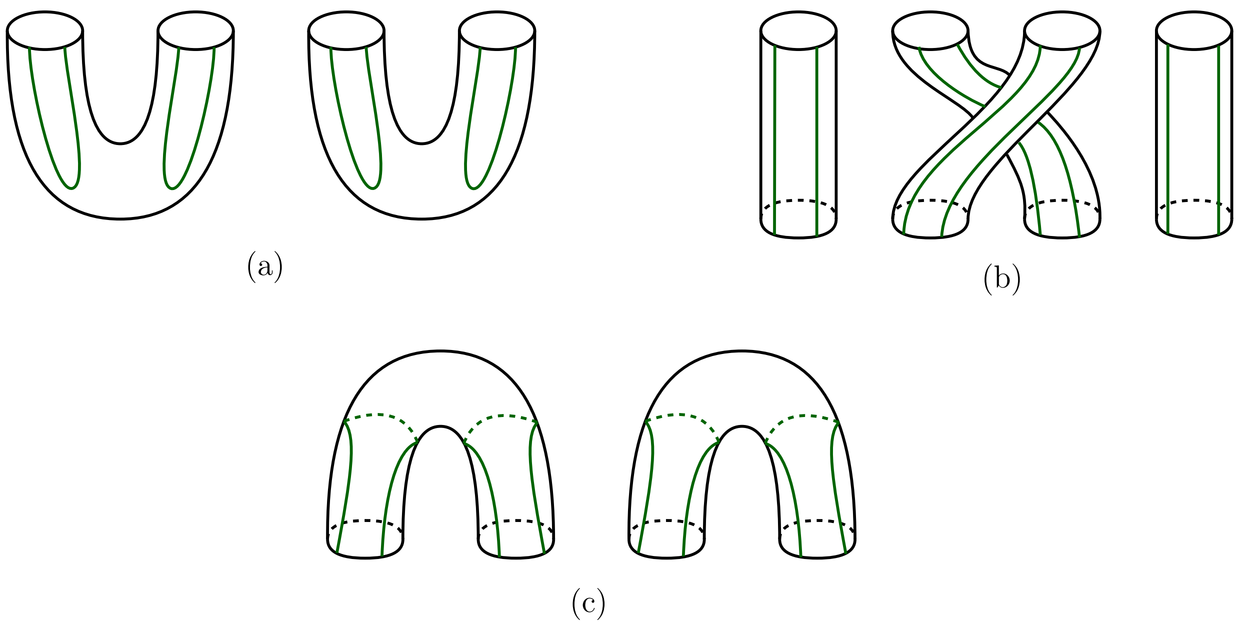

FIGURE 1. **Elementary bordered-sutured tangles.** (a) The bottom plat closure. (b) A braid generator. (c) The top plat closure.

- A 1-submanifold Γ of the vertical boundary of $X(T)$ with $\partial\Gamma$ equal to the endpoints of the intervals in $\mathcal{Z}_1 \amalg \mathcal{Z}_2$ and such that Γ divides the vertical boundary into two pieces (with one piece on each side of each component of Γ).

From now on, we will abuse notation and let T denote both the tangle and the bordered-sutured structure on $X(T)$; we refer to this data as a *bordered-sutured tangle*. If we want to keep track of the bordering we will write $T\colon \mathcal{Z}_1 \to \mathcal{Z}_2$ where $-\mathcal{Z}_1$ is the pointed matched circle parameterizing $X(T)\cap(\{0\}\times DD^2)$ and $\mathcal{Z}_2$ is the pointed matched circle parameterizing $X(T)\cap(\{1\}\times\mathbb{D}^2)$.

Zarev associates an algebra $\mathcal{A}(\mathcal{Z})$ to each arc diagram $\mathcal{Z}$, and a differential $\mathcal{A}(\mathcal{Z}_1)$-$\mathcal{A}(\mathcal{Z}_2)$-bimodule $BSDD(T)$ to each bordered-sutured tangle $T\colon \mathcal{Z}_1 \to \mathcal{Z}_2$.

We are interested in computing $\widehat{HFK}(K)$ for knots $K \subset S^3$. Let K_n be the bordered-sutured tangle obtained by viewing K as lying in $[0,1]\times\mathbb{D}^2$ and placing $2n$ meridional sutures on the exterior of K. The connection between Zarev's invariants and knot Floer homology comes from the following:

- If $\mathcal{Z}$ is the (unique) arc diagram parameterizing $\mathbb{D}^2$ then $\mathcal{A}(\mathcal{Z}) = \mathbb{F}_2$. In particular, $BSDD(K_n)$ is just a chain complex over $\mathbb{F}_2$.
- $H_*(BSDD(K_n)) \cong \widehat{HFK}(K)\otimes(\mathbb{F}_2\oplus\mathbb{F}_2)^{\otimes n}$.

