2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

On Applications of the Spectral Theory of the Koopman Operator in Dynamical Systems and Control Theory

Igor Mezić

***Abstract*— Recent contributions have extended the applicability of Koopman operator theory from dynamical systems to control. Stability theory got reformulated in terms of spectral properties of the Koopman operator [1], providing a nice link between the way we treat linear systems and nonlinear systems and opening the door for the use of classical linear e.g. pole placement theory in the fully nonlinear setting. New concepts such as isostables proved useful in the context of optimal control. Here, using Kato Decomposition we develop Koopman expansion for general LTI systems. We also interpret stable and unstable subspaces in terms of zero level sets of Koopman eigenfunctions. We then utilize conjugacy properties of Koopman eigenfunctions to extend these results to globally stable systems. In conclusion, we discuss how the classical Hamilton-Jacobi-Bellman setting for optimal control can be reformulated in operator-theoretic terms and point the applicability of spectral operator theory in max-plus algebra to it. Geometric theories such as differential positivity have been also related to spectral theories of the Koopman operator [2], in cases when the attractor is a fixed point or a limit cycle, pointing the way to the more general case of quasiperiodic and chaotic attractors.**

I. INTRODUCTION

In the last 20 years, an opertor-theoretic point of view emerged as an appropriate one to treat certain types of problems in dynamical systems in control. It has its roots in 1930's through the work of Koopman and von Neumann [3], [4]. Koopman was working with square-integrable observables on state space. For a dynamical system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

defined on a state-space M (i.e. $\mathbf{x} \in M$ - where we by slight abuse of notation identify a point in a manifold M with its vector representation $\mathbf{x}$ in $\mathbb{R}^m$, m being the dimension of the manifold), where $\mathbf{x}$ is a vector and $\mathbf{F}$ is a possibly nonlinear vector-valued smooth function, of the same dimension as its argument $\mathbf{x}$, denote by $\mathbf{S}^t(\mathbf{x}_0)$ the position at time t of trajectory of (1) that starts at time 0 at point $\mathbf{x}_0$ (see Figure 1). We call $\mathbf{S}^t(\mathbf{x}_0)$ the flow. Denote by $\mathbf{g}$ an arbitrary,

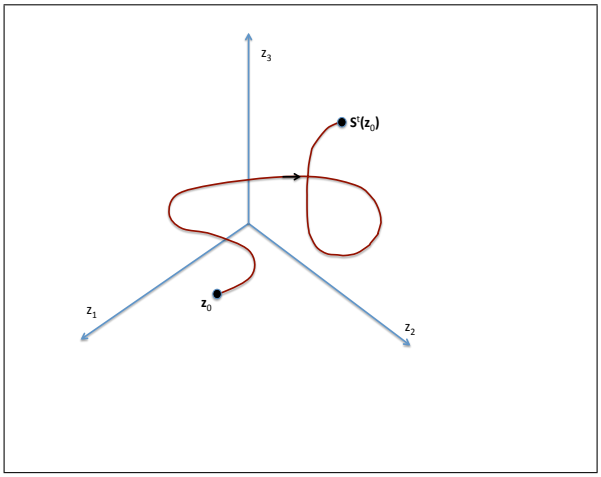

Fig. 1. Trajectory of a dynamical system in $\mathbb{R}^3$.

vector-valued observable from M to $\mathbb{R}^k$. The value of this observable $\mathbf{g}$ that the system trajectory starting from $\mathbf{x}_0$ at

Igor Mezić, Departments of Mechanical Engineering and Mathematics, University of California, Santa Barbara, CA 93106. Email: mezic@engineering.ucsb.edu, Phone: 805-893-7603

978-1-4799-7886-1/15/$31.00 ©2015 IEEE

time 0 sees at time t is

$$\mathbf{g}(t, \mathbf{x}_0) = \mathbf{g}(\mathbf{S}^t(\mathbf{x}_0)). \tag{2}$$

Note that the space of all observables $\mathbf{g}$ is a linear vector space. The family of operators U^t, acting on the space of observables parametrized by time t is defined by

$$U^t\mathbf{g}(\mathbf{x}_0) = \mathbf{g}(\mathbf{S}^t(\mathbf{x}_0)). \tag{3}$$

