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THE SWING UP CONTROL FOR THE PENDUBOT BASED ON ENERGY CONTROL APPROACH

Xin XIN Masahiro KANEDA Toshitaka OKI

Department of Communication Engineering
Faculty of Computer Science and System Engineering
Okayama Prefectural University
111 Kuboki, Soja, Okayama 719-1197, JAPAN
Email: xxin@c.oka-pu.ac.jp

Abstract: This paper studies the energy based control of an underactuated two-link robot called the Pendubot. After having investigated the characteristics of the closed-loop system with the energy based control law (Fantoni *et al.*, 2000) for swinging the Pendubot up, this paper proposes a sufficient condition about parameters in the control law such that the total energy of the Pendubot will converge to the potential energy of its top upright position. This paper gives an answer to the unsolved issue in (Fantoni *et al.*, 2000) whether the total energy of the Pendubot will converge to the potential energy of its top upright position. Moreover, with the aid of the proposed condition, the parameters in the control law are easy to be chosen. *Copyright © 2002 IFAC*

Keywords: Robot control, Stability, Lyapunov methods, Energy control, Attractor

1. INTRODUCTION

The Pendubot as shown in Fig. 1 is a two-degree-of-freedom planar robot with single actuator at the shoulder of the first link; the joint of two links is unactuated and allowed to swing free. In addition to other mechanical systems such as inverted pendulum (Åström and Furuta, 2000), the Acrobot (Spong, 1995), (Berkemeier and Fearing, 1999), (Olfati-Saber and Megretski, 1998), (Zergeroglu *et al.*, 1999), (Brown and Passino, 1997), and brachiating robot (Nakanishi *et al.*, 1999), such robot is used for research as an example of underactuated mechanical systems (Kolmanovsky and McClamroch, 1995) and for control and robot education, (Spong and Block, 1995).

The swing up control problem for the Pendubot is to swing the Pendubot up to its unstable inverted position (top unstable equilibrium) and balance it about the vertical. For solving such problem, (Spong and Block, 1995) uses partial feedback linearization techniques for the swing up control (swing up phase), and performs linearization about the desired equilibrium point and then uses linear quadratic regulator (LQR) or pole placement technique for the balancing control (balancing phase). However, no stability analysis is provided there.

Without using the standard techniques of feedback linearization or partial linearization, (Fantoni *et al.*, 2000) proposes a novel energy based control solution to the swing control problem of the Pendubot. The control algorithm and stability analysis are given based on Lyapunov stability theory. When initial conditions of the Pendubot and parameters in the proposed control law satisfy certain conditions, (Fantoni *et al.*, 2000) shows that the total energy of the Pendubot converges to a constant. If such constant is equal to the potential energy of the position in which both links are at vertical, (Fantoni *et al.*, 2000) shows that link 1 is at rest at the vertical and link 2 moves according to a homoclinic orbit which contains the point corresponding to link 2 being at rest at vertical. Otherwise, (Fantoni *et al.*, 2000) shows that the Pendubot can be brought close to the top unstable equilibrium if the control input torque is small. Furthermore, (Fantoni *et al.*, 2000) shows that the torque can be guaranteed to be small *if a parameter in the control is chosen sufficiently small.*

However, (Fantoni *et al.*, 2000) does not show which of the forementioned two cases will occur for a given initial condition of the Pendubot and given parameters in the control law. Also, for the latter case, i.e., the total energy of the Pendubot converges to a constant which is not equal to the potential energy of its top upright position, (Fantoni *et al.*, 2000) does not make clear how small the parameter should be chosen. If the parameter is chosen to be too small, the solution of the closed-loop systems will converge slowly. In this respect, the latter case is somewhat undesirable. Therefore, with the anxiety for possible occurence of the latter case, it is not easy to choose the parameter appropriately to bring the Pendubot closely to the top equilibrium.