(Philosophically, one of the n copies of $\mathbb{F}_2\oplus\mathbb{F}_2$ comes from the fact that we are in $[0,1]\times\mathbb{D}^2$; the other $(n-1)$ copies come from the fact that we have K_n rather than K_1.)

There is also a composition formula for the $BSDD$, corresponding to gluing tangles. We will discuss this composition formula in Section 3.1. Given this, the algorithm to compute $\widehat{HFK}$ is as follows:

(1) Compute the modules associated to the plat closures in Figure 1. These are given in Section 4. (In fact, because of the form of the composition law, we will only need the invariant of the bottom closure.)
(2) Compute the bimodules associated to the braid generators. These are actually compositions of simpler mapping class groupoid elements, arc-slides. The modules associated to arc-slides are described in Section 5.
(3) Compose the bimodules. The composition law is described in Section 3.

Before turning to these steps, we fix some notation for the relevant arc diagrams and algebras.

2. The relevant arc diagrams and algebras

An arc diagram $\mathcal{Z}$ is a tuple $(Z, \mathbf{a}, M)$ where Z is a collection of oriented intervals, $\mathbf{a}$ is a finite set of points in Z, and $M\colon \mathbf{a} \to \mathbf{a}$ is a fixed-point free involution (a "matching"). These are required to satisfy the condition that performing surgery on $\mathbf{a}$ according to M does not give any closed components.

The arc diagrams relevant to computing $\widehat{HFK}$ are shown in Figure 2. We will start and end with the left-most diagram, but pass through the next two when decomposing elementary braid generators as compositions of arc-slides.

I will not explain the bordered-sutured algebra associated to an arc diagram—see [Zar09]—but will review some parts to introduce some notation.

For any subset $\mathbf{s} \subset \{\alpha\text{-arcs}\} = \{1, \dots, 2k\}$ there is a corresponding indecomposable idempotent $I(\mathbf{s})$.

A *chord* in $\mathcal{Z}$ is an interval in Z with boundary in $\mathbf{a}$. Associated to any chord ξ is an algebra element $a(\xi)$, the sum of all ways of adding horizontal strands to ξ. That is, if ξ has initial point i and terminal point j then $a(\xi)$ is the sum

$$\sum_S (S, S \setminus \{i\} \cup \{j\}, \phi_S\colon S \to S \setminus \{i\} \cup \{j\})$$

where the sum is over all S such that:

- $i \in S$,
- $j \notin S$,
- $S \cap M(S) = \emptyset$, and
- writing $T = S \setminus \{i\} \cup \{j\}$, $T \cap M(T) = \emptyset$.

The map ϕ_S is given by $\phi_S(i) = j$ and $\phi|_{S \setminus \{i\}} = \mathbb{I}$.

This definition extends in an obvious way to sets of chords, as long as no two of the chords start (respectively end) on the same matched pair; if ξ is a set of chords, we will still denote the corresponding element by $a(\xi)$.

We name the chords in the diagram $\mathcal{Z}_k$ for the disk with k holes as shown in Figure 2.

2.0.1. *Computer talk.* In the code on my web page, pointed matched circles are encoded via a class `PMC`. Elements of $\mathcal{A}(\mathcal{Z})$ are encoded as the class `AlgElt`; each `AlgElt` is a set (really, a list) of `Strand_Diagram`'s. (Recall that a set is the same as a linear combination over $\mathbb{F}_2$.)

3. Modules of type D and DD, and Mor

Type D modules are like chain complexes of free modules. More precisely, let $\mathcal{A}$ be a *dg* algebra over $\mathcal{I} = \bigoplus_{i=0}^{N} \mathbb{F}_2$.

