Operations Research I: Models and Applications

Quiz for Week 3 (Integer Programming)

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. A manufacturer can sell product 1 at a price of \$5 per unit and product 2 at a price of \$7 per unit. Nine units of raw material are needed to manufacture one unit of product 1, and seven units of raw material are needed to manufacture one unit of product 2. A total of 120 units of raw material are available. The setup costs for producing products 1 and 2 are \$30 and \$40, respectively.

   To formulate an integer program that maximize the total profit, let

$$x_i = \text{sales quantity of product } i \text{ and}$$
$$w_i = \begin{cases} 0 & \text{if } x_i = 0 \\ 1 & \text{if } x_i > 0 \end{cases}$$

   be the decision variables. The formulation with some parts missing (labeled as A, B, and C) is

$$\begin{array}{rl} A & 5x_1 - 30w_1 + 7x_2 - 40w_2 \\ \text{s.t.} & 9x_1 + 7x_2 \leq 120 \\ & x_1 \leq Bw_1 \\ & x_2 \leq \frac{120}{7}w_2 \\ & x_i \geq 0 \quad \forall i = 1, 2. \\ & w_i \in C \quad \forall i = 1, 2. \end{array}$$

   (a) $A = \min$, $B = \frac{120}{9}$, and $C = \{0, 1\}$.
   (b) $A = \max$, $B = \frac{120}{7}$, and $C = \{0, 1\}$.
   (c) $A = \max$, $B = \frac{120}{9}$, and $C = [0, 1]$.
   (d) $A = \min$, $B = \frac{120}{7}$, and $C = [0, 1]$.
   (e) None of the above.

2. Following from the above problem, suppose that now when both products are produced for positive amounts, there is a saving of \$20 in the setup cost (i.e., in total only \$50 must be paid to set up the production processes). If the objective function is modified to

$$\max\ 5x_1 - 30w_1 + 7x_2 - 40w_2 + 20z,$$

   where z is a newly added decision variable, which of the following constraints should be added to make the formulation work? Check all correct answers.

   (a) $z \leq w_1$ and $z \leq w_2$.
   (b) $z \leq w_1 + w_2$.
   (c) $2z \leq w_1 + w_2$.
   (d) $z \geq w_1$ and $z \geq w_2$.
   (e) $2z \geq w_1 + w_2$.

3. Consider the following integer program

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 - 6 \leq M_1 z \\
& x_1 + 2x_2 - 8 \leq M_2(1 - z) \\
& x_1 \leq 10 \\
& x_2 \leq 10 \\
& x_i \geq 0 \quad \forall i = 1, 2 \\
& z \in \{0, 1\},
\end{aligned}
$$

where M_1 and M_2 are parameters and x_1, x_2, and z are variables. The binary variable z is to select at least one constraint to be satisfied.

To make the program work correctly, what are the minimum possible values of M_1 and M_2 that can enable z to do the "at-least-one" selection?

(a) $M_1 = 3$ and $M_2 = 3$.
(b) $M_1 = 999999$ and $M_2 = 999999$.
(c) $M_1 = 24$ and $M_2 = 22$.
(d) $M_1 = 30$ and $M_2 = 30$.
(e) None of the above.

4. Ten jobs should be scheduled on one single machine. The processing times for these jobs are 6, 9, 3, 5, 10, 6, 3, 9, 7, and 10 (in hours). The due times for these jobs are 50, 53, 55, 56, 59, 60, 62, 67, 68, and 70 (in hours). We want to schedule the jobs to minimize total tardiness, which is the completion time minus the due time if positive or zero otherwise.

To formulate the scheduling problem as an integer program, let processing time and due time of job j be p_j and d_j, respectively. The decision variables include

$$
\begin{aligned}
C_j &= \text{completion time of job } j,\ j \in J, \\
T_j &= \text{tardiness of job } j,\ j \in J, \text{ and} \\
z_{ij} &= \begin{cases} 1 & \text{if job } j \text{ is scheduled before job } i \\ 0 & \text{otherwise} \end{cases}, i \in I, j \in J, i < j.
\end{aligned}
$$

Finally, let $M = \sum_{j \in J} p_j$ be a large enough number. An incomplete integer program that has the potential to minimize the total tardiness is

$$
\begin{aligned}
\min \quad & \sum_{j \in J} T_j \\
\text{s.t.} \quad & T_j \geq C_j - d_j & \forall j \in J \\
& C_i + p_j - C_j \leq M z_{ij} & \forall i \in J, j \in J, i < j \\
& T_j \geq 0, C_j \geq 0 & \forall j \in J \\
& z_{ij} \in \{0, 1\} & \forall i \in J, j \in J, i < j.
\end{aligned}
$$

Which of the following constraint should be added into the above program to make it complete?

(a) $C_j + p_i - C_i \leq M z_{ij}$ for all $i \in J, j \in J, i < j$.
(b) $C_j + p_i - C_i \leq M(1 - z_{ij})$ for all $i \in J, j \in J, i < j$.
(c) $C_j + p_i - C_i \geq M z_{ij}$ for all $i \in J, j \in J, i < j$.
(d) $C_j + p_i - C_i \geq M(1 - z_{ij})$ for all $i \in J, j \in J, i < j$.
(e) All of the above.

5. Following from the above problem, suppose that two precedence rules must be followed: Job 1 must be finished before job 5 can start, and either job 5 or job 6 must be finished before job 7 can start. Which of the following constraints should be added into the above program to ensure the precedence rules are satisfied?

   (a) $z_{15} = 1$ and $z_{56} + z_{57} \leq 1$.
   (b) $z_{15} = 1$ and $z_{56} + z_{57} \geq 1$.
   (c) $z_{15} = 0$ and $z_{56} + z_{57} \leq 1$.
   (d) $z_{15} = 0$ and $z_{56} + z_{57} \geq 1$.
   (e) None of the above.