Thus, for a fixed time τ, U^τ maps the vector-valued observable $\mathbf{g}(\mathbf{x}_0)$ to $\mathbf{g}(\tau, \mathbf{x}_0)$. We will call the family of operators U^t indexed by time t the Koopman operator of the continuous-time system (1). This family was defined for the first time in [3], for Hamiltonian systems. In operator theory, such operators, when defined for general dynamical systems, are often called composition operators, since U^t acts on observables by composing them with the mapping $\mathbf{S}^t$ [5]. In discrete-time the definition is even simpler: if

$$\mathbf{x}' = \mathbf{T}(\mathbf{x}), \tag{4}$$

is a discrete-time dynamical system defined on a set M then the Koopman operator U associated with it is defined by

$$U\mathbf{g}(\mathbf{x}) = \mathbf{g} \circ \mathbf{T}(\mathbf{x}).$$

The operator U is linear, as shown here for the discrete case:

$$\begin{aligned} U(c_1\mathbf{g}_1(\mathbf{x}) + c_2\mathbf{g}_2(\mathbf{x})) &= c_1\mathbf{g}_1(\mathbf{T}(\mathbf{x})) + c_2\mathbf{g}_2(\mathbf{T}(\mathbf{x})) \\ &= c_1U\mathbf{g}_1(\mathbf{x}) + c_2U\mathbf{g}_2(\mathbf{x}). \end{aligned} \tag{5}$$

In the continuous-time case, a similar calculation also shows linearity of members of the Koopman family for each time t.

It was only in the 1990's that potential for wider applications of the operator-theoretic approach has been realized [6], [7]. In this century the trend of applications of this approach has continued, as summarized in [8]. This is partially due to the fact that strong connections have been made between the spectral properties the Koopman operator for dissipative systems and the geometry of the state space. In fact, the hallmark of the work on the operator-theoretic approach in the last two decades is the linkage between geometrical properties of dynamical systems - whose study has been advocated and strongly developed by Poincaré and followers - with the geometrical properties of the level sets of Koopman eigenfunctions [7], [9], [10]. The operator-theoretic approach has been shown capable of detecting object of key importance in geometric study, such as invariant sets, but doing so globally, as opposed to locally as in the geometric approach. It also provides an opportunity for study of high-dimensional evolution equations in terms of dynamical systems concepts [11], [12] via a spectral decomposition, and links with associated numerical methods for such evolution equations [13].

II. Continuous-time Linear Systems with Simple Spectrum

In the case when the sys is linear, and given by $\dot{\mathbf{x}} = A\mathbf{x}$, its matrix eigenvalues are eigenvalues of the associated Koopman operator. The associated Koopman eigenfunctions are given by ([12])

$$\phi_j(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w}_j \rangle, \qquad j = 1, \ldots, n \tag{6}$$

where $\mathbf{w}_j$ are eigenvectors of the adjoint A^* (that is, $A^*\mathbf{w}_j = \lambda_j^c\mathbf{w}_j$), normalized so that $\langle \mathbf{v}_j, \mathbf{w}_k \rangle = \delta_{jk}$, where $\mathbf{v}_j$ is an eigenvector of A, and $\langle \cdot, \cdot \rangle$ denotes an inner product on a linear space M in which the evolution is taking place. This is easily seen by observing

$$\dot{\phi}_j = \langle \dot{\mathbf{x}}, \mathbf{w}_j \rangle = \langle A\mathbf{x}, \mathbf{w}_j \rangle = \langle \mathbf{x}, A^*\mathbf{w}_j \rangle = \lambda_j \langle \mathbf{x}, \mathbf{w}_j \rangle = \lambda_j\phi_j, \tag{7}$$

and thus $\phi_j(t, \mathbf{x}_0) = U^t \phi_j(\mathbf{x}_0) = \exp(\lambda_j t)\phi_j(\mathbf{x}_0)$. Now, for any $\mathbf{x} \in M$, as long as $\mathbf{A}$ has a full set of eigenvectors at distinct eigenvalues λ_j, we may write

$$\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{w}_j \rangle \, \mathbf{v}_j = \sum_{j=1}^{n} \phi_j(\mathbf{x}) \mathbf{v}_j.$$

Thus,

$$\begin{aligned} U^t \mathbf{x}(\mathbf{x}_0) &= \mathbf{x}(t) = \exp(At)\mathbf{x}_0 \\ &= \sum_{j=1}^{n} \langle \exp(At)\mathbf{x}_0, \mathbf{w}_j \rangle \, \mathbf{v}_j \\ &= \sum_{j=1}^{n} \langle \mathbf{x}_0, \exp(A^* t)\mathbf{w}_j \rangle \, \mathbf{v}_j \\ &= \sum_{j=1}^{n} \exp(\lambda_j t) \, \langle \mathbf{x}_0, \mathbf{w}_j \rangle \, \mathbf{v}_j, \\ &= \sum_{j=1}^{n} \exp(\lambda_j t) \phi_j(\mathbf{x}_0) \mathbf{v}_j, \end{aligned} \tag{8}$$