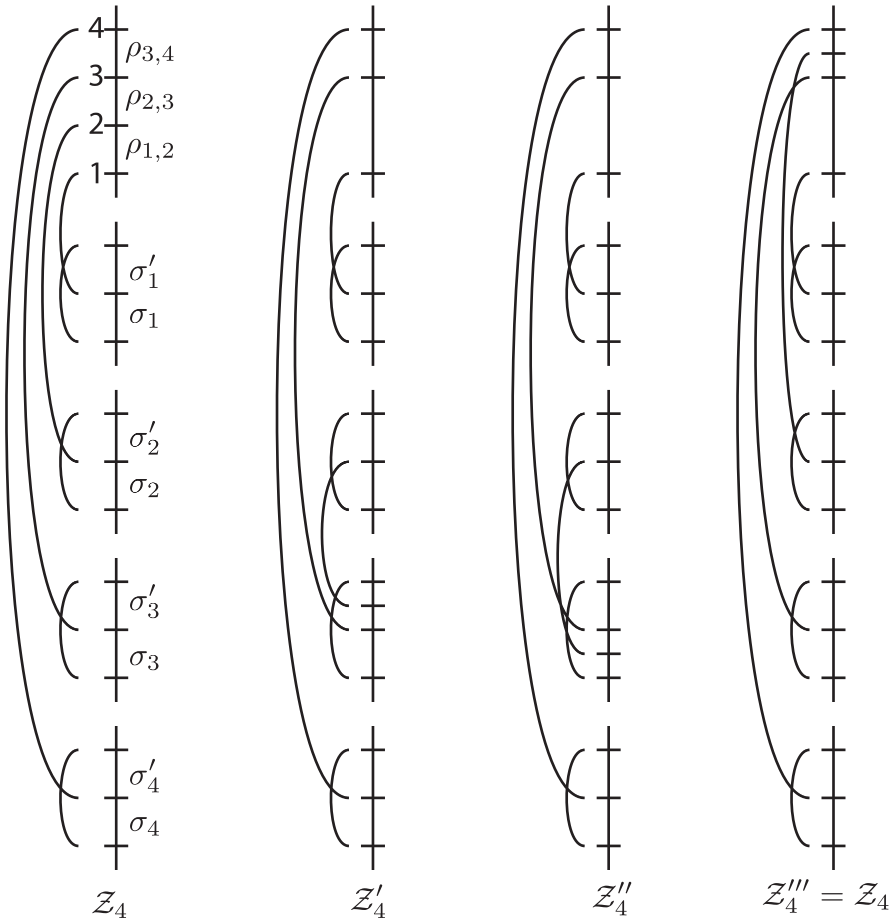

FIGURE 2. **Arc diagrams for a planar surface.** The case of a 4-times punctured disk is shown. Each of the arc diagrams shown is obtained from the previous one by an arc-slide; the composition of these arc-slides is the braid generator σ_2.

Definition 3.1. *A* type D module *over $\mathcal{A}$ is a dg module P which is written as $P = \mathcal{A} \otimes_{\mathcal{I}} V$ for some $\mathcal{I}$-module V.*

Given type D modules P_1 and P_2, define $\mathrm{Mor}_{\mathcal{A}}(P_1, P_2)$ to be the space of $\mathcal{A}$-linear maps $P_1 \to P_2$. There is a differential on $\mathrm{Mor}_{\mathcal{A}}(P_1, P_2)$ given by

$$(\partial f)(x) = f(\partial x) + \partial(f(x)).$$

See Example 4.1, below, for an example of this morphism complex.

There are natural generalizations to bimodules. Fix another *dg* algebra $\mathcal{B}$ over another ring of idempotents $\mathcal{I}' = \bigoplus_{i=0}^{N} \mathbb{F}_2$.

Definition 3.2. *A* type DD module *over $\mathcal{A}$ and $\mathcal{B}$ is a dg $(\mathcal{A}, \mathcal{B})$-bimodule Q which is written as $Q = \mathcal{A} \otimes_{\mathcal{I}} W \otimes_{\mathcal{I}'} \mathcal{B}$ for some $(\mathcal{I}, \mathcal{I}')$-bimodule W.*

Given a type D module P over $\mathcal{A}$ and a type DD module Q over $(\mathcal{A}, \mathcal{B})$ there are a couple of natural notions of the space of morphisms from Q to P. The most obvious one is to take

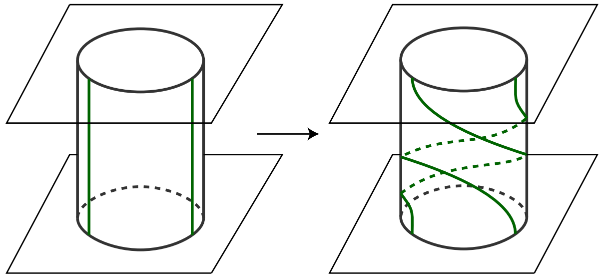

FIGURE 3. **A full twist on a boundary component.**

the space of $\mathcal{A}$-linear maps from Q to P. This retains an action by $\mathcal{B}$, but is *not* in general a type D module over $\mathcal{B}$.