This paper gives an answer to the issue when *the former case will occur*, i.e., when the total energy of the Pendubot will converge to the potential energy of its top upright position. This result implies how to exclude the possibility of occurence of the latter case. With the aid of this result, the control parameter in the control developed in (Fantoni *et al.*, 2000) is easy to be chosen. To explain specifically, first we present simple formulae of the energy of the Pendubot when the latter case occurs. Then, we show that if two parameters in the control law of (Fantoni *et al.*, 2000) satisfy a linear inequality, then the former case will occur. In this way, the characteristics of the solution to closed-loop systems with the energy based control law for swing up phase is illustrated further.

2. PRELIMINARIES

We recall the result of (Fantoni *et al.*, 2000) further for describing our result in the next section.

With the notation and conventions shown in Fig. 1, from (Spong and Vidyasagar, 1989), (Fantoni *et al.*, 2000), the equations of motion of the Pendubot are:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{1}$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \tau = \begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} \tag{2}$$

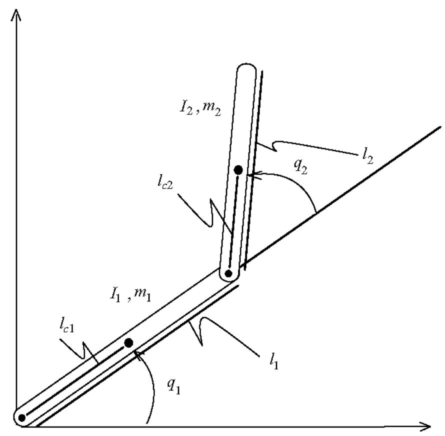

Fig. 1. The Pendubot.

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \theta_2 + \theta_3 \cos q_2 \\ \theta_2 + \theta_3 \cos q_2 & \theta_2 \end{bmatrix} \tag{3}$$

$$C(q,\dot{q}) = \theta_3 \begin{bmatrix} -\dot{q}_2 & -\dot{q}_2 - \dot{q}_1 \\ \dot{q}_1 & 0 \end{bmatrix} \sin q_2 \tag{4}$$

$$G(q) = \begin{bmatrix} \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) \\ \theta_5 g \cos(q_1 + q_2) \end{bmatrix} \tag{5}$$

with

$$\begin{aligned} \theta_1 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\ \theta_2 &= m_2 l_{c2}^2 + I_2, \quad \theta_3 = m_2 l_1 l_{c2} \\ \theta_4 &= m_1 l_{c1} + m_2 l_1, \quad \theta_5 = m_2 l_{c2} \end{aligned}$$

The object of control is to swing the Pendubot up and balance it to

$$q_1 = \pi/2, \quad q_2 = 0 \tag{6}$$

with

$$\dot{q}_1 = 0, \quad \dot{q}_2 = 0 \tag{7}$$

where (6) holds in the meaning of modulo 2π.

The total energy of the Pendubot is given by

$$E = \frac{1}{2}\dot{q}^T D(q)\dot{q} + \theta_4 g \sin q_1 + \theta_5 g \sin(q_1 + q_2) \tag{8}$$

The total of energy when the Pendubot is at rest at the vertical, i.e., (6) and (7) hold, is

$$E_{\text{top}} = \theta_4 g + \theta_5 g \tag{9}$$

Define the following Lyapunov function candidate

$$V = \frac{1}{2}k_E \tilde{E}^2 + \frac{1}{2}k_D \dot{q}_1^2 + \frac{1}{2}k_P \tilde{q}_1^2 \tag{10}$$

where $k_E > 0$, $k_D > 0$, $k_P > 0$ and

$$\tilde{E} = E - E_{\text{top}}, \quad \tilde{q}_1 = q_1 - \pi/2 \tag{11}$$

The main result in (Fantoni *et al.*, 2000) is summarized as follows:

LEMMA 1. (Fantoni *et al.*, 2000) Consider the Pendubot system (1). Take the Lyapunov function candidate (10) with strictly positive constants k_E, k_D and k_P. Provided that for some $\epsilon > 0$

$$|\tilde{E}(0)| < c := \min\left(2\theta_4 g, 2\theta_5 g, \frac{k_D - \epsilon}{k_E\theta_1}\right) \quad (12)$$

$$V(0) \le \frac{1}{2}c^2 k_E \quad (13)$$

hold for initial conditions $q(0)$ and $\dot{q}(0)$. Then the solution of the closed-loop system with the control law

$$\tau_1 = \frac{-k_D F(q,\dot{q}) - (\theta_1\theta_2 - \theta_3^2\cos^2 q_2)(\dot{q}_1 + k_P\tilde{q}_1)}{(\theta_1\theta_2 - \theta_3^2\cos^2 q_2)k_E\tilde{E} + k_D\theta_2} \quad (14)$$

where

$$F(q,\dot{q}) = \theta_2\theta_3(\dot{q}_1+\dot{q}_2)^2\sin q_2 + \theta_3^2\dot{q}_1^2\cos q_2\sin q_2$$
$$-\theta_2\theta_4 g\cos q_1 + \theta_3\theta_5 g\cos q_2\cos(q_1+q_2) \quad (15)$$

converges to the invariant set M given by the homoclinic orbit

$$\frac{1}{2}\theta_2\dot{q}_2^2 = \theta_5 g(1-\cos q_2) \quad (16)$$

with $(q_1, \dot{q}_1) = (\pi/2, 0)$ and the interval

$$(q_1, \dot{q}_1, q_2, \dot{q}_2) = (\pi/2 - \varepsilon, 0, \varepsilon, 0)$$

where $|\varepsilon| < \varepsilon^*$ and ε^* is arbitrarily small.

REMARK 1. Though the condition that k_P is sufficiently small is not stated explicitly in Lemma 1, such condition is found necessary in the proof of Lemma 1 which will be explained briefly as follow.

To begin with, we explain the derivation of control law (14) in (Fantoni *et al.*, 2000). Since the time derivative of V in (10) along (1) under control law (14) satisfies

$$\dot{V} = \dot{q}_1(k_E\tilde{E}\tau_1 + k_D\ddot{q}_1 + k_P\tilde{q}_1) \quad (17)$$

τ_1 is chosen (if possible) such that

$$-\dot{q}_1 = k_E\tilde{E}\tau_1 + k_D\ddot{q}_1 + k_P\tilde{q}_1 \quad (18)$$

which yields

$$\dot{V} = -\dot{q}_1^2 \quad (19)$$

To obtain τ_1 in (14), one just needs to put the formula of $\ddot{q}_1$ calculated from (1) as

$$\ddot{q}_1 = \frac{\theta_2\tau_1 + F(q,\dot{q})}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2} \quad (20)$$

into (18).

Next, note that under conditions (12) and (13), the denominator of the control law (14) is not zero for all time. Indeed, together with (19), we have $V(t) \le V(0)$ and $|\tilde{E}(t)| < c$. Thus,

$$|k_E\tilde{E}(t)| < k_E c < \frac{k_D}{\theta_1} \le \frac{k_D\theta_2}{\theta_1\theta_2 - \theta_3^2\cos^2 q_2} \quad (21)$$

Again under these two conditions, the Pendubot can not get stuck at any equilibrium other than $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (\pi/2, 0, 0, 0)$.

Now, it follows from (19) that $\dot{V}(t) = 0$ and $\dot{q}_1(t) = 0$ holds as $t \to \infty$. In this case,

$$\tilde{q}_1 = \text{constant}, \quad \tilde{E} = \text{constant} \quad (22)$$

Finally, the following two cases are discussed in (Fantoni *et al.*, 2000) separately.

