HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

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In recent years the theory of Hilbert space and its linear transformations has come into prominence.[1] It has been recognized to an increasing extent that many of the most important departments of mathematical physics can be subsumed under this theory. In classical physics, for example in those phenomena which are governed by linear conditions—linear differential or integral equations and the like, in those relating to harmonic analysis, and in many phenomena due to the operation of the laws of chance, the essential rôle is played by certain linear transformations in Hilbert space. And the importance of the theory in quantum mechanics is known to all. It is the object of this note to outline certain investigations of our own in which the domain of this theory has been extended in such a way as to include classical Hamiltonian mechanics, or, more generally, systems defining a steady n-dimensional flow of a fluid of positive density.

Consider the dynamical system of n degrees of freedom, the canonical equations of which are formed from the Hamiltonian $H(q, p) = H(q_1, \ldots, q_n, p_1, \ldots, p_n)$, which we will assume to be single-valued, real, and analytic in a certain $2n$-dimensional region R of the real qp-space. The solutions, or equations of motion, are $q_k = f_k(q^\circ, p^\circ, t)$, $p_k = g_k(q^\circ, p^\circ, t)$, $(k = 1, \ldots, n)$, these functions being single-valued, real and analytic for all (q°, p°) in R and for t in a real interval containing $t = 0$ dependent on (q°, p°). It is shown that the transformation $S_t : (q^\circ, p^\circ) \longrightarrow (q, p)$ defined by these equations for suitably restricted t has the formal properties: $S_{t_1}S_{t_2} = S_{t_1 + t_2}$, $S_0 = I$. The system admits the "integral of energy" $H(q, p) = \text{const.}$; hence, if Ω denote a variety $H(q, p) = C$ of points of R, a path curve of S_t having one point on Ω will remain on Ω as long as the curve remains in R. We shall assume that C is such that this is the case for all values of t; this will be the situation, for example, if Ω consists of a closed set of interior points of R. It is shown that under these conditions f_k and g_k are analytic for all (q°, p°) on Ω and for $-\infty < t < +\infty$, so that S_t effectuates a one-parameter group of analytic automorphisms of Ω. Furthermore, S_t leaves invariant the value of a certain integral $\int \rho d\omega$ taken over an arbitrary region of Ω; here, ρ is a positive, single-valued, analytic function on Ω. This is a consequence of the fact that $\int dq_1 \ldots dq_n, dp_1 \ldots dp_n$ is an integral invariant of the system. In the special case where there are m further integrals $F_j(q, p) = C_j$ of the system, we take Ω as the $(2n - m - 1)$-dimensional locus in R of these equations together with $H(q, p) = C$; but there is no further change to be made in the case treated above.

The starting point of our investigation is the N-dimensional variety Ω and the group of automorphisms S_t having the positive integral invariant $\int \rho d\omega$, and these are considered without reference to the problem which gave them origin. Let $\varphi = \varphi(A)$ be a complex-valued function of the point A of Ω, restricted only as follows: (i) φ is single-valued; (ii) φ is measurable; (iii) the Lebesgue integrals $\int_\Omega \rho|\varphi|d\omega$ and $\int_\Omega \rho|\varphi|^2 d\omega$ are finite. The totality of such functions φ constitutes the aggregate of points of a Hilbert space $\mathfrak{H}$: the metric of which is determined by the "inner product" $(\varphi, \psi) = \int_\Omega \rho\varphi\bar{\psi}d\omega$.

Let the transformation U_t of the "points" φ of $\mathfrak{H}$ be defined by $U_t\varphi(A) = \varphi(S_tA)$. This transformation is evidently defined for all φ of $\mathfrak{H}$ and is continuous in φ; and it is defined and continuous for all real t. The kinematical interpretation of U_t is immediately obvious: If t represents the time, S_t specifies the steady flow of a fluid of density ρ occupying the space Ω; and if the values of $\varphi(A)$ be regarded as being attached to the respective points A of the fluid when $t = 0$, in the course of the flow these values will be carried into those of the function $U_{-t}\varphi(A)$. Thus, $U_t\varphi$ has at A the value which φ has at the point S_tA into which A flows after the lapse of the time t.