Instead, we define $\mathrm{Mor}(Q,P)$ in two steps. There is a dual type DD structure $\overline{Q}$ to Q, a $(\mathcal{B},\mathcal{A})$-bimodule, defined as follows. The differential on Q induces a map $\delta\colon W \to \mathcal{A} \otimes_{\mathcal{I}} W \otimes_{\mathcal{I}'} \mathcal{B}$ by $\delta(w) = \partial(1 \otimes w \otimes 1)$. Write W^* for the dual vector space to W. Then the map δ transposes to a map $\overline{\delta}\colon W^* \to \mathcal{B} \otimes W^* \otimes \mathcal{A}$, and $\overline{Q}$ is defined to be $\mathcal{B} \otimes W^* \otimes \mathcal{A}$ with differential

$$\partial(b \otimes w \otimes a) = (db) \otimes w \otimes a + b \cdot \overline{\delta}(w) \cdot a + b \otimes w \otimes da.$$

Define

$$\mathrm{Mor}(Q,P) = \overline{Q} \otimes_{\mathcal{A}} P.$$

The complex $\mathrm{Mor}(Q,P)$ retains the structure of a type D module over $\mathcal{B}$.

3.0.2. *Computer talk.* By the Leibniz rule, the differential of an element $a \otimes v$ is given by $\partial(a \otimes v) = (da) \otimes v + a \cdot \partial(1 \otimes v)$. So, we can encode a type D structure via the following data:

- A basis $\{v_1, \dots, v_n\}$ for V. Each basis element should be equipped with an idempotent $I(v_i) \in \mathcal{I}$.
- For each pair (i,j) an element $a_{i,j} \in \mathcal{A}$, which is the coefficient of v_j in $\partial(1 \otimes v_i)$.

This is how my program keeps track of type D structures.

Similarly, a basis for $\mathrm{Mor}(P_1,P_2)$ is given by triples $(v_i, a_{i,j}, w_j)$ where v_i is a basis element for P_1, w_j is a basis element for P_2, and $a_{i,j}$ is a basis element for $\mathcal{A}$. This is how my program computes Mor complexes.

One can keep track of type DD modules and their morphism complexes similarly.

3.1. **The** Hom **pairing theorem.** At this point I would like to tell you that if $T_1\colon \mathcal{Z}_2 \to \mathcal{Z}_1$ and $T_2\colon \mathcal{Z}_2 \to \mathcal{Z}_3$ are tangles then

$$BSDD((-T_1) \cup_{\mathcal{Z}_2} T_2) = \mathrm{Mor}_{\mathcal{A}(\mathcal{Z}_2)}(BSDD(T_1), BSDD(T_2)). \tag{3.3}$$

Unfortunately, this isn't quite true. Rather, the right-hand side picks up an extra twist around each boundary component (as indicated in Figure 3).

One solution is to:

(1) Start at the bottom of the tangle and work up.
(2) Use sutures on the bottom plat-closure piece that are invariant under the extra twisting, for instance the sutures shown in Figure 1.

For the sutures arising this way, Formula (3.3) does hold. In the end, we will get K_n, where n is the arc index, instead of K_1; such is life.

The presence of the minus sign in Formula (3.3) means that we cap off at the top by taking Mor from the invariant of the bottom closure (part (a) of Figure 1); the invariant of the top closure is never used.

4. Invariants of plat closures

4.1. **The bottom closure.** The type D module for the bottom closure is given as follows. Let B_n denote the bottom closure with n plats, i.e., the closure for braids on $2n$ strands. Then $\mathit{BSD}(B_n)$ has 2^n generators, in bijection with a certain subset of the idempotents of $\mathcal{A}(\mathcal{Z}_{2n})$. Specifically, $\mathcal{Z}_n$ has one long arc with $2n$ points $1,\dots,2n$ on it, and $2n$ shorter arcs with points on each; label these points $(i,0)$, $(i,1)$ and $(i,2)$ where $i\in\{1,\dots,n\}$; the matching identifies $(i,1)\sim i$ and $(i,0)\sim(i,2)$. Consider subsets $\mathbf{s}\subset(\mathbf{a}/M)$ containing all of the matched pairs $\{(i,0),(i,2)\}$ and, for each $j=1,\dots,n$, exactly one of $\{2j-1,(2j-1,1)\}$ or $\{2j,(2j,1)\}$. There is one generator $\mathbf{x}(\mathbf{s})$ for each $\mathbf{s}$ of this form, with corresponding idempotent $I(\mathbf{s})$.

(For interest / my reference later, this module is associated to the Heegaard diagram shown in Figure 4.)