Case 1 $\quad \tilde{E} = 0$

From (22) and (18), we obtain $\tilde{q}_1 = 0$, i.e., $q_1 = \pi/2$. Together with $\dot{q}_1 = 0$, it follows from (8), (9) and (11) that $\tilde{E} = 0$ is equivalent to (16). In this case, the solution of the closed-loop system converges to $\tilde{q}_1 = 0$ and the homoclinic orbit (16).

Case 2 $\quad \tilde{E} \neq 0$

Owing to (22), (18) is reduced to be

$$k_E\tilde{E}\tau_1 + k_P\tilde{q}_1 = 0 \quad (23)$$

Since $\tilde{q}_1$ is constant, (Fantoni *et al.*, 2000) (p. 728) points out that if one chooses k_P close to zero and k_E not too small, then $|\tilde{E}\tau_1|$ will be small. Under the Case 2, (Fantoni *et al.*, 2000) concludes that if k_P is small, τ_1 will be small. Furthermore, (Fantoni *et al.*, 2000) shows that sufficiently small k_P implies that q_2 and $\tilde{q}_1$ are both arbitrarily close to zero.

However, for a given initial condition of the Pendubot and given parameters in the control law, (Fantoni *et al.*, 2000) does not show which of Case 1 and Case 2 will occur. Also, for Case 2, (Fantoni *et al.*, 2000) does not make clear how small one should choose k_P. If k_P the parameter is too small, the solution of the closed-loop systems will convergent slowly. Hence, Case 2 is undesirable from this respect. Therefore, due to incapability of determination of occurence of Case 1 or Case 2, it is not easy to choose k_P appropriately for goal of bringing the Pendubot closely to the top equilibrium.

In what follows, we will show that how to choose the control parameters in (14) such that only Case 1 will occur and Case 2 will not occur at all.

3. CHOICE OF CONTROL PARAMETERS FOR SWING UP PHASE

Suppose that the solution of the closed-loop system of the Pendubot converges to Case 2. Then we can obtain the following result.

LEMMA 2. Consider the Pendubot system (1). Let $t_0 > 0$ be sufficiently large. Suppose that for $t > t_0$, q_1 and $\tilde{E}$ are constant with $\tilde{E} \neq 0$, and $\dot{q}_2$ is bounded, and (23) holds. Then q_2 is also constant, and

$$\tau_1 = \theta_4 g \cos q_1 \tag{24}$$

$$\cos(q_1 + q_2) = 0 \tag{25}$$

Furthermore,

$$\tilde{E} = \theta_4 g(\sin q_1 - 1) \tag{26}$$

for $\sin(q_1 + q_2) = 1$, and

$$\tilde{E} = \theta_4 g(\sin q_1 - 1) - 2\theta_5 g \tag{27}$$

for $\sin(q_1 + q_2) = -1$

Proof. Since q_1 and $\tilde{E}$ are constant with $\tilde{E} \neq 0$, from (23) we know that τ_1 is constant too. Using the fact that q_1 is constant, we obtain the following relations from (1)

$$(\theta_2 + \theta_3 \cos q_2)\ddot{q}_2 - \theta_3 \dot{q}_2^2 \sin q_2$$

$$= \tau_1 - \theta_4 g \cos q_1 - \theta_5 \cos(q_1 + q_2) \tag{28}$$

$$\theta_2 \ddot{q}_2 = -\theta_5 \cos(q_1 + q_2) \tag{29}$$

Putting (29) into (28) yields

$$\theta_3 \ddot{q}_2 \cos q_2 - \theta_3 \dot{q}_2^2 \sin q_2 = \tau_1 - \theta_4 g \cos q_1 =: \alpha_1 \tag{30}$$

which follows that

$$\theta_3 \frac{d(\dot{q}_2 \cos q_2)}{dt} = \alpha_1 \tag{31}$$

Since α_1 is constant, integrating the above equation with respect to time t yields

$$\theta_3 \dot{q}_2 \cos q_2 = \alpha_1 t + \alpha_2, \quad t > t_0 \tag{32}$$

where α_2 is a constant to be determined. Rewriting (32) as

$$\alpha_1 = \frac{\theta_3 \dot{q}_2 \cos q_2 - \alpha_2}{t}, \quad t > t_0 \tag{33}$$

Since (33) holds for $\forall t > t_0$ and $\dot{q}_2$ is bounded, then

$$\alpha_1 = \lim_{t\to\infty} \frac{\theta_3 \dot{q}_2 \cos q_2 - \alpha_2}{t} = 0 \tag{34}$$

which follows from (30) that (24) holds.