where $\mathbf{x}(\mathbf{x}_0)$ is the vector function that associates Cartesian coordinates with a point $\mathbf{x}_0$ (the initial condition) in state space. This is an expansion of the dynamics of observables - in this case the coordinate functions $\mathbf{x}(\mathbf{x}_0)$ in terms of spectral quantities (eigenvalues, eigenfunctions) of the Koopman family U^t. Something interesting comes out: the quantity we know as eigenvector $\mathbf{v}_j$ is *not* associated with the Koopman operator, but rather with the observable - if we changed the observable to, for example $\mathbf{y} = C\mathbf{x}$, C being an $m \times n$ matrix, then the expansion would read

$$U^t \mathbf{y}(\mathbf{x}_0) = \sum_{j=1}^{n} \exp(\lambda_j t) \phi_j(\mathbf{x}_0) C \mathbf{v}_j, \tag{9}$$

and we would call $\phi_j(\mathbf{x}_0) C\mathbf{v}_j$ the $j-th$ Koopman modes of observable $\mathbf{y}$. Assume now that the space of observables on $\mathbb{R}^n$ we are considering is the space of complex linear combinations of $\mathbf{x}(\mathbf{x}_0)$. Then, $\phi_j(\mathbf{x}_0) C\mathbf{v}_j$ is the projection of the observable $C\mathbf{x}$ onto the eigenspace of the Koopman family spanned by the eigenfunction $\phi_j(\mathbf{x}_0) = \langle \mathbf{x}, \mathbf{w}_j \rangle$.

Note that what changed between expansions (8) and (9) is the vector $\mathbf{v}_j$. On the other hand, the eigenvalues and eigenfunctions used in the expansion do not change. Thus, what changes with change in observables is their contribution to the overall evolution in the observable, encoded in $C\mathbf{v}_j$. These properties persist in the fully nonlinear case, with the change that the spectral expansion is typically infinite and can have a continuous spectrum part.

Note also that the evolution of coordinate functions can be written in terms of evolution of Koopman eigenfunctions, by

$$U^t \mathbf{x}(\mathbf{x}_0) = \sum_{j=1}^{n} \phi_j(t, \mathbf{x}_0) \mathbf{v}_j. \tag{10}$$

III. Continuous-time Linear Systems: General Case

In general, the matrix A can have repeated eigenvalues and this can lead to a lack of eigenvectors. Recall that algebraic multiplicity of an eigenvalue λ_j of A is the exponent (m_j) of the polynomial factor $(\lambda - \lambda_j)^{m_j}$ of the characteristic polynomial $\det(A - \lambda I)$. In other words, it is the number of repeat appearences of λ_j as a zero of the characteristic polynomial. An eigenvalue that repeats m_j times does not necessarily have m_j eigenvectors associated with it. Indeed - the algebraic multiplicity m_j of λ_j is bigger than or equal to geometric multiplicity, which is the number of eigenvectors associated with λ_j. A very elegant way of seeing this is to use the so-called Kato Decomposition. Kato Decomposition is an example of a spectral decomposition, where a linear operator is decomposed into a sum of terms consisting of scalar multiples of projection and nilpotent operators. For a finite-dimensional linear operator A it reads

$$U = \sum_{h=1}^{s} \lambda_h P_h + D_h, \tag{11}$$

Each P_h is a projection operator on the *algebraic eigenspace* M_h that can be defined as the null space of $(U - \lambda_h I)^{m_h}$,

and D_h is a nilpotent operator. We now use the spectral decomposition theorem for finite-dimensional linear operators to provide an easy, elegant proof of Hirsch-Smale theorem [14] on solutions of ordinary differential equations. Consider a linear ordinary differential equation on $\mathbb{R}^m$, $\dot{\mathbf{x}} = A\mathbf{x}$ where A is an $n \times n$ matrix. It is well-known that the solution of this equation reads $\mathbf{x}(t) = \exp(At)\mathbf{x}_0$, where $\mathbf{x}_0$ is the initial condition. The exponentiation of the matrix A reads

$$\exp(At) = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}. \tag{12}$$

Now, from the Kato decomposition we obtain

$$A^k = \sum_{h=1}^{s} \lambda_h^k P_h + \sum_{j=1}^{k} \binom{k}{j} \sum_{h=1}^{s} \lambda_h^{k-j} D_h^j,$$

where $\lambda_h, h = 1, ..., s$ are eigenvalues of A. Now we rewrite $\exp(At)$ as

$$\begin{aligned} & I + \sum_{k=1}^{\infty} \frac{\sum_{h=1}^{s} \lambda_h^k P_h + \sum_{j=1}^{k} \binom{k}{j} \sum_{h=1}^{s} \lambda_h^{k-j} D_h^j}{k!} t^k, \\ = & \sum_{h=1}^{s} P_h \sum_{k=0}^{\infty} \frac{\lambda_h^k t^k}{k!} + \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{k} \binom{k}{j} \sum_{h=1}^{s} \lambda_h^{k-j} D_h^j}{k!} t^k, \\ = & \sum_{h=1}^{s} e^{\lambda_h t} P_h + \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{k} \binom{k}{j} \sum_{h=1}^{s} \lambda_h^{k-j} D_h^j}{k!} t^k, \end{aligned} \tag{13}$$