For example, in the case $n=1$ the generators correspond to the two subsets

$$\begin{aligned}\mathbf{s}_1&=\big\{\{(1,0),(1,2)\},\{(2,0),(2,2)\},\{1,(1,1)\}\big\}\\ \mathbf{s}_2&=\big\{\{(1,0),(1,2)\},\{(2,0),(2,2)\},\{2,(2,1)\}\big\}.\end{aligned}$$

In the case $n=2$ the generators correspond to the four subsets

$$\begin{aligned}&\big\{\{(1,0),(1,2)\},\{(2,0),(2,2)\},\{(3,0),(3,2)\},\{(4,0),(4,2)\},\{1,(1,1)\},\{3,(3,1)\}\big\}\\ &\big\{\{(1,0),(1,2)\},\{(2,0),(2,2)\},\{(3,0),(3,2)\},\{(4,0),(4,2)\},\{1,(1,1)\},\{4,(4,1)\}\big\}\\ &\big\{\{(1,0),(1,2)\},\{(2,0),(2,2)\},\{(3,0),(3,2)\},\{(4,0),(4,2)\},\{2,(2,1)\},\{3,(3,1)\}\big\}\\ &\big\{\{(1,0),(1,2)\},\{(2,0),(2,2)\},\{(3,0),(3,2)\},\{(4,0),(4,2)\},\{2,(2,1)\},\{4,(4,1)\}\big\}.\end{aligned}$$

The differential on $\mathit{BSD}(B_n)$ is given as follows. Consider the element $\alpha\in\mathcal{A}(\mathcal{Z}_{2n})$ given by

$$\sum_{i=1}^{n} a(\rho_{2i-1,2i})+a(\sigma_{2i})a(\sigma'_{2i})a(\rho_{2i-1,2i})a(\sigma_{2i-1})a(\sigma'_{2i-1}).$$

Then

$$\partial\mathbf{x}(\mathbf{s})=\sum_{\mathbf{x}(\mathbf{t})} I(\mathbf{s})\cdot\alpha\cdot\mathbf{x}(\mathbf{t}).$$

(Notice that many of the terms in this expression vanish because the idempotents do not match up.)

For example, in the case $n=1$ we have

$$\begin{aligned}\partial\mathbf{x}_{\mathbf{s}_1}&=\big(a(\rho_{1,2})+a(\sigma_2)a(\sigma'_2)a(\rho_{1,2})a(\sigma_1)a(\sigma'_1)\big)\mathbf{x}_{\mathbf{s}_2}\\ \partial\mathbf{x}_{\mathbf{s}_2}&=0.\end{aligned}$$

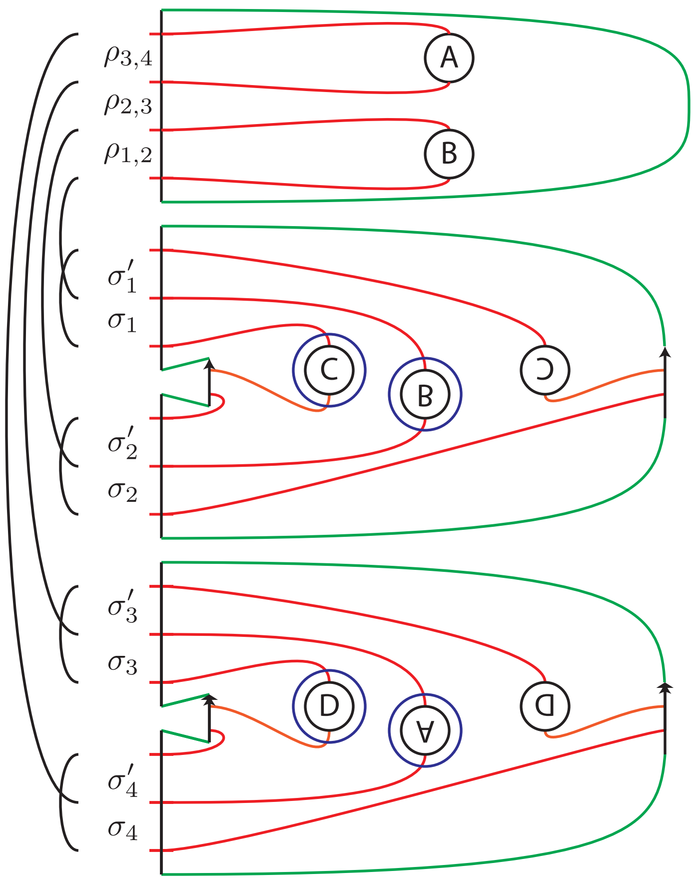

FIGURE 4. **Heegaard diagram representing the bottom closure.** The case $k = 4$ is shown.