Rewriting (32) with $\alpha_1 = 0$, we have

$$\dot{q}_2 \cos q_2 = \frac{d(\sin q_2)}{dt} = \alpha_2/\theta_3 \tag{35}$$

which follows that

$$\sin q_2 = \frac{\alpha_2}{\theta_3} t + \alpha_3, \quad t > t_0 \tag{36}$$

where α_3 is a constant.

$$\alpha_2 = \frac{\theta_3(\sin q_2 - \alpha_3)}{t}, \quad t > t_0$$

Similar to the proof of $\alpha_1 = 0$, we obtain

$$\alpha_2 = \lim_{t\to\infty} \frac{\theta_3(\sin q_2 - \alpha_3)}{t} = 0 \tag{37}$$

Thus,

$$\sin q_2 = \alpha_3 \tag{38}$$

which is a constant. Therefore, q_2 is constant.

Finally, it follows directly from (28) and (29) that (24) and (25) hold. Then, $\sin(q_1 + q_2) = \pm 1$. Consequently, (26) or (27) holds owing to

$$-\tilde{E} = \theta_4 g(\sin q_1 - 1) + \theta_5 g(\sin(q_1 + q_2) - 1) \tag{39}$$

for $\dot{q}_1 = \dot{q}_2 = 0$. ■

Now we are ready to present the main result of this paper.

THEOREM 1. Consider the Pendubot system (1). Take the Lyapunov function candidate (10) with strictly positive constants k_E, k_D and k_P. Provided that

$$V(0) \leq \frac{1}{2} k_E c_1^2 \tag{40}$$

holds for initial conditions $q(0)$ and $\dot{q}(0)$, where

$$c_1 := \min\left(2\theta_5 g, \frac{k_D - \epsilon}{k_E \theta_1}\right) \tag{41}$$

for some $\epsilon > 0$. Define

$$\eta(x) = \frac{(\cos x - 1)\sin x}{x} \tag{42}$$

and

$$\eta^* = \max_{x\in[\pi \quad 3\pi/2]} \eta(x) \tag{43}$$

Under the control law given in (14), if

$$k_P > \eta^* k_E \theta_4^2 g^2 \tag{44}$$

then,

(i) the following relations hold:

$$\lim_{t\to\infty} \tilde{E}(t) = 0,\ \lim_{t\to\infty} \tilde{q}_1(t) = 0,\ \lim_{t\to\infty} V(t) = 0 \quad (45)$$

(ii) the solution of the closed-loop system converges to the invariant set M given by the homoclinic orbit (16) with $(q_1, \dot{q}_1) = (\pi/2, 0)$.

Proof. (i) According to the analysis of Case 1 given in Section 2, it suffices to show that $\tilde{E}$ will converge to 0 under initial condition (40) and control law given in (14).

On the contrary, assume that $\tilde{E}$ will converge to a nonzero constant, i.e., $\tilde{E} \neq 0$. We can use Lemma 2. Note that from (40) $|\tilde{E}(t)| < c_1$ holds for $\forall t \geq 0$. If $\sin(q_1 + q_2) = -1$, we have $|\tilde{E}| = |\theta_4 g(\sin q_1 - 1) - 2\theta_5 g| \geq 2\theta_5 g$ which contradicts $|\tilde{E}(0)| < c_1$.Therefore, $\sin(q_1 + q_2) = 1$. It yields that (26) holds.