The transformation U_t is unitary, that is, it is a one to one transformation of $\mathfrak{H}$ into itself which is linear:

$$\begin{aligned} U_t[a\varphi(A) + b\psi(A)] &= a\varphi(S_tA) + b\psi(S_tA) \\ &= aU_t\varphi(A) + bU_t\psi(A) \end{aligned}$$

and such that $(U_t\varphi, U_t\psi) = (\varphi, \psi)$. This last is an easily established consequence of the invariance of $\int \rho d\omega$ under S_t. And since[2] U_t is a one-parameter group of unitary transformations in Hilbert space, it will have on infinitesimal generator P:

$$\left[\frac{\partial}{\partial t} U_t\varphi(A)\right]_{t=0} = iP\varphi(A),$$

$i = \sqrt{-1}$, and P is a self-adjoint linear operator defined in a set of points everywhere dense in $\mathfrak{H}$: $(P\varphi, \psi) = (\varphi, P\psi)$.

In a sufficiently small region of Ω, P may be given a simple interpretation. Let $\xi_1, \ldots, \xi_N$ be a set of Gaussian coördinates of the region, the differential equations of the flow will be $d\xi_k/dt = \Xi_k(\xi_1, \ldots, \xi_N)(k = 1, \ldots, N)$; then for any differentiable $\varphi(A) = \varphi(\xi_1, \ldots, \xi_N)$:

$$P\varphi = i\sum_{k=1}^{N} \Xi_k \frac{\partial\varphi}{\partial\xi_k}.$$

We recognize the Lie operator. The property of $\rho = \rho(\xi_1, \ldots, \xi_N)$ is expressed by the "equation of continuity"

$$\sum_{k=1}^{N} \frac{\partial(\rho\Xi_k)}{\partial\xi_k} = 0,$$

which might have been used to prove formally that P is self-adjoint.

The recent results in the spectral theory of linear operators establish the existence of a "canonical resolution of the identity" E_l corresponding with U_t, which consists of a family of self-adjoint operators defined throughout $\mathfrak{H}$, with the properties that

$$E_{-\infty} = 0,\ E_{\infty} = I,\ E_lE_h = E_hE_l = E_h, \text{ when } h \leqq l \ (^1).$$

This E_l effectuates the spectral resolution of U_t and P:

$$U_t\varphi = \int_{-\infty}^{\infty} e^{itl}\, dE_l\varphi \ (^2),\ P\varphi = \int_{-\infty}^{\infty} l dE_l\varphi.$$

These are Lebesgue-Stieltjes integrals and they exist in the sense of convergence in the mean. Thus, e.g.,

$$\int_{\Omega} \rho \left| \sum_{j=1}^{J} e^{itl_j} (E_{l_j}\varphi - E_{l_{j-1}}\varphi) - U_t\varphi \right|^2 d\omega \longrightarrow 0$$

as the net determined by the points $(l_0, l_1, \ldots, l_J)$ becomes "indefinitely fine" in an arbitrary interval of the l-axis. It follows from a corollary of the Riesz-Fischer Theorem that a set $\Omega_1 \subset \Omega$ exists having $\int_{\Omega_1} \rho d\omega = \int_{\Omega} \rho d\omega$, at every point A of which

$$U_t(A) = \lim \sum_{j=1}^{J} e^{itl_j}[E_{l_j}\varphi(A) - E_{l_{j-1}}\varphi(A)]$$

as the set of values $(l_0, l_1, \ldots, l_j)$ is replaced successively by the members of a sequence of sets, which sequence forms a properly chosen sub-sequence of the totality of sets considered in the previous formula. In particular, since the values on Ω of the dynamical variables q_k and p_k are among the points of $\mathfrak{H}$, it follows that throughout such a set Ω_1 these variables may be represented as limits of trigonometrical sums of the above type.