Example 4.1. We compute the complex $\mathrm{Mor}(BSD(B_1), BSD(B_1))$. Write $\mathbf{x}_i$ for $\mathbf{x}_{\mathbf{s}_i}$. A basis for this complex is given by

$$
\begin{array}{lll}
\mathbf{x}_1 \mapsto \mathbf{x}_1 & \mathbf{x}_1 \mapsto a(\sigma_2)a(\sigma'_2)\mathbf{x}_1 & \mathbf{x}_1 \mapsto a(\sigma_{11'})\mathbf{x}_1 \\
\mathbf{x}_1 \mapsto a(\{\sigma_1, \sigma'_1\})\mathbf{x}_1 & \mathbf{x}_1 \mapsto a(\sigma_2)a(\sigma'_2)a(\sigma_{11'})\mathbf{x}_1 & \mathbf{x}_1 \mapsto a(\sigma_2)a(\sigma'_2)a(\{\sigma_1, \sigma'_1\})\mathbf{x}_1 \\
\mathbf{x}_1 \mapsto a(\rho_{12})\mathbf{x}_2 & \mathbf{x}_1 \mapsto a(\sigma_2)a(\sigma'_2)a(\rho_{12})\mathbf{x}_2 & \mathbf{x}_1 \mapsto a(\sigma_{11'})a(\rho_{12})\mathbf{x}_2 \\
\mathbf{x}_1 \mapsto a(\sigma_{11'})a(\sigma_{22'})a(\rho_{12}) & & \\
\mathbf{x}_2 \mapsto \mathbf{x}_2 & \mathbf{x}_2 \mapsto a(\sigma_1)a(\sigma'_1)\mathbf{x}_2 & \mathbf{x}_2 \mapsto a(\sigma_{22'})\mathbf{x}_2 \\
\mathbf{x}_2 \mapsto a(\{\sigma_2, \sigma'_2\})\mathbf{x}_2 & \mathbf{x}_2 \mapsto a(\sigma_1)a(\sigma'_1)a(\sigma_{22'})\mathbf{x}_2 & \mathbf{x}_2 \mapsto a(\sigma_1)a(\sigma'_1)a(\{\sigma_2, \sigma'_2\})\mathbf{x}_2.
\end{array}
$$

Here, the notation $\mathbf{x}_1 \mapsto a(\sigma_2)a(\sigma'_2)\mathbf{x}_1$, say, means the map $\mathcal{A} \otimes V \to \mathcal{A} \otimes V$ that sends $b\mathbf{x}_1$ to $ba(\sigma_2)a(\sigma'_2)\mathbf{x}_1$ and sends $b\mathbf{x}_2$ to 0 (for any $b \in \mathcal{A}$). We let $\sigma_{11'}$ denote the chord which goes from the start of σ_1 to the end of σ'_1; the differential of $a(\sigma_{11'})$ is $a(\{\sigma_1, \sigma'_1\})$.

There are three sources of differentials on this complex:

(1) Differentials on the algebra. For example, this contributes a term of the form

$$\big(\mathbf{x}_1 \mapsto a(\sigma_{11'})\mathbf{x}_1\big) \xrightarrow{\partial} \big(\mathbf{x}_1 \mapsto a(\{\sigma_1, \sigma'_1\})\mathbf{x}_1\big).$$

(2) Post-composing with differentials on the module $BSD(B_1)$. For example, this contributes terms:

$$(\mathbf{x}_1 \mapsto \mathbf{x}_1) \xrightarrow{\partial} a(\rho_{1,2})\mathbf{x}_2 + a(\sigma_2)a(\sigma_2')a(\rho_{1,2})a(\sigma_1)a(\sigma_1'))\mathbf{x}_2.$$

(Note that the second term on the right is equal to $\mathbf{x}_1 \mapsto a(\sigma_{11'})a(\sigma_{22'})a(\rho_{12})$.)

(3) Pre-composing with differentials on the module $BSD(B_1)$. For example,

$$(\mathbf{x}_2 \mapsto \mathbf{x}_2) \xrightarrow{\partial} (\mathbf{x}_1 \mapsto \rho_{12}\mathbf{x}_2).$$

Cancelling all differentials of type (1) leaves the following generators:

$$\begin{array}{lll}
\mathbf{x}_1 \mapsto \mathbf{x}_1 & \mathbf{x}_1 \mapsto a(\sigma_2)a(\sigma_2')\mathbf{x}_1 & \mathbf{x}_1 \mapsto a(\rho_{12})\mathbf{x}_2 \\
\mathbf{x}_1 \mapsto a(\sigma_2)a(\sigma_2')a(\rho_{12})\mathbf{x}_2 & \mathbf{x}_1 \mapsto a(\sigma_{11'})a(\rho_{12})\mathbf{x}_2 & \mathbf{x}_1 \mapsto a(\sigma_{11'})a(\sigma_{22'})a(\rho_{12})\mathbf{x}_2 \\
\mathbf{x}_2 \mapsto \mathbf{x}_2 & \mathbf{x}_2 \mapsto a(\sigma_1)a(\sigma_1')\mathbf{x}_2. &
\end{array}$$