Putting (24) and (26) into (23), and letting

$$\Delta(q_1) := k_E \theta_4^2 g^2 (\sin q_1 - 1)\cos q_1 + k_P(q_1 - \pi/2) \quad (46)$$

we have

$$\Delta(q_1) = 0 \quad (47)$$

It is obvious that $q_1 = \pi/2$ is a root of equation $\Delta(q_1) = 0$. In what follows, we will show that $q_1 = \pi/2$ is the unique root of $\Delta(q_1) = 0$ under the condition (44). To begin with, define $x = \tilde{q}_1 = q_1 - \pi/2$ and

$$f(x) = k_0 x - (\cos x - 1)\sin x \quad (48)$$

with $k_0 = k_P/(k_E \theta_4^2 g^2) > 0$. It is easy to see that $q_1 = \pi/2$ is the unique root of $\Delta(q_1) = 0$ if and only if $x = 0$ is the unique root of $f(x) = 0$. Now, since $f(-x) = -f(x)$ holds, it suffices to consider $x > 0$. First, we consider $x \in (0\ \ 2\pi]$ as followings.

For $x \in (0\ \ \pi)$, since $(\cos x - 1)\sin x < 0$, $f(x) \geq k_0 x > 0$, $f(x) = 0$ has no solution.

For $x \in [\pi\ \ 3\pi/2]$, since $(\cos x - 1)\sin x \geq 0$, it is possible that $f(x) = 0$ has solution(s). It is straightforward to see that $f(x)$ has no solution in $x \in [\pi\ \ 3\pi/2]$ if and only if $k_0 > \eta^*$, i.e., (44) holds.

For $x \in [3\pi/2\ \ 2\pi]$, since $(\cos x - 1)\sin x \geq 0$, it is possible that $f(x) = 0$ has solution(s). Note that $f(x)$ has no solution in $x \in [\pi\ \ 3\pi/2]$ if and only if

$$k_0 > \eta_s^* := \max_{x\in[3\pi/2\ \ 2\pi]} \eta(x).$$

Since $\eta_s^* < \eta^*$, $f(x)$ has no solution in $x \in [3\pi/2\ \ 2\pi]$ if (44) holds.

Next, as to $x > 2\pi$, via a similar analysis, we can show that $f(x)$ has no solution in $x \in [3\pi/2\ \ 2\pi]$ if (44) holds.

Therefore, $\Delta(q_1) = 0$ has the unique root $q_1 = \pi/2$ under (44). It yields from (26) that $\tilde{E} = 0$ which contradicts the assumption that $\tilde{E} \neq 0$.

Therefore, we can conclude that $\tilde{E} = 0$. It follows from (23) and (10) that the rest equations in (45) hold.

Consequently, (ii) holds. This completes the proof of Theorem 1. ∎

REMARK 2. Direct numerical calculation of η^* yields that $\eta^* = 0.3146$.

From the above discussion, we have found that the Pendubot cannot get stuck at the equilibrium $(-\pi/2, 0, \pi, 0)$. Note that condition (41) is weaker than condition (12) owing to the fact $c_1 \geq c$.

4. SIMULATION RESULTS

We simulated the Pendubot using the same parameters as those given in (Block, 1996), i.e.,

$$\theta_1 = 0.0799, \quad \theta_2 = 0.0244, \quad \theta_3 = 0.0205$$
$$\theta_4 = 0.42126, \quad \theta_5 = 0.10630, \quad g = 9.8$$

According to (40) and (44), for an initial condition

$$q_1(0) = \frac{\pi}{3},\ q_2(0) = -\frac{\pi}{8},\ \dot{q}_1(0) = 0,\ \dot{q}_2(0) = 0$$

we choose $k_e = 1$, $k_p = 5.5 > \eta^* k_e \theta_4^2 g^2 = 5.3618$ and $k_d = 0.5$. The simulation results under (14) with the above control parameters are depicted in Fig. 2 and Fig. 3.