Note now that

$$t^l e^{\lambda_h t} = t^l \sum_{k=0}^{\infty} \frac{\lambda_h^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{\lambda_h^k t^{k+l}}{k!} = \sum_{m=l}^{\infty} \frac{\lambda_h^{m-l} t^m}{(m-l)!}.$$

We can rewrite the second sum in the last line of (13) as

$$\begin{aligned} & \sum_{j<m_h} \sum_{k=j}^{\infty} \binom{k}{j} \sum_{h=1}^{s} \lambda_h^{k-j} D_h^j \frac{t^k}{k!} \\ = & \sum_{h=1}^{s} \sum_{j<m_h} \sum_{k=j}^{\infty} \binom{k}{j} \lambda_h^{k-j} D_h^j \frac{t^k}{k!}, \end{aligned} \tag{14}$$

leading further to

$$\begin{aligned} = & \sum_{h=1}^{s} \sum_{j<m_h} \sum_{k=j}^{\infty} \frac{k \cdot (k-1) \cdot ... \cdot (k-j+1)}{j!} \lambda_h^{k-j} D_h^j \frac{t^k}{k!}, \\ = & \sum_{h=1}^{s} \sum_{j<m_h} \frac{D_h^j}{j!} \sum_{k=j}^{\infty} \lambda_h^{k-j} \frac{t^k}{(k-j)!}, \\ = & \sum_{h=1}^{s} \sum_{j<m_h} \frac{D_h^j}{j!} t^j e^{\lambda_h t}. \end{aligned} \tag{15}$$

Thus we get

$$\exp(At) = \sum_{h=1}^{s} (e^{\lambda_h t} P_h + \sum_{j<m_h} \frac{t^j e^{\lambda_h t}}{j!} D_h^j), \tag{16}$$

Let us now connect this expansion to the formula we obtained previously, given by (8). In that case, we assumed that algebraic multiplicities of all eigenvalues are 1, and there is a full set of associated eigenvectors $\mathbf{v}_j$. Thus, the nilpotent part $D_j = 0$, and the projection of a vector $\mathbf{x}_0$ on the $j-th$ eigenspace is $P_j \mathbf{x}_0 = \langle \mathbf{x}_0, \mathbf{w}_j \rangle \mathbf{v}_j = \phi_j(\mathbf{x}_0)\mathbf{v}_j$. Using this with (16), we obtain (8).

More generally, let the dimension of each geometric eigenspace be equal to 1, let $j = 1, ..., s$ be the counter of distinct eigenvalues of A and $m_1, ..., m_s$ their multiplicities (or equivalently dimensions of algebraic eigenspaces corresponding to eigenvalues). Label the basis of the generalized eigenspace E_j by $\mathbf{v}_j^1, ..., \mathbf{v}_j^{m_j}$, where $\mathbf{v}_j^i$ are chosen so that $(A - \lambda_j I)^i \mathbf{v}_j^i = 0$. In other words, $\mathbf{v}_j^1$ is a standard eigenvector of A at λ_j and the generalized eigenvectors $\mathbf{v}_j^i, i = 2, ..., m_j$ satisfy $A\mathbf{v}_j^i = \lambda_j \mathbf{v}_j^i + \mathbf{v}_j^{i-1}$. Now let $\phi_j^i(\mathbf{x}) = \langle \mathbf{x}, \mathbf{w}_j^i \rangle$ where $\mathbf{w}_j^i$ is the dual basis vector to $\mathbf{v}_j^i$ and satisfies

$$\begin{aligned} A^* \mathbf{w}_j^i &= \lambda_j^c \mathbf{w}_j^i + \mathbf{w}_j^{i+1}, i < m_j. \\ A^* \mathbf{w}_j^i &= \lambda_j^c \mathbf{w}_j^i, i = m_j. \end{aligned} \tag{17}$$

Note that for $i > 1$.

$$\begin{aligned}\dot{\phi}_j^i(\mathbf{x}) &= \left\langle \dot{\mathbf{x}}, \mathbf{w}_j^i \right\rangle = \left\langle A\mathbf{x}, \mathbf{w}_j^i \right\rangle \\ &= \left\langle \mathbf{x}, A^*\mathbf{w}_j^i \right\rangle = \left\langle \mathbf{x}, \lambda_j^c \mathbf{w}_j^i + \mathbf{w}_j^{i+1} \right\rangle \\ &= \lambda_j \left\langle \mathbf{x}, \mathbf{w}_j^i \right\rangle + \left\langle \mathbf{x}, \mathbf{w}_j^{i+1} \right\rangle \\ &= \lambda_j \phi_j^i(\mathbf{x}) + \phi_j^{i+1}(\mathbf{x}). \end{aligned} \tag{18}$$

We call $\phi_j^i(\mathbf{x}), 1 \leq i < m_j$ generalized eigenfunctions of the Koopman operator at eigenvalue λ_j.