The remaining nontrivial differentials are

$$\begin{aligned}
\partial(\mathbf{x}_1 \mapsto \mathbf{x}_1) &= (\mathbf{x}_1 \mapsto \rho_{12}\mathbf{x}_2) + (\mathbf{x}_1 \mapsto \sigma_{11'}\sigma_{22'}\rho_{12}\mathbf{x}_2) \\
\partial(\mathbf{x}_1 \mapsto \sigma_2\sigma_2'\mathbf{x}_1) &= (\mathbf{x}_1 \mapsto \sigma_2\sigma_2'\rho_{12}\mathbf{x}_2) \\
\partial(\mathbf{x}_2 \mapsto \mathbf{x}_2) &= (\mathbf{x}_1 \mapsto \rho_{12}\mathbf{x}_2) + (\mathbf{x}_1 \mapsto \sigma_{11'}\sigma_{22'}\rho_{12}\mathbf{x}_2) \\
\partial(\mathbf{x}_2 \mapsto \sigma_1\sigma_1'\mathbf{x}_2) &= (\mathbf{x}_1 \mapsto \sigma_{11'}\rho_{12}\mathbf{x}_2).
\end{aligned}$$

(The first two lines come from differentials of type (2), while the last two lines come from differentials of type (3).)

Thus, the homology of $\mathrm{Mor}(BSD(B_1), BSD(B_1))$ is 2-dimensional, generated by $(\mathbf{x}_1 \mapsto \mathbf{x}_1) + (\mathbf{x}_2 \mapsto \mathbf{x}_2)$ and $(\mathbf{x}_1 \mapsto \rho_{12}\mathbf{x}_2) = (\mathbf{x}_1 \mapsto \sigma_{11'}\sigma_{22'}\rho_{12}\mathbf{x}_2)$. This is what we expected: twice the rank of $\widehat{HFK}$ of the unknot (which is rank 1).

4.2. **The top closure.** As mentioned earlier, we will not need the invariant of the top closure: to close at the top we compute $\mathrm{Mor}(BSD(B_n), \cdot)$. (Equivalently, for those who know what type A invariants are, the type A invariant of the top closure is the dual of $BSD(B_n)$; the type D invariant could be obtained from this by tensoring with $BSDD(\mathbb{I})$.)

5. Invariants of arc-slides

Instead of computing directly the bimodule associated to a braid generator σ_i, we factor this braid generator as a product of arc slides; such a factorization is illustrated in Figure 2. There is a bimodule $BSDD(\phi)$ associated to each arc-slide ϕ. The definition of $BSDD(\phi)$ is a straightforward adaptation of the case discussed in [LOT10, Section 4]; see also the introduction to [LOT10], which is intended to give a readable description of the relevant bimodules. (In the language of that paper, all of the relevant arc-slides can be treated as "underslides"; the bimodules associated to underslides are a little simpler to describe than the bimodules associated to overslides.)

In an arc-slide, we slide one element of a matched pair over another matched pair. Call the foot that is moving b_1 and the foot it slides over c_1. Let b_2 be the position matched to b_1, c_2 the position matched to c_1, and b_1' the image of b_1 after sliding.

We can represent the arc-slide by a tri-valent graph as in Figure 5.

Let $\mathcal{Z}$ denote the initial arc diagram and $\mathcal{Z}'$ the result of performing the arcslide on $\mathcal{Z}$. There is an obvious identification between matched pairs of $\mathcal{Z}$ and matched pairs of $\mathcal{Z}'$; in particular, this identifies $\{b_1, b_2\}$ and $\{b_1', b_2\}$. By *complementary idempotents* we mean a pair