An evident property of U_t is that, for an arbitrary single-valued function of several variables, F,

$$U_tF(\varphi_1, \varphi_2, \ldots) = F(U_t\varphi_1, U_t\varphi_2, \ldots),$$

an equation which plays an important part in the developments of our theory.

If φ is a characteristic function of $U_t : U_t\varphi = e^{i\lambda t}\varphi$, it will follow from the preceding equation with $F(\varphi) = |\varphi|$ that $U_t|\varphi| = |U_t\varphi| = |\varphi|$; and hence, when $P|\varphi|$ is defined, $P|\varphi| = 0$; thus $|\varphi|$, which remains constant along the path curves of S_t, has the property of a single-valued integral of the equations of motion. In the important case where no such integral exists we may take $\varphi = e^{i\theta}$. Then $\theta = \theta(A)$ is a real function on Ω, in general infinitely multiply-valued, its branches differing by multiples of 2π, each branch being measurable; and since $U_t\varphi = e^{iU_t\theta} = e^{i\lambda t + i\theta}$, we shall have $U_t\theta = \theta + \lambda t$ and when $P\theta$ is defined, $P\theta = \lambda$. Thus, φ and θ are *analytic* along each given path curve. If they are continuous throughout Ω, $\theta =$ const. would represent a "surface of section" of the dynamical system.

If $\varphi_1, \ldots, \varphi_k$ are characteristic functions corresponding to the characteristic numbers $\lambda_1, \ldots, \lambda_k$, we shall have for any set of integers $m_1, \ldots, m_k$,

$$\begin{aligned} U_t(\varphi_1^{m_1}\ldots\varphi_k^{m_k}) &= (U_t\varphi_1)^{m_1}\ldots(U_t\varphi_k)^{m_k} \\ &= e^{i(m_1\lambda_1 + \ldots + m_k\lambda_k)}\varphi_1^{m_1}\ldots\varphi_k^{m_k}, \end{aligned}$$

so that $\varphi = \varphi_1^{m_1}\ldots\varphi_k^{m_k}$ (if it is in $\mathfrak{H}$, as will be true if, as above, $\varphi_j = e^{i\theta_j}$) is a characteristic function of characteristic number $\lambda = m_1\lambda_1 + \ldots + m_k\lambda_k$. If in particular there is the Diophantine relation $m_1\lambda_1 + \ldots + m_k\lambda_k = 0$, φ will be an integral in the above sense of the equations of motion. Hence, in the non-integrable case, $\varphi_1^{m_1}\ldots\varphi_k^{m_k} =$ constant; hence, if the λ's have the "basis" $\lambda_1, \ldots, \lambda_h : \lambda_j = m_1^j\lambda_1 + \ldots + m_h^j\lambda_h$, we have $\varphi_j = l\varphi_1^{m_{1j}}\ldots\varphi_h^{m_{jh}}$.

In the case where U_t has a pure point spectrum, the characteristic functions form a complete system, and the formula $U_t\varphi = \int_{-\infty}^{\infty} e^{itl}dE_l\varphi$ above provides us with a trigonometrical series convergent in the mean on Ω by means of which to represent the coördinates and momenta (q, p).

A study is made of the structure of E_l in the general case, with a view to relating the dynamical properties with the structure of the spectrum. In particular, the group of unitary transformations V of $\mathfrak{H}$ is considered which transform E_l as follows: $VE_lV^{-1} = E_{l+\sigma}$. Various generalizations of the above results are made.

[1] Cf. J. v. Neumann, *Math. Annalen*, **102**, 49–131 (1929).

[2] Cf. M. H. Stone, these PROCEEDINGS, **16**, 173–174 (1930).