Example 1 *To justify the name generalized eigenfunctions, consider the following simple example: let $m_j = 2$. Then $\dot{\phi}_j^1 = \lambda_j \phi_j^1 + \phi_j^2$, where ϕ_2^1 is an eigenfunction of U^t at λ_j satisfying $\dot{\phi}_j^2 = \lambda_j \phi_j^2$. Then*

$$(d/dt - \lambda_j I)^2 \phi_j^1 = 0.$$

Thus, ϕ_j^1 is in the nullspace of the differential operator $(\frac{d}{dt} - \lambda_j I)^2$.

Expanding from the example, for m_j arbitrary, generalized eigenfunctions ϕ satisfy $(d/dt - \lambda_j I)^{m_j}\phi = 0$. By integrating (18), the time evolution of the generalized eigenfunctions reads

$$\phi_j^i(t) = \sum_{n=0}^{m_j - i} \frac{t^n}{n!} e^{\lambda_j t} \sum_{l=m_j}^{i+n} \phi_j^l(0). \tag{19}$$

(in fact by directly differentiating (19), one can easily find out that it satisfies (18)). Now writing

$$\mathbf{x} = \sum_{j=1}^{s} \sum_{i=1}^{m_j} \left\langle \mathbf{x}, \mathbf{w}_j^i \right\rangle \mathbf{v}_j^i,$$

we get

$$\begin{aligned} U^t \mathbf{x}(\mathbf{x}_0) &= \mathbf{x}(t) = \exp(At)\mathbf{x}_0 \\ &= \sum_{j=1}^{s} \sum_{i=1}^{m_j} \left\langle \exp(At)\mathbf{x}_0, \mathbf{w}_j^i \right\rangle \mathbf{v}_j^i \end{aligned} \tag{20}$$

leading further to

$$\begin{aligned} &= \sum_{j=1}^{s} \sum_{i=1}^{m_j} \left\langle \mathbf{x}_0, \exp(A^*t)\mathbf{w}_j^i \right\rangle \mathbf{v}_j^i \\ &= \sum_{j=1}^{s} \sum_{i=1}^{m_j} e^{\lambda_j t} \Big(\sum_{k=i}^{m_j} \frac{t^{k-i}}{(k-i)!} \left\langle \mathbf{x}_0, \mathbf{w}_j^k \right\rangle \Big) \mathbf{v}_j^i, \\ &= \sum_{j=1}^{s} e^{\lambda_j t} \sum_{i=1}^{m_j} \Big(\sum_{k=i}^{m_j} \frac{t^{k-i}}{(k-i)!} \phi_j^k(\mathbf{x}_0) \Big) \mathbf{v}_j^i \\ &= \sum_{j=1}^{s} \left[e^{\lambda_j t} \left(\sum_{k=1}^{m_j} \phi_j^k(\mathbf{x}_0) \mathbf{v}_j^k \right) \right. \\ &+ \left. \sum_{i=1}^{m_j - 1} \frac{t^i}{i!} e^{\lambda_j t} \left(\sum_{k=i+1}^{m_j} \phi_j^k(\mathbf{x}_0) \mathbf{v}_j^{k-i} \right) \right]. \end{aligned} \tag{21}$$

We connect the formula we just obtained with the expansion (16). Comparing the two, it is easy to see that

$$P_j \mathbf{x} = \sum_{k=1}^{m_j} \phi_j^k(\mathbf{x}) \mathbf{v}_j^k, \tag{22}$$

and

$$D_j^i \mathbf{x} = \sum_{k=i+1}^{m_j} \phi_j^k(\mathbf{x}) \mathbf{v}_j^{k-i} \tag{23}$$

The above discussion also shows that, as long as we restrict the space of observables on $\mathbb{R}^m$ to linear ones, $f(\mathbf{x}) = \langle \mathbf{c}, \mathbf{x} \rangle$, where $\mathbf{c}$ is a vector in $\mathbb{R}^m$, then the generalized eigenfunctions and associated eigenvalues of the Koopman operator are obtainable in a straightforward fashion from the standard linear analysis of A and its transpose.

Generalized eigenfuctions are also preserved under conjugation: for $i > 1$ we have

$$\begin{aligned} \textstyle\sum_{n=0}^{m_j - i} \quad & \frac{t^n}{n!} e^{\lambda_j t} \sum_{l=m_j}^{i+n} \phi_j^l(\mathbf{h}(\mathbf{y})) = U_S^t \phi_j^i(\mathbf{h}(\mathbf{y})) \\ = \quad & \phi_j^i(S^t \mathbf{h}(\mathbf{y})) = \phi_j^i(\mathbf{h}(T^t \mathbf{y})) = U_T^t \phi_j^i(\mathbf{h}(\mathbf{y})) \end{aligned}$$

Thus, $\phi_j^i \circ \mathbf{h}$ is a function that evolves in time according to the evolution equation (19) and thus is a a generalized eigenfunction. Together with the fact this is easily proved for ordinary eigenfunctions, we get

Proposition 2 *Let $\mathbf{S}^t, U_S^t$ be the family of mappings and the Koopman operator associated with $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\mathbf{T}^t, U_T^t$ a family of mappings and the Koopman operator associated with $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$. In addition, let $\mathbf{h}$ be a mapping such that $\mathbf{S}^t\mathbf{h}(\mathbf{y}) = \mathbf{h}(\mathbf{T}^t(\mathbf{y}))$, i.e. the two dynamical systems are conjugate. If ϕ is a (generalized) eigenfunction at λ of U_S^t, then the composition $\phi \circ \mathbf{h}$ is a (generalized) eigenfunction of U_T^t at λ.*

It is easy to see that the most general case, in which dimension of geometric eigenspaces is not necessarily 1, is easily treated by considering geometric eigenspace of dimension say 2 as two geometric eigenspaces of dimension 1. Keeping in mind that these correspond to -numerically - the same eigenvalue, we can define generalized eigenvectors corresponding to each eigenvector in - now separate - 1-dimensional geometric eigenspaces.

IV. Stable, Unstable and Center Subspace

Let us recall the definition of stable, unstable and center subspaces of $\dot{\mathbf{x}} = A\mathbf{x}$: the *stable* subspace of the fixed point 0 is the location of all the points that go to the fixed point at the origin as $t \to \infty$. The stable subspace is classically obtained as the span of (generalized) eigenvectors corresponding to eigenvalues of negative real part. In the same way, the *unstable* subspace of the fixed point 0 is the location of all the points that go to the fixed point at the origin as $t \to -\infty$, and is classically obtained as the span of (generalized) eigenvectors corresponding to eigenvalues of positive real part. The *center* subspace is usually not defined by its asymptotics (but could be, as we will see that it is the location of all the points in the state space that stay at the finite distance from the origin as $t \to \infty$ or grow slowly (algebraically)), but rather as the span of (generalized) eigenvectors associated with eigenvalues of zero real part.

Looking at equation (8) it is interesting to note that one can extract the geometrical location of stable, unstable and center subspaces from the eigenfunctions of the Koopman operator. Order eigenvalues $\lambda_j, j = 1, .., n$ from the largest to the smallest, where we do not pay attention to the possible repeat of eigenvalues. Let s, c, u be the number of negative real part eigenvalues, 0 and positive real part eigenvalues. Let $\lambda_1, ...\lambda_u$ be positive real part eigenvalues, $\lambda_{u+1}, ..., \lambda_{u+c}$ 0 real part eigenvalues, and $\lambda_{u+c+1}, ..., \lambda_s$ be negative real part eigenvalues. Then the joint level set of (generalized) eigenfunctions $\phi_1 = 0, ..., \phi_{u+c} = 0$ is the stable subspace E^s, $\phi_1 = 0, ..., \phi_u = 0, \phi_{u+c+1} = 0..., \phi_{u+c+s} = 0$ is the center subspace E^c, and $\phi_{u+1} = 0, ..., \phi_{u+c+s} = 0$ the unstable subspace E^u. This generalizes nicely to non-linear systems (see below) while we know that the standard definition, where e.g. the unstable space is the span of $\mathbf{v}_1, ..., \mathbf{v}_u$ does not - even when the system is of the form $\dot{\mathbf{x}} = A\mathbf{x} + \epsilon\mathbf{f}$ for $\mathbf{f}$ bounded, $\mathbf{f}(0) = 0$, and ϵ small, we can only show existence of the unstable, stable and center manifolds that are tangent to the unstable, stable and center subspace E^u, E^s, E^c, respectively. So, the joint zero level sets of Koopman eigenfunctions define dynamically important geometric objects- invariant subspaces - of linear dynamical systems. This is not an isolated incident. Rather, in general the *level sets of Koopman eigenfunctions* reveal important geometrical information about the underlying dynamical system.

V. Systems with Globally Stable Equilibria

Non-degenerate linear systems (i.e. those with $\det A \neq 0$) have a single equilibrium at the origin as the distinguished solution. As the natural first extension to nonlinear realm, it is interesting to consider a class of nonlinear systems that (at least locally) have an equilibrium as the only special solution, and consider what spectral theory of the Koopman operator

for such systems can say.

For systems that are stable to an equilibrium from an open attracting set, we develop in this section a theory that strongly resembles that of linear systems - as could be expected once it is understood how Koopman eigenfunctions change under conjugacy - the topic we discuss first. Geometric notions that were discussed in the previous LTI context, such as stable, unstable and center manifolds are developed in this nonlinear context.

VI. Koopman Eigenfunctions Under Conjugacy

Spectral properties of the Koopman operator transform nicely. Consider a nonlinear system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$, globally stable (in its domain of definition) to an equilibrium. Let $\mathbf{S}^t, U_S^t$ be the family of mappings and the Koopman operator associated with $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\mathbf{T}^t, U_T^t$ a family of mappings and the Koopman operator associated with $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$. Assume that $\phi(\mathbf{x})$ is an eigenfunction of U_S^t associated with eigenvalue λ. In addition, let $\mathbf{h}$ be a mapping such that $\mathbf{S}^t\mathbf{h}(\mathbf{y}) = \mathbf{h}(\mathbf{T}^t(\mathbf{y}))$, i.e. the two dynamical systems are conjugate. Then we have

$$\exp(\lambda t)\phi\circ\mathbf{h}(\mathbf{y}) = \phi(\mathbf{S}^t\mathbf{h}(\mathbf{y})) = \phi(\mathbf{h}(\mathbf{T}^t(\mathbf{y}))) = U_T^t(\phi\circ\mathbf{h}(\mathbf{y})), \tag{24}$$

i.e. if ϕ is an eigenfunction at λ of U_S^t, then the composition $\phi \circ \mathbf{h}$ is an eigenfunction of U_T^t at λ. In other words, if we can find a global conjugacy to a linear system then the spectrum of the Koopman operator can be determined from the spectrum of the linearization at the fixed point.

VII. Eigenmode Expansion

The Koopman semigroup of operators U^t describes the evolution of a (vector-valued) *observable* $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{C}^m$ along the trajectories of the system and is rigorously defined as the composition $U^t\mathbf{f}(\mathbf{x}) = \mathbf{f} \circ \phi(t, \mathbf{x})$. In particular, for an analytic observable, the spectral decomposition of the operator yields [15]

$$U^t\mathbf{f}(\mathbf{x}) = \sum_{\{k_1,\ldots,k_n\}\in\mathbb{N}^n} s_1^{k_1}(\mathbf{x})\cdots s_n^{k_n}(\mathbf{x})\overline{\mathbf{v}}_{k_1\cdots k_n}\, e^{(k_1\lambda_1+\cdots+k_n\lambda_n)t}. \tag{25}$$

The functions $s_j(\mathbf{x})$, $j = 1,\ldots,n$, are the smooth eigenfunctions of U^t associated with the eigenvalues λ_j, i.e.

$$U^t s_j(\mathbf{x}) = s_j(\phi(t,\mathbf{x})) = s_j(\mathbf{x})e^{\lambda_j t}, \tag{26}$$

and the vectors $\overline{\mathbf{v}}_{k_1\cdots k_n}$ are the so-called Koopman modes [16], i.e. the projections of the observable $\mathbf{f}$ onto $s_1^{k_1}(\mathbf{x})\cdots s_n^{k_n}(\mathbf{x})$. For the particular observable $\mathbf{f}(\mathbf{x}) = \mathbf{x}$, (25) corresponds to the expression of the flow and can be rewritten as

$$U^t\mathbf{x} = \mathbf{x}^* + \sum_{j=1}^{n} s_j(\mathbf{x})\mathbf{v}_j\, e^{\lambda_j t} + \sum_{\substack{\{k_1,\ldots,k_n\}\in\mathbb{N}_0^n \\ k_1+\cdots+k_n>1}} s_1^{k_1}(\mathbf{x})\cdots s_n^{k_n}(\mathbf{x})\cdot\mathbf{v}_{k_1\cdots k_n} e^{(k_1\lambda_1+\cdots+k_n\lambda_n)t}$$

The first part of the expansion is similar to the linear case. We recognize here that the operator formalism we are developing leads to a striking realization: the only difference in the representation of dynamics of linear and nonlinear systems with equilibria on state space is that in the linear case, the expansion is finite, while in the nonlinear case it is infinite. In linear systems, we are expandding the state $\mathbf{x}(p)$ which itself is a linear function on the state-space, in terms of eigenfunctions of the Koopman operator that are also linear in state $\mathbf{x}(p)$. In the nonlinear case, this changes - the Koopman eigenfunctions are *nonlinear* as functions of state $\mathbf{x}(p)$ and the expansion is infinite. It is also useful to observe that the expansion is *asymptotic* in nature - namely, there are terms that describe evolution close to an equilibrium point, and terms that have higher expansion or decay rates.

VIII. Stable, Unstable and Center Manifolds

Let $\mathbf{x}^*$ be a unique equilibrium point of an analytic nonlinear system, with eigenvalues $\lambda_j, j = 1, .., s$ associated with the linearization at $\mathbf{x}^*$. As shown in the previous section, the Koopman operator associated with the system has spectrum composed of products of the *core* eigenvalues $\lambda_j, j = 1, .., s$. Just like in linear systems, order $\lambda_j, j = 1, .., n$ from the largest to the smallest, where we do not pay attention to the possible repeat of eigenvalues. Let s, c, u be the number of negative real part eigenvalues, 0 and positive eigenvalues. Let $\lambda_1, ...\lambda_u$ be positive real part eigenvalues, $\lambda_{u+1}, ..., \lambda_{u+c}$ 0 real part eigenvalues, and $\lambda_{u+c+1}, ..., \lambda_{u+c+s}$ be negative real part eigenvalues. Then the joint level set of (generalized) eigenfunctions $\phi_1 = 0, ..., \phi_{u+c} = 0$ is the stable manifold W^s, $\phi_1 = 0, ..., \phi_u = 0, \phi_{u+c+1} = 0..., \phi_s = 0$ is the center manifold W^c, and $\phi_{u+1} = 0, ..., \phi_s = 0$ the unstable manifold W^u of the fixed point $\mathbf{x}^*$.

IX. Conclusions

We have applied Koopman operator theory to LTI and nonlinear globally stable systems. The formalism reveals an interesting way to characterize stable, center and unstable subspaces in both the linear and nonlinear case based on the same concept: the zero level sets of certain Koopman eigenfunctions. It is interesting to consider the extentsion of Koopman operator theory in the optimal control context. The generator of the Koopman operator family for a vector field $\mathbf{f}$ is $\mathbf{f} \cdot \nabla$. In the case of uptime control, the operator of interest is $\max(\mathbf{f} + \mathbf{u}) \cdot \nabla$. This is not a linear operator in the usual function algebra, however it is linear in the so-called max plus algebra [17]. This opens up the possibility for using spectral expansion and geometric methods in level set theory of (appropriately modified) operator eigenfunctions in optimal control theory.

References

[1] Alexandre Mauroy and Igor Mezić. A spectral operator-theoretic framework for global stability. In *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*, pages 5234–5239. IEEE, 2013.

[2] A Mauroy, F Forni, and R Sepulchre. An operator-theoretic approach to differential positivity. *arXiv preprint arXiv:1504.01548*, 2015.

[3] B.O. Koopman. Hamiltonian systems and transformation in Hilbert space. *Proceedings of the National Academy of Sciences of the United States of America*, 17(5):315, 1931.

[4] BO Koopman and J v Neumann. Dynamical systems of continuous spectra. *Proceedings of the National Academy of Sciences of the United States of America*, 18(3):255, 1932.

[5] RK Singh and JS Manhas. *Composition operators on function spaces*. Number 179. North Holland, 1993.

[6] A. Lasota and M. C. Mackey. *Chaos, Fractals and Noise*. Springer-Verlag, New York, 1994.

[7] I. Mezić. *On Geometrical and Statistical Properties of Dynamical Systems: Theory and Applications*. PhD thesis, California Institute of Technology, 1994.

[8] Marko Budisić, Ryan Mohr, and Igor Mezić. Applied koopmanism. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(4):047510, 2012.

[9] Alexandre Mauroy and Igor Mezić. On the use of fourier averages to compute the global isochrons of (quasi) periodic dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(3):033112, 2012.

[10] Alexandre Mauroy, Igor Mezić, and Jeff Moehlis. Isostables, isochrons, and koopman spectrum for the action–angle representation of stable fixed point dynamics. *Physica D: Nonlinear Phenomena*, 261:19–30, 2013.

[11] Igor Mezić. Spectral properties of dynamical systems, model reduction and decompositions. *Nonlinear Dynamics*, 41(1-3):309–325, 2005.

[12] C.W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D.S. Henningson. Spectral analysis of nonlinear flows. *Journal of Fluid Mechanics*, 641(1):115–127, 2009.

[13] Peter J Schmid. Dynamic mode decomposition of numerical and experimental data. *Journal of Fluid Mechanics*, 656:5–28, 2010.

[14] M.W. Hirsch and S. Smale. *Differential equations, dynamical systems, and linear algebra*. Academic Press Inc, 1974.

[15] I. Mezić. Analysis of fluid flows via spectral properties of Koopman operator. *Annual Review of Fluid Mechanics*, 45, January 2013.

[16] C. W. Rowley, I. Mezic, S. Bagheri, P. Schlatter, and D. S. Henningson. Spectral analysis of nonlinear flows. *Journal of Fluid Mechanics*, 641:115–127, 2009.

[17] William M McEneaney. *Max-plus methods for nonlinear control and estimation*. Springer Science & Business Media, 2006.