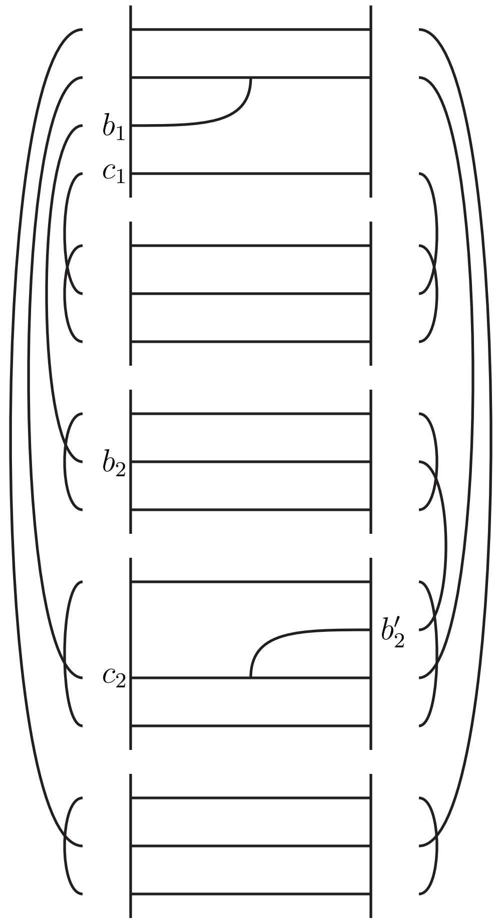

FIGURE 5. **Trivalent graph representing an arc-slide.** This is the first in the sequence of arc-slides from Figure 2 giving a braid generator.

of idempotents $I(\mathbf{s})\otimes I(\mathbf{t}) \in \mathcal{A}(\mathcal{Z})\otimes\mathcal{A}(\mathcal{Z}')$ such that the matched pairs in $\mathbf{s}$ are disjoint from the matched pairs in $\mathbf{t}$. By *almost complementary idempotents* we mean a pair of idempotents $I(\mathbf{s}) \otimes I(\mathbf{t}) \in \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}')$ so that $\mathbf{s}\cap\mathbf{t} = \{c_1, c_2\}$, $\{b_1, b_2\} \notin \mathbf{s}$ and $\{b_1', b_2\} \notin \mathbf{t}$. By *near complementary idempotents* we mean a pair of idempotents $I(\mathbf{s}) \otimes I(\mathbf{t}) \in \mathcal{A}(\mathcal{Z}) \otimes \mathcal{A}(\mathcal{Z}')$ which are either complementary or almost complementary.

The bimodule $BSDD(\phi)$ has one generator $\mathbf{x} = \mathbf{x}_{\mathbf{s},\mathbf{t}}$ for each pair of near complementary idempotents $I(\mathbf{s}) \otimes I(\mathbf{t})$. These interact with the idempotents in the obvious way:

$$I(\mathbf{s}) \cdot \mathbf{x}_{\mathbf{s},\mathbf{t}} \cdot I(\mathbf{t}) = \mathbf{x}_{\mathbf{s},\mathbf{t}}.$$

By a *near chord* we mean one of the six kinds of pairs (ξ, ξ') shown in [LOT10, Figure 17, p. 46] (or the results of rotating one of those pictured by 180°). Here, each of ξ (respectively ξ') consists of 0, 1 or 2 chords in $\mathcal{Z}$ (respectively $\mathcal{Z}'$).

The differential on $BSDD(\phi)$ is given by

$$\partial\mathbf{x}_{\mathbf{s},\mathbf{t}} = \sum_{U,V \text{ near complementary}} \sum_{(\xi,\xi') \text{ a near chord}} I(\mathbf{s})a(\xi)\mathbf{x}_{U,V}a(\xi')I(\mathbf{t})$$

and the Leibniz rule.

6. Gradings

The Alexander grading should probably be obvious...

7. Some more questions and programming problems

Problem 1. Write a computer program to compute (the rank of) $\widehat{HFK}$, using the algorithm described above.

Question 2. Are the algebras $\mathcal{A}(\mathcal{Z}_k)$ formal, i.e., quasi-isomorphic to their homologies?

Question 3. Are the algebras $\mathcal{A}(\mathcal{Z}_k)$ π-formal in the sense of [LT12]?

Problem 4. Each element of $\mathrm{Mor}(P_1, P_2)$ is a morphism from P_1 to P_2. Currently, when working with the chain complex of morphisms my program doesn't remember what morphism they correspond to for very long. It would often be useful to keep track of the exact morphisms as one simplifies the chain complex (and so on). Some reasons to do this:

(1) The cycles in the Mor complex are the chain maps (i.e., homomorphisms). One is often interested in knowing these.
(2) The homology of Mor is the chain maps up to chain homotopy equivalence.
(3) If one kept track of the morphisms throughout, it would be fairly easy to check if P_1 and P_2 are homotopy equivalent: you compute bases for $H_*(\mathrm{Mor}(P_1, P_2))$ and $H_*(\mathrm{Mor}(P_2, P_1))$, and see if you can get $\mathbb{I}_{P_1} \in H_*(\mathrm{Mor}(P_1, P_1))$ and $\mathbb{I}_{P_2} \in H_*(\mathrm{Mor}(P_2, P_2))$ by composing them.
(4) One can compute invariants of 4-dimensional cobordisms by understanding the composition map on the space of morphisms. (We have a paper in progress about this.)

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