From Fig. 2, we know that the first link converges to $q_1 = \pi/2$ and the second link remains swinging while approaching closer and closer to the vertical. From Fig. 3, we can observe that the Lyapunov function V and $\tilde{E}$ converges to zero, while τ_1 does not converge. Also, from Fig. 3, the second link converges to homoclinic orbit (16).

In contrast to implicit condition that k_p should be chosen sufficient small, we can determine k_p easier according to (44) together with (40).

5. CONCLUSIONS

This paper has studied the swing up control for the pendubot based on energy based control approach. It has given an answer to the unsolved issue in (Fantoni *et al.*, 2000) whether the total energy of the Pendubot will converge to the potential energy of its top upright position.

After having investigated the characteristics of the closed-loop systems with the energy based control law (Fantoni *et al.*, 2000) for swinging the Pendubot up, this paper has proposed a sufficient condition about parameters in the control law such that the total energy of the Pendubot will converge to the potential energy of its top upright position. It guarantees the solution to closed-loop systems converges to be that link 1 is at rest at the vertical and link 2 moves according to the homoclinic orbit. In this way, the characteristics of the closed-loop systems with the energy based control law has been illustrated clearer. Moreover, with the aid of the proposed condition, the parameters in the control law are easy to be chosen.

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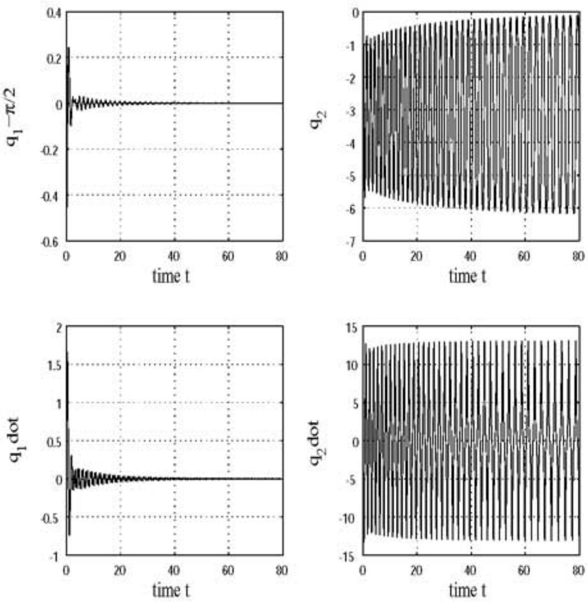

Fig. 2. Time response of states of the Pendubot.

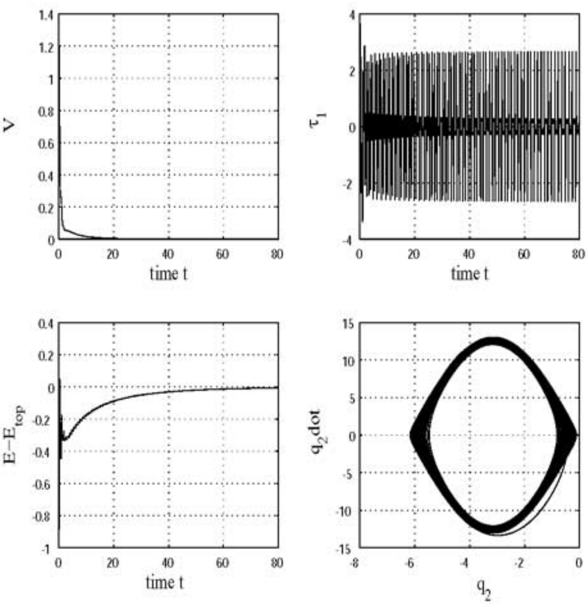

Fig. 3. Time responses of V, τ_1, $\tilde{E}$, and phase plot of